Probab. Theory Relat. Fields (2009) 143:285–328
DOI 10.1007/s00440-007-0127-5

Stochastic heat equation driven by fractional noise and local time

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Received: 16 February 2007 / Revised: 26 November 2007 / Published online: 4 January 2008

Abstract The aim of this paper is to study the d-dimensional stochastic heat equation with a multiplicative Gaussian noise which is white in space and has the covariance of a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ in time. Two types of equations are considered. First we consider the equation in the Itô-Skorohod sense, and later in the Stratonovich sense. An explicit chaos expansion for the solution is obtained. On the other hand, the moments of the solution are expressed in terms of the exponential moments of some weighted intersection local time of the Brownian motion.

Mathematics Subject Classification (2000) 60H15 · 60H07

1 Introduction

This paper deals with the d-dimensional stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamond \frac{\partial^2 W^H}{\partial t \partial x} \tag{1.1}$$

driven by a Gaussian noise W^H which is a white noise in the spatial variable and a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ in the time variable (see

Y. Hu is supported by the National Science Foundation under DMS0504783.
D. Nualart is supported by the National Science Foundation under DMS0604207.

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(2.1) in the next section for a precise definition of this noise). The initial condition u_0 is a bounded continuous function on $\mathbb{R}^d$, and the solution will be a random field $\{u_{t,x}, t \geq 0, x \in \mathbb{R}^d\}$. The symbol $\diamond$ in Eq. (1.1) denotes the Wick product. For $H = \frac{1}{2}$, $\frac{\partial^2 W^H}{\partial t \partial x}$ is a space-time white noise, and in this case, Eq. (1.1) coincides with the stochastic heat equation considered by Walsh (see [20]). We know that in this case the solution exists only in dimension one ($d = 1$).

There has been some recent interest in studying stochastic partial differential equations driven by a fractional noise. Linear stochastic evolution equations in a Hilbert space driven by an additive cylindrical fBm with Hurst parameter H were studied by Duncan et al. [3] in the case $H \in (\frac{1}{2}, 1)$ and by Tindel et al. [18] in the general case, where they provide necessary and sufficient conditions for the existence and uniqueness of an evolution solution. In particular, the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\Delta u + \frac{\partial^2 W^H}{\partial t \partial x}$$

on $\mathbb{R}^d$ has a unique solution if and only if $H > \frac{d}{4}$. The same result holds when one adds to the above equation a nonlinearity of the form $b(t, x, u)$, where b satisfies the usual linear growth and Lipschitz conditions in the variable u, uniformly with respect to (t, x) (see Maslowski and Nualart [9]). The stochastic heat equation on $[0, \infty) \times \mathbb{R}^d$ with a multiplicative fractional white noise of Hurst parameter $H = (H_0, H_1, \ldots, H_d)$ has been studied by Hu [5] under the conditions $\frac{1}{2} < H_i < 1$ for $i = 0, \ldots, d$ and $\sum_{i=0}^d H_i < d - \frac{2}{2H_0 - 1}$. Another important and relevant paper is [11].

The main purpose of this paper is to find conditions on H and d for the solution to Eq. (1.1) to exist as a real-valued stochastic process, and to relate the moments of the solution to the exponential moments of weighted intersection local times. This relation is based on Feynman-Kac's formula applied to a regularization of Eq. (1.1). In order to illustrate this fact, consider the particular case $d = 1$ and $H = \frac{1}{2}$. It is known that there is no Feynman-Kac's formula for the solution of the one-dimensional stochastic heat equation driven by a space-time white noise. Nevertheless, using an approximation of the solution by regularizing the noise we can establish the following formula for the moments:

$$E\left[u_{t,x}^k\right] = E^B\left[\prod_{j=1}^k u_0(x + B_t^j) \exp\left(\sum_{i,j=1, i<j}^k \int_0^t \delta_0\left(B_s^i - B_s^j\right) ds\right)\right], \tag{1.2}$$

for all $k \geq 2$, where B_t is a k-dimensional Brownian motion independent of the space-time white noise $W^{\frac{1}{2}}$. In the case $H > \frac{1}{2}$ and $d \geq 1$, a similar formula holds but $\int_0^t \delta_0(B_s^i - B_s^j) ds$ has to be replaced by the weighted intersection local time

$$L_t(i, j) = H(2H - 1) \int_0^t \int_0^t |s - r|^{2H-2}\, \delta_0\left(B_s^i - B_r^j\right) ds dr, \tag{1.3}$$

where $\{B^j, j \geq 1\}$ are independent d-dimensional Brownian motions (see Theorem 5.3).

The solution of Eq. (1.1) has a formal Wiener chaos expansion $u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x))$. Then, for the existence of a real-valued square integrable solution we need

$$\sum_{n=0}^{\infty} n! \, \|f_n(\cdot, t, x)\|^2_{\mathcal{H}_d^{\otimes n}} < \infty, \tag{1.4}$$

where $\mathcal{H}_d$ is the Hilbert space associated with the covariance of the noise W^H (see (2.2) in the next section). It turns out that, if $H > \frac{1}{2}$, the asymptotic behavior of the norms $\|f_n(\cdot, t, x)\|_{\mathcal{H}_d^{\otimes n}}$ is similar to the behavior of the nth moment of the random variable L_t defined in (1.3). More precisely, if u_0 is a constant K, for all $n \geq 1$ we have

$$(n!)^2 \, \|f_n(\cdot, t, x)\|^2_{\mathcal{H}_d^{\otimes n}} = K^2 E(L_t^n).$$

These facts lead to the following results:

(i) If $d = 1$ and $H > \frac{1}{2}$, the series (1.4) converges, and there exists a solution to Eq. (1.1) which has moments of all orders that can be expressed in terms of the exponential moments of the weighted intersection local times L_t. In the case $H = \frac{1}{2}$ we just need the local time of a one-dimensional standard Brownian motion (see 1.2).

(ii) If $H > \frac{1}{2}$ and $d < 4H$, the norms $\|f_n(\cdot, t, x)\|_{\mathcal{H}_d^{\otimes n}}$ are finite and $E(L_t^n) < \infty$ for all n. In the particular case $d = 2$, the series (1.4) converges if t is small enough, and the solution exists in a small time interval. Similarly, if $d = 2$ the random variable L_t satisfies $E(\exp \lambda L_t) < \infty$ if λ and t are small enough.

(iii) If $d = 1$ and $\frac{3}{8} < H < \frac{1}{2}$, the norms $\|f_n(\cdot, t, x)\|_{\mathcal{H}_d^{\otimes n}}$ are finite and $E(L_t^n) < \infty$ for all n.

A natural problem is to investigate what happens if we replace the Wick product by the ordinary product in Eq. (1.1), that is, we consider the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \frac{\partial^2 W^H}{\partial t \partial x}. \tag{1.5}$$

In terms of the mild formulation, the Wick product leads to the use of Itô-Skorohod stochastic integrals, whereas the ordinary product requires the use of Stratonovich integrals. For this reason, if we use the ordinary product we must assume $d = 1$ and $H > \frac{1}{2}$. In this case we show that the solution exists and its moments can be computed in terms of exponential moments of weighted intersection local times and weighted self-intersection local times in the case $H > \frac{3}{4}$.

The paper is organized as follows. Section 2 contains some preliminaries on the fractional noise W^H and the Skorohod integral with respect to it. In Sect. 3 we present the results on the moments of the weighted intersection local times assuming $H \geq \frac{1}{2}$. Section 4 is devoted to study the Wiener chaos expansion of the solution to Eq. (1.1). The case $H < \frac{1}{2}$ is more involved because it requires the use of fractional derivatives. We show here that if $\frac{3}{8} < H < \frac{1}{2}$, the norms $\|f_n(\cdot, t, x)\|_{\mathcal{H}_d^{\otimes n}}$ are finite and they are related to the moments of a fractional derivative of the intersection local time. We derive the formulas for the moments of the solution in the case $H \geq \frac{1}{2}$ in Sect. 5. Finally, Sect. 6 deals with equations defined by using ordinary product and Stratonovich integrals.

2 Preliminaries

Suppose that $W^H = \left\{W^H(t, A), t \geq 0, A \in \mathcal{B}(\mathbb{R}^d), |A| < \infty\right\}$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra of $\mathbb{R}^d$, is a zero mean Gaussian family of random variables with the covariance function

$$E(W^H(t, A)W^H(s, B)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)|A \cap B|, \tag{2.1}$$

defined in a complete probability space $(\Omega, \mathcal{F}, P)$, where $H \in (0, 1)$, and $|A|$ denotes the Lebesgue measure of A. Thus, for each Borel set A with finite Lebesgue measure, $\{W^H(t, A), t \geq 0\}$ is a fractional Brownian motion (fBm) with Hurst parameter H and variance $t^{2H}|A|$, and the fractional Brownian motions corresponding to disjoint sets are independent.

Then, the multiplicative noise $\frac{\partial^2 W^H}{\partial t \partial x}$ appearing in Eq. (1.1) is the formal derivative of the random measure $W^H(t, A)$:

$$W^H(t, A) = \int_A \int_0^t \frac{\partial^2 W^H}{\partial s \partial x} ds dx.$$

We know that there is an integral representation of the form

$$W^H(t, A) = \int_0^t \int_A K_H(t, s) W(ds, dx),$$

where W is a space-time white noise, and the square integrable kernel K_H is given by

$$K_H(t, s) = c_H \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du\right),$$

for some constant c_H. We will set $K_H(t, s) = 0$ if $s > t$.

Denote by $\mathcal{E}$ the space of step functions on $\mathbb{R}_+$. Let $\mathcal{H}$ be the closure of $\mathcal{E}$ with respect to the inner product induced by

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = K_H(t, s).$$

The operator $K_H^* : \mathcal{E} \to L^2(\mathbb{R}_+)$ defined by $K_H^*(\mathbf{1}_{[0,t]})(s) = K_H(t, s)$ provides a linear isometry between $\mathcal{H}$ and $L^2(\mathbb{R}_+)$.

Define the tensor product

$$\mathcal{H}_d = \mathcal{H} \otimes L^2(\mathbb{R}^d). \tag{2.2}$$

The mapping $\mathbf{1}_{[0,t]\times A} \to W^H(t, A)$ extends to a linear isometry between $\mathcal{H}_d$ and the Gaussian space spanned by W^H. We will denote this isometry by W^H. Then, for each $\varphi \in \mathcal{H}_d$ we have

$$W^H(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \left(K_H^* \otimes I\right) \varphi(t, x) W(dt, dx).$$

We will make use of the notation $W^H(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi dW^H$.

If $H = \frac{1}{2}$, then $\mathcal{H} = L^2(\mathbb{R}_+)$, and the operator K_H^* is the identity. In this case, we have $\mathcal{H}_d = L^2(\mathbb{R}_+ \times \mathbb{R}^d)$.

Suppose now that $H > \frac{1}{2}$. The operator K_H^* can be expressed as a fractional integral operator composed with power functions (see [13]). More precisely, for any function $\varphi \in \mathcal{E}$ with support included in the time interval $[0, T]$ we have

$$\left(K_H^* \varphi\right)(t) = c'_H t^{\frac{1}{2}-H} I_{T-}^{H-\frac{1}{2}} \left(\varphi(s) s^{H-\frac{1}{2}}\right)(t),$$

where $I_{T-}^{H-\frac{1}{2}}$ is the right-sided fractional integral operator defined by

$$I_{T-}^{H-\frac{1}{2}} f(t) = \frac{1}{\Gamma\left(H-\frac{1}{2}\right)} \int_t^T (s-t)^{H-\frac{3}{2}} f(s) ds.$$

In this case the space $\mathcal{H}$ is not a space of functions (see [16]) because it contains distributions. Denote by $|\mathcal{H}|$ the space of measurable functions on $\mathbb{R}_+$ such that

$$\int_0^\infty \int_0^\infty |r-u|^{2H-2} |\varphi_r| |\varphi_u| dr du < \infty.$$

Then, $|\mathcal{H}| \subset \mathcal{H}$ and the inner product in the space $\mathcal{H}$ can be expressed in the following form for $\varphi, \psi \in |\mathcal{H}|$

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^\infty \int_0^\infty \phi(r, u) \varphi_r \varphi_u dr du, \tag{2.3}$$

where $\phi(s, t) = H(2H-1)|t-s|^{2H-2}$.

Using Hölder and Hardy-Littlewood inequalities, one can show (see [10]) that

$$\|\varphi\|_{\mathcal{H}_d} \leq \beta_H \, \|\varphi\|_{L^{\frac{1}{H}}(\mathbb{R}_+; L^2(\mathbb{R}^d))}, \tag{2.4}$$

and this easily implies that

$$\|\varphi\|_{\mathcal{H}_d^{\otimes n}} \le \beta_H^n \|\varphi\|_{L^{\frac{1}{H}}(\mathbb{R}_+^n; L^2(\mathbb{R}^{nd}))}. \tag{2.5}$$

If $H < \frac{1}{2}$, the operator K_H^* can also be expressed as a fractional derivative operator composed with power functions. More precisely, for any function $\varphi \in \mathcal{E}$ with support included in the time interval $[0, T]$ we have

$$\left(K_H^* \varphi\right)(t) = c''_H t^{\frac{1}{2}-H} D_{T-}^{\frac{1}{2}-H}\left(\varphi(s) s^{H-\frac{1}{2}}\right)(t),$$

where $D_{T-}^{\frac{1}{2}-H}$ is the right-sided fractional derivative operator defined by

$$D_{T-}^{\frac{1}{2}-H} f(t) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left(\frac{f(t)}{(T-t)^{\frac{1}{2}-H}} - \left(\frac{1}{2}-H\right)\int_t^T \frac{f(s)-f(t)}{(s-t)^{H-\frac{3}{2}}} ds\right).$$

Moreover, for any $\gamma > \frac{1}{2} - H$ and any $T > 0$ we have $C^\gamma([0, T]) \subset \mathcal{H} = I_{T-}^{\frac{1}{2}-H}\left(L^2(\mathbb{R}_+)\right)$.

If φ is a function with support on $[0, T]$, we can express the operator K_H^* in the following form

$$K_H^* \varphi(t) = K_H(T, t)\varphi(t) + \int_t^T [\varphi(s) - \varphi(t)] \frac{\partial K_H}{\partial s}(s, t) ds. \tag{2.6}$$

Let us now present some preliminaries on the Skorohod integral and the Wick product. The nth Wiener chaos, denoted by $\mathbf{H}_n$, is defined as the closed linear span of the random variables of the form $H_n(W^H(\varphi))$, where φ is an element of $\mathcal{H}_d$ with norm one and H_n is the nth Hermite polynomial. We denote by I_n the linear isometry between $\mathcal{H}_d^{\otimes n}$ (equipped with the modified norm $\sqrt{n!}\,\|\cdot\|_{\mathcal{H}_d^{\otimes n}}$) and the nth Wiener chaos $\mathbf{H}_n$, given by $I_n(\varphi^{\otimes n}) = n! H_n(W^H(\varphi))$, for any $\varphi \in \mathcal{H}_d$ with $\|\varphi\|_{\mathcal{H}_d} = 1$. Any square integrable random variable, which is measurable with respect to the σ-field generated by W^H, has an orthogonal Wiener chaos expansion of the form

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where f_n are symmetric elements of $\mathcal{H}_d^{\otimes n}$, uniquely determined by F.

Consider a random field $u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}^d\}$ such that $E\left(u_{t,x}^2\right) < \infty$ for all t, x. Then, u has a Wiener chaos expansion of the form

$$u_{t,x} = E(u_{t,x}) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)), \tag{2.7}$$

where the series converges in $L^2(\Omega)$.

Definition 2.1 We say the random field u satisfying (2.7) is Skorohod integrable if $E(u) \in \mathcal{H}_d$, for all $n \geq 1$, $f_n \in \mathcal{H}_d^{\otimes(n+1)}$, and the series

$$W^H(E(u)) + \sum_{n=1}^{\infty} I_{n+1}(\widetilde{f_n})$$

converges in $L^2(\Omega)$, where $\widetilde{f_n}$ denotes the symmetrization of f_n.We will denote the sum of this series by $\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u \delta W^H$.

The Skorohod integral coincides with the adjoint of the derivative operator. That is, if we define the space $\mathbb{D}^{1,2}$ as the closure of the set of smooth and cylindrical random variables of the form

$$F = f\left(W^H(h_1), \ldots, W^H(h_n)\right),$$

$h_i \in \mathcal{H}_d$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth) under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathcal{H}_d}^2)},$$

where

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\left(W^H(h_1), \ldots, W^H(h_n)\right) h_j,$$

then, the following duality formula holds

$$E(\delta(u)F) = E\left(\langle DF, u\rangle_{\mathcal{H}_d}\right), \tag{2.8}$$

for any $F \in \mathbb{D}^{1,2}$ and any Skorohod integrable process u.

If $F \in \mathbb{D}^{1,2}$ and h is a function which belongs to $\mathcal{H}_d$, then Fh is Skorohod integrable and, by definition, the Wick product equals to the Skorohod integral of Fh:

$$\delta(Fh) = F \diamond W^H(h). \tag{2.9}$$

This formula justifies the use of the Wick product in the formulation of Eq. (1.1).

Finally, let us remark that in the case $H = \frac{1}{2}$, if $u_{t,x}$ is an adapted stochastic process such that $E\left(\int_0^\infty \int_{\mathbb{R}^d} u_{t,x}^2 dx dt\right) < \infty$, then u is Skorohod integrable and $\delta(u)$

coincides with the Itô stochastic integral:

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u_{t,x} W(dt, dx).$$

3 Weighted intersection local times for standard Brownian motions

In this section we will introduce different kinds of weighted intersection local times which are relevant in computing the moments of the solutions of stochastic heat equations with multiplicative fractional noise.

Suppose first that B^1 and B^2 are independent d-dimensional standard Brownian motions. Consider a nonnegative measurable function $\eta(s, t)$ on $\mathbb{R}_+^2$. We are interested in the weighted intersection local time formally defined by

$$I = \int_0^T \int_0^T \eta(s, t)\delta_0(B_s^1 - B_t^2)dsdt. \tag{3.1}$$

We will make use of the following conditions on the weight η:

(C1) For all $T > 0$

$$\|\eta\|_{1,T} := \max\left(\sup_{0\le t\le T} \int_0^T \eta(s, t)ds, \sup_{0\le s\le T} \int_0^T \eta(s, t)dt \right) < \infty.$$

(C2) For all $T > 0$ there exist constants $\gamma_T > 0$ and $H \in (0, 1)$ such that

$$\eta(s, t) \le \gamma_T \left|s - t\right|^{2H-2},$$

for all $s, t \le T$.

Clearly, when $H > \frac{1}{2}$, (C2) is stronger than (C1). We will denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}$. Consider the approximation of the weighted intersection local time (3.1) defined by

$$I_\varepsilon = \int_0^T \int_0^T \eta(s, t)p_\varepsilon(B_s^1 - B_t^2)dsdt. \tag{3.2}$$

Let us compute the kth moment of I_ε, where $k \geq 1$ is an integer. For this we will follow the approach used in Sect. 2 of [6]. We can write

$$E\left(I_\varepsilon^k\right) = \int_{[0,T]^{2k}} \prod_{i=1}^{k} \eta(s_i, t_i)\psi_\varepsilon(\mathbf{s}, \mathbf{t}) d\mathbf{s} d\mathbf{t}, \tag{3.3}$$

where $\mathbf{s} = (s_1, \ldots, s_k)$, $\mathbf{t} = (t_1, \ldots, t_k)$ and

$$\psi_\varepsilon(\mathbf{s}, \mathbf{t}) = E\left(p_\varepsilon\left(B_{s_1}^1 - B_{t_1}^2\right) \cdots p_\varepsilon\left(B_{s_k}^1 - B_{t_k}^2\right)\right). \tag{3.4}$$

We denote by $\psi(\mathbf{s}, \mathbf{t})$ the density at the origin of the kd-dimensional Gaussian vector $\left(B_{s_1}^1 - B_{t_1}^2, \ldots, B_{s_k}^1 - B_{t_k}^2\right)$, that is,

$$\psi(\mathbf{s}, \mathbf{t}) = (2\pi)^{-\frac{kd}{2}} \left[\det M(\mathbf{s}, \mathbf{t})\right]^{-\frac{d}{2}}, \tag{3.5}$$

where $M(\mathbf{s}, \mathbf{t})$ is the $k \times k$ matrix whose entries are $M_{ij}(\mathbf{s}, \mathbf{t}) = s_i \wedge s_j + t_i \wedge t_j$. We claim that

$$\psi_\varepsilon(\mathbf{s}, \mathbf{t}) \leq \psi(\mathbf{s}, \mathbf{t}). \tag{3.6}$$

In fact, notice first that

$$\psi_\varepsilon(\mathbf{s}, \mathbf{t}) = \left(E\left(p_\varepsilon\left(b_{s_1}^1 - b_{t_1}^2\right) \cdots p_\varepsilon\left(b_{s_k}^1 - b_{t_k}^2\right)\right)\right)^d, \tag{3.7}$$

where b_t^i, $i = 1, 2$, are independent one-dimensional Brownian motions, and in (3.7) p_ε denotes the one-dimensional heat kernel. Using the Fourier transform of the heat kernel, with the notation $\iota = \sqrt{-1}$, we can write

$$\begin{aligned}
&E\left(p_\varepsilon\left(b_{s_1}^1 - b_{t_1}^2\right) \cdots p_\varepsilon\left(b_{s_k}^1 - b_{t_k}^2\right)\right) \\
&= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} E\left(\exp\left(\iota \sum_{j=1}^{k} \left(\xi_j \left(b_{s_j}^1 - b_{t_j}^2\right) - \frac{\varepsilon}{2}\xi_j^2\right)\right)\right) d\xi \\
&\leq \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} e^{-\frac{1}{2}\sum_{j,l=1}^{k} \xi_j (s_j \wedge s_l + t_j \wedge t_l)\xi_l} d\xi \\
&= (2\pi)^{-\frac{k}{2}} \left[\det(M(\mathbf{s}, \mathbf{t}))\right]^{-\frac{1}{2}},
\end{aligned} \tag{3.8}$$

where $\xi = (\xi_1, \ldots, \xi_k)$. In view of (3.5) and (3.7), this implies (3.6). From (3.3) and (3.6) we obtain

$$E\left(I_\varepsilon^k\right) \leq \alpha_k, \tag{3.9}$$

where

$$\alpha_k = \int_{[0,T]^{2k}} \prod_{i=1}^{k} \eta(s_i, t_i)\psi(\mathbf{s}, \mathbf{t}) d\mathbf{s} d\mathbf{t}. \tag{3.10}$$

Then, if $\alpha_k < \infty$ for all $k \geq 1$, the family I_ε converges in L^p, for all $p \geq 2$, to a limit I and $E(I^k) = \alpha_k$. In fact,

$$\lim_{\varepsilon,\delta \downarrow 0} E(I_\varepsilon I_\delta) = \alpha_2,$$

so I_ε converges in L^2, and the convergence in L^p follows from the boundedness in L^q for $q > p$. Then the following result holds.

Proposition 3.1 *Suppose that* (C1) *holds and* $d = 1$. *Then, for all* $\lambda > 0$ *the random variable defined in* (3.2) *satisfies*

$$\sup_{\varepsilon>0} E\left(\exp\left(\lambda I_\varepsilon\right)\right) \leq 1 + \Phi\left(\sqrt{\frac{T}{2}}\, \|\eta\|_{1,T}\, \lambda\right), \tag{3.11}$$

where $\Phi(x) = \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(\frac{k}{2}+1)}$. *Also,* I_ε *converges in* L^p *for all* $p \geq 2$, *and the limit, denoted by* I, *satisfies the estimate* (3.11).

Proof Taking into account the above discussion on the convergence of I_ε, it suffices to show the estimate (3.11). We have, using (3.9)

$$E\left(\exp\left(\lambda I_\varepsilon\right)\right) = \sum_{k=0}^{\infty} \frac{\lambda^k E(I_\varepsilon^k)}{k!} \leq 1 + \sum_{k=1}^{\infty} \frac{\lambda^k \alpha_k}{k!}, \tag{3.12}$$

where α_k has been introduced in (3.10). Thus, in order to show the inequality (3.11) we have to estimate the terms α_k. For any $\mathbf{s} = (s_1, \ldots, s_k)$ we denote by $M(\mathbf{s})$ the $k \times k$ matrix whose entries are $M_{ij}(\mathbf{s}) = s_i \wedge s_j$. If $\mathbf{s}$ and $\mathbf{t}$ are two elements in $(0, \infty)^k$ with pairwise distinct components, the associated matrices $M(\mathbf{s})$ and $M(\mathbf{t})$ are positive definite, and by A8, (viii) in [12] we have

$$\det M(\mathbf{s}, \mathbf{t}) \geq \det M(\mathbf{s}) + \det M(\mathbf{t}),$$

which implies

$$\psi(\mathbf{s}, \mathbf{t}) \leq (2\pi)^{-\frac{k}{2}} \left[\det M(\mathbf{s}) \det M(\mathbf{t})\right]^{-\frac{1}{4}}.$$

We recall that $(2\pi)^{-\frac{k}{2}} \left[\det M(\mathbf{s})\right]^{-\frac{1}{2}}$ is the density at the origin of the Gaussian vector $(b^1_{s_1}, \ldots, b^1_{s_k})$. This density is equal to $(2\pi)^{-\frac{k}{2}} \beta(\mathbf{s})^{-\frac{1}{2}}$, where $\beta(\mathbf{s}) = s_{\sigma(1)}(s_{\sigma(2)} - s_{\sigma(1)}) \cdots (s_{\sigma(k)} - s_{\sigma(k-1)})$, and we denote by σ the permutation of $\{1, \ldots, k\}$ such that $s_{\sigma(1)} < \cdots < s_{\sigma(k)}$. Hence,

$$\psi(\mathbf{s}, \mathbf{t}) \leq (2\pi)^{-\frac{k}{2}} \left[\beta(\mathbf{s})\beta(\mathbf{t})\right]^{-\frac{1}{4}}. \tag{3.13}$$

Therefore, from (3.13) and (3.10) we obtain

$$\alpha_k \leq (2\pi)^{-\frac{k}{2}} \int\limits_{[0,T]^{2k}} \prod_{i=1}^{k} \eta(s_i, t_i) \left[\beta(\mathbf{s})\beta(\mathbf{t})\right]^{-\frac{1}{4}} d\mathbf{s}d\mathbf{t}. \tag{3.14}$$

Applying Cauchy–Schwarz inequality yields

$$\begin{aligned} \alpha_k &\leq (2\pi)^{-\frac{k}{2}} \left\{ \int\limits_{[0,T]^{2k}} \prod_{i=1}^{k} \eta(s_i, t_i) \beta(\mathbf{s})^{-\frac{1}{2}} d\mathbf{s}d\mathbf{t} \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \int\limits_{[0,T]^{2k}} \prod_{i=1}^{k} \eta(s_i, t_i) \beta(\mathbf{t})^{-\frac{1}{2}} d\mathbf{s}d\mathbf{t} \right\}^{\frac{1}{2}} \\ &\leq \left((2\pi)^{-\frac{1}{2}} \|\eta\|_{1,T} \right)^k k! \int\limits_{T_k} \beta(\mathbf{s})^{-\frac{1}{2}} d\mathbf{s} \\ &= \frac{k! 2^{-\frac{k}{2}} T^{\frac{k}{2}} \|\eta\|_{1,T}^k}{\Gamma(\frac{k}{2}+1)}, \end{aligned} \tag{3.15}$$

where $T_k = \{\mathbf{s} = (s_1, \ldots, s_k) : 0 < s_1 < \cdots < s_k < T\}$. Substituting (3.15) into (3.12) leads to the estimate (3.11). □

This result can be extended to the case of a d-dimensional Brownian motion under the stronger condition (C2):

Proposition 3.2 *Suppose that (C2) holds and* $2 \leq d < 4H$. *Then,* $\lim_{\varepsilon \downarrow 0} I_\varepsilon = I$, *exists in* L^p, *for all* $p \geq 2$. *Moreover, if* $d = 2$ *and* $\lambda < \lambda_0(T)$, *where*

$$\lambda_0(T) = \frac{H(2H-1)2\pi}{\gamma_T \beta_H^2 \Gamma\left(1 - \frac{1}{2H}\right)^{2H}} \left(1 - \frac{1}{2H}\right)^{2H-1} T^{1-2H}, \tag{3.16}$$

and β_H *is the constant appearing in the inequality (2.4), then*

$$\sup_{\varepsilon > 0} E\left(\exp\left(\lambda I_\varepsilon\right)\right) < \infty, \tag{3.17}$$

and I *satisfies* $E\left(\exp\left(\lambda I\right)\right) < \infty$.

Proof As in the proof of Proposition 3.1, using condition (C2) and inequality (2.5) we obtain the estimates

$$\begin{aligned}\alpha_k &\le \gamma_T^k (2\pi)^{-\frac{kd}{2}} \int\limits_{[0,T]^{2k}} \prod_{i=1}^k |t_i - s_i|^{2H-2} [\beta(\mathbf{s})\beta(\mathbf{t})]^{-\frac{d}{4}} \, d\mathbf{s}d\mathbf{t} \\ &\le \gamma_T^k (2\pi)^{-\frac{kd}{2}} \alpha_H^k \left(\int\limits_{[0,T]^k} \beta(\mathbf{s})^{-\frac{d}{4H}} d\mathbf{s} \right)^{2H} \\ &= \left(\gamma_T \alpha_H \ (2\pi)^{-\frac{d}{2}}\right)^k (k!)^{2H} \frac{\Gamma\left(1 - \frac{d}{4H}\right)^{k2H} T^{k\left(1-\frac{d}{4H}\right)2H}}{\Gamma\left(k\left(1-\frac{d}{4H}\right)+1\right)^{2H}} \\ &= c_{H,d,T}^k \frac{(k!)^{2H}}{\Gamma\left(k\left(1-\frac{d}{4H}\right)+1\right)^{2H}}, \end{aligned} \tag{3.18}$$

where $\alpha_H = \frac{\beta_H^2}{H(2H-1)}$ and $c_{H,d,T} = \gamma_T \alpha_H (2\pi)^{-\frac{d}{2}} \Gamma\left(1 - \frac{d}{4H}\right)^{2H} T^{\left(2H - \frac{d}{2}\right)}$. For any $a \in (0,1)$ we have

$$\lim_{k\to\infty} \frac{\Gamma(ak+1)}{a^{ak} k^{\frac{1}{2}-\frac{a}{2}} (k!)^a} = c_a, \tag{3.19}$$

where c_a is a positive constant. Therefore, from (3.18) and (3.19) we deduce that there exists a constant $k_{H,d}$ such that

$$\alpha_k \le k_{H,d} c_{H,d,T}^k \left(1 - \frac{d}{4H}\right)^{\left(\frac{d}{2}-2H\right)k} k^{\frac{d}{4}} (k!)^{\frac{d}{2}}.$$

Combining this estimate with (3.12) allows us to conclude the proof. □

If $d=2$ and $\eta(s,t)=1$ it is known that the intersection local time $\int_0^T \int_0^t \delta_0\left(B_s^1 - B_t^2\right) dsdt$ exists and it has finite exponential moments up to a critical exponent λ_0 (see [1,7]).

Consider now a one-dimensional standard Brownian motion B, and the weighted self-intersection local time

$$I = \int_0^T \int_0^T \eta(s,t) \delta_0(B_s - B_t) dsdt.$$

As before, set

$$I_\varepsilon = \int_0^T \int_0^T \eta(s,t) p_\varepsilon (B_s - B_t) \, dsdt.$$

Proposition 3.3 *Suppose that* (C2) *holds. If $H > \frac{1}{2}$, then we have*

$$\sup_{\varepsilon>0} E\left(\exp\left(\lambda \left[I_\varepsilon - E\left(I_\varepsilon\right)\right]\right)\right) < \infty, \tag{3.20}$$

for all $\lambda > 0$*. Moreover, the normalized local time* $I - E(I)$ *exists as a limit in* L^p *of* $I_\varepsilon - E(I_\varepsilon)$*, for all* $p \geq 2$*, and it has exponential moments of all orders.*

If $H > \frac{3}{4}$*, then we have for all* $\lambda > 0$

$$\sup_{\varepsilon > 0} E(\exp(\lambda I_\varepsilon)) < \infty, \tag{3.21}$$

for all $\lambda > 0$*, and the local time* I *exists as a limit in* L^p *of* I_ε*, for all* $p \geq 2$*, and it is exponentially integrable.*

Proof We will follow the ideas of Le Gall in [7]. Suppose first that $H > \frac{1}{2}$ and let us show (3.20). To simplify the proof we assume $T = 1$. It suffices to show these results for

$$J_\varepsilon := \int_0^1 \int_0^t \eta(s,t) p_\varepsilon(B_s - B_t) ds dt.$$

Denote, for $n \geq 1$, and $1 \leq k \leq 2^{n-1}$

$$A_{n,k} = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]$$

(Fig. 1).

Set

$$\alpha_{n,k}^\varepsilon = \int_{A_{n,k}} \eta(s,t) p_\varepsilon(B_s - B_t) ds dt$$

and

$$\bar{\alpha}_{n,k}^\varepsilon = \alpha_{n,k}^\varepsilon - E\left(\alpha_{n,k}^\varepsilon\right).$$

Notice that the random variables $\alpha_{n,k}^\varepsilon$, $1 \leq k \leq 2^{n-1}$, are independent. We have

$$J_\varepsilon = \sum_{n=1}^\infty \sum_{k=1}^{2^{n-1}} \alpha_{n,k}^\varepsilon,$$

and

$$J_\varepsilon - E(J_\varepsilon) = \sum_{n=1}^\infty \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}^\varepsilon.$$

We can write

$$\begin{aligned}
\alpha_{n,k}^\varepsilon &= 2^{-2n} \int_0^1 \int_0^1 \eta\left(\frac{2k-1}{2^n} - \frac{s}{2^n}, \frac{2k-1}{2^n} + \frac{t}{2^n}\right) p_\varepsilon\left(B_{\frac{2k-1}{2^n} - \frac{s}{2^n}} - B_{\frac{2k-1}{2^n} + \frac{t}{2^n}}\right) ds dt \\
&\leq \gamma_1 2^{-2n-(2H-2)n} \int_0^1 \int_0^1 |t+s|^{2H-2} p_\varepsilon\left(B_{\frac{2k-1}{2^n} - \frac{s}{2^n}} - B_{\frac{2k-1}{2^n} + \frac{t}{2^n}}\right) ds dt,
\end{aligned}$$

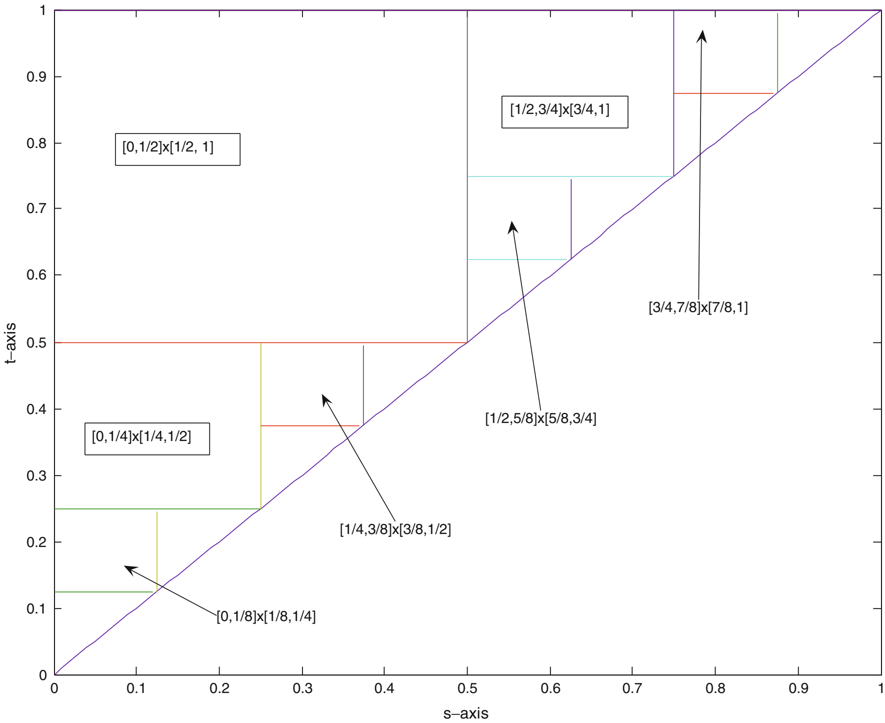

Fig. 1 Plot of the domains $A_{n,k}$

which has the same distribution as

$$\beta_{n,k}^{\varepsilon} = \gamma_1 2^{\left(\frac{1}{2}-2H\right)n} \int_0^1 \int_0^1 |t+s|^{2H-2} p_{\varepsilon 2^n}\left(B_s^1 - B_t^2\right) ds dt,$$

where B^1 and B^2 are independent one-dimensional Brownian motions. Notice that $E\left(\bar{\alpha}_{n,k}^{\varepsilon}\right) = 0$, and for any integer $j \geq 2$,

$$\begin{aligned} E\left(\left(\bar{\alpha}_{n,k}^{\varepsilon}\right)^j\right) &\leq 2^{j-1}\left[E\left(\left(\alpha_{n,k}^{\varepsilon}\right)^j\right) + \left(E\left(\alpha_{n,k}^{\varepsilon}\right)\right)^j\right] \\ &\leq 2^j E\left(\left(\alpha_{n,k}^{\varepsilon}\right)^j\right) = 2^j E\left(\left(\beta_{n,k}^{\varepsilon}\right)^j\right). \end{aligned}$$

Thus,

$$E\left(\exp\left(\lambda \bar{\alpha}_{n,k}^{\varepsilon}\right)\right) = 1 + \sum_{j=2}^{\infty} \frac{\lambda^j}{j!} E\left(\left(\bar{\alpha}_{n,k}^{\varepsilon}\right)^j\right)$$

$$\leq 1 + \sum_{j=2}^{\infty} \frac{(2\lambda)^j}{j!} E\left(\left(\beta_{n,k}^{\varepsilon}\right)^j\right). \tag{3.22}$$

The moment of order j of $\beta_{n,k}^{\varepsilon}$ can be estimated from (3.15) with the weight $\widehat{\eta}(s,t) = \gamma_1 2^{\left(\frac{1}{2}-2H\right)n}|t+s|^{2H-2}$, which satisfies $\|\widehat{\eta}\|_{1,1} \leq \; C_1 2^{\left(\frac{1}{2}-2H\right)n}$, where $C_1 = \frac{\gamma_1}{2H-1}$. Thus,

$$E\left(\left(\beta_{n,k}^{\varepsilon}\right)^j\right) \leq \frac{j!2^{-\frac{j}{2}}\left(C_1 2^{\left(\frac{1}{2}-2H\right)n}\right)^j}{\Gamma(\frac{j}{2}+1)}, \tag{3.23}$$

and substituting (3.23) into (3.22) yields

$$E\left(\exp\left(\lambda\bar{\alpha}_{n,k}^{\varepsilon}\right)\right) \leq 1 + c_{\lambda,n}, \tag{3.24}$$

where

$$c_{\lambda,n} = \sum_{j=2}^{\infty} \frac{\left(C_2 2^{\left(\frac{1}{2}-2H\right)n}\lambda\right)^j}{\Gamma\left(\frac{j}{2}+1\right)},$$

with $C_2 = 2^{-\frac{1}{2}}C_1$. Notice that $c_{\lambda,n}$ is finite for any $\lambda > 0$, because the radius of convergence of this power series is infinity.

Fix $a > 0$ such that $a < \min\left(\frac{1}{2} - 2H, 4H - 2\right)$. For any $N \geq 2$ define

$$b_N = \prod_{j=2}^{N}\left(1 - 2^{-a(j-1)}\right),$$

and notice that $\lim_{N\to\infty} b_N = b_\infty > 0$. Then, by Hölder's inequality, for all $N \geq 2$ we have

$$\begin{aligned} &E\left[\exp\left(\lambda b_N \sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}^{\varepsilon}\right)\right] \\ &\leq \left\{E\left[\exp\left(\frac{\lambda b_N}{1-2^{-a(N-1)}}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}^{\varepsilon}\right)\right]\right\}^{1-2^{-a(N-1)}} \\ &\quad\times \left\{E\left[\exp\left(\lambda b_N 2^{a(N-1)}\sum_{k=1}^{2^{N-1}} \bar{\alpha}_{N,k}^{\varepsilon}\right)\right]\right\}^{2^{-a(N-1)}} \end{aligned}$$

$$
\begin{aligned}
&\le \left\{ E\left[\exp\left(\lambda b_{N-1} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}^{\varepsilon} \right) \right] \right\} \\
&\quad \times \left\{ E\left[\exp\left(\lambda b_N 2^{a(N-1)} \bar{\alpha}_{N,k}^{\varepsilon} \right) \right] \right\}^{2^{(1-a)(N-1)}} \\
&= A_N B_N.
\end{aligned}
$$

Using (3.24), the second factor in the above expression can be dominated by

$$
B_N \le \left(1 + c_{\lambda b_N 2^{a(N-1)},N}\right)^{2^{(1-a)(N-1)}}.
$$

Taking into account that $a < 2H - \frac{1}{2}$ we can write

$$
c_{\lambda b_N 2^{a(N-1)},N} = \sum_{j=2}^{\infty} \frac{\left(C_2 2^{\left(\frac{1}{2}-2H+a\right)N-a} b_N \lambda \right)^j}{\Gamma\left(\frac{j}{2}+1\right)} \le C_3\, 2^{(1-4H+2a)N},
$$

for some constant C_3. Hence, using that $\log(1+x) \le x$ we obtain

$$
B_N \le \exp\left(C_4 2^{(a+2-4H)N} \right),
$$

where $C_4 = C_3 2^{a-1}$. Thus by induction we have

$$
\begin{aligned}
E\left[\exp\left(\lambda b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k} \right) \right] &\le \exp\left\{ \sum_{n=2}^{N} C_4 2^{(a+2-4H)n} \right\} E\left(\exp \bar{\alpha}_{1,1}\right) \\
&\le \exp\left(C_4 \left(1 - 2^{a+2-4H}\right)^{-1} \right) \\
&\quad \times E\left(\exp\left(\bar{\alpha}_{1,1}\right)\right) < \infty,
\end{aligned}
$$

because $a < 4H - 2$. By Fatou lemma we see that

$$
\sup_{\varepsilon > 0} E\ \left(\exp\left(\lambda b_\infty \left(J_\varepsilon - E\left(J_\varepsilon\right)\right)\right)\right) < \infty,
$$

and (3.20) follows.

On the other hand, one can easily show that

$$\begin{aligned}
&\lim_{\varepsilon,\delta\downarrow 0} E((J_\varepsilon - E(J_\varepsilon))(J_\delta - E(J_\delta))) \\
&= \frac{1}{2\pi} \int\limits_{s<t<1, s'<t'<1} \eta(s,t)\eta(s',t') \\
&\times \left[\left(\det \begin{bmatrix} t-s & |[s,t]\cap[s',t']| \\ |[s,t]\cap[s',t']| & t'-s' \end{bmatrix} \right)^{-\frac{1}{2}} \right. \\
&\left. - \left((t-s)(t'-s')\right)^{-\frac{1}{2}} \right] ds\,dt\,ds'\,dt' < \infty,
\end{aligned}$$

which implies the convergence of $J_\varepsilon - E(J_\varepsilon)$ in L^2. The convergence in L^p for $p \geq 2$ and the estimate (3.20) follow immediately.

The proof of the inequality (3.21) is similar. The estimate (3.24) is replaced by

$$E\left(\exp\left(\lambda\left(\alpha^\varepsilon_{n,k}\right)\right)\right) \leq 1 + d_{\lambda,n}, \tag{3.25}$$

where

$$d_{\lambda,n} = \sum_{j=1}^{\infty} \frac{\left(C_2 2^{\left(\frac{1}{2}-2H\right)n}\lambda\right)^j}{\Gamma\left(\frac{j}{2}+1\right)},$$

and, assuming that $a < 2H - \frac{1}{2}$ and using that $H > \frac{3}{4}$, we obtain

$$\begin{aligned}
&E\left[\exp\left(\lambda b_N \sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}} \alpha^\varepsilon_{n,k}\right)\right] \\
&\leq \exp\left\{\sum_{n=2}^{N} C_4 2^{\left(\frac{3}{2}-2H\right)n}\right\} E\left(\exp\left(\alpha_{1,1}\right)\right) \\
&\leq \exp\left\{ C_4\left(1-2^{\left(\frac{3}{2}-2H\right)}\right)^{-1}\right\} E\left(\exp\left(\alpha_{1,1}\right)\right) < \infty.
\end{aligned}$$

By Fatou lemma we see that

$$\sup_{\varepsilon>0} E\left(\exp\left(\lambda b_\infty J_\varepsilon\right)\right) < \infty,$$

which implies (3.21). The convergence in L^p of J_ε is proved as usual. □

Notice that condition $H > \frac{3}{4}$ cannot be improved because

$$E\left(\int_0^T\int_0^T |t-s|^{-\frac{1}{2}}\delta_0(B_s - B_t)\,ds\,dt\right) = \frac{1}{\sqrt{2\pi}}\int_0^T\int_0^T |t-s|^{-1}\,ds\,dt = \infty.$$

4 Stochastic heat equation in the Itô-Skorohod sense

In this section we study the stochastic partial differential equation (1.1) on $\mathbb{R}^d$, where W^H is a zero mean Gaussian family of random variables with the covariance function (2.1), defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and the initial condition u_0 belongs to $C_b(\mathbb{R}^d)$. First we give the definition of a solution using the Skorohod integral, which corresponds formally to the Wick product appearing in Eq. (1.1).

For any $t \geq 0$, we denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s, A), 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^d), |A| < \infty\}$ and the P-null sets. A random field $u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}\}$ is adapted if for any (t, x), $u_{t,x}$ is $\mathcal{F}_t$-measurable.

For any bounded Borel function φ on $\mathbb{R}$ we write $p_t\varphi(x) = \int_{\mathbb{R}^d} p_t(x-y)\varphi(y)dy$.

Definition 4.1 An adapted random field $u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}^d\}$ such that $E(u_{t,x}^2) < \infty$ for all (t, x) is a solution to Eq. (1.1) if for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u_{s,y}\mathbf{1}_{[0,t]}(s), s \geq 0, y \in \mathbb{R}^d\}$ is Skorohod integrable, and the following equation holds:

$$u_{t,x} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u_{s,y}\delta W_{s,y}^H. \tag{4.1}$$

The fact that Eq. (1.1) contains a multiplicative Gaussian noise allows us to find recursively an explicit expression for the Wiener chaos expansion of the solution. This approach has extensively used in the literature. For instance, we refer to the papers by Hu [5], Buckdahn and Nualart [2], Nualart and Zakai [15], Nualart and Rozovskii [14], and Tudor [19], among others.

4.1 General chaos expansions

Suppose that $u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}^d\}$ is a solution to Eq. (1.1). Then, for any fixed (t, x), the random variable $u_{t,x}$ admits the following Wiener chaos expansion

$$u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)), \tag{4.2}$$

where for each (t, x), $f_n(\cdot, t, x)$ is a symmetric element in $\mathcal{H}_d^{\otimes n}$. To find the explicit form of f_n we substitute (4.2) in the Skorohod integral appearing in (4.1), and we obtain

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u_{s,y}\delta W_{s,y}^H &= \sum_{n=0}^{\infty} \int_0^t \int_{\mathbb{R}^d} I_n(p_{t-s}(x-y)f_n(\cdot, s, y))\, \delta W_{s,y}^H \\ &= \sum_{n=0}^{\infty} I_{n+1}(p_{t-s}(x \widetilde{- y)f}_n(\cdot, s, y)).\end{aligned}$$

Here, $\widetilde{p_{t-s}(x-y)f_n}(\cdot,s,y)$ denotes the symmetrization of the function

$$p_{t-s}(x-y)f_n(s_1,x_1,\dots,s_n,x_n,s,y)$$

in the variables $(s_1,x_1),\dots,(s_n,x_n),(s,y)$, that is,

$$\begin{aligned}\widetilde{p_{t-s}(x-y)f_n}(\cdot,s,y)=\frac{1}{n+1}\Bigg[&p_{t-s}(x-y)f_n(s_1,x_1,\dots,s_n,x_n,s,y)\\&+\sum_{j=1}^{n}p_{t-s_j}(x-y_j)\\&\times f_n(s_1,x_1,\dots,s_{j-1},x_{j-1},s,y,s_{j+1},\\&x_{j+1},\dots,s_n,y_n,s_j,y_j)\Bigg].\end{aligned}$$

Thus, Eq. (4.1) is equivalent to say that $f_0(t,x)=p_tu_0(x)$, and

$$f_{n+1}(\cdot,t,x)=\widetilde{p_{t-s}(x-y)f_n}(\cdot,s,y) \tag{4.3}$$

for all $n\geq 0$. Notice that, the adaptability property of the random field u implies that $f_n(s_1,x_1,\dots,s_n,x_n,t,x)=0\ \text{ if } s_j>t$ for some j.

This leads to the following formula for the kernels f_n, for $n\geq 1$

$$\begin{aligned}f_n(s_1,x_1,\dots,s_n,x_n,t,x)&=\frac{1}{n!}\\&\times p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)})p_{s_{\sigma(1)}}u_0(x_{\sigma(1)}),\end{aligned} \tag{4.4}$$

where σ denotes the permutation of $\{1,2,\dots,n\}$ such that $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t$. This implies that there is a unique solution to Eq. (4.1), and the kernels of its chaos expansion are given by (4.4). In order to show the existence of a solution, it suffices to check that the kernels defined in (4.4) determine an adapted random field satisfying the conditions of Definition 4.1. This is equivalent to show that for all (t,x) we have

$$\sum_{n=1}^{\infty}n!\,\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_d^{\otimes n}}<\infty. \tag{4.5}$$

4.2 Case $H\geq\frac{1}{2}$

Suppose first that $H=\frac{1}{2}$ and $d=1$. In this case it is easy to show that (4.5) holds. In fact, we have, assuming $|u_0|\leq K$, and with the notation $\mathbf{x}=(x_1,\dots,x_n)$, and $\mathbf{s}=(s_1,\dots,s_n)$:

$$
\begin{aligned}
\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}} &= \frac{1}{(n!)^2}\int\limits_{[0,t]^n}\int\limits_{\mathbb{R}^n} p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})^2\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)})^2 \\
&\quad\times p_{s_{\sigma(1)}}u_0(x_{\sigma(1)})^2\, d\mathbf{x}d\mathbf{s} \\
&\le K^2\frac{(4\pi)^{-\frac{n}{2}}}{(n!)^2}\int\limits_{[0,t]^n}\prod_{j=1}^n (s_{\sigma(j+1)}-s_{\sigma(j)})^{-\frac12}d\mathbf{s} \\
&= \frac{K^2\,(4\pi)^{-\frac{n}{2}}}{n!}\int\limits_{T_n}\prod_{j=1}^n (s_{j+1}-s_j)^{-\frac12}d\mathbf{s},
\end{aligned}
$$

where $T_n = \{(s_1,\ldots,s_n)\in[0,t]^n : 0<s_1<\cdots<s_n<t\}$ and by convention $s_{n+1}=t$. Hence,

$$\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}} \le \frac{K^2 2^{-n} t^{\frac{n}{2}}}{n!\Gamma(\frac{n+1}{2})},$$

which implies (4.5). On the other hand, if $H=\frac12$, $u_0=1$, and $d\ge 2$, these norms are infinite.

Notice that if $u_0=1$, then $(n!)^2\,\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}}$ coincides with the moment of order n of the local time at zero of the one-dimensional Brownian motion with variance $2t$, that is,

$$(n!)^2\,\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}} = E\left[\left(\int_0^t \delta_0(B_{2s})ds\right)^n\right].$$

To handle the case $H>\frac12$, we need the following technical lemma.

Lemma 4.2 *Set*

$$g_{\mathbf{s}}(x_1,\ldots,x_n) = p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)})). \tag{4.6}$$

Then,

$$\langle g_{\mathbf{s}}, g_{\mathbf{t}}\rangle_{L^2(\mathbb{R}^{nd})} = \psi(\mathbf{s},\mathbf{t}),$$

where $\psi(\mathbf{s},\mathbf{t})$ is defined in (3.5).

Proof By Plancherel's identity

$$\langle g_{\mathbf{s}}, g_{\mathbf{t}}\rangle_{L^2(\mathbb{R}^{nd})} = (2\pi)^{-dn}\,\langle \mathcal{F}g_{\mathbf{s}}, \mathcal{F}g_{\mathbf{t}}\rangle_{L^2(\mathbb{R}^{nd})}\,,$$

where $\mathcal{F}$ denotes the Fourier transform, given by

$$
\begin{aligned}
\mathcal{F}g_{\mathbf{s}}(\xi_1,\ldots,\xi_n) &= (2\pi)^{-\frac{nd}{2}}\prod_{j=1}^n (s_{\sigma(j+1)}-s_{\sigma(j)})^{-\frac{d}{2}} \\
&\quad\times\int\limits_{\mathbb{R}^{nd}}\prod_{j=1}^n \exp\left(i\langle\xi_j,x_j\rangle - \frac{\left|x_{\sigma(j+1)}-x_{\sigma(j)}\right|^2}{2\left(s_{\sigma(j+1)}-s_{\sigma(j)}\right)}\right)d\mathbf{x},
\end{aligned}
$$

with the convention $x_{n+1} = x$ and $s_{n+1} = t$. Making the change of variables $u_j = x_{\sigma(j+1)} - x_{\sigma(j)}$ if $1 \le j \le n-1$, and $u_n = x - x_{\sigma(n)}$, we obtain

$$\begin{aligned}\mathcal{F}g_{\mathbf{s}}(\xi_1,\ldots,\xi_n) &= (2\pi)^{-\frac{nd}{2}} \prod_{j=0}^{n} (s_{\sigma(j+1)} - s_{\sigma(j)})^{-\frac{d}{2}} \\ &\quad \times \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \exp\Bigg(i\left\langle \xi_{\sigma(j)}, x - u_n - \cdots - u_j \right\rangle \\ &\quad - \frac{|u_j|^2}{2\left(s_{\sigma(j+1)} - s_{\sigma(j)}\right)} \Bigg) d\mathbf{u} \\ &= E\left(\prod_{j=1}^{n} \exp\left(i\left\langle \xi_{\sigma(j)}, x - B_t - B_{s_{\sigma(j)}} \right\rangle\right)\right) \\ &= E\left(\prod_{j=1}^{n} \exp\left(i\left\langle \xi_j, x - B_t - B_{s_j} \right\rangle\right)\right).\end{aligned}$$

As a consequence,

$$\langle g_{\mathbf{s}}, g_{\mathbf{t}} \rangle_{L^2(\mathbb{R}^{nd})} = (2\pi)^{-nd} \int_{\mathbb{R}^{nd}} E\left(\prod_{j=1}^{n} \exp\left(i\left\langle \xi_j, B^1_{s_j} - B^2_{t_j} \right\rangle\right)\right) d\xi,$$

which implies the desired result. □

In the case $H > \frac{1}{2}$, and assuming that $u_0 = 1$, the next proposition shows that the norm $(n!)^2 \left\| f_n(\cdot,t,x) \right\|^2_{\mathcal{H}_d^{\otimes n}}$ coincides with the nth moment of the intersection local time of two independent d-dimensional Brownian motions with weight $\phi(t,s) = H(2H-1)|t-s|^{2H-2}$.

Proposition 4.3 *Suppose that $H > \frac{1}{2}$ and $d < 4H$. Then, for all $n \ge 1$*

$$(n!)^2 \left\| f_n(\cdot,t,x) \right\|^2_{\mathcal{H}_d^{\otimes n}} \le \|u_0\|^2_\infty E\left[\left(\int_0^t \int_0^t \phi(s,r)\delta_0(B^1_s - B^2_r)ds dr \right)^n \right] < \infty, \tag{4.7}$$

with equality if u_0 is constant. Moreover, we have:

1. *If $d = 1$, there exists a unique solution to Eq.* (4.1).
2. *If $d = 2$, then there exists a unique solution in an interval* $[0,T]$ *provided $T < T_0$, where*

$$T_0 = \frac{2H}{2H-1}\left(\frac{2H(2H-1)\pi}{\gamma_T \beta_H^2} \left(\Gamma\left(1 - \frac{1}{2H}\right)\right)^{2H}\right)^{1/(2H-1)}. \tag{4.8}$$

Proof From (4.4) we deduce

$$(n!)^2 \|f_n(\cdot,t,x)\|^2_{\mathcal{H}_d^{\otimes n}} \le \|u_0\|^2_\infty \int_{[0,t]^{2n}} \prod_{j=1}^n \phi(s_j,t_j) \langle g_{\mathbf{s}}, g_{\mathbf{t}}\rangle_{L^2(\mathbb{R}^{nd})}\, d\mathbf{s}d\mathbf{t}, \tag{4.9}$$

where $g_{\mathbf{s}}$ is defined in (4.6). Then the results follow easily from from Lemma 4.2, Eq. (3.10) and Proposition 3.2. □

4.3 Case $H < \frac{1}{2}$ and $d = 1$

We know that in this case, the norm in the space $\mathcal{H}$ is defined in terms of fractional derivatives. The aim of this section is to show that $\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}}$ is related to the nth moment of a fractional derivative of the self-intersection local time of two independent one-dimensional Brownian motions, and these moments are finite for all $n \ge 1$, provided $\frac{3}{8} < H < \frac{1}{2}$.

Consider the operator $\left(K_H^*\right)^{\otimes 2}$ on functions of two variables defined as the action of the operator K_H^* on each coordinate. That is, using the notation (2.6) we have

$$\begin{aligned}\left(K_H^*\right)^{\otimes 2} f(r_1,r_2) &= K_H(T,r_1)K_H(T,r_2)f(r_1,r_2)\\ &\quad + K_H(T,r_1)\int_{r_2}^t \frac{\partial K_H}{\partial s}(s,r_2)\left(f(r_1,s)-f(r_1,r_2)\right)ds\\ &\quad + K_H(T,r_2)\int_{r_1}^t \frac{\partial K_H}{\partial s}(v,r_1)\left(f(v,r_2)-f(r_1,r_2)\right)dv\\ &\quad + \int_{r_2}^t\int_{r_1}^t \frac{\partial K_H}{\partial s}(s,r_2)\frac{\partial K_H}{\partial v}(v,r_1)\left[f(v,s)-f(r_1,s)-f(v,r_2)\right.\\ &\quad \left. + f(r_1,r_2)\right]dsdv.\end{aligned}$$

Suppose that $f(s,t)$ is a continuous function on $[0,T]^2$. Define the Hölder norms

$$\begin{aligned}\|f\|_{1,\gamma} &= \sup\left\{\frac{|f(s_1,t)-f(s_2,t)|}{|s_1-s_2|^\gamma}, s_1,s_2,t\in[0,T], s_1\ne s_2\right\},\\ \|f\|_{2,\gamma} &= \sup\left\{\frac{|f(s,t_1)-f(s,t_2)|}{|t_1-t_2|^\gamma}, t_1,t_2,s\in[0,T], t_1\ne t_2\right\}\end{aligned}$$

and

$$\|f\|_{1,2,\gamma} = \sup\frac{|f(s_1,t_1)-f(s_1,t_2)-f(s_2,t_1)+f(s_2,t_2)|}{|s_1-s_2|^\gamma|t_1-t_2|^\gamma},$$

where the supremum is taken in the set $\{t_1, t_2, s_2, s_2 \in [0, T], s_1 \neq s_2, t_1 \neq t_2\}$. Set

$$\|f\|_{0,\gamma} = \|f\|_{1,\gamma} + \|f\|_{2,\gamma} + \|f\|_{1,2,\gamma} .$$

Then, $\left(K_H^*\right)^{\otimes 2} f$ is well defined if $\|f\|_{0,\gamma} < \infty$ for some $\gamma > \frac{1}{2} - H$. As a consequence, if B^1 and B^2 are two independent one-dimensional Brownian motions, the following random variable is well defined for all $\varepsilon > 0$

$$J_\varepsilon = \int_0^T \left[\left(K_H^*\right)^{\otimes 2} q_\varepsilon\right](r, r) dr, \tag{4.10}$$

where $q_\varepsilon(s, t) = p_\varepsilon(B_s^1 - B_t^2)$. The next theorem asserts that J_ε converges in L^p for all $p \geq 2$ to a fractional derivative of the intersection local time of B^1 and B^2.

Proposition 4.4 *Suppose that* $\frac{3}{8} < H < \frac{1}{2}$.*Then, for any integer* $k \geq 1$ *and,* $T > 0$ *we have* $E\left(J_\varepsilon^k\right) \geq 0$ *and*

$$\sup_{\varepsilon > 0} E\left(J_\varepsilon^k\right) < \infty. \tag{4.11}$$

Moreover, for all $p \geq 2$, J_ε *converges in* L^p *as* ε *tends to zero to a random variable denoted by*

$$\int_0^T \left(K_H^*\right)^{\otimes 2} \delta_0(B_\cdot^1 - B_\cdot^2)(r, r) dr.$$

Proof Fix $k \geq 1$. The proof of the estimate (4.11) is technical will be done in several steps. Before proceeding to the proof, we will give the main ideas.

In order to compute the k moment of J^ε we have to apply the operator K_H^* to all the coordinates of the function $\psi_\varepsilon(\mathbf{s}, \mathbf{t})$. First we show that we can replace $\psi_\varepsilon(\mathbf{s}, \mathbf{t})$ by $\psi(\mathbf{s}, \mathbf{t})$, and it suffices to consider the restriction of this function to T_k^2, which equals (up to a constant) to $\prod_{j=1}^k (s_j - s_{j-1} + t_j - t_{j-1})^{-\frac{1}{2}}$. Then, although the operator K_H^* is not positivity preserving, we show that we get the same estimates if we work with $\prod_{j=1}^k ((s_j - s_{j-1})(t_j - t_{j-1}))^{-\frac{1}{4}}$. Thus we have to estimate $(K_H^*)^{\otimes k}\left(\prod_{j=1}^k (s_j - s_{j-1})^{-\frac{1}{4}} \mathbf{1}_{\{\mathbf{s} \in T_k\}}\right)$. The operator K_H^* behaves as fractional derivative of order $\frac{1}{2} - H$, and when we apply it to a product of functions, it will give rise to several terms. The worst case is when it acts on both coordinates of one of the factors $(s_j - s_{j-1})^{-\frac{1}{4}}$. In this case $(K_H^*)^{\otimes 2}\left((s_j - s_{j-1})^{-\frac{1}{4}}\right)$ behaves as $(s_j - s_{j-1})^\gamma$, where $\gamma = -\frac{1}{4} + 2(H - \frac{1}{2}) = 2H - \frac{5}{4}$. Twice this quantity, $4H - \frac{5}{2}$, must be larger that -1, which leads to the condition $H > \frac{3}{8}$.

Step 1. Let us first compute the moment of order k of J_ε. We can write

$$E\left(J_\varepsilon^k\right) = \int_{[0,T]^k} E\left(\prod_{i=1}^k \left[\left(K_H^*\right)^{\otimes 2} q_\varepsilon\right](r_i, r_i)\right) d\mathbf{r}. \tag{4.12}$$

Using the definition of $\psi_\varepsilon(\mathbf{s},\mathbf{t})$ given in (3.4), we obtain

$$E\left(\prod_{i=1}^k\left[\left(K_H^*\right)^{\otimes 2} q_\varepsilon\right](r_i,r_i)\right)=\left[\left(K_H^*\right)^{\otimes 2k}\psi_\varepsilon\right](\mathbf{r},\mathbf{r}),$$

where $\left(K_H^*\right)^{\otimes 2k}\psi_\varepsilon(\mathbf{s},\mathbf{t})$ is defined as the action of the operator $\left(K_H^*\right)^{\otimes 2}$ in the coordinates $(s_1,t_1),\ldots,(s_k,t_k)$. The operator K_H^* does not preserve positivity. However, we claim that this expression is nonnegative, and

$$\left[\left(K_H^*\right)^{\otimes 2k}\psi_\varepsilon\right](\mathbf{r},\mathbf{r})\le\left[\left(K_H^*\right)^{\otimes 2k}\psi\right](\mathbf{r},\mathbf{r}). \tag{4.13}$$

In fact, from (3.8) we obtain

$$\begin{aligned}
&\left[\left(K_H^*\right)^{\otimes 2k}\psi_\varepsilon\right](\mathbf{r},\mathbf{r})\\
&=(2\pi)^{-k}\int_{\mathbb{R}^k}\left[\left(K_H^*\right)^{\otimes 2k}e^{-\frac12\sum_{j,l=1}^k\xi_j\xi_l(s_j\wedge s_l+t_j\wedge t_l)}\right](\mathbf{r},\mathbf{r})e^{-\frac{\varepsilon}{2}|\xi|^2}d\xi\\
&=(2\pi)^{-k}\int_{\mathbb{R}^k}\left[\left(K_H^*\right)^{\otimes k}e^{-\frac12\sum_{j,l=1}^k\xi_j\xi_l(s_j\wedge s_l)}\right]^2(\mathbf{r})e^{-\frac{\varepsilon}{2}|\xi|^2}d\xi\\
&\le(2\pi)^{-k}\int_{\mathbb{R}^k}\left[\left(K_H^*\right)^{\otimes k}e^{-\frac12\sum_{j,l=1}^k\xi_j\xi_l(s_j\wedge s_l)}\right]^2(\mathbf{r})d\xi,
\end{aligned}\tag{4.14}$$

which leads to (4.13). Therefore, it suffices to show that

$$\int_{[0,T]^k}\left[\left(K_H^*\right)^{\otimes 2k}\psi\right](\mathbf{r},\mathbf{r})d\mathbf{r}<\infty. \tag{4.15}$$

Step 2. If the points $\mathbf{s}$ and $\mathbf{t}$ belong to the simplex $T_k=\{0<t_1<\cdots<t_k<T\}$, then the function $\psi(\mathbf{s},\mathbf{t})$ has the simple expression

$$\psi(\mathbf{s},\mathbf{t})=(2\pi)^{-\frac{k}{2}}\prod_{j=1}^k\left[s_j-s_{j-1}+t_j-t_{j-1}\right]^{-\frac12}.$$

We are going to show that in order to prove (4.15) we can replace ψ by its restriction to T_k^2.

Set $g(\xi,\mathbf{s})=e^{-\frac12\sum_{j,l=1}^k\xi_j\xi_l(s_j\wedge s_l)}$. Then, we have

$$\left(K_H^*\right)^{\otimes k}g(\xi,\mathbf{s})=\sum_\sigma\left[\left(K_H^*\right)^{\otimes k}g(\xi,\mathbf{s})\mathbf{1}_{\{\sigma(\mathbf{s})\in T_k\}}\right](\mathbf{r})$$

$$= \sum_{\sigma} \left[\left(K_H^*\right)^{\otimes k} g(\sigma(\xi), \sigma(\mathbf{s})) \mathbf{1}_{\{\sigma(\mathbf{s}) \in T_k\}} \right] (\mathbf{r})$$
$$= \sum_{\sigma} \left[\left(K_H^*\right)^{\otimes k} g(\sigma(\xi), \mathbf{s}) \mathbf{1}_{\{\mathbf{s} \in T_k\}} \right] (\sigma(\mathbf{r})),$$

where σ runs over all permutations of $\{1, \ldots, k\}$, $s_{\sigma(1)}, \ldots, s_{\sigma(k)}$, and $\sigma(\xi)$ and $\sigma(\mathbf{r})$ are defined in the same way. In fact, the second equality follows from $g(\xi, \mathbf{s}) = g(\sigma(\xi), \sigma(\mathbf{s}))$, and the third one is a consequence of the definition of the tensor product $\left(K_H^*\right)^{\otimes k}$. Then, from (4.14) we obtain

$$\begin{aligned}
&\int_{[0,T]^k} \left[\left(K_H^*\right)^{\otimes 2k} \psi \right] (\mathbf{r}, \mathbf{r}) d\mathbf{r} \\
&= (2\pi)^{-k} \int_{[0,T]^k} \int_{\mathbb{R}^k} \left(\sum_{\sigma} \left[\left(K_H^*\right)^{\otimes k} g(\sigma(\xi), \mathbf{s}) \mathbf{1}_{\{\mathbf{s} \in T_k\}} \right] (\sigma(\mathbf{r})) \right)^2 d\xi d\mathbf{r} \\
&\leq (2\pi)^{-k} k! \sum_{\sigma} \int_{[0,T]^k} \int_{\mathbb{R}^k} \left[\left(K_H^*\right)^{\otimes k} g(\sigma(\xi), \mathbf{s}) \mathbf{1}_{\{\mathbf{s} \in T_k\}} \right]^2 (\sigma(\mathbf{r})) d\xi d\mathbf{r} \\
&= (2\pi)^{-k} (k!)^2 \int_{[0,T]^k} \int_{\mathbb{R}^k} \left[\left(K_H^*\right)^{\otimes k} g(\xi, \mathbf{s}) \mathbf{1}_{\{\mathbf{s} \in T_k\}} \right]^2 (\mathbf{r}) d\xi d\mathbf{r}. \qquad (4.16)
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{\mathbb{R}^k} \left[\left(K_H^*\right)^{\otimes k} g(\xi, \mathbf{s}) \mathbf{1}_{\{\mathbf{s} \in T_k\}} \right]^2 (\mathbf{r}) d\xi \\
&= \left[\left(K_H^*\right)^{\otimes 2k} \left(\mathbf{1}_{\{\mathbf{s}, \mathbf{t} \in T_k\}} \int_{\mathbb{R}^k} g(\xi, \mathbf{s}) g(\xi, \mathbf{t}) d\xi \right) \right] (\mathbf{r}) \\
&= (2\pi)^{\frac{k}{2}} \left[\left(K_H^*\right)^{\otimes 2k} \left(\mathbf{1}_{\{\mathbf{s}, \mathbf{t} \in T_k\}} \prod_{j=1}^{k} \left[s_j - s_{j-1} + t_j - t_{j-1} \right]^{-\frac{1}{2}} \right) \right] (\mathbf{r}). \qquad (4.17)
\end{aligned}$$

Substituting (4.17) into (4.16) it suffices to show that the integral

$$\int_{T_k} \left[\left(K_H^*\right)^{\otimes 2k} \left(\mathbf{1}_{\{\mathbf{s}, \mathbf{t} \in T_k\}} \prod_{j=1}^{k} \left[s_j - s_{j-1} + t_j - t_{j-1} \right]^{-\frac{1}{2}} \right) \right] (\mathbf{r}) d\mathbf{r} \qquad (4.18)$$

is finite.

Step 3. Again, although the operator $\left(K_H^*\right)^{\otimes 2k}$ is not positivity preserving, we claim that the proof that (4.18) is finite can be reduced to show that

$$\int_{T_k} \left[\left(K_H^*\right)^{\otimes k} \left(\mathbf{1}_{\{\mathbf{s}\in T_k\}} \prod_{j=1}^k \left[s_j - s_{j-1}\right]^{-\frac{1}{4}} \right) \right]^2 (\mathbf{r}) d\mathbf{r} \tag{4.19}$$

is finite. In order to show this claim, we fix a constant a and we are going to compute

$$\left(K_H^*\right)^{\otimes k} \left(\mathbf{1}_{\{\mathbf{t}\in T_k\}} \prod_{j=1}^k \left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \right) (\mathbf{r}). \tag{4.20}$$

To do this we need some notation. Let Δ_j and I_j be the operators defined on a function $f(t_1, \dots, t_k)$ by

$$\Delta_j f = f - f|_{t_j=r_j},$$

and

$$I_j f = f|_{t_j=r_j}.$$

The operator K_H^* is the sum of two components (see 2.6), and it suffices to consider only the second one because the first one is easy to control. In this way, substituting the expression

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(H - \frac{1}{2}\right) t^{H-\frac{1}{2}} s^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}$$

in (4.20), and considering only the second component of the operator K_H^*, we obtain, up to a constant, a term of the form

$$\int_{T_k} \left[\int_{[0,T]^k} \Delta_1 \cdots \Delta_k \left(\prod_{j=1}^k t_j^{H-\frac{1}{2}} \left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} \right) \right.$$
$$\left. \times \prod_{j=1}^k (t_j - r_j)^{H-\frac{3}{2}} r_j^{\frac{1}{2}-H} \mathbf{1}_{\{r_j<t_j\}} \right]^2 d\mathbf{r}.$$

Because $t_j^{H-\frac{1}{2}} r_j^{\frac{1}{2}-H} \le 1$, we can disregard the factors $r_j^{\frac{1}{2}-H}$ and $t_j^{H-\frac{1}{2}}$. Using the rule

$$\begin{aligned}
\Delta_j(FG) &= F(t_j)G(t_j) - F(r_j)G(r_j) \\
&= \left[F(t_j) - F(r_j)\right] G(t_j) + F(r_j) \left[G(t_j) - G(r_j)\right] \\
&= \Delta_j F G + I_j F \Delta_j G,
\end{aligned}$$

we obtain

$$
\begin{aligned}
&\Delta_1 \cdots \Delta_k \left(\prod_{i=1}^{k} \left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} \right) \\
&\quad = \sum_S \prod_{j=1}^{k} S_j \left(\left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} \right),
\end{aligned}
$$

where S_j is an operator of the form:

$$
II_j,\, I\Delta_j,\, \Delta_{j-1}I_j,\, \Delta_{j-1}\Delta_j,
$$

where I denotes the identity, and for each j, Δ_j must appear only once in the product $\prod_{j=1}^{k} S_j$. Let us estimate each one of the possible four terms. Fix $\varepsilon > 0$ such that $H - \frac{3}{8} > 2\varepsilon$.

1. Term II_j:

$$
II_j \left(\left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} \right) = \left[r_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<r_j\}},
$$

2. Term $I\Delta_j$:

$$
\begin{aligned}
&\left| I\Delta_j \left(\left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} \right) \right| \\
&\quad = \left| \left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} - \left[r_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<r_j\}} \right| \\
&\quad \le C \left[t_j - r_j\right]^{\frac{1}{2}-H+\varepsilon} \left[r_j - t_{j-1} + a\right]^{H-1-\varepsilon} \mathbf{1}_{\{t_{j-1}<r_j\}} \\
&\qquad + C \left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{r_j<t_{j-1}\}}.
\end{aligned}
$$

3. Term $\Delta_{j-1}I$:

$$
\begin{aligned}
&\left| \Delta_{j-1}I \left(\left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} \right) \right| \\
&\quad = \left| \left[t_j - t_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{t_{j-1}<t_j\}} - \left[t_j - r_{j-1} + a\right]^{-\frac{1}{2}} \mathbf{1}_{\{r_{j-1}<t_j\}} \right| \\
&\quad \le C \left[t_{j-1} - r_{j-1}\right]^{\frac{1}{2}-H+\varepsilon} \left[t_j - t_{j-1} + a\right]^{H-1-\varepsilon} \mathbf{1}_{\{r_{j-1}<t_{j-1}<t_j\}}.
\end{aligned}
$$

4. Term $\Delta_{j-1}\Delta_j$:

$$\begin{aligned}
&\left|\Delta_{j-1}\Delta_j\left(\left[t_j-t_{j-1}+a\right]^{-\frac{1}{2}}\mathbf{1}_{\{t_{j-1}<t_j\}}\right)\right| \\
&= \left|\left[t_j-t_{j-1}+a\right]^{-\frac{1}{2}}\mathbf{1}_{\{t_{j-1}<t_j\}}-\left[r_j-t_{j-1}+a\right]^{-\frac{1}{2}}\mathbf{1}_{\{t_{j-1}<r_j\}}\right. \\
&\quad \left.-\left[t_j-r_{j-1}+a\right]^{-\frac{1}{2}}\mathbf{1}_{\{r_{j-1}<t_j\}}+\left[r_j-r_{j-1}+a\right]^{-\frac{1}{2}}\mathbf{1}_{\{r_{j-1}<r_j\}}\right| \\
&\le C\left[t_j-r_j\right]^{\frac{1}{2}-H+\varepsilon}\left[t_{j-1}-r_{j-1}\right]^{\frac{1}{2}-H+\varepsilon}\left[r_j-t_{j-1}+a\right]^{2H-\frac{3}{2}-2\varepsilon}\mathbf{1}_{\{t_{j-1}<r_j<t_j\}} \\
&\quad +C\left[t_{j-1}-r_{j-1}\right]^{\frac{1}{2}-H+\varepsilon}\left[t_j-t_{j-1}+a\right]^{H-1-\varepsilon}\;\mathbf{1}_{\{r_j<t_{j-1}<t_j\}} \\
&\quad +C\left[r_j-r_{j-1}+a\right]^{-\frac{1}{2}}\mathbf{1}_{\{r_{j-1}<r_j<t_{j-1}<t_j\}}.
\end{aligned}$$

If we replace the constant a by $s_j - s_{j-1}$ and we treat the term $s_j - s_{j-1}$ in the same way, using the inequality

$$(a+b)^{-\alpha} \le a^{-\frac{\alpha}{2}} b^{-\frac{\alpha}{2}},$$

we obtain the same estimates as if we had started with (4.19) instead of (4.18).

Step 4. As a consequence of the previous estimates, in order to show that (4.19) is finite it suffices to control the following integral

$$\int_{T_k}\left(\int_{T_k}\prod_{j=1}^{k}A_j^{a,b}(\mathbf{t},\mathbf{r})d\mathbf{t}\right)^2 d\mathbf{r}, \tag{4.21}$$

where $a, b \in \{0, 1\}$, and A_j has one of the following forms

$$\begin{aligned}
A_j^{0,0} &= \left[r_j-t_{j-1}\right]^{-\frac{1}{4}}\mathbf{1}_{\{t_{j-1}<r_j\}}, \\
A_{j,1}^{0,1} &= \left[t_j-r_j\right]^{-1+\varepsilon}\left[r_j-t_{j-1}\right]^{H-\frac{3}{4}-\varepsilon}\mathbf{1}_{\{t_{j-1}<r_j\}} \\
A_{j,2}^{0,1} &= \left[t_j-t_{j-1}\right]^{-\frac{1}{4}}\left[t_j-r_j\right]^{H-\frac{3}{2}}\mathbf{1}_{\{r_j<t_{j-1}\}}, \\
A_j^{1,0} &= \left[t_{j-1}-r_{j-1}\right]^{-1+\varepsilon}\left[t_j-t_{j-1}\right]^{H-\frac{3}{4}-\varepsilon}\mathbf{1}_{\{r_{j-1}<t_{j-1}<t_j\}}, \\
A_{j,1}^{1,1} &= \left[t_j-r_j\right]^{-1+\varepsilon}\left[t_{j-1}-r_{j-1}\right]^{-1+\varepsilon}\left[r_j-t_{j-1}\right]^{2H-\frac{5}{4}-2\varepsilon}\mathbf{1}_{\{t_{j-1}<r_j<t_j\}}, \\
A_{j,2}^{1,1} &= \left[t_{j-1}-r_{j-1}\right]^{-1+\varepsilon}\left[t_j-t_{j-1}\right]^{H-\frac{5}{4}-\varepsilon}\;\left[t_j-r_j\right]^{H-\frac{3}{2}}\mathbf{1}_{\{r_j<t_{j-1}<t_j\}}, \\
A_{j,3}^{1,1} &= \left[r_j-r_{j-1}\right]^{-\frac{1}{4}}\left[t_j-r_j\right]^{H-\frac{3}{2}}\left[t_{j-1}-r_{j-1}\right]^{H-\frac{3}{2}}\mathbf{1}_{\{r_{j-1}<r_j<t_{j-1}<t_j\}},
\end{aligned}$$

and with the convention that any term of the form $A_j^{0,1}$ or $A_j^{1,1}$ must be followed by $A_j^{0,0}$ or $A_j^{0,1}$ and any term of the form $A_j^{0,0}$ or $A_j^{1,0}$ must be followed by $A_j^{1,0}$ or $A_j^{1,1}$.

It is not difficult to check that the integral (4.21) is finite. For instance, for a product of the form $A^{0,0}_{j-1}A^{1,1}_{j,1}$ we get

$$\int\limits_{\{r_{j-1}<t_{j-1}<r_j<t_j\}} \left[r_{j-1}-t_{j-2}\right]^{-\frac{1}{4}} \left[t_{j-1}-r_{j-1}\right]^{-1+\varepsilon} \left[r_j-t_{j-1}\right]^{2H-\frac{5}{4}-2\varepsilon}$$
$$\times \left[t_j-r_j\right]^{-1+\varepsilon} \, dt_{j-1}$$
$$= \left[r_{j-1}-t_{j-2}\right]^{-\frac{1}{4}} \left[r_j-r_{j-1}\right]^{2H-\frac{5}{4}-\varepsilon} \left[t_j-r_j\right]^{-1+\varepsilon},$$

and the integral in the variable r_j of the square of this expression will be finite because $4H-\frac{5}{2}-2\varepsilon>-1$.

So, we have proved that $\sup_\varepsilon E(J_\varepsilon^k)<\infty$ for all k. Notice that all these moments are positive. It holds that $\lim_{\varepsilon,\delta\downarrow 0} E(J_\varepsilon J_\delta)$ exists, and this implies the convergence in L^2, and also in L^p, for all $p\geq 2$. □

On the other hand, if the initial condition of Eq. (1.1) is a constant K, then for all $n\geq 1$ we have

$$(n!)^2 \left\| f_n(\cdot,t,x)\right\|^2_{\mathcal{H}_1^{\otimes n}} = K^2 E\left[\left(\int\limits_0^t \left(K_H^*\right)^{\otimes 2} \delta_0(B^1_\cdot - B^2_\cdot)(r,r)dr\right)^n\right] < \infty,$$

provided $H\in\left(\frac{3}{8},\frac{1}{2}\right)$. In fact, by Lemma 4.2 we have

$$(n!)^2 \left\| f_n(\cdot,t,x)\right\|^2_{\mathcal{H}_1^{\otimes n}} = K^2 \int\limits_{[0,t]^{2n}} \langle g_{\mathbf{s}}, g_{\mathbf{t}}\rangle_{L^2(\mathbb{R}^n)} \prod_{i=1}^n K_H^*(dt_i,r_i)$$
$$\times \prod_{i=1}^n K_H^*(ds_i,r_i) d\mathbf{s}d\mathbf{t}$$
$$= K^2 \int\limits_{[0,t]^{2n}} \psi(\mathbf{s},\mathbf{t}) \prod_{i=1}^n K_H^*(dt_i,r_i) \prod_{i=1}^n K_H^*(ds_i,r_i) d\mathbf{s}d\mathbf{t},$$

and it suffices to apply the above proposition. Here $\int_0^\infty \varphi(t)K_H^*(dt,r)$ is a notation for $(K_H^*\varphi)(r)$.

However, we do not know the rate of convergence of the sequence $\left\| f_n(\cdot,t,x)\right\|^2_{\mathcal{H}_1^{\otimes n}}$ as n tends to infinity, and for this reason we are not able to show the existence of a solution to Eq. (1.1) in this case.

5 Moments of the solution

In this section we introduce an approximation of the Gaussian noise W^H by means of an approximation of the identity. In the space variable we choose the heat kernel to define this approximation and in the time variable we choose a rectangular kernel. In this way, for any $\varepsilon > 0$ and $\delta > 0$ we set

$$\dot{W}^{\varepsilon,\delta}_{t,x} = \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s) p_\varepsilon(x-y) dW^H_{s,y}, \tag{5.1}$$

where

$$\varphi_\delta(t) = \frac{1}{\delta}\mathbf{1}_{[0,\delta]}(t).$$

Now we consider the approximation of Eq. (1.1) defined by

$$\frac{\partial u^{\varepsilon,\delta}_{t,x}}{\partial t} = \frac{1}{2}\Delta u^{\varepsilon,\delta}_{t,x} + u^{\varepsilon,\delta}_{t,x} \diamond \dot{W}^{\varepsilon,\delta}_{t,x}. \tag{5.2}$$

We recall that the Wick product $u^{\varepsilon,\delta}_{t,x} \diamond \dot{W}^{\varepsilon,\delta}_{t,x}$ is well defined as a square integrable random variable provided the random variable $u^{\varepsilon,\delta}_{t,x}$ belongs to the space $\mathbb{D}^{1,2}$ (see 2.9), and in this case we have

$$u^{\varepsilon,\delta}_{s,y} \diamond \dot{W}^{\varepsilon,\delta}_{s,y} = \int_0^s \int_{\mathbb{R}^d} \varphi_\delta(s-r) p_\varepsilon(y-z) u^{\varepsilon,\delta}_{s,y} \delta W^H_{r,z}. \tag{5.3}$$

The mild or evolution version of Eq. (5.2) will be

$$u^{\varepsilon,\delta}_{t,x} = p_t u_0(y) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u^{\varepsilon,\delta}_{s,y} \diamond \dot{W}^{\varepsilon,\delta}_{s,y} ds dy. \tag{5.4}$$

Substituting (5.3) into (5.4), and formally applying Fubini's theorem yields

$$u^{\varepsilon,\delta}_{t,x} = p_t u_0(y) + \int_0^t \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \varphi_\delta(s-r) p_\varepsilon(y-z) u^{\varepsilon,\delta}_{s,y} ds dy \right) \delta W^H_{r,z}. \tag{5.5}$$

This leads to the following definition.

Definition 5.1 An adapted random field $u^{\varepsilon,\delta} = \{u^{\varepsilon,\delta}_{t,x}, t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to Eq. (5.2) if for each $(r, z) \in \mathbb{R}_+ \times \mathbb{R}^d$ the integral

$$Y^{t,x}_{r,z} = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \varphi_\delta(s-r) p_\varepsilon(y-z) u^{\varepsilon,\delta}_{s,y} ds dy$$

exists and $Y^{t,x}$ is a Skorohod integrable process such that (5.5) holds for each (t, x).

The above definition is equivalent to saying that $u^{\varepsilon,\delta}_{t,x} \in L^2(\Omega)$, and for any random variable $F \in \mathbb{D}^{1,2}$, we have

$$E(Fu^{\varepsilon,\delta}_{t,x}) = E(F)p_t u_0(y) + \left\langle \left(\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\varphi_\delta(s-\cdot)p_\varepsilon(y-\cdot)u^{\varepsilon,\delta}_{s,y}dsdy \right), DF \right\rangle_{\mathcal{H}_d}, \tag{5.6}$$

where $\mathcal{H}_d = \mathcal{H} \otimes L^2(\mathbb{R}^d)$ (see 2.2).

Our aim is to construct a solution of Equation (5.2) using a suitable version of Feynman-Kac's formula. Suppose that $B = \{B_t, t \geq 0\}$ is a d-dimensional Brownian motion starting at 0, independent of W. Set

$$\begin{aligned}\int_0^t \dot{W}^{\varepsilon,\delta}_{t-s,x+B_s}ds &= \int_0^t \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s-r)p_\varepsilon(B_s+x-y)dW^H_{r,y}ds \\ &= \int_0^t \int_{\mathbb{R}^d} A^{\varepsilon,\delta}_{r,y}dW^H_{r,y},\end{aligned}$$

where

$$A^{\varepsilon,\delta}_{r,y} = \int_0^t \varphi_\delta(t-s-r)p_\varepsilon(B_s+x-y)ds. \tag{5.7}$$

Define

$$u^{\varepsilon,\delta}_{t,x} = E^B\left(u_0(x+B_t)\exp\left(\int_0^t \int_{\mathbb{R}^d} A^{\varepsilon,\delta}_{r,y}dW^H_{r,y} - \frac{1}{2}\alpha^{\varepsilon,\delta}\right)\right), \tag{5.8}$$

where $\alpha^{\varepsilon,\delta} = \left\| A^{\varepsilon,\delta} \right\|^2_{\mathcal{H}_d}$.

Proposition 5.2 *The random field $u^{\varepsilon,\delta}_{t,x}$ given by (5.8) is a solution to Eq. (5.2).*

Proof It suffices to show that (5.6) holds for a random variable of the form F_φ, where for any $\varphi \in \mathcal{H}_1$ we set

$$F_\varphi = \exp\left(W^H(\varphi) - \frac{1}{2}\left\|\varphi\right\|^2_{\mathcal{H}_d}\right),$$

because the set of these random variables spans a dense subset of $\mathbb{D}^{1,2}$. We will make use of the notation $S_{t,x}(\varphi) = E\left(u_{t,x}^{\varepsilon,\delta} F_\varphi\right)$. From (5.8) we have

$$\begin{aligned} S_{t,x}(\varphi) &= EE^B\left(u_0(x+B_t)\exp\left(W^H(A^{\varepsilon,\delta}+\varphi) - \frac{1}{2}\alpha^{\varepsilon,\delta} - \frac{1}{2}\|\varphi\|^2_{\mathcal{H}_d}\right)\right) \\ &= E^B\left(u_0(x+B_t)\exp\left(\langle A^{\varepsilon,\delta},\varphi\rangle_{\mathcal{H}_d}\right)\right) \\ &= E^B\left(u_0(x+B_t)\exp\left(\int_0^t \langle \varphi_\delta(t-s-\cdot)p_\varepsilon(B_s+x-\cdot),\varphi\rangle_{\mathcal{H}_d}\,ds\right)\right). \end{aligned}$$

By the classical Feynman-Kac's formula, $S_{t,x}(\varphi)$ satisfies the heat equation with potential $V(t,x) = \langle \varphi_\delta(t-\cdot)p_\varepsilon(x-\cdot),\varphi\rangle_{\mathcal{H}_d}$, that is,

$$\frac{\partial S_{t,x}(\varphi)}{\partial t} = \frac{1}{2}\Delta S_{t,x}(\varphi) + S_{t,x}(\varphi)\,\langle \varphi_\delta(t-\cdot)p_\varepsilon(x-\cdot),\varphi\rangle_{\mathcal{H}_d}.$$

As a consequence,

$$S_{t,x}(\varphi) = p_t u_0(x) + \int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)S_{s,y}(\varphi)\,\langle \varphi_\delta(s-\cdot)p_\varepsilon(y-\cdot),\varphi\rangle_{\mathcal{H}_d}\,dsdy,$$

which implies Eq. (5.6) because $DF_\varphi = \varphi F_\varphi$. □

The next theorem says that the random variables $u_{t,x}^{\varepsilon,\delta}$ have moments of all orders, uniformly bounded in ε and δ, and converge to the solution to Eq. (1.1), which is unique by Proposition 4.3, as δ and ϵ tend to zero. Moreover, it provides an expression for the moments of the solution to Eq. (1.1).

Theorem 5.3 *Suppose that $H \geq \frac{1}{2}$ and $d = 1$. Then, for any integer $k \geq 1$ we have*

$$\sup_{\varepsilon,\delta} E\left[\left|u_{t,x}^{\varepsilon,\delta}\right|^k\right] < \infty, \tag{5.9}$$

and the limit $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} u_{t,x}^{\varepsilon,\delta}$ exists in L^p, for all $p \geq 1$, and it coincides with the solution $u_{t,x}$ of Eq. (4.1). Furthermore, if $U_0^B(t,x) = \prod_{j=1}^k u_0(x+B_t^j)$, where B^j are independent d-dimensional Brownian motions, we have for any $k \geq 2$

$$E\left[u_{t,x}^k\right] = E^B\left[U_0^B(t,x)\exp\left(\sum_{i<j}\int_0^t \delta_0(B_s^i - B_s^j)ds\right)\right], \tag{5.10}$$

if $H = \frac{1}{2}$, *and*

$$E\left[u_{t,x}^k\right] = E^B\left[U_0^B(t,x)\exp\left(\sum_{i<j}\int_0^t\int_0^t \phi(s,r)\delta_0(B_s^i - B_r^j)dsdr\right)\right], \quad (5.11)$$

if $H > \frac{1}{2}$.

In the case $d = 2$, *for any integer* $k \geq 2$ *there exists* $t_0(k) > 0$ *such that for all* $t < t_0(k)$ (5.9) *holds. If* $t < t_0(M)$ *for some* $M \geq 3$ *then the limit* $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} u_{t,x}^{\varepsilon,\delta}$ *exists in* L^p *for all* $2 \leq p < M$, *and it coincides with the solution* $u_{t,x}$ *of Eq.* (1.1). *Moreover* (5.11) *holds for all* $1 \leq k \leq M-1$.

Proof Fix an integer $k \geq 2$. Suppose that $B^i = \left\{B_t^i, t \geq 0\right\}$, $i = 1, \ldots, k$ are independent d-dimensional standard Brownian motions starting at 0, independent of W^H. Then, using (5.8) we have

$$E\left[\left(u_{t,x}^{\varepsilon,\delta}\right)^k\right] = E\left(\prod_{j=1}^k E^B\left[u_0(x+B_t^j)\exp\left(\int_0^t\int_{\mathbb{R}} A_{r,y}^{\varepsilon,\delta,B^j}dW_{r,y}^H - \frac{1}{2}\alpha^{\varepsilon,\delta,B^j}\right)\right]\right),$$

where $A_{r,y}^{\varepsilon,\delta,B^j}$ and $\alpha^{\varepsilon,\delta,B^j}$ are computed using the Brownian motion B^j. Therefore,

$$\begin{aligned} E\left[\left(u_{t,x}^{\varepsilon,\delta}\right)^k\right] &= E^B\left[\exp\left(\frac{1}{2}\left\|\sum_{j=1}^k A^{\varepsilon,\delta,B^j}\right\|_{\mathcal{H}_d}^2 - \frac{1}{2}\sum_{j=1}^k \alpha^{\varepsilon,\delta,B^j}\right)\prod_{j=1}^k u_0(x+B_t^j)\right] \\ &= E^B\left[\exp\left(\sum_{i<j}\left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_d}\right)\prod_{j=1}^k u_0(x+B_t^j)\right]. \end{aligned}$$

That is, the correction term $\frac{1}{2}\alpha^{\varepsilon,\delta}$ in (5.8) due to the Wick product produces a cancellation of the diagonal elements in the square norm of $\sum_{j=1}^k A^{\varepsilon,\delta,B^j}$. The next step is to compute the scalar product $\left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_d}$ for $i \neq j$. We consider two cases.

Case 1. Suppose first that $H = \frac{1}{2}$ and $d = 1$. In this case we have

$$\begin{aligned} \left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_1} &= \int_{\mathbb{R}}\int_0^t\int_0^t\int_0^t \varphi_\delta(t-s_1-r)p_\varepsilon(B_{s_1}^i + x - y) \\ &\quad\times\varphi_\delta(t-s_2-r)p_\varepsilon(B_{s_2}^j + x - y)ds_1ds_2drdy \\ &= \int_0^t\int_0^t\int_0^t \varphi_\delta(t-s_1-r)\varphi_\delta(t-s_2-r) \\ &\quad\times p_{2\varepsilon}(B_{s_1}^i - B_{s_2}^j)ds_1ds_2dr. \end{aligned}$$

We have

$$\begin{aligned}
&\int_0^t \varphi_\delta(t-s_1-r)\varphi_\delta(t-s_2-r)dr \\
&\quad = \delta^{-2}\left[(t-s_1)\wedge(t-s_2)-(t-s_1-\delta)^+\vee(t-s_2-\delta)^+\right]^+ \\
&\quad = \eta_\delta(s_1,s_2).
\end{aligned}$$

It is easy to check that η_δ is a a symmetric function on $[0,t]^2$ such that for any continuous function g on $[0,t]^2$,

$$\lim_{\delta\downarrow 0}\int_0^t\int_0^t \eta_\delta(s_1,s_2)g(s_1,s_2)ds_1ds_2=\int_0^t g(s,s)ds.$$

As a consequence the following limit holds almost surely

$$\lim_{\delta\downarrow 0}\left\langle A^{\varepsilon,\delta,B^i},A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_1}=\int_0^t p_{2\varepsilon}(B_s^i-B_s^j)ds,$$

and by the properties of the local time of the one-dimensional Brownian motion we obtain that, almost surely.

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\left\langle A^{\varepsilon,\delta,B^i},A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_1}=\int_0^t \delta_0(B_s^i-B_s^j)ds.$$

The function η_δ satisfies

$$\sup_{0\le r\le t}\int_0^t \eta_\delta(s,r)ds\le 1,$$

and, as a consequence, the estimate (3.11) implies that for all $\lambda>0$

$$\sup_{\varepsilon,\delta} E^B\left[\lambda\exp\left\langle A^{\varepsilon,\delta,B^i},A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_1}\right]<\infty.$$

Hence (5.9) holds and $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}u_{t,x}^{\varepsilon,\delta}:=v_{t,x}$ exists in L^p, for all $p\ge 1$. Moreover, $E(v_{t,x}^k)$ equals to the right-hand side of Eq. (5.10). Finally, Eq. (5.5) and the duality relationship (2.8) imply that for any random variable $F\in\mathbb{D}^{1,2}$ with zero mean we have

$$E\left(Fu_{t,x}^{\varepsilon,\delta}\right)=E\left(\left\langle DF,\left(\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\varphi_\delta(s-\cdot)p_\varepsilon(y-\cdot)u_{s,y}^{\varepsilon,\delta}dsdy\right)\right\rangle_{\mathcal{H}_1}\right),$$

and letting δ and ε tend to zero we get

$$E\left(Fv_{t,x}\right) = E\left(\left\langle DF, vp_{t-\cdot}(x-\cdot)\right\rangle_{\mathcal{H}_1}\right)$$

which implies that the process v is the solution of Eq. (1.1), and by the uniqueness of solution proved in Proposition 4.3, $v_{t,x} = u_{t,x}$.

Case 2. Consider now the case $H > \frac{1}{2}$ and $d = 2$. We have

$$\begin{aligned}\left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_d} &= \int_{\mathbb{R}^2}\int_0^t\int_0^t\int_0^t\int_0^t \varphi_\delta(t-s_1-r_1)p_\varepsilon(B^i_{s_1}+x-y)\\ &\quad\times\varphi_\delta(t-s_2-r_2)p_\varepsilon(B^j_{s_2}+x-y)ds_1ds_2\,\phi(r_1,r_2)dr_1dr_2dy\\ &= \int_0^t\int_0^t\int_0^t\int_0^t \varphi_\delta(t-s_1-r_1)\varphi_\delta(t-s_2-r_2)\\ &\quad\times p_{2\varepsilon}(B^i_{s_1}-B^j_{s_2})ds_1ds_2\phi(r_1,r_2)dr_1dr_2.\end{aligned}$$

This scalar product can be written in the following form

$$\left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\right\rangle_{\mathcal{H}_d} = \int_0^t\int_0^t \eta_\delta(s_1,s_2)p_{2\varepsilon}(B^i_{s_1}-B^j_{s_2})ds_1ds_2,$$

where

$$\eta_\delta(s_1,s_2) = \int_0^t\int_0^t \varphi_\delta(t-s_1-r_1)\varphi_\delta(t-s_2-r_2)\,\phi(r_1,r_2)dr_1dr_2. \tag{5.12}$$

We claim that there exists a constant γ such that

$$\eta_\delta(s_1,s_2) \le \gamma|s_1-s_2|^{2H-2}. \tag{5.13}$$

In fact, we can assume that $s = s_2 - s_1 \ge 0$. We have

$$\begin{aligned}\eta_\delta(s_1,s_2) &\le H(2H-1)\delta^{-2}\int_{s_1}^{s_1+\delta}\int_{s_2}^{s_2+\delta}|u-v|^{2H-2}dudv\\ &= \delta^{-2}E\left[\left(B^H_{s_2+\delta}-B^H_{s_2}\right)\left(B^H_{s_1+\delta}-B^H_{s_1}\right)\right]\\ &= \frac{1}{2\delta^2}\left[(s+\delta)^{2H}-|s-\delta|^{2H}-2s^{2H}\right],\end{aligned}$$

where B^H is a fractional Brownian motion with Hurst parameter H. Then, if $s \geq \delta$

$$\eta_\delta(s_1, s_2) \leq H\delta^{-2} \int_s^{s+\delta} \left(y^{2H-1} - (y-\delta)^{2H-1}\right) dy \leq H\delta^{2H-2} \leq H2^{2-2H} s^{2H-2},$$

which implies (5.13). On the other hand, if $s < \delta$, clearly $(s+\delta)^{2H} - |s-\delta|^{2H} - 2s^{2H}$ is bounded by a constant times δ^{2H}, and again (5.13) holds.

It it easy to check that for any continuous function g on $[0, t]^2$,

$$\lim_{\delta \downarrow 0} \int_0^t \int_0^t \eta_\delta(s_1, s_2) g(s_1, s_2) ds_1 ds_2 = \int_0^t \int_0^t \phi(s_1, s_2) g(s_1, s_2) ds_1 ds_2.$$

As a consequence the following limit holds almost surely

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \right\rangle_{\mathcal{H}_d} = \int_0^t \int_0^t \phi(s_1, s_2) \delta_0(B^i_{s_1} - B^j_{s_2}) ds_1 ds_2.$$

From (5.13) and the estimate (3.17) we get

$$\sup_{\varepsilon,\delta} E^B \left[\exp\left(\lambda \left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \right\rangle_{\mathcal{H}_d}\right)\right] < \infty, \tag{5.14}$$

if $\lambda < \lambda_0(t)$, where $\lambda_0(t)$ is defined in (3.16) with γ_T replaced by γ.

Hence, for any integer $k \geq 2$, if $t < t_0(k)$, where $\frac{k(k-1)}{2} = \lambda_0(t_0(k))$, then (5.9) holds because

$$E\left[\left(u^{\varepsilon,\delta}_{t,x}\right)^k\right] \leq \|u_0\|^k \left(E^B\left[\exp\left(\frac{k(k-1)}{2} \left\langle A^{\varepsilon,\delta,B^1}, A^{\varepsilon,\delta,B^2} \right\rangle_{\mathcal{H}_d}\right)\right]\right)^{\frac{2}{k(k-1)}}.$$

Finally, if $t < t_0(M)$ and $M \geq 3$, the limit $\lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} u^{\varepsilon,\delta}_{t,x} := v_{t,x}$ exists in L^p, for all $2 \leq p < M$ and it is equal to the right-hand side of Eq. (5.11). Finally, we conclude that $v_{t,x} = u_{t,x}$ by the same arguments as in the case 1. □

6 Pathwise heat equation

In this section we consider the one-dimensional stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \dot{W}^H_{t,x}, \tag{6.1}$$

where the product between the solution u and the noise $\dot{W}^H_{t,x}$ is now an ordinary product. We first introduce a notion of solution using the Stratonovich integral and a weak formulation of the mild solution. Given a random field $v = \{v_{t,x}, t \geq 0, x \in \mathbb{R}\}$ such that $\int_0^T \int_{\mathbb{R}} |v_{t,x}| \, dxdt < \infty$ a.s. for all $T > 0$, the Stratonovich integral

$$\int_0^T \int_{\mathbb{R}} v_{t,x} \circ dW_{t,x}^H$$

is defined as the following limit in probability if it exists

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_0^T \int_{\mathbb{R}} v_{t,x} \dot{W}_{t,x}^{\varepsilon,\delta} dxdt,$$

where $W_{t,x}^{\varepsilon,\delta}$ is the approximation of the noise W^H introduced in (5.1). This generalized integral has been defined through the regularization method studied, among others, by Russo and Vallois in [17].

Definition 6.1 A random field $u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}\}$ is a weak solution to Eq. (6.1) if for any C^∞ function φ with compact support on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} u_{t,x}\varphi(x)dx = \int_{\mathbb{R}} u_0(x)\varphi(x)dx + \int_0^t \int_{\mathbb{R}} u_{s,x}\varphi''(x)dxds + \int_0^t \int_{\mathbb{R}} u_{s,x}\varphi(x) \circ dW_{s,x}^H.$$

Consider the approximating stochastic heat equation

$$\frac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{1}{2}\Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta} \dot{W}_{t,x}^{\varepsilon,\delta}. \tag{6.2}$$

Theorem 6.2 *Suppose that $H > \frac{3}{4}$. For any $p \geq 2$, the limit*

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} u_{t,x}^{\varepsilon,\delta} = u_{t,x}$$

exists in L^p, and defines a weak solution to Eq. (6.2) in the sense of Definition 6.1. Furthermore, for any positive integer k

$$E\left(u_{t,x}^k\right) = E^B\left[U_0^B(t,x)\exp\left(\sum_{i,j=1}^k \int_0^t \int_0^t \phi(s_1,s_2)\delta(B_{s_1}^i - B_{s_2}^j)ds_1ds_2\right)\right],$$

where $U_0^B(t,x)$ has been defined in Theorem (5.3).

Proof By Feynman-Kac's formula we can write

$$u_{t,x}^{\varepsilon,\delta} = E^B \left\{ u_0(x+B_t) \exp\left(\int_0^t \int_{\mathbb{R}} A_{r,y}^{\varepsilon,\delta} dW_{r,y}^H \right) \right\}, \tag{6.3}$$

where $A_{r,y}^{\varepsilon,\delta}$ has been defined in (5.7). We will first show that for all $k \geq 1$

$$\sup_{\delta,\varepsilon} E\left[\left| u_{t,x}^{\varepsilon,\delta} \right|^k \right] < \infty. \tag{6.4}$$

Suppose that $B^i = \left\{ B_t^i, t \geq 0 \right\}$, $i = 1, \ldots, k$ are independent standard Brownian motions starting at 0, independent of W^H. Then, we have, as in the proof of Theorem 5.3

$$E\left(\left(u_{t,x}^{\varepsilon,\delta} \right)^k \right) = E^{\,B} \left[\exp\left(\frac{1}{2} \sum_{i,j=1}^k \left\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j} \right\rangle_{\mathcal{H}_1} \right) U_0^B(t,x) \right]. \tag{6.5}$$

Notice that

$$\left\langle A^{\varepsilon,\delta B^i}, A^{\varepsilon,\delta B^j} \right\rangle_{\mathcal{H}_1} = \int_0^t \int_0^t \eta_\delta(s_1,s_2) p_{2\varepsilon}(B_{s_1}^i - B_{s_2}^j) ds_1 ds_2,$$

where $\eta_\delta(s_1,s_2)$ satisfies (5.13). As a consequence, the inequalities (3.11) and (3.21) and the fact that $H > \frac{3}{4}$, imply that for all $\lambda > 0$, and all i,j we have

$$\sup_{\varepsilon,\delta} E\left(\exp \lambda \left\langle A^{\varepsilon,\delta B^i}, A^{\varepsilon,\delta B^j} \right\rangle_{\mathcal{H}_1} \right) < \infty.$$

Thus, (6.4) holds, and

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} E\left[\left(u_{t,x}^{\varepsilon,\delta} \right)^k \right] \\ &= E^B \exp\left[U_0^B(t,x) \exp\left(\frac{1}{2} \sum_{i,j=1}^k \int_0^t \int_0^t \phi(s_1,s_2) \delta_0(B_{s_1}^i - B_{s_2}^j) ds_1 ds_2 \right) \right]. \end{aligned} \tag{6.6}$$

In a similar way we can show that the limit $\lim_{\varepsilon,\varepsilon'\downarrow 0} \lim_{\delta,\delta'\downarrow 0} E\left(u_{t,x}^{\varepsilon,\delta} u_{t,x}^{\varepsilon',\delta'} \right)$ exists. Therefore, the iterated limit $\lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} u_{t,x}^{\varepsilon,\delta}$ exists in L^2. Furthermore, the convergence also holds in L^p, for all $p \geq 2$, and from (6.5) it follows that this convergence is uniform in (t,x), if $0 \leq t \leq T$ and $|x| \leq K$.

Finally, in order to show that u solves Eq. (6.1) we need to show that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\left(\int_0^t\int_{\mathbb{R}} u_{s,x}\varphi(x)\circ dW_{s,x}^H - \int_0^t\int_{\mathbb{R}} u_{s,x}^{\varepsilon,\delta}\varphi(x)\dot{W}_{s,x}^{\varepsilon,\delta}dsdx\right) = 0,$$

in probability.

From the weak equation satisfied by $u_{t,x}^{\varepsilon,\delta}$, it follows that $\int_0^t\int_{\mathbb{R}} u_{s,x}^{\varepsilon,\delta}\varphi(x)\dot{W}_{s,x}^{\varepsilon,\delta}dsdx$ converges in L^2 to some random variable G as $\delta\downarrow 0$ and $\varepsilon\downarrow 0$. Hence, if

$$B_{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}} \left(u_{s,x}^{\varepsilon,\delta} - u_{s,x}\right)\varphi(x)\dot{W}_{s,x}^{\varepsilon,\delta}dsdx \tag{6.7}$$

converges in L^2 to zero, $u_{s,x}\varphi(x)$ will be Stratonovich integrable and

$$\int_0^t\int_{\mathbb{R}} u_{s,x}\varphi(x)\circ dW_{s,x}^H = G.$$

In order to show the convergence to zero of (6.7), we will express the product $\left(u_{s,x}^{\varepsilon,\delta} - u_{s,x}\right)\dot{W}_{s,x}^{\varepsilon,\delta}$ as the sum of a divergence integral plus a trace term (see, for instance, Proposition 1.3.3 in [13]). In this way we obtain

$$\begin{aligned}\left(u_{s,x}^{\varepsilon,\delta} - u_{s,x}\right)\dot{W}_{s,x}^{\varepsilon,\delta} &= \int_0^t\int_{\mathbb{R}} \left(u_{s,x}^{\varepsilon,\delta} - u_{s,x}\right)\varphi_\delta(s-r)p_\varepsilon(x-z)\delta W_{r,z}^H\\ &\quad + \left\langle D\left(u_{s,x}^{\varepsilon,\delta} - u_{s,x}\right), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\right\rangle_{\mathcal{H}_1}.\end{aligned} \tag{6.8}$$

Substituting (6.8) into (6.7) yields

$$\begin{aligned}B_{\varepsilon,\delta} &= \int_0^t\int_{\mathbb{R}} \phi_{r,z}^{\varepsilon,\delta}\delta W_{r,z}^H\\ &\quad + \int_0^t\int_{\mathbb{R}} \varphi(x)\left\langle D\left(u_{s,x}^{\varepsilon,\delta} - u_{s,x}\right), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\right\rangle_{\mathcal{H}_1} dsdx\\ &= B_{\varepsilon,\delta}^1 + B_{\varepsilon,\delta}^2,\end{aligned}$$

where

$$\phi_{r,z}^{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}} \left(u_{s,x}^{\varepsilon,\delta} - u_{s,x}\right)\varphi(x)\varphi_\delta(s-r)p_\varepsilon(x-z)dsdx.$$

For the term $B^1_{\varepsilon,\delta}$, using the estimates on the L^2 norm of the Skorohod integral (see (1.47) in [13]), we obtain

$$E\left[\left(B^1_{\varepsilon,\delta}\right)^2\right] \le E\left(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}_1}\right) + E\left(\|D\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}_1\otimes\mathcal{H}_1}\right).$$

We have

$$\begin{aligned}
E\left(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}_1}\right) &= \int_0^t\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} E\left(\left(u^{\varepsilon,\delta}_{s,x} - u_{s,x}\right)\left(u^{\varepsilon,\delta}_{r,y} - u_{r,y}\right)\right)\varphi(x)\varphi(y)\\
&\quad \times \langle \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot), \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}_1}\, dsdxdrdz\\
&= \int_0^t\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} E\left(\left(u^{\varepsilon,\delta}_{s,x} - u_{s,x}\right)\left(u^{\varepsilon,\delta}_{r,y} - u_{r,y}\right)\right)\varphi(x)\varphi(y)\\
&\quad \times \eta_\delta(s-r)p_{2\varepsilon}(x-y)dsdxdrdz,
\end{aligned}$$

where $\eta_\delta(s-r)$ has been defined in (5.12). Then, applying the estimate (5.13) and assuming that $\varphi(x) = 0$ if $|x| > K$, yields

$$E\left(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}_1}\right) \le C \sup_{\substack{0\le s\le t\\ |x|\le K}} E\left(\left(u^{\varepsilon,\delta}_{s,x} - u_{s,x}\right)^2\right),$$

which converges to zero as $\delta \downarrow 0$, and $\varepsilon \downarrow 0$. On the other hand, we have

$$D\left(u^{\varepsilon,\delta}_{s,x}\right) = E^{\,B}\left\{u_0(x+B_t)\exp\left(\int_0^t\int_{\mathbb{R}} A^{\varepsilon,\delta}_{s,y}dW^H_{s,y}\right)A^{\varepsilon,\delta}\right\}$$

and similarly to (6.6) we can show that

$$\begin{aligned}
&\lim_{\varepsilon,\varepsilon'\downarrow 0}\lim_{\delta.\delta'\downarrow 0} E\left(\left\langle D\left(u^{\varepsilon,\delta}_{s,x}\right), D\left(u^{\varepsilon',\delta'}_{s,x}\right)\right\rangle^2_{\mathcal{H}_1}\right)\\
&= E^{\,B}\left[u_0(x+B^1_t)u_0(x+B^2_t)\exp\left(\sum_{i,j=1}^2\int_0^t\int_0^t \phi(s_1,s_2)\delta_0(B^i_{s_1} - B^j_{s_2})ds_1ds_2\right)\right.\\
&\qquad\left.\times\int_0^t\int_0^t \phi(s_1,s_2)\delta_0(B^1_{s_1} - B^2_{s_2})ds_1ds_2\right].
\end{aligned}$$

This implies that $u^{\varepsilon,\delta}_{s,x}$ converges in the space $\mathbb{D}^{1,2}$ to $u_{s,x}$ as $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, and the convergence is uniform in (t,x), whenever $0 \le t \le T$ and $|x| \le K$. Then,

$$E\left(\left\| D\phi^{\varepsilon,\delta}\right\|^2_{\mathcal{H}_1\otimes\mathcal{H}_1}\right)$$
$$= \int_0^t \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} E\left(\left\langle D\left(u^{\varepsilon,\delta}_{s,x} - u_{s,x}\right), D\left(u^{\varepsilon,\delta}_{r,y} - u_{r,y}\right)\right\rangle_{\mathcal{H}_1}\right) \varphi(x)\varphi(y)$$
$$\times \langle \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot), \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}_1}\, dsdxdrdz$$

converges to zero as $\delta \downarrow 0$ and $\varepsilon \downarrow 0$. Finally, notice that

$$\left\langle A^{\varepsilon,\delta}, \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\right\rangle_{\mathcal{H}_1} = \int_0^t \eta_\delta(t-s,s)p_{2\varepsilon}(B_s)ds,$$

and

$$\left\langle D\left(u^{\varepsilon,\delta}_{s,x}\right), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\right\rangle_{\mathcal{H}_1}$$
$$= E^B\left\{u_0(x+B_t)\left[\exp\left(\int_0^t\int_{\mathbb{R}} A^{\varepsilon,\delta}_{s,y} dW^H_{s,y}\right)\int_0^t \eta_\delta(t-s,s)p_{2\varepsilon}(B_s)ds\right]\right\}.$$

From this expression, it follows easily that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} E\left[(B^2_{\varepsilon,\delta})^2\right] = 0.$$

This completes the proof. □

Since the solution is square integrable it admits a Wiener-Itô chaos expansion. The explicit form of the Wiener chaos coefficients are given below.

Theorem 6.3 *The solution to* (6.1) *is given by*

$$u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)) \tag{6.9}$$

where

$$f_n(t_1,x_1,\ldots,t_n,x_n,t,x)$$
$$= E^B\left[u_0(x+B_t)\exp\left(\frac{1}{2}\int_0^t\int_0^t \phi(s_1,s_2)\delta_0(B_{s_1}-B_{s_2})ds_1ds_2\right)\right.$$
$$\left.\times\delta_0(B_{t_1}+x-x_1)\cdots\delta_0(B_{t_n}+x-x_n)\right]. \tag{6.10}$$

Proof From the Feynman-Kac formula it follows that

$$
\begin{aligned}
u_{t,x}^{\varepsilon,\delta} &= E^B\left(u_0(x+B_t)\exp\left(\int_{\mathbb{R}^2} A_{r,y}^{\varepsilon,\delta} dW_{r,y}^H\right)\right) \\
&= E^B\left\{u_0(x+B_t)\exp\left(\frac{1}{2}\|A^{\varepsilon,\delta}\|_{\mathcal{H}_1}^2\right)\exp\left(\int_0^t\int_{\mathbb{R}} A_{r,y}^{\varepsilon,\delta} dW_{r,y}^H - \frac{1}{2}\|A^{\varepsilon,\delta}\|_{\mathcal{H}_1}^2\right)\right\} \\
&= \sum_{n=0}^{\infty} I_n(f_n^{\varepsilon,\delta}(t,x)),
\end{aligned}
$$

where

$$
f_n^{\varepsilon,\delta}(t_1,x_1,\ldots,t_n,x_n,t,x) = E^B\left[u_0(x+B_t)\exp\left(\frac{1}{2}\|A^{\varepsilon,\delta}\|_{\mathcal{H}_1}^2\right)A_{t_1,x_1}^{\varepsilon,\delta}\cdots A_{t_n,x_n}^{\varepsilon,\delta}\right].
$$

Letting δ and ε go to 0, we obtain the chaos expansion of $u_{t,x}$. □

Consider the stochastic partial differential equation (6.1) and its approximation (6.2). Assume that the initial condition $u_0(x)$ is nonnegative and not identically zero. We shall study the strict positivity of the solution. In particular we shall show that $E\left[u_t(x)^{-p}\right] < \infty$.

Theorem 6.4 *Let $H > 3/4$. Assume that $u_0 \geq 0$, and u_0 is not identically zero. Then for any $0 < p < \infty$, we have that*

$$
E\left(u_{t,x}^{-p}\right) < \infty \tag{6.11}
$$

and moreover,

$$
\begin{aligned}
E\left[u_t(x)^{-p}\right] \leq (Eu_0(x+B_t))^{-p-1}\, E^{\,B}\Bigg[&|u_0(x+B_t)| \\
&\times\exp\left(\frac{p^2}{2}\int_0^t\int_0^t \delta(B_{s_1}-B_{s_2})\phi(s_1,s_2)ds_1ds_2\right)\Bigg].
\end{aligned} \tag{6.12}
$$

Proof Denote $\kappa_p = \left(E^{\,B}u_0(x+B_t)\right)^{-p-1}$. Then, Jensen's inequality applied to the equality $u_{t,x}^{\varepsilon,\delta} = E^{\,B}\left\{u_0(x+B_t)\exp\left(\int_0^t\int_{\mathbb{R}} A_{r,y}^{\varepsilon,\delta}dW_{r,y}^H\right)\right\}$ implies that

$$
\left(u_{t,x}^{\varepsilon,\delta}\right)^{-p} \leq \kappa_p E^{\,B}\left\{u_0(x+B_t)\exp\left(-p\int_0^t\int_{\mathbb{R}} A_{r,y}^{\varepsilon,\delta}dW_{r,y}^H\right)\right\}.
$$

Therefore

$$E\left[\left(u_{t,x}^{\varepsilon,\delta}\right)^{-p}\right] \leq \kappa_p E E^B \left\{u_0(x+B_t) E\left[\exp\left(-p\int_0^t \int_{\mathbb{R}} A_{r,y}^{\varepsilon,\delta} dW_{r,y}^H\right)\right]\right\}$$
$$= \kappa_p E^B \left\{u_0(x+B_t) E\left[\exp\left(\frac{p^2}{2}\left\|A^{\varepsilon,\delta}\right\|_{\mathcal{H}_1}^2\right)\right]\right\},$$

and we can conclude as in the proof of Theorem 6.2. □

Using the theory of rough path analysis (see [8]) and p-variation estimates, Gubinelli et al. [4] have proved that for $H > \frac{3}{4}$, the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sigma(u)\dot{W}_{t,x}^H$$

had a unique mild solution up to a random explosion time $T > 0$, provided $\sigma \in C_b^2(\mathbb{R})$. In this sense, the restriction $H > \frac{3}{4}$, that we found in the case $\sigma(x) = x$ is natural, and in this particular case, using chaos expansion and Feynman-Kac's formula we have been able to show the existence of a solution for all times.

Acknowledgments We thank Carl Mueller for discussions. We would like to thank an anonymous referee for his remarks that have helped us to improve this paper.

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