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The exact asymptotic of the collision time tail distribution for independent Brownian particles with different drifts

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Abstract In this note we consider the time of the collision τ for n independent Brownian motions $X_t^1, \ldots, X_t^n$ with drifts $a_1, \ldots, a_n$, each starting from $\boldsymbol{x} = (x_1, \ldots, x_n)$, where $x_1 < \cdots < x_n$. We show the exact asymptotics of $\mathbb{P}_{\boldsymbol{x}}(\tau > t) = Ch(\boldsymbol{x})t^{-\alpha}\mathrm{e}^{-\gamma t}(1 + o(1))$ as $t \to \infty$ and identify $C, h(\boldsymbol{x}), \alpha, \gamma$ in terms of the drifts.

Keywords Brownian motion with drift · Collision time · Karlin–McGregor formula · Stable partition

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1 Introduction and results

Let $W = \{\boldsymbol{y} : y_1 < \cdots < y_n\}$ be the Weyl chamber. Consider $\boldsymbol{X}_t = (X_t^1, \ldots, X_t^n)$, wherein coordinates are independent Brownian motions with unit variance parameter, drift vector $\boldsymbol{a} = (a_1, \ldots, a_n)$ and starting point $\boldsymbol{X}_0 = \boldsymbol{x} \in W$. In this paper we study

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the collision time τ, which is the exit time of $\boldsymbol{X}_t$ from the Weyl chamber, i.e.,

$$\tau = \inf\{t > 0 : \boldsymbol{X}_t \notin W\}.$$

For identical drifts $a_1 = \cdots = a_n$, say $a_i \equiv 0$, the celebrated Karlin–McGregor formula states (see [7])

$$\mathbb{P}(\tau > t; \boldsymbol{X}_t \in \mathrm{d}\boldsymbol{y}) = \det\left[p_t(x_i, y_j)\right] \mathrm{d}\boldsymbol{y}, \tag{1.1}$$

where $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{2t}}$, which yields the tail distribution of τ:

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \int_W \det\left[p_t(x_i, y_j)\right] \mathrm{d}\boldsymbol{y}.$$

For the use of Karlin–McGregor formula it is essential that processes $X_t^1, \ldots, X_t^n$ are independent copies of the same strong Markov, with skip-free realisations process, starting at $t = 0$ from $\boldsymbol{x} \in W$. In this case the asymptotic of $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$ was first studied by Grabiner [5] (for the Brownian case) (see also proofs by Doumerc and O'Connell [4] and Puchała [9]). Later Puchała and Rolski [10]) showed that this asymptotic is also true for the Poisson and continuous time random walk case. The above mentioned asymptotics is:

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) \sim D\Delta(\boldsymbol{x})t^{-n(n-1)/4}, \tag{1.2}$$

where $\Delta(\boldsymbol{x}) = \det\left[\left(x_i^{j-1}\right)_{i,j=1}^n\right]$ is the Vandermonde determinant, and

$$D = (2\pi)^{-n/2} c_n \int_W \mathrm{e}^{-\frac{|\boldsymbol{y}|^2}{2}} \Delta(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}, \tag{1.3}$$

for $t \to \infty$ and

$$c_n = \frac{1}{\prod_{j=1}^{n-1} j!}. \tag{1.4}$$

In this note we study the same problem, however for Brownian motions with different drifts. For this we derive first, in Sect. 2, a formula for $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$ by the change of measure. It is apparent that possible results must depend on the form of drift vector $\boldsymbol{a}$. For example we can analyse all cases for $n = 2$, because in this case the collision equals to the first passage to zero of the Brownian process $X_t^2 - X_t^1$ with variance parameter 2, for which the density function is known (see, e.g., [3]). Hence

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \int_{t/2}^{\infty} \frac{x}{\sqrt{2\pi} s^{3/2}} \exp\left[-\frac{(x + as)^2}{2s}\right] \mathrm{d}s,$$

where $x = x_2 - x_1$ and $a = a_2 - a_1$. This yields

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \begin{cases} \frac{2^{5/2}}{a^2\sqrt{2\pi}} x \mathrm{e}^{ax} t^{-3/2} \mathrm{e}^{-ta^2/4}\ (1 + o(1)), & a_1 > a_2 \\ \frac{2^{\frac{3}{2}}}{\sqrt{2\pi}} x t^{-\frac{1}{2}}\ (1 + o(1)), & a_1 = a_2 \\ 1 - \mathrm{e}^{-ax} + o(1) & a_1 < a_2. \end{cases}$$

For general n the situation is much more complex and different scenarios are possible. For example the drifts can be diverging and then $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$ tends to a positive constant, whose situation was analysed by Biane et al. [2]. Another case is when all drifts are equal, in which the case the probability $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$ is polynomially decaying, as it was found by Grabiner [5]. However there are various situations when the probabilities are exponentially decaying with polynomial prefactors. The full characterisation depends on a concept of the stable partition of the drift vector, which the notion is introduced in Sect. 3. In Sect. 4 we state the main theorem, which shows all possible exact asymptotics of $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$ in form of $Ch(\boldsymbol{x})t^{-\alpha}\mathrm{e}^{-\gamma t}$, where formulas for C, α and γ are given in terms of the stable partition of the drift vector.

2 Formula for $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$

We note our basic probabilistic space with natural history filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_{\boldsymbol{x}})$ and consider on it process $\boldsymbol{X}_t$ as defined in the Introduction. Unless otherwise stated we tacitly assume that $\boldsymbol{x} \in W$. We start off a lemma on the change of measure for the Brownian case, which the proof can be found for example in Asmussen [1], Theorem 3.4. Let $M_t = \mathrm{e}^{<\boldsymbol{\alpha},\boldsymbol{X}_t>}/\mathbb{E}\mathrm{e}^{<\boldsymbol{\alpha},\boldsymbol{X}_t>}$ be a Wald martingale. For a probability measure $\mathbb{P}_{\boldsymbol{x}}$ its restriction to $\mathcal{F}_t$ we denote by $\mathbb{P}_{\boldsymbol{x}|t}$. Let $\tilde{\mathbb{P}}_{\boldsymbol{x}}$ be a probability measure obtained by the change of measure $\mathbb{P}_{\boldsymbol{x}}$ with the use of martingale M_t, that is defined by a family of measures $\tilde{\mathbb{P}}_{\boldsymbol{x}|t} = M_t\,\mathrm{d}\mathbb{P}_{\boldsymbol{x}|t}$, $t \geq 0$. For the theory we refer, e.g., to Sect. XIII.3 in [1].

Lemma 2.1 *If $\boldsymbol{X}_t$ is a Brownian motion with drift $\boldsymbol{a}$ under $\mathbb{P}_{\boldsymbol{x}}$, then this process is a Brownian motion with drift $\boldsymbol{a} + \boldsymbol{\alpha}$ under $\tilde{\mathbb{P}}_{\boldsymbol{x}}$.*

The sought for formula for the tail distribution of the collision time is given in the next proposition.

Proposition 2.2

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = (2\pi)^{-n/2}\mathrm{e}^{-<\boldsymbol{a},\boldsymbol{x}>-||\boldsymbol{x}||^2/2t} \int_W \mathrm{e}^{-||\boldsymbol{y}-\boldsymbol{a}\sqrt{t}||^2/2} \det\left[\mathrm{e}^{x_i y_j/\sqrt{t}}\right] \mathrm{d}\boldsymbol{y}. \quad (2.5)$$

Proof We use $\boldsymbol{\alpha} = -\boldsymbol{a}$ to eliminate the drift under $\tilde{\mathbb{P}}_{\boldsymbol{x}}$. Thus $\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \tilde{\mathbb{E}}_{\boldsymbol{x}}[M_t^{-1}; \tau > t]$. Now by Karlin–McGregor formula (1.1) we write

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \tilde{\mathbb{E}}_{\boldsymbol{x}}\left[\mathrm{e}^{<\boldsymbol{a},X_t>}\mathbb{E}_{\boldsymbol{x}}\mathrm{e}^{<-\boldsymbol{a},X_t>}; \tau > t\right]$$
$$= \mathrm{e}^{<-\boldsymbol{a},\boldsymbol{x}>-||\boldsymbol{a}||^2 t/2} \int\limits_{\boldsymbol{y}\in W} \mathrm{e}^{<\boldsymbol{a},\boldsymbol{y}>} \det[p_t(x_i, y_j)]\, \mathrm{d}\boldsymbol{y},$$

and next, algebraic manipulations yield (2.5). □

In the paper we use the following vector notations. For a vector $\boldsymbol{a} \in \mathbb{R}^n$ we denote $\boldsymbol{a}_{[i;j]} = (a_i, a_{i+1}, \ldots, a_j)$ and $\bar{a}_{[i;j]} = (a_i + a_{i+1} + \cdots + a_j)/(j - i + 1)$. We also use $\boldsymbol{a}_{(i;j]} = (a_{i+1}, \ldots, a_j)$ and $\boldsymbol{a}_{(i;j)} = (a_{i+1}, \ldots, a_{j-1})$. By $\boldsymbol{z}^{\boldsymbol{k}}$, where $\boldsymbol{z} = (z_1, \ldots, z_m)$ and $\boldsymbol{k} = (k_1, \ldots, k_m)$ we denote $\prod_{j=1}^m z_j^{k_j}$. Vector $\boldsymbol{s}_{(n)} \in \mathbb{R}^{n-1}$ is obtained from $\boldsymbol{s} \in \mathbb{R}^n$ by deleting the n^{th} coordinate and $\boldsymbol{A}_{(n)}$ is an $n \times n$ matrix $\boldsymbol{A}$ without nth row and nth column.

3 Stable partition of *a*

Let $\boldsymbol{a} \in \mathbb{R}^n$. Our aim is to make a suitable partition

$$(a_1, \ldots, a_{\nu_1})(a_{\nu_1+1}, \ldots, a_{\nu_1+\nu_2}), \ldots, (a_{\nu_1+\cdots+\nu_{q-1}+1}, \ldots, a_{\nu_1+\cdots+\nu_q}). \tag{3.6}$$

of $\boldsymbol{a}$, where $\nu_i > 0$. For short we denote $m_1 = \nu_1, m_2 = \nu_1 + \nu_2, \ldots, m_q = \nu_1 + \cdots + \nu_q = n$. We also set $m_0 = 0$.

We say that sequence $\boldsymbol{a}$ is *irreducible* if

$$\left.\begin{array}{lcl} \bar{a}_{[1;1]} & > & \bar{a}_{[2;n]} \\ \bar{a}_{[1;2]} & > & \bar{a}_{[3;n]} \\ \vdots & \vdots & \vdots \\ \bar{a}_{[1;n-1]} & > & \bar{a}_{[n;n]} \end{array}\right\}. \tag{3.7}$$

Suppose we have a partition defined by $m_1, \ldots, m_q$. The mean of the ith sub-vector is denoted by $f^i = \bar{a}_{(m_{i-1};m_i]}$. Furthermore we define a vector $\boldsymbol{f} = (f_1, \ldots, f_n)$ by

$$f_i = f^k, \quad \text{if} \quad m_{k-1} < i \le m_k.$$

It is said that partition (3.6) of vector $\boldsymbol{a}$ is *stable* if

$$f^1 \le f^2 \le \cdots \le f^q \tag{3.8}$$

and each vector $\boldsymbol{a}_{(m_{i-1};m_i]}$ is irreducible ($i = 1, \ldots, q$). Remark that a stable partition is defined if we know $\boldsymbol{m} = (m_1, \ldots, m_q)$ for which (3.8) hold and each $\boldsymbol{a}_{(m_{i-1};m_i]}$ is irreducible ($i = 1, \ldots, q$). In the sequel, for a given stable partition of $\boldsymbol{a}$, characters $q, \boldsymbol{f}, \boldsymbol{m}$ are reserved for it.

Consider now $f_{m_1}, f_{m_2}, \ldots, f_{m_q}$ and define a subsequence $\boldsymbol{m}' = (m'_1, \ldots, m'_{q'})$ of $\boldsymbol{m} = (m_1, m_2, \ldots, m_q)$ as follows. Let q' be the number of strict inequalities in

$f^1 \le f^2 \le \cdots \le f^q$ plus 1. Furthermore we define inductively by $m'_0 = 0$ and for $i = 1, \ldots, q' - 1$

$$m'_i = \inf\{m_j > m'_{i-1} : m_j \in \boldsymbol{m}, f_{m_j} < f_{m_{j+1}}\},$$

and finally we set $m_{q'} = n$. We also define a subsequence of indices $i_0, i_1, \ldots, i_{q'}$ inductively by $i_0 = 0$ and for $k = 1, \ldots, q' - 1$

$$i_k = \inf\{j > i_{k-1} : f_{m_j} < f_{m_{j+1}}\}$$

and $i_{q'} = q$. Hence we have

$$f_{m'_1} < f_{m'_2} < \cdots < f_{m'_{q'}}.$$

In this case we say that $(m'_1, \ldots, m'_{q'})$ is a strong representation of the stable partition of $\boldsymbol{a}$ and q', $(m'_1, \ldots, m'_{q'})$ are characters reserved for it. Set $\nu'_i = m'_i - m'_{i-1}$, $(i = 1, \ldots, q')$.

Example 1 Suppose that $\boldsymbol{a} = (3, 1, 2, 5, 1)$. Then $q = 3$ and $m_1 = 2, m_2 = 3, m_3 = 5$ define the stable partition $(3, 1)(2)(5, 1)$ with means $f^1 = 2, f^2 = 2, f^3 = 3$. Furthermore $q' = 2, m'_1 = 3, m'_2 = 5$ and $i_1 = 2, i_2 = 3$.

Proposition 3.1 *For each vector* $\boldsymbol{a}$*, there exists its unique stable partition.*

Before we state a proof of Proposition 3.1 we prove few lemmas.

Lemma 3.2 *If* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *is irreducible, then*

$$\left.\begin{array}{ccccc} \bar{a}_{[1;1]} & > \bar{a}_{(0;n]} & > & \bar{a}_{[2;n]} \\ \bar{a}_{[1;2]} & > \bar{a}_{(0;n]} & > & \bar{a}_{[3;n]} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{[1;n-1]} & > \bar{a}_{(0;n]} & > & \bar{a}_{[n;n]} \end{array}\right\}. \tag{3.9}$$

Proof Quantity $\bar{a}_{(0;n]}$ is a nontrivial weighted mean of every pair $\bar{a}_{[1;i]}$ and $\bar{a}_{[i+1;n]}$. □

Lemma 3.3 *In a stable partition, for each element* $\boldsymbol{a}_{(m_{i-1};m_i]}$

$$\bar{a}_{(m_{i-1};m_{i-1}+k]} \ge f_{m_i}.$$

Proof The case $k \le m_i - m_{i-1}$ follows from Lemma 3.2. Clearly for $k = m_i - m_{i-1}$ we have equality. Consider now $k > m_i - m_{i-1}$. Than $\bar{a}_{(m_{i-1};m_{i-1}+k]}$ is a weighted mean of f_{m_i} and $\bar{a}_{(m_i;m_i+k-(m_i-m_{i-1})]}$ and the later term is greater or equal than f_{m_i} by (3.8) and (3.9). □

In the next lemma we consider two vectors $\boldsymbol{a}^1 \in \mathbb{R}^{n_1}$ and $\boldsymbol{a}^2 \in \mathbb{R}^{n_2}$. We consider a situation of creating a new vector $(\boldsymbol{a}^1, \boldsymbol{a}^2) = (a_1 \ldots, a_{n_1+n_2}) \in \mathbb{R}^{n_1+n_2}$.

Lemma 3.4 *Suppose that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are irreducible and $\bar{a}^1_{[1,n_1]} > \bar{a}^2_{[1,n_2]}$. Then vector $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is irreducible.*

Proof Recall that $(\boldsymbol{a}_1, \boldsymbol{a}_2) = (a_1 \ldots, a_{n_1+n_2}) \in \mathbb{R}^{n_1+n_2}$. Suppose $1 \le k \le n_1$. By Lemma 3.2 we have $\bar{a}^1_{[1;k-1]} > \bar{a}^1_{[1,n_1]} > \bar{a}^1_{[k;n_1]}$, also $\bar{a}^1_{[1;k-1]} > \bar{a}^1_{[1,n_1]} > \bar{a}^2_{[1,n_2]}$. Hence $\bar{a}_{[1;k-1]} > \bar{a}_{[k;n_1+n_2]}$ because $\bar{a}_{[k;n_1+n_2]}$ is a weighted mean of $\bar{a}_{[k;n_1]}$ and $\bar{a}^2_{[1,n_2]}$. Suppose now $n_1 < k$. Then $\bar{a}_{[1;k-1]}$ is a weighted mean of $\bar{a}^1_{[1,n_1]}$ and $\bar{a}^2_{[1,k-n_1]} = \bar{a}_{[n_1+1;k]}$ and both by Lemma 3.2 are greater than $\bar{a}_{[k;n_1+n_2]}$, which completes the proof. □

Proof of Proposition 3.1. The existence part is by induction with respect n. For $n = 2$ we have two situations

1. if $a_1 \le a_2$, than $q = 2$ with $m_1 = 1,\ m_2 = 2$ is a stable partition,
2. if $a_1 > a_2$, than $q = 1$ with $m_1 = 2$ is a stable partition.

Assume that there exists a *stable* partition with q partition vectors of a vector $\boldsymbol{a} \in R^n$. We add a new element a_{n+1} at the end of vector $\boldsymbol{a}$ to create new one $(\boldsymbol{a}, a_{n+1}) = (a_1, \ldots, a_{n+1})$.

We have two situations.

1. If $a_{n+1} \ge f^q$ than in a stable partition a_{n+1} is alone in the $q+1$ partition vector.
2. If $a_{n+1} < f^q$, than we proceed inductively as follow. We use Lemma 3.4 with $\boldsymbol{a}_1 = \boldsymbol{a}_{[m_{q-1};m_q]}$ and $\boldsymbol{a}_2 = (a_{n+1})$ and let f^q and $f^{q+1} = a_{n+1}$ are means of these partition vectors. In result $(\boldsymbol{a}_{(m_{q-1};m_q]}, a_{n+1})$ form an irreducible vector, for which we have to check whether condition (3.8) holds. If yes, then we end with a stable partition, otherwise we join the $q-1$ partition vector with the new q partition vectors and repeat the procedure. In the worst case we end up with one partition vector.

For the uniqueness proof, suppose that we have two different *stable* partitions: $m^1_1 < m^1_2 < \cdots < m^1_{q_1}$ and $m^2_1 < m^2_2 < \cdots < m^2_{q_2}$. The means of fs are $(f^1)^1, \ldots, (f^1)^{q_1}$ for the first partition vector and $(f^2)^1, \ldots, (f^2)^{q_2}$ for the second respectively. Since partitions are supposedly different, there exists i such that $m^1_i \ne m^2_i$. We take the minimal i with this property and without loss of generality we can assume $m^2_i > m^1_i$. Set $k = m^2_i - m^1_i$. We have to analyse the following cases.

1. $(m^2_i = m^1_{i+1})$. We have

$$\bar{a}_{(m^2_{i-1};m^1_i]} > \bar{a}_{(m^1_i;m^2_i]}.$$

On the other hand $(f^1)^i = \bar{a}_{(m^1_{i-1};m^1_i]} = \bar{a}_{(m^2_{i-1};m^1_i]} > \bar{a}_{[m^1_i;m^2_i]} = \bar{a}_{[m^1_i;m^1_{i+1}]} = (f^1)^{i+1}$ and this contradicts with $(f^1)^i \le (f^1)^{i+1}$.

2. $(m^2_i > m^1_{i+1})$. We have by Lemma 3.3

$$(f^1)^{i+1} = \bar{a}_{(m^1_i;m^1_{i+1}]} > \bar{a}_{(m^1_{i+1};m^2_i]} \ge (f^1)^{i+2},$$

which is a contradiction.

3. $(m_i^2 < m_{i+1}^1)$. We have by Lemma 3.3

$$(f^1)^i = \bar{a}_{(m_{i-1}^1;m_i^1]} > \bar{a}_{(m_i^1;m_i^2]} \geq (f^1)^{i+1},$$

which is a contradiction.

The proof is completed. □

Remark The stable partition can be obtained by considering the following simple deterministic dynamical system. We have n particles starting from $x_1 < x_2 < \cdots < x_n$. The ith particle has speed a_i. Each particle moves with a constant speed on the real line until it collides with one of its neighbouring particle (if it happens). Then both the particles coalesce and from this time on they move with the proportional speed which is the mean of speed of colliding particles, and so on. Ultimately the particles will form never colliding groups, which are the same as in the stable partition of $\boldsymbol{a}$. Notice that resulted grouping do not depend on a starting position $\boldsymbol{x}$.

4 The theorem and examples

We begin introducing some notations. Suppose that $\boldsymbol{a}$ has a stable partition with characteristics q, (m_i), q', (m_i') respectively. In the sequel we will use the following notations:

$$\gamma = \frac{1}{2}\sum_{\ell=1}^{q}\left(\frac{1}{\nu_\ell}\sum_{m_{l-1}<u<v\leq m_l}(a_u - a_v)^2\right), \tag{4.10}$$

$$\alpha = \frac{1}{2}\left(\sum_{j=1}^{q'}\binom{\nu_j'}{2} + (n-q) + \sum_{j=1}^{q}\binom{\nu_j}{2}\right), \tag{4.11}$$

$$h(\boldsymbol{x}) = \mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>}\det\left[\mathrm{e}^{x_i f_j} x_i^{\sum_{l=1}^{q'}(j-m_{l-1}'-1)\mathbf{1}_{\{m_{l-1}'<j\leq m_l'\}}}\right]. \tag{4.12}$$

The following polynomial will play an important role:

$$V_{\boldsymbol{a}}(s_1,\ldots,s_{n-1}) = \sum_{l=1}^{q}\left(\frac{1}{\nu_l}\sum_{m_{l-1}<u<v\leq m_l}(s_u+\cdots+s_{v-1})(a_u - a_v)\right). \tag{4.13}$$

We also see that if $m_{i-1} < k < m_i$, then the coefficient at s_k is

$$\frac{(m_i-k)(k-m_{i-1})}{m_i - m_{i-1}}\left(\frac{a_{m_{i-1}+1}+\cdots+a_k}{k-m_{i-1}} - \frac{a_{k+1}+\cdots+a_{m_i}}{m_i-k}\right)$$

and it is strictly positive by the definition of the stable partition. It is important to notice that $V_{\boldsymbol{a}}(\boldsymbol{s})$ depends only on those s_i, where $i \notin \{m_1,\ldots m_q\}$.

Moreover we define a function

$$I(\boldsymbol{a}, t) = \int\limits_{W - \boldsymbol{f}\sqrt{t}} \mathrm{e}^{-\frac{1}{2}|z|^2} \mathrm{e}^{-\frac{1}{2}\sum_{l=1}^{q}\left(\frac{2\sqrt{t}}{\nu_l}\sum_{m_{l-1}<u<v\leq m_l}(z_u - z_v)(a_v - a_u)\right)} \prod_{j=1}^{q'} \Delta\left(\boldsymbol{z}_{(m'_{j-1};m'_j]}\right) \mathrm{d}\,\boldsymbol{z}. \tag{4.14}$$

Remark that from Lemma 5.1 it will follow

$$\frac{1}{\nu_l}\sum_{m_{l-1}<u<v\leq m_l}(a_u - a_v)^2 = \sum_{m_{l-1}<u<v\leq m_l}(a_u - \bar{a}^l)^2,$$

where

$$\bar{a}^l = \frac{1}{\nu_l}\sum_{u=m_{l-1}+1}^{m_l} a_u.$$

Using this notation we now state a proposition which is useful for calculations in some cases. Recall the definition of c_n from (1.4).

Proposition 4.1

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}(\tau > t) = (2\pi)^{-n/2}\prod_{j=1}^{q'} c_{\nu'_j}\mathrm{e}^{-\gamma t}t^{-\frac{1}{2}\sum_{j=1}^{q'}\binom{\nu'_j}{2}} \\ \times \mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>}\det\left[\mathrm{e}^{x_k f_j} x_k^{\sum_l^{q'}(j-m'_{l-1}-1)\mathbf{I}_{\{m'_{l-1}<j\leq m'_l\}}}\right] \\ \times I(\boldsymbol{a}, t)\ (1 + o(1)).\end{aligned} \tag{4.15}$$

Remark that formula (4.15) does not give us straightforward asymptotic because integral $I(\boldsymbol{a}, t)$ depends on t. However in some cases this dependence vanishes and this is why Proposition 4.1 can be sometimes useful. The next theorem gives us asymptotic for all cases.

Theorem 4.2 *For some C given below, as $t \to \infty$*

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = Ch(\boldsymbol{x})t^{-\alpha}\mathrm{e}^{-\gamma t}(1 + o(1)),$$

γ, α, and $h(\boldsymbol{x})$ are defined in (4.10),(4.11),(4.12) *respectively.*

To define C we need few more definitions. Let

$$H(s_1, \ldots, s_\ell) = \prod_{1\leq i<j\leq \ell+1}(s_i + \cdots + s_{j-1}). \tag{4.16}$$

Define now

$$C = A_1 \times A_2 \times A_3, \tag{4.17}$$

where

$$A_1 = (2\pi)^{-n/2}\sqrt{2\pi n}\prod_{j=1}^{q'} c_{\nu'_j},$$

$$A_2 = \int\limits_{\xi_i>0: i\notin\{m_1,\ldots,m_q\}} \cdots \int \mathrm{e}^{-V_{\boldsymbol{a}}(\boldsymbol{\xi})} \prod_{i=1}^{q} H\left(\boldsymbol{\xi}_{(m_{i-1};m_i)}\right) \prod_{i\notin\{m_1,\ldots,m_q\}} \mathrm{d}\xi_i,$$

and

$$A_3 = \int\limits_{\substack{\xi_i>0: i=1,\ldots,n-1 \\ i\in\{m_1,\ldots,m_{q-1}\}\setminus\{m'_1,\ldots,m'_{q'-1}\}}} \cdots \int \int\limits_{\substack{\xi_i>-\infty: i=1,\ldots,n-1 \\ i\in\{m'_1,\ldots,m'_{q'-1}\}}} \cdots \int \mathrm{e}^{-\frac{1}{2}\left(\sum_{k,l\in\{m_1,\ldots,m_{q-1}\}} S_{kl}\xi_k\xi_l\right)}$$
$$\times \prod_{k=0}^{q-1} \prod_{\substack{i:\{i,i+1,\ldots,i+k\} \\ \in\{1,\ldots,q-1\}\setminus\{i_1,\ldots,i_{q'-1}\}}} \left(\sum_{j=0}^{k} \xi_{m_{i+j}}\right)^{\nu_i\nu_{i+k+1}} \prod_{i\in\{m_1,\ldots,m_{q-1}\}} \mathrm{d}\xi_i,$$

where $\boldsymbol{S}$ is $n-1\times n-1$ matrix with entries $S_{kl}=(n-k)l/n$ for $k\le l$ and $S_{kl}=S_{lk}$, $1\le k,l\le n-1$. Since for $V_{\boldsymbol{a}}$ defined in (4.13), all coefficients are positive, the integral is convergent.

In the remaining part of this section we display some special cases.

Example 2 $(a_1=a_2=\cdots=a_n)$ This is no drift case. Here $q=n$ and $m_1=1, m_2=2,\ldots,m_n=n$, also $q'=1$ and $m'_1=n$. In result $f_{m_1}=a_1,\ldots,f_{m_n}=a_n$. Let a be the common value of the drift. Using Proposition 4.1 we have

$$\mathbb{P}_{\boldsymbol{x}}(\tau>t) = (2\pi)^{-n/2}c_n\mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>}\det\left[\mathrm{e}^{x_k f_j}x_k^{j-1}\right]t^{-\frac{1}{2}\binom{n}{2}}$$
$$\times\int\limits_{W-\boldsymbol{f}\sqrt{t}} \mathrm{e}^{-\frac{|z|^2}{2}}\Delta(\boldsymbol{z}_{[1;n]})\,\mathrm{d}\boldsymbol{z}\,(1+o(1)).$$

First we notice that since all the coordinates in vector $\boldsymbol{f}$ are the same, we have

$$\det\left[\mathrm{e}^{x_k f_j}x_k^{j-1}\right] = \mathrm{e}^{<\boldsymbol{x},\boldsymbol{f}>}\det\left[x_k^{j-1}\right] = \mathrm{e}^{<\boldsymbol{x},\boldsymbol{a}>}\Delta(\boldsymbol{x}).$$

Furthermore $W-\boldsymbol{f}\sqrt{t}=W$ because $y_1<y_2<\cdots<y_n$ if and only if $y_1+a\sqrt{t}<y_2+a\sqrt{t}<\cdots<y_n+a\sqrt{t}$. Finally we write

$$\mathbb{P}_{\boldsymbol{x}}(\tau>t) = C\,h(\boldsymbol{x})\,t^{-\alpha}\,(1+o(1)),$$

where

$$\alpha = \frac{1}{2}\binom{n}{2},$$
$$h(\boldsymbol{x}) = \Delta(\boldsymbol{x}),$$
$$C = (2\pi)^{-n/2} c_n \int\limits_W \mathrm{e}^{-\frac{|z|^2}{2}} \Delta(z)\,\mathrm{d}\boldsymbol{z}.$$

Before we state the next example we prove the following lemma.

Lemma 4.3 *If $\boldsymbol{a} \in W$, then $\{W - \boldsymbol{a}t\} \to \mathbb{R}^n$ as $t \to \infty$.*

Proof Let $\boldsymbol{a} \in W$. We show that for all $\boldsymbol{y} \in \mathbb{R}^n$ there exists $s > 0$, such that for all $t > s$, $\boldsymbol{y} \in \{W - \boldsymbol{a}t\}$. Let $y \in \mathbb{R}^n$. We note $b_i = y_{i+1} - y_i$ and $d_i = a_{i+1} - a_i$. Condition $\boldsymbol{a} \in W$ implies $d_i > 0$ for all $i = 1, 2, \ldots, n-1$. We take $s = \max\{-b_i, 0\}/\min\{d_i\}$ and $t > s$. Set $z_i = y_i + ta_i$, then we get that $\boldsymbol{z} \in W$, because

$$z_{i+1} - z_i = y_{i+1} + ta_{i+1} - y_i - ta_i = b_i + td_i > b_i + sd_i \geq b_i + \max\{-b_i, 0\} \geq 0.$$

Thus for $t > s$ we have $\boldsymbol{y} = \boldsymbol{z} - t\boldsymbol{a}$, where $\boldsymbol{z} \in W$, and so $\boldsymbol{y} \in \{W - ta\}$ for all $t > s$. □

Example 3 ($a_1 < a_2 < \cdots < a_n$) This is the case of non-colliding drifts. Here $q = q' = n$, $m_1 = m'_1 = 1, \ldots, m_n = m'_n = n$, $f_{m_1} = a_1, \ldots, f_{m_n} = a_n$. Using Proposition 4.1 we have

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = (2\pi)^{-n/2} \mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>} \det\left[\mathrm{e}^{x_k a_j}\right] \int\limits_{W - \boldsymbol{a}\sqrt{t}} \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{d}\boldsymbol{z}\,(1 + o(1)).$$

By Lemma 4.3 we have that

$$\lim_{t\to\infty} \int\limits_{W - \boldsymbol{a}\sqrt{t}} \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{d}\boldsymbol{z} = \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{d}\boldsymbol{z} = (2\pi)^{n/2}.$$

Finally we write

$$\lim_{t\to\infty} \mathbb{P}_{\boldsymbol{x}}(\tau > t) = \mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>} \det\left[\mathrm{e}^{x_k a_j}\right].$$

This result was derived earlier by Biane et al. [2].

Example 4 Case when $q = q' = 1$. This is the case of a one irreducible drift vector. Here $m_1 = m'_1 = n$, $f_1 = f_2 \cdots = f_n = \bar{a}_{[1;n]} = \frac{a_1 + \cdots + a_n}{n}$. Using Theorem 4.2 we have

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = Ch(\boldsymbol{x}) t^{-\alpha} \mathrm{e}^{-\gamma t}(1 + o(1)),$$

where

$$\gamma = \frac{1}{2}\left(\frac{1}{n}\sum_{0<u<v\le n}(a_u - a_v)^2\right),$$
$$\alpha = \frac{(n-1)(n+1)}{2},$$
$$h(\boldsymbol{x}) = \mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}-\bar{a}_{[1;n]}>}\Delta(\boldsymbol{x}),$$
$$C = (2\pi)^{-(n-1)/2}\sqrt{n}c_n \times \int_{\xi_i>0:i=1,2,\dots,n-1}\cdots\int \mathrm{e}^{-\left(\frac{1}{n}\sum_{0<u<v\le n}(\xi_u+\cdots+\xi_{v-1})(a_u-a_v)\right)} H(\boldsymbol{\xi}_{[1;n-1]})\prod_{i=1}^{n-1}\mathrm{d}\xi_i.$$

We now analyse a remaining situation for $n = 3$.

Example 5 ($a_1 > a_2$ and $\frac{a_1+a_2}{2} < a_3$). This is the case of two subsequences. Thus $q = 2,\ q' = 2$ and $m_1 = m_1' = 2,\ m_2 = m_2' = 3$. By Theorem 4.2 we have

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = Ch(\boldsymbol{x})\mathrm{e}^{-\frac{t}{4}(a_2-a_1)^2}t^{-\frac{3}{2}},$$

where

$$\gamma = \frac{(a_2-a_1)^2}{4},$$
$$\alpha = \frac{3}{2},$$
$$h(\boldsymbol{x}) = \mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>}\begin{vmatrix} \mathrm{e}^{x_1\frac{a_1+a_2}{2}} & \mathrm{e}^{x_1\frac{a_1+a_2}{2}}x_1 & \mathrm{e}^{x_1 a_3} \\ \mathrm{e}^{x_2\frac{a_1+a_2}{2}} & \mathrm{e}^{x_2\frac{a_1+a_2}{2}}x_2 & \mathrm{e}^{x_2 a_3} \\ \mathrm{e}^{x_3\frac{a_1+a_2}{2}} & \mathrm{e}^{x_3\frac{a_1+a_2}{2}}x_3 & \mathrm{e}^{x_3 a_3} \end{vmatrix},$$
$$C = \frac{3}{2\sqrt{\pi}(a_1-a_2)^2}.$$

5 Auxiliary results

For the proof we need a set of lemmas and propositions, presented in sections below.

5.1 Useful lemmas

We need a few technical lemmas, which we state without proofs.

Lemma 5.1 *For* $\boldsymbol{a} \in \mathbb{R}^m$

$$\sum_{i=1}^{m}\left(\bar{a}_{[1;m]} - a_i\right)^2 = \frac{1}{m}\sum_{1\le u<v\le m}(a_u - a_v)^2.$$

Lemma 5.2 *For* $\boldsymbol{a}, \boldsymbol{z} \in \mathbb{R}^m$

$$\sum_{i=1}^{m} z_i \left(\bar{a}_{[1;m]} - a_i\right) = \frac{1}{m} \sum_{u<v} (z_v - z_u)(a_u - a_v).$$

The proof of the following lemma follows easily from Lemmas 5.1 and 5.2.

Lemma 5.3 *For* $\boldsymbol{a}, \boldsymbol{f} \in R^n$ *such that* $\boldsymbol{f}$ *is a vector obtained from the stable partition of* $\boldsymbol{a}$*, and* $z \in R^n$*, we have*

$$|\boldsymbol{f}\sqrt{t} - \boldsymbol{a}\sqrt{t} + \boldsymbol{z}|^2 = |\boldsymbol{z}|^2 + \sum_{l=1}^{q} \left(\frac{t}{\nu_l} \sum_{m_{l-1}<u<v\le m_l} (a_u - a_v)^2 \right.$$
$$\left. + \frac{2\sqrt{t}}{\nu_l} \sum_{m_{l-1}<u<v\le m_l} (z_v - z_u)(a_u - a_v) \right).$$

Lemma 5.4 *If*

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 \dots & 0 & 0 \\ 0 & -1 & 1 \dots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \dots & -1 & 1 \\ 1 & 1 & 1 \dots & 1 & 1 \end{pmatrix},$$

is $n \times n$ *matrix, then*

$$(\boldsymbol{A}^{-1})^T \boldsymbol{A}^{-1} = \frac{1}{n} \begin{pmatrix} n-1 & & & & & \\ n-2 & 2(n-2) & & & & \\ n-3 & 2(n-3) & 3(n-3) & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ 1 & 2 & 3 & \dots & n-1 & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Note that $(\boldsymbol{A}^{-1})^T \boldsymbol{A}^{-1}$ *is symmetric and positive definite.*

By Proposition 2.2 we have

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = (2\pi)^{-n/2} \mathrm{e}^{-<\boldsymbol{a},\boldsymbol{x}>-||\boldsymbol{x}||^2/2t} \int_W \mathrm{e}^{-||\boldsymbol{y}-\boldsymbol{a}\sqrt{t}||^2/2} \det(\mathrm{e}^{x_i y_j/\sqrt{t}}) \, \mathrm{d}\boldsymbol{y}.$$

We now introduce new variable $\boldsymbol{z}$ by

$$\boldsymbol{y} = \boldsymbol{f}\sqrt{t} + \boldsymbol{z},$$

where $\boldsymbol{f} = (f_1, \dots, f_n)$ is a vector obtained from the stable partition of $\boldsymbol{a}$.

Finally we rewrite formula (2.5) in new variables by the use of Lemma 5.3:

Lemma 5.5

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = (2\pi)^{-n/2} \mathrm{e}^{-<\boldsymbol{a},\boldsymbol{x}>-||\boldsymbol{x}||^2/2t} \mathrm{e}^{-\gamma t} \int\limits_{W-\boldsymbol{f}\sqrt{t}} \mathrm{e}^{-\frac{|z|^2}{2}} \mathrm{e}^{-\frac{1}{2}\sum_{l=1}^{q} \frac{2\sqrt{t}}{\nu_l} \sum_{m_{l-1}<u<v\le m_l} (z_v - z_u)(a_u - a_v)} \det\left[\mathrm{e}^{x_i(z_j/\sqrt{t}+f_j)}\right] \mathrm{d}\boldsymbol{z}. \tag{5.18}$$

5.2 Asymptotic behaviour of determinant

The following lemma is an extension of Lemma 2 from Puchała [9].

We define functions

$$g_{\boldsymbol{k}}(z) = \frac{\det[z_i^{k_j}]}{\det[z_i^{j-1}]},$$

for $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $0 \le k_1 < \cdots < k_n$ Functions g corresponds to Schur functions $g_{\boldsymbol{k}} = s_{\boldsymbol{k}-(0,1,\ldots,n-1)}$; see, e.g., Macdonald [8], Chap. 1.3.

Lemma 5.6 *Let* $k_0 = \sum_{j=1}^{q'} \binom{\nu'_j}{2}$

$$\det\left[\mathrm{e}^{x_i(z_j/\sqrt{t}+f_j)}\right] = \sum_{k=k_0}^{\infty} t^{-k/2} T_k, \tag{5.19}$$

where

$$T_k = \prod_{j=1}^{q'} \Delta\left(\boldsymbol{z}_{(m'_{j-1};m'_l]}\right) \times \sum_{\substack{k_1+\cdots+k_n=k \\ k_1<\cdots<k_{m'_1}; \\ \vdots \\ k_{m'_{q'-1}+1}<\ldots<k_{m'_{q'}}}} \times \frac{g_{\boldsymbol{k}_{(m'_0;m'_1]}}(\boldsymbol{z}_{(m'_0;m'_1]})}{k_1!\ldots k_{m'_1}!} \cdots \frac{g_{\boldsymbol{k}_{(m'_{q'-1};m'_{q'}]}}(\boldsymbol{z}_{(m'_{q'-1};m'_{q'}]})}{k_{m'_{q'-1}+1}!\ldots k_{m'_{q'}}!} \det\left[\mathrm{e}^{x_i f_j} x_i^{k_j}\right].$$

In particular as $t \to \infty$

$$\det\left[e^{x_i(z_j/\sqrt{t}+f_j)}\right] = t^{-\frac{1}{2}\sum_{j=1}^{q'}\binom{\nu'_j}{2}} \prod_{j=1}^{q'} c_{\nu'_j}\,\Delta\left(z_{(m'_{j-1},m'_l]}\right)$$

$$\times \det\left[e^{x_k f_j} x_k^{\sum_l^{q'}(j-m'_{l-1}-1)\mathbf{I}_{\{m'_{l-1}<j\le m'_l\}}}\right](1+o(1)).$$

Proof By S_n we denote the group of permutations on n-set. We write

$$\begin{aligned}
&\det\left[e^{x_i(z_j/\sqrt{t}+f_j)}\right]\\
&= \sum_{\sigma\in S_n}(-1)^\sigma e^{\sum x_i f_{\sigma(i)}} e^{\sum x_i z_{\sigma(i)}/\sqrt{t}}\\
&= \sum_{\sigma\in S_n}(-1)^\sigma e^{\sum x_i f_{\sigma(i)}} \sum_{k=0}^{\infty} t^{-k/2}(x_1 z_{\sigma(1)}+\cdots+x_n z_{\sigma(n)})^k/k!\\
&= \sum_{k=0}^{\infty}\frac{t^{-k/2}}{k!}\sum_{\sigma\in S_n}(-1)^\sigma e^{\sum x_i f_{\sigma(i)}}(x_1 z_{\sigma(1)}+\cdots+x_n z_{\sigma(n)})^k.
\end{aligned}$$

Now the coefficient at $t^{-k/2}$ is equal to

$$\begin{aligned}
T_k = T_k(z) &= \frac{1}{k!}\sum_{\sigma\in S_n}(-1)^\sigma e^{\sum x_i f_{\sigma(i)}}(x_1 z_{\sigma(1)}+\cdots+x_n z_{\sigma(n)})^k\\
&= \frac{1}{k!}\sum_{\sigma\in S_n}(-1)^\sigma e^{\sum x_i f_{\sigma(i)}} \sum_{k_1+\cdots+k_n=k}\frac{k!}{k_1!\ldots k_n!}(x_1 z_{\sigma(1)})^{k_{\sigma(1)}}\ldots(x_n z_{\sigma(n)})^{k_{\sigma(n)}}\\
&= \sum_{k_1+\cdots+k_n=k}\frac{1}{k_1!\ldots k_n!}\sum_{\sigma\in S_n}(-1)^\sigma e^{\sum x_i f_{\sigma(i)}}(x_1 z_{\sigma(1)})^{k_{\sigma(1)}}\ldots(x_n z_{\sigma(n)})^{k_{\sigma(n)}}\\
&= \sum_{k_1+\cdots+k_n=k}\frac{z^{\boldsymbol{k}}}{k_1!\ldots k_n!}\det\left[e^{x_i f_j} x_i^{k_j}\right].
\end{aligned}$$

Recall that

$$f_1 = \cdots = f_{m'_1} < f_{m'_1+1} = \cdots = f_{m'_2} < \cdots < f_{m'_{q'-1}+1} = \cdots = f_{m'_{q'}}.$$

If $k_i = k_j$ and $f_i = f_j$, then the determinant $\det\left[e^{x_i f_j} x_i^{k_j}\right]$ is 0. Thus we have a non-zero determinant if k_i are different for those i such that f_i are equal. Thus index

k such that T_k is non-zero must be at least

$$k = \sum_{j=1}^{n} k_j \geq k_0 = \sum_{j=1}^{q'} \binom{\nu'_j}{2}.$$

Moreover we get all nonzero $\det\left[e^{x_i f_j} x_i^{k_j}\right]$ putting in each subsequence

$$\left(\boldsymbol{k}_{(m'_0;m'_1]}, \ldots, \boldsymbol{k}_{(m'_{q'-1};m'_{q'}]}\right),$$

all possible permutations of strictly ordered numbers from $\mathbb{Z}_+$ such that all sum up to k. Thus we have

$$T_k = \sum_{\substack{k_1+\cdots+k_n=k \\ k_1<\cdots<k_{m'_1}; \\ \vdots \quad \vdots \\ k_{m'_{q'-1}+1}<\cdots<k_{m'_{q'}}}} \sum_{\sigma_1 \in S_{\nu'_1}} \cdots \sum_{\sigma_{q'} \in S_{\nu'_{q'}}} \frac{\boldsymbol{z}_{(m'_0;m'_1]}^{\sigma_1(\boldsymbol{k}_{(m'_0;m'_1]})}}{k_1! \ldots k_{m'_1}!} \cdots \frac{\boldsymbol{z}_{(m'_{q'-1};m'_{q'}]}^{\sigma_{q'}(\boldsymbol{k}_{(m'_{q'-1};m'_{q'}]})}}{k_{m'_{q'-1}+1}! \ldots k_{m'_{q'}}!}$$
$$\times \det\left[e^{x_i f_j} x_i^{\sum_{l=1}^{q'} \sigma_l(k_j) 1_{m'_{l-1}<j\leq m'_l}}\right]$$

Again we notice that permutations in the determinant influence only by the change of sign. These signs and sums over the group of permutations form determinants, thus we have

$$T_k = \sum_{\substack{k_1+\cdots+k_n=k \\ k_1<\cdots<k_{m'_1}; \\ \vdots \quad \vdots \\ k_{m'_{q'-1}+1}<\cdots<k_{m'_{q'}}}} \frac{\det\left[\left\{z_i^{k_j}\right\}_{i,j=1}^{m'_1}\right]}{k_1! \ldots k_{m'_1}!} \cdots \frac{\det\left[\left\{z_i^{k_j}\right\}_{i,j=m'_{q'-1}+1}^{m'_{q'}}\right]}{k_{m'_{q'-1}+1}! \ldots k_{m'_{q'}}!} \det\left[e^{x_i f_j} x_i^{k_j}\right].$$

□

Remark Using Itzykson–Zuber integral (see, e.g., [6]) we can write

$$\frac{\det\left[e^{x_i(z_j/\sqrt{t}+f_j)}\right]}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{z}/\sqrt{t}+\boldsymbol{f})} = c_n \int_{\boldsymbol{U}(n)} e^{\mathrm{Tr}\,\mathrm{diag}(\boldsymbol{x})U\mathrm{diag}(\boldsymbol{z}/\sqrt{t}+\boldsymbol{f})U^*} \mu(\mathrm{d}U),$$

where $\mu(\mathrm{d}U)$ is (normalised) Haar measure on the unitary group $\boldsymbol{U}(n)$. Now letting $t \to \infty$,

$$\int_{\boldsymbol{U}(n)} \mathrm{e}^{\mathrm{Tr}\mathrm{diag}(\boldsymbol{x})U\mathrm{diag}(\boldsymbol{z}/\sqrt{t}+\boldsymbol{f})U^*} \mu(\mathrm{d}U) \to \int_{\boldsymbol{U}(n)} \mathrm{e}^{\mathrm{Tr}(\mathrm{diag}(\boldsymbol{x})U\mathrm{diag}(\boldsymbol{f})U^*)} \mu(\mathrm{d}U)$$
$$= \frac{\det\left[\mathrm{e}^{x_i f_j}\right]}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{f})}$$

and

$$\Delta(\boldsymbol{z}/\sqrt{t}+\boldsymbol{f}) = t^{-\sum_{i=1}^{q'}\binom{\nu_i'}{2}} \prod_{i=1}^{q'} \Delta(\boldsymbol{z}_{(m_{i-1}';m_i']}) \prod_{1\le k<l\le n} (f^l - f^k)^{\nu_k'\nu_l'}(1+o(1)).$$

Hence, as $t \to \infty$

$$\det\left[\mathrm{e}^{x_i(z_j/\sqrt{t}+f_j)}\right]$$
$$\to t^{-\sum_{i=1}^{q'}\binom{\nu_i'}{2}} c_n \prod_{i=1}^{q'} \Delta(\boldsymbol{z}_{(m_{i-1}';m_i']}) \prod_{1\le k<l\le n} (f^l - f^k)^{\nu_k'\nu_l'} \frac{\det\left[\mathrm{e}^{x_i f_j}\right]}{\Delta(\boldsymbol{f})}.$$

This is a less detailed version of the formula from Lemma 5.6. Remark that if $\Delta(\boldsymbol{f}) = 0$ at some $\boldsymbol{f}$ (that is components of $\boldsymbol{f}$ are not strictly increasing), then we have to interpret $\det[\mathrm{e}^{x_i f_i}]/\Delta(\boldsymbol{f})$ as a limiting value at $\boldsymbol{f}$.

6 Proof of the theorem

Using (5.19) and formula (5.18) we write

$$\begin{aligned}&\mathbb{P}_{\boldsymbol{x}}(\tau > t)\\ &\quad= (2\pi)^{-n/2}\mathrm{e}^{-||\boldsymbol{x}||^2/2t}\mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>}\mathrm{e}^{-\gamma t}\\ &\quad\times \sum_{k=k_0}^{\infty} \int_{W-\boldsymbol{f}\sqrt{t}} \mathrm{e}^{-\frac{1}{2}|\boldsymbol{z}|^2}\mathrm{e}^{-\frac{1}{2}\sum_{l=1}^{q}\left(\frac{2\sqrt{t}}{\nu_l}\sum_{m_{l-1}<u<v\le m_l}(z_u-z_v)(a_v-a_u)\right)} t^{-k/2}T_k(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z},\end{aligned} \tag{6.20}$$

First we will analyse the above expression by taking only the first term in the sum (6.20), and then we show that it gives the right asymptotic. Thus the first term equals to

$$
\begin{aligned}
&(2\pi)^{-n/2}\mathrm{e}^{-||\boldsymbol{x}||^2/2t}\mathrm{e}^{-\gamma t}\mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>}\int\limits_{W-\boldsymbol{f}\sqrt{t}}\mathrm{e}^{-\frac{1}{2}|\boldsymbol{z}|^2}\\
&\quad\times\mathrm{e}^{-\frac{1}{2}\sum_{l=1}^{q}\left(\frac{2\sqrt{t}}{\nu_l}\sum_{m_{l-1}<u<v\le m_l}(z_u-z_v)(a_v-a_u)\right)}\\
&\quad\times\prod_{j=1}^{q'}c_{\nu'_j}\Delta(\boldsymbol{z}_{(m'_{j-1};m'_j]})\\
&\quad\times\det\left[\mathrm{e}^{x_k f_j}x_k^{\sum_l^{q'}(j-m'_{l-1}-1)\mathbf{I}_{\{m'_{l-1}<j\le m'_l\}}}\right]t^{-\frac{1}{2}k_0}\,\mathrm{d}\boldsymbol{z}\\
&=(2\pi)^{-n/2}\mathrm{e}^{-||\boldsymbol{x}||^2/2t}\mathrm{e}^{-\gamma t}\\
&\quad\times\mathrm{e}^{-<\boldsymbol{x},\boldsymbol{a}>}\det\left[\mathrm{e}^{x_k f_j}x_k^{\sum_l^{q'}(j-m'_{l-1}-1)\mathbf{I}_{\{m'_{l-1}<j\le m'_l\}}}\right]t^{-\frac{1}{2}k_0}\\
&\quad\times\prod_{j=1}^{q'}c_{\nu'_j}I(\boldsymbol{a},t),
\end{aligned}
$$

where $I(\boldsymbol{a},t)$ was introduced in (4.14).

6.1 Asymptotic behaviour of integral

If $\boldsymbol{s}=\boldsymbol{A}\boldsymbol{z}$, where

$$
\boldsymbol{A}=\begin{pmatrix}-1 & 1 & 0\dots & 0 & 0\\ 0 & -1 & 1\dots & 0 & 0\\ \vdots & \vdots & \vdots\ \ddots & \vdots & \vdots\\ 0 & 0 & 0\dots & -1 & 1\\ 1 & 1 & 1\dots & 1 & 1\end{pmatrix},
$$

than $z_u-z_v=s_v+s_{v+1}+\cdots+s_{u-1}$ and

$$
|\boldsymbol{z}|^2=\boldsymbol{z}^T\boldsymbol{z}=(\boldsymbol{A}^{-1}\boldsymbol{s})^T(\boldsymbol{A}^{-1}\boldsymbol{s})=\boldsymbol{s}^T(\boldsymbol{A}^{-1})^T\boldsymbol{A}^{-1}\boldsymbol{s}.
$$

Recall

$$
\boldsymbol{S}=(S_{ij})=((\boldsymbol{A}^{-1})^T\boldsymbol{A}^{-1})_{(n)},\tag{6.21}
$$

which is a positive definite matrix. By Lemma 5.4 we have

$$
|\boldsymbol{z}|^2=\frac{1}{n}s_n^2+\boldsymbol{s}_{(n)}^T((\boldsymbol{A}^{-1})^T\boldsymbol{A}^{-1})_{(n)}\boldsymbol{s}_{(n)},
$$

where $\boldsymbol{s}_{(n)}$ is obtained from $\boldsymbol{s}$ by deleting the nth coordinate and $\boldsymbol{A}_{(n)}$ is matrix $\boldsymbol{A}$ without nth row and nth column. Recall

$$V_{\boldsymbol{a}}(\boldsymbol{s}_{(n)}) = \sum_{l=1}^{q} \left(\frac{1}{\nu_l} \sum_{m_{l-1}<u<v\le m_l} (s_u + \cdots + s_{v-1})(a_u - a_v) \right).$$

It is important to notice that $V_{\boldsymbol{a}}(\boldsymbol{s}_{(n)})$ depends only on those s_i, where $i \notin \{m_1, \ldots m_q\}$. We also see that if $m_{i-1} < k < m_i$, then the coefficient at s_k in $V_{\boldsymbol{a}}$ are strictly positive.

After substitution $\boldsymbol{s} = \boldsymbol{A}\boldsymbol{z}$, integral $I(\boldsymbol{a}, t)$ is

$$\begin{aligned} I(\boldsymbol{a}, t) = & \underset{\substack{s_i > (f_i - f_{i+1})\sqrt{t}, \\ \text{for } i=1,\ldots,n-1}}{\int \cdots \int} \int_{s_n \in \mathbb{R}} \mathrm{e}^{-\frac{1}{2}\left(\frac{s_n^2}{n} + \boldsymbol{s}_{(n)}^T ((\boldsymbol{A}^{-1})^T \boldsymbol{A}^{-1})_{(n)} \boldsymbol{s}_{(n)}\right)} \\ & \times \mathrm{e}^{-\sqrt{t} V_{\boldsymbol{a}}(\boldsymbol{s}_{(n)})} \\ & \times \prod_{k=1}^{q'} H(\boldsymbol{s}_{(m'_{k-1};m'_k)})\, \mathrm{d}\boldsymbol{s}_{(n)}\, \mathrm{d}s_n \qquad (6.22) \\ = & \sqrt{2\pi n} \underset{\substack{s_i > (f_i - f_{i+1})\sqrt{t}, \\ \text{for } i=1,\ldots,n-1}}{\int \cdots \int} \mathrm{e}^{-\frac{1}{2}\left(\boldsymbol{s}_{(n)}^T ((\boldsymbol{A}^{-1})^T \boldsymbol{A}^{-1})_{(n)} \boldsymbol{s}_{(n)}\right)} \\ & \times \mathrm{e}^{-\sqrt{t} V_{\boldsymbol{a}}(\boldsymbol{s}_{(n)})} \\ & \times \prod_{k=1}^{q'} H(\boldsymbol{s}_{(m'_{k-1};m'_k)})\, \mathrm{d}\boldsymbol{s}_{(n)}. \qquad (6.23) \end{aligned}$$

Note that polynomials H in integral $I(\boldsymbol{a}, t)$ depends only on s_j, where $j \notin \{m'_1, \ldots, m'_{q'}\}$.

We now introduce new variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n-1})$ by

$$\xi_j = \begin{cases} \sqrt{t} s_j, & \text{for } j \notin \{m_1, \ldots, m_{q-1}\} \\ s_j, & \text{for } j \in \{m_1, \ldots, m_{q-1}\}. \end{cases} \qquad (6.24)$$

We define subspaces $E_1 = \{\boldsymbol{x} \in \mathbb{R}^{n-1} : x_j = 0 \text{ for } j \notin \{m_1, \ldots, m_{q-1}\}\}$ and $E_2 = \{\boldsymbol{x} \in \mathbb{R}^{n-1} : x_j = 0 \text{ for } j \in \{m_1, \ldots, m_{q-1}\}\}$. We can write (6.24) in short $\boldsymbol{s}_{(n)} = \boldsymbol{\xi}(t) = \boldsymbol{\xi}_1 + \frac{1}{\sqrt{t}} \boldsymbol{\xi}_2$, where $\boldsymbol{\xi}_1 \in E_1$ and $\boldsymbol{\xi}_2 \in E_2$.

We define function K by $K\left(\boldsymbol{\xi}_{(m'_{k-1};m'_k)}, t\right) = H\left(\boldsymbol{s}_{(m'_{k-1};m'_k)}\right)$.

Consider now $H\left(\boldsymbol{s}_{(1;m'_1)}\right)$. Since $\boldsymbol{m}'$ is a subsequence of $\boldsymbol{m}$, we recall that i_1 is such that $m_{i_1} = m'_1$. Similarly are defined $i_1, \ldots, i_{q'}$. We now factorise $H\left(\boldsymbol{s}_{(1;m'_1)}\right)$ into parts in which there in none of m_i, there is exactly one m_i, exactly two and so on.

Thus

$$H\left(\boldsymbol{s}_{(m'_0;m'_1)}\right)=\prod_{k=1}^{i_1}H\left(\boldsymbol{s}_{(m_{k-1};m_k)}\right)$$
$$\times\prod_{k=1}^{i_1-1}\prod_{\substack{m_{k-1}<i\le m_k\\ m_k<j\le m_{k+1}}}(s_i+\cdots+s_{j-1})$$
$$\times\prod_{k=1}^{i_1-2}\prod_{\substack{m_{k-1}<i\le m_k\\ m_{k+1}<j\le m_{k+2}}}(s_i+\cdots+s_{j-1})$$
$$\vdots$$
$$\times\prod_{\substack{m_0<i\le m_1\\ m_{i_1-1}<j\le m_{i_1}}}(s_i+\cdots+s_{j-1}).$$

We make analogous factorisation for other $H(\boldsymbol{s}_{(m'_{k-1};m'_k)})$.

Lemma 6.1 *As* $t\to\infty$

$$\prod_{k=1}^{q'}K(\boldsymbol{\xi}_{(m'_{k-1};m'_k)},t)=t^{-\frac{1}{2}\sum_{i=1}^{q}\binom{\nu_i}{2}}\prod_{i=1}^{q}H(\boldsymbol{\xi}_{(m_{i-1};m_i)})$$
$$\times\prod_{k=0}^{q-1}\prod_{\substack{i:\{i,i+1,\dots,i+k\}\\ \in\{1,\dots,q\}\setminus\{i_1,\dots,i_{q'}\}}}\left(\sum_{j=0}^{k}\xi_{m_{i+j}}\right)^{\nu_i\nu_{i+k+1}}(1+o(1)).$$

Proof After the substitution we get

$$K(\boldsymbol{\xi}_{(m'_0;m'_1)},t)=\prod_{k=1}^{i_1}H\left(\boldsymbol{\xi}_{(m_{k-1};m_k)}/\sqrt{t}\right)$$
$$\times\prod_{k=1}^{i_1-1}\prod_{\substack{m_{k-1}<i\le m_k\\ m_k<j\le m_{k+1}}}\left(\sum_{r=i,r\ne m_k}^{j-1}\xi_r/\sqrt{t}+\xi_{m_k}\right)$$
$$\times\prod_{k=1}^{i_1-2}\prod_{\substack{m_{k-1}<i\le m_k\\ m_{k+1}<j\le m_{k+2}}}\left(\sum_{r=i,r\notin\{m_k,m_{k+1}\}}^{j-1}\xi_r/\sqrt{t}+\xi_{m_k}+\xi_{m_{k+1}}\right)$$
$$\vdots$$

$$\times \prod_{\substack{m_0<i\le m_1\\ m_{i_1-1}<j\le m_{i_1}}} \left(\sum_{r=i, r\notin\{m_1,\dots,m_{i_1-1}\}}^{j-1} \xi_r/\sqrt{t} + \sum_{k=1}^{i_1-1} \xi_{m_k} \right).$$

It is not difficult to see that asymptotic behaviour of the above expression is

$$\begin{aligned} K(\boldsymbol{\xi}_{(m'_0;m'_1)}, t) = t^{-\frac{1}{2}\sum_{l=1}^{i_1}\binom{\nu_l}{2}} &\prod_{k=1}^{i_1} H\left(\boldsymbol{\xi}_{(m_{k-1};m_k)}\right) \\ &\times \prod_{k=1}^{i_1-1} (\xi_{m_k})^{\nu_k\nu_{k+1}} \\ &\times \prod_{k=1}^{i_1-2} (\xi_{m_k}+\xi_{m_{k+1}})^{\nu_k\nu_{k+2}} \\ &\vdots \\ &\times \left(\sum_{k=1}^{i_1-1}\xi_{m_k}\right)^{\nu_1\nu_{i_1}} (1+o(1)). \end{aligned}$$

We make similar considerations for $K(\boldsymbol{\xi}_{(m'_i;m'_{i+1})}, t)$ for $i = 1, \dots, q'-1$. Denote

$$k(\boldsymbol{\xi}) = \prod_{i=1}^{q} H\left(\boldsymbol{\xi}_{(m_{i-1};m_i)}\right) \prod_{k=0}^{q-1} \prod_{\substack{i:\{i,i+1,\dots,i+k\}\\ \in\{1,\dots,q\}\setminus\{i_1,\dots,i_{q'}\}}} \left(\sum_{j=0}^{k} \xi_{m_{i+j}}\right)^{\nu_i\nu_{i+k+1}}.$$

Then the whole polynomial is asymptotically

$$\prod_{k=1}^{q'} K\left(\boldsymbol{\xi}_{(m'_{k-1};m'_k)}, t\right) = t^{-\frac{1}{2}\sum_{i=1}^{q}\binom{\nu_i}{2}} \left(k(\boldsymbol{\xi}) + \frac{1}{\sqrt{t}} R(\boldsymbol{\xi}, t)\right). \tag{6.25}$$

□

For substitution (6.24), we have $\mathrm{d}\boldsymbol{s}_{(n)} = t^{-(n-q)/2}\,\mathrm{d}\boldsymbol{\xi}$. Note that $f_{k+1} = f_k$ for $k \ne m_i$, and hence the integration on the kth coordinate starts from 0. On the other hand if $k = m_i$ for some i, and $k \ne m'_j$ for every j, then we also have $f_{k+1} = f_k$ and therefore the integration starts from 0. Finally if $k = m'_j$ for some j, then $f_{k+1} > f_k$ and the integrations starts from $(f_k - f_{k+1})\sqrt{t}$.

So we can clearly see that $\prod_{k=1}^{q'} K(\boldsymbol{\xi}_{(m'_{k-1};m'_k)}, t)$ depends only on ξ_i's such that $i \notin \{m'_1, \dots, m'_{q'}\}$ and it can be factorised into a part which depends only on $i \notin \{m_1, \dots, m_q\}$ and a part that depends on $i \in \{m_1, \dots, m_q\}\setminus\{m'_1, \dots, m'_{q'}\}$. Hence we have after the substitution

$$I(\boldsymbol{a},t) = t^{-(n-q)/2}\sqrt{2\pi n} \underset{\substack{\xi_i > (f_i - f_{i+1})\sqrt{t} \\ \text{for } i=1,\dots,n-1}}{\int \cdots \int} \mathrm{e}^{-\frac{1}{2}\left(\left(\boldsymbol{\xi}_1 + \frac{1}{\sqrt{t}}\boldsymbol{\xi}_2\right)^T \boldsymbol{S}\left(\boldsymbol{\xi}_1 + \frac{1}{\sqrt{t}}\boldsymbol{\xi}_2\right)\right)}$$

$$\times\, \mathrm{e}^{-V_{\boldsymbol{a}}(\boldsymbol{\xi})} t^{-\frac{1}{2}\sum_{i=1}^{q}\binom{v_i}{2}} \left(k(\boldsymbol{\xi}) + \frac{1}{\sqrt{t}} R(\boldsymbol{\xi},t)\right) \mathrm{d}\boldsymbol{\xi} \tag{6.26}$$

$$= t^{-(n-q)/2}\sqrt{2\pi n} \underset{\substack{\xi_i > (f_i - f_{i+1})\sqrt{t} \\ \text{for } i=1,\dots,n-1}}{\int \cdots \int} \mathrm{e}^{-\frac{1}{2}\left(\left(\boldsymbol{\xi}_1 + \frac{1}{\sqrt{t}}\boldsymbol{\xi}_2\right)^T \boldsymbol{S}\left(\boldsymbol{\xi}_1 + \frac{1}{\sqrt{t}}\boldsymbol{\xi}_2\right)\right)}$$

$$\times\, \mathrm{e}^{-V_{\boldsymbol{a}}(\boldsymbol{\xi})} t^{-\frac{1}{2}\sum_{i=1}^{q}\binom{v_i}{2}} k(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}$$

$$+ t^{-(n-q)/2}\sqrt{2\pi n} \underset{\substack{\xi_i > (f_i - f_{i+1})\sqrt{t} \\ \text{for } i=1,\dots,n-1}}{\int \cdots \int} \mathrm{e}^{-\frac{1}{2}\left(\left(\boldsymbol{\xi}_1 + \frac{1}{\sqrt{t}}\boldsymbol{\xi}_2\right)^T \boldsymbol{S}\left(\boldsymbol{\xi}_1 + \frac{1}{\sqrt{t}}\boldsymbol{\xi}_2\right)\right)}$$

$$\times\, \mathrm{e}^{-V_{\boldsymbol{a}}(\boldsymbol{\xi})} t^{-\frac{1}{2}\sum_{i=1}^{q}\binom{v_i}{2}} \frac{1}{\sqrt{t}} R(\boldsymbol{\xi},t)\, \mathrm{d}\boldsymbol{\xi}$$

$$= I_1(\boldsymbol{a},t) + I_2(\boldsymbol{a},t). \tag{6.27}$$

In (6.26) function $R(\boldsymbol{\xi}, t)$ is a polynomial of the form

$$\sum_{\boldsymbol{i}\in\mathcal{J}} t^{-b(\boldsymbol{i})/2} a(\boldsymbol{i}) \xi_1^{i_1} \dots \xi_{n-1}^{i_{n-1}},$$

where a set of indices $\boldsymbol{i} = (i_1, \dots, i_{n-1})$ runs through a finite set $\mathcal{J}$, $a(\boldsymbol{i}) \in \mathbb{R}$ and $b(\boldsymbol{i}) \in \{0, 1, \dots\}$.

To derive majorants we need the following lemma.

Lemma 6.2 *There exists* $0 < d < 1$ *such that*

$$(\boldsymbol{\xi}_1 + a\boldsymbol{\xi}_2)^T \boldsymbol{S}(\boldsymbol{\xi}_1 + a\boldsymbol{\xi}_2) \geq d\boldsymbol{\xi}_1^T \boldsymbol{S}\boldsymbol{\xi}_1, \quad 0 < a \leq 1, \quad \boldsymbol{\xi}_1 \in E_1, \quad \boldsymbol{\xi}_2 \in E_2.$$

Proof Since $\boldsymbol{S}$ is positive definite we can define a scalar product in $\mathbb{R}^{n-1}$ by $(\boldsymbol{x}_1, \boldsymbol{x}_2)_S = \boldsymbol{x}_1^T \boldsymbol{S}\boldsymbol{x}_2$. Let $||\boldsymbol{x}||_S = \sqrt{(\boldsymbol{x},\boldsymbol{x})_S}$. Function $f(a) = (\boldsymbol{\xi}_1 + a\boldsymbol{\xi}_2)^T \boldsymbol{S}(\boldsymbol{\xi}_1 + a\boldsymbol{\xi}_2)$, $0 < a \leq 1$ reaches its minimum at

$$a_0 = -\frac{(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)_S}{||\boldsymbol{\xi}_2||_S^2}$$

and $f(a_0) = ||\boldsymbol{\xi}_1||_S^2(1 - (\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2)_S^2)$, where $\bar{\boldsymbol{\xi}}_i = \boldsymbol{\xi}_i/||\boldsymbol{\xi}_i||_S$. Let

$$1 - d = \sup_{\substack{||\bar{\boldsymbol{\xi}}_i||=1, i=1,2 \\ \bar{\boldsymbol{\xi}}_i \in E_i}} (\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2)_S^2 = (\bar{\boldsymbol{\eta}}_1, \bar{\boldsymbol{\eta}}_2)_S^2.$$

Suppose that $(\bar{\boldsymbol{\eta}}_1, \bar{\boldsymbol{\eta}}_2)_S^2 = 1 = ||\boldsymbol{\eta}_1||_S||\boldsymbol{\eta}_2||_S$. We have the equality in Schwartz inequality if $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are linearly dependent, which is impossible because $\boldsymbol{\eta}_i \in E_i$. Thus $d > 0$. □

Note now that function

$$\mathrm{e}^{-\frac{1}{2}d\boldsymbol{\xi}_1^T S\boldsymbol{\xi}_1 - V_{\boldsymbol{a}}(\boldsymbol{\xi})}k(\boldsymbol{\xi})1_{\{\xi_i>0 \text{ for } i\notin\{m'_1,\dots,m'_{q'-1}\}\}}$$

is an integrable majorant to derive an asymptotics for $I_1(\boldsymbol{a}, t)$. Similar consideration yields that $I_2(\boldsymbol{a}, t) \to 0$.

Hence

$$\begin{aligned} I(\boldsymbol{a}, t) &= t^{-(n-q)/2} t^{-\frac{1}{2}\sum_{i=1}^q \binom{\nu_i}{2}} \sqrt{2\pi n} \\ &\times \underset{\substack{\xi_i>0:i=1,\dots,n-1 \\ i\notin\{m_1,\dots,m_{q-1}\}}}{\int\cdots\int} \mathrm{e}^{-\frac{1}{2}\sum_{l=1}^q \left(\frac{2}{\nu_l}\sum_{m_{l-1}<u<v\le m_l}(\xi_u+\dots+\xi_{v-1})(a_u-a_v)\right)} \\ &\times \prod_{i=1}^q H(\boldsymbol{\xi}_{(m_{i-1};m_i)}) \prod_{i\notin\{m_1,\dots,m_{q-1}\}} \mathrm{d}\xi_i \\ &\times \underset{\xi_i>0:i\in\{m_1,\dots,m_{q-1}\}\setminus\{m'_1,\dots,m'_{q'-1}\}}{\int\cdots\int} \ \underset{\xi_i>-\infty:i\in\{m'_1,\dots,m'_{q'-1}\}}{\int\cdots\int} \mathrm{e}^{-\frac{1}{2}\left(\sum_{k,l\in\{m_1,\dots,m_{q-1}\}} S_{kl}\xi_k\xi_l\right)} \\ &\times \prod_{k=0}^{q-1} \prod_{\substack{i:\{i,i+1,\dots,i+k\} \\ \in\{1,\dots,q-1\}\setminus\{i_1,\dots,i_{q'-1}\}}} \left(\sum_{j=0}^k \xi_{m_{i+j}}\right)^{\nu_i\nu_{i+k+1}} \prod_{i\in\{m_1,\dots,m_{q-1}\}} \mathrm{d}\xi_i \ (1+o(1)). \end{aligned} \tag{6.28}$$

Concluding we have

$$I(\boldsymbol{a}, t) = C_1 t^{-(n-q)/2} t^{-\frac{1}{2}\sum_{i=1}^q \binom{\nu_i}{2}} (1+o(1)),$$

where C_1 depends only on drift vector $\boldsymbol{a}$.

6.2 Proof of Theorem 4.2

Following considerations of Sect. 6.1, notice first-that it suffices to take the first term from the sum (6.20) for asymptotic analysis because next terms consists of positive rank polynomials of variable $\boldsymbol{z}$ and therefore they will tend to zero faster after substitution (6.24). For the proof of the main theorem we have to plug the asymptotics (6.28) to integral (6.20).

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