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Critical behavior and the limit distribution for long-range oriented percolation. I

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Abstract We consider oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ whose bond-occupation probability is $pD(\,\cdot\,)$, where p is the percolation parameter and D is a probability distribution on $\mathbb{Z}^d$. Suppose that $D(x)$ decays as $|x|^{-d-\alpha}$ for some $\alpha > 0$. We prove that the two-point function obeys an infrared bound which implies that various critical exponents take on their respective mean-field values above the upper-critical dimension $d_c = 2(\alpha \wedge 2)$. We also show that, for every k, the Fourier transform of the normalized two-point function at time n, with a proper spatial scaling, has a convergent subsequence to $e^{-c|k|^{\alpha\wedge 2}}$ for some $c > 0$.

1 Introduction

Oriented percolation is a model that exhibits a phase transition when the percolation parameter p in the bond-occupation probability $pD(\,\cdot\,)$ changes its value, where D is a given probability distribution on $\mathbb{Z}^d$. It has been proved using the lace expansion [16,20] that finite-variance oriented percolation, where the tail of D decays fast enough to ensure finite variance $\sigma^2 = \sum_x |x|^2 D(x)$ in particular, exhibits the critical behavior for (finite-range) branching random walk, if $d > 4$ and $\sigma^2 \gg 1$ or $d \gg 4$; it has also been proved that, for every $p \le p_c$ for finite-range oriented percolation [20] and for general (possibly infinite-range) finite-variance oriented percolation at $p = p_c$ [16],

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the Fourier transform of the normalized two-point function at time n, spatially scaled by $\sqrt{n}$, converges to $e^{-c|k|^2}$ for some constant $c \in (0, \infty)$.

In this paper, we consider long-range oriented percolation with index $\alpha > 0$, where $D(x)$ decays as $|x|^{-d-\alpha}$ for large $|x|$. In [8], Chen and Shieh studied a long-range model with $\alpha = 1$ and proved that, if $d > 2$ (and a certain spread-out parameter $L \gg 1$), the standard susceptibility exponent γ and a couple of other critical exponents take on their respective mean-field values. The goal of this paper is to investigate the α-dependence of the critical behavior and the limit distribution. We prove that the model exhibits the mean-field behavior if $d > 2(\alpha \wedge 2)$ (and a spread-out parameter $L \gg 1$). Furthermore, we prove that, for every $p \leq p_{\mathrm{c}}$, the Fourier transform of the normalized two-point function at time n, spatially scaled by $n^{\frac{1}{\alpha\wedge 2}}$ if $\alpha \neq 2$ or by $\sqrt{n \log n}$ if $\alpha = 2$, is bounded from below by $e^{-c|k|^{\alpha\wedge 2}}$ and from above by $e^{-c'|k|^{\alpha\wedge 2}}$ in $n \uparrow \infty$, where $c, c' \in (0, \infty)$ and $c/c' = 1 + O(L^{-d})$. We stress that, although we do not prove convergence in this paper, our results hold for $p \leq p_{\mathrm{c}}$ for general finite-variance oriented percolation, which is not completely covered in the aforementioned results in [16,20].

Our proof is based on the lace expansion for oriented percolation. We analyze the lace expansion for all $\alpha > 0$ simultaneously to discover a potential crossover in the critical behavior by changing the value of α. However, since our D does not have finite variance when $\alpha \leq 2$, the standard Taylor-expansion analyses for the Fourier transform of the expansion coefficients for finite-variance oriented percolation do not always work. To overcome this difficulty, we use the trigonometric techniques that were first developed in [6] for percolation on finite graphs and later in [25] for finite-range self-avoiding walk on $\mathbb{Z}^d$. We adapt these techniques for the time-oriented setting (to analyze the Fourier–Laplace transform of the expansion coefficients).

1.1 Model

We define the model more precisely. A bond is an ordered pair $((x, n), (y, n + 1))$ of vertices in space-time $\mathbb{Z}^d \times \mathbb{Z}_+$, where $\mathbb{Z}_+ \equiv \{0\} \dot{\cup} \mathbb{N}$ is the set of nonnegative integers. Each bond is, independently of the other bonds, occupied (resp., vacant) with probability $pD(y-x)$ (resp., $1 - pD(y-x)$), where D is a probability distribution on $\mathbb{Z}^d$. The percolation parameter $p \in [0, \|D\|_\infty^{-1}]$ equals the average number of occupied bonds per vertex. We say that (x, l) is connected to (y, n), and write $(x, l) \to (y, n)$, if either $(x, l) = (y, n)$ or there is a time-oriented path of occupied bonds from (x, l) to (y, n). Let $\mathbb{P}_p$ be the probability distribution of the bond variables, and denote its expectation by $\mathbb{E}_p$.

Our D is defined as follows. Let h be a bounded probability distribution on $\mathbb{R}^d$ that is invariant under rotations by $\pi/2$ and reflections in the coordinate hyperplanes. Suppose that h is piecewise continuous, so that $\int_{\mathbb{R}^d} d^d x\, h(x) \equiv 1$ can be approximated by the Riemann sum $\frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} h(x/L)$ for large $L < \infty$. We define

$$D(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)}, \tag{1.1}$$

where $x/L = (x_1/L, \ldots, x_d/L)$. Note that the denominator is $O(L^d)$.

Fix $\alpha > 0$ throughout this paper. We assume that there is an $\ell < \infty$ such that

$$h(x) \asymp |x|^{-d-\alpha} \qquad (|x| \geq \ell), \tag{1.2}$$

where $f(x) \asymp g(x)$ means that $f(x)/g(x)$ is bounded away from zero and infinity. We note that the rth moment $\sum_{x \in \mathbb{Z}^d} |x|^r D(x)$ does not exist if $r \geq \alpha$, but exists and equals $O(L^r)$ if $r \in (0, \alpha)$. A simple example of h that satisfies the above assumptions is

$$h(x) = \frac{1}{\mathcal{N}} (|x| \vee 1)^{-d-\alpha}, \tag{1.3}$$

where $\mathcal{N}$ is the normalization constant. In this case, D equals

$$D(x) = \frac{(|\frac{x}{L}| \vee 1)^{-d-\alpha}}{\sum_{y \in \mathbb{Z}^d} (|\frac{y}{L}| \vee 1)^{-d-\alpha}}. \tag{1.4}$$

The main properties of D are summarized as follows:

Proposition 1.1 *Let $\lambda = L^{-d}$, and denote by $D^{\star n}$ and $\hat{D}$, respectively, the n-fold convolution and the Fourier transform of D:*

$$D^{\star n}(x) = \begin{cases} D(x) & (n = 1), \\ \sum_{y \in \mathbb{Z}^d} D^{\star(n-1)}(y)\, D(x-y) & (n \geq 2), \end{cases} \qquad \hat{D}(k) = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} D(x). \tag{1.5}$$

Then, for $L \gg 1$, there are $C < \infty$ and $\Delta \in (0, 1)$ such that

$$\|D^{\star n}\|_\infty \leq C \lambda n^{-\frac{d}{\alpha \wedge 2}}, \qquad 1 - \hat{D}(k) \begin{cases} < 2 - \Delta & (k \in [-\pi, \pi]^d), \\ > \Delta & (\|k\|_\infty > (\ell L)^{-1}). \end{cases} \tag{1.6}$$

Moreover, when $\|k\|_\infty \leq (\ell L)^{-1}$,

$$1 - \hat{D}(k) \asymp \begin{cases} (L|k|)^{\alpha \wedge 2} & (\alpha \neq 2), \\ (L|k|)^2 \log \frac{\pi}{2\ell L|k|} & (\alpha = 2). \end{cases} \tag{1.7}$$

We will prove Proposition 1.1 in Appendix A.

1.2 Main results

We investigate the following two-point function:

$$\varphi_p(y - x, n - l) = \mathbb{P}_p((x, l) \to (y, n)), \tag{1.8}$$

where we have used the fact that the right-hand side depends only on $y-x$ and $n-l$, due to the translation invariance of the model. Assuming summability of the two-point function, we define, for $k \in [-\pi, \pi]^d$ and $z \in \mathbb{C}$,

$$Z_p(k;n) = \sum_{x\in\mathbb{Z}^d} \varphi_p(x,n)\, e^{ik\cdot x}, \quad \hat{\varphi}_p(k,z) = \sum_{n\in\mathbb{Z}_+} Z_p(k;n)\, z^n. \tag{1.9}$$

Let $\mathcal{C}_n$ be the set of vertices at time n that are connected from $(o,0)$, and let $\mathcal{C} = \bigcup_{n\geq 0}\mathcal{C}_n$. The quantities in (1.9) for $k=0$ and $(k,z)=(0,1)$ can be described as

$$Z_p(0;n) = \mathbb{E}_p[|\mathcal{C}_n|], \quad \chi_p \equiv \hat{\varphi}_p(0,1) = \mathbb{E}_p[|\mathcal{C}|], \tag{1.10}$$

where $|\mathcal{A}|$ is the cardinality of a set $\mathcal{A}$, and χ_p is called the susceptibility. Since $Z_p(0;n)$ is sub-multiplicative, i.e., for $l, n \geq 0$,

$$\begin{aligned} Z_p(0;l+n) &= \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\Bigg(\bigcup_{y\in\mathbb{Z}^d} \{\{(o,0)\to(y,l)\}\cap\{(y,l)\to(x,l+n)\}\}\Bigg) \\ &\leq Z_p(0;l)\, Z_p(0;n), \end{aligned} \tag{1.11}$$

the radius m_p of convergence of the series $\hat{\varphi}_p(0,z)$ is well-defined and satisfies (cf., e.g., [9, Appendix II])

$$m_p^{-1} = \lim_{n\uparrow\infty} Z_p(0;n)^{1/n} = \inf_{n\geq 1} Z_p(0;n)^{1/n}. \tag{1.12}$$

This implies that $\hat{\varphi}_p(0,m)$ for $m \in \mathbb{R}$ diverges as $m \uparrow m_p$ for every $p > 0$, because

$$\hat{\varphi}_p(0,m) = \sum_{n\in\mathbb{Z}_+} Z_p(0;n)\, m^n \geq \sum_{n\in\mathbb{Z}_+} \left(\frac{m}{m_p}\right)^n = \frac{m_p}{m_p - m}. \tag{1.13}$$

This also implies that $m_p > 1$ if and only if $\chi_p < \infty$. Since $\hat{\varphi}_0(0,m) = 1$ for any $m \geq 0$, we define $m_0 = \infty$. It is known [1,2,5,10] that there is a unique critical point $p_c \geq 1$ such that

$$\chi_p \begin{cases} <\infty, & \text{if } p < p_c, \\ =\infty, & \text{if } p \geq p_c, \end{cases} \qquad \Theta_p \equiv \mathbb{P}_p(|\mathcal{C}| = \infty) \begin{cases} =0, & \text{if } p \leq p_c, \\ >0, & \text{if } p > p_c, \end{cases} \tag{1.14}$$

and that $\lim_{p\uparrow p_c}\chi_p = \infty$ (hence $m_{p_c} \leq 1$) and $\lim_{p\downarrow p_c}\Theta_p = 0$.

Our first result is about an upper bound on $|\hat{\varphi}_p(k,z)|$ for $p < p_c$ and $|z| < m_p$.

Theorem 1.2 *Let $d > 2(\alpha \wedge 2)$ and $L \gg 1$. Then, there is a $C < \infty$ such that*

$$|\hat{\varphi}_p(k,z)| \leq \frac{C}{p(m_p - |z|) + |\arg(z)| + 1 - \hat{D}(k)}, \tag{1.15}$$

for any $p \in (0, p_c)$, $k \in [-\pi, \pi]^d$ *and* $z \in \mathbb{C}$ *with* $|z| < m_p$.

To prove this theorem and the other results throughout this paper, we use the lace expansion for oriented percolation. We will briefly review it in Sect. 3.

It has been proved [19,20] that (1.15) holds for finite-variance oriented percolation (for which, $1 - \hat{D}(k) \asymp |k|^2$) if $d > 4$ and $\sigma^2 \gg 1$ or $d \gg 4$, hence

$$\int\limits_{[-\pi,\pi]^{d+1}} \frac{d^d k}{(2\pi)^d} \frac{d\theta}{2\pi} \left|\hat{\varphi}_p(k, me^{i\theta})\right|^3 \tag{1.16}$$

is bounded uniformly in $p < p_c$ and $m < m_p$. By the dimension-independent results in [2,3], this implies that the critical exponents β, γ and δ defined as

$$\Theta_p \underset{p \downarrow p_c}{\asymp} (p - p_c)^{\beta}, \quad \chi_p \underset{p \uparrow p_c}{\asymp} (p_c - p)^{-\gamma}, \quad \mathbb{P}_{p_c}(|\mathcal{C}| \ge n) \underset{n \uparrow \infty}{\asymp} n^{-1/\delta}, \tag{1.17}$$

exist and take on their mean-field values for $d > 4$: $\beta = \gamma = 1$ and $\delta = 2$. Since our $1 - \hat{D}(k)$ satisfies (1.7), the integral (1.16) is bounded uniformly in $p < p_c$ and $m < m_p$ when $d > 2(\alpha \wedge 2)$. Let τ and η be the critical exponents for $m_p - m_{p_c}$ and $Z_{p_c}(0; n)$, respectively:

$$m_p - m_{p_c} \underset{p \uparrow p_c}{\asymp} (p_c - p)^{\tau}, \quad Z_{p_c}(0; n) \underset{n \uparrow \infty}{\asymp} n^{\eta}. \tag{1.18}$$

Corollary 1.3 *Let* $d > 2(\alpha \wedge 2)$ *and* $L \gg 1$, *so that Theorem 1.2 holds. Then,* $m_{p_c} = 1$ *and the critical exponents* β, γ, δ *and* τ *exist and take on their respective mean-field values:* $\beta = \gamma = \tau = 1$ *and* $\delta = 2$.

The identity $\tau = 1$ follows immediately from $\gamma = 1$ and the inequality

$$\frac{m_p}{\chi_p} \le m_p - 1 \le \frac{C}{p\chi_p} \qquad (0 < p < p_c). \tag{1.19}$$

The lower bound is due to (1.13) for $m = 1$, and the upper bound is due to Theorem 1.2 for $(k, z) = (0, 1)$. By the continuity of χ_p^{-1} in p, we obtain $m_{p_c} = \lim_{p \uparrow p_c} m_p = 1$. It may be worth pointing out that the trivial bound $Z_p(0; n) \le p^n$ and the inequality (1.19) with $\chi_p \ge 1$ imply $m_p \asymp p^{-1}$ for all $p \in (0, 1)$.

The mean-field result on the exponent η is in Theorem 1.5 below.

The critical exponents are generally believed to be universal in the sense that their values depend only on d and α, but not on the microscopic details of the model, such as the value of $L < \infty$. However, the value of p_c is not universal and changes depending on the value of L. In [15], an asymptotic estimate of p_c as $L \to \infty$ was investigated for various finite-variance models, such as self-avoiding walk, percolation, oriented percolation and the contact process, above the model-dependent upper-critical dimension. Using Proposition 1.1 and Theorem 1.2, we obtain the same asymptotic estimate of p_c for our long-range oriented percolation for $d > 2(\alpha \wedge 2)$, as follows:

Theorem 1.4 *Let $d > 2(\alpha \wedge 2)$. Then, as $L \to \infty$,*

$$p_{\mathrm{c}} = 1 + \frac{1}{2} \sum_{n=2}^{\infty} D^{\star 2n}(o) + O(\lambda^2), \tag{1.20}$$

where the sum of the $2n$-fold convolutions over $n \geq 2$ is $O(\lambda)$ if $d > \alpha \wedge 2$.

Our last results are about asymptotic estimates of the expected number $Z_p(0; n)$ of vertices at time n connected from $(o, 0)$ and the Fourier transform of the normalized two-point function $Z_p(\,\cdot\,; n)/Z_p(0; n)$. For finite-range oriented percolation with $d > 4$ and $\sigma^2 \gg 1$ or $d \gg 4$, Nguyen and Yang [20] used Tauberian estimates to prove that, for any $p \in (0, p_{\mathrm{c}}]$ and $k \in \mathbb{R}^d$, there are $c_1, c_2 = 1 + O(\lambda)$ such that $Z_p(0; n) \sim c_1 m_p^{-n}$ and $Z_p(k/\sqrt{n}; n)/Z_p(0; n) \sim e^{-c_2|k|^2}$; sharper error estimates for general finite-variance oriented percolation at $p = p_{\mathrm{c}}$ were obtained in [16] by an inductive analysis of the lace expansion. In this paper, we follow the line of [20] using Tauberian estimates to prove the following theorem for long-range oriented percolation:

Theorem 1.5 *Let $d > 2(\alpha \wedge 2)$ and $L \gg 1$, so that Theorem 1.2 holds. Fix $\epsilon \in (0, 1 \wedge \frac{d-2(\alpha\wedge 2)}{\alpha \wedge 2})$. Then, the following* (i)–(ii) *hold for any $p \in (0, p_{\mathrm{c}}]$ and $k \in \mathbb{R}^d$:*

(i) *There is a $C_1 = 1 + O(\lambda)$ such that*

$$Z_p(0; n) = C_1 m_p^{-n} \left(1 + O(n^{-\epsilon})\right) \qquad (n \geq 1). \tag{1.21}$$

In particular, the critical exponent η takes on its mean-field value: $\eta = 0$.

(ii) *Suppose that there is an L-dependent constant $v_\alpha \in (0, \infty)$ such that*

$$1 - \hat{D}(k) \underset{|k| \to 0}{\sim} \begin{cases} v_\alpha |k|^{\alpha \wedge 2} & (\alpha \neq 2), \\ v_2 |k|^2 \log \frac{1}{|k|} & (\alpha = 2). \end{cases} \tag{1.22}$$

Let

$$k_n = k \times \begin{cases} (v_\alpha n)^{-\frac{1}{\alpha \wedge 2}} & (\alpha \neq 2), \\ (v_2 n \log \sqrt{n})^{-\frac{1}{2}} & (\alpha = 2). \end{cases} \tag{1.23}$$

Then, there are C_2 and C_2', both equal to $1 + O(\lambda)$, such that

$$e^{-C_2|k|^{\alpha\wedge 2}} \leq \liminf_{n \to \infty} \frac{Z_p(k_n; n)}{Z_p(0; n)} \leq \limsup_{n \to \infty} \frac{Z_p(k_n; n)}{Z_p(0; n)} \leq e^{-C_2'|k|^{\alpha \wedge 2}}. \tag{1.24}$$

We note that our D satisfies the bound (1.7) on $1 - \hat{D}(k)$ for small k. The assumption (1.22) identifies the coefficient of the leading term of $1 - \hat{D}(k)$.

In the proof of the above theorem, we estimate fractional moments for the *time* variable of the lace-expansion coefficients. In the ongoing work (LC Chen and A Sakai, In preparation), we have been able to show that the limit of $Z_p(k_n; n)/Z_p(0; n)$ exists for $\alpha > 2$ and $d > 6$ by crude fractional-moment estimates for the *spatial* variable of the expansion coefficients. The difficulty in proving existence of the limit for all $\alpha > 0$ and $d > 2(\alpha \wedge 2)$ is due to the fact that the support of our D is unbounded, so that we cannot simply bound $|x|^r \varphi_p(x, n)$ for some $r > 0$, which may show up in the fractional-moment analysis, by a multiple of $n^r \varphi_p(x, n)$, as done in [20] for finite-range oriented percolation. To squeeze the bounds in (1.24) in order to identify the limit of $Z_p(k_n; n)/Z_p(0; n)$, we may have to improve the aforementioned fractional-moment estimates for the *spatial* variable. We expect that the idea may also be extended to investigate $\xi_p^{(r)}(n) \equiv \sum_x |x|^r \varphi_p(x, n)/Z_p(0; n)$. Nguyen and Yang proved in [20] that $\xi_p^{(2)}(n) \asymp n$ for any $p \in (0, p_c]$ for sufficiently spread-out finite-range oriented percolation for $d > 4$. We are aiming to show that $\xi_p^{(r)}(n) \asymp n^{\frac{r}{\alpha \wedge 2}}$ for any $p \in (0, p_c]$ and $r < \alpha$ for our long-range oriented percolation for $d > 2(\alpha \wedge 2)$.

1.3 Organization

The rest of this paper is organized as follows. In Sect. 2, we prove the above three theorems assuming a couple of key propositions. These propositions are proved in Sects. 4–6. Finally, in the Appendix, we prove Proposition 1.1.

2 Proof of the main results

In Sects. 2.2–2.4, we prove Theorems 1.2, 1.4 and 1.5, respectively, assuming several key ingredients. The most important ingredient is the lace expansion.

2.1 Lace expansion

The idea of the lace expansion was initiated by Brydges and Spencer in [7] for investigating weakly self-avoiding walk for $d > 4$. Later, the lace expansion was applied to various stochastic-geometrical models, such as strictly self-avoiding walk for $d > 4$ (e.g., [13]), lattice trees/animals for $d > 8$ (e.g., [12]), percolation for $d > 6$ (e.g., [11]), oriented percolation for $d > 4$ (e.g., [19]) and the contact process for $d > 4$ (e.g., [22]). Application to the Ising model was recently reported in [23]. See [25] for a complete list of references up to 2005.

The derivation of the lace expansion, the definition of the expansion coefficients and their diagrammatic bounds in terms of two-point functions depend on which model is concerned, but are independent of the specific choice of D. Therefore, we can apply the standard lace expansion for oriented percolation to the current long-range setting. We will briefly review the expansion in Sect. 3.

The result of the lace expansion is a recursion equation similar to that for the random-walk two-point function

$$P_p(x, n) = \delta_{x,o}\delta_{n,0} + p^n D^{\star n}(x) \mathbb{1}_{\{n \geq 1\}} = \delta_{x,o}\delta_{n,0} + (q_p * P_p)(x, n), \tag{2.1}$$

where $\mathbb{1}_{\{\cdots\}}$ is the indicator function and

$$q_p(x,n) = pD(x)\delta_{n,1}. \tag{2.2}$$

For oriented percolation, we have (see Proposition 3.1 below)

$$\varphi_p(x,n) = \pi_p(x,n) + (\pi_p * q_p * \varphi_p)(x,n) \qquad (0 \le p < \infty), \tag{2.3}$$

where $\pi_p(x,n)$ is the alternating sum of the nonnegative lace-expansion coefficients $\pi_p^{(N)}(x,n)$:

$$\pi_p^{(N)}(x,n) \ge 0 \quad (N = 0, 1, \dots), \quad \pi_p(x,n) = \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(x,n). \tag{2.4}$$

If $n = 0$, then $\pi_p^{(N)}(x,0) = \delta_{x,o}\delta_{N,0}$, hence $\pi_p(x,0) = \delta_{x,o}$, due to the definition (3.10) of $\pi_p^{(N)}(x,n)$ below. Comparing (2.1) and (2.3), we are naturally led to expect that $\varphi_p(x,n)$ behaves similarly to $P_p(x,n)$, if $\pi_p(x,n) - \delta_{x,o}\delta_{n,0}$ is small.

2.2 Infrared bound

We prove Theorem 1.2 by comparing $\hat{\varphi}_p(k,z)$, where $k \in [-\pi,\pi]^d$ and $z \in \mathbb{C}$ with $|z| < m_p$, with the Fourier transform of the random-walk Green's function with a certain rate $\mu = \mu_p(z) \in \mathbb{C}$:

$$\hat{G}_\mu(k) \equiv \sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+} P_\mu(x,n)\, e^{ik\cdot x} = \frac{1}{1-\mu\hat{D}(k)} \qquad (|\mu| < 1). \tag{2.5}$$

It is not hard to see that $\hat{G}_\mu(k)$ obeys the following infrared bound:

$$|\hat{G}_\mu(k)| \le \frac{c}{(1-|\mu|) + |\arg(\mu)| + 1 - \hat{D}(k)}, \tag{2.6}$$

where $c < \infty$ is independent of μ and k.

Let

$$\mu_p(z) = \Big(1 - \hat{\varphi}_p(0,|z|)^{-1}\Big)\, e^{i\arg(z)}, \tag{2.7}$$

where $|\mu_p(z)| < 1$ for $|z| < m_p$ and $\mu_p(m) \uparrow 1$ as $m \uparrow m_p$. Inspired by the bootstrapping hypotheses used in [6] for percolation on finite graphs and in [25] for finite-range self-avoiding walk on $\mathbb{Z}^d$, we define

$$f(p,m) = \max_{i=1,2,3} f_i(p,m) \qquad (p < p_c,\ m < m_p), \tag{2.8}$$

where

$$f_1(p,m)=p(m\vee 1),\quad f_2(p,m)=\sup_{\substack{k\in[-\pi,\pi]^d\\ z\in\mathbb{C}:|z|\in\{m,1\}}}\left|\frac{\hat{\varphi}_p(k,z)}{\hat{G}_{\mu_p(z)}(k)}\right|,\tag{2.9}$$

$$f_3(p,m)=\sup_{\substack{k,l\in[-\pi,\pi]^d\\ z\in\mathbb{C}:|z|\in\{m,1\}}}\frac{\hat{G}_{\mu_p(m\vee 1)}(k)\,|\hat{\varphi}_p(l,z)-\frac{1}{2}(\hat{\varphi}_p(l+k,z)+\hat{\varphi}_p(l-k,z))|}{K\sum_{(j,j')=(0,\pm 1),(1,-1)}|\hat{G}_{\mu_p(z)}(l+jk)\,\hat{G}_{\mu_p(z)}(l+j'k)|}.\tag{2.10}$$

for some large but finite constant $K>0$ whose precise value is unimportant for the moment and will be determined in Sect. 4.2. These functions will be used in the bootstrapping argument, as stated in Proposition 2.1 below. We emphasize that, although the work in [6,25] did not concern the long-range models, the definition of f_3 is well-adapted to the long-range setting, especially for $\alpha\le 2$; since we are not using the Taylor expansion for the numerator of (2.10), we do not have to assume convergence of the second moment for the spatial variable of the two-point function. We use similar functions in the bootstrapping argument in the ongoing work (M Heydenreich et al., in preparation) to investigate the critical behavior for the long-range Ising model, percolation and self-avoiding walk on $\mathbb{Z}^d$.

We prove below Theorem 1.2 using the following proposition:

Proposition 2.1 (i) *Let $d>2(\alpha\wedge 2)$ and $L\gg 1$ and fix $p<p_c$ and $m<m_p$. Then, $f(p,m)\le 3$ implies that there is a (p,m)-independent constant $C<\infty$ such that*

$$\sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{N}} n^r\pi_p^{(N)}(x,n)m^n\le (C\lambda)^{N\vee 1}\quad (N\ge 0,\ r=0,1),\tag{2.11}$$

$$\sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+}(1-\cos(k\cdot x))\,|\pi_p(x,n)|m^n\le C\,\lambda\,\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}\quad (k\in[-\pi,\pi]^d).\tag{2.12}$$

(ii) *Let $d>2(\alpha\wedge 2)$ and $L\gg 1$ and fix $p<p_c$ and $m<m_p$. Then, (2.11), (2.12) and $f(p,m)\le 3$ imply the stronger bound $f(p,m)\le 2$.*

(iii) *The function $f(p,m)$ is continuous in $m<m_p$ for every $p<p_c$, and $f(p,1)$ is continuous in $p<p_c$, with $f(0,1)=1$.*

We will prove Proposition 2.1 in Sect. 4.

Proof of Theorem 1.2 assuming Proposition 2.1 Note that Proposition 2.1(i), (ii) imply $f(p,m)\notin[2,3)$ for every $p<p_c$ and $m<m_p$. With the help of the continuity in Proposition 2.1(iii), we conclude that indeed $f(p,m)\le 2$ holds for all $p<p_c$ and $m<m_p$. In particular, by (2.6) and the definition of f_2, we have

$$|\hat{\varphi}_p(k,z)|\le\frac{2c}{(1-|\mu_p(z)|)+|\arg(z)|+1-\hat{D}(k)}\qquad (p<p_c,\ |z|<m_p).\tag{2.13}$$

To complete the proof of Theorem 1.2, it suffices to show that

$$1-|\mu_p(z)|\equiv\hat{\varphi}_p(0,|z|)^{-1}\geq\frac{1}{2}p(m_p-|z|)\qquad(0<p<p_{\mathrm{c}}).\tag{2.14}$$

Before proving (2.14), we note that $\hat{\varphi}_p(0,m)$ diverges as $m\uparrow m_p$ for every $p>0$ (cf., (1.13)) and that, by using (2.3),

$$\begin{aligned}1\leq\sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+}\varphi_p(x,n)\,m^n&=\hat{\varphi}_p(0,m)\\&=\frac{\hat{\pi}_p(0,m)}{1-pm\hat{\pi}_p(0,m)}<\infty\quad(m<m_p).\end{aligned}\tag{2.15}$$

By (2.11) for $r=0$, $|\hat{\pi}_p(0,m)-1|$ is uniformly bounded by $O(\lambda)$. Moreover, by monotone convergence and (2.11) for $r=1$,

$$\begin{aligned}m_p|\hat{\pi}_p(0,m_p)-\hat{\pi}_p(0,m)|&\leq\sum_{(x,n)}|\pi_p(x,n)|\,m_p(m_p^n-m^n)\\&\leq(m_p-m)\sum_{(x,n)}n|\pi_p(x,n)|\,m_p^n\\&\leq(m_p-m)\sum_{(x,n)}\sum_{N=0}^{\infty}n\,\pi_p^{(N)}(x,n)\,m_p^n\\&=(m_p-m)\lim_{m\uparrow m_p}\sum_{(x,n)}\sum_{N=0}^{\infty}n\,\pi_p^{(N)}(x,n)\,m^n\\&\leq O(\lambda)(m_p-m),\end{aligned}\tag{2.16}$$

where the $O(\lambda)$ term is independent of m, so that $\hat{\pi}_p(0,m_p)=\lim_{m\uparrow m_p}\hat{\pi}_p(0,m)$. Therefore, for $\hat{\varphi}_p(0,m)$ to diverge as $m\uparrow m_p$, the denominator in (2.15) should be nonnegative and vanish as $m\uparrow m_p$, and hence

$$pm_p\hat{\pi}_p(0,m_p)=1\qquad(0<p<p_{\mathrm{c}}).\tag{2.17}$$

Now we continue with the proof of (2.14). Since $\hat{\pi}_p(0,|z|)=1+O(\lambda)>0$ as explained above, we obtain

$$\hat{\varphi}_p(0,|z|)^{-1}=\left(\frac{\hat{\pi}_p(0,|z|)}{1-p|z|\hat{\pi}_p(0,|z|)}\right)^{-1}=\hat{\pi}_p(0,|z|)^{-1}-p|z|.\tag{2.18}$$

By repeated use of (2.17), $\hat{\varphi}_p(0,|z|)^{-1}$ is rewritten as

$$\begin{aligned}\hat{\varphi}_p(0,|z|)^{-1} &= \hat{\pi}_p(0,|z|)^{-1} - p|z| + pm_p - \hat{\pi}_p(0,m_p)^{-1}\\ &= p(m_p-|z|) + \frac{\hat{\pi}_p(0,m_p)-\hat{\pi}_p(0,|z|)}{\hat{\pi}_p(0,|z|)\,\hat{\pi}_p(0,m_p)}\\ &= p\left((m_p-|z|) + \frac{m_p(\hat{\pi}_p(0,m_p)-\hat{\pi}_p(0,|z|))}{\hat{\pi}_p(0,|z|)}\right).\end{aligned} \tag{2.19}$$

By (2.16), we have arrived at

$$\hat{\varphi}_p(0,|z|)^{-1} \ge (1-O(\lambda))\,p(m_p-|z|). \tag{2.20}$$

This completes the proof of Theorem 1.2 assuming Proposition 2.1. □

2.3 Asymptotic estimate of p_c

We begin with the identity (2.15) for $m=1$:

$$1 \le \chi_p \equiv \hat{\varphi}_p(0,1) = \frac{\hat{\pi}_p(0,1)}{1-p\hat{\pi}_p(0,1)} < \infty \qquad (p<p_c). \tag{2.21}$$

By (2.11) for $m=1$ and $r=0$, $|\hat{\pi}_p(0,1)-1|$ is bounded by $O(\lambda)$ uniformly in $p<p_c$. Since $\chi_p\uparrow\infty$ and $m_p\downarrow 1$ as $p\uparrow p_c$, we have

$$1 = p_c\hat{\pi}_{p_c}(0,1) \equiv p_c \lim_{p\uparrow p_c}\hat{\pi}_p(0,1), \tag{2.22}$$

and therefore $p_c = \hat{\pi}_{p_c}(0,1)^{-1} = 1+O(\lambda)$.

To improve this estimate, we use the following proposition:

Proposition 2.2 *Let $d>2(\alpha\wedge 2)$ and $L\gg 1$. Then, there is a $C<\infty$ such that, for $p\in(1,p_c)$,*

$$|\partial_p\hat{\pi}_p(0,1)| \le C\lambda. \tag{2.23}$$

We will prove Proposition 2.2 in Sect. 5.

Proof of Theorem 1.4 assuming Proposition 2.2 First we rewrite (2.22) as

$$\begin{aligned}1 = p_c\left(\hat{\pi}_{p_c}(0,1)-\hat{\pi}_1(0,1)\right) + (p_c-1)\left(\hat{\pi}_1(0,1)-1\right)\\ +\left(\hat{\pi}_1(0,1)-1\right)+p_c.\end{aligned} \tag{2.24}$$

We already know $(p_c-1)(\hat{\pi}_1(0,1)-1)=O(\lambda^2)$. By the mean-value theorem and Proposition 2.2,

$$|\hat{\pi}_{p_c}(0,1)-\hat{\pi}_1(0,1)| = (p_c-1)|\partial_p\hat{\pi}_p(0,1)| \le O(\lambda^2). \tag{2.25}$$

Moreover, by (2.11) for $(p, m) = (1, 1)$ and $r = 0$, we have $\hat{\pi}_1^{(N)}(0, 1) \le O(\lambda)^N$ for $N \ge 2$. Therefore,

$$p_c = 1 + \hat{\pi}_1^{(1)}(0, 1) - \big(\hat{\pi}_1^{(0)}(0, 1) - 1\big) + O(\lambda^2). \tag{2.26}$$

To complete the proof of Theorem 1.4, it suffices to show that, for $d > 2(\alpha \wedge 2)$,

$$\hat{\pi}_1^{(1)}(0, 1) - \big(\hat{\pi}_1^{(0)}(0, 1) - 1\big) = \frac{1}{2}\sum_{n=2}^{\infty} D^{\star 2n}(o) + O(\lambda^2), \tag{2.27}$$

where the sum is $O(\lambda)$ if $d > \alpha \wedge 2$, because of Proposition 1.1. In fact, (2.27) follows from the same argument as in [15, Sect. 3.1] and using Proposition 1.1. The main point is that, since $p = 1$, we can estimate $\hat{\pi}_1^{(i)}(0, 1)$ with random walks. For example, $\hat{\pi}_p^{(0)}(0, 1) - 1$ is the sum over $(x, n) \in \mathbb{Z}^d \times \mathbb{N}$ of the probability that there are at least *two* bond-disjoint connections from $(o, 0)$ to (x, n) (cf., the definition (3.2) of $\pi_p^{(0)}(x, n)$ below). Since $p = 1$, each of these bond-disjoint connections can be approximated by a random-walk path from o to x in n steps. Therefore, the main contribution to $\hat{\pi}_1^{(0)}(0, 1) - 1$ is

$$\frac{1}{2}\sum_{n=2}^{\infty}\sum_{x\in\mathbb{Z}^d}\big(D^{\star n}(x)\big)^2 = \frac{1}{2}\sum_{n=2}^{\infty} D^{\star 2n}(o), \tag{2.28}$$

where the combinatorial factor $\frac{1}{2}$ is due to the symmetry between the two bond-disjoint connections (cf., [15, (3.11)]), which is absent in the main contribution to $\hat{\pi}_1^{(1)}(0, 1)$ (cf., [15, (3.22)]), leading to the factor $\frac{1}{2}$ in the difference (2.27). The corrections to $\hat{\pi}_1^{(0)}(0, 1) - 1$ and $\hat{\pi}_1^{(1)}(0, 1)$ can be estimated as $O(\lambda^2)$ by applying Proposition 1.1 to the error terms in [15, Sect. 3.1]. For example, [15, (3.29)] is replaced by

$$\begin{aligned}\sum_{\substack{t,s,s'\in\mathbb{Z}_+:\\ 0\le s<s'\le t}} \frac{O(\lambda)}{(1\vee t)^{d/(\alpha\wedge 2)}}\,\frac{O(\lambda)}{(1\vee(s'-s))^{d/(\alpha\wedge 2)}} \\ \le \sum_{t=0}^{\infty}\frac{O(\lambda^2)}{(1\vee t)^{d/(\alpha\wedge 2)-1}} \le O(\lambda^2),\end{aligned} \tag{2.29}$$

where we have used $d > 2(\alpha\wedge 2)$. This completes the proof of Theorem 1.4 assuming Proposition 2.2. ◻

2.4 Limit distribution

Assuming the lace expansion (2.3) and the bounds in Proposition 2.1 on the expansion coefficients, we have that, for $p \in (0, p_c)$, $k \in [-\pi, \pi]^d$ and $m < m_p$,

$$\hat{\varphi}_p(k, m)^{-1} = \hat{\pi}_p(k, m)^{-1} - pm\hat{D}(k), \tag{2.30}$$

where $\hat{\pi}_p(k,m) = 1 + O(\lambda)$. In the course of the proof of Theorems 1.2 and 1.4 in Sects. 2.2 and 2.3, we obtained $pm_p \equiv \hat{\pi}_p(0,m_p)^{-1} = 1 + O(\lambda)$ for $p \in (0, p_c]$ and $m_{p_c} = 1$, as stated in Corollary 1.3. For $m < 1$, $\hat{\pi}_{p_c}(k,m) \equiv \lim_{p\uparrow p_c} \hat{\pi}_p(k,m)$ is well-defined, due to (2.30) and the continuity of $\hat{\varphi}_p(k,m)$ in $p < p_c$ for every $m < 1$, as well as the uniform bound on $\hat{\pi}_p(k,m)$.

Using these facts and Tauberian estimates, we first derive an asymptotic formula of $Z_p(k;n)$ for every $p \in (0, p_c]$. Then, by using this formula, we will prove Theorem 1.5.

Since $\hat{\pi}_p(k,m) = 1 + O(\lambda)$ and $\hat{\pi}_p(0,m_p)^{-1} = pm_p$, we can reorganize (2.30) for $m < m_p$ as

$$\begin{aligned}
\hat{\varphi}_p(k,m)^{-1} &= \underbrace{\hat{\pi}_p(k,m)^{-1} - pm\hat{D}(k) - \Big(\hat{\pi}_p(k,m_p)^{-1} - pm_p\hat{D}(k)\Big)}_{p(m_p-m)\hat{A}_p(k,m)} \\
&\quad + \underbrace{\hat{\pi}_p(k,m_p)^{-1} - pm_p\hat{D}(k) - \Big(\hat{\pi}_p(0,m_p)^{-1} - pm_p\Big)}_{pm_p\hat{B}_p(k)} \\
&= pm_p\left(\left(1 - \tfrac{m}{m_p}\right)\hat{A}_p(k,m) + \hat{B}_p(k)\right), \qquad (2.31)
\end{aligned}$$

where

$$\hat{A}_p(k,m) = \hat{D}(k) - \frac{\hat{\pi}_p(k,m_p)^{-1} - \hat{\pi}_p(k,m)^{-1}}{p(m_p - m)}, \qquad (2.32)$$

$$\hat{B}_p(k) = 1 - \hat{D}(k) + \frac{\hat{\pi}_p(k,m_p)^{-1} - \hat{\pi}_p(0,m_p)^{-1}}{pm_p}. \qquad (2.33)$$

Similarly to (2.16), we can show that the second term in $\hat{A}_p(k,m)$ is $O(\lambda)$ and the last term in $\hat{B}_p(k)$ is $O(\lambda)\hat{G}_{\mu_p(m_p\vee 1)}(k) \equiv O(\lambda)(1-\hat{D}(k))$ for $p \le p_c$, $k \in [-\pi,\pi]^d$ and $m < m_p$. Then, we decompose $\hat{A}_p(k,m)$ as $\hat{A}_p(k,m) = \hat{A}_p^{(1)}(k) + \hat{A}_p^{(2)}(k,m)$, where

$$\hat{A}_p^{(1)}(k) = \hat{D}(k) - \frac{m_p\,\partial_m\hat{\pi}_p(k,m_p)^{-1}}{pm_p}, \qquad (2.34)$$

$$\hat{A}_p^{(2)}(k,m) = \frac{m_p\,\partial_m\hat{\pi}_p(k,m_p)^{-1}}{pm_p} - \frac{\hat{\pi}_p(k,m_p)^{-1} - \hat{\pi}_p(k,m)^{-1}}{p(m_p - m)}, \qquad (2.35)$$

where $\partial_m\hat{\pi}_p(k,m_p)^{-1}$ is an abbreviation for $\partial_m\hat{\pi}_p(k,m)^{-1}|_{m=m_p}$. Again, similarly to (2.16), we can show that the common term in (2.34) and (2.35) is $O(\lambda)$ for any $p \le p_c$ and $k \in [-\pi,\pi]^d$. In particular, $\hat{A}_p^{(1)}(k)$ is continuous at $k = 0$, and $\hat{A}_p^{(1)}(k) + \hat{B}_p(k) =$

$1 + O(\lambda)$. Using these quantities, we can rewrite (2.31) as

$$\begin{aligned} pm_p\hat{\varphi}_p(k,m) &= \frac{1}{\left(1-\frac{m}{m_p}\right)\hat{A}_p(k,m)+\hat{B}_p(k)} \\ &= \frac{1}{\left(1-\frac{m}{m_p}\right)\hat{A}_p^{(1)}(k)+\hat{B}_p(k)} + \hat{\Phi}_p(k,m), \end{aligned} \tag{2.36}$$

where

$$\hat{\Phi}_p(k,m) = \frac{-\left(1-\frac{m}{m_p}\right)\hat{A}_p^{(2)}(k,m)}{\left(\left(1-\frac{m}{m_p}\right)\hat{A}_p(k,m)+\hat{B}_p(k)\right)\left(\left(1-\frac{m}{m_p}\right)\hat{A}_p^{(1)}(k)+\hat{B}_p(k)\right)}. \tag{2.37}$$

The first term of the rightmost expression in (2.36) can be expanded in powers of $\frac{m}{m_p}$ as

$$\begin{aligned} &\frac{1}{\hat{A}_p^{(1)}(k)+\hat{B}_p(k)-\frac{m}{m_p}\hat{A}_p^{(1)}(k)} \\ &\quad= \frac{1}{\hat{A}_p^{(1)}(k)+\hat{B}_p(k)}\sum_{n=0}^{\infty}\left(\frac{m}{m_p}\right)^n\left(\frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k)+\hat{B}_p(k)}\right)^n. \end{aligned} \tag{2.38}$$

In Sect. 6, we will prove the following bound on $\hat{\Phi}_p(k,m)$:

Proposition 2.3 *Let $d > 2(\alpha \wedge 2)$ and $L \gg 1$, and fix an $\epsilon \in \left(0, 1 \wedge \frac{d-2(\alpha\wedge 2)}{\alpha\wedge 2}\right)$. Then, there is an ϵ-dependent constant $C_\epsilon < \infty$ such that*

$$|\partial_\zeta \hat{\Phi}_p(k, m_p\zeta)| \le C_\epsilon |1-\zeta|^{-2+\epsilon} \tag{2.39}$$

holds for $p \in (0, p_c]$, $k \in [-\pi, \pi]^d$ and $\zeta \in \mathbb{C}$ with $|\zeta| < 1$.

By this result and [18, Lemma 6.3.3(ii)], the coefficient of $\zeta^n \equiv \left(\frac{m}{m_p}\right)^n$ in $\hat{\Phi}_p(k,m)$ is bounded by $O(n^{-\epsilon'})$ for any $\epsilon' < \epsilon$. Together with (2.36) and (2.38) and using $pm_p = 1 + O(\lambda)$, we finally obtain

$$\begin{aligned} Z_p(k;n) = &\frac{m_p^{-n}}{pm_p(\hat{A}_p^{(1)}(k)+\hat{B}_p(k))}\left(\frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k)+\hat{B}_p(k)}\right)^n \\ &+O(m_p^{-n}n^{-\epsilon'}) \quad (n \ge 1). \end{aligned} \tag{2.40}$$

Proof of Theorem 1.5 using (2.40) When $k = 0$, since $\hat{B}_p(0) \equiv 0$, we immediately obtain from (2.40) that

$$Z_p(0;n) = C_1 m_p^{-n} + O(m_p^{-n}n^{-\epsilon'}) \qquad (n \ge 1), \tag{2.41}$$

where $C_1 \equiv (pm_p\hat{A}_p^{(1)}(0))^{-1} = 1+O(\lambda)$. This completes the proof of Theorem 1.5(i).

To prove Theorem 1.5(ii) using (2.40), it suffices to investigate

$$\left(\frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k)+\hat{B}_p(k)}\right)^n = \left(\left(1+\frac{\hat{B}_p(k)}{\hat{A}_p^{(1)}(k)}\right)^{\frac{\hat{A}_p^{(1)}(k)}{\hat{B}_p(k)}}\right)^{-\frac{n\left(1-\hat{D}(k)\right)}{\hat{A}_p^{(1)}(k)}\frac{\hat{B}_p(k)}{1-\hat{D}(k)}} \tag{2.42}$$

for small k, for which $\hat{A}_p^{(1)}(k)$ is bounded away from 0 and $\hat{B}_p(k)$ is close to 0. For k_n defined in (1.23),

$$\left(1+\frac{\hat{B}_p(k_n)}{\hat{A}_p^{(1)}(k_n)}\right)^{\frac{\hat{A}_p^{(1)}(k_n)}{\hat{B}_p(k_n)}} \underset{n\uparrow\infty}{\to} e, \qquad \frac{n(1-\hat{D}(k_n))}{\hat{A}_p^{(1)}(k_n)} \underset{n\uparrow\infty}{\to} \frac{|k|^{\alpha\wedge 2}}{\hat{A}_p^{(1)}(0)}, \tag{2.43}$$

where we have used the continuity: $\hat{A}_p^{(1)}(k_n) \to \hat{A}_p^{(1)}(0) = 1+O(\lambda)$. By (2.12) and (2.33), $\hat{B}_p(k)/(1-\hat{D}(k)) = 1+O(\lambda)$ uniformly in k. This completes the proof of Theorem 1.5(ii) using (2.40). □

3 Review of the lace expansion

3.1 Derivation of the expansion

In this section, we briefly explain the lace expansion (2.3) for oriented percolation. In the literature, there are currently three different ways to obtain (2.3) and different representations for $\pi_p(x,n)$. One is based on an algebraic approach using the Markov property [19], another one is to use inclusion–exclusion and nested expectations [17], and the other is to use inclusion–exclusion and the Markov property [22]. Here, we provide a quick overview of the third approach, which is thought to be conceptually simplest. The readers who are familiar to the lace expansion for oriented percolation may skip this section and immediately go to Sect. 4.

Recall that $\varphi_p(x,n)$ is the probability that $(o,0)$ is connected to (x,n). In order for this event to occur, there are two disjoint events depending on whether there is or is not a pivotal bond for $\{(o,0)\to(x,n)\}$. If a bond b is pivotal for $\{(o,0)\to(x,n)\}$, then (x,n) is not contained in the set of sites connected from $(o,0)$ without using b. For $(v,l)\in\mathbb{Z}^d\times\mathbb{Z}_+$, let

$$\tilde{\mathcal{C}}^b(v,l) = \{(y,n)\in\mathbb{Z}^d\times\mathbb{Z}_+ : (v,l)\to(y,n) \text{ without using } b\}. \tag{3.1}$$

If there is no pivotal bond for $\{(o,0)\to(x,n)\}$, then $(o,0)=(x,n)$ or there are at least two bond-disjoint nonzero occupied paths from $(o,0)$ to (x,n). We denote this event by $\{(o,0)\rightrightarrows(x,n)\}$ and define

$$\pi_p^{(0)}(x,n) = \mathbb{P}_p((o,0)\rightrightarrows(x,n)). \tag{3.2}$$

Then, by taking the first pivotal bond b (if it exists) for $\{(o,0) \to (x,n)\}$, we obtain

$$\varphi_p(x,n) = \pi_p^{(0)}(x,n) + \sum_b \mathbb{P}_p\Big((o,0) \rightrightarrows b \to (x,n) \notin \tilde{\mathcal{C}}^b(o,0)\Big), \tag{3.3}$$

where, by denoting $b = (\underline{b}, \overline{b})$, we have used the abbreviation

$$\begin{aligned}\{(o,0) \rightrightarrows b \to (x,n)\} &= \{(o,0) \rightrightarrows \underline{b}\} \cap \{b \to (x,n)\}\\ &= \{(o,0) \rightrightarrows \underline{b}\} \cap \{b \text{ is occupied}\} \cap \{\overline{b} \to (x,n)\}.\end{aligned} \tag{3.4}$$

By inclusion–exclusion in terms of the condition $(x,n) \notin \tilde{\mathcal{C}}^b(o,0)$, the second term in (3.3) is

$$\begin{aligned}&\sum_b \mathbb{P}_p((o,0) \rightrightarrows b \to (x,n)) - \sum_b \mathbb{P}_p\Big((o,0) \rightrightarrows b \to (x,n) \in \tilde{\mathcal{C}}^b(o,0)\Big)\\ &\quad = (\pi_p^{(0)} * q_p * \varphi_p)(x,n) - R_p^{(1)}(x,n)\end{aligned} \tag{3.5}$$

where we have applied the Markov property for the first term, and

$$R_p^{(1)}(x,n) = \sum_b \mathbb{P}_p\Big((o,0) \rightrightarrows b \to (x,n) \in \tilde{\mathcal{C}}^b(o,0)\Big). \tag{3.6}$$

Therefore,

$$\varphi_p(x,n) = \pi_p^{(0)}(x,n) + (\pi_p^{(0)} * q_p * \varphi_p)(x,n) - R_p^{(1)}(x,n). \tag{3.7}$$

This completes the first step of the full expansion (2.3).

To proceed the expansion further, it suffices to consider $R_p^{(1)}(x,n)$. Given a set $\mathcal{C}$ of vertices, we define

$$E(b,(x,n);\mathcal{C}) = \{b \to (x,n) \in \mathcal{C}\} \cap \Big\{\nexists b' \text{pivotal for } \{\overline{b} \to (x,n)\} \text{ satisfying } \underline{b}' \in \mathcal{C}\Big\}, \tag{3.8}$$

and, for $N \ge 1$ and $\overrightarrow{b}_N = (b_1, \ldots, b_N)$,

$$\tilde{E}^{(N)}_{\overrightarrow{b}_N}(x,n) = \{(o,0) \rightrightarrows \underline{b}_1\} \cap \bigcap_{i=1}^N E\Big(b_i, \underline{b}_{i+1}; \tilde{\mathcal{C}}^{b_i}(\overline{b}_{i-1})\Big), \tag{3.9}$$

with the convention $\overline{b}_0 = (o, 0)$ and $\underline{b}_{N+1} = (x, n)$. For $N \geq 0$, we define

$$\pi_p^{(N)}(x,n) = \begin{cases} \mathbb{P}_p((o,0) \rightrightarrows (x,n)) & (N=0), \\ \sum_{\vec{b}_N} \mathbb{P}_p\left(\tilde{E}^{(N)}_{\vec{b}_N}(x,n)\right) & (N \geq 1), \end{cases} \tag{3.10}$$

$$R_p^{(N+1)}(x,n) = \sum_{\vec{b}_{N+1}} \mathbb{P}_p\left(\tilde{E}^{(N)}_{\vec{b}_N}(\underline{b}_{N+1}) \cap \left\{b_{N+1} \to (x,n) \in \tilde{\mathcal{C}}^{b_{N+1}}(\overline{b}_N)\right\}\right), \tag{3.11}$$

which are consistent with (3.2) and (3.6). It has been proved [14,22] that

$$R_p^{(N)}(x,n) = \pi_p^{(N)}(x,n) + (\pi_p^{(N)} * q_p * \varphi_p)(x,n) - R_p^{(N+1)}(x,n). \tag{3.12}$$

We note that $R_p^{(N)}(x,n)$ involves the sum over $b_1, \ldots, b_N$ with $\overline{b}_{j-1} < \overline{b}_j$ for $j = 2, \ldots, N$, hence $R_p^{(N)}(x,n) = 0$ if $N > n$. Repeatedly using (3.12), we arrive at the following conclusion:

Proposition 3.1 ([14,22])

$$\varphi_p(x,n) = \pi_p(x,n) + (\pi_p * q_p * \varphi_p)(x,n), \tag{3.13}$$

where

$$\pi_p(x,n) = \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(x,n). \tag{3.14}$$

Extending the above idea, we obtain the following representation[1] of $\partial_p \pi_p(x,n)$ for $p \in (0, p_{\mathrm{c}})$, which will be used in Sect. 5 to prove Proposition 2.2.

Proposition 3.2 ([14]) *For $p \in (0, p_{\mathrm{c}})$,*

$$\partial_p \pi_p(x,n) = \frac{1}{p} \sum_{N=1}^{\infty} (-1)^N \Pi_p^{(N)}(x,n), \tag{3.15}$$

[1] Proposition 3.2 is a result of applying Russo's formula [21] to $\varphi_p(x,n)$ and compare the result with the derivative of (3.13). Since Russo's formula can be used only for finite systems, we should first approximate $\varphi_p(x,n)$ by a finite-volume version $\varphi_{p,R}(x,n) \equiv \mathbb{P}_p((o,0) \to (x,n)$ in $\Lambda_R)$, where $\Lambda_R = (\mathbb{Z} \cap [-R,R])^d \times \mathbb{Z}_+$, and then apply Russo's formula. This strategy is explained in [14, Sect. 3.2], where a sort of finite-confinement argument of random-walk paths is used. Since the tail of the underlying random walk in the current setting does not decay fast, we restrict p to $p < p_{\mathrm{c}}$ and use the fact that $\chi_p < \infty$ and $\tilde{\chi}_{p,R} \equiv \sum_{(x,n) \notin \Lambda_R} \varphi_p(x,n) \to 0$ as $R \to \infty$. Then, the corresponding quantities to the first and second lines of [14, (3.58)] are bounded respectively by $\tilde{\chi}_{p,R}$ and $\chi_p^3 \tilde{\chi}_{p,R}$, both of which tend to zero as $R \to \infty$, hence we obtain (3.15), (3.16).

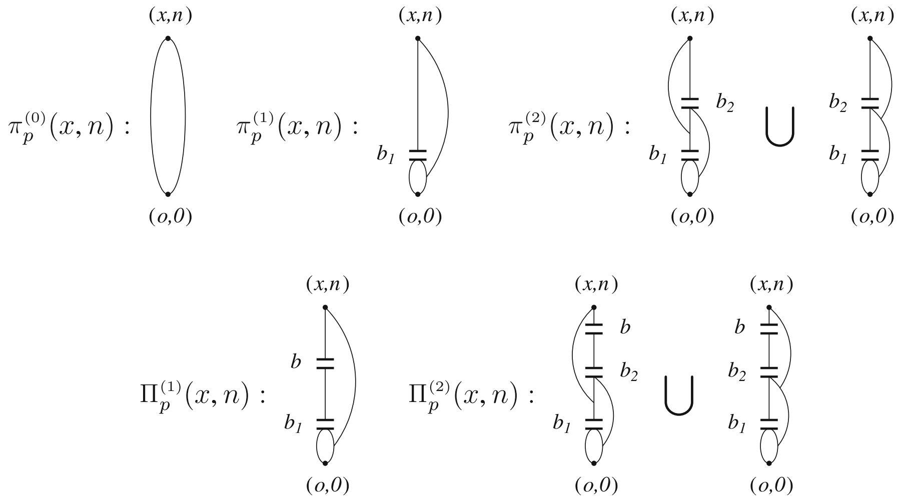

Fig. 1 Schematic representations of $\pi_p^{(N)}(x,n)$ for $N = 0, 1, 2$ and $\Pi_p^{(N)}(x,n)$ for $N = 1, 2$. The b's are bonds that are summed over

where

$$\Pi_p^{(N)}(x,n) = \sum_{\vec{b}_N, b} \sum_{j=1}^{N} \mathbb{P}_p \left(\tilde{E}^{(N)}_{\vec{b}_N}(x,n) \cap \Big\{ b = b_j \text{ or } b \text{ is pivotal for } \{\overline{b}_j \to \underline{b}_{j+1}\} \Big\} \right), \tag{3.16}$$

with the convention $\underline{b}_{N+1} = (x,n)$.

3.2 Diagrammatic bounds on the expansion coefficients

In this section, we provide diagrammatic bounds on $\pi_p^{(N)}(x,n)$ and $\Pi_p^{(N)}(x,n)$. These bounds consist of two-point functions, and are results of applications of the BK inequality [4] and

$$\varphi_p(x,n) \le (q_p * \varphi_p)(x,n) \qquad (n \ge 1). \tag{3.17}$$

For example, $\pi_p^{(0)}(x,n)$ is bounded as

$$\begin{aligned} \pi_p^{(0)}(x,n) \le \varphi_p(x,n)^2 &= \delta_{x,o}\delta_{n,0} + \big((1 - \delta_{x,o}\delta_{n,0})\,\varphi_p(x,n)\big)^2 \\ &\le \delta_{x,o}\delta_{n,0} + (q_p * \varphi_p)(x,n)^2. \end{aligned} \tag{3.18}$$

The other terms are bounded similarly.

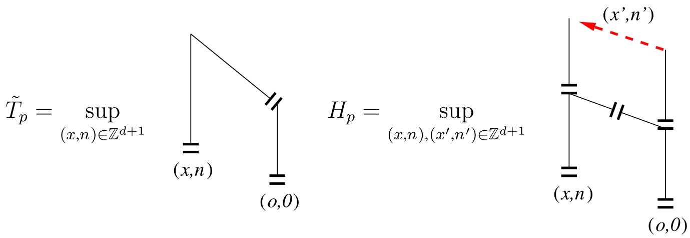

Fig. 2 Schematic representations of $\tilde{T}_p$ and H_p

Let $\varphi_p^{(m)}(x,n) = \varphi_p(x,n)m^n$ and define the weighted bubble $W_p^{(m)}(k)$, the triangles $T_p^{(m)}$ and $\tilde{T}_p$, the square $S_p^{(m)}$ and the H-shaped diagrams H_p as (see Fig. 2)

$$W_p^{(m)}(k) = \sup_{(x,n)} \sum_{(y,t)} (1-\cos(k\cdot y)) \times \begin{cases} (q_p * \varphi_p)(y,t)\cdot(mq_p * \varphi_p^{(m)})(y-x,t-n), & \text{if } m<1, \\ (mq_p * \varphi_p^{(m)})(y,t)\cdot(q_p * \varphi_p)(y-x,t-n), & \text{if } m\geq 1, \end{cases} \tag{3.19}$$

$$T_p^{(m)} = \sup_{(x,n)} \sum_{(y,t)} (q_p * \varphi_p * \varphi_p)(y,t)\cdot(mq_p * \varphi_p^{(m)})(y-x,t-n), \tag{3.20}$$

$$S_p^{(m)} = \sup_{(x,n)} \sum_{(y,t)} (q_p * \varphi_p * \varphi_p * \varphi_p)(y,t)\cdot(mq_p * \varphi_p^{(m)})(y-x,t-n), \tag{3.21}$$

$$\tilde{T}_p = \sup_{(x,n)} \sum_{(y,t)} (q_p * \varphi_p * q_p * \varphi_p)(y,t)\cdot(q_p * \varphi_p)(y-x,t-n), \tag{3.22}$$

$$\begin{aligned} H_p = \sup_{(x,n),(x',n')} \sum_{(y_i,t_i),\, i=1,2,3} & (q_p * \varphi_p)(y_1,t_1)\cdot(\varphi_p * q_p * \varphi_p)(y_2-y_1,t_2-t_1) \\ & \times(q_p * \varphi_p)(y_2-x,t_2-n)\cdot(q_p * \varphi_p)(y_3-y_1,t_3-t_1) \\ & \times(q_p * \varphi_p)(x'+y_3-y_2,n'+t_3-t_2). \end{aligned} \tag{3.23}$$

The expansion coefficients obey the following bounds:

Proposition 3.3 (i) *For $N \geq 0$ and $r = 0, 1, 2$,*

$$\sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{N}} n^r \pi_p^{(N)}(x,n) m^n \leq (N+1)^r (1+2T_p^{(m)})(2T_p^{(m)})^{(N-1)\vee 0} \times \begin{cases} T_p^{(m)} & (r=0,1), \\ S_p^{(m)} & (r=2), \end{cases} \tag{3.24}$$

$$\sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+} (1-\cos(k\cdot x))\, \pi_p^{(N)}(x,n) m^n \leq 3(N+1)^2(1+2T_p^{(m)}) \times (2T_p^{(m)})^{(N-1)\vee 0} W_p^{(m)}(k). \tag{3.25}$$

(ii) *For $N \geq 1$,*

$$\sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+} \Pi_p^{(N)}(x,n) \leq N(1+2T_p^{(1)}) \Big((T_p^{(1)} + \tilde{T}_p)(2T_p^{(1)})^{N-1} + H_p (2T_p^{(1)})^{(N-2)\vee 0} \Big). \tag{3.26}$$

The proof of the above proposition is irrelevant in this paper, and is found in [24].

4 Proof of Proposition 2.1

In this section, we prove Proposition 2.1 that was the key for the proof of Theorem 1.2. First, in Sect. 4.1, we prove Proposition 2.1(iii) that is nothing to do with the lace expansion. Then, in Sect. 4.2, we prove Proposition 2.1(ii) using the trigonometric technique in [25, Sect. 5.1]. Finally, in Sect. 4.3, we prove Proposition 2.1(i) using the diagrammatic bounds on the expansion coefficients in Sect. 3.2.

4.1 Proof of Proposition 2.1(iii)

First we prove $f(0,1) = 1$. When $p = 0$, by definition we have $f_1(0,1) = 0$, $\hat{\varphi}_0(k,z) \equiv 1$, $\mu_0(z) \equiv 0$ (cf., (2.7)) and hence $\hat{G}_{\mu_0(z)}(k) \equiv 1$. Therefore, $f_2(0,1) = 1$ and $f_3(0,1) = 0$.

Next we discuss the continuity of $f(p,m)$. Since $f_1(p,m) \equiv p(m \vee 1)$ is obviously continuous in p and m, we only need to investigate $f_2(p,m)$ and $f_3(p,m)$.

Fix $p < p_c$. To prove the continuity of $f(p,m)$ in $m < m_p$, it suffices to show that $f(p,m)$ is continuous in $m \in [0, \tilde{m}]$ for every $\tilde{m} < m_p$. To prove this for $f_2(p,m)$, it suffices to show that the derivative

$$\partial_m \frac{\hat{\varphi}_p(k, me^{i\theta})}{\hat{G}_{\mu_p(me^{i\theta})}(k)} = \frac{\partial_m \hat{\varphi}_p(k, me^{i\theta})}{\hat{G}_{\mu_p(me^{i\theta})}(k)} - \hat{\varphi}_p(k, me^{i\theta}) \frac{\partial_m \hat{G}_{\mu_p(me^{i\theta})}(k)}{\hat{G}_{\mu_p(me^{i\theta})}(k)^2} \tag{4.1}$$

is bounded uniformly in $(k,\theta) \in [-\pi,\pi]^{d+1}$ and $m \in [0,\tilde{m}]$ (cf., [25, Lemma 5.13]). However, by $n\varphi_p(x,n) \le (q_p * \varphi_p * \varphi_p)(x,n)$ (cf., [22, (5.17)]), we have

$$|\partial_m \hat{\varphi}_p(k, me^{i\theta})| \le \sum_{(x,n)} n\varphi_p(x,n) m^{n-1} \le p\hat{\varphi}_p(0,m)^2 \le p\hat{\varphi}_p(0,\tilde{m})^2. \tag{4.2}$$

Since $|\hat{G}_{\mu_p(me^{i\theta})}(k)| \ge \frac{1}{2}$, the first term on the right-hand side of (4.1) is indeed uniformly bounded. Also, since $\hat{\varphi}_p(0,m)$ (≥ 1) is nondecreasing in m, we obtain

$$\left|\frac{\partial_m \hat{G}_{\mu_p(me^{i\theta})}(k)}{\hat{G}_{\mu_p(me^{i\theta})}(k)^2}\right| = |\hat{D}(k)\,\partial_m \mu_p(me^{i\theta})| \le \frac{\partial_m \hat{\varphi}_p(0,m)}{\hat{\varphi}_p(0,m)^2}, \tag{4.3}$$

which is uniformly bounded by p, as described in (4.2). Consequently, (4.1) is uniformly bounded by $p\hat{\varphi}_p(0,\tilde{m})(2\hat{\varphi}_p(0,\tilde{m})+1)$. This completes the proof of the continuity of $f_2(p,m)$ in $m \in [0,\tilde{m}]$.

Similarly to the above, we can easily show that the derivative

$$\partial_m \frac{\hat{G}_{\mu_p(m)}(k)\ \left(\hat{\varphi}_p\left(l, me^{i\theta}\right) - \frac{1}{2}\left(\hat{\varphi}_p\left(l+k, me^{i\theta}\right) + \hat{\varphi}_p\left(l-k, me^{i\theta}\right)\right)\right)}{\hat{G}_{\mu_p(me^{i\theta})}(l+jk)\,\hat{G}_{\mu_p(me^{i\theta})}(l+j'k)} \tag{4.4}$$

is bounded uniformly in $(k,\theta) \in [-\pi,\pi]^{d+1}$, $(j,j') = (0,\pm 1), (1,-1)$ and $m \in [0,\tilde{m}]$. This justifies the continuity of $f_3(p,m)$ in $m \in [0,\tilde{m}]$.

To prove the continuity of $f(p,1)$ in $p < p_c$, it suffices to show that $f(p,1)$ is continuous in $p \in [0,\tilde{p}]$ for every $\tilde{p} < p_c$. First we note that, by Russo's formula [21] (see also Footnote 1) and the fact that $\chi_p \equiv \hat{\varphi}_p(0,1)$ (≥ 1) is nondecreasing in p, we have, for $|z| = 1$,

$$|\partial_p \hat{\varphi}_p(k,z)| \le \sum_{(x,n)} \partial_p \varphi_p(x,n) \le \sum_{(x,n)} (\varphi_p * q_1 * \varphi_p)(x,n) \le \chi_p^2, \tag{4.5}$$

$$\left|\frac{\partial_p \hat{G}_{\mu_p(z)}(k)}{\hat{G}_{\mu_p(z)}(k)^2}\right| = |\hat{D}(k)\,\partial_p \mu_p(z)| \le \frac{\partial_p \chi_p}{\chi_p^2} \le 1. \tag{4.6}$$

Since $|\hat{G}_{\mu_p(z)}(k)| \ge \frac{1}{2}$, we obtain

$$\left|\partial_p \frac{\hat{\varphi}_p(k,z)}{\hat{G}_{\mu_p(z)}(k)}\right| \le \left|\frac{\partial_p \hat{\varphi}_p(k,z)}{\hat{G}_{\mu_p(z)}(k)}\right| + |\hat{\varphi}_p(k,z)| \left|\frac{\partial_p \hat{G}_{\mu_p(z)}(k)}{\hat{G}_{\mu_p(z)}(k)^2}\right| \le \chi_{\tilde{p}}(2\chi_{\tilde{p}}+1), \tag{4.7}$$

uniformly in $k \in [-\pi,\pi]^d$, $|z| = 1$ and $p \in [0,\tilde{p}]$. This implies the continuity of $f_2(p,1)$ in $p \in [0,\tilde{p}]$ for every $\tilde{p} < p_c$.

The continuity of $f_3(p,1)$ can be proved in a similar way. This completes the proof of Proposition 2.1(iii). □

4.2 Proof of Proposition 2.1(ii)

In this section, we prove that, for every $p < p_c$ and $m < m_p$, the weaker bound $f(p,m) \le 3$ and (2.11), (2.12) imply the stronger bound $f(p,m) \le 2$ if $L \gg 1$.

First, by (2.17) (recall that this is a consequence of the assumed bound (2.11) and the fact that $\hat{\varphi}_p(0,m)$ diverges as $m \uparrow m_p$) and (2.11), we immediately obtain

$$f_1(p,m) \equiv p(m \vee 1) \le pm_p = \hat{\pi}_p(0,m_p)^{-1} = 1 + O(\lambda) \le 2. \tag{4.8}$$

Next we consider $f_2(p,m)$. First we rewrite $\hat{\varphi}_p(k,z)/\hat{G}_{\mu_p(z)}(k)$ as

$$\begin{aligned}\frac{\hat{\varphi}_p(k,z)}{\hat{G}_{\mu_p(z)}(k)} &= \hat{\pi}_p(k,z) + \hat{\varphi}_p(k,z)\left(\frac{1}{\hat{G}_{\mu_p(z)}(k)} - \frac{\hat{\pi}_p(k,z)}{\hat{\varphi}_p(k,z)}\right)\\ &= \hat{\pi}_p(k,z) + \hat{\varphi}_p(k,z)\left(pz\hat{\pi}_p(k,z) - \mu_p(z)\right)\hat{D}(k)\\ &= \hat{\pi}_p(k,z) + \hat{\varphi}_p(k,z)\left(p|z|\hat{\pi}_p(k,z) - 1 + \frac{1}{\hat{\varphi}_p(0,|z|)}\right)\\ &\quad \times e^{i\arg(z)}\hat{D}(k),\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}&p|z|\hat{\pi}_p(k,z) - 1 + \frac{1}{\hat{\varphi}_p(0,|z|)}\\ &\quad = p|z|\left(\hat{\pi}_p(k,z) - \hat{\pi}_p(0,|z|)\right) - \big(\underbrace{1 - p|z|\hat{\pi}_p(0,|z|)}_{\hat{\pi}_p(0,|z|)/\hat{\varphi}_p(0,|z|)}\big) + \frac{1}{\hat{\varphi}_p(0,|z|)}\\ &\quad = p|z|\left(\pi_p(k,z) - \pi_p(0,|z|)\right) + \frac{1 - \hat{\pi}_p(0,|z|)}{\hat{\varphi}_p(0,|z|)}.\end{aligned} \tag{4.10}$$

We note that $|\hat{\pi}_p(k,z) - 1| = O(\lambda)$, due to (2.11) for $r = 0$, and that $|\hat{\varphi}_p(k,z)/\hat{\varphi}_p(0,|z|)| \le 1$ by definition. To complete the proof of $f_2(p,m) = 1 + O(\lambda) \le 2$, it thus suffices to show that

$$|\hat{\varphi}_p(k,z)|\left(|\pi_p(k,z) - \pi_p(0,z)| + |\pi_p(0,z) - \pi_p(0,|z|)|\right) = O(\lambda), \tag{4.11}$$

uniformly in $k \in [-\pi,\pi]^d$ and $z \in \mathbb{C}$ with $|z| = m$ or 1. However, by (2.11), (2.12) and denoting $\theta = \arg(z)$, we have

$$|\pi_p(k,z)-\pi_p(0,z)| \le O(\lambda)\,\hat{G}_{\mu_p(m\vee 1)}(k)^{-1} \le O(\lambda)\,\Big(1-\mu_p(m\vee 1)+1-\hat{D}(k)\Big), \tag{4.12}$$

$$|\pi_p(0,z)-\pi_p(0,|z|)| = \Bigg|\sum_{(x,n)}\pi_p(x,n)|z|^n(e^{i\theta n}-1)\Bigg| \le |\theta|\sum_{(x,n)} n|\pi_p(x,n)||z|^n = O(\lambda)|\theta|. \tag{4.13}$$

On the other hand, by $f_2(p,m) \le 3$, (2.6) and $|\mu_p(z)| \le \mu_p(m\vee 1)$ for $|z| = m$ or 1 (cf., (2.7)),

$$|\hat{\varphi}_p(k,z)| \le \frac{3c}{1-\mu_p(m\vee 1)+|\theta|+1-\hat{D}(k)}. \tag{4.14}$$

This completes the proof of (4.11), and hence $f_2(p,m) \le 2$.

For $f_3(p,m)$, we introduce the following notation for $\hat{f}(l) \equiv \sum_{x\in\mathbb{Z}^d} f(x)e^{il\cdot x}$:

$$\Delta_k \hat{f}(l) = \hat{f}(l+k) + \hat{f}(l-k) - 2\hat{f}(l). \tag{4.15}$$

We note that $-\frac{1}{2}\Delta_k \hat{f}(l)$ is the Fourier transform of $(1-\cos(k\cdot x))f(x)$:

$$\begin{aligned} -\frac{1}{2}\Delta_k \hat{f}(l) &= \sum_{x\in\mathbb{Z}^d} f(x)\left(e^{il\cdot x} - \frac{e^{i(l+k)\cdot x}+e^{i(l-k)\cdot x}}{2}\right) \\ &= \sum_{x\in\mathbb{Z}^d} f(x)\,(1-\cos(k\cdot x))\,e^{il\cdot x}. \end{aligned} \tag{4.16}$$

Recall the definition of $f_3(p,m)$ whose numerator contains $-\frac{1}{2}\Delta_k\hat{\varphi}_p(l,z)$. Let

$$\hat{a}_p(l,z) = pz\hat{D}(l)\,\hat{\pi}_p(l,z) \equiv \sum_{(x,n)} (q_p * \pi_p)(x,n)z^n\cos(l\cdot x), \tag{4.17}$$

so that $\hat{\varphi}_p(l,z) = \hat{\pi}_p(l,z)/(1-\hat{a}_p(l,z))$. Then, we have

$$\begin{aligned} \Delta_k\hat{\varphi}_p(l,z) = \frac{\Delta_k\hat{\pi}_p(l,z)}{1-\hat{a}_p(l,z)} &+ \sum_{j=\pm 1}\frac{(\hat{\pi}_p(l+jk,z)-\hat{\pi}_p(l,z))(\hat{a}_p(l+jk,z)-\hat{a}_p(l,z))}{(1-\hat{a}_p(l,z))(1-\hat{a}_p(l+jk,z))} \\ &+\hat{\pi}_p(l,z)\,\Delta_k\frac{1}{1-\hat{a}_p(l,z)}, \end{aligned} \tag{4.18}$$

where, by (2.11), (2.12) and $f_2(p,m) \le 2$,

$$\left|\frac{\Delta_k\hat{\pi}_p(l,z)}{1-\hat{a}_p(l,z)}\right| = \left|\frac{\Delta_k\hat{\pi}_p(l,z)}{\hat{\pi}_p(l,z)}\right| |\hat{\varphi}_p(l,z)| \le O(\lambda)\,\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}|\hat{G}_{\mu_p(z)}(l)|. \tag{4.19}$$

The second term of (4.18) can be bounded as follows. First, by $|e^{il\cdot x}(e^{ijk\cdot x}-1)|\leq |\sin(k\cdot x)|+1-\cos(k\cdot x)$ for $j=\pm 1$,

$$\begin{aligned}|\hat{\pi}_p(l+jk,z)-\hat{\pi}_p(l,z)| \leq \sum_{(x,n)} |\sin(k\cdot x)||\pi_p(x,n)||z|^n \\ +\sum_{(x,n)}(1-\cos(k\cdot x))\,|\pi_p(x,n)||z|^n,\end{aligned} \tag{4.20}$$

where the second term is bounded by $O(\lambda)\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}$, due to (2.12). By the Cauchy–Schwarz inequality and using (2.11), (2.12), the first term is bounded by

$$\begin{aligned}&\Bigg(\sum_{(x,n):x\neq o}|\pi_p(x,n)||z|^n\Bigg)^{1/2}\Bigg(\sum_{(x,n):x\neq o}\sin^2(k\cdot x)|\pi_p(x,n)||z|^n\Bigg)^{1/2}\\ &\leq O(\lambda)^{1/2}\Bigg(\sum_{(x,n)}(1-\cos(k\cdot x))\,|\pi_p(x,n)||z|^n\Bigg)^{1/2}\\ &\leq O(\lambda)\,\hat{G}_{\mu_p(m\vee 1)}(k)^{-1/2}.\end{aligned} \tag{4.21}$$

Therefore, $|\hat{\pi}_p(l+jk,z)-\hat{\pi}_p(l,z)|\leq O(\lambda)\hat{G}_{\mu_p(m\vee 1)}(k)^{-1/2}$. Similarly, we can show $|\hat{a}_p(l+jk,z)-\hat{a}_p(l,z)|\leq O(1)\hat{G}_{\mu_p(m\vee 1)}(k)^{-1/2}$, where we use

$$\begin{aligned}&\sum_{(x,n)}(1-\cos(k\cdot x))\,(q_p*|\pi_p|)(x,n)|z|^n\\ &\leq 5p|z|\Bigg(\underbrace{\sum_{y}(1-\cos(k\cdot y))\,D(y)}_{1-\hat{D}(k)}\ \underbrace{\sum_{(x,n)}|\pi_p(x-y,n-1)||z|^{n-1}}_{1+O(\lambda)}\\ &\qquad+\sum_{y}D(y)\underbrace{\sum_{(x,n)}(1-\cos(k\cdot(x-y)))\,|\pi_p(x-y,n-1)||z|^{n-1}}_{O(\lambda)\,\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}}\Bigg)\\ &\leq 10\,(2+O(\lambda))\,\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}.\end{aligned} \tag{4.22}$$

Here, the first inequality is due to $1-\cos(X+Y)\leq 5(1-\cos X)+5(1-\cos Y)$ (cf., [25, (4.50)]), and the second inequality is due to $f_1(p,m)\leq 2$ and $1-\hat{D}(k)$

$\leq 2\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}$ (since $\mu_p(m \vee 1) \in [0, 1]$). Therefore, for $j = \pm 1$,

$$\begin{aligned}\left|\frac{(\hat{\pi}_p(l+jk,z)-\hat{\pi}_p(l,z))(\hat{a}_p(l+jk,z)-\hat{a}_p(l,z))}{(1-\hat{a}_p(l,z))(1-\hat{a}_p(l+jk,z))}\right| \\ \leq O(\lambda)\,\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}|\hat{G}_{\mu_p(z)}(l)\,\hat{G}_{\mu_p(z)}(l+jk)|. \qquad (4.23)\end{aligned}$$

To complete bounding $\Delta_k\hat{\varphi}_p(l,z)$, it remains to investigate $\Delta_k(1-\hat{a}_p(l,z))^{-1}$ in the last term of (4.18). Let

$$\hat{a}_p^{\cos}(l,z;k) = \sum_{(x,n)} (q_p * \pi_p)(x,n)z^n \cos(l\cdot x)\cos(k\cdot x), \qquad (4.24)$$

$$\hat{a}_p^{\sin}(l,z;k) = \sum_{(x,n)} (q_p * \pi_p)(x,n)z^n \sin(l\cdot x)\sin(k\cdot x). \qquad (4.25)$$

Then, by [6, Lemma 5.3],

$$\begin{aligned}\Delta_k\frac{1}{1-\hat{a}_p(l,z)} = \frac{\hat{\varphi}_p(l,z)}{\hat{\pi}_p(l,z)}\Bigg(\sum_{j=\pm1}\frac{\hat{\varphi}_p(l+jk,z)}{\hat{\pi}_p(l+jk,z)}\left(\hat{a}_p^{\cos}(l,z;k)-\hat{a}_p(l,z)\right) \\ +2\prod_{j=\pm1}\frac{\hat{\varphi}_p(l+jk,z)}{\hat{\pi}_p(l+jk,z)}\,\hat{a}_p^{\sin}(l,z;k)^2\Bigg), \qquad (4.26)\end{aligned}$$

where, by (4.22),

$$\begin{aligned}|\hat{a}_p^{\cos}(l,z;k)-\hat{a}_p(l,z)| &\leq \sum_{(x,n)}(1-\cos(k\cdot x))\,(q_p*|\pi_p|)(x,n)|z|^n \\ &\leq 10\,(2+O(\lambda))\;\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}. \qquad (4.27)\end{aligned}$$

Moreover, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\hat{a}_p^{\sin}(l,z;k)^2 &\leq \left(\sum_{(x,n)}(q_p*|\pi_p|)(x,n)|z|^n\sin^2(l\cdot x)\right) \\ &\quad\times\sum_{(x,n)}(q_p*|\pi_p|)(x,n)|z|^n\sin^2(k\cdot x) \\ &\leq 2^2\left(\sum_{(x,n)}(1-\cos(l\cdot x))\,(q_p*|\pi_p|)(x,n)|z|^n\right) \\ &\quad\times\sum_{(x,n)}(1-\cos(k\cdot x))\,(q_p*|\pi_p|)(x,n)|z|^n \\ &\leq 20^2(2+O(\lambda))^2\hat{G}_{\mu_p(m\vee 1)}(l)^{-1}\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}. \qquad (4.28)\end{aligned}$$

As a result, since $f_2(p,m) \le 2$ and $|\hat G_{\mu_p(z)}(l)| \le \hat G_{\mu_p(m\vee 1)}(l)$ for $|z|=m$ or 1, we obtain

$$\begin{aligned}\left|\Delta_k \frac{1}{1-\hat a_p(l,z)}\right| &\le \hat G_{\mu_p(m\vee 1)}(k)^{-1}\Bigg(40\,(2+O(\lambda))\sum_{j=\pm 1}|\hat G_{\mu_p(z)}(l)\,\hat G_{\mu_p(z)}(l+jk)|\\ &\qquad + 80^2\,(2+O(\lambda))^2\,|\hat G_{\mu_p(z)}(l+k)\,\hat G_{\mu_p(z)}(l-k)|\Bigg)\\ &\le 2K\,(1+O(\lambda))\,\hat G_{\mu_p(m\vee 1)}(k)^{-1}\\ &\qquad\times \sum_{(j,j')=(0,\pm 1),(1,-1)}|\hat G_{\mu_p(z)}(l+jk)\,\hat G_{\mu_p(z)}(l+j'k)|, \end{aligned}\tag{4.29}$$

where $K=2\cdot 80^2$.

Finally, by summarizing (4.18), (4.19), (4.23) and (4.29), we arrive at

$$\frac{\hat G_{\mu_p(m\vee 1)}(k)\,|\frac{1}{2}\Delta_k\hat\varphi_p(l,z)|}{K\sum_{(j,j')=(0,\pm 1),(1,-1)}|\hat G_{\mu_p(z)}(l+jk)\,\hat G_{\mu_p(z)}(l+j'k)|}\le 1+O(\lambda)\le 2. \tag{4.30}$$

This completes the proof of Proposition 2.1(ii). □

4.3 Proof of Proposition 2.1(i)

Proposition 2.1(i) is an immediate consequence of Proposition 3.3(i) and the following lemma:

Lemma 4.1 *Let $d>2(\alpha\wedge 2)$ and $L\gg 1$, and fix $p<p_{\mathrm{c}}$ and $m<m_p$. Then, $f(p,m)\le 3$ implies that there are (p,m)-independent constants $C_T, C_W<\infty$ such that*

$$T_p^{(m)}\le C_T\lambda,\quad W_p^{(m)}(k)\le C_W\lambda\hat G_{\mu_p(m\vee 1)}(k)^{-1}. \tag{4.31}$$

Proof Note that the Fourier transform of $\varphi_p^{(m)}(x,n)\equiv\varphi_p(x,n)m^n$ for $m<m_p$ is

$$\begin{aligned}\hat\varphi_p^{(m)}(k,e^{i\theta}) &= \sum_{(x,n)}\varphi_p^{(m)}(x,n)e^{ik\cdot x}e^{i\theta n}\\ &= \sum_{(x,n)}\varphi_p(x,n)e^{ik\cdot x}(me^{i\theta})^n=\hat\varphi_p(k,me^{i\theta}). \end{aligned}\tag{4.32}$$

By $f_1(p,m) \vee f_2(p,m) \le 3$ and (2.6), $T_p^{(m)}$ is bounded as

$$
\begin{aligned}
T_p^{(m)} &\le p^2 m \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d}\, \hat{D}(k)^2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, |\hat{\varphi}_p(k,e^{i\theta})|^2 |\hat{\varphi}_p^{(m)}(k,e^{-i\theta})| \\
&\le 3^2 \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d}\, \hat{D}(k)^2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left(\frac{3c}{\frac{1}{\hat{\varphi}_p(0,1)} + |\theta| + 1 - \hat{D}(k)} \right)^2 \\
&\quad \times \frac{3c}{\frac{1}{\hat{\varphi}_p(0,m)} + |\theta| + 1 - \hat{D}(k)} \\
&\le O(1) \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{\hat{D}(k)^2}{(1-\hat{D}(k))^2} = O(1) \sum_{n=2}^{\infty} (n-1)\, D^{\star n}(o) \le O(\lambda),
\end{aligned}
\tag{4.33}
$$

where the last inequality is due to (1.6) and $d > 2(\alpha \wedge 2)$.

To prove the bound on $W_p^{(m)}(k)$, we first note that, by $(q_p * \varphi_p)(y,t) \le (q_p * q_p * \varphi_p)(y,t)$ for $t \ge 2$,

$$
\begin{aligned}
&\sum_{(y,t)} (1-\cos(k\cdot y))\, (q_p * \varphi_p)(y,t) \cdot (q_p * \varphi_p)(y-x,t-n) \\
&\quad = p \sum_{y \in \mathbb{Z}^d} (1-\cos(k\cdot y))\, D(y) \cdot (q_p * \varphi_p)(y-x,1-n) \\
&\qquad + \sum_{(y,t):t\ge 2} (1-\cos(k\cdot y))\, (q_p * q_p * \varphi_p)(y,t) \cdot (q_p * \varphi_p)(y-x,t-n).
\end{aligned}
\tag{4.34}
$$

In the first sum on the right-hand side of (4.34), $1-n$ must be larger than or equal to 1. If $1-n=1$, then, since $(q_p * \varphi_p)(y-x,1) \equiv pD(y-x) \le p\|D\|_\infty \le Cp\lambda$ (see (1.6)), $f_1(p,m) \le 3$ and $1-\hat{D}(k) \le 2\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}$ (see below (4.22)), we obtain

$$
\begin{aligned}
&p \sum_y (1-\cos(k\cdot y))\, D(y) \cdot (q_p * \varphi_p)(y-x,1) m \le 3^2 C\lambda \left(1-\hat{D}(k)\right) \\
&\quad \le 18 C\lambda \hat{G}_{\mu_p(m\vee 1)}(k)^{-1}.
\end{aligned}
\tag{4.35}
$$

If $1-n \ge 2$, then we use $(q_p * \varphi_p)(y,1-n) \le (q_p * q_p * \varphi_p)(y,1-n)$, $f_1(p,m) \vee f_2(p,m) \le 3$, (2.6) and $1-\hat{D}(k) \le 2\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}$ to obtain that, for $m < 1$,

$$
\begin{aligned}
&p \sum_{y} (1-\cos(k\cdot y))\, D(y) \cdot (mq_p * mq_p * \varphi_p^{(m)})(y-x, 1-n) \\
&\quad \le 3\Big(1-\hat{D}(k)\Big) \int\limits_{[-\pi,\pi]^d} \frac{d^d l}{(2\pi)^d}\, \hat{D}(l)^2 \int\limits_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \frac{3^3 c}{\frac{1}{\hat{\varphi}_p(0,m)} + |\theta| + 1 - \hat{D}(l)} \\
&\quad \le O(1)\, \hat{G}_{\mu_p(m\vee 1)}(k)^{-1} \int\limits_{[-\pi,\pi]^d} \frac{d^d l}{(2\pi)^d}\, \frac{\hat{D}(l)^2}{1-\hat{D}(l)} \le O(\lambda)\, \hat{G}_{\mu_p(m\vee 1)}(k)^{-1},
\end{aligned} \tag{4.36}
$$

where the last inequality is due to (1.6) and $d > \alpha \wedge 2$. The other case of $m \ge 1$ can be estimated in the same way.

To complete the proof of the bound on $W_p^{(m)}(k)$, it remains to show that the second sum on the right-hand side of (4.34) is bounded by a multiple of $\lambda \hat{G}_{\mu_p(m\vee 1)}(k)^{-1}$. Using $1 - \cos \sum_{j=1}^3 X_j \le 7 \sum_{j=1}^3 (1-\cos X_j)$ (cf., [25, (4.50)]), we have

$$
\begin{aligned}
&(1-\cos(k\cdot y))\, (q_p * q_p * \varphi_p)(y,t) \\
&\quad \le 7p^2 \sum_{u,v\in\mathbb{Z}^d} \Big((1-\cos(k\cdot u))\, D(u) D(v-u) \varphi_p(y-v, t-2) \\
&\qquad\qquad + D(u)\, (1-\cos(k\cdot(v-u)))\, D(v-u) \varphi_p(y-v, t-2) \\
&\qquad\qquad + D(u) D(v-u)\, (1-\cos(k\cdot(y-v)))\, \varphi_p(y-v, t-2)\Big).
\end{aligned} \tag{4.37}
$$

Recalling (4.16) and using $f_1(p,m) \le 3$, we obtain that, for $m < 1$,

$$
\begin{aligned}
&\sum_{(y,t)} (1-\cos(k\cdot y))\, (q_p * q_p * \varphi_p)(y,t) \cdot (mq_p * \varphi_p^{(m)})(y-x, t-n) \\
&\quad \le 7\cdot 3^3 \left(2\Big(1-\hat{D}(k)\Big) \int\limits_{[-\pi,\pi]^d} \frac{d^d l}{(2\pi)^d}\, \hat{D}(l)^2 \int\limits_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, |\hat{\varphi}_p(l, e^{i\theta})\, \hat{\varphi}_p^{(m)}(l, e^{-i\theta})| \right. \\
&\qquad \left. + \int\limits_{[-\pi,\pi]^d} \frac{d^d l}{(2\pi)^d}\, |\hat{D}(l)|^3 \int\limits_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \Big|\tfrac{1}{2}\Delta_k \hat{\varphi}_p(l, e^{i\theta})\Big|\, |\hat{\varphi}_p^{(m)}(l, e^{-i\theta})| \right).
\end{aligned} \tag{4.38}
$$

Similarly to the above, by using $1 - \hat{D}(k) \le 2\hat{G}_{\mu_p(m\vee 1)}(k)^{-1}$, $f_2(p,m) \le 3$ and (2.6), the first term on the right-hand side of (4.38) is bounded by a multiple of $\lambda \hat{G}_{\mu_p(m\vee 1)}(k)^{-1}$ when $d > \alpha \wedge 2$. For the second term on the right-hand side of (4.38), we use $f_2(p,m) \vee f_3(p,m) \le 3$ to obtain

$$\int\limits_{[-\pi,\pi]^d} \frac{d^d l}{(2\pi)^d} |\hat{D}(l)|^3 \int\limits_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left|\tfrac{1}{2}\Delta_k \hat{\varphi}_p(l, e^{i\theta})\right| |\hat{\varphi}_p^{(m)}(l, e^{-i\theta})|$$
$$\le 3^2 K \hat{G}_{\mu_p(m\vee 1)}(k)^{-1} \sum_{(j,j')=(0,\pm 1),(1,-1)} \int\limits_{[-\pi,\pi]^d} \frac{d^d l}{(2\pi)^d} \hat{D}(l)^2$$
$$\times \int\limits_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\hat{G}_{\mu_p(e^{i\theta})}(l + jk)||\hat{G}_{\mu_p(e^{i\theta})}(l + j'k)||\hat{G}_{\mu_p(me^{-i\theta})}(l)|. \tag{4.39}$$

By using (2.6) as in (4.33), the summand is bounded by a multiple of λ for any k (the worst case is when $k = 0$) as long as $d > 2(\alpha \wedge 2)$. This completes the proof of the bound on $W_p^{(m)}(k)$ and of Lemma 4.1. □

5 Proof of Proposition 2.2

In this section, we prove Proposition 2.2 that was used in Sect. 2.3 to prove Theorem 1.4. First we note that, by (3.15),

$$|\partial_p \hat{\pi}_p(0, 1)| \le \frac{1}{p} \sum_{N=1}^{\infty} \sum_{(x,n)} \Pi_p^{(N)}(x, n), \tag{5.1}$$

where $\Pi_p^{(N)}(x, n)$ obeys the diagrammatic bound (3.26), with $T_p^{(1)} \le C_T \lambda$ as in (4.31). Therefore, to complete the proof of Proposition 2.2, it suffices to prove the following lemma:

Lemma 5.1 *Let $d > 2(\alpha \wedge 2)$ and $L \gg 1$. Then, there are $C_{\tilde{T}}, C_H < \infty$ such that, for $p \in (1, p_c)$,*

$$\tilde{T}_p \le C_{\tilde{T}} \lambda, \quad H_p \le C_H \lambda^2. \tag{5.2}$$

Proof The bound on $\tilde{T}_p$ can be proved in the same way as in (4.33). Taking the Fourier transform and using Theorem 1.2 and $p_c = 1 + O(\lambda) \le 2$, we can bound H_p as

$$H_p \le p^5 \int\limits_{[-\pi,\pi]^{2d}} \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \hat{D}(k_1)^2 \hat{D}(k_2)^2 \left|\hat{D}(k_1 - k_2)\right| \int\limits_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \left|\hat{\varphi}_p(k_1, e^{i\theta_1})\right|^2$$
$$\times \int\limits_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \left|\hat{\varphi}_p(k_2, e^{i\theta_2})\right|^2 \left|\hat{\varphi}_p(k_1 - k_2, e^{i(\theta_1-\theta_2)})\right|^2$$
$$\le 2^5 \int\limits_{[-\pi,\pi]^{2d}} \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \hat{D}(k_1)^2 \hat{D}(k_2)^2 \int\limits_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \left(\frac{C}{|\theta_1| + 1 - \hat{D}(k_1)}\right)^2$$

$$
\begin{aligned}
&\times \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \bigg(\frac{C}{|\theta_2| + 1 - \hat{D}(k_2)}\bigg)^2 \bigg(\frac{C}{|\theta_1 - \theta_2| + 1 - \hat{D}(k_1 - k_2)}\bigg)^2 \\
&\le \int_{[-\pi,\pi]^{2d}} \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{\hat{D}(k_1)^2 \hat{D}(k_2)^2}{(1 - \hat{D}(k_2))^2} \\
&\quad \times \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \frac{O(1)}{(|\theta_1| + 1 - \hat{D}(k_1))^2 (|\theta_1| + 1 - \hat{D}(k_1 - k_2))} \\
&\le O(1) \int_{[-\pi,\pi]^d} \frac{d^d k_2}{(2\pi)^d} \frac{\hat{D}(k_2)^2}{(1 - \hat{D}(k_2))^2} \int_{[-\pi,\pi]^d} \frac{d^d k_1}{(2\pi)^d} \frac{\hat{D}(k_1)^2}{(1 - \hat{D}(k_1) \vee \hat{D}(k_1 - k_2))^2}.
\end{aligned}
\tag{5.3}
$$

To prove the bound on H_p in (5.2), it suffices to show that the last integral with respect to k_1 is $O(\lambda)$ for every k_2. Since this is trivial if $\hat{D}(k_1) \ge \hat{D}(k_1 - k_2)$ (then the integrals in (5.3) are decoupled, each of them is $O(\lambda)$), it is sufficient to prove that

$$
\int_{[-\pi,\pi]^d} \frac{d^d k_1}{(2\pi)^d} \frac{\hat{D}(k_1)^2}{(1 - \hat{D}(k_1 - k_2))^2} = O(\lambda). \tag{5.4}
$$

However, by (1.6), the integral over $\|k_1 - k_2\|_\infty > (\ell L)^{-1}$ is bounded as

$$
\begin{aligned}
\int_{\|k_1 - k_2\|_\infty > (\ell L)^{-1}} \frac{d^d k_1}{(2\pi)^d} \frac{\hat{D}(k_1)^2}{(1 - \hat{D}(k_1 - k_2))^2} &\le \frac{1}{\Delta^2} \int_{[-\pi,\pi]^d} \frac{d^d k_1}{(2\pi)^d} \hat{D}(k_1)^2 \\
&\le \frac{\|D\|_\infty}{\Delta^2} = O(\lambda).
\end{aligned}
\tag{5.5}
$$

Moreover, by (1.7), the integral over $\|k_1 - k_2\|_\infty \le (\ell L)^{-1}$ is bounded as, for $\alpha \ne 2$,

$$
\begin{aligned}
\int_{\|k_1 - k_2\|_\infty \le (\ell L)^{-1}} \frac{d^d k_1}{(2\pi)^d} \frac{\hat{D}(k_1)^2}{(1 - \hat{D}(k_1 - k_2))^2} &\le O(L^{-2(\alpha \wedge 2)}) \int_0^{(\ell L)^{-1}} dr\, r^{d-1-2(\alpha \wedge 2)} \\
&= O(\lambda).
\end{aligned}
\tag{5.6}
$$

The case for $\alpha = 2$ can be estimated similarly, since the log divergence as $|k| \to 0$ in (1.7) is unimportant in (5.6) as long as $d > 4$. This completes the proof of Lemma 5.1. □

6 Proof of Proposition 2.3

In this section, we prove Proposition 2.3 that was used in Sect. 2.4 to show (2.40), the key ingredient for the proof of Theorem 1.5.

First we derive an expression for $\partial_\zeta \hat{\Phi}_p(k, m_p\zeta)$. Since $\hat{A}_p(k,z) = \hat{A}_p^{(1)}(k) + \hat{A}_p^{(2)}(k,z)$, where $\hat{A}_p^{(1)}(k)$ is independent of z, we have

$$\begin{aligned}
\partial_\zeta \hat{\Phi}_p(k, m_p\zeta) &\equiv \partial_\zeta \frac{-(1-\zeta)\hat{A}_p^{(2)}(k, m_p\zeta)}{\Big((1-\zeta)\hat{A}_p(k, m_p\zeta) + \hat{B}_p(k)\Big)\Big((1-\zeta)\hat{A}_p^{(1)}(k) + \hat{B}_p(k)\Big)} \\
&= \frac{\hat{A}_p^{(2)}(k, m_p\zeta) - (1-\zeta)\partial_\zeta \hat{A}_p^{(2)}(k, m_p\zeta)}{\Big((1-\zeta)\hat{A}_p(k, m_p\zeta) + \hat{B}_p(k)\Big)\Big((1-\zeta)\hat{A}_p^{(1)}(k) + \hat{B}_p(k)\Big)} \\
&\quad + \frac{-(1-\zeta)\hat{A}_p^{(2)}(k, m_p\zeta)}{(1-\zeta)\hat{A}_p^{(1)}(k) + \hat{B}_p(k)} \\
&\qquad \times \frac{\hat{A}_p^{(1)}(k) + \hat{A}_p^{(2)}(k, m_p\zeta) - (1-\zeta)\partial_\zeta \hat{A}_p^{(2)}(k, m_p\zeta)}{\Big((1-\zeta)\hat{A}_p(k, m_p\zeta) + \hat{B}_p(k)\Big)^2} \\
&\quad + \frac{-(1-\zeta)\hat{A}_p^{(2)}(k, m_p\zeta)}{(1-\zeta)\hat{A}_p(k, m_p\zeta) + \hat{B}_p(k)} \, \frac{\hat{A}_p^{(1)}(k)}{\Big((1-\zeta)\hat{A}_p^{(1)}(k) + \hat{B}_p(k)\Big)^2}.
\end{aligned} \tag{6.1}$$

Recall that

$$\hat{A}_p^{(2)}(k, m_p\zeta) = \frac{\partial_\zeta \hat{\pi}_p(k, m_p)^{-1}}{pm_p} - \frac{\hat{\pi}_p(k, m_p)^{-1} - \hat{\pi}_p(k, m_p\zeta)^{-1}}{pm_p(1-\zeta)}, \tag{6.2}$$

where $\partial_\zeta \hat{\pi}_p(k, m_p)^{-1}$ is an abbreviation of $\partial_\zeta \hat{\pi}_p(k, m_p\zeta)^{-1}|_{\zeta=1} \equiv m_p \partial_z \hat{\pi}_p (k,z)^{-1}|_{z=m_p}$, so that

$$\begin{aligned}
(1-\zeta)\, \partial_\zeta \hat{A}_p^{(2)}(k, m_p\zeta) &= (1-\zeta)\, \partial_\zeta \left(-\frac{\hat{\pi}_p(k, m_p)^{-1} - \hat{\pi}_p(k, m_p\zeta)^{-1}}{pm_p(1-\zeta)} \right) \\
&= \frac{\partial_\zeta \hat{\pi}_p(k, m_p\zeta)^{-1}}{pm_p} - \frac{\hat{\pi}_p(k, m_p)^{-1} - \hat{\pi}_p(k, m_p\zeta)^{-1}}{pm_p(1-\zeta)} \\
&= \frac{\partial_\zeta \hat{\pi}_p(k, m_p\zeta)^{-1} - \partial_\zeta \hat{\pi}_p(k, m_p)^{-1}}{pm_p} + \hat{A}_p^{(2)}(k, m_p\zeta).
\end{aligned} \tag{6.3}$$

Therefore, $\hat{A}_p^{(2)}(k, m_p\zeta) - (1-\zeta)\partial_\zeta \hat{A}_p^{(2)}(k, m_p\zeta)$ in (6.1) can be replaced by $\hat{a}_p^{(2)}(k, m_p\zeta)$, which is

$$\hat{a}_p^{(2)}(k, m_p\zeta) = \frac{\partial_\zeta \hat{\pi}_p(k, m_p)^{-1} - \partial_\zeta \hat{\pi}_p(k, m_p\zeta)^{-1}}{pm_p}. \tag{6.4}$$

Now, Proposition 2.3 is an immediate consequence of the following lemma:

Lemma 6.1 *Let $d > 2(\alpha \wedge 2)$, $\epsilon \in (0, 1 \wedge \frac{d-2(\alpha\wedge 2)}{\alpha\wedge 2})$ and $L \gg 1$. Then, the following hold uniformly in $p \in (0, p_{\mathrm{c}}]$, $k \in [-\pi, \pi]^d$ and $\zeta \in \mathbb{C}$ with $|\zeta| < 1$:*

(i) *There is a positive constant c such that*

$$\left.\begin{array}{l}|(1-\zeta)\hat{A}_p(k, m_p\zeta) + \hat{B}_p(k)| \\ |(1-\zeta)\hat{A}_p^{(1)}(k) + \hat{B}_p(k)|\end{array}\right\} \geq c|1-\zeta|. \tag{6.5}$$

(ii) *There is a finite constant c_ϵ such that*

$$\left.\begin{array}{l}|A_p^{(2)}(k, m_p\zeta)| \\ |\hat{a}_p^{(2)}(k, m_p\zeta)|\end{array}\right\} \leq c_\epsilon|1-\zeta|^\epsilon. \tag{6.6}$$

In the following proof, the constant in the $O(\,\cdot\,)$ term is independent of p, k and ζ.

Proof of Lemma 6.1(i) Since both bounds can be proved in the same way, we only prove the bound on $|(1-\zeta)\hat{A}_p^{(1)}(k) + \hat{B}_p(k)|$.

We consider the following four cases: (a) $\Re\zeta \leq 0$; (b) $\Re\zeta \geq 0$ with $\Re(1-\zeta) \geq \Im(1-\zeta)$; (c) $\Re\zeta \geq 0$ with $\Re(1-\zeta) \leq \Im(1-\zeta)$ and $\hat{D}(k) \geq 1-\Delta$; (d) $\Re\zeta \geq 0$ and $\hat{D}(k) \leq 1-\Delta$. Note that these four cases exhaust all $\zeta \in \mathbb{C}$ with $|\zeta| \leq 1$. For the moment, we abbreviate $\hat{A}_p^{(1)}(k)$ to A and $\hat{B}_p(k)$ to B.

(a) Since $A, B \in \mathbb{R}$ and $|w| \geq |\Re w|$ for any $w \in \mathbb{C}$,

$$|(1-\zeta)A + B| = |A + B - A\zeta| \geq |A + B + A\Re(-\zeta)|. \tag{6.7}$$

Since $\hat{D}(k) > -1 + \Delta$ holds for all $k \in [-\pi, \pi]^d$ (cf., (1.6)), we have $A = \hat{D}(k) + O(\lambda) \geq -1 + \Delta - O(\lambda)$. Since $A + B = 1 + O(\lambda)$ and $\Re(-\zeta) \geq 0$, we obtain

$$|A + B + A\Re(-\zeta)| = A + B + A\Re(-\zeta) \geq \Delta - O(\lambda), \tag{6.8}$$

uniformly in the concerned ζ.

(d) Using $\Re\zeta \geq 0$ and $\hat{D}(k) \leq 1 - \Delta$, we can prove (6.8) similarly.

(b) Since $A+B = 1+O(\lambda)$, $B \geq 0$, $\Re\zeta \geq 0$, and $\Re(1-\zeta) \geq \frac{1}{\sqrt{2}}|1-\zeta|$, we obtain

$$\begin{aligned}|(1-\zeta)A + B| &= |(1-\zeta)(A+B) + B\zeta| \\ &\geq |(A+B)\,\Re(1-\zeta) + B\Re\zeta| \\ &\geq (A+B)\,\Re(1-\zeta) \geq \frac{1-O(\lambda)}{\sqrt{2}}\,|1-\zeta|. \end{aligned} \tag{6.9}$$

(c) Since $A = \hat{D}(k) + O(\lambda) \geq 1 - \Delta - O(\lambda)$ and $|\Im(1-\zeta)| \geq \frac{1}{\sqrt{2}}|1-\zeta|$, by using the imaginary part (i.e., $|w| \geq |\Im w|$ for $w \in \mathbb{C}$) we obtain

$$|(1-\zeta)A + B| \geq |A\Im(1-\zeta)| = A|\Im(1-\zeta)| \geq \frac{1-\Delta-O(\lambda)}{\sqrt{2}}\,|1-\zeta|. \tag{6.10}$$

This completes the proof of Lemma 6.1(i). □

Proof of Lemma 6.1(ii) First, by adding and subtracting, we can rewrite $\hat{A}_p^{(2)}(k, m_p\zeta)$ in (6.2) as

$$\begin{aligned}\hat{A}_p^{(2)}(k, m_p\zeta) &= -\frac{\partial_\zeta \hat{\pi}_p(k, m_p)}{pm_p\hat{\pi}_p(k, m_p)^2} + \frac{\hat{\pi}_p(k, m_p) - \hat{\pi}_p(k, m_p\zeta)}{pm_p(1-\zeta)\hat{\pi}_p(k, m_p)\hat{\pi}_p(k, m_p\zeta)}\\ &= -\frac{\partial_\zeta \hat{\pi}_p(k, m_p) - \frac{\hat{\pi}_p(k, m_p) - \hat{\pi}_p(k, m_p\zeta)}{1-\zeta}}{pm_p\hat{\pi}_p(k, m_p)^2}\\ &\quad + \frac{\big(\hat{\pi}_p(k, m_p) - \hat{\pi}_p(k, m_p\zeta)\big)^2}{pm_p(1-\zeta)\hat{\pi}_p(k, m_p)^2\hat{\pi}_p(k, m_p\zeta)},\end{aligned} \tag{6.11}$$

and $\hat{a}_p^{(2)}(k, m_p\zeta)$ in (6.4) as

$$\begin{aligned}\hat{a}_p^{(2)}(k, m_p\zeta) &= -\frac{\partial_\zeta \hat{\pi}_p(k, m_p)}{pm_p\hat{\pi}_p(k, m_p)^2} + \frac{\partial_\zeta \hat{\pi}_p(k, m_p\zeta)}{pm_p\hat{\pi}_p(k, m_p\zeta)^2}\\ &= -\frac{\partial_\zeta \hat{\pi}_p(k, m_p) - \partial_\zeta \hat{\pi}_p(k, m_p\zeta)}{pm_p\hat{\pi}_p(k, m_p)^2}\\ &\quad + \frac{\big(\hat{\pi}_p(k, m_p)^2 - \hat{\pi}_p(k, m_p\zeta)^2\big)\,\partial_\zeta \hat{\pi}_p(k, m_p\zeta)}{pm_p\hat{\pi}_p(k, m_p)^2\hat{\pi}_p(k, m_p\zeta)^2}.\end{aligned} \tag{6.12}$$

Since $pm_p\hat{\pi}_p(k, m_p)^2 = 1 + O(\lambda)$, $\hat{\pi}_p(k, m_p\zeta)^2 = 1 + O(\lambda)$, $|\partial_\zeta \hat{\pi}_p(k, m_p\zeta)| = O(\lambda)$ and $|\hat{\pi}_p(k, m_p) - \hat{\pi}_p(k, m_p\zeta)| = O(\lambda)|1-\zeta|$, the second terms in (6.11) and (6.12) are $O(|1-\zeta|)$. To prove (6.6), it thus suffices to show that the numerator of the first term in (6.11) and that in (6.12) are both bounded by $O_\epsilon(1)|1-\zeta|^\epsilon$, where the constant in the $O_\epsilon(1)$ term may depend on ϵ. Since both can be proved similarly, we only prove that $|\partial_\zeta \hat{\pi}_p(k, m_p) - \partial_\zeta \hat{\pi}_p(k, m_p\zeta)| \leq O_\epsilon(1)|1-\zeta|^\epsilon$.

Note that

$$\begin{aligned}|\partial_\zeta \hat{\pi}_p(k, m_p) - \partial_\zeta \hat{\pi}_p(k, m_p\zeta)| &= \left|\sum_{(x,n):n\geq 2} n(1-\zeta^{n-1})\pi_p(x, n)e^{ik\cdot x}m_p^n\right|\\ &\leq \sum_{(x,n):n\geq 2} n|1-\zeta^{n-1}|\,|\pi_p(x, n)|m_p^n.\end{aligned} \tag{6.13}$$

For $n \geq 2$, we have

$$
\begin{aligned}
|1-\zeta^{n-1}| &= \left|(1-\zeta^{n-1})^{1-\epsilon}\left(\frac{1-\zeta^{n-1}}{1-\zeta}\right)^{\epsilon}(1-\zeta)^{\epsilon}\right| \\
&\leq 2^{1-\epsilon}\left|\sum_{l=0}^{n-2}\zeta^{l}\right|^{\epsilon}|1-\zeta|^{\epsilon} \leq 2|1-\zeta|^{\epsilon}n^{\epsilon}. \qquad (6.14)
\end{aligned}
$$

Moreover, for $\epsilon \in (0,1)$, we have (cf., [18, Sect. 6.3])

$$
n^{1+\epsilon} = \frac{n^2}{(1-\epsilon)\,\Gamma(1-\epsilon)}\int_0^\infty e^{-n\rho^{1/(1-\epsilon)}}\,d\rho. \qquad (6.15)
$$

Applying these to (6.13) and using the diagrammatic bound (3.24) for $r = 2$ and $T_p^{(\tilde{m}_\rho)} \leq C_T\lambda$ with $\tilde{m}_\rho = m_p e^{-\rho^{1/(1-\epsilon)}}$, we have

$$
\begin{aligned}
|\partial_\zeta\hat{\pi}_p(k,m_p) - \partial_\zeta\hat{\pi}_p(k,m_p\zeta)| &\leq \frac{2|1-\zeta|^{\epsilon}}{(1-\epsilon)\,\Gamma(1-\epsilon)}\int_0^\infty d\rho\,\sum_{(x,n)} n^2|\pi_p(x,n)|\,\tilde{m}_\rho^n \\
&\leq \frac{2(1+2C_T\lambda)|1-\zeta|^{\epsilon}}{(1-\epsilon)\,\Gamma(1-\epsilon)} \\
&\quad \times \sum_{N=0}^\infty (N+1)^2(2C_T\lambda)^{(N-1)\vee 0}\int_0^\infty d\rho\; S_p^{(\tilde{m}_\rho)}, \qquad (6.16)
\end{aligned}
$$

where, by (1.15) and $p \leq pm_p = 1 + O(\lambda)$,

$$
\begin{aligned}
\int_0^\infty d\rho\; S_p^{(\tilde{m}_\rho)} &\leq p^2 m_p \int \frac{d^dk}{(2\pi)^d}\,\hat{D}(k)^2 \int \frac{d\theta}{2\pi}\left(\frac{C}{p(m_p-1)+|\theta|+1-\hat{D}(k)}\right)^3 \\
&\quad \times \int_0^\infty d\rho\,\frac{Ce^{-\rho^{1/(1-\epsilon)}}}{pm_p(1-e^{-\rho^{1/(1-\epsilon)}})+|\theta|+1-\hat{D}(k)} \\
&\leq \int \frac{d^dk}{(2\pi)^d}\,\frac{O(1)}{(1-\hat{D}(k))^2}\int_0^\infty d\rho\,\frac{e^{-\rho^{1/(1-\epsilon)}}}{1-e^{-\rho^{1/(1-\epsilon)}}+1-\hat{D}(k)}. \qquad (6.17)
\end{aligned}
$$

However, since

$$
\begin{aligned}
\int_0^\infty d\rho\, \frac{e^{-\rho^{1/(1-\epsilon)}}}{1-e^{-\rho^{1/(1-\epsilon)}}+1-\hat{D}(k)} &= \int_0^\infty ds\, \frac{(1-\epsilon)s^{-\epsilon}e^{-s}}{1-e^{-s}+1-\hat{D}(k)} \qquad (\because \rho = s^{1-\epsilon}) \\
&\le \frac{1-\epsilon}{1-e^{-1}}\left(\int_1^\infty ds\, e^{-s} + \int_0^1 ds\, \frac{s^{-\epsilon}}{s+1-\hat{D}(k)}\right) \\
&\le \frac{1-\epsilon}{1-e^{-1}}\left(1 + \int_0^{1-\hat{D}(k)} ds\, \frac{s^{-\epsilon}}{1-\hat{D}(k)} + \int_{1-\hat{D}(k)}^1 ds\, s^{-1-\epsilon}\right) \\
&\le \frac{1-\epsilon}{1-e^{-1}}\left(1+\frac{(1-\hat{D}(k))^{-\epsilon}}{1-\epsilon} + \frac{(1-\hat{D}(k))^{-\epsilon}}{\epsilon}\right),
\end{aligned}
\tag{6.18}
$$

we obtain that

$$
\int_0^\infty d\rho\, S_p^{(\tilde{m}_\rho)} \le \int \frac{d^dk}{(2\pi)^d}\, \frac{O_\epsilon(1)}{(1-\hat{D}(k))^{2+\epsilon}} < \infty, \tag{6.19}
$$

as long as $d > (2+\epsilon)(\alpha \wedge 2)$, due to (1.7). By (6.16), this completes the proof of $|\partial_\zeta \hat{\pi}_p(k, m_p) - \partial_\zeta \hat{\pi}_p(k, m_p\zeta)| \le O_\epsilon(1)|1-\zeta|^\epsilon$ and of Lemma 6.1(ii). □

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A Proof of Proposition 1.1

In this section, we prove the bounds on D summarized in Proposition 1.1. Since the bounds on $1-\hat{D}(k)$ in (1.6) are equivalent to [16, (1.20)–(1.21)] whose proofs are independent of the range of D (see [16, Appendix A]), it thus remains to prove the bound on $\|D^{\star n}\|_\infty$ in (1.6) and the bounds on $1-\hat{D}(k)$ for $\|k\|_\infty \le (\ell L)^{-1}$ in (1.7).

First we prove the bound on $\|D^{\star n}\|_\infty$ assuming (1.7). By definition, it is trivial when $n = 1$. For $n \ge 2$, we let

$$
R = \{k \in [-\pi, \pi]^d : |k| \le (\ell L)^{-1},\ \hat{D}(k) \ge 0\}, \tag{A.1}
$$

so that $|\hat{D}(k)| = 1-(1-\hat{D}(k)) \le e^{-(1-\hat{D}(k))}$ for $k \in R$, and that $0 \le |\hat{D}(k)| < 1-\Delta$ for $k \notin R$, due to the bound on $1-\hat{D}(k)$ in (1.6). Therefore, for any $x \in \mathbb{Z}^d$,

$$\begin{aligned} D^{\star n}(x) &\le \int\limits_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d}\, |\hat{D}(k)|^n \\ &\le \int\limits_{R} \frac{d^dk}{(2\pi)^d}\, e^{-n(1-\hat{D}(k))} + (1-\Delta)^{n-2} \int\limits_{R^c} \frac{d^dk}{(2\pi)^d}\, \hat{D}(k)^2, \end{aligned} \tag{A.2}$$

where the integral over $k \in R^c \equiv [-\pi,\pi]^d \setminus R$ is bounded by $\|D\|_\infty (1-\Delta)^{n-2} \le O(\lambda)\, n^{-d/(\alpha\wedge 2)}$. For the integral over $k \in R$, we use the bounds on $1-\hat{D}(k)$ in (1.7). If $\alpha \ne 2$, then

$$\int\limits_R \frac{d^dk}{(2\pi)^d}\, e^{-n(1-\hat{D}(k))} \le c'\lambda \int\limits_0^\infty \frac{dr}{r}\, r^d e^{-cnr^{\alpha\wedge 2}} = \frac{c'\Gamma(\frac{d}{\alpha\wedge 2})\lambda}{(\alpha\wedge 2)(cn)^{d/(\alpha\wedge 2)}}, \tag{A.3}$$

for some $c, c' \in (0,\infty)$, where $r = \ell L|k|$. If $\alpha = 2$, then

$$\begin{aligned} \int\limits_R \frac{d^dk}{(2\pi)^d}\, e^{-n(1-\hat{D}(k))} &\le c'\lambda \int\limits_0^1 \frac{dr}{r}\, r^d e^{-cnr^2 \log\frac{\pi}{2r}} \\ &\le c'\lambda \int\limits_0^\infty \frac{dr}{r}\, r^d e^{-c''nr^2} = \frac{c'\Gamma(\frac{d}{2})\lambda}{2(c''n)^{d/2}}, \end{aligned} \tag{A.4}$$

where $c'' = c\log\frac{\pi}{2} > 0$. This completes the proof of the bound on $\|D^{\star n}\|_\infty$ in (1.6).

Next we prove the bounds on $1-\hat{D}(k)$ for $|k| \le (\ell L)^{-1}$ with $L \gg 1$. Since $\|k\|_\infty \le |k|$, this is sufficient for the proof of (1.7). First we note that, by the Riemann sum approximation,

$$\frac{1}{L^d} \sum_{x\in\mathbb{Z}^d} h(x/L) = \int\limits_{\mathbb{R}^d} d^dx\; h(x) + o(1) = 1 + o(1), \tag{A.5}$$

where $o(1) \to 0$ as $L \to \infty$. Therefore,

$$1-\hat{D}(k) = (1+o(1))\,(I_1+I_2+I_3), \tag{A.6}$$

where

$$I_1 = L^{-d} \sum_{x\in\mathbb{Z}^d:|x|<\ell L} h(x/L)\,(1-\cos(k\cdot x))\,, \tag{A.7}$$

$$I_2 = L^{-d} \sum_{\substack{x\in\mathbb{Z}^d:\\ \ell L\le|x|<\frac{\pi}{2|k|}}} h(x/L)\,(1-\cos(k\cdot x))\,, \tag{A.8}$$

$$I_3 = L^{-d} \sum_{x\in\mathbb{Z}^d:|x|\ge\frac{\pi}{2|k|}} h(x/L)\,(1-\cos(k\cdot x))\,. \tag{A.9}$$

However, by (1.2) and using $1-\cos(k\cdot x) \asymp |k|^2|x|^2$ if $|x| \le \frac{\pi}{2|k|}$ and $1-\cos(k\cdot x) \le 2$ otherwise, we obtain

$$I_1 \le O(L^{-d}|k|^2) \sum_{x\in\mathbb{Z}^d:|x|<\ell L} |x|^2 = O((L|k|)^2), \tag{A.10}$$

$$I_2 \asymp O(L^{\alpha}|k|^2) \sum_{\substack{x\in\mathbb{Z}^d:\\ \ell L\le|x|<\frac{\pi}{2|k|}}} |x|^{-d-\alpha+2} = \begin{cases} O((L|k|)^{\alpha\wedge 2}) & (\alpha\ne 2),\\ O\left((L|k|)^2\log\frac{\pi}{2\ell L|k|}\right) & (\alpha=2), \end{cases} \tag{A.11}$$

$$I_3 \le O(L^{\alpha}) \sum_{x\in\mathbb{Z}^d:|x|\ge\frac{\pi}{2|k|}} |x|^{-d-\alpha} = O((L|k|)^{\alpha}). \tag{A.12}$$

This completes the proof of (1.7). □

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