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Threshold $\theta \geq 2$ contact processes on homogeneous trees

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Abstract We study the threshold $\theta \geq 2$ contact process on a homogeneous tree $\mathbb{T}_b$ of degree $\kappa = b + 1$, with infection parameter $\lambda \geq 0$ and started from a product measure with density p. The corresponding mean-field model displays a discontinuous transition at a critical point $\lambda_c^{\mathrm{MF}}(\kappa, \theta)$ and for $\lambda \geq \lambda_c^{\mathrm{MF}}(\kappa, \theta)$ it survives iff $p \geq p_c^{\mathrm{MF}}(\kappa, \theta, \lambda)$, where this critical density satisfies $0 < p_c^{\mathrm{MF}}(\kappa, \theta, \lambda) < 1$, $\lim_{\lambda\to\infty} p_c^{\mathrm{MF}}(\kappa, \theta, \lambda) = 0$. For large b, we show that the process on $\mathbb{T}_b$ has a qualitatively similar behavior when λ is small, including the behavior at and close to the critical point $\lambda_c(\mathbb{T}_b, \theta)$. In contrast, for large λ the behavior of the process on $\mathbb{T}_b$ is qualitatively distinct from that of the mean-field model in that the critical density has $p_c(\mathbb{T}_b, \theta, \infty) := \lim_{\lambda\to\infty} p_c(\mathbb{T}_b, \theta, \lambda) > 0$. We also show that $\lim_{b\to\infty} b\lambda_c(\mathbb{T}_b, \theta) = \Phi_\theta$, where $1 < \Phi_2 < \Phi_3 < \cdots$, $\lim_{\theta\to\infty} \Phi_\theta = \infty$, and $0 < \liminf_{b\to\infty} b^{\theta/(\theta-1)} p_c(\mathbb{T}_b, \theta, \infty) \leq \limsup_{b\to\infty} b^{\theta/(\theta-1)} p_c(\mathbb{T}_b, \theta, \infty) < \infty$.

Keywords Threshold Contact process · Homogeneous trees · Critical points · Critical density · Phase diagram · Discontinuous transition

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1 Introduction and results

1.1 Preliminaries

Threshold contact processes form a natural class of interacting particle systems (see, e.g., [8] for background to the area). They are most naturally defined on a graph or oriented graph $G = (V, E)$. In both cases V is an arbitrary countable set, whose elements are called vertices or sites. When G is a graph, $E \subset \{\{v, u\} : v, u \in V\}$ is its set of edges, also called bonds. When G is an oriented graph, $E \subset V \times V$ is its set of oriented edges, also called oriented bonds. We denote the influence neighborhood of $v \in V$ in G by

$$\mathcal{N}_{G,v} = \begin{cases} \{u \in V : \{v, u\} \in E\}, & \text{if } G \text{ is a graph,} \\ \{u \in V : (v, u) \in E\}, & \text{if } G \text{ is an oriented graph.} \end{cases}$$

The degree of the site $v \in V$ is the cardinality of $\mathcal{N}_{G,v}$. The threshold θ contact process on G is now defined as the Markov process on $\{0, 1\}^V$ with flip rates at $v \in V$ at time $t \geq 0$ given by

- 1 flips to 0 at rate 1.
- 0 flips to 1 at rate λ in case there are at least θ sites of $\mathcal{N}_{G,v}$ in state 1 at time t, and at rate 0 otherwise.

The parameter $\lambda \geq 0$ is called the infection rate. The state of the process at each site at each time is called the spin at that site at that time. A spin 0 is interpreted as a vacant or healthy site, while a spin 1 is interpreted as an occupied or infected site. It is well known (see, e.g., Chapt. I of [8]) that such rates define a unique Markov process. Note that the flip rates above are attractive (see Chap. III of [8]), a property that has many consequences.

When $\theta = 1$, the threshold contact process is easier to analyze, among other reasons because it has an additive dual process. (In this dual process infected sites become healthy at rate 1, and they infect simultaneously all their neighbors at rate λ.) The behavior of the process is not expected then to be qualitatively different from that of the much studied (linear) contact process (see Chap. VI of [8] and Part I of [9]). For this reason we will focus in this paper on the cases $\theta \geq 2$, which are more challenging and do present a different qualitative behavior. As we will explain in the next subsection, this difference in behavior is indicated by the associated mean-field model. In the end of this introduction we will review some results about threshold $\theta \geq 2$ and related models, from [2,4–6,10].

For coupling purposes, it is convenient to construct the process using a system of Poisson marks. For this purpose, associate to each site in V two independent Poisson processes: one with rate 1, and one with rate λ. Mark the arrival times of the former with symbols D (for "down") and those of the latter with symbols U (for "up"). Make these Poisson processes independent from site to site. Use the marks now in the obvious way, to define the process: A spin 1 at site v flips to 0 when it encounters a D mark there; a spin 0 at site v flips to 1 when it encounters an U mark there and at least θ neighbors of v have spin 1 at that time. The probability space on which these Poisson processes are defined will be large enough to accommodate the process started from arbitrary initial configurations.

We will denote by $(\eta^{\mu}_{G,\theta,\lambda;t})_{t\geq 0}$ the process started from a random distribution picked according to law μ at time 0. When μ is product measure with density p we will use the notation $(\eta^{p}_{G,\theta,\lambda;t})$. When there is no risk of confusion, G, θ and λ may be omitted from the notation.

The point mass on the configuration with all sites in state $i \in \{0, 1\}$ will be denoted $\delta_{G,i}$. The distribution $\delta_{G,0}$ is trivially invariant for the threshold θ contact process, when $\theta > 0$. By attractivity, $\eta^{1}_{G,\theta,\lambda;t} \Rightarrow \nu_{G,\theta,\lambda}$, as $t \to \infty$, where $\Rightarrow$ denotes convergence in distribution, and $\nu_{G,\theta,\lambda}$ is called the upper invariant measure.

We say that the process started from the distribution μ dies out when $\eta^{\mu}_{G,\theta,\lambda;t} \Rightarrow \delta_{G,0}$, as $t \to \infty$. When this happens for every μ, we simply say that the process dies out. Attractivity implies that the process dies out precisely when $\nu_{G,\theta,\lambda} = \delta_{G,0}$. When the process does not die out, we will say that it survives.

For $v \in V$, set

$$\rho^{\mu}_{G,\theta,\lambda;t}(v) = \mathbb{P}\left(\eta^{\mu}_{G,\theta,\lambda;t}(v) = 1\right).$$

We will use for $\rho^{\mu}_{G,\theta,\lambda;t}(v)$ the same conventions on notation as for $\eta^{\mu}_{G,\theta,\lambda;t}$. Also, when G is such that $\rho^{\mu}_{G,\theta,\lambda;t}(v)$ does not depend on v, we will omit v from this notation. The critical point for the threshold θ contact process on G is defined by

$$\begin{aligned}\lambda_{\mathrm{c}}(G,\theta) &= \sup\{\lambda : \nu_{G,\theta,\lambda} = \delta_{G,0}\}\\ &= \sup\left\{\lambda : \text{for each } v \in V,\ \rho^{1}_{G,\theta,\lambda;t}(v) \to 0 \text{ as } t \to \infty\right\}.\end{aligned}$$

Clearly, the convergence of $\rho^{1}_{G,\theta,\lambda;t}(v)$ to 0 cannot be faster then exponential. Explicitly:

$$\rho^{1}_{G,\theta,\lambda;t}(v) \geq \mathbb{P}(\text{there is no } D \text{ mark at } v \text{ from time 0 to } t) \geq e^{-t}, \tag{1.1}$$

for $t \geq 0$. It is natural to define

$$\lambda_{\exp}(G,\theta) = \sup\left\{\lambda : \text{for each } v \in V,\ \rho^{1}_{G,\theta,\lambda;t}(v) \to 0 \text{ exponentially fast as } t \to \infty\right\}.$$

Obviously

$$\lambda_{\exp}(G,\theta) \leq \lambda_{\mathrm{c}}(G,\theta),$$

and it is interesting to decide when equality holds.

Even when the process survives, it may happen that for small $p > 0$, $\eta^{p}_{G,\theta,\lambda;t} \Rightarrow \delta_{G,0}$, as $t \to \infty$. By attractivity, if this happens for some value of p, it will also happen for smaller values of p. This leads to the following definitions:

$$\begin{aligned} p_{\mathrm{c}}(G,\theta,\lambda) &= \sup\left\{p\in[0,1] : \eta^p_{G,\theta,\lambda;t} \Rightarrow \delta_{G,0}, \text{ as} t\to\infty\right\} \\ &= \sup\left\{p\in[0,1] : \text{for each } v\in V,\ \rho^p_{G,\theta,\lambda;t}(v)\to 0 \text{ as } t\to\infty\right\}, \\ p_{\exp}(G,\theta,\lambda) &= \sup\Big\{p\in[0,1] : \text{for each } v\in V,\ \rho^p_{G,\theta,\lambda;t}(v) \\ &\qquad \to 0 \text{ exponentially fast as } t\to\infty\Big\}. \end{aligned}$$

As above, obviously

$$p_{\exp}(G,\theta,\lambda) \le p_{\mathrm{c}}(G,\theta,\lambda),$$

and again it is interesting to decide when equality holds.

When $(\eta^p_{G,\theta,\lambda;t})$ survives, it is natural to ask if it converges in distribution as $t\to\infty$. This is in general a difficult question, but the following provides a partial answer and, as a by-product, an estimate on $p_{\mathrm{c}}(G,\theta,\lambda)$.

Proposition 1 *For any G, θ and λ, for $p\ge\lambda/(\lambda+1)$,*

$$\eta^p_{G,\theta,\lambda;t} \Rightarrow \nu_{G,\theta,\lambda}, \quad \text{as } t\to\infty. \tag{1.2}$$

Therefore, if the process survives for a certain value of λ, then

$$p_c(G,\theta,\lambda) \le \frac{\lambda}{\lambda+1}. \tag{1.3}$$

Proof Let $\beta_{G,p}$ be the product measure with density p. For any $\theta\ge 0$, the threshold θ contact process is stochastically dominated by the threshold 0 contact process. This latter process is simply an independent flip process whose unique invariant distribution is $\beta_{G,\lambda/(\lambda+1)}$. Hence, for $p\ge\lambda/(\lambda+1)$,

$$\nu_{G,\theta,\lambda} \le \beta_{G,\lambda/(\lambda+1)} \le \beta_{G,p} \le \delta_{G,1}, \quad \text{stochastically.}$$

But as $t\to\infty$, we know that $\eta^{\nu_{G,\theta,\lambda}}_{G,\theta,\lambda;t} \Rightarrow \nu_{G,\theta,\lambda}$ and $\eta^1_{G,\theta,\lambda;t} \Rightarrow \nu_{G,\theta,\lambda}$. Therefore (1.2) also holds. □

Remark on case $\theta=1$: In that case, duality can easily be used to show that in Proposition 1, (1.2) holds for every $p>0$ and hence (1.3) can be replaced by $p_{\mathrm{c}}(G,\theta,\lambda)=0$.

1.2 The mean-field model

When G is regular of degree κ (i.e., each site has κ neighbors), it is natural to compare the evolution of the threshold contact process on G with a corresponding "mean-field" evolution. By this we mean the evolution of a deterministic density $(\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t})_{t\geq 0}$, which is governed by

$$\frac{d}{dt}\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t} = -\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t} + \lambda\left(1-\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t}\right)\mathrm{Bin}\left(\kappa, \rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t}, \theta\right), \tag{1.4}$$

with $\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;0} = p$, and where

$$\mathrm{Bin}(\kappa, x, \theta) = \sum_{i=\theta}^{\kappa}\binom{\kappa}{i}x^i(1-x)^{\kappa-i}$$

is the probability that a binomial random variable with κ attempts and probability of success x is larger than or equal to θ.

To explain the origin of this mean-field evolution and its relationship with the threshold contact process on G, we observe that from the definition of that process, for each $v \in V$,

$$\begin{aligned}\frac{d}{dt}\rho^{p}_{G,\theta,\lambda;t}(v) &= -\rho^{p}_{G,\theta,\lambda;t}(v)\\ &+ \lambda\mathbb{P}\left(\eta^{p}_{G,\theta,\lambda;t}(v) = 0,\ \#\left\{u \in \mathcal{N}_{G,v} : \eta^{p}_{G,\theta,\lambda;t}(u) = 1\right\} \geq \theta\right).\end{aligned}$$

One then obtains (1.4) if one pretends that the $\kappa + 1$ random variables $\eta^{p}_{G,\theta,\lambda;t}(v)$, $\eta^{p}_{G,\theta,\lambda;t}(u)$, $u \in \mathcal{N}_{G,v}$, are i.i.d., with common density $\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t}$.

Trivially, $\rho^{\mathrm{MF},0}_{\kappa,\theta,\lambda;t} = 0$, for all $t \geq 0$. When $p \in (0, 1]$, it clearly follows from (1.4) that $\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t} \geq pe^{-t} > 0$, for all $t \geq 0$. It will then be convenient to rewrite (1.4) as

$$\frac{d}{dt}\log\left(\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t}\right) = H\left(\kappa, \theta, \lambda; \rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t}\right), \tag{1.5}$$

where

$$H(\kappa, \theta, \lambda; x) = -1 + \lambda\frac{1-x}{x}\mathrm{Bin}(\kappa, x, \theta), \quad x \in (0, 1].$$

Note that when $\theta \geq 2$, then $0 \leq \mathrm{Bin}(\kappa, x, \theta) \leq (\kappa^2/2)x^2$, and hence

$$\lim_{x\searrow 0} H(\kappa, \theta, \lambda; x) = -1. \tag{1.6}$$

Also important are the elementary facts that $H(\kappa, \theta, \lambda; x)$ is continuous in x and in λ, $H(\kappa, \theta, \lambda; 1) = -1$, $H(\kappa, \theta, 0; x) = -1$, and, provided $\kappa \geq \theta$, for each $x \in (0, 1)$, $H(\kappa, \theta, \lambda; x)$ is strictly increasing in λ, with $\lim_{\lambda\to\infty} H(\kappa, \theta, \lambda; x) = \infty$.

These facts motivate the definition of the critical point

$$\lambda_c^{\mathrm{MF}}(\kappa, \theta) = \sup\left\{\lambda \geq 0 : \sup_{x\in(0,1]} H(\kappa, \theta, \lambda; x) < 0\right\},$$

and the critical density

$$p_c^{\mathrm{MF}}(\kappa, \theta, \lambda) = \inf\{x \in (0, 1] : H(\kappa, \theta, \lambda; x) \geq 0\}.$$

Note that the facts above imply that, when $\kappa \geq \theta \geq 2$,

$$0 < \lambda_c^{\mathrm{MF}}(\kappa, \theta) < \infty,$$

and

$$0 < p_c^{\mathrm{MF}}(\kappa, \theta, \lambda) < 1 \quad \text{for } \lambda \geq \lambda_c^{\mathrm{MF}}(\kappa, \theta).$$

Moreover $p_c^{\mathrm{MF}}(\kappa, \theta, \lambda)$ is strictly decreasing in $\lambda \geq \lambda_c^{\mathrm{MF}}(\kappa, \theta)$, with

$$\lim_{\lambda\to\infty} p_c^{\mathrm{MF}}(\kappa, \theta, \lambda) = 0. \tag{1.7}$$

Define

$$\mathcal{D}^{\mathrm{MF}}(\kappa, \theta) = \left\{(\lambda, p) \in [0, \infty] \times [0, 1] : \lambda < \lambda_c^{\mathrm{MF}}(\kappa, \theta) \text{ or } p < p_c^{\mathrm{MF}}(\kappa, \theta, \lambda)\right\}.$$

Proposition 2 *For every $\kappa \geq \theta \geq 2$, the following dichotomy holds.*
In case $(\lambda, p) \in (\mathcal{D}^{MF}(\kappa, \theta))^c$,

$$\rho^{MF,p}_{\kappa,\theta,\lambda;t} \geq p_c^{MF}(\kappa, \theta, \lambda) > 0, \quad \textit{for all } t \geq 0. \tag{1.8}$$

In case $(\lambda, p) \in \mathcal{D}^{MF}(\kappa, \theta)$, then for some $C \in (0, \infty)$,

$$pe^{-t} \leq \rho^{MF,p}_{\kappa,\theta,\lambda;t} \leq Ce^{-t}, \quad \textit{for all } t \geq 0. \tag{1.9}$$

In particular, a discontinuous transition happens at $\lambda_c^{\mathrm{MF}}(\kappa, \theta)$.

Proof The only statement that requires explanation is the upper bound in (1.9). To prove it, note first that (1.5) implies that when $(\lambda, p) \in \mathcal{D}^{\mathrm{MF}}(\kappa, \theta)$ we have $\lim_{t\to\infty} \rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t} = 0$. Using (1.5) a second time now, this time in combination with (1.6), shows that for any $\epsilon > 0$, there is $C' \in (0, \infty)$ such that

$$\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t} \leq C' e^{-(1-\epsilon)t}, \quad \text{for all } t \geq 0.$$

Using this estimate in combination with (1.4) now yields, since $\theta \geq 2$,

$$\frac{d}{dt}\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t} \leq -\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t} + \lambda\frac{\kappa^2}{2}(C')^2 e^{-2(1-\epsilon)t}.$$

Multiplying by e^t, we obtain

$$\frac{d}{dt}\left(e^t\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;t}\right) \leq \lambda\frac{\kappa^2}{2}(C')^2 e^{-(1-2\epsilon)t}.$$

Supposing $\epsilon < 1/2$, integration in t from 0 to s yields

$$\begin{aligned} e^s\rho^{\mathrm{MF},p}_{\kappa,\theta,\lambda;s} - p &\leq \int_0^s \lambda\frac{\kappa^2}{2}(C')^2 e^{-(1-2\epsilon)t}\,dt \\ &\leq \int_0^\infty \lambda\frac{\kappa^2}{2}(C')^2 e^{-(1-2\epsilon)t}\,dt = C'' < \infty. \end{aligned}$$

(1.9) follows, with $C = C'' + p$. ◻

It is easy to see that $\lambda_c^{\mathrm{MF}}(\kappa,\theta) \to 0$, as $\kappa \to \infty$. The rate at which this convergence occurs is also easily identified from standard facts about convergence of binomial distributions to Poisson distributions. For this purpose extend the definition of $\mathrm{Bin}(\kappa, x, \theta)$ to be 1 when $x > 1$, and observe that straightforward computations then yield

$$\mathrm{Bin}(\kappa,\gamma/\kappa,\theta) \to \mathrm{Poisson}(\gamma,\theta) := \sum_{i\geq\theta} e^{-\gamma}\frac{\gamma^i}{i!}, \quad \text{as } \kappa \to \infty, \tag{1.10}$$

uniformly in $\gamma > 0$. The corresponding limit for the function $H(\kappa,\theta,\lambda;x)$ is

$$H(\kappa,\theta,\phi/\kappa;\gamma/\kappa) \to -1 + \phi\frac{\mathrm{Poisson}(\gamma,\theta)}{\gamma}, \quad \text{as } \kappa \to \infty,$$

uniformly in $\gamma > 0$, for each $\phi > 0$. From this it is easy to derive

$$\lim_{\kappa\to\infty} \kappa\lambda_c^{\mathrm{MF}}(\kappa,\theta) = \Phi_\theta := \inf_{\gamma>0}\frac{\gamma}{\mathrm{Poisson}(\gamma,\theta)}. \tag{1.11}$$

The constants Φ_θ can easily be shown to satisfy

$$1 < \Phi_2 < \Phi_3 < \cdots \quad \text{and} \quad \lim_{\theta\to\infty}\Phi_\theta = \infty.$$

Remark on case $\theta = 1$: In this case, in contrast to (1.6), we have $\lim_{x\searrow 0} H(\kappa,\theta,\lambda;x) = -1 + \lambda\kappa$. It is an instructive exercise to use this fact and the bound $H(\kappa,\theta,\lambda;x) <$

$-1 + \lambda\kappa$, for $x > 0$, to analyze the behavior of the mean-field model in this case. In contrast to Proposition 2, one finds a continuous transition at $\lambda_c^{\mathrm{MF}}(\kappa, 1) = 1/\kappa$, with $p_c^{\mathrm{MF}}(\kappa, 1, \lambda) = 0$ for all $\lambda > \lambda_c^{\mathrm{MF}}(\kappa, 1)$. The analogue of (1.11) is also true, with $\Phi_1 = 1$.

1.3 Results for the process on homogeneous trees

We will study threshold contact processes on the homogeneous tree $\mathbb{T}_b$, of degree $b + 1$.

Our first result provides conditions for survival of the process on $\mathbb{T}_b$ based on the survival of the mean-field model with $\kappa = b$ (note: not $\kappa = b + 1$) and $\lambda/(\lambda + 1)$ in place of λ.

Theorem 1 *If $b \geq 2$ and $\theta \geq 2$ are such that $\lambda_c^{MF}(b, \theta) < 1$, then*

$$\lambda_c(\mathbb{T}_b, \theta) \leq \frac{\lambda_c^{MF}(b, \theta)}{1 - \lambda_c^{MF}(b, \theta)} < \infty, \tag{1.12}$$

and for $\lambda > \lambda_c^{MF}(b, \theta)/(1 - \lambda_c^{MF}(b, \theta))$,

$$p_c(\mathbb{T}_b, \theta, \lambda) \leq p_c^{MF}(b, \theta, \lambda/(\lambda + 1)) < 1. \tag{1.13}$$

Since we know that $\lim_{b\to\infty} \lambda_c^{\mathrm{MF}}(b, \theta) = 0$, the hypothesis of Theorem 1 are satisfied when b is large. Moreover, combining this theorem with (1.11), we learn that

$$\limsup_{b\to\infty} b\lambda_c(\mathbb{T}_b, \theta) \leq \Phi_\theta. \tag{1.14}$$

This result will be sharpened in Theorem 3 below. In our approach to the proof of Theorem 3, we will prove first the somewhat technical Theorem 2 below. Note that thanks to (1.14), this theorem covers the behavior near $\lambda_c(\mathbb{T}_b, \theta)$, when b is large. This theorem should be compared to Proposition 2.

Theorem 2 *For each $\theta \geq 2$ and $A \in (0, \infty)$, there are $b_0, \delta \in (0, \infty)$ such that if $b \geq b_0$ and $\lambda \leq A/b$, then for every $p \in [0, 1]$ the following dichotomy holds. Either*

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \rho^p_{\mathbb{T}_b,\theta,\lambda;t}\, dt \geq \frac{\delta}{b}, \tag{1.15}$$

or, for some $C \in (0, \infty)$,

$$\rho^p_{\mathbb{T}_b,\theta,\lambda;t} \leq Ce^{-t}, \quad \textit{for all } t \geq 0. \tag{1.16}$$

Moreover, for $b \geq b_0$ the set

$$\mathcal{D}_A(\mathbb{T}_b, \theta) = \{(\lambda, p) \in [0, A/b] \times [0, 1] : \textit{alternative } (1.16) \textit{ holds}\}$$

is an open subset of $[0, A/b] \times [0, 1]$ *in the relative topology induced by the Euclidean topology of* $\mathbb{R}^2$.

Note that alternative (1.15) implies that the process survives when it starts with density p. For $p = 1$, (1.15) is equivalent to each one of the statements

$$\rho^1_{\mathbb{T}_b,\theta,\lambda;t} \geq \frac{\delta}{b}, \qquad \text{for all } t \geq 0,$$

and

$$\rho^1_{\mathbb{T}_b,\theta,\lambda;\infty} := \lim_{t\to\infty} \rho^1_{\mathbb{T}_b,\theta,\lambda;t} \geq \frac{\delta}{b}.$$

The following is immediate from Theorem 2, (1.14) and (1.3).

Corollary 1 *For each* $\theta \geq 2$, *the following statements hold when b is large.*

$$0 < \lambda_{\exp}(\mathbb{T}_b, \theta) = \lambda_c(\mathbb{T}_b, \theta) < \infty,$$

and the process survives at this critical point. Moreover for $\lambda \geq \lambda_c(\mathbb{T}_b, \theta)$ *close to this critical point,*

$$0 < p_{\exp}(\mathbb{T}_b, \theta, \lambda) = p_c(\mathbb{T}_b, \theta, \lambda) < 1,$$

and the process started from this critical density $p_c(\mathbb{T}_b, \theta, \lambda)$ *survives.*

Note that in particular, under the conditions in Corollary 1,

$$p_c(\mathbb{T}_b, \theta, \lambda_c(\mathbb{T}_b, \theta)) < 1.$$

Theorem 2 and Corollary 1 show qualitative similarities between the behavior of the threshold contact process on $\mathbb{T}_b$ and the corresponding mean-field model, when b is large and λ is small. The next theorem shows a related quantitative similarity.

Theorem 3 *For each* $\theta \geq 2$,

$$\lim_{b\to\infty} b\lambda_{\exp}(\mathbb{T}_b, \theta) = \lim_{b\to\infty} b\lambda_c(\mathbb{T}_b, \theta) = \Phi_\theta. \tag{1.17}$$

Remark on case $\theta = 1$: In that case, duality can easily be used to show that (1.17) also holds, with $\Phi_1 = 1$.

In contrast to the results above, for large values of λ, the process on $\mathbb{T}_b$ and the mean-field model behave differently, as the comparison between the following theorem and (1.7) shows. Set

$$p_{\exp}(\mathbb{T}_b, \theta, \infty) = \lim_{\lambda\to\infty} p_{\exp}(\mathbb{T}_b, \theta, \lambda), \quad p_c(\mathbb{T}_b, \theta, \infty) = \lim_{\lambda\to\infty} p_c(\mathbb{T}_b, \theta, \lambda).$$

Theorem 4 *For each $b \geq 2$ and $\theta \geq 2$,*

$$p_c(\mathbb{T}_b, \theta, \infty) \geq p_{\exp}(\mathbb{T}_b, \theta, \infty) > 0.$$

We do not know if the critical densities $p_c(\mathbb{T}_b, \theta, \infty)$ and $p_{\exp}(\mathbb{T}_b, \theta, \infty)$ are identical to each other, but the next theorem shows that at least they display similar behavior as $b \to \infty$.

Theorem 5 *For each $\theta \geq 2$,*

$$0 < \liminf_{b\to\infty} b^{\theta/(\theta-1)} p_{\exp}(\mathbb{T}_b, \theta, \infty) \leq \limsup_{b\to\infty} b^{\theta/(\theta-1)} p_c(\mathbb{T}_b, \theta, \infty) < \infty. \tag{1.18}$$

1.4 Results for the process on oriented homogeneous trees

As a tool, for comparison purposes and for its own sake, we will also study the threshold contact process on an oriented graph $\vec{\mathbb{T}}_b$, obtained from $\mathbb{T}_b$ in a fashion described next.

First we introduce some notation. We embed a copy of $\mathbb{Z}$ in $\mathbb{T}_b$ and use the notation $\mathcal{L} = \mathcal{L}_0$ to denote the set of vertices of $\mathbb{T}_b$ covered by this embedding. Abusing notation, we will denote the elements of $\mathcal{L}$ by the names of the elements of $\mathbb{Z}$ that they represent in this embedding. We will also refer to site 0 as the root of $\mathbb{T}_b$. Define, inductively in $n \geq 1$, $\mathcal{L}_n$ as the set of vertices which are neighbors to some vertex in $\mathcal{L}_{n-1}$ and are not in $\cup_{i=0}^{n-1} \mathcal{L}_i$.

With this notation, include (v, u) in the set of oriented edges of $\vec{\mathbb{T}}_b$ if $v, u \in \mathcal{L}$ and $u = v + 1$, or if $v \in \mathcal{L}_{n-1}$ and $u \in \mathcal{L}_n$, for some $n \geq 1$.

Theorems 1 to 5 have analogues for the threshold θ contact process on $\vec{\mathbb{T}}_b$. Theorem 1 for $\mathbb{T}_b$ is actually a corollary to the same statement for $\vec{\mathbb{T}}_b$. Theorem 2 and Corollary 1 admit much stronger versions for $\vec{\mathbb{T}}_b$; those are stated as Theorem 6 and Corollary 2 below. The analogues of Theorems 3, 4 and 5 for $\vec{\mathbb{T}}_b$ are also true and can either be obtained from the corresponding results for $\mathbb{T}_b$, or can more easily be proved directly.

Theorem 6 *For each $\theta \geq 2$ and $b \geq 2$, the following dichotomy holds for every $\lambda \geq 0$ and $p \in [0, 1]$. Either*

$$\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} \geq p_c^{MF}(b, \theta, \lambda), \quad \textit{for all } t \geq 0, \tag{1.19}$$

or, for some $C \in (0, \infty)$,

$$\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} \leq Ce^{-t}, \quad \textit{for all } t \geq 0. \tag{1.20}$$

Moreover, for every $\theta \geq 2$ and $b \geq 2$, the set

$$\mathcal{D}(\vec{\mathbb{T}}_b, \theta) = \{(\lambda, p) \in [0, \infty) \times [0, 1] : \textit{alternative } (1.20) \textit{ holds}\}$$

is an open subset of $[0, \infty) \times [0, 1]$ *in the relative topology induced by the Euclidean topology of* $\mathbb{R}^2$.

Corollary 2 *For each* $\theta \geq 2$ *and* $b \geq 2$ *for which* $\lambda_{\exp}(\vec{\mathbb{T}}_b, \theta) < \infty$*, the following statements hold.*

$$\lambda_c^{MF}(b, \theta) \leq \lambda_{\exp}(\vec{\mathbb{T}}_b, \theta) = \lambda_c(\vec{\mathbb{T}}_b, \theta),$$

and the process survives at this critical point. Moreover for $\lambda \geq \lambda_c(\vec{\mathbb{T}}_b, \theta)$*,*

$$p_c^{MF}(b, \theta, \lambda) \leq p_{\exp}(\vec{\mathbb{T}}_b, \theta, \lambda) = p_c(\vec{\mathbb{T}}_b, \theta, \lambda) < 1,$$

and the process started from this critical density $p_c(\vec{\mathbb{T}}_b, \theta, \lambda)$ *survives.*

We do not know if the first inequality in each display in Corollary 2 holds for the threshold contact process on $\mathbb{T}_b$. But it is known that the second of these does not hold for the process on $\mathbb{Z}^d$, $d \geq 3$, with $\theta = 2$, as reviewed in the next subsection.

1.5 Related previous results

Threshold contact processes with $\theta = 2$ and closely related models have been studied in [2,4–6,10], sometimes under the name "sexual contact process", and mostly on $\mathbb{Z}^d$. In [10] discrete time versions were studied, and contour arguments were used to show survival. In [6] these contour methods were adapted to continuous time; their main result can be stated as follows using our terminology. Let $\vec{\mathbb{Z}}^2$ be the oriented graph obtained from $\mathbb{Z}^2$ by setting $\mathcal{N}_{\vec{\mathbb{Z}}^2, v} = \{v + (1, 0), v + (0, 1)\}$. It is proved in [6] that $\lambda_c(\vec{\mathbb{Z}}^2, 2) < \infty$. Note that, by stochastic domination, this implies that $\lambda_c(\mathbb{Z}^d, 2) < \infty$, for $d \geq 2$. In [2] a renormalization procedure was introduced, which can replace the contour methods in proving survival. In [4] and [5] continuous time models which can be seen as modified threshold $\theta = 2$ contact processes on $\mathbb{Z}^d$ were studied. In one of these modified models the flip rates at $v \in \mathbb{Z}^d$ at time $t \geq 0$ are given by

- 1 flips to 0 at rate 1.
- 0 flips to 1 at rate λ in case there are at least 2 sites of $\mathcal{N}_{\mathbb{Z}^d, v}$ that are separated from each other by Euclidean distance $\sqrt{2}$ and are in state 1 at time t.

The most important result from [4,5] in connection to the current paper is the fact that for this modified model in $d \geq 3$, when λ is large, survival occurs starting from any positive density p. Since the threshold 2 contact process dominates that modified model, we learn that when $d \geq 3$ and λ is large, $p_c(\mathbb{Z}^d, 2, \lambda) = 0$. This means that the qualitative behavior of the $\theta = 2$ threshold contact process on $\mathbb{Z}^d$, $d \geq 3$, deviates from that of the corresponding mean-field models (for which $p_c^{MF}(2d, 2, \lambda) > 0$, for all $\lambda > 0$), but this deviation is in the "opposite direction" of the deviation observed in the corresponding models on homogeneous trees (for which, contrary to the mean-field model, $\lim_{\lambda \to \infty} p_c(\mathbb{T}_{2d-1}, 2, \lambda) > 0$).

The only results that we are aware of for threshold $\theta \geq 2$ contact processes on trees are the following ones, from [6]. There the authors consider the model with $\theta = 2$

on $\vec{\mathbb{T}}_2$. They state that the contour methods used in that paper can be used to prove that this process survives when λ is large. They then prove, using this result, that the transition at $\lambda_c(\vec{\mathbb{T}}_2, 2)$ is discontinuous (our Corollary 2 extends this result).

1.6 Organization of the paper

In Sect. 2 we prove the results about the threshold contact process on $\vec{\mathbb{T}}_b$ (obtaining Theorem 1 as a corollary). In Sect. 3 we prove the results about the threshold contact process on $\mathbb{T}_b$ when b is large and λ is small, namely, Theorems 2 and 3. In Sect, 4 we prove the results about the threshold contact process on $\mathbb{T}_b$ when λ is large, namely, Theorems 4 and 5; for this purpose the bootstrap percolation model will be introduced as a tool.

2 Comparison between the model on $\vec{\mathbb{T}}_b$ and the mean-field model

In this section we will prove the analogue of Theorem 1 for $\vec{\mathbb{T}}_b$ and Theorem 6. Both theorems result from a fairly direct comparison with the mean-field model. This comparison is based on writing down, for an arbitrary site v, the differential equation

$$\begin{aligned}\frac{d}{dt}\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} &= -\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}\\ &\quad + \lambda\mathbb{P}\left(\eta^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}(v) = 0, \#\left\{u \in \mathcal{N}_{\vec{\mathbb{T}}_b,v} : \eta^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}(u) = 1\right\} \geq \theta\right),\end{aligned} \tag{2.1}$$

and noticing that $\vec{\mathbb{T}}_b$ has the special property that the random variables $\eta^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}(u)$, $u \in \mathcal{N}_{\vec{\mathbb{T}}_b,v}$, are independent and have the same distribution as $\eta^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}(v)$. If the random variable $\eta^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}(v)$ were also independent of those, (2.1) would reduce to the mean-field equation (1.4), but this independence does not hold. In each one of the two proofs below we deal with this lack of independence in a different way.

Proof of Theorem 1 and its analogue for $\vec{\mathbb{T}}_b$ We will prove that for arbitrary $b \geq 2$ and $\theta \geq 2$, if $\lambda_c^{\mathrm{MF}}(b,\theta) < 1$, and $\lambda > \lambda_c^{\mathrm{MF}}(b,\theta)/(1-\lambda_c^{\mathrm{MF}}(b,\theta))$, then

$$p_c(\vec{\mathbb{T}}_b, \theta, \lambda) \leq p_c^{\mathrm{MF}}(b, \theta, \lambda/(\lambda+1)) < 1. \tag{2.2}$$

This suffices, since it obviously implies

$$\lambda_c(\vec{\mathbb{T}}_b, \theta) \leq \frac{\lambda_c^{\mathrm{MF}}(b,\theta)}{1-\lambda_c^{\mathrm{MF}}(b,\theta)} < \infty, \tag{2.3}$$

and (1.12) and (1.13) follow respectively from (2.3) and (2.2), since the threshold θ contact process on $\mathbb{T}_b$ stochastically dominates the threshold θ contact process on $\vec{\mathbb{T}}_b$.

Only the first inequality in (2.2) needs to be proved. For this purpose consider the arbitrary site v that appears in (2.1) and define the following event E: $\eta^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;0}(v) = 0$, and either there is no U nor D mark at v between times 0 and t, or else, the last such mark is a D mark. By using the time reversibility of Poisson processes, a standard computation gives

$$\mathbb{P}(E) \geq \frac{1}{\lambda+1}(1-p). \tag{2.4}$$

The event E is clearly independent of the random variables $\eta^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t}(u)$, $u \in \mathcal{N}_{\vec{\mathbb{T}}_{b},v}$. Since also $E \subset \{\eta^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t}(v) = 0\}$, it follows from (2.1), (2.4) and the observation after (2.1) that

$$\begin{aligned} \frac{d}{dt}\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} &\geq -\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} + \lambda\mathbb{P}\left(E, \#\left\{u \in \mathcal{N}_{\vec{\mathbb{T}}_{b},v} : \eta^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t}(u) = 1\right\} \geq \theta\right) \\ &= -\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} + \lambda\mathbb{P}\left(E\right)\mathbb{P}\left(\#\left\{u \in \mathcal{N}_{\vec{\mathbb{T}}_{b},v} : \eta^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t}(u) = 1\right\} \geq \theta\right) \\ &\geq -\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} + \frac{\lambda}{\lambda+1}(1-p)\mathrm{Bin}\left(b, \rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t}, \theta\right). \end{aligned} \tag{2.5}$$

It is convenient to rewrite (2.5) as

$$\frac{d}{dt}\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} \geq L\left(b, \theta, \lambda, p; \rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t}\right), \tag{2.6}$$

where

$$L(b, \theta, \lambda, p; x) = xH(b, \theta, \lambda/(\lambda+1); x) + \frac{\lambda}{\lambda+1}\mathrm{Bin}(b, x, \theta)(x-p), \quad x \in [0, 1].$$

When $\lambda_{\mathrm{c}}^{\mathrm{MF}}(b,\theta) < 1$ and $\lambda > \lambda_{\mathrm{c}}^{\mathrm{MF}}(b,\theta)/(1-\lambda_{\mathrm{c}}^{\mathrm{MF}}(b,\theta))$, then $\lambda/(\lambda+1) > \lambda_{\mathrm{c}}^{\mathrm{MF}}(b,\theta)$. So $p_{\mathrm{c}}^{\mathrm{MF}}(b,\theta,\lambda/(\lambda+1)) < 1$, and for $p > p_{\mathrm{c}}^{\mathrm{MF}}(b,\theta,\lambda/(\lambda+1))$ arbitrarily close to $p_{\mathrm{c}}^{\mathrm{MF}}(b,\theta,\lambda/(\lambda+1))$ we have $H(b,\theta,\lambda/(\lambda+1); p) > 0$. It follows then that also $L(b,\theta,\lambda,p;p) > 0$. We claim that

$$\inf_{t\geq 0}\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} \geq p > 0. \tag{2.7}$$

Indeed, set

$$t_p = \inf\left\{t \geq 0 : \rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} < p\right\}.$$

If (2.7) were false, we would have $t_p < \infty$. Then by the continuity of $\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t}$ in t, we would have $\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t_p} = p$, and $d/dt\rho^{p}_{\vec{\mathbb{T}}_{b},\theta,\lambda;t} \leq 0$, at $t = t_p$. But (2.6) implies

$d/dt\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} \geq L(b,\theta,\lambda,p;p) > 0$, at $t = t_p$. This contradiction proves (2.7), which implies

$$p_c(\vec{\mathbb{T}}_b,\theta,\lambda) \leq p.$$

Since p can be taken arbitrarily close to $p_c^{\mathrm{MF}}(b,\theta,\lambda/(\lambda+1))$, the proof of (2.2) is complete. □

Proof of Theorem 6 Applying Harris' inequality to (2.1), we obtain

$$\frac{d}{dt}\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} \leq -\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} + \lambda\mathbb{P}\left(\eta^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}(v)=0\right) \times \mathbb{P}\left(\#\left\{u\in\mathcal{N}_v : \eta^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}(u)=1\right\}\geq\theta\right).$$

From the observation after (2.1), now

$$\frac{d}{dt}\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} \leq -\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} + \lambda\left(1-\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}\right)\mathrm{Bin}\left(b,\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t},\theta\right), \tag{2.8}$$

which for $p > 0$ is equivalent to

$$\frac{d}{dt}\log\left(\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}\right) \leq H\left(b,\theta,\lambda;\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t}\right). \tag{2.9}$$

If $p = 0$, then (1.20) holds. Suppose that $p > 0$ and (1.19) fails. Then (2.9) implies that $\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} \to 0$ as $t\to\infty$. The proof that (1.20) hold then can be completed as the proof of Proposition 2. This shows that for each λ and p either (1.19) or (1.20) hold.

The statement about the set $\mathcal{D}(\vec{\mathbb{T}}_b,\theta)$ follows now from the fact that the negation of (1.19) is a "finite-time condition":

$$\rho^p_{\vec{\mathbb{T}}_b,\theta,\lambda;t} < p_c^{\mathrm{MF}}(b,\theta,\lambda), \quad \text{for some } t\geq 0. \tag{2.10}$$

If (2.10) holds for some (λ, p), then, by continuity, it also holds close to this point, with the same t. □

3 The regime of small λ

In this section we will prove Theorems 2 and 3. We will abbreviate the notation, omitting $\mathbb{T}_b$, θ and λ for instance in:

$$\eta^p_{\mathbb{T}_b,\theta,\lambda;t} = \eta^p_t, \quad \rho^p_{\mathbb{T}_b,\theta,\lambda;t} = \rho^p_t, \quad \mathcal{N}_{\mathbb{T}_b,v} = \mathcal{N}_v.$$

We will compare the threshold contact process on $\mathbb{T}_b$ with the similar process in which the spin of one of the neighbors of the root is frozen in the state 1. Recall the definition of $\mathcal{L}$ from Sect. 1.4, and the corresponding terminology and notation. In our modified process the flip rates are as in the threshold θ contact process on $\mathbb{T}_b$, except for the site -1, where the spin is kept frozen in the state 1. We will start this modified process with each site other than site -1 taking independently the value +1 with probability p, and being in state 0, otherwise. The notation $\eta_t^{*,p}$ will denote this process. Define also

$$\sigma_t^{l,p} = \mathbb{P}(\eta_t^{*,p}(l) = 1), \quad l \in \mathbb{Z},$$

and abbreviate $\sigma_t^p = \sigma_t^{0,p}$.

For comparison, we will also consider the trivial threshold 0 contact process on $\mathbb{T}_b$, i.e., the process in which the spin of each site flips independently of anything else, with 0 flipping to 1 at rate λ, and 1 flipping to 0 at rate 1. Let π_t^p be the probability that in this process a given site is in state 1 at time t, when at time 0 this probability is set to $\pi_0^p = p$. It is elementary that for every $p \in [0,1]$, $\pi_t^p \le \pi_t^1 \searrow \lambda/(\lambda+1)$ as $t \to \infty$. In particular, there is $\tilde{t}(\lambda)$ such that

$$\pi_t^p \le \lambda, \quad \text{for } t \ge \tilde{t}(\lambda). \tag{3.1}$$

By attractivity, for any $p \in [0,1]$, $t \ge 0$ and $0 \le l_1 \le l_2$,

$$\rho_t^p \le \sigma_t^{l_2,p} \le \sigma_t^{l_1,p} \le \pi_t^p. \tag{3.2}$$

Lemma 1 *For arbitrary $b \ge 2$, $\theta \ge 2$, $p \in [0,1]$ and $t \ge 0$,*

$$\frac{d}{dt}\sigma_t^{l,p} \le -\sigma_t^{l,p} + \lambda b \sigma_t^{l+1,p}, \quad l \ge 0. \tag{3.3}$$

And

$$\frac{d}{dt}\sigma_t^{l,p} \le -\sigma_t^{l,p} + \lambda\left\{\pi_t^p b \sigma_t^p + Bin(b, \sigma_t^p, \theta)\right\}, \quad l \ge 1. \tag{3.4}$$

Proof We will use the following terminology. For each site v of $\mathbb{T}_b$, the b sites in $\mathcal{N}_{\vec{\mathbb{T}}_b,v}$ will be called forward neighbors of v and the single site in $\mathcal{N}_{\mathbb{T}_b,v} \backslash \mathcal{N}_{\vec{\mathbb{T}}_b,v}$ will be called the backward neighbor of v.

From the definition of $(\eta_t^{*,p})$ and $\sigma_t^{l,p}$,

$$\begin{aligned}\frac{d}{dt}\sigma_t^{l,p} &= -\sigma_t^{l,p} + \lambda\mathbb{P}\left(\eta_t^{*,p}(l) = 0, \#\left\{u \in \mathcal{N}_l : \eta_t^{*,p}(u) = 1\right\} \ge \theta\right) \\ &\le -\sigma_t^{l,p} + \lambda\mathbb{P}\left(\#\left\{u \in \mathcal{N}_l : \eta_t^{*,p}(u) = 1\right\} \ge \theta\right).\end{aligned} \tag{3.5}$$

Inequality (3.3) follows from (3.5) and the observation that, since $\theta \ge 2$, for the site l to have at least θ occupied neighbors, it must have at least one occupied forward neighbor.

To derive (3.4) from (3.5), we compare the process $\eta_t^{*,p}$ with a further modified process in which the spins at the sites -1 and l are both frozen in the state 1, while the spins at other sites evolve as in the threshold θ contact process. In this modified process, the spins at the neighbors of l evolve independently of each other. Note that by attractivity, the distribution of $(\eta_t^{*,p}(u))_{u\in\mathcal{N}_l}$ is therefore stochastically dominated by a product measure in which the forward neighbors of l have probability σ_t^p of being occupied, while the backward neighbor of l has probability π_t^p of being occupied. The probability in the r.h.s. of (3.5) is now estimated from above by the probability that either the backward neighbor of l and at least one of its forward neighbors are both occupied (recall $\theta \geq 2$), or else that at least θ of its forward neighbors are occupied. □

Lemma 2 *For arbitrary $\theta \geq 2$ and $A \in (0,\infty)$, there are $b^*, \delta^*, t^* \in (0,\infty)$, such that if $b \geq b^*$ and $\lambda \leq A/b$, then for every $p \in [0,1]$ the following dichotomy holds. Either*

$$\sigma_t^p \geq \frac{\delta^*}{b}, \quad \textit{for all } t \geq t^*, \tag{3.6}$$

or, for some $C \in (0,\infty)$,

$$\sigma_t^p \leq Ce^{-0.6t}, \quad \textit{for all } t \geq 0. \tag{3.7}$$

Remark The exponential rate 0.6 in (3.7), could be replaced with any rate smaller than 1, with minor modifications in the proof and a larger value for b^*. In our proof of Theorem 2, all that we will need about this rate is that it is larger than 1/2.

Proof For later convenience, we take $b^* = 9A^3$. We will use Lemma 1, and for this purpose we need to estimate π_t^p. Under the assumptions in the lemma that we are proving, we have $\lambda \leq A/b \leq A/b^* = 1/(9A^2) =: \hat{\lambda}$. Let $(\hat{\pi}_t^p)_{t\geq 0}$ be defined in the same way as $(\pi_t^p)_{t\geq 0}$, but with $\hat{\lambda}$ replacing λ. Clearly $\pi_t^p \leq \hat{\pi}_t^p$. So, by (3.1), there is $t^* = \tilde{t}(\hat{\lambda})$ which depends on A, but not on b or λ (once they satisfy the conditions in the lemma), such that $\pi_t^p \leq \hat{\lambda} = 1/(9A^2)$, for every $p \in [0,1]$ and $t \geq t^*$.

We use now the two inequalities in Lemma 1. The first one with $l = 0$ and the second one with $l = 1$. We suppose that $t \geq t^*$, so that we can use the estimate above on π_t^p. Since also $\lambda \leq A/b$, these inequalities read then

$$\begin{aligned}
\frac{d}{dt}\sigma_t^p &\leq -\sigma_t^p + A\sigma_t^{1,p}, \\
\frac{d}{dt}\sigma_t^{1,p} &\leq -\sigma_t^{1,p} + \frac{1}{9A}\sigma_t^p + \frac{A}{b}\mathrm{Bin}(b, \sigma_t^p, \theta).
\end{aligned}$$

Multiply the first of these inequalities by $1/(3\sqrt{A})$ and the second one by $\sqrt{A}$, and add the resulting inequalities to obtain

$$\frac{d}{dt}x_t \leq -\frac{2}{3}x_t + \frac{A^{3/2}}{b}\mathrm{Bin}(b, \sigma_t^p, \theta), \quad t \geq t^*, \tag{3.8}$$

where

$$x_t = \frac{1}{3\sqrt{A}}\sigma_t^p + \sqrt{A}\sigma_t^{1,p}.$$

By (3.2) $\sigma_t^{1,p} \leq \sigma_t^p$. Therefore we obtain the following comparison between x_t and σ_t^p:

$$\frac{1}{3\sqrt{A}}\sigma_t^p \leq x_t \leq \left\{\frac{1}{3\sqrt{A}} + \sqrt{A}\right\}\sigma_t^p \tag{3.9}$$

From (3.8), the fact that $\mathrm{Bin}(b, \sigma_t^p, \theta) \leq b^2(\sigma_t^p)^2/2$ (since $\theta \geq 2$) and the first inequality in (3.9),

$$\frac{d}{dt}x_t \leq -\frac{2}{3}x_t + \frac{3}{2}A^{5/2}b(x_t)^2 =: G(x_t), \quad t \geq t^*.$$

Note that $G(x) < 0$ for $0 < x < 4/(9A^{5/2}b)$, and $\lim_{x \searrow 0} G(x)/x = -2/3 < -0.6$. One can use these facts, as in the proof of Proposition 2, to conclude that if there is some $\tilde{t} \geq t^*$ such that

$$x_{\tilde{t}} < \frac{4}{9A^{5/2}b}, \tag{3.10}$$

then there is some $C' < \infty$ such that

$$x_t \leq C'e^{-0.6t}, \quad \text{for all } t \geq 0. \tag{3.11}$$

But from (3.9), the condition (3.10) is implied by

$$\sigma_{\tilde{t}}^p < \frac{4}{3A^2(1+3A)b},$$

and (3.11) implies (3.7) with $C = 3\sqrt{A}C'$. This completes the proof of the lemma, with $\delta^* = 4/(3A^2(1+3A))$. □

Lemma 3 *For arbitrary $\theta \geq 2$ and $A \in (0, \infty)$, there is $b_1 \in (0, \infty)$, such that if $b \geq b_1$ and $\lambda \leq A/b$, then for every $p \in [0, 1]$ and $l \geq 0$,*

$$\rho_t^p \geq \sigma_t^{l,p} - \left(\frac{1}{\sqrt{b}}\right)^{l+1} \tag{3.12}$$

Proof To derive (3.12) we consider the discrepancies between the processes (η_t^p) and $(\eta_t^{*,p})$. We construct these two processes using the same structure of Poisson marks, and for every site v of $\mathbb{T}_b$ we set

$$\delta_t(v) = \eta_t^{*,p}(v) - \eta_t^p(v).$$

In this fashion, δ_t takes the value 1 at the sites where the processes disagree at time t, and 0 at the other sites.

Observe that a D mark eliminates a discrepancy, while a U mark will possibly create a discrepancy at a site v only if at least one neighbor of v has a discrepancy at the time of this mark. Therefore $\delta_t \le \zeta_t^{*,0}$, where $(\zeta_t^{*,0})$ is a process in which the spin at the site -1 is frozen in the state 1 while all other spins are initially set to 0, and evolve with the following rules at each site $v \neq -1$:

- 1 flips to 0 at Poisson D marks at v.
- 0 flips to 1 at Poisson U marks at v if and only if at least one of the spins in $\mathcal{N}_v$ is in state 1 at the time of that mark.

Note that these flip rules are those of the threshold 1 contact process on $\mathbb{T}_b$, with infection parameter λ.

The threshold 1 contact process is stochastically dominated by the contact process, in which spins flip at rates:

- 1 flips to 0 at rate 1.
- 0 flips to 1 at rate λ times the number of sites of $\mathcal{N}_v$ in state 1 at time t.

We denote by $(\xi_t^{*,0})$ the process on $\mathbb{T}_b$, in which the spin at the site -1 is frozen in the state 1 while all other spins are initially set to 0, and then allowed to evolve according to the flip rates of this contact process.

The chain of comparisons presented above implies that

$$\sigma_t^{l,p} - \rho_t^p = \mathbb{P}\big(\delta_t(l) = 1\big) \le \mathbb{P}\big(\zeta_t^{*,0}(l) = 1\big) \le \mathbb{P}\big(\xi_t^{*,0}(l) = 1\big).$$

Let $(\xi_t^{\{v\}})$ denote the contact process started from the configuration in which only the site v is occupied (and no spin is frozen). Self-duality for the contact process implies

$$\begin{aligned}
\mathbb{P}\big(\xi_t^{*,0}(l) = 1\big) &= \mathbb{P}\big(\xi_s^{\{l\}}(-1) = 1, \text{ for some } s \in [0,t]\big) \\
&\le \mathbb{P}\big(\xi_s^{\{l\}}(-1) = 1, \text{ for some } s \ge 0\big) \\
&= \mathbb{P}\big(\xi_s^{\{0\}}(l+1) = 1, \text{ for some } s \ge 0\big) =: u(l+1).
\end{aligned}$$

The function $u(\cdot)$ has played an important role in the study of the contact process on $\mathbb{T}_b$. The proof of (3.12) will be complete once we argue that under our hypothesis,

$$u(l) \le \left(\frac{1}{\sqrt{b}}\right)^l.$$

For this purpose we refer to results in Chap. 4 in Part I of [9], where references to the original contributions can be found. The contact process on $\mathbb{T}_b$, $b \ge 2$, has two critical points $0 < \lambda_1(b) < \lambda_2(b) < \infty$. Theorem 4.1 in Part I of [9] tells us that $\lambda_2(b) \ge 1/(2\sqrt{b})$. Therefore we can find b_1 so that $\lambda \le A/b < \lambda_2(b)$, when $b \ge b_1$. Display (4.49) of Part I of [9] tells us that $u(l) \le (\beta(\lambda))^l$, where $\beta(\lambda) := \lim_{l\to\infty}(u(l))^{1/l}$. Finally Theorem 4.65 in Part I of [9] tells us that $\beta(\lambda) \le 1/\sqrt{b}$ when $\lambda \le \lambda_2(b)$. This completes the proof of (3.12). $\square$

Lemma 4 *For arbitrary $\theta \geq 2$ and $A \in (0, \infty)$, there are $b_0, \delta, \delta^*, t^* \in (0, \infty)$, such that if $b \geq b_0$ and $\lambda \leq A/b$, then for every $p \in [0, 1]$ the following dichotomy holds. Either* (3.6) *and* (1.15) *both hold, or else* (3.7) *and* (1.16) *both hold.*

Proof Let b^*, δ^* and t^* be as in Lemma 2. We will take $b_0 \geq b^*$, so that under the hypothesis of the lemma that we are proving we know from Lemma 2 that either (3.6) or (3.7) holds.

Suppose first that (3.6) holds. Define

$$\bar{\rho}^p = \liminf_{T\to\infty} \frac{1}{T} \int_0^T \rho_t^p dt, \quad \bar{\sigma}^{l,p} = \liminf_{T\to\infty} \frac{1}{T} \int_0^T \sigma_t^{l,p} dt, \quad \bar{\sigma}^p = \bar{\sigma}^{0,p}.$$

From (3.3) and $\lambda \leq A/b$, for each $l \geq 0$,

$$A \frac{1}{T} \int_0^T \sigma_t^{l+1,p} dt \geq \frac{1}{T} \int_0^T \sigma_t^{l,p} dt + \frac{\sigma_T^{l,p} - \sigma_0^{l,p}}{T}.$$

Hence,

$$A\bar{\sigma}^{l+1,p} \geq \bar{\sigma}^{l,p}.$$

By induction in l and (3.6), we obtain now

$$\bar{\sigma}^{l,p} \geq \frac{\bar{\sigma}^p}{A^l} \geq \frac{\delta^*}{A^l b}. \tag{3.13}$$

We use now Lemma 3, and for this suppose that $b \geq \max\{b^*, b_1\}$. Then, from (3.12) and (3.13), we obtain

$$\bar{\rho}^p \geq \bar{\sigma}^{l,p} - \left(\frac{1}{\sqrt{b}}\right)^{l+1} \geq \frac{\delta^*}{A^l b} - \left(\frac{1}{\sqrt{b}}\right)^{l+1} \geq \frac{\delta^*}{A^l b}\left(1 - \frac{A}{\delta^*}\left(\frac{A}{\sqrt{b}}\right)^{l-1}\right).$$

Taking $b_0 \geq \max\{b^*, b_1\}$ large enough, we have $A/\sqrt{b} \leq 1/2$, when $b \geq b_0$. Hence there is $\hat{l}$ such that

$$\bar{\rho}^p \geq \frac{\delta^*}{2A^{\hat{l}} b},$$

for all $b \geq b_0$. We conclude that (1.15) holds then with $\delta = \delta^*/(2A^{\hat{l}})$.

Suppose now that (3.7) holds. From the definition of (η_t^p) and ρ_t^p,

$$\begin{aligned} \frac{d}{dt}\rho_t^p &= -\rho_t^p + \lambda \mathbb{P}\left(\eta_t^p(0) = 0, \#\left\{u \in \mathcal{N}_l : \eta_t^p(0) = 1\right\} \geq \theta\right) \\ &\leq -\rho_t^p + \lambda \mathbb{P}\left(\#\left\{u \in \mathcal{N}_l : \eta_t^p(0) = 1\right\} \geq \theta\right). \end{aligned} \tag{3.14}$$

We compare the process (η_t^p) with the modified process in which the spin at the root is frozen in the state 1, while the spins at other sites evolve as in the threshold θ contact process. In this modified process, the spins at the neighbors of the root evolve independently of each other. Note that by attractivity, the distribution of $(\eta_t^p(u))_{u\in\mathcal{N}_0}$ is therefore stochastically dominated by a product measure with density σ_t^p. Therefore,

$$\frac{d}{dt}\rho_t^p + \rho_t^p \le \lambda \mathrm{Bin}(b+1, \sigma_t^p, \theta) \le \lambda \frac{(b+1)^2}{2}(\sigma_t^p)^2 \le C' e^{-1.2t},$$

for some $C' < \infty$, where in the last step we used (3.7). Multiplying both sides of this differential inequality by e^t and integrating yields (1.16) (see the end of the proof of Proposition 2 for an identical estimate). □

Proof of Theorem 2 Lemma 4 established the claimed dichotomy. It also implied that under the hypothesis of Theorem 2, (1.16) is equivalent to the negation of (3.6), i.e.,

$$\sigma_t^p < \frac{\delta^*}{b}, \quad \text{for some } t \ge t^*. \tag{3.15}$$

The statement about the set $\mathcal{D}_A(\mathbb{T}_b, \theta)$ follows then from the fact that if (3.15) holds for some (λ, p), then it also holds close to this point (with the same t). (As in the proof of Theorem 6, this is a typical "finite-time condition" argument.) □

Proof of Theorem 3 Since (1.14) has already been proved, we only have to prove that

$$\liminf_{b\to\infty} b\lambda_{\exp} \ge \Phi_\theta. \tag{3.16}$$

For this purpose, let $A < \Phi_\theta$ and $\lambda = A/b$. We will show that then

$$\rho_\infty^1 \le \frac{1}{b^{3/2}}, \quad \text{when } b \text{ is large.} \tag{3.17}$$

From Theorem 2 and the remarks after that theorem, we know that this implies that, when b is large, alternative (1.16) must hold and hence $\lambda \le \lambda_{\exp}$. Therefore $b\lambda_{\exp} \ge A$, and since A can be taken arbitrarily close to Φ_θ, (3.17) implies (3.16).

From the proof of Proposition 1, in the introduction, we know that if $p \ge \lambda/(\lambda+1)$, then $\rho_\infty^1 \le \rho_t^p$, for every $t \ge 0$. Therefore (3.17) will follow once we show that

$$\inf_{t\ge 0} \rho_t^\lambda \le \frac{1}{b^{3/2}}, \quad \text{when } b \text{ is large.} \tag{3.18}$$

To prove this claim, we use again (3.14), but this time we compare the process $(\eta_t^p)_{t\ge 0}$ with the modified process in which the spin at the root is frozen in the state 0, while the spins at other sites evolve as in the threshold θ contact process. We denote this modified process by $(\eta_t^{\odot,p})_{t\ge 0}$. Let F_T be the event that the origin is vacant at time 0 and that between time 0 and time T there is no U mark at the origin. Then, for $0 \le t \le T$,

$$\begin{aligned}\mathbb{P}\left(\#\left\{u \in \mathcal{N}_l : \eta_t^p(0) = 1\right\} \ge \theta\right) &\le \mathbb{P}\left(\#\left\{u \in \mathcal{N}_l : \eta_t^p(u) = 1\right\} \ge \theta,\, F_T\right) + \mathbb{P}((F_T)^c) \\ &= \mathbb{P}\left(\#\left\{u \in \mathcal{N}_l : \eta_t^{\odot,p}(u)=1\right\} \ge \theta,\, F_T\right) + \mathbb{P}((F_T)^c) \\ &\le \mathbb{P}\left(\#\left\{u \in \mathcal{N}_l : \eta_t^{\odot,p}(u) = 1\right\} \ge \theta\right) + p + \lambda T.\end{aligned}$$

In the modified process $(\eta_t^{\odot,p})_{t\ge 0}$, the spins at the neighbors of the root evolve independently of each other. Note that by attractivity, the distribution of $(\eta_t^{\odot,p}(u))_{u\in\mathcal{N}_0}$ is therefore stochastically dominated by a product measure with density ρ_t^p. Therefore, (3.14) yields

$$\frac{d}{dt}\rho_t^p \le -\rho_t^p + \lambda\left(\mathrm{Bin}(b+1, \rho_t^p, \theta) + p + \lambda T\right), \quad 0 \le t \le T.$$

We will use this inequality with $p = \lambda = A/b$ and $T = b^{1/4}$. We also change variables to $x_t = (b+1)\rho_t^p$. The inequality above then implies

$$\frac{d}{dt}x_t \le -x_t + A\mathrm{Bin}\left(b+1, \frac{x_t}{b+1}, \theta\right) + \frac{A}{b} + \frac{2A^2}{b^{3/4}}, \quad 0 \le t \le b^{1/4}.$$

If (3.18) were false, then there would be arbitrarily large b for which

$$x_t > \frac{1}{b^{1/2}}, \quad \text{for all } t \ge 0. \tag{3.19}$$

We would then have, for the b for which (3.19) holds,

$$\frac{d}{dt}\log(x_t) \le -1 + \frac{A}{x_t}\mathrm{Bin}\left(b+1, \frac{x_t}{b+1}, \theta\right) + \frac{A}{b^{1/2}} + \frac{2A^2}{b^{1/4}}, \quad 0 \le t \le b^{1/4}. \tag{3.20}$$

Thanks to (1.10), as $b \to \infty$, the right hand side of this inequality, as a function of x_t, converges uniformly to

$$F(x_t) = -1 + A\frac{\mathrm{Poisson}(x_t, \theta)}{x_t}.$$

Since $A < \Phi_\theta$, we have $\sup_{x>0} F(x) = -C$, for some $C > 0$. Therefore (3.20) yields, when b is large,

$$\frac{d}{dt}\log(x_t) \le -\frac{C}{2}, \quad 0 \le t \le b^{1/4}.$$

This implies

$$x_{b^{1/4}} \le x_0 e^{-(C/2)b^{1/4}} = A\frac{b+1}{b} e^{-(C/2)b^{1/4}},$$

which for large b contradicts (3.19). This contradiction proves (3.18), and completes the proof of (3.16). $\square$

4 The regime of large λ

In this section we will prove Theorems 4 and 5. Our lower bounds on $p_{\mathrm{exp}}(\mathbb{T}_b, \theta, \infty)$ will be obtained by comparison with a bootstrap percolation model that we describe next.

The continuous time bootstrap percolation model on the graph or oriented graph $G = (V, E)$ with threshold θ and infection parameter λ can be defined by taking the threshold contact process $(\eta_{G,\theta,\lambda;t})_{t\ge 0}$ on G, with same threshold θ, and suppressing all the flips from 1 to 0. In other words, the bootstrap percolation process has flip rates at $v \in V$ at time $t \ge 0$ given by

- 1 flips to 0 at rate 0.
- 0 flips to 1 at rate λ in case there are at least θ sites of $\mathcal{N}_{G,v}$ in state 1 at time t, and at rate 0 otherwise.

We will denote by $(\zeta^{\mu}_{G,\theta,\lambda;t})_{t\ge 0}$ the resulting process, started from a random distribution picked according to law μ at time 0. If one uses the same Poisson system of D and U marks to construct $(\eta^{\mu}_{G,\theta,\lambda;t})$ and $(\zeta^{\mu}_{G,\theta,\lambda;t})$, then, clearly

$$\eta^{\mu}_{G,\theta,\lambda;t} \le \zeta^{\mu}_{G,\theta,\lambda;t}, \quad \text{for all } t \ge 0. \tag{4.1}$$

It is also clear that $\zeta^{\mu}_{G,\theta,\lambda;t}$ is increasing in time, and therefore has a limit, $\zeta^{\mu}_{G,\theta,\lambda;\infty}$. Also note that $\zeta^{\mu}_{G,\theta,\lambda;\infty}$ does not depend on $\lambda > 0$, and that it can be obtained by the following iteration. Let S_0 be the set of sites which at time 0 are in state 1. Recursively define then

$$S_n = S_{n-1} \cup \{v \in (S_{n-1})^c : \#\{\mathcal{N}_{G,v} \cap S_{n-1}\} \ge \theta\}, \quad n \ge 1.$$

The sets S_n increase, and their limit is $\cup_n S_n = S_\infty = \zeta^{\mu}_{G,\theta,\lambda;\infty}$. (This iteration is often taken as the definition of bootstrap percolation in discrete time.)

From (4.1), it follows that, for any $\lambda > 0$, the sites that are vacant in S_∞ are vacant in the process $(\eta^{\mu}_{G,\theta,\lambda;t})$ at all times.

When $G = \mathbb{T}_b$, the observation in the last paragraph can be used as follows. Consider the clusters of occupied sites in S_∞, i.e., the connected components of the subgraph of $\mathbb{T}_b$ induced by the sites in S_∞. It is easy to see that since $\theta \ge 2$, the sites that belong to finite clusters of S_∞ will eventually be in state 0 in the process $\eta^{\mu}_{G,\theta,\lambda;t}$. Suppose that μ is product measure with density p. If S_∞ contains almost surely only finite clusters, then for any $\lambda > 0$ the process $(\eta^{p}_{\mathbb{T}_b,\theta,\lambda;t})_{t\ge 0}$ dies out. Therefore

$p \le p_c(\mathbb{T}_b, \theta, \infty)$. The next lemma provides a stronger conclusion under a stronger assumption. In its statement and its proof, we will use the following terminology and notation for $\mathbb{T}_b$. The distance between two sites is the length of the path that connects them. We will use B_n for the ball of radius n and center at the origin. The outside neighbors of a site v are the neighbors of v that are farther apart from the origin than v (each site $v \neq 0$ has b outside neighbors, and the origin has $b+1$ outside neighbors). Denote by R_n the event that the site 0 and some site separated from it by distance n are in the same cluster of S_∞.

Lemma 5 *Suppose that for bootstrap percolation on $\mathbb{T}_b$, started from product measure with density p, $\mathbb{P}(R_n)$ decays exponentially with n. Then $p \le p_{\exp}(\mathbb{T}_b, \theta, \infty)$.*

Proof Let $(\eta^{B_n}_{\mathbb{T}_b,\theta,\lambda;t})$ be the threshold contact process started with the ball of radius n around the origin fully occupied and all other sites vacant. From the observations above,

$$\mathbb{P}\left(\eta^{p}_{\mathbb{T}_b,\theta,\lambda;t}(0) = 1, (R_n)^c\right) \le \mathbb{P}\left(\eta^{B_n}_{\mathbb{T}_b,\theta,\lambda;t}(0) = 1\right). \tag{4.2}$$

The process $(\eta^{B_n}_{\mathbb{T}_b,\theta,\lambda;t})$ is stochastically dominated by the process started from the same configuration, in which a spin 0 never flips and a spin 1 flips to 0, at rate 1, iff all its outside neighbors are in state 0. For this process, let τ_v, $v \in B_n$, be the random amount of time needed for the spin at v to flip to 0 after the moment when it became allowed to flip. Clearly the τ_v, $v \in B_n$, are i.i.d., with exponential distribution with mean 1. A simple induction argument, starting from the sites at distance n from the root, and moving inwards, shows that the root will then flip to 0 at the random time

$$\max_{\pi \in \Pi_n} \sum_{v \in \pi} \tau_v,$$

where Π_n is the set of $(b+1)b^{n-1}$ paths from 0 to the sites that are at distance n from it.

We obtain therefore, from (4.2), and the hypothesis of the lemma

$$\begin{aligned}
\mathbb{P}\left(\eta^{p}_{\mathbb{T}_b,\theta,\lambda;t}(0) = 1\right) &\le \mathbb{P}(R_n) + \mathbb{P}\left(\eta^{p}_{\mathbb{T}_b,\theta,\lambda;t}(0) = 1, (R_n)^c\right) \\
&\le C_1 e^{-C_2 n} + \mathbb{P}\left(\max_{\pi \in \Pi_n} \sum_{v \in \pi} \tau_v \ge t\right) \\
&\le C_1 e^{-C_2 n} + (b+1)b^{n-1}\mathbb{P}\left(\sum_{v \in \tilde{\pi}} \tau_v \ge t\right),
\end{aligned}$$

where C_1, C_2 are positive finite constants and $\tilde{\pi}$ is an arbitrary element of Π_n. Taking $n = \lfloor \epsilon t \rfloor$, for some $\epsilon > 0$ small enough, a standard large deviation estimate for

Poisson random variables (see, e.g., (A.1) in the Appendix of [7]) shows that

$$\mathbb{P}\left(\sum_{v\in\tilde{\pi}} \tau_v \geq t\right) \leq C_3 e^{-C_4 n},$$

where $C_3 \in (0, \infty)$ and C_4 is large enough that $e^{-C_4} < b$. The last two displayed inequalities imply then that $\mathbb{P}\left(\eta^p_{\mathbb{T}_b,\theta,\lambda;t}(0) = 1\right)$ decays exponentially with t, completing the proof. □

Bootstrap percolation on homogeneous trees has been studied in [3] and [1]. Below we could build on some of their estimates. Nevertheless, for the reader's benefit, and at little extra cost, we will present a self-contained approach to our problem of estimating $\mathbb{P}(R_n)$, in order to use Lemma 5.

To study the bootstrap percolation process on $\mathbb{T}_b$, it is convenient to study also, as a tool, the bootstrap percolation processes on its subgraph $\mathbb{T}_b^+$, induced by the following subset V_b^+ of vertices. The set V_b^+ is the minimal set of vertices of $\mathbb{T}_b$ with the properties that $0 \in V_b^+$ and if $v \in V_b^+$ then $\mathcal{N}_{\vec{\mathbb{T}}_b,v} \subset V_b^+$. We will also consider bootstrap percolation on the oriented graph $\vec{\mathbb{T}}_b^+$, which has as set of vertices also V_b^+, and defined then by $\mathcal{N}_{\vec{\mathbb{T}}_b^+,v} = \mathcal{N}_{\vec{\mathbb{T}}_b,v}$.

First we observe that bootstrap percolation on $\mathbb{T}_b^+$ and on $\vec{\mathbb{T}}_b^+$ are strongly related to each other in the following way. If we start them from a same set of occupied sites, S_0, then for $n \geq 0$, either both will have the root in S_n or neither one will have it. To see this, given a set of sites R of $\mathbb{T}_b^+$, say that a site $v \in R$ is hidden from the root in R if there is another site $u \in R$ which belongs to the path which connects the root to v. Observe that if in the iteration which defines S_n for either one of the two processes that we are considering we eliminate all the sites that are hidden from the root in S_{n-1}, we do not change the truth or falsehood of the statement that the root belongs to S_n. But with this modification, the sets S_n are the same for both processes.

It is easy to write down a recursion for the probability p_n^+ that the root belongs to S_n in the bootstrap percolation process on $\mathbb{T}_b^+$ or $\vec{\mathbb{T}}_b^+$, started from product distribution with density p. In the last paragraph we argued that p_n^+ is the same for both processes. Now, for the process on $\vec{\mathbb{T}}_b^+$, the root will belong to S_n in case it belongs to S_0, or in case it does not belong to S_0, but at least θ of the sites in $\mathcal{N}_{\vec{\mathbb{T}}_b^+,0}$ are in S_{n-1}. This observation and some obvious facts about the geometry of $\vec{\mathbb{T}}_b^+$ yield:

$$p_n^+ = p + (1-p)\mathrm{Bin}(b, p_{n-1}^+, \theta), \tag{4.3}$$

with initial condition $p_0^+ = p$.

The right-hand-side of (4.3) is an increasing continuous function of p_n^+. Therefore

$$p_n^+ \nearrow p_\infty^+ := \inf\{x > 0 : x = p + (1-p)\mathrm{Bin}(b, x, \theta)\}. \tag{4.4}$$

The limit p_∞^+ is the probability that the root belongs to S_∞ in this bootstrap percolation process on $\mathbb{T}_b^+$ or $\vec{\mathbb{T}}_b^+$. Since $\theta \geq 2$, $p_\infty^+ \leq \inf\{x > 0 : x = p + (b/2)^2 x^2\}$. It is easy to use this observation to conclude that

$$p_\infty^+ \searrow 0 \quad \text{as } p \searrow 0. \tag{4.5}$$

The following concept will be used in the proof of Theorem 4. Consider bootstrap percolation on $G = (V, E)$, and let $W \subset V$ and $w \in W$. We will say that "w is eventually W-internally occupied" if w becomes eventually occupied in the bootstrap percolation process restricted to W. To make the definition precise, set $S_0^W = S_0 \cap W$,

$$S_n^W = S_{n-1}^W \cup \{v \in W \backslash S_{n-1}^W : \#\{\mathcal{N}_{G,v} \cap S_{n-1}^W\} \geq \theta\}, \quad n \geq 1.$$

We now say that w is eventually W-internally occupied in case $w \in S_\infty^W := \cup_n S_n^W$. Note that this event depends only on the initial configuration of occupied sites in W.

Proof of Theorem 4 We will use Lemma 5, and for this purpose we need to estimate $\mathbb{P}(R_n)$. Consider bootstrap percolation on $\mathbb{T}_b$ and let $\{0 \leftrightarrow n\}$ denote the event that the sites 0 and n belong to the same cluster of S_∞. Note that this is the same as the event that the sites $0, 1, \ldots, n$ are all eventually occupied in this bootstrap percolation process.

We will use the definition in the last paragraph before this proof in the case $G = \mathbb{T}_b$, $W = \mathcal{L}^c$. Note that the subgraph of $\mathbb{T}_b$ induced by W (i.e., the subgraph of $\mathbb{T}_b$ obtained by removing $\mathcal{L}$ from the set of vertices, along with the edges incident to these vertices) is an infinite collection of copies of $\mathbb{T}_b^+$. The roots of these copies of $\mathbb{T}_b^+$ are neighbors to the sites in $\mathcal{L}$, with each site in $\mathcal{L}$ being neighbor to $b-2$ of these roots. For $j \in \mathcal{L}$, define X_j as the number of neighbors of the site j which are eventually W-internally occupied. When bootstrap percolation on $\mathbb{T}_b$ is started from product measure with density p, it follows from the remarks above that each X_j has a binomial distribution corresponding to $b-2$ attempts each with probability p_∞^+ of success, where p_∞^+ is given by (4.4). Clearly the X_j are also mutually independent.

For each $j \in \mathcal{L}$ define a grade as follows. If the site j is in state 1 at time 0, give this site grade A. If not, give this site the grade according to: also grade A if $X_j \geq \theta$, grade B if $X_j = \theta - 1$, grade C if $X_j = \theta - 2$, grade F if $X_j \leq \theta - 3$. The probability of obtaining grades A, B or C are then, respectively:

$$\begin{aligned} p_A &= p + (1-p)\text{Bin}(b-2, p_\infty^+, \theta), \\ p_B &= (1-p)\text{Bin}(b-2, p_\infty^+, \theta - 1), \\ p_C &= (1-p)\text{Bin}(b-2, p_\infty^+, \theta - 2). \end{aligned} \tag{4.6}$$

Observe that if $\{0 \leftrightarrow n\}$ occurs, then the following must happen:

(i) No site in $0, 1, \ldots, n$ can have grade F.
(ii) If the sites j and k, with $0 \leq j < k \leq n$ both have grade C, then there must exist a site i with $j < i < k$ with grade A.

For a given realization of the process, denote by n_A, n_B and n_C, respectively, the number of sites in $\{0, \ldots, n\}$ which receive grades A, B, and C. Then (i) implies

$$n_A + n_B + n_C = n + 1, \tag{4.7}$$

while (ii) implies

$$n_A \geq n_C - 1. \tag{4.8}$$

Since there are 3^{n+1} ways to assign grades A, B and C to the sites in $\{0, \ldots, n\}$, it follows that

$$\begin{aligned}
\mathbb{P}(0 \leftrightarrow n) &\leq 3^{n+1} \max_{\substack{n_A, n_B, n_C \\ n_A+n_B+n_C=n+1 \\ n_A \geq n_C-1}} (p_A)^{n_A} (p_B)^{n_B} (p_C)^{n_C} \\
&\leq 3^{n+1} \max_{\substack{n_A, n_B, n_C \\ 2n_A+n_B \geq n}} (p_A)^{n_A} (p_B)^{n_B} (p_C)^{n_C} \\
&\leq 3^{n+1} \max_{\substack{n_A, n_B \\ n_A+n_B \geq n/2}} (p_A)^{n_A} (p_B)^{n_B} \\
&\leq 3^{n+1} (\max\{p_A, p_B\})^{n/2}.
\end{aligned} \tag{4.9}$$

From the geometry of $\mathbb{T}_b$ and (4.9) we obtain

$$\mathbb{P}(R_n) \leq (b+1) b^{n-1} \mathbb{P}(0 \leftrightarrow n) \leq 3 \frac{b+1}{b} \left[3b (\max\{p_A, p_B\})^{1/2} \right]^n .$$

But (4.5) and (4.6) imply that for $p > 0$ small enough, $\max\{p_A, p_B\} < 1/(3b)^2$, and hence

$$\mathbb{P}(R_n) \to 0, \quad \text{exponentially fast as } n \to \infty.$$

Lemma 5 now implies $p_{\exp}(\mathbb{T}_b, \theta, \infty) \geq p > 0$. □

We turn now to the proof of Theorem 5. The origin of the exponent $\theta/(\theta-1)$ there is the following. $\mathrm{Bin}(b, \gamma/b^\alpha, \theta)$ is of order $1/b^\alpha$ for large b iff $\alpha = \theta/(\theta-1)$. The precise version of this statement that we need below is the following one, which can be checked by elementary computations. For arbitrary $\bar{\gamma} > 0$,

$$\frac{\theta!}{\gamma^\theta} b^{\theta/(\theta-1)} \mathrm{Bin}(b, \gamma/b^{\theta/(\theta-1)}, \theta) \to 1, \quad \text{as } b \to \infty, \tag{4.10}$$

uniformly in $\gamma \in (0, \bar{\gamma}]$. This has the following consequence for the mean-field model. For arbitrary $\bar{\gamma} > 0$,

$$H(b, \theta, \lambda; \gamma/b^{\theta/(\theta-1)}) \to -1 + \lambda \frac{\gamma^{\theta-1}}{\theta!}, \quad \text{as } b \to \infty,$$

uniformly in $\gamma \in (0, \bar{\gamma}]$. Therefore, for arbitrary $\lambda > 0$,

$$b^{\theta/(\theta-1)} p_c^{\mathrm{MF}}(b, \theta, \lambda) \to (\theta!/\lambda)^{1/(\theta-1)}, \quad \text{as } b \to \infty. \tag{4.11}$$

Proof of Theorem 5 From (1.13), in Theorem 1,

$$p_c(\mathbb{T}_b, \theta, \infty) \le p_c^{\mathrm{MF}}(b, \theta, 1).$$

Combined with (4.11), this implies

$$\limsup_{b\to\infty} b^{\theta/(\theta-1)} p_c(\mathbb{T}_b, \theta, \infty) \le (\theta!)^{1/(\theta-1)},$$

which provides the upper bound in (1.18).

The proof of the lower bound in (1.18),

$$\liminf_{b\to\infty} b^{\theta/(\theta-1)} p_c(\mathbb{T}_b, \theta, \infty) > 0, \tag{4.12}$$

builds on the proof of Theorem 4. We will use the same notation as in that proof. Define also $\gamma = pb^{\theta/(\theta-1)}$, $\gamma_n = p_n^+ b^{\theta/(\theta-1)}$, and $\gamma_\infty = p_\infty^+ b^{\theta/(\theta-1)}$. The recursion (4.3) implies

$$\gamma_n \le \gamma + b^{\theta/(\theta-1)} \mathrm{Bin}\big(b, \gamma_{n-1}/b^{\theta/(\theta-1)}, \theta\big) \tag{4.13}$$

Set

$$\bar{\gamma} = \inf\big\{x > 0 : x = \gamma + 2x^\theta/\theta!\big\}.$$

(In this definition the factor 2 is arbitrary; any number larger than 1 could be used instead.) Since $\theta \ge 2$, for small $\gamma > 0$ we have $0 < \bar{\gamma} < \infty$, and

$$\bar{\gamma} = \gamma + 2\frac{(\bar{\gamma})^\theta}{\theta!}. \tag{4.14}$$

Moreover, similarly to (4.5),

$$\bar{\gamma} \searrow 0 \quad \text{as } \gamma \searrow 0. \tag{4.15}$$

Our next goal is to prove that when b is large

$$p_\infty^+ \le \frac{\bar{\gamma}}{b^{\theta/(\theta-1)}} \tag{4.16}$$

From (4.10), there is $\bar{b}$ such that for $0 \le y \le \bar{\gamma}$ and $b \ge \bar{b}$,

$$b^{\theta/(\theta-1)} \mathrm{Bin}\big(b, y/b^{\theta/(\theta-1)}, \theta\big) \le \frac{2y^\theta}{\theta!}. \tag{4.17}$$

Since $\gamma + 2x^\theta/\theta!$ is increasing in x, when $y < \bar{\gamma}$, we have $\gamma + 2y^\theta/\theta! \le \bar{\gamma}$. Note that $\gamma \le \bar{\gamma}$. Hence (4.13) and (4.17) imply, by induction on n, that $\gamma_n \le \bar{\gamma}$. Therefore $\gamma_\infty \le \bar{\gamma}$ and (4.16) follows.

Combining (4.6), (4.16) and (4.15) we obtain, when b is large,

$$\begin{aligned} p_A &\leq \frac{\gamma_A}{b^{\theta/(\theta-1)}}, \\ p_B &\leq \frac{\gamma_B}{b}, \\ p_C &\leq \frac{\gamma_C}{b^{(\theta-2)/(\theta-1)}}, \end{aligned} \tag{4.18}$$

where $\lim_{\gamma\searrow 0}\gamma_A = \lim_{\gamma\searrow 0}\gamma_B = \lim_{\gamma\searrow 0}\gamma_C = 0$. Note that (4.18) implies the following technical estimate:

$$\max\left\{(p_A)^2, (p_B)^2, p_C\, p_A\right\} \leq \frac{\gamma'}{b^2}, \tag{4.19}$$

where $\lim_{\gamma\searrow 0}\gamma' = 0$. This estimate is useful in combination with the following one:

$$\mathbb{P}(0 \leftrightarrow n) \leq 3^{n+1}\left(\max\left\{(p_A)^2, (p_B)^2, p_C\, p_A\right\}\right)^{(n/2)-1}. \tag{4.20}$$

To prove this inequality, one can match pairs of sites in $\{0, \dots, n\}$ in the following way. Recall that if the event $\{0 \leftrightarrow n\}$ happens, then the facts (i) and (ii) in the proof of Theorem 4 must happen. From fact (ii) we know that each site which receives a grade C is followed eventually by a site with grade A, except possibly for the last site with grade C. Pair each site with a grade C with the first site with grade A after it, leaving possibly one unmatched site with grade C. Considering the sites with grade A which are unmatched to any site with grade C, we match the first of these sites to the second one, the third to the fourth, etc, leaving at most one unmatched site with grade A. Finally we match the first site with grade B to the second such site, the third site with grade B to the fourth such site, etc, leaving at most one unmatched site with grade B. Since the number of sites in $\{0, \dots, n\}$ is $n+1$ and there are at most 3 unmatched sites, the number of matched pairs is at least $((n+1)-3)/2 = (n/2)-1$. The estimate (4.20) now follows from the fact that the number of ways to assign grades A, B and C to the sites in $\{0, \dots, n\}$, is 3^{n+1}.

Combining (4.19) with (4.20), we obtain, when b is large,

$$\begin{aligned} \mathbb{P}(R_n) &\leq (b+1)b^{n-1}\mathbb{P}(0 \leftrightarrow n) \leq (b+1)b^{n-1}3^{n+1}\left(\frac{\gamma'}{b^2}\right)^{(n/2)-1} \\ &= 27(b+1)b\left(3\sqrt{\gamma'}\right)^{n-2}. \end{aligned}$$

By taking $\gamma > 0$ sufficiently small, we can make $3\sqrt{\gamma'} < 1$. Then

$$\mathbb{P}(R_n) \to 0, \quad \text{exponentially fast as } n \to \infty.$$

Lemma 5 now implies $b^{\theta/(\theta-1)} p_{\exp}(\mathbb{T}_b, \theta, \infty) \geq b^{\theta/(\theta-1)} p = \gamma > 0$. This proves (4.12). □

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