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Poisson boundary of a relativistic diffusion

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Abstract In this article, we study the asymptotic behaviour of a random motion in Minkowski spacetime, representing the random evolution of an object (or signal) traveling at a speed strictly less than the speed of the light, introduced by Dudley in his article (Ark Mat 6:241–268, 1966). We determine its invariant σ-algebra and give an explicit description of the Poisson boundary of its differential generator.

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1 Introduction

In his article [14], Dudley determined a class of Markov processes in Minkowski space $\mathbb{R} \times \mathbb{R}^3$, representing the random evolution in spacetime of an object (or signal) traveling at a speed strictly less than the speed of light, and whose law is invariant under the action of the group of isometries of the space. This group is the group of affine isometries of the quadratic form

$$q(\xi) = \left(\xi^0\right)^2 - \left(\left(\xi^1\right)^2 + \left(\xi^2\right)^2 + \left(\xi^3\right)^2\right),$$

where $(\xi^0, \dots, \xi^3)$ are the coordinates of ξ in the canonical basis of $\mathbb{R} \times \mathbb{R}^3$.

The sample paths $\{\xi_s\}_{s \geqslant 0}$ of the random motions in $\mathbb{R} \times \mathbb{R}^3$ are all made on the same model. $\{\xi_s\}_{s \geqslant 0}$ is of the form

$$\xi_s = \xi_0 + \int_0^s \dot{\xi}_r \, dr, \tag{1.1}$$

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for some càdlàg process $\{\dot{\xi}_s\}_{s\geqslant 0}$ on the (half-)hyperboloid

$$\mathbb{H} = \{\xi \in \mathbb{R} \times \mathbb{R}^3 ;\ q(\xi) = 1,\ \xi^0 > 0\}.$$

So, strictly speaking, Dudley's Markov processes are not processes in spacetime $\mathbb{R}^{1,3}$, but are processes with state space the phase space $\mathbb{H} \times \mathbb{R}^{1,3}$, where speed and position of an object are recorded.

As the restriction of q to any tangent space of the half unit pseudo-sphere $\mathbb{H}$ is negative definite, it endows $\mathbb{H}$ with a Riemannian structure of constant curvature equal to -1. The space $\mathbb{H}$ is (a model of) the three-dimensional hyperbolic space.

Dudley considers Markov processes $\{\dot{\xi}_s\}_{s\geqslant 0}$ on $\mathbb{H}$ with a law invariant under the action of isometries of $\mathbb{H}$. Roughly speaking, he shows that $\{\dot{\xi}_s\}_{s\geqslant 0}$ is a mixture of an $\mathbb{H}$-Brownian motion and jump processes, with radial jump laws. This is the analogue in $\mathbb{H}$ of a Lévy process in $\mathbb{R}^3$ whose law is invariant under the action of Euclidean affine isometries.

Apart from in another article of Dudley [15] where the asymptotic direction of the speed $\{\dot{\xi}_s\}_{s\geqslant 0}$ of the process is shown to converge, nothing has been written about this class of processes.

The preceding family of processes (essentially) contains a unique diffusion: $\{\dot{\xi}_s\}_{s\geqslant 0}$ is a Brownian motion on $\mathbb{H}$, and $\{\xi_s\}_{s\geqslant 0}$ its integral. As the geometrical framework is that of special relativity, the preceding uniqueness property justifies our calling this diffusion on $\mathbb{H} \times \mathbb{R}^{1,3}$ the *relativistic diffusion*. Given $(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,d}$, denote by $\mathbb{P}_{\dot{\xi},\xi}$ the law of the relativistic diffusion started for $(\dot{\xi}, \xi)$.

The object of this article is to determine the probabilistic information on the asymptotic behaviour of the relativistic diffusion[1] encoded in its invariant σ-algebra $Inv\big((\dot{\xi}, \xi)\big)$. More precisely:

"Find an $Inv\big((\dot{\xi}, \xi)\big)$-measurable random variable X such that the two σ-algebras $Inv\big((\dot{\xi}, \xi)\big)$ and $\sigma(X)$ are indistinguishable under any $\mathbb{P}_{\dot{\xi},\xi}$."

Denote by $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ the canonical basis of $\mathbb{R} \times \mathbb{R}^3$. Any point of $\dot{\zeta} \in \mathbb{H}\backslash\{\varepsilon_0\}$ can be uniquely written

$$(\dot{\zeta}^0, \ldots, \dot{\zeta}^3) = \big(\mathrm{ch}\rho, (\mathrm{sh}\rho)\sigma\big),$$

using polar coordinates $(\rho, \sigma) \in \mathbb{R}_*^+ \times \mathbb{S}^2$ of $\dot{\zeta} \in \mathbb{H}$. Let us denote by (ρ_s, σ_s) the polar coordinates of the point $\dot{\xi}_s \in \mathbb{H}$.

Theorem 1 *Let* $(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}$ *be given.*

1. *The following limits exist* $\mathbb{P}_{\dot{\xi},\xi}$*-almost surely.*

$$\begin{aligned} \lim_{s\to+\infty} \sigma_s &\equiv \sigma_\infty, \\ \lim_{s\to+\infty} q(\xi_s, \varepsilon_0 + \sigma_\infty) &\equiv R_\infty^{\sigma_\infty}. \end{aligned} \tag{1.2}$$

[1] Encoded in the tail σ-algebra.

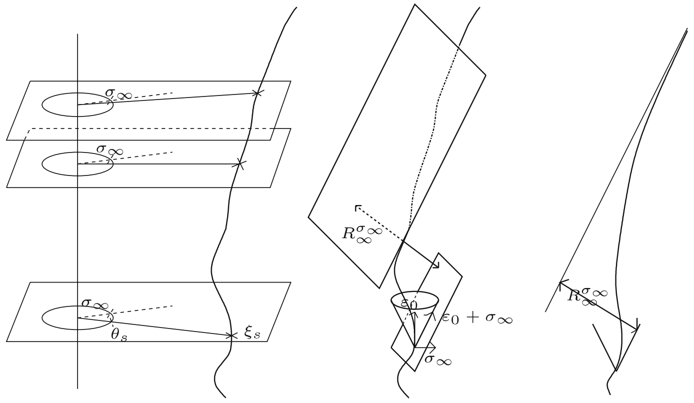


Fig. 1 Asymptotic behaviour of a typical trajectory

2. *The invariant σ-algebra of the relativistic diffusion coincide $\mathbb{P}_{\dot{\xi},\xi}$-almost surely with $\sigma(\sigma_\infty, R_\infty^{\sigma_\infty})$.*

The two asymptotic quantities σ_∞ and $R_\infty^{\sigma_\infty}$ can be interpreted geometrically using only the spacetime part $\{\xi_s\}_{s\geqslant 0}$ of the diffusion Fig. 1.

- On the left drawing, each hyperplane corresponds to the set of events of spacetime with constant time. The trajectory $\{\xi_s\}_{s\geqslant 0}$ hits each of these hyperplanes at a unique point. If one parametrizes this point using the polar coordinates defined on the hyperplane, then the polar angle converges towards σ_∞. So, in some sense, σ_∞ is the asymptotic direction in which an immobile observer sees $\{\xi_s\}_{s\geqslant 0}$ go towards infinity.
- We can associate to each direction $\sigma \in \mathbb{S}^2$, the hyperplane of tangent vectors to the cone $\{q = 0\}$ at point $\varepsilon_0 + \sigma$. Point *1* of theorem **1** asserts the existence of a random hyperplane $\left\{\xi \in \mathbb{R} \times \mathbb{R}^3 ;\ q(\xi, \varepsilon_0 + \sigma_\infty) = R_\infty^{\sigma_\infty}\right\}$, parallel to the hyperplane associated to $\sigma_\infty \in \mathbb{S}^2$, such that the point ξ_s goes to infinity as it approaches this (random) hyperplane.

The main tool of the proof of Theorem **1** is the coupling technique. This technique is not easily implemented in a non-elliptic framework, where we have fewer driving Brownian motions than the dimension of the diffusion, and the general problem remains largely unsolved. Fortunately, the construction of a quasi-coupling will suffice for our purposes.

The organization of the article is the following. The necessary geometric background is recalled in Sect. **2**. In this framework, one constructs in Sect. **3** the relativistic diffusion and a related diffusion on the group of affine isometries. The analysis of their analytical properties is made in Sect. **4**, where the method used to prove Theorem **1** is explained. Section **5** is dedicated to this proof. The last section (Sect. **6**) ends this article with a series of remarks.

2 Geometric background

Denote by $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ the canonical basis of $\mathbb{R} \times \mathbb{R}^3$, and $\left(\xi^0, \xi^1, \xi^2, \xi^3\right)$ the coordinates of a point $\xi \in \mathbb{R} \times \mathbb{R}^3$ in this basis. The vector space $\mathbb{R} \times \mathbb{R}^3$ is endowed with the quadratic form

$$q(\xi) = \left(\xi^0\right)^2 - \left(\left(\xi^1\right)^2 + \left(\xi^2\right)^2 + \left(\xi^3\right)^2\right).$$

To distinguish the usual Euclidean space $\mathbb{R}^4$ from $(\mathbb{R} \times \mathbb{R}^3, q)$, we shall write $\mathbb{R}^{1,3}$ for this latter space.

In this section we reacll some necessary facts from hyperbolic geometry. We refer the reader to the book [18] of Helgason for details.

2.1 Isometries of q

The set of affine isometries of $\mathbb{R}^{1,3}$ is well known. The set of direct linear isometries is a subgroup $SO(1,3)$ of $GL(\mathbb{R}^4)$, with Lie algebra $so(1,3) = \left\{ \begin{pmatrix} 0 & c^* \\ c & A \end{pmatrix} ; c \in \mathbb{R}^3, A \in so(3) \right\}$. Any direct affine isometry φ of $\mathbb{R}^{1,3}$ can be uniquely written

$$\xi \in \mathbb{R}^{1,3}, \quad \varphi(\xi) = g(\xi) + b,$$

for an element $g \in SO(1,3)$ and $b \in \mathbb{R}^{1,3}$. Write $\varphi = (g, b)$. The group of direct affine isometries of $\mathbb{R}^{1,3}$ is the semi-direct product $SO(1,3) \times \mathbb{R}^{1,3}$, with product:

$$(g, b)(g', b') = (gg', b + gb').$$

Definition 2 The connected component containing (Id, 0) in the set of affine isometries of $\mathbb{R}^{1,3}$ is called the **Poincaré group**, and denoted by $\mathcal{G}$.

Notation We write $SO_0(1,3) = \exp\big((so(1,3)\big)$; this is the conected component of the identity in $SO(1,3)$. The Poincaré group is equal to $SO_0(1,3) \times \mathbb{R}^{1,3}$.

Definition 3 The unit pseudo-sphere of $\mathbb{R}^{1,3}$ has two components; we denote by

$$\mathbb{H} = \left\{\xi \in \mathbb{R}^{1,3}\,;\ q(\xi) = 1,\ \xi^0 > 0\right\},$$

the component corresponding to positive ξ^0.

2.2 Hyperbolic space

Polar coordinates One can write $\mathbb{H} = \{(\mathrm{ch}\rho, (\mathrm{sh}\rho)\sigma);\, \rho \geqslant 0, \sigma \in \mathbb{S}^2\}$. Apart from the point $\varepsilon_0 \in \mathbb{H}$, every point of $\mathbb{H} \subset \mathbb{R}^{1+3}$ can be uniquely written as $(\mathrm{ch}\rho, (\mathrm{sh}\rho)\sigma) \in \mathbb{R} \times \mathbb{R}^3$. The pair (ρ, σ) is said to be the *polar coordinates of the point* $(\mathrm{ch}(\rho), (\mathrm{sh}\rho)\sigma) \in \mathbb{H}$.

The q-norm of the velocity of a $\mathcal{C}^1$ path $\gamma_s = (\mathrm{ch}(\rho_s), (\mathrm{sh}\rho_s)\sigma_s) \in \mathbb{H}$ is

$$q(\dot{\gamma}_s) = -\left(\dot{\rho}_s^2 + (\mathrm{sh}^2\rho_s)\, ||\dot{\sigma}_s||^2_{\mathrm{Eucl}}\right);$$

so q induces a Riemannian metric on $\mathbb{H}$, given in (ρ, σ) coordinates by the $(0, 2)$-tensor: $d\rho^2 + (\mathrm{sh}^2\rho)||d\sigma||^2_{\mathrm{Eucl}}$. These coordinates are the Riemannian exponential coordinates associated with the point $\varepsilon_0 \in \mathbb{H}$. Note their bad behaviour at ε_0; they parametrize $\mathbb{H}\backslash\{\varepsilon_0\}$. Given $\sigma \in \mathbb{S}^2$, the path $\{(\rho, \sigma)\}_{\rho \in \mathbb{R}}$ is a geodesic.

Halfspace coordinates Endow the halfspace $\mathbb{R}_+^* \times \mathbb{R}^2$ with the Riemannian metric defined at a point $(y, x) \in \mathbb{R}_+^* \times \mathbb{R}^2$ by the formula[2]

$$X, Y \in \mathbb{R} \times \mathbb{R}^2, \quad (X, Y)_{(y,x)} = \frac{\langle X, Y\rangle_{\mathrm{Eucl}}}{y^2}.$$

One can check by a direct calculation that

Proposition 4 *The restriction to $\mathbb{H}$ of the following application is an isometry between $(\mathbb{H}, q)$ and the half-space $\left(\mathbb{R}_*^+ \times \mathbb{R}^2, (\cdot, \cdot)_\cdot\right)$.*

$$(\xi^0, \xi^1, \xi^2, \xi^3) \in \mathbb{R}^{1,3} \overset{\psi^{-1}}{\longmapsto} \left(\frac{1}{\xi^0 - \xi^1}, \frac{\xi^2}{\xi^0 - \xi^1}, \ldots, \frac{\xi^3}{\xi^0 - \xi^1}\right) \in \mathbb{R}_+^* \times \mathbb{R}^2. \tag{2.1}$$

Its inverse is given by the formula

$$\left(y, (x_1, x_2)\right) \in \mathbb{R}_+^* \times \mathbb{R}^2 \overset{\psi}{\mapsto} \left(\frac{|x|^2 + y^2 + 1}{2y}, \frac{|x|^2 + y^2 - 1}{2y}, \frac{x_1}{y}, \frac{x_2}{y}\right) \in \mathbb{H}. \tag{2.2}$$

where $|x|$ is Euclidean norm of $x \in \mathbb{R}^2$.

The image by ψ of the level hypersurface $\{y = constant\}$ is the intersection of the hyperplane $\xi^0 - \xi^1 = \frac{1}{y}$ with $\mathbb{H}$.

Definition 5 Given a frame $g \in SO_0(1, 3)$ of spacetime, we define coordinates on $\mathbb{H}$ by setting $(y, x) = \psi^{-1}\left(g^{-1}(\dot{\xi})\right)$. These coordinates are said to be the *halfspace coordinates associated with the frame g.*

[2] $\langle X, Y\rangle_{\mathrm{Eucl}}$ denotes Euclidean scalar product of $X, Y \in \mathbb{R}^3$.

2.3 Sphere at infinity, action on this sphere

(i) We see in Definition 3 that $SO_0(1,3)$ is the set of direct isometries of $\mathbb{H}$ and that it acts transitively on it. It also acts transitively on the unit tangent bundle of $\mathbb{H}$:

$$\big\{(\dot\xi, V) \in T\mathbb{H}\,;\ \dot\xi \in \mathbb{H},\ V \in T_{\dot\xi}\mathbb{H},\ q(V) = -1\big\}.$$

(ii) Adopt polar coordinates on $\mathbb{H}$. As a non-parametrized path, the geodesic $\big\{(\rho,\sigma)\big\}_{\rho\in\mathbb{R}}$ is the intersection of $\mathbb{H}$ with the two-dimensional vector space $\langle\varepsilon_0,\sigma\rangle \subset \mathbb{R}^{1,3}$. The intersection of $\Big(\mathbb{R}_+^*\Big)\,\varepsilon_0 + \big(\mathbb{R}_+^*\big)\,\sigma$ with the null cone is the half-line $\mathbb{R}_+^*\,(\varepsilon_0+\sigma)$. The geodesic $\big\{(\rho,\sigma)\big\}_{\rho\in\mathbb{R}}$ is said to have *asymptotic direction* $\sigma \in \mathbb{S}^2$. Now, given a geodesic γ, started from $\dot\xi$, in direction V, we know from (i) that there exists some isometry $g \in SO_0(1,3)$ such that

$$\dot\xi = g(\varepsilon_0) \quad \text{and} \quad V = g(\varepsilon_1);$$

g sends the null half-line $\mathbb{R}_+^*\,(\varepsilon_0+\varepsilon_1)$ to another null half-line $\mathbb{R}_+^*\,(\varepsilon_0+\sigma)$. This line $\mathbb{R}_+^*\,(\varepsilon_0+\sigma)$ being the intersection of the null cone with $\big(\mathbb{R}_+^*\big)\,\dot\xi + \big(\mathbb{R}_+^*\big)\,V \subset \mathbb{R}^{1,3}$, the direction $\sigma \in \mathbb{S}^2$ is uniquely determined by the geodesic γ. One says that γ has asymptotic direction σ. The set of asymptotic directions is $\mathbb{S}^2$.

(iii) In our framework, $\mathbb{H}$ will be seen as a set of velocity vectors. As a translation does not change the speed of a path, and an isometry $g \in SO_0(1,3)$ transforms a geodesic γ of $\mathbb{H}$ to the geodesic $g(\gamma)$, we define the action of an affine isometry $(g,b) \in \mathcal{G}$ on the set $\mathbb{S}^2$ of asymptotic directions by the formula

$$(g,b).\sigma = \sigma', \quad \text{if } g\big(\mathbb{R}_+^*(\varepsilon_0+\sigma)\big) = \mathbb{R}_+^*(\varepsilon_0+\sigma').$$

3 Relativistic diffusion

The relativistic diffusion $\big\{(\dot\xi_s, \xi_s)\big\}_{s\geqslant 0}$ on $\mathbb{H}\times\mathbb{R}^{1,3}$ is defined in Sect. 3.1.2. We shall see in Sect. 3.2 that this diffusion can be seen as the natural projection of a $\mathcal{G}$-valued left invariant diffusion whose support is determined in Proposition 8.

3.1 Brownian motion on $\mathbb{H}$ and relativistic diffusion

3.1.1 Brownian motion on $\mathbb{H}$ and rough asymptotic behaviour

(a) **Brownian motion on** $\mathbb{H}$ Hyperbolic Brownian motion has infinitesimal generator half the hyperbolic Laplacian. This operator has a simple expression in halfspace coordinates.

$$\triangle^{\mathbb{H}} = y^2(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_y^2) - y\,\partial_y. \qquad (3.1)$$

So if w^y is a real Brownian motion independent of the two-dimensional Brownian motion w^x, the diffusion solving the equations

$$\begin{aligned} dy_s &= y_s dw_s^y - \frac{1}{2} y_s ds, \\ dx_s &= y_s dw_s^x, \end{aligned} \tag{3.2}$$

is a Brownian motion on $\mathbb{H}$.

Notation We write $\mathbb{P}_{\dot{\xi}_0}$ the law of Brownian motion started from $\dot{\xi}_0 \in \mathbb{H}$. We may also write $\mathbb{P}_{(y_0,x_0)}$ for $\mathbb{P}_{\dot{\xi}_0}$ if $\dot{\xi}_0$ has halfspace coordinates $(y_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R}^2$.
The process $\{y_s\}_{s\geqslant 0}$ is explicit under $\mathbb{P}_{(y_0,x_0)}$:

$$\begin{aligned} y_s &= y_0\, e^{w_s^y - s}, \\ x_s &= x_0 + \int_0^s y_r dw_r^x. \end{aligned} \tag{3.3}$$

We see in Eqs. (3.3) that both y_s and x_s converge; the former to 0 and the latter to a random point $x_\infty \in \mathbb{R}^2$. Using polar coordinates (ρ_s, σ_s) to describe the evolution of the point $\dot{\xi}_s \in \mathbb{H}$, the preceding convergences means that

$$\rho_s \underset{s\to+\infty}{\longrightarrow} +\infty, \quad \text{and} \quad \sigma_s \underset{s\to+\infty}{\longrightarrow} \sigma_\infty,$$

for some random $\sigma_\infty \in \mathbb{S}^2$.

The invariance of $\triangle^{\mathbb{H}}$ under the action of isometries of $\mathbb{H}$([3]) gives Brownian motion an analogous property: for any $\dot{\xi} \in \mathbb{H}$ and any isometry φ of $\mathbb{H}$, the image by φ of a Brownian motion started from $\dot{\xi}$ has the same law as a Brownian motion started from $\varphi(\dot{\xi})$. In particuler, if $\varphi(\dot{\xi}) = \dot{\xi}$, the process $\left\{\varphi(\dot{\xi}_s)\right\}_{s\geqslant 0}$ has under $\mathbb{P}_{\dot{\xi}}$ the same law as $\{\dot{\xi}_s\}_{s\geqslant 0}$. So, if Brownian motion is started from ε_0, its asymptotic direction $\sigma_\infty \in \mathbb{S}^2$ has a law invariant under the action of $SO(3)$; that is, the law of σ_∞ is uniform. It is not difficult to deduce from this fact the law of x_∞ (and σ_∞) under any $\mathbb{P}_{\dot{\xi}_0}$.

(b) **Rough asympotic behaviour of** $\{\dot{\xi}_s\}_{s\geqslant 0}$ Let us use halfspace coordinates. Suppose Brownian motion is started from $(y_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R}^2$. If one looks at the point $\dot{\xi}_s$ in the exponential coordinates $(\widetilde{\rho}, \widetilde{\sigma})$ *associated with* (y_0, x_0), we have just seen that the law of the asymptotic direction $\widetilde{\sigma}_\infty$ of $\dot{\xi}_s$ is the uniform probability on $\mathbb{S}^2$. So, the law of x_∞ is the image of the uniform probablity on $\mathbb{S}^2$ by the function which associates to a direction $V \in \mathbb{S}^2$ the point $x_\infty \in \mathbb{R}^2$ which is "at the end" of the geodesic started from (y_0, x_0) in direction V. It has a good expression if one uses on $\mathbb{R}^2$ polar coordinates *centered at* x_0, and denoted by (r, α), $\alpha \in \mathbb{S}^1$. Denote by $d\alpha$ the uniform probability on $\mathbb{S}^1$.

$$\mathbb{P}_{(y_0,x_0)}\big(r(x_\infty) \in r + dr,\ \alpha(x_\infty) \in \alpha + d\alpha\big) = \frac{2y_0}{y_0^2 + r^2} \mathbf{1}_{r>0}\, dr\, d\alpha.$$

[3] $\triangle^{\mathbb{H}}$ is defined in purely metric terms: $\triangle^{\mathbb{H}} f = \mathrm{div}(\mathrm{grad} f)$.

This formula shows that the law of x_∞ has a smooth density with respect to Lebesgue measure dx on $\mathbb{R}^2$, depending smoothly on $\big((y_0, x_0), x\big) \in \big(\mathbb{R}_+^* \times \mathbb{R}^2\big) \times \mathbb{R}^2$.

Notation (Density of σ_∞) Returning to polar coordinates (ρ, σ), this implies that the law of σ_∞ has, under any $\mathbb{P}_{\dot{\xi}_0}$, a smooth density $h^\sigma(\dot{\xi}_0)$ with respect to the uniform probability $d\sigma$ on $\mathbb{S}^2$, with $h^\sigma(\dot{\xi}_0)$ depending smoothly on $(\dot{\xi}_0, \sigma) \in \mathbb{H} \times \mathbb{S}^2$. I is well known that, for a fixed $\sigma \in \mathbb{S}^2$, $h^\sigma(.)$ is a $\triangle^{\mathbb{H}}$-harmonic function.[4]

3.1.2 Relativistic diffusion

Definition 6 Let $\{\dot{\xi}_s\}_{s\geqslant 0}$ be a Brownian motion on $\mathbb{H}$ started from $\dot{\zeta}_0 \in \mathbb{H}$. Let $\zeta_0 \in \mathbb{R}^{1,3}$. The **relativistic diffusion on** $\mathbb{H} \times \mathbb{R}^{1,3}$, started from $(\dot{\zeta}_0, \zeta_0)$, is the process $\big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ on $\mathbb{H} \times \mathbb{R}^{1,3}$ where

$$\xi_s = \zeta_0 + \int_0^s \dot{\xi}_r \, dr.$$

Notation We shall denote by $\mathbb{P}_{\dot{\xi}_0,\xi_0}$ the law of the diffusion started from $(\dot{\xi}_0, \xi_0) \in \mathbb{H} \times \mathbb{R}^{1,3}$.

- The infinitesimal generator of the process $\big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ is given by the formula

$$Lf(\dot{\xi}, \xi) = \frac{\triangle^{\mathbb{H}}_{\dot{\xi}} f}{2} + \partial_\xi f(\dot{\xi}, \xi).\dot{\xi},$$

  where $\partial_\xi f(\dot{\xi}, \xi).\dot{\xi}$ is the differential of f with respect to ξ, in the direction $\dot{\xi}$.

From now on, we write $\triangle^{\mathbb{H}}$ for $\triangle^{\mathbb{H}}_{\dot{\xi}}$. Note that the functions $\{h^\sigma(\dot{\xi}),\ \sigma \in \mathbb{S}^2\}$ on $\mathbb{H}$ satisfying the relation $\triangle^{\mathbb{H}} h^\sigma = 0$ also satisfy the relation $Lh^\sigma = 0$, when considered as functions of $(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}$.

3.2 Relativistic diffusion as the projection of a Brownian motion on $\mathcal{G}$

(a) **A diffusion $\big\{(g_s, \xi_s)\big\}_{s\geqslant 0}$ on $\mathcal{G}$**

(i) Following Eells and Elworthy, one usually constructs Brownian motion on a Riemannian manifold $\mathbb{M}$ of dimension n as the projection of a singular diffusion on the orthonormal frame bundle $\mathbb{OM}$ of $\mathbb{M}$. Precisely, there exist canonical horizontal vector fields $\{H_i\}_{i=1,\ldots,n}$ on $\mathbb{OM}$ such that the differential operator $\sum_{i=1}^n H_i^2$ on $\mathbb{OM}$ induces Laplacian $\triangle^{\mathbb{M}}$ on $\mathbb{M}$, in the sense that if we denote by π the natural

[4] See [22], Chapts. 7 and 9, for a more general statement.

projection $\mathbb{OM} \to \mathbb{M}$, we have for all $f \in \mathcal{C}^2(\mathbb{M})$,

$$\left(\sum_{i=1}^{n} H_i^2\right) f \circ \pi = (\Delta^{\mathbb{M}} f) \circ \pi. \tag{3.4}$$

Denote by $(x, \mathbf{f})$ a generic element of $\mathbb{OM}$: $x \in \mathbb{M}$ and $\mathbf{f} = (f_1, \ldots, f_n)$ is an orthonormal frame of $T_x\mathbb{M}$. Given $j \in \{1, \ldots, n\}$, one defines a motion $\{(x(s), \mathbf{f}(s))\}$ in $\mathbb{OM}$ by asking that $\frac{dx(s)}{ds} = f_j(s)$, and that $\mathbf{f}$ should be transported parallely along $\{x(s)\}$. One defines a vector field V_j on $\mathbb{OM}$ using the infinitesimal motion of all the points according to the preceding dynamics. These vector fields ($j = 1, \ldots, n$) are the *canonical horizontal vector fields*.

(ii) In our situation, $\mathbb{H}$ being the half unit pseudo-sphere of $\mathbb{R}^{1,3}$, a point in $\mathbb{OH}$ is an orthonormal frame of $\mathbb{R}^{1,3}$. So $\mathbb{OH}$ can be identified with the set of orthonormal bases (g^0, g^1, g^2, g^3) of $\mathbb{R}^{1,3}$, with $g^0 \in \mathbb{H}$, and the natural projection is $g \mapsto g^0$. The horizontal vector fields H_i are:

$$H_i(g) = gE_i, \quad i = 1, \ldots, 3,$$

where the E_i are the matrices $E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
$E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

These left invariant vector fields give rise to the left invariant differential operator $H_1^2 + H_2^2 + H_3^2$. If g belongs to $SO_0(1, 3)$ (that is, if $\left(g^1, g^2, g^3\right)$ is a direct basis of $T_{g^0}\mathbb{H}$), the diffusion $\{g_s\}_{s \geqslant 0}$ on $O(1, 3)$ solution of the following stochastic differential equation, with initial condition g, remains in $SO_0(1, 3)$.

$$dg_s = g_s E_1 \circ dw_s^1 + g_s E_2 \circ dw_s^2 + g_s E_3 \circ dw_s^3.$$

The w^i are real independent Brownian motions. By construction, $\{g_s^0\}_{s \geqslant 0}$ is a Brownian motion on $\mathbb{H}$, started from g^0.

(iii) So, the natural framework to construct the relativistic diffusion seems to be $SO_0(1, 3) \times \mathbb{R}^{1,3}$, the equations of motion being

$$\begin{aligned} dg_s &= g_s E_i \circ dw_s^i, \\ d\xi_s &= g_s^0\, ds. \end{aligned} \tag{3.5}$$

We use Einstein's conventions on summations. Set $\widetilde{E}_i = (E_i, 0)$, for $i = 1, \ldots, 3$, $\widetilde{E}_0 = (0, \varepsilon_0)$, and define left invariant vector fields V_i, $i = 0, \ldots, 3$, on the group $\mathcal{G}$, setting

$$V_i\big((g,\xi)\big) = (g,\xi)\widetilde{E}_i.$$

It is elementary to see that (3.5) is equivalent to

$$d\big((g_s,\xi_s)\big) = V_i\big((g_s,\xi_s)\big) \circ dw_s^i + V_0\big((g_s,\xi_s)\big)\, ds. \tag{3.6}$$

The group $\mathcal{G}$ naturally projects to $\mathbb{H}\times\mathbb{R}^{1,3}$: $(g,\xi)\mapsto(g^0,\xi)$; the diffusion $\{(g_s,\xi_s)\}_{s\geqslant 0}$ is the natural lift of the relativistic diffusion to $\mathcal{G}$. Its infinitesimal generator is

$$\widetilde{L} \equiv \frac{1}{2}\sum_{i=1,\dots,3} V_i^2 + V_0. \tag{3.7}$$

The algebraic framework provided by the group $\mathcal{G}$ will be useful in Sect. **5.2.2** to establish the uniform continuity of some functions.

(b) **Support of the diffusion** $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$**—Notation**—Denote by $\mathbb{P}_{(g,\xi)}$ the law of the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ on $\mathcal{G}$, started from (g,ξ).

We determine its suppport using the support theorem, as presented in Theorem 8.2 in [19]. Denote by $\|.\|_T$ the uniform norm on $\mathcal{C}\big([0,T],\mathcal{G}\big)$.

Theorem 7 (Stroock, Varadhan) *If ϕ^i, $i=1,\dots,d$, are piecewise smooth, continuous controls, and if $\varphi\big(t;(g,\xi)\big)$ is the solution to the equation*

$$d\big(\varphi(t)\big) = V_i\big(\varphi(t)\big)\phi_t^i dt + V_0\big(\varphi(t)\big)dt, \quad \varphi(0) = (g,\xi), \tag{3.8}$$

on $\mathcal{G}$, then one has

$$\forall\, T>0,\ \mathbb{P}_{(g,\xi)}\big(\|(g_s,\xi_s) - \varphi\big(s;(g,\xi)\big)\|_T < \varepsilon \ \mid\ \|w-\phi\|_T < \delta\big) \to 1,\ \ as\ \delta \searrow 0. \tag{3.9}$$

Proposition 8 (Support of the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ on $\mathcal{G}$) *Let $(g_0,\xi_0)\in\mathcal{G}$, $(\underline{g},\underline{\xi})\in\mathcal{G}$ be such that $q(\underline{\xi}-\xi_0)>0$, and let $\mathcal{V}_1\times\mathcal{V}_2$ be a product neighbourhood of $(\underline{g},\underline{\xi})$. Then*

$$\mathbb{P}_{(g_0,\xi_0)}\left(\int_0^\infty \mathbf{1}_{\mathcal{V}_1\times\mathcal{V}_2}\big((g_s,\xi_s)\big)\, ds > 0\right) > 0.$$

◁ From the support Theorem 7 it is sufficient to find smooth controls $\phi^1,\dots,\phi^d$, defined on some interval $[0,T]$, such that the path $\big\{\big(g(s),\xi(s)\big)\big\}_{0\leqslant s\leqslant T}$ solving the equation

$$\begin{aligned} &d\big(g(s),\xi(s)\big) = V_i\big(\big(g(s),\xi(s)\big)\big)\phi^i(s)ds + V_0\big(\big(g(s),\xi(s)\big)\big)ds,\ s\in[0,T],\\ &\big(g(0),\xi(0)\big) = \big((\varepsilon_0,\dots,\varepsilon_3),0\big),\end{aligned}$$

satisfies $\big(g(T),\xi(T)\big)\in\mathcal{V}_1\times\mathcal{V}_2$.

(1) First, find a smooth timelike path $\gamma = \{\varphi(s)\}_{0\leqslant s\leqslant T}$ on $\mathbb{R}^{1,3}$, with $\dot\varphi(s) = \frac{d\varphi(s)}{ds} \in \mathbb{H}$, defined on an unprescribed interval $[0, T]$, such that
- $\varphi(0) = 0$, $\dot\varphi(0) = \varepsilon_0$, $\dot\varphi(T) = \underline{g}^0$, and
- $\varphi(T)$ is not far from $\underline{\xi}$.

Such a path exists. Parallel transport in $\mathbb{H}$ of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ along $\{\dot\varphi(s)\}_{s\in[0,T]}$ defines a lift $\{g_s^{\dot\varphi}\}_{s\in[0,T]}$ of $\{\dot\varphi(s)\}_{s\in[0,T]}$ to $SO_0(1,3)(\simeq \mathbb{OH})$. It gives us a unique 3-uple $\big(\varphi_1(s), \varphi_2(s), \varphi_3(s)\big)$ of smooth functions defined on $[0, T]$, and such that

$$d\big(\dot\varphi(s)\big) = (g_s^{\dot\varphi})^i \varphi_i(s)\, ds.$$

(2) Now, consider the following equation on $\mathcal{G}$.

$$d\big(\psi(s)\big) = \sum_{i=1}^{3} V_i\big(\psi(s)\big)\varphi_i(s)ds + V_0\big(\psi(s)\big)ds, \quad \psi(0) = (\varepsilon_0, \ldots, \varepsilon_d).$$

Its solution ψ is such that $\psi(T)$ is of the form

$$\big((\dot\varphi(T), g^1, g^2, g^3), \varphi(T)\big) = \big((\underline{g}^0, g^1, g^2, g^3), \varphi(T)\big),$$

where (g^1, g^2, g^3) is an orthonormal basis of $T_{\underline{g}^0}\mathbb{H}$. This basis need not be near $\left(\underline{g}^1, \underline{g}^2, \underline{g}^3\right)$. The geometry of $\mathbb{H}$ allows one to change a little $\dot\varphi$ (*i.e.* the φ_i's) so that $\psi(T)$ is near $\left(\underline{g}, \underline{\xi}\right)$.

Proposition 9 [7, p. 641] *Let $\dot\xi \in \mathbb{H}$ be given. One can obtain any orthonormal basis of $T_{\dot\xi}\mathbb{H}$ by parallel transport of a fixed orthonormal basis of $T_{\dot\xi}\mathbb{H}$ along loops contained in an arbitrarily small neighbourhood of $\dot\xi \in \mathbb{H}$.*

(3) We take advantage of this fact to add to the path $\dot\varphi$ a small loop at its end, along which one the parallel transport application maps $\big(g^1, g^2, g^3\big)$ to $\big(\underline{g}^1, \underline{g}^2, \underline{g}^3\big)$. For $0 < \varepsilon < 1$, define $\dot\phi$ by requiring that
- $\dot\phi_s = \dot\varphi_{(1+\varepsilon)sT}$, for $s \leqslant \frac{1}{1+\varepsilon}$, and
- $\dot\phi_s$ describes the loop for $\frac{1}{1+\varepsilon} \leqslant s \leqslant 1$.

If ε is small enough, $\phi(1)$ remains near $\underline{\xi}$. The lift of $\dot\phi$ to $\mathcal{G}$ provides the required ϕ^i.

4 Analytic framework, Poisson boundary of L

The operators $\widetilde{L}-\partial_s$ and $L-\partial_s$ are hypoelliptic (**4.1**). This smoothing property enables one to give an analytical counterpart to the probabilistic problem of the determination of the invariant σ-algebra of $\big\{(\dot\xi_s, \xi_s)\big\}_{s\geqslant 0}$: determine the Poisson boundary of its infinitesimal generator L. Using a variation on a theorem of Bony about some special class of hypoelliptic differential operators, we first obtain in Theorem **17** a compactness property of the set of non negative L-harmonic functions. This theorem is needed to apply Choquet's theorem on integral representation in well capped cones. Its use in Sect. 4.3 to determine the Poisson boundary of L is based on a theorem which will be proved in Sect. 5.

4.1 Infinitesimal generator

(a) From a semi-group point of view, the diffusions $\{(\dot{\xi}_s, \xi_s)\}_{s\geqslant 0}$ and $\{(g_s, \xi_s)\}_{s\geqslant 0}$ on $\mathbb{H}\times\mathbb{R}^{1,3}$ and $\mathcal{G}$ are characterized by their infinitesimal generators, respectively

$$Lf(\dot{\xi},\xi) = \frac{\triangle^{\mathbb{H}} f(\dot{\xi},\xi)}{2} + \partial_\xi f(\dot{\xi},\xi).\dot{\xi}, \quad f\in\mathcal{C}_0^\infty\left(\mathbb{H}\times\mathbb{R}^{1,d}\right),$$

and

$$\widetilde{L}\widetilde{f} = \frac{1}{2}\sum_{i=1,\dots,3} V_i^2\widetilde{f} + V_0\widetilde{f}, \quad \widetilde{f}\in\mathcal{C}_0^\infty(\mathcal{G}).$$

The two operators are linked: if the function $\widetilde{f}(g,\xi)$ depends only on g^0 and ξ, then

$$\widetilde{L}\widetilde{f} = L\widetilde{f}. \tag{4.1}$$

Notation Let $Haar$ be a (left) Haar measure on $\mathcal{G}$. Write $\widetilde{L}^*$ the $\mathbb{L}^2(Haar)$ adjoint of $\widetilde{L}$.

The vector fields V_i being left invariant, we have for any $\widetilde{f},\widetilde{g}\in\mathcal{C}_0^\infty(\mathcal{G})$,

$$\int \widetilde{f}\big(V_i^2\widetilde{g}\big)Haar = \int \big(V_i^2\widetilde{f}\big)\widetilde{g}Haar;$$

so $\widetilde{L}^* = \frac{1}{2}\sum_{i=1,\dots,3} V_i^2 - V_0$. Set

$$E_{12} = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&-1&0&0\\ 0&0&0&0\end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0\end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&-1&0\end{pmatrix}.$$

Proposition 10 *The differential operators $\widetilde{L}-\partial_s$ and $\widetilde{L}^*-\partial_s$ on $\mathcal{G}\times\mathbb{R}_+^*$ are hypoelliptic.*

◁ Write $\widetilde{E}_{pq} = (E_{pq}, 0)$, $1\leqslant p<q\leqslant 3$, and $\widetilde{A}_i = (0,\varepsilon_i)$, $1\leqslant i\leqslant 3$. The brackets between the $\widetilde{E}_i$'s and $\widetilde{A}_0$ are given by the following relations

$$[\widetilde{E}_i, \widetilde{E}_j] = \widetilde{E_{ij}}, \quad [\widetilde{E}_i, \widetilde{A_0}] = \widetilde{A}_i, \quad [\widetilde{E}_i, \widetilde{A}_i] = \widetilde{A_0}.$$

As the family $\{\widetilde{E}_i,\ \widetilde{E}_{jk},\ \widetilde{A}_\ell;\ i=1,\dots,3,\ 1\leqslant j<k\leqslant 3,\ \ell=0,\dots,3\}$ is a basis of the Lie algebra of $\mathcal{G}$, Hörmander's theorem ensures the hypoellipticity of $\widetilde{L}-\partial_s$ and $\widetilde{L}^*-\partial_s$. ▷

Definition 11 • Let L' be a second order differential operator. A $\mathcal{C}^2$ *function* f *satisfying the relation* $L'f = 0$ *is said to be* L'*-harmonic*.
• For historical reasons, the set of bounded L'-harmonic functions is called the *Poisson boundary of* L'.

Notation Denote by $\{\widetilde{P}_s\}_{s\geqslant 0}$ the semi-group of the process $\{(g_s, \xi_s)\}_{s\geqslant 0}$ on $\mathcal{G}$, and $\{P_s\}_{s\geqslant 0}$ that of the process $\{(\dot{\xi}_s, \xi_s)\}_{s\geqslant 0}$ on $\mathbb{H}\times\mathbb{R}^{1,3}$.
(b) From probability to analysis—It follows from the hypoellipticity of $\widetilde{L} - \partial_s$ that for any bounded Borel function $\widetilde{f}$ on $\mathcal{G}$, the function $\widetilde{P}_s\widetilde{f}\big((g,\xi)\big)$ is a smooth function of s and (g,ξ) which is $\big(\widetilde{L} - \partial_s\big)$-harmonic. From relation (4.1), $P_s f$ is a smooth $(L-\partial_s)$-harmonic function on $(\mathbb{H}\times\mathbb{R}^{1,3})\times\mathbb{R}_+^*$ if f is any bounded Borel function on $\mathbb{H}\times\mathbb{R}^{1,3}$. We can write

$$\widetilde{P}_s\widetilde{f}\big(g,\xi\big) = \int_{\mathcal{G}} \widetilde{f}(g',\xi')\,\widetilde{p}_s\big((g,\xi),(g',\xi')\big)Haar\big(d(g',\xi')\big).$$

Proposition **10** ensures that $\widetilde{p}_s\big((g,\xi),(g',\xi')\big)$ is a smooth function of $\big(s,(g,\xi)\big)$ and $\big(s,(g',\xi')\big)$, separately. Remark that the left invariance of the vector fields V_i implies the left invariance of $\widetilde{p}_s$:

$$\forall\,(\underline{g},\underline{\xi})\in\mathcal{G},\quad \widetilde{p}_s\big((\underline{g},\underline{\xi})(g,\xi),(\underline{g},\underline{\xi})(g',\xi')\big) = \widetilde{p}_s\big((g,\xi),(g',\xi')\big).$$

So, $\widetilde{p}_.(.,.)$ is jointly continuous in $\big(s,(g,\xi),(g',\xi')\big)$.
Notation (Invariant σ-algebra of $\{(\dot{\xi}_s,\xi_s)\}_{s\geqslant 0}$) Denote by $Inv\big((\dot{\xi}_s,\xi_s)\big)$ the invariant σ-algebra of the relativistic diffusion $\{(\dot{\xi}_s,\xi_s)\}_{s\geqslant 0}$.

The following classical proposition states that it is the same problem to determine $Inv\big((\dot{\xi}_s,\xi_s)\big)$ or to determine the Poisson boundary of L.

Proposition 12 (Probability/Analysis correspondence)

• *For any bounded* $Inv\big((\dot{\xi}_s,\xi_s)\big)$*-measurable random variable* X*, the formula*

$$(\dot{\xi},\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}\mapsto \mathbb{E}_{\dot{\xi},\xi}[X] \tag{4.2}$$

defines an L*-harmonic bounded function.*

• *Conversely, any bounded* L*-harmonic function on* $\mathbb{H}\times\mathbb{R}^{1,3}$ *is of this form, for some bounded* $Inv\big((\dot{\xi}_s,\xi_s)\big)$*-measurable random variable* X.

$\lhd$ • Given an $Inv\big((\dot{\xi}_s,\xi_s)\big)$-measurable bounded random variable X, the measurable bounded function $h : (\dot{\xi},\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}\mapsto\mathbb{E}_{\dot{\xi},\xi}[X]$, satisfies the identities

$$\begin{aligned}P_s h(\dot{\xi},\xi) &= \mathbb{E}_{\dot{\xi},\xi}\big[h(\dot{\xi}_s,\xi_s)\big] = \mathbb{E}_{\dot{\xi},\xi}\big[\mathbb{E}_{\dot{\xi}_s,\xi_s}[X]\big] = \mathbb{E}_{\dot{\xi},\xi}[X\circ\theta_s]\\ &= \mathbb{E}_{\dot{\xi},\xi}[X] = h(\dot{\xi},\xi).\end{aligned}$$

So it follows from point (b) that h is smooth and L-harmonic, since it does not depend on s.

- Reciprocally, given a bounded L-harmonic function h, define the $Inv\big((\dot{\xi}_s, \xi_s)\big)$-measurable random variable

$$X = \lim_{s \to +\infty} h(\dot{\xi}_s, \xi_s), \quad \text{when it exists, 0 elsewhere.}$$

For $(\dot{\xi}_0, \xi_0) \in \mathbb{H} \times \mathbb{R}^{1,3}$, the process $\big\{h(\dot{\xi}_s, \xi_s)\big\}_{s \geqslant 0}$ is, under $\mathbb{P}_{\dot{\xi}_0, \xi_0}$, a bounded martingale, so

$$h(\dot{\xi}_0, \xi_0) = \mathbb{E}_{\dot{\xi}_0, \xi_0}[X].$$

▷

We are going to describe the Poisson boundary of L. Before explaining in Sect. 4.3 how to proceed, we need a compactness result on the set of non negative L-harmonic functions.

4.2 Compactness matters

In this paragraph, we shall use a result of Bony to obtain a Harnack compactness principle for non-negative $\widetilde{L}$-harmonic functions; it is stated in Theorem 17. Here is the result we would like to use.

Theorem 13 (Bony's Harnack Inequality [8]) *Let $L' = \sum_{i=1}^{r} V_i^2 + V$ be a differential operator on a connected manifold $\mathbb{M}$ of dimension n such that the V_i cannot be null all at the same time. Suppose the Lie algebra $\mathcal{L}(V_1, \ldots, V_r)$ generated by the V_i has rank n at every point. Then, for any compact K contained in a chart $\{x^i\}$, every point $y_0 \in \mathbb{M}$ and every multi-index p, there exists a constant λ, depending on K, $\{x^i\}$ and p, such that every nonnegative L'-harmonic function h satisfies*

$$\sup_{x \in K} \left| \frac{\partial^p h(x)}{\partial x^p} \right| \leqslant \lambda\, h(y_0). \tag{4.3}$$

Before giving its proof, recall Bony demonstrated in [8], Corollary 5.2, the following fact.

Proposition 14 *If the V_i's are not null all at the same time and rank$\big(\mathcal{L}(V_1, \ldots, V_r, V)\big) = n$, everywhere, then there exists a basis of the topology made up of open sets $\mathcal{U}$ for which the Dirichlet problem*

$$\text{for } f \in \mathcal{C}(\partial\mathcal{U}) \text{ find } u \in \mathcal{C}^2(\mathcal{U}) \cap \mathcal{C}(\overline{\mathcal{U}}) \text{ such that } L'u = 0 \text{ on } \mathcal{U} \text{ and } u_{|\partial\mathcal{U}} = f$$

has a unique solution. These open sets are called elementary open sets.

Here is Bony's proof of Theorem 13.

◁ Let x_0 be a point in K. On the one hand, one knows that given two relatively compact open neighbourhoods $\mathcal{O} \Subset \mathcal{O}'$ of x_0, small enough to be in a chart, and a multi-index p, there exists a constant c' such that any L'-harmonic function h satisfies the inequality[5]

$$\sup_{x\in\mathcal{O}}\left|\frac{\partial^p h(x)}{\partial x^p}\right| \leqslant c' \int_{\mathcal{O}'} h(x)dx.$$

On the other hand, one knows that when we are in an elementary open set $\mathcal{U}$, one has

$$h(y_0) \geqslant \beta \int_{\mathcal{U}} g_\beta^{\mathcal{U}}(y_0, x)h(x)dx,$$

where $g_\beta^{\mathcal{U}}$ is the Green function of the operator $L' - \beta$ on $\mathcal{U}$ with respect to the measure dx.[6] So, if we could suppose y_0, $\mathcal{O}$ and $\mathcal{O}'$ to be in an elementary open set $\mathcal{U}$, and $g_\beta^{\mathcal{U}}(y_0, \mathcal{O}') \geqslant c'' > 0$,[7] we would have

$$\sup_{x\in\mathcal{O}}\left|\frac{\partial^p h(x)}{\partial x^p}\right| \leqslant \frac{c'}{c''\beta} h(y_0).$$

The hypothesis on the rank of the Lie algebra is made so as to ensure that the Strong Minimum Principle holds, and with it the fact that $g_\beta^{\mathcal{U}}(., x_0)$ being > 0 somewhere is > 0 everywhere; in particular $g_\beta^{\mathcal{U}}(y_0, x_0) > 0$. The continuity of $g_\beta^{\mathcal{U}}(y_0, .)$ gives a neighbourhood $\mathcal{O}'$ of x_0 and a positive constant c'' such that $g_\beta^{\mathcal{U}}(y_0, \mathcal{O}') \geqslant c''$. The constant c' is determined as soon as $\mathcal{O}'$, $\mathcal{O}$, p and the coordinates x are chosen.
To obtain the Harnak inequality not just for y_0, $\mathcal{O}$, $\mathcal{O}'$, in an elementary open set, one uses connectedness of $\mathbb{M}$ and compactness of K. ▷

This argument cannot be applied without any change to our situation, where the operator $\widetilde{L} = \frac{1}{2}\sum_{i=1}^d V_i^2 + V_0$ on $\mathcal{G}$ is such that $\mathrm{rank}\big(\mathcal{L}(V_1, \ldots, V_d)\big) < \dim \mathcal{G}$. Yet, the rank hypothesis is made just to ensure the positivity of $g_\beta^{\mathcal{U}}(y_0, x_0)$. We can get this positivity thanks to the support Theorem **7**.
Notation Until the end this paragraph, we denote by $\mathbf{e}$ a generic element of $\mathcal{G}$ instead of (g, ξ).

Given piecewise smooth, continuous controls, ϕ^i, $i = 1, \ldots, 3$, and $\mathbf{e}_0 \in \mathcal{G}$, the equation

$$d\big(\varphi(t)\big) = V_i\big(\varphi(t)\big)\phi_t^i dt + V_0\big(\varphi(t)\big)dt, \quad \varphi(0) = \mathbf{e_0}, \tag{4.4}$$

is the control equation associated with the ϕ^i and $\mathbf{e}_0$. We denote by $\varphi(t; \mathbf{e}_0)$ its solution.

5 This is a quantitative version of Hörmander's theorem on hypoellipticity.

6 The existence of $g_\beta^{\mathcal{U}}$ is proved in [8].

7 $g_\beta^{\mathcal{U}}(y_0, \mathcal{O}') \geqslant c''$ means $\inf\{g_\beta^{\mathcal{U}}(y_0, x);\, x \in \mathcal{O}'\} \geqslant c''$.

Definition 15 • Let $\mathcal{U}$ be an open set of $\mathcal{G}$ and let $\mathbf{e}_0 \in \mathcal{U}$. A point $\mathbf{z}$ is said *to be in the future of* $\mathbf{e}_0$ in $\mathcal{U}$ if there exists controls ϕ^i, and some $t > 0$, such that $z = \varphi(t\ \mathbf{e}_0)$, and if $\varphi\big([0,t]; \mathbf{e}_0\big) \subset \mathcal{U}$. We shall write $\varphi(t; \mathbf{e}_0) = \varphi_{\mathbf{z}}(t; \mathbf{e}_0)$ to emphasize the fact that $\varphi(t; \mathbf{e}_0)$ is associated with $\mathbf{z}$.

- A set S is said to be in the future of $\mathbf{e}_0$ in $\mathcal{U}$ if each of its points is in the future of $\mathbf{e}_0$ in $\mathcal{U}$. The future of $\mathbf{e}_0$ is its future in $\mathcal{G}$.

The future of $\mathbf{e}_0$ (in $\mathcal{G}$) is the interior of the support of the Green function of the diffusion $\big\{(g_s, \xi_s)\big\}_{s\geqslant 0}$, started from $\mathbf{e}_0 = (g_0, \xi_0)$. From Proposition 8, it is equal to $\big\{(g, \xi) \in \mathcal{G}\,;\ q(\xi - \xi_0) > 0\big\}$.

Suppose $\mathbf{e_0}$ and $\mathcal{O}'$ are in an elementary open set $\mathcal{U}$ and $\mathcal{O}'$ is in the future of $\mathbf{e_0}$ in $\mathcal{U}$. If one takes $\mathbf{z}$ in $\mathcal{O}'$, the path $\varphi_{\mathbf{z}}(\cdot; \mathbf{e_0})$ spends a positive amount of time in $\mathcal{O}'$. Choose $\varepsilon > 0$ small enough so that $B(\mathbf{z}, \varepsilon) \subset \mathcal{O}'$, and that the ε-neighbourhood of $\varphi_{\mathbf{z}}([0,t]; \mathbf{e_0})$ should be included in $\mathcal{U}$. Then the diffusion $\{\mathbf{e}_s\}$, started from $\mathbf{e}_0$, spends a positive amount of time in $\mathcal{O}'$ with a positive probability, before leaving $\mathcal{U}$ (support theorem); that is

$$g_\beta^{\mathcal{U}}(\mathbf{e}_0, \mathcal{O}') > 0.$$

The function $g_\beta^{\mathcal{U}}(\mathbf{e}_0, .)$ being continuous, we can suppose, without loss of generality, that $g_\beta^{\mathcal{U}}(\mathbf{e}_0, \mathcal{O}') \geqslant c'' > 0$.

Now, suppose K is a compact in the interior of the future of $\mathbf{e_0}$ in $\mathcal{G}$. Let $\mathbf{z}_0 \in K$ and $\varphi_{\mathbf{z}_0}(.; \mathbf{e}_0)$ be a solution of (4.4) such that $\varphi_{\mathbf{z}_0}(T; \mathbf{e}_0) = \mathbf{z}_0$, for some $T > 0$. One can find a finite sequence $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_s$ of points of $\varphi_{\mathbf{z}_0}\big([0,T]; \mathbf{e}_0\big)$, elementary open sets $\mathcal{U}_i, i = 0, \ldots, (s-1)$, open sets $\mathcal{O}'_i, i = 1, \ldots, s$, such that $\{\mathbf{e}_i\}$ and $\mathcal{O}'_{i+1}$ are included in $\mathcal{U}_i$, $\mathcal{O}'_{i+1}$ is in the future of $\mathbf{e}_i$, and $\mathbf{e}_{i+1} \in \mathcal{O}'_{i+1}$. The conclusion of Bony's Harnack inequality applies in each $\mathcal{U}_i$. If one takes $p = 0$ in (4.3), for $i = 1, \ldots, (s-1)$, and use (4.3) with p for $i = s$, one obtains the existence of a constant $\lambda_{\mathbf{z}_0}$, and of some neighbourhood $\mathcal{O}_{\mathbf{z}_0}$ of $\mathbf{z}_0$, such that the inequality

$$\sup_{x\in\mathcal{O}_{\mathbf{z}_0}} \left|\frac{\partial^p h(x)}{\partial x^p}\right| \leqslant \lambda_{\mathbf{z}_0} h(\mathbf{e}_0),$$

holds for any nonnegative $\widetilde{L}$-harmonic function h. The compactness of K then yields the following version of Bony's result.

Theorem 16 (Harnack's inequality) *Let K be a compact subset of $\mathcal{G}$, located in the future of a point $\mathbf{e}_0 \in \mathcal{G}$, small enough to be in a chart $\{\mathbf{e}^i\}$, and let p be a mutli-index. There exists a constant λ, depending on K, $\{\mathbf{e}^i\}$ and p, such that every nonnegative $\widetilde{L}$-harmonic function h satisfies*

$$\sup_{\mathbf{e}\in K} \left|\frac{\partial^p h(\mathbf{e})}{\partial \mathbf{e}^p}\right| \leqslant \lambda\, h(\mathbf{e}_0).$$

The smallness restriction is not a real one since any compact subset K can be covered by finitely many charts.

Harnack inequality and Ascoli's theorem justify

Theorem 17 (Harnack compactness principle) • *Let* $\{h_n\}_{n\geqslant 0}$ *be a sequence of non-negative* $\widetilde{L}$*-harmonic functions. If there is a point* $\mathbf{e}_0 \in \mathcal{G}$ *such that the sequence* $\left\{h_n(\mathbf{e}_0)\right\}_{n\geqslant 0}$ *is bounded, then* $\{h_n\}_{n\geqslant 0}$ *has a subsequence that* $\mathcal{C}^\infty$*-converges uniformly and locally on*[8]

$$\left\{(g,\xi)\in\mathcal{G}\,;\; q(\xi-\xi_0)>0\right\}.$$

- *An analogue result holds for* L*-harmonic functions: if there exists a point* $(\dot\xi_0,\xi_0)\in\mathbb{H}\times\mathbb{R}^{1,3}$ *such that the sequence* $\left\{h_n(\dot\xi_0,\xi_0)\right\}_{n\geqslant 0}$ *is bounded, then* $\{h_n\}_{n\geqslant 0}$ *has a subsequence that* $\mathcal{C}^\infty$*-converges uniformly and locally on the set*

$$\left\{(\dot\xi,\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}\,;\; q(\xi-\xi_0)>0\right\}.$$

4.3 Representation theorem and consequences

(a) **Representation theorem** Denote by (ρ_s,σ_s) the polar coordinates of $\dot\xi_s\in\mathbb{H}$. Recall the notation $h^\sigma(\dot\xi,\xi)$ for the density of the law of σ_∞ under $\mathbb{P}_{\dot\xi,\xi}$ with respect to the uniform probability $d\sigma$ on $\mathbb{S}^2$. Section **5** is dedicated to the proof of the following theorem.

Theorem 18 *Given* $(\dot\xi,\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}$,

1. *the limits*

$$\sigma_\infty=\lim_{s\to+\infty}\sigma_s,\quad R_\infty^{\sigma_\infty}=\lim_{s\to+\infty}q(\xi_s,\varepsilon_0+\sigma_\infty),$$

   exist $\mathbb{P}_{\dot\xi,\xi}$*-almost surely*;
2. *the law of the pair* $(\sigma_\infty,R_\infty^{\sigma_\infty})$ *has, under* $\mathbb{P}_{\dot\xi,\xi}$*, a density with respect to the product measure* $d\sigma d\ell$ *on* $\mathbb{S}^2\times\mathbb{R}$ *of the form*

$$h^\sigma(\dot\xi,\xi)h_\ell^\sigma(\dot\xi,\xi),$$

   for some explicit function $h_\ell^\sigma(\dot\xi,\xi)$ *depending smoothly on* $\sigma\in\mathbb{S}^2$, $\ell\in\mathbb{R}$ *and* $(\dot\xi,\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}$;
3. *for any* $\sigma\in\mathbb{S}^2$, $\ell\in\mathbb{R}$*, the nonnegative function* $h^\sigma h_\ell^\sigma(.)$ *is a minimal* L*-harmonic function.*

This theorem provides a decomposition of the constant function **1** as a mean of L-harmonic minimal functions:

$$\mathbf{1}=\int_{\mathbb{S}^2\times\mathbb{R}} h^\sigma h_\ell^\sigma\, d\sigma d\ell. \tag{4.5}$$

[8] "$\mathcal{C}^\infty$-convergence uniform and local", means that for any $p\geqslant 0$, all the derivatives of h of order $\leqslant p$ uniformly converge on compacta.

(b) **Poisson boundary of** L The preceding formula and Harnack compactness principle (Theorem 17) are all we need to determine the Poisson boundary of L. This paragraph is dedicated to the description of this boundary. We follow the classical Martin's scheme, as used in the study of heat equation (see for instance Doob [13], Chap. XIX, or [1], and the references therein).

The following lemma enables us to define a well capped cone adapted to the situation.

Lemma 19 *For $n \in \mathbb{N}$, set $(\dot{\zeta}_n, \zeta_n) \equiv (\varepsilon_0, -n\varepsilon_0) \in \mathbb{H} \times \mathbb{R}^{1,3}$. One has*

$$\sum_{n \geqslant 0} h^\sigma h^\sigma_\ell (\dot{\zeta}_n, \zeta_n) < +\infty,$$

for any $\sigma \in \mathbb{S}^2$, $\ell \in \mathbb{R}$.

◁ Using the explicit expression of $h^\sigma h^\sigma_\ell$ given in (5.23), we read in the formula

$$h^\sigma h^\sigma_\ell (\dot{\zeta}_n, \zeta_n) = (\ell + n)^{-3} \exp\left(-\frac{2}{\ell + n}\right),$$

the convergence of the series. ▷

Choose a sequence of positive numbers p_n such that $\sum_{n \geqslant 0} p_n = 1$ and set $\nu \equiv \sum_{n \geqslant 0} p_n \delta_{(\dot{\zeta}_n, \zeta_n)}$. The set of $\mathbb{L}^1(\nu)$-integrable functions contains the set of bounded functions. Write $\langle f, \nu \rangle \equiv \int f d\nu$, and define the following subsets of L-harmonic functions.

$$\mathbf{C}_\nu \equiv \{h \geqslant 0; Lh = 0, \langle h, \nu \rangle < +\infty\}, \ \mathbf{K}_\nu \equiv \{h \in C_\nu\,;\ \langle h, \nu \rangle \leqslant 1\}, \quad \text{and}$$
$$\mathbf{K}^1_\nu \equiv \{h \in C_\nu\,;\ \langle h, \nu \rangle = 1\}. \tag{4.6}$$

$\mathbf{C}_\nu$ is a convex cone containing the functions $h^\sigma h^\sigma_\ell$, and $\mathbf{K}_\nu$ and $\mathbf{K}^1_\nu$ are convex subsets of $\mathbf{C}_\nu$. As a consequence of Harnack inequality (Theorem 16), the only function $h \in \mathbf{C}_\nu$ such that $\langle h, \nu \rangle = 0$ is the zero function. Endow $\mathbf{C}_\nu$ with the topology of uniform convergence on compacta.

Proposition 20 $\mathbf{K}_\nu$ *is compact.*

◁ The topology being metrisable, we check that any sequence of points of $\mathbf{K}_\nu$ has a convergent subsequence. Any function $h \in \mathbf{K}_\nu$ satisfying the relation $\langle h, \nu \rangle = \sum_{n \geqslant 0} p_n h(\dot{\zeta}_n, \zeta_n) \leqslant 1$, one has $h(\dot{\zeta}_n, \zeta_n) \leqslant \frac{1}{p_n}$, for any integer. Fix n. The compactness principle (Theorem 17) enables us to extract from any sequence $\{h_p\}$ of points of $\mathbf{K}_\nu$ a subsequence that uniformly locally $\mathcal{C}^\infty$-converges on the set $\{(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}\,;\ q(\xi - \zeta_n) \geqslant 0\}$. So, a diagonal extraction provides a subsequence uniformly locally converging on the set $\bigcup_{n \geqslant 0} \{(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}\,;\ q(\xi - \zeta_n) \geqslant 0\}$. The choice of the u_n was made so as to ensure that this set is equal to $\mathbb{H} \times \mathbb{R}^{1,3}$. The limit function belongs to $\mathbf{K}_\nu$ because $0 \leqslant \langle \underline{\lim}\, h_p, \nu \rangle \leqslant \underline{\lim}\, \langle h_p, \nu \rangle \leqslant 1$. ▷

Proposition 21 $\mathbf{C}_\nu$ *is a lattice with respect to its own order.*

$\lhd$ This is so because the set of nonnegative L-harmonic functions is itself a lattice with respect to its own order, and if we have $0 \leqslant h' \leqslant h$, with $h \in \mathbf{C}_\nu$, then $h' \in \mathbf{C}_\nu$. $\rhd$

In this context, Choquet's representation theory[9] applies.

Theorem 22 (Choquet's representation theorem)

1. *Any point h of $\mathbf{K}^1_\nu$ can be uniquely written as*

$$h = \int \underline{h}\ \mu_h(d\underline{h}),$$

where μ_h is a probability measure supported on the set of extremal points of $\mathbf{K}^1_\nu$.

2. • *Associate to each non-null $h \in \mathbf{C}_\nu$, first, the point $\frac{h}{\langle h,\nu\rangle} \in \mathbf{K}^1_\nu$, and then the probability $\mu_{\frac{h}{\langle h,\nu\rangle}}$ on the set of extremal points of $\mathbf{K}^1_\nu$. The application*

$$h \in \mathbf{C}_\nu \mapsto \widetilde{\mu}_h \equiv \langle h, \nu\rangle \mu_{\frac{h}{\langle h,\nu\rangle}} \tag{4.7}$$

is a lattice isomorphism between $\mathbf{C}_\nu$ and the set of Borel finite measures on $\mathbf{K}^1_\nu$.[10]
• *In particular, if $h \leqslant h'$, then $\widetilde{\mu}_h \leqslant \widetilde{\mu}_{h'}$; so $\widetilde{\mu}_h = F\widetilde{\mu}_{h'}$, for some Borel function F such that one has $0 \leqslant F \leqslant 1$, $\widetilde{\mu}_{h'}$-almost everywhere.*

As one has $0 < \langle h^\sigma h^\sigma_\ell, \nu\rangle < +\infty$ for all σ, ℓ, one can rewrite the integral representation (4.5) as

$$\mathbf{1} = \int_{\mathbb{S}^2\times\mathbb{R}} \frac{h^\sigma h^\sigma_\ell}{\langle h^\sigma h^\sigma_\ell, \nu\rangle} \langle h^\sigma h^\sigma_\ell, \nu\rangle\, d\sigma d\ell,$$

writing the function $\mathbf{1}$ as a average of extremal points of $\mathbf{K}^1_\nu$.

The preceding decomposition is the unique Choquet's representation of $\mathbf{1}$ in $\mathbf{K}_\nu$:

$$\widetilde{\mu}_{\mathbf{1}} = \langle h^\sigma h^\sigma_\ell, \nu\rangle\, d\sigma d\ell.$$

As $\langle h^\sigma h^\sigma_\ell, \nu\rangle$ is a positive continuous function of σ and ℓ,[11] any $\widetilde{\mu}_{\mathbf{1}}$-almost sure equality is a $(d\sigma d\ell)$-almost sure equality.
Let $f \in \mathbf{C}_\nu$ be any (non null) nonnegative L-harmonic bounded function. Since

- $f \leqslant \|f\|_\infty \mathbf{1}$, we have $\widetilde{\mu}_f = F\widetilde{\mu}_{\|f\|_\infty \mathbf{1}}$, for some function F on $\mathbb{S}^2 \times \mathbb{R}$ such that $0 \leqslant F \leqslant 1$, $(d\sigma d\ell)$-almost everywhere,
- $\widetilde{\mu}_{\|f\|_\infty \mathbf{1}} = \|f\|_\infty \widetilde{\mu}_{\mathbf{1}} = \|f\|_\infty \langle h^\sigma h^\sigma_\ell, \nu\rangle\, d\sigma d\ell$,

9 As explained in the books of Choquet [10,11], Phelps [21], or Becker [6].

10 Endowed with its natural lattice structure: if $f = \frac{d\mu}{d(\mu+\mu')}$ and $g = \frac{d\mu'}{d(\mu+\mu')}$, $\frac{d(\mu\wedge\mu')}{d(\mu+\mu')} = f \wedge g$.

11 It does not depend on σ.

we deduce from Theorem 22 that

$$f = \int \frac{h^\sigma h^\sigma_\ell}{\langle h^\sigma h^\sigma_\ell, \nu\rangle} F(\sigma,\ell)\|f\|_\infty \langle h^\sigma h^\sigma_\ell, \nu\rangle \, d\sigma d\ell.$$

One sees in this expression that f is of the form

$$\int h^\sigma h^\sigma_\ell \mathbf{F}(\sigma,\ell)\, d\sigma d\ell$$

for some Borel function $\mathbf{F}$, bounded $(d\sigma d\ell)$-almost everywhere.

Conversely, for any ($(d\sigma d\ell)$-almost everywhere) bounded Borel function $\mathbf{F}$ on $\mathbb{S}^2 \times \mathbb{R}$, the function

$$\int_{\mathbb{S}^2\times\mathbb{R}} \mathbf{F}(\sigma,\ell) h^\sigma h^\sigma_\ell(.)\, d\sigma d\ell = \mathbb{E}_.[F(\sigma_\infty, R^{\sigma_\infty}_\infty)]$$

is bounded and L-harmonic. This provides a complete description of the Poisson boundary of L.

Theorem 23 (Poisson boundary of $(L, \mathbb{H}\times\mathbb{R}^{1,3})$ *Any bounded L-harmonic function is of the form*

$$\int_{\mathbb{S}^2\times\mathbb{R}} \mathbf{F}(\sigma,\ell) h^\sigma(.) h^\sigma_\ell(.)\, d\sigma d\ell,$$

for some bounded Borel function $\mathbf{F}$ *on* $\mathbb{S}^2\times\mathbb{R}$. *Conversely, such a formula defines a bounded L-harmonic function.*

The description of the invariant σ-algebra of the diffusion $\{(\dot\xi_s, \xi_s)\}_{s\geqslant 0}$ given in Theorem 1 follows from Proposition 4.2. Indeed, given a bounded $Inv\big((\dot\xi_s,\xi_s)\big)$-measurable random variable X, the harmonic function (Proposition 4.2)

$$(\dot\xi,\xi) \in \mathbb{H}\times\mathbb{R}^{1,3} \mapsto \mathbb{E}_{\dot\xi,\xi}[X]$$

is of the form

$$\int_{\mathbb{S}^2\times\mathbb{R}} h^\sigma(\dot\xi,\xi) h^\sigma_\ell(\dot\xi,\xi)\mathbf{F}(\sigma,\ell)\, d\sigma d\ell,$$

for some bounded measurable function $\mathbf{F}$ on $\mathbb{S}^2\times\mathbb{R}$. That is,

$$\mathbb{E}_{\dot\xi,\xi}[X] = \mathbb{E}_{\dot\xi,\xi}\big[\mathbf{F}(\sigma_\infty, R^{\sigma_\infty}_\infty)\big], \quad (\dot\xi,\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}.$$

Let $(\dot{\xi}_0, \xi_0) \in \mathbb{H} \times \mathbb{R}^{1,3}$ be given. Since the process $\{\mathbb{E}_{\dot{\xi}_s,\xi_s}[X]\}_{s\geqslant 0}$ is under $\mathbb{P}_{\dot{\xi}_0,\xi_0}$ a bounded martingale, it almost surely converges towards X, and towards $\mathbf{F}(\sigma_\infty, R_\infty^{\sigma_\infty})$; both quantities are therefore $\mathbb{P}_{\dot{\xi}_0,\xi_0}$-almost surely equal.

Corollary 24 (Invariant σ-algebra of the relativistic diffusion) *For any* $(\dot{\xi}_0, \xi_0) \in \mathbb{H} \times \mathbb{R}^{1,3}$*, the* σ*-algebras* $Inv\big((\dot{\xi}_s, \xi_s)\big)$ *and* $\sigma\big(\sigma_\infty, R_\infty^{\sigma_\infty}\big)$ *coincide up to* $\mathbb{P}_{\dot{\xi}_0,\xi_0}$*-null sets.*

5 Representation formula

This section is dedicated to the proof of Theorem 18. It is organized as follows. We first define h_ℓ^σ in Sect. 5.1. To reduce the dimension of the problem, we shall show that bounded $L^{h^\sigma h_\ell^\sigma}$-harmonic functions depend only on two coordinates. It is easier to show this property first for bounded L^{h^σ}-harmonic functions, and then for $L^{h^\sigma h_\ell^\sigma}$-harmonic functions. This is what we do in Sect. 5.2, using a coupling. Then we shall see in Sect. 5.3 how another coupling enables one to show that bounded $L^{h^\sigma h_\ell^\sigma}$-harmonic functions are constant.

Recall that $\{\varepsilon_0, \ldots, \varepsilon_3\}$ is the canonical basis of $\mathbb{R}^{1,3}$. Let $\sigma \in \mathbb{S}^2$. Choose a (q-orthonormal) basis $\{\epsilon_0, \ldots, \epsilon_3\}$ of $\mathbb{R}^{1,3}$ such that

$$\epsilon_0 = \varepsilon_0 \quad \text{and} \quad \epsilon_1 = \sigma.$$

Throughout this section, the direction σ and the basis $\{\epsilon_0, \ldots, \epsilon_3\}$ are fixed.
Notations • We adopt the notation $(\xi^0, \ldots, \xi^3)$ for the coordinates of a point $\xi \in \mathbb{R}^{1,3}$ *in the basis* $\{\epsilon_0, \ldots, \epsilon_3\}$. Although in the preceding section the same notation was used for the coordinates of ξ in the canonical basis there will be no confusion since we shall not use $\{\varepsilon_0, \ldots, \varepsilon_3\}$-linear coordinates in this section. • In the same way, we also write

$$\psi : \big(y, (x_1, x_2)\big) \in \mathbb{R}_+^* \times \mathbb{R}^2 \mapsto \left(\frac{|x|^2 + y^2 + 1}{2y}\epsilon_0 + \frac{|x|^2 + y^2 - 1}{2y}\epsilon_1 + \frac{x_1}{y}\epsilon_2 + \frac{x_2}{y}\epsilon_3\right) \in \mathbb{H}.$$

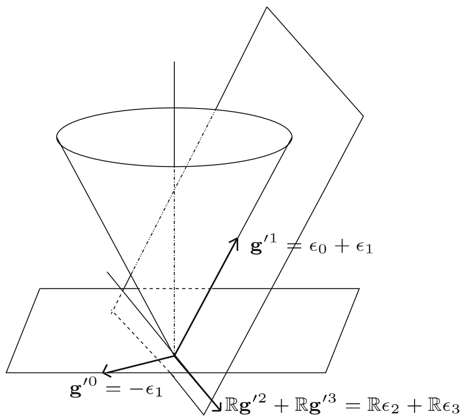
• In the sequel, we systematically use $\{\epsilon_0, \ldots, \epsilon_3\}$-halfspace coordinates (y, x) on $\mathbb{H}$ (see Definition 5).

5.1 h^{ϵ_1}-process

(i) The hyperbolic Laplacian has the same expression (3.1) in $\{\epsilon_0, \ldots, \epsilon_3\}$-halfspace coordinates (y, x) as in $\{\varepsilon_0, \ldots, \varepsilon_3\}$-halfspace coordinates. In these coordinates, the harmonic function $h^{\epsilon_1}\big((y, x), \xi\big)$ is proportional to y^2. So,

$$L^{h^{\epsilon_1}} f = \frac{y^2}{2}(\partial_x^2 + \partial_y^2) f + \frac{3}{2} y \partial_y f + \partial_\xi f(\dot{\xi}, \xi).\dot{\xi};$$

Fig. 2 New coordinates on $\mathbb{R}^{1,3}$



like L, the operator $L^{h^{\epsilon_1}}$ is hypoelliptic. Indeed, suppose $L^{h^{\epsilon_1}} f = \frac{L(h^{\epsilon_1} f)}{h^{\epsilon_1}}$ is of class $\mathcal{C}^\infty$, then $L(h^{\epsilon_1} f)$ is $\mathcal{C}^\infty$ (since h^{ϵ_1} is $\mathcal{C}^\infty$), so $h^{\epsilon_1} f$ is smooth; since h^{ϵ_1} is positive and smooth, f is smooth.

(ii) The evolution of the h^{ϵ_1}-process is determined by the following stochastic differential system.

$$\begin{aligned} dy_s &= y_s dw_s^y + \frac{3}{2} y_s ds, \\ dx_s &= y_s dw_s^x, \\ d\xi_s &= \dot{\xi}_s\, ds = \psi\big((y_s, x_s)\big) ds. \end{aligned} \tag{5.1}$$

where w^y is a real Brownian motion and w^x a two-dimensional Brownian motion independent of w^y.

We get a better insight into the evolution of ξ_s by changing coordinates in $\mathbb{R}^{1,3}$. Take the coordinates associated with the basis

$$\mathbf{g}' = \{-\epsilon_1, \epsilon_0 + \epsilon_1, \epsilon_2, \epsilon_3\}. \tag{5.2}$$

Notation (new coordinates on $\mathbb{R}^{1,3}$) Denote by $(\xi'^0, \xi'^1, \xi'^2, \xi'^3)$ the coordinates of a point $\xi \in \mathbb{R}^{1,3}$ in the basis $\mathbf{g}'$ Fig. 2.

In these coordinates, the equation

$$d\xi_s = \psi\big((y_s, x_s)\big) ds$$

takes the form

$$\begin{aligned} d\xi_s'^0 &= \frac{ds}{y_s}, \\ d\xi_s'^1 &= \frac{|x_s|^2 + y_s^2 + 1}{2y_s} ds, \\ d\xi_s'^j &= \frac{x_s^{j-1}}{y_s} ds, \;\; j \in \{2, 3\}. \end{aligned} \tag{5.3}$$

Notation of an element of $\mathbb{H}\times\mathbb{R}^{1,3}$ From now on, we use $\mathbf{g}'$-linear coordinates on $\mathbb{R}^{1,3}$, and $\{\epsilon_0,\dots,\epsilon_3\}$-halfspace coordinates on $\mathbb{H}$. So, a generic point $u\in\mathbb{H}\times\mathbb{R}^{1,3}$ is denoted either $u=(\dot{\xi},\xi)$ when no coordinates are needed, or $u=\big((y,x),(\xi'^0,\dots,\xi'^3)\big)$.

The process $\{y_s\}_{s\geqslant 0}$ has under $\mathbb{P}_u^{\epsilon_1}$ an explicit expression:

$$y_s = y\, e^{w_s^y + s},$$

and

$$\xi_s'^0 = \xi'^0 + \frac{1}{y}\int_0^s e^{-w_r^y - r}\, dr$$

converges $\mathbb{P}_u^{\epsilon_1}$-almost surely as $s\to+\infty$, towards some random variable $R_\infty^{\epsilon_1}$. One has a coordinate free expression of $R_\infty^{\epsilon_1}$, since

$$\xi_s'^0 = q(\xi_s,\epsilon_0+\epsilon_1),$$

so that

$$R_\infty^{\epsilon_1} = \lim_{s\to+\infty} q(\xi_s,\epsilon_0+\epsilon_1).$$

As $L^{h^{\epsilon_1}}$ is hypoelliptic, Proposition 12 says that, given $\ell\in\mathbb{R}$, the function

$$h_{\geqslant\ell}^{\epsilon_1}(u) = \mathbb{P}_u^{\epsilon_1}\big(R_\infty^{\epsilon_1}\geqslant \ell\big),$$

is smooth and $L^{h^{\epsilon_1}}$-harmonic. Examining the expression

$$h_{\geqslant\ell}^{\epsilon_1}(u) = \mathbb{P}_u^{\epsilon_1}\left(\xi'^0 + \frac{1}{y}\int_0^{+\infty} e^{-w_r^y - r}\, dr \geqslant \ell\right) = \mathbb{P}_u^{\epsilon_1}\left(\int_0^{+\infty} e^{-w_r^y - r}\, dr \geqslant y\big(\ell-\xi'^0\big)\right)$$

one sees that $h_{\geqslant\ell}^{\epsilon_1}$ is identically equal to 1 on the half space $\big\{(\dot{\xi},\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}\,;\ \ell-\xi'^0\leqslant 0\big\}$. Let w be a Brownian motion under some probability $\mathbb{P}$ and set

$$G(t) \equiv \mathbb{P}\left(\int_0^{+\infty} e^{-w_r - r}\, dr \geqslant t\right).$$

Then,

$$h_{\geqslant\ell}^{\epsilon_1}(u) = G\big(y(\ell-\xi'^0)\big). \tag{5.4}$$

Since $h_{\geqslant\ell}^{\epsilon_1}$ is smooth, G is also smooth; it satisfies the following differential equation on the open half space $\big\{(\dot{\xi},\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}\,;\ \ell-\xi'^0>0\big\}$, reflecting the $L^{h^{\epsilon_1}}$-harmonicity of $h_{\geqslant\ell}^{\epsilon_1}$.

$$\frac{\big(y(\ell-\xi'^0)\big)^2}{2} G''\big(y(\ell-\xi'^0)\big) + \left(\frac{3\big(y(\ell-\xi'^0)\big)}{2} - 1\right) G'\big(y(\ell-\xi'^0)\big) = 0;$$

that is,

$$G''(r) + \left(\frac{3}{r} - \frac{2}{r^2}\right) G'(r) = 0. \tag{5.5}$$

We find

$$G'(r) = -C\,\frac{e^{-2/r}}{r^3}\mathbf{1}_{r>0}, \tag{5.6}$$

or $G(s) = C\int_s^{+\infty} \frac{e^{-2/r}}{r^3}\,dr$, for $s>0$, where C is such that $G(0)=1$; $G\equiv 1$ on $(-\infty,0]$. So, $h^{\epsilon 1}_{\geqslant \ell}(u) = \mathbb{P}_u\big(R^{\epsilon 1}_\infty \geqslant \ell\big) = C\int_{\max(y(\ell-\xi'^0),0)}^{+\infty} \frac{e^{-2/r}}{r^3}\,dr$, and the density of the law of $R^{\epsilon 1}_\infty$ under $\mathbb{P}^{\epsilon 1}_u$, with respect to Lebesgue measure $d\ell$ on $\mathbb{R}$, is equal to

$$h^{\epsilon 1}_\ell(u) = y\,\frac{e^{-\frac{2}{y(\ell-\xi'^0)}}}{\big(y(\ell-\xi'^0)\big)^3}\mathbf{1}_{\ell>\xi'^0}.\text{[12]} \tag{5.7}$$

Each $h^{\epsilon 1}_\ell$ being $L^{h^{\epsilon 1}}$-harmonic, the function $h^{\epsilon 1}h^{\epsilon 1}_\ell$ is L-harmonic.

To show the minimality of $h^{\epsilon 1}h^{\epsilon 1}_\ell$, we shall show that the only bounded $L^{h^{\epsilon 1}h^{\epsilon 1}_\ell}$-harmonic functions are the constants. As announced above, this will be done in two steps.

1. Using couplings, we shall show that any bounded $L^{h^{\epsilon 1}h^{\epsilon 1}_\ell}$-harmonic function depends only on y and ξ'^0.
2. Another coupling argument will show that they are constant.

To prove point *1* we first show that bounded $L^{h^{\epsilon 1}}$-harmonic functions depend only on y and ξ'^0. The next section is dedicated to the proof of this fact, stated in Theorem **38**.

5.2 Bounded $L^{h^{\epsilon 1}}$-harmonic functions

The proof of Theorem **38** will come from the following two results, proved in Sects. **5.2.1** and **5.2.2**, respectively.

1. Let $u_0 = \big((Y_0,x),(Z_0,\xi'^{\geqslant 1})\big)$ and $\underline{u}_0 = \big((Y_0,\underline{x}),(Z_0,\underline{\xi}'^{\geqslant 1})\big)$ be two points in $\mathbb{H}\times\mathbb{R}^{1,3}$, with the same y and ξ'^0 coordinates.[13]

Theorem 25 (Coupling theorem) *Pick an $\varepsilon>0$. We can couple two $h^{\epsilon 1}$-processes, started from u_0 and $\underline{u}_0$, such that after the coupling time,*

[12] It is interesting to notice that the preceding lines give a new proof of the fact that the law of Dufresne's integral $\int_0^\infty e^{w_r - r}\,dr$ has the same law as $\frac{2}{\gamma}$, where γ is a gamma random variable with parameter 2. Consult [5] for an explanation of this result, as well as for references on Dufresne's integral.

[13] We note $\xi'^{\geqslant 1}$ for $(\xi'^1,\xi'^2,\xi'^3)\in\mathbb{R}^3$.

- $\dot{\underline{\xi}}_s = \dot{\xi}_s$ *and*
- $\underline{\xi}_s = \xi_s + C_1 \mathbf{g}'^1 + c_2 \mathbf{g}'^2 + c_3 \mathbf{g}'^3$,
where C_1 is a (random) constant and c_2, c_3 are (random) constants such that $|c_i| \leqslant \varepsilon$, $i \in \{2, 3\}$.

2. Recall we can regard the diffusion $\{(\dot{\xi}_s, \xi_s)\}_{s \geqslant 0}$ on $\mathbb{H} \times \mathbb{R}^{1,3}$ as the projection of a diffusion $\{(g_s, \xi_s)\}_{s \geqslant 0}$ on the Poincaré group $\mathcal{G}$ of affine isometries of $\mathbb{R}^{1,3}$, with generator $\widetilde{L}$ written in (3.7).

Theorem 26 (Uniform continuity) *Bounded $\widetilde{L}^{h^{\epsilon_1}}$-harmonic functions are right uniformly continuous.*

5.2.1 Coupling

Fix $Y_0 \in \mathbb{R}_+^*$, $Z_0 \in \mathbb{R}$ and $\varepsilon > 0$. Let $u_0 = \big((Y_0, x), (Z_0, \xi'^{\geqslant 1})\big)$ and $\underline{u}_0 = \big((Y_0, \underline{x}), (Z_0, \underline{\xi}'^{\geqslant 1})\big)$ be two points of $\mathbb{H} \times \mathbb{R}^{1,3}$ having the same y and ξ'^0 coordinates. We shall couple two h^{ϵ_1}-process, started from u_0 and $\underline{u}_0$, respectively.

Let w^y be a real Brownian motion, and $w^x, \underline{w}^x$ be two $\mathbb{R}^2$-Brownian motions, defined on some measurable space $(\Omega, \mathcal{F})$. Consider the system

$$\begin{aligned}
dy_s &= y_s dw_s^y + \frac{3}{2} y_s ds, & d\underline{y}_s &= \underline{y}_s dw_s^y + \frac{3}{2} \underline{y}_s ds, \\
dx_s &= y_s dw_s^x, & d\underline{x}_s &= \underline{y}_s d\underline{w}_s^x, \\
d\xi_s'^0 &= \frac{ds}{y_s}, & d\underline{\xi}_s'^0 &= \frac{ds}{\underline{y}_s}, \\
d\xi_s'^1 &= \frac{x_s^2 + y_s^2 + 1}{2y_s} ds, & d\underline{\xi}_s'^1 &= \frac{\underline{x}_s^2 + \underline{y}_s^2 + 1}{2\underline{y}_s} ds, \\
d\xi_s'^{\geqslant 2} &= \frac{x_s}{y_s} ds, & d\underline{\xi}_s'^{\geqslant 2} &= \frac{\underline{x}_s}{\underline{y}_s} ds,
\end{aligned} \tag{5.8}$$

with initial conditions u_0 and $\underline{u}_0$, respectively.

Remarks

1. Since u_0 and $\underline{u}_0$ have the same y-coordinate, and $\{y_s\}_{s \geqslant 0}$ and $\{\underline{y}_s\}_{s \geqslant 0}$ are driven by the same Brownian motion w^y, we have $\underline{y}_s = y_s$, for all $s \geqslant 0$.
2. As $\underline{\xi}_0'^0 = \xi_0'^0$, we also have $\underline{\xi}_s'^0 = \xi_s'^0$, for all $s \geqslant 0$.

Define on $(\Omega, \mathcal{F})$ the filtration $\{\mathfrak{F}_s\}_{s \geqslant 0}$ generated by $\{u_s\}_{s \geqslant 0}$ and $\{\underline{u}_s\}_{s \geqslant 0}$:

$$\mathfrak{F}_s = \sigma\big((u_r, \underline{u}_r);\ r \leqslant s\big).$$

Given independent w^y and w^x, we construct a Brownian motion $\underline{w}^x$, and an $\{\mathfrak{F}_s\}_{s \geqslant 0}$-stopping time T, such that if one notes $\mathbf{P}_{u_0, \underline{u}_0}^{\epsilon_1}$ the law of the solution of system (5.8), one has $\mathbf{P}_{u_0, \underline{u}_0}^{\epsilon_1}$-almost surely

- T is finite,
- after time T, $\dot{\underline{\xi}}_s = \dot{\xi}_s$, and $\underline{\xi}_s = \xi_s + C_1\mathbf{g}'^1 + c_2\mathbf{g}'^2 + c_3\mathbf{g}'^3$, where C_1 is a (random) constant and c_2, c_3 are (random) constants such that $|c_i| \leqslant \varepsilon$, $i \in \{2, 3\}$.

Remark that such a construction provides us with two diffusions $\{u_s\}_{s\geqslant 0}$ and $\{\underline{u}_s\}_{s\geqslant 0}$, with respective laws $\mathbb{P}^{\epsilon_1}_{u_0}$ and $\mathbb{P}^{\epsilon_1}_{\underline{u}_0}$.
(**a**) To make things clearer, we first consider an analogous of system (5.8), where x and $\xi'^{\geqslant 2}$ one-dimensional. We write ξ'^2 instead of $\xi'^{\geqslant 2}$. The solutions $\{u_s\}_{s\geqslant 0}$ and $\{\underline{u}_s\}_{s\geqslant 0}$ of the modified system live in $\left(\mathbb{R}_+^* \times \mathbb{R}\right) \times (\mathbb{R} \times \mathbb{R}^2)$.

Denote by $\mathbb{P}^{\epsilon_1}_{u_0,\underline{u}_0}$ the law of the pair $\left\{(u_s, \underline{u_s})\right\}_{s\geqslant 0}$, solution of system (5.8), *with* $\underline{w}^x$ *independent of* w^x *and* w^y, and with initial conditions $(u_0, \underline{u}_0)$. Set

$$T = \inf\left\{s \geqslant 0\,;\ \underline{x}_s = x_s\ ,\ \left|\underline{\xi}_s'^2 - \xi_s'^2\right| \leqslant \varepsilon\right\},$$

with the convention that $\inf \emptyset = +\infty$. T is an $\{\mathfrak{F}_s\}_{s\geqslant 0}$-stopping time.

Theorem 27 *T is $\mathbb{P}^{\epsilon_1}_{u_0,\underline{u}_0}$-almost surely finite.*

Taking this for granted and setting $\underline{w}_s = w_s$, for $s \geqslant T$, the two stochastic differential equations of (5.8) are identical for $s \geqslant T$; so after time T, $\underline{x}_s = x_s$ and $\underline{\xi}_s'^2 - \xi_s'^2$ is constant, with absolute value smaller than or equal to ε.

Proof of Theorem 27 Write $z_s \equiv \frac{\underline{x}_s - x_s}{y_s}$. The objective is to show that the $\mathbb{R}^2$-valued process $\left\{(z_s, \underline{\xi}'^2 - \xi'^2 + \int_0^s z_u\,du)\right\}_{s\geqslant 0}$ reaches the set

$$\left\{(z, Z) \in \mathbb{R}^2\,;\ z = 0\,,\ |Z| \leqslant \varepsilon\right\}$$

in a $\mathbb{P}^{\epsilon_1}_{u_0,\underline{u}_0}$-almost surely finite time.

Lemma 28 *The process $\{z_s\}_{s\geqslant 0}$ is a diffusion, which is positive recurrent.*

$\lhd$ Use formulas (5.8) and Itô's formula to get

$$dz_s = (d\underline{w}_s^x - dw_s^x) - z_s dw_s^y - \frac{3}{2} z_s\,ds.$$

The infinitesimal generator of $\{z_s\}_{s\geqslant 0}$ is

$$f \mapsto \frac{2 + z^2}{2} f'' - \frac{3}{2} z f'.$$

It has a unique invariant probability, which has a density with respect to Lebesgue measure on $\mathbb{R}$ proportional to $(2+z^2)^{-\frac{5}{2}}$.[14] ▷

So we can look at its successive excursions outside $\{0\}$, of height $\geqslant 1$. Write

$$S_0 \equiv \inf\{s \geqslant 0;\ z_s = 0\}$$

and

$$S_n = \inf\left\{ s \geqslant S_{n-1}\,;\ \sup_{S_{n-1}\leqslant u\leqslant s} |z_u| \geqslant 1 \text{ and } z_s = 0 \right\}.$$

Because of the Strong Markov Property, the excursions $\{z_s\}_{S_{n-1}\leqslant s\leqslant S_n}$, $n \geqslant 1$, are independent and identical in law. So are the integrals $\left\{\int_{S_{n-1}}^{S_n} z_u\,du\right\}_{n\geqslant 1}$. So, using the following well known criterion[15] it will suffice to show that the random variable $\int_{S_0}^{S_1} z_u\,du$ is integrable and that the support of its law is non-lattice to prove Theorem 27.

Proposition 29 (Chung, Fuchs) *Let μ be a probability on $\mathbb{R}$ having a moment of order 1. Let $\{X_n\}_{n\geqslant 1}$ be a sequence of independent random variables, of law μ. Then the random walk $\left\{\sum_{i=1}^n X_k\right\}_{n\geqslant 1}$ is recurrent in the closed subgroup generated by the support of μ if, and only if, $\int x\,\mu(dx) = 0$.*

We first show that the integrability condition is fulfilled.

Lemma 30 $\mathbb{E}^{\epsilon_1}_{u_0,\underline{u}_0}\left[\left|\int_{S_0}^{S_1} z_u\,du\right|\right] < +\infty$.

◁ To evaluate this average, we cut the integral into two parts: the first one corresponds to the integral of z between time S_0 and the first time H when $\{z_s\}_{s\geqslant 0}$ hits ± 1, the second one is $\int_H^{S_1} z_u\,du$. We show that

$$\mathbb{E}^{\epsilon_1}_{u_0,\underline{u}_0}\left[\left|\int_{S_0}^{H} z_u\,du\right|\right] < +\infty, \quad \text{and} \quad \mathbb{E}^{\epsilon_1}_{u_0,\underline{u}_0}\left[\left|\int_{H}^{S_1} z_u\,du\right|\right] < +\infty.$$

[14] We know from differential equations theory that any invariant probability has a smooth density with respect to Lebesgue measure dz on $\mathbb{R}$. Denote it by $m(z)$. It must satisfy the relation

$$\left(\frac{2+z^2}{2} m(z)\right)'' = -\frac{3}{2}\big(zm(z)\big)'.$$

If we define $u(z) = \frac{2+z^2}{2} m(z)$, this function satisfies the equation $u''(z) = -\frac{3}{2}\left(\frac{2}{2+z^2}u(z)\right)'$, *i.e.*, up to an additive constant $u'(z) = -\frac{3}{2}\frac{2}{2+z^2}u(z)$. So, $u(z) = (2+z^2)^{-\frac{3}{2}}$, and $m(z)$ is proportional to $(2+z^2)^{-\frac{5}{2}}$.

[15] See [12], Theorem 8.3.4, p. 251, for instance.

- For any $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the law of $\{z_s\}_{s\geqslant 0}$ started from x. Since $|z_s| \leqslant 1$ on $[S_0, H]$,[16]

$$\mathbb{E}^{\epsilon_1}_{u_0,\underline{u}_0}\left[\left|\int_{S_0}^{H} z_u\,du\right|\right] \leqslant \mathbb{E}_0[H] < +\infty.$$

- The second integral $\int_H^{S_1} z_u\,du$ is handled as follows.

Denote by $\tau_0 \equiv \inf\{s \geqslant 0\ ;\ z_s = 0\}$ the hitting time of $\{0\}$ by $\{z_s\}_{s\geqslant 0}$, and by $g^0(x, y)$ the Green function of the process $\{z_s\}_{s\geqslant 0}$ *killed at time* τ_0.

$$\mathbb{E}^{\epsilon_1}_{u_0,\underline{u}_0}\left[\int_H^{S_1} z_u\,du\right] = \mathbb{E}_1\left[\int_0^{\tau} z_u\,du\right] = \int_0^{+\infty} g^0(1, y)y\,dy.$$

We can find an explicit formula of the Green function $g^0(z, y)$, approximating it by the Green function $g^{0,b}(z, y)$ of the process $\{z_s\}$ *killed at time* $\tau_{0,b} = \inf\{s > 0;\ z_s \in \{0, b\}\}$, $b > 0$. Let φ be any nonnegative smooth function on $\mathbb{R}^*_+$, with compact support. The monotone convergence theorem justifies the equalities $\overset{\bullet}{=}$ below.

$$\begin{aligned}\int_0^{+\infty} g^0(z, y)\varphi(y)\,dy &= \mathbb{E}_z\left[\int_0^{\tau_0}\varphi(z_u)\,du\right] \overset{\bullet}{=} \lim_{b\to+\infty}\mathbb{E}_z\left[\int_0^{\tau_{0,b}}\varphi(z_u)\,du\right]\\ &\overset{\bullet}{=} \lim_{b\to+\infty}\int_0^{+\infty} g^{0,b}(z, y)\varphi(y)\,dy\\ &= \int_0^{+\infty}\lim_{b\to+\infty} g^{0,b}(z, y)\,\varphi(y)\,dy.\end{aligned}$$

But the function $\int_0^{\infty} g^{0,b}(x, y)\varphi(y)\,dy$ is the unique solution of the following equation.

$$\begin{aligned}&\frac{2+z^2}{2} f''(z) - \frac{d}{2} z f'(z) = -\varphi(z)\\ &f(0^+) = 0,\ f(b^-) = 0\end{aligned} \tag{5.9}$$

This equation can be solved explicitly, since it is a first order equation in f'. The set of solutions of the equation

[16] The diffusion $\{z_s\}_{s\leqslant H}$ is uniformly elliptic and has bounded drift. See [9], Theorem 16.24, p. 359.

$$\frac{2+z^2}{2}u'(z) - \frac{3}{2}zu(z) = -\varphi(z)$$

is found by the method of variation of the parameters. Any solution is of the form

$$u(z) = \lambda(2+z^2)^{\frac{3}{2}} + 2\,(2+z^2)^{\frac{3}{2}} \int_0^z \frac{-\varphi(r)\,dr}{(2+r^2)^2}, \tag{5.10}$$

where λ is a real constant.
Using the boundary conditions in (5.9), we find

$$\int_0^{+\infty} g^{0,b}(z,y)\varphi(y)\,dy = 2\frac{\int_0^b (2+u^2)^{\frac{3}{2}} \int_0^u \frac{\varphi(r)\,dr}{(2+r^2)^2}\,du}{\int_0^b (2+u^2)^{\frac{3}{2}}\,du} \int_0^z (2+u^2)^{\frac{3}{2}}\,du$$
$$-2\int_0^z (2+u^2)^{\frac{3}{2}} \int_0^u \frac{\varphi(r)\,dr}{(2+r^2)^2}\,du. \tag{5.11}$$

As

$$\lim_{b\to+\infty} \frac{\int_0^b (2+u^2)^{\frac{3}{2}} \int_0^u (2+r^2)^{-2}\varphi(r)\,dr\,du}{\int_0^b (2+u^2)^{\frac{3}{2}}\,du} = \int_0^{+\infty} (2+r^2)^{-2}\varphi(r)\,dr,$$

we deduce from Eq. (5.11) that

$$g^0(z,y) = 2(2+y^2)^{-2} \int_0^{z\wedge y} (2+u^2)^{\frac{3}{2}}\,du.$$

We see in this formula that

$$\mathbb{E}^{\epsilon_1}_{u_0,\underline{u}_0}\left[\int_H^{S_1} z_u\,du\right] = \int_0^{+\infty} g^0(1,y)y\,dy < \infty.$$

$\triangleright$

As noted above, it remains to prove that the closed group generated by the support of the law of $\int_{S_0}^{S_1} z_u\,du$ is non-lattice to end the proof of Theorem **27**.

Lemma 31 *The closed group generated by the support of the law of $\int_{S_0}^{S_1} z_u\,du$ is non-lattice.*

$\triangleleft$ We can suppose $S_0 = 0$. Let $a > 0$ and $\varphi : [0,1] \to \mathbb{R}$ be a continuous function such that

- $\varphi(0) = 0, \varphi(1) = -\varepsilon < 0,$

- $\varphi(t) = 0$ only for $t = 0$ and some t_0 near 1,
- $\max_{s\in[0,1]} |\varphi(s)| > 1$,
- $\int_0^1 \varphi(r)\,dr \notin a\mathbb{Z}$.

The support Theorem 7 asserts that, for any $\eta > 0$, the event $\left\{ \sup_{r\in[0,1]} |z_r - \varphi(r)| \leqslant \eta \right\}$ has positive probability. Choose $\eta > 0$ small enough so that

- S_1 is close to t_0,
- $\max_{r\in[0,S_1]} |z_r| \geqslant 1$,
- $\int_0^{S_1} z_r\,dr$ is near $\int_0^1 \varphi(r)\,dr$.

Reducing $\eta > 0$ if necessary, we have $\left| \int_0^{S_1} z_r\,dr - \int_0^1 \varphi(r)\,dr \right| < \mathrm{dist}\left(\int_0^1 \varphi(r)\,dr,\ a\mathbb{Z} \right)$, and $\int_0^{S_1} z_r\,dr \notin a\mathbb{Z}$ on a set of positive probability. $\triangleright$

(**b**) Coupling in $\mathbb{H} \times \mathbb{R}^{1,3}$ Write • x^1, x^2 the coordinates of $x \in \mathbb{R}^2$,

- $(w^x)^1, (w^x)^2$ the coordinates of w^x (resp. $\left(\underline{w}^x\right)^i$ for $\underline{w}$), and
- $\widehat{\mathbb{P}}^{\epsilon_1}_{u_0,\underline{u}_0}$ the law of the diffusion solving system (5.8), when w^y, w^x, $\underline{w}^x$ are independent.

First, use the coupling Theorem 27 to find a $\widehat{\mathbb{P}}^{\epsilon_1}_{u_0,\underline{u}_0}$-almost surely finite $\{\mathfrak{F}_s\}_{s\geqslant 0}$-stopping time T_1 such that

$$\underline{x}^1_{T_1} = x^1_{T_1}, \quad \text{and } \left|\underline{\xi}'^2_{T_1} - \xi'^2_{T_1}\right| \leqslant \varepsilon.$$

From that time on, set $\left(\underline{w}^x\right)^1 = (w^x)^1$, $\left(\underline{w}^x\right)^2$ remaining independent of $(w^x)^2$, $(w^x)^1$, and w^y. For $s \geqslant T_1$, $\underline{x}^1_s = x^1_s$ and $\left(\underline{\xi}'^2_s - \xi'^2_s\right)$ is constant with an absolute value less than or equal to ε.

Then look at the processes $\left\{(\underline{y}_s, \underline{x}^2_s, \underline{\xi}'^3_s)\right\}_{s\geqslant T_1}$ and $\left\{(y_s, x^2_s, \xi'^3_s)\right\}_{s\geqslant T_1}$. Thanks to the strong Markov property, we can use the coupling Theorem 27 to couple the second coordinate $\underline{x}^2_s$ of $\underline{x}_s$ with x^2_s at some time T_2(note that $\underline{y}_{T_1} = y_{T_1}$).

Set $\left(\underline{w}^x_s\right)^2 = \left(w^x_s\right)^2$, for $s \geqslant T_2$. After time T_2

- $\underline{x}^1_s = x^1_s$, and $\underline{x}^2_s = x^2_s$,
- $\underline{\xi}'^3_s - \xi'^3_s$ is constant with an absolute value less than or equal to ε.

Thus, we have shown the following theorem.

Theorem 32 (Coupling) *Let $\varepsilon > 0$ and $u_0 = (\dot{\xi}_0, \xi_0)$ and $\underline{u}_0 = (\dot{\underline{\xi}}_0, \underline{\xi}_0)$ be two points of $\mathbb{H} \times \mathbb{R}^{1,3}$ such that $\dot{\xi}_0$ and $\dot{\underline{\xi}}_0$ have the same y-coordinate, and $\xi'^0_0 = \underline{\xi}'^0_0$. We can find*

- *a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathfrak{F}_s\}_{s\geqslant 0}, \mathbb{P}^{\epsilon_1}_{u_0,\underline{u}_0}\right)$,*
- *paths space adapted random variables $\{u_s\}_{s\geqslant 0}$ and $\{\underline{u}_s\}_{s\geqslant 0}$, on $\left(\Omega, \mathcal{F}, \{\mathfrak{F}_s\}_{s\geqslant 0}\right)$,*
- *an $\{\mathfrak{F}_s\}_{s\geqslant 0}$-stopping time T,*

such that

- *the law of* $\{u_s\}_{s\geqslant 0}$ *under* $\mathbb{P}^{\epsilon_1}_{u_0,\underline{u}_0}$ *is* $\mathbb{P}^{\epsilon_1}_{u_0}$*, and that of* $\{\underline{u}_s\}_{s\geqslant 0}$ *is* $\mathbb{P}^{\epsilon_1}_{\underline{u}_0}$*,*
- *after the coupling time* T*,* $\dot{\underline{\xi}}_s = \dot{\xi}_s$ *and* $\underline{\xi}_s = \xi_s + C_1\mathbf{g}'_1 + c_2\mathbf{g}'^2 + c_3\mathbf{g}'^3$*,*

where C_1 *is a (random) constant and* c_2, c_3 *(random) constants such that* $|c_i| \leqslant \varepsilon$*,* $i \in \{2, 3\}$.

We need two ingredients to show that bounded $L^{h^{\varepsilon_1}}$-harmonic functions depend only on y and ξ'^0. The first one is the preceding theorem, the second one is a uniform continuity property of bounded $L^{h^{\epsilon_1}}$-harmonic functions.

To get this property, we lift to the group $\mathcal{G}$, where the diffusion $\{(g_s, \xi_s)\}_{s\geqslant 0}$ defined in Sect. 3.2 lives. Its generator is denoted by $\widetilde{L}$. A function f defined on $\mathbb{H}\times\mathbb{R}^{1,3}$ naturally extends to $\mathcal{G}$ setting $f\big((g,\xi)\big) = f(g\varepsilon_0, \xi)$. For smooth functions f on $\mathcal{G}$, set

$$\widetilde{L}^{h^{\epsilon_1}} f = \frac{\widetilde{L}(h^{\epsilon_1} f)}{h^{\epsilon_1}}.$$

We establish in paragraph 5.2.2 that bounded $\widetilde{L}^{h^{\epsilon_1}}$-harmonic functions enjoy a uniform continuity property. Being on a non-commutative group, one must differentiate between right and left uniform continuity. We show in Theorem 36 that any bounded $\widetilde{L}^{h^{\epsilon_1}}$-harmonic function h is *right* uniformly continuous: given $\varepsilon > 0$, there is a neighbourhood $\mathcal{V}_\varepsilon$ of Id in $\mathcal{G}$ such that

$$\forall\, \mathbf{e} \in \mathcal{G}, \forall\, \mathbf{e}' \in \mathcal{V}_\varepsilon, \quad \big|h(\mathbf{e}\mathbf{e}') - h(\mathbf{e})\big| \leqslant \varepsilon.$$

Notation We shall write $\mathbf{e} = (g, \xi)$ an element of $\mathcal{G}$. Denote by $\{\widetilde{P}^{\epsilon_1}_s(\mathbf{e}, d\mathbf{e}')\}_{s\geqslant 0}$ the transition kernels of the $\widetilde{L}^{h^{\epsilon_1}}$-diffusion

5.2.2 Uniform continuity

We shall obtain the right uniform continuity of bounded $\widetilde{L}^{h^{\varepsilon_1}}$-harmonic functions from the following three lemmas.

Recall that we defined in Sect. 2.3, (iii) the action of an element $\mathbf{e} = (g, \xi)$ of $\mathcal{G}$ on $\mathbb{S}^2$:

$$(g, \xi).\sigma = \sigma', \text{ if } \; g\big(\mathbb{R}(\varepsilon_0 + \sigma)\big) = \mathbb{R}(\varepsilon_0 + \sigma').$$

Lemma 33 *Each element* $\mathbf{e} \in \mathcal{G}$ *can be written* $\mathbf{e} = \underline{\mathbf{e}}\,\widehat{\mathbf{e}}$*, with* $\underline{\mathbf{e}}$ *fixing* ϵ_1*, and* $\widehat{\mathbf{e}}$ *in a* compact *subset* $\widehat{\mathcal{G}}$ *of* $\mathcal{G}$*.*

◁ Let $\mathbf{e} = (g, \xi) \in \mathcal{G}$. Writing $\mathbf{e} = (\text{Id}, \xi)(g, 0)$, we can drop (Id, ξ) apart. Use the halfspace model of $\mathbb{H}$ to describe the set of isometries of $\mathbb{H}$. It is well known that

- any isometry of $\mathbb{H}$ can be *uniquely* written as a product

$$g = t\,\lambda\, r,$$

where t is a translation in a direction of the form $(0, \tau) \in \mathbb{R}_+^* \times \mathbb{R}^2$, λ is the isometry $(y, x) \mapsto (\lambda y, \lambda x)$ and r is a hyperbolic rotation with center $(1, 0)$,

- the set of hyperbolic rotations with center $(1, 0)$ is isomorphic to the *compact* group $SO(3)$.

Each transform t, λ leaves ϵ_1 fixed, not r. Take $\underline{\mathbf{e}} = (t\lambda, \xi)$ and $\widehat{\mathbf{e}} = (r, 0)$. ▷

Notation Given $\mathbf{e} \in \mathcal{G}$, denote by $L_{\mathbf{e}}$ the left translation by $\mathbf{e}$ on $\mathcal{G}$: $L_{\mathbf{e}}(\mathbf{e}') = \mathbf{e}\mathbf{e}'$.

Lemma 34 *Let $\underline{\mathbf{e}}$ be an element of $\mathcal{G}$ fixing ϵ_1. Then one has*

$$\widetilde{P}_t^{\epsilon_1}(f) \circ L_{\underline{\mathbf{e}}} = \widetilde{P}_t^{\epsilon_1}(f \circ L_{\underline{\mathbf{e}}}),$$

for any compactly supported smooth function f.

◁ Write $\underline{\mathbf{e}} = (t\lambda, \xi)$, with t, λ as in the preceding lemma. Recall h^{ϵ_1} is a multiple of y^2. For any $\dot{\xi} \in \mathbb{H}$,

$$h^{\epsilon_1} \circ L_{\underline{\mathbf{e}}}(\dot{\xi}) = h^{\epsilon_1}\big(\mathbf{e}(\dot{\xi})\big) = h^{\epsilon_1}\big(t(\lambda(\dot{\xi}))\big) = \lambda^2 h^{\epsilon_1}(\dot{\xi}). \tag{5.12}$$

The identity of the lemma is the integrated version of the identity

$$\frac{\widetilde{L}(h^{\epsilon_1} f)}{h^{\epsilon_1}} \circ \widetilde{L}_{\underline{\mathbf{e}}} = \frac{\widetilde{L}\big((h^{\epsilon_1} f) \circ \widetilde{L}_{\underline{\mathbf{e}}}\big)}{h^{\epsilon_1} \circ \widetilde{L}_{\underline{\mathbf{e}}}} = \frac{\widetilde{L}\big(h^{\epsilon_1} f \circ \widetilde{L}_{\underline{\mathbf{e}}}\big)}{h^{\epsilon_1}}. \tag{5.13}$$

The first equality is a consequence of the left invariance of $\widetilde{L}$ by *any* translation, the second comes from identity (5.12). Here, f is any smooth function on $\mathcal{G}$ with compact support. We get from (5.13)

$$\widetilde{P}_t^{\epsilon_1}(f) \circ L_{\underline{\mathbf{e}}} = \left(e^{-t\widetilde{L}^{h^{\epsilon_1}}} f\right) \circ L_{\underline{\mathbf{e}}} = e^{-t\widetilde{L}^{h^{\epsilon_1}}}\left(f \circ L_{\underline{\mathbf{e}}}\right) = \widetilde{P}_t^{\epsilon_1}(f \circ L_{\underline{\mathbf{e}}}).$$

▷

Notation Denote by $\widetilde{p}_1^{\epsilon_1}(\mathbf{e}, \mathbf{e}')$ the density of the measure $\widetilde{P}_1^{\epsilon_1}(\mathbf{e}, d\mathbf{e}')$ with respect to a Haar measure $Haar(d\mathbf{e}')$ on $\mathcal{G}$. It is a continuous function of $(\mathbf{e}, \mathbf{e}')$.[17]

Lemma 34 means that for any $\mathbf{e} \in \mathcal{G}$, with decomposition $\mathbf{e} = \underline{\mathbf{e}}\widehat{\mathbf{e}}$, and any $\mathbf{e}' \in \mathcal{G}$,

$$\forall\, \mathbf{e}, \mathbf{e}' \in \mathcal{G}, \quad \widetilde{p}_1(\mathbf{e}, \mathbf{e}') = \widetilde{p}_1(\widehat{\mathbf{e}}, \underline{\mathbf{e}}^{-1}\mathbf{e}').$$

Because any bounded $\widetilde{L}^{h^{\epsilon_1}}$-harmonic function satisfies the relation

$$h(\mathbf{e}) = \int h(\mathbf{e}')\, \widetilde{p}_1^{\epsilon_1}(\mathbf{e}, \mathbf{e}') Haar(d\mathbf{e}'),$$

[17] See Sect. **4.1, b)**.

we have for any $\widetilde{\mathbf{e}} \in \mathcal{G}$,

$$\left|h(\mathbf{e}) - h(\mathbf{e}\widetilde{\mathbf{e}})\right| \leqslant 2\|h\|_\infty \int_{\mathcal{G}} \left|\widetilde{p}_1^{\epsilon_1}(\mathbf{e}, \mathbf{e}') - \widetilde{p}_1^{\epsilon_1}(\mathbf{e}\widetilde{\mathbf{e}}, \mathbf{e}')\right| Haar(d\mathbf{e}'). \tag{5.14}$$

Let $\mathbf{e} = \underline{\mathbf{e}}\,\widehat{\mathbf{e}}$ be the decomposition of $\mathbf{e}$ given by Lemma 33, with $\underline{\mathbf{e}}$ and $\underline{\mathbf{e}}^{-1}$ fixing ϵ_1.

$$\begin{aligned}\int_{\mathcal{G}} \left|\widetilde{p}_1^{\epsilon_1}(\mathbf{e}, \mathbf{e}') - \widetilde{p}_1^{\epsilon_1}(\mathbf{e}\widetilde{\mathbf{e}}, \mathbf{e}')\right| Haar(d\mathbf{e}') &= \int_{\mathcal{G}} \Big|\widetilde{p}_1^{\epsilon_1}(\underline{\mathbf{e}}^{-1}\mathbf{e}, \underline{\mathbf{e}}^{-1}\mathbf{e}') \\ &\quad - \widetilde{p}_1^{\epsilon_1}(\underline{\mathbf{e}}^{-1}\mathbf{e}\,\widetilde{\mathbf{e}}, \underline{\mathbf{e}}^{-1}\mathbf{e}')\Big| Haar(d\mathbf{e}') \\ &= \int_{\mathcal{G}} \left|\widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})\right| Haar(d\mathbf{a}).\end{aligned} \tag{5.15}$$

Lemma 35 *The function*

$$\widetilde{\mathbf{e}} \in \mathcal{G} \mapsto \sup_{\widehat{\mathbf{e}} \in \widehat{\mathcal{G}}} \int_{\mathcal{G}} \left|\widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})\right| Haar(d\mathbf{a})$$

converges to 0 *as* $\widetilde{\mathbf{e}} \to \mathrm{Id}$.

◁ Let $\mathcal{U}$ be a compact neighbourhood of $\mathrm{Id} \in \mathcal{G}$, and $\varepsilon > 0$. The family of probabilities $\left\{\widetilde{P}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, .)\right\}_{\widehat{\mathbf{e}} \in \widehat{\mathcal{G}}, \widetilde{\mathbf{e}} \in \mathcal{U}}$ is tight. Let $\underline{\mathcal{G}}$ be a compact subset of $\mathcal{G}$ such that for any $\widehat{\mathbf{e}} \in \widehat{\mathcal{G}}, \widetilde{\mathbf{e}} \in \mathcal{U}$,

$$\widetilde{P}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \underline{\mathcal{G}}) \geqslant 1 - \varepsilon.$$

Then

$$\int_{\mathcal{G}} \left|\widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})\right| Haar(d\mathbf{a}) \leqslant 2\,\varepsilon + \int_{\underline{\mathcal{G}}} \left|\widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})\right| Haar(d\mathbf{a}). \tag{5.16}$$

As $\widehat{\mathbf{e}}, \widetilde{\mathbf{e}}$ and $\mathbf{a}$ are in compact subsets of $\mathcal{G}$ and the function $\widetilde{p}_1^{\epsilon_1}(.,.)$ is continuous, $\widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}, \mathbf{a})$ and $\widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})$ are bounded by a constant when $\widehat{\mathbf{e}} \in \widehat{\mathcal{G}}, \widetilde{\mathbf{e}} \in \mathcal{U}$ and $\mathbf{a} \in \underline{\mathcal{G}}$. So, the function

$$(\widehat{\mathbf{e}}, \widetilde{\mathbf{e}}) \in \widehat{\mathcal{G}} \times \mathcal{U} \mapsto \int_{\underline{\mathcal{G}}} \left|\widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})\right| Haar(d\mathbf{a})$$

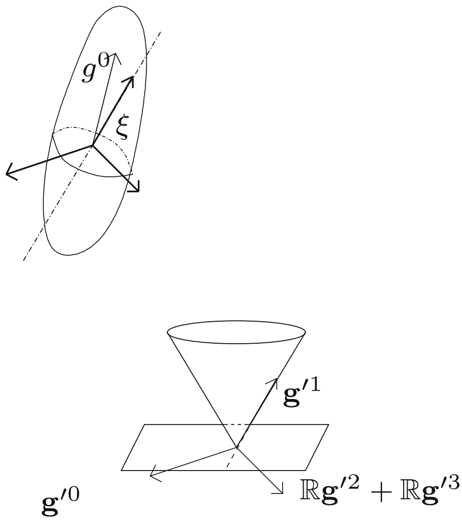


Fig. 3 Trace on $\mathbb{R}^{1,3}$ of a neigbourhood $\mathbf{e}\mathcal{V}$ of $\mathbf{e} = (g, \xi) \in \mathcal{G}$

is uniformly continuous. Since it is equal to zero on $\widehat{\mathcal{G}} \times \{0\}$, we get from (5.16)

$$\overline{\lim_{\widetilde{\mathbf{e}} \to \mathrm{Id}}} \int_{\mathcal{G}} \left| \widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}},\ \mathbf{a}) - \widetilde{p}_1^{\epsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a}) \right| Haar(d\mathbf{a}) \leqslant 2\,\varepsilon.$$

As $\varepsilon > 0$ was chosen arbitrarily, the statement of the lemma follows. $\triangleright$

Inequality (5.14) together with Lemma **35** imply that

Theorem 36 *Any bounded $\widetilde{L}^{h^{\epsilon_1}}$-harmonic function is right uniformly continuous.*

From now on the notation $\underline{\mathbf{e}}$ will no longer refer to the decomposition of Lemma **33**.

5.2.3 $L^{h^{\epsilon_1}}$-harmonic bounded functions

The usefulness of Theorem **36** appears in the following lemma, which is illustrated in Fig. 3. It roughly says that a bounded $L^{h^{\epsilon_1}}$-harmonic function $h(\dot{\xi}, \xi)$ does not vary much if $\dot{\xi}$ remains near g^0 and ξ moves in the pictured "ellipsoid".

Lemma 37 *Let $\mathcal{V}$ be a neighbourhood of Id $\in \mathcal{G}$.*

1. *Let $\mathbf{e} = (g, \xi) \in \mathcal{G}$ and $(y, x) \in \mathbb{R}_+^* \times \mathbb{R}^2$ be the halfspace coordinates of g^0. Identify ξ with 0 on the straight line $\xi + \mathbb{R}\mathbf{g}'^1$. The intersection of $\xi + \mathbb{R}\mathbf{g}'^1$ with $\mathbf{e}\mathcal{V}$ contains an interval $]a(y), b(y)[$ with the property that $a(y) \to -\infty$ and $b(y) \to +\infty$ as $y \to +\infty$.*
2. *For $\xi \in \mathbb{R}^{1,3}$, identify ξ to 0 in $\xi + \left(\mathbb{R}\mathbf{g}'^2 + \mathbb{R}\mathbf{g}'^3\right)$. There exists a constant $\varepsilon > 0$ depending only on $\mathcal{V}$, such that for any $\mathbf{e} = (g, \xi) \in \mathcal{G}$, the intersection of $\xi + \left(\mathbb{R}\mathbf{g}'^2 + \mathbb{R}\mathbf{g}'^3\right)$ with $\mathbf{e}\mathcal{V}$ contains the Euclidean ball $B(0, \varepsilon)$.*

$\triangleleft$ **1)** For any $\mathbf{e} = (g, \xi)$ and $\underline{\mathbf{e}} = (\underline{g}, \underline{\xi})$ in $\mathcal{G}$, we have $\mathbf{e}\underline{\mathbf{e}} = (g\underline{g}, \xi + g\underline{\xi})$. Let $\varepsilon > 0$ small enough such that $\mathcal{V}$ contains a product neighbourhood

$\mathcal{V}_1 \times]-\varepsilon, \varepsilon[^4 \subset SO_0(1,3) \times \mathbb{R}^{1,3}$. We need to examine the set $g\,.\,]-\varepsilon, \varepsilon[^4$.

$$g\,\underline{\xi} = \underline{\xi}^0 g^0 + \underline{\xi}^1 g^1 + \underline{\xi}^2 g^2 + \underline{\xi}^3 g^3.$$

In the basis $\mathbf{g}'$, g^0 has coordinates $\left(\frac{1}{y}, \frac{|x|^2+y^2+1}{2y}, \frac{x}{y}\right)$. As the set $g\,.\,]-\varepsilon, \varepsilon[^4 \cap \mathbb{R}\mathbf{g}'^1$ contains the segment $\frac{|x|^2+y^2+1}{2y}\ ]-\varepsilon, \varepsilon[$, the statement holds with $a(y) = -\varepsilon\frac{y^2+1}{2y}$ and $b(y) = \varepsilon\frac{y^2+1}{2y}$.

2) Let (y, x) be the halfspace coordinates of g^0. Any $V \in T_{(y,x)}\left(\mathbb{R}_+^* \times \mathbb{R}^2\right)$, with hyperbolic norm equal to 1, is of the form $y(u, v) \in \mathbb{R} \times \mathbb{R}^2$, where $u^2 + |v|_{\mathrm{Eucl}} = 1$. Note that since $\psi : \mathbb{R}_+^* \times \mathbb{R}^2 \to \mathbb{H}$ is an isometry, each g^i, $i = 1, \ldots, 3$, is of the form $D_{(y,x)}\psi\big(y(u, v)\big)$ for some (u, v), t follows that

$$\left\{\underline{\xi}^1 g^1 + \underline{\xi}^2 g^2 + \underline{\xi}^3 g^3\ ;\ \underline{\xi} \in]-\varepsilon, \varepsilon[^3\right\} = D_{(y,x)}\psi\big(yB_{\mathbb{R}^3}(0, \varepsilon)\big),$$

where $B_{\mathbb{R}^3}(0, \varepsilon)$ is the open Euclidean ball of $\mathbb{R}^3$ with center 0 and radius ε. In particular, the vector $D_{(y,x)}\psi\big(y(0, v)\big)$ belongs to $\left\{\underline{\xi}^1 g^1 + \underline{\xi}^2 g^2 + \underline{\xi}^3 g^3\ ;\ \underline{\xi} \in]-\varepsilon, \varepsilon[^3\right\}$ if $v \in \mathbb{R}^2$, $|v| \leqslant \varepsilon$. A direct calculation gives the $\mathbf{g}'$-coordinates of $D_{(y,x)}\psi\big(y(0, v)\big)$: $\big(\langle x, v\rangle, \langle xv\rangle, v\big)$. The second point of the lemma is now straightforward. ▷

We can now prove that

Theorem 38 *Any bounded $L^{h^{\epsilon_1}}$-harmonic function only depends on y and ξ'^0.*

◁ Let h be a bounded $L^{h^{\epsilon_1}}$-harmonic function, considered as a $\widetilde{L}^{h^{\epsilon_1}}$-harmonic function. Let $\eta > 0$ be given. Because of the right uniform continuity of h, there exists a neighbourhood $\mathcal{V}$ of $\mathrm{Id} \in \mathcal{G}$ such that for any $\widetilde{\mathbf{e}} \in \mathcal{V}$,

$$\forall\, \mathbf{e} \in \mathcal{G},\ \big|h(\mathbf{e}) - h(\mathbf{e}\widetilde{\mathbf{e}})\big| \leqslant \eta.$$

Take two points $\mathbf{e}_0 = \big((g_0^0, \ldots, g_0^3), \xi_0\big)$ and $\underline{\mathbf{e}}_0 = \big((\underline{g}_0^0, \ldots, \underline{g}_0^3), \underline{\xi}_0\big)$ in $\mathcal{G}$, with g_0^0 and $\underline{g}_0^0$ ($\in \mathbb{H}$) having the same y-coordinate, and $\underline{\xi}_0'^0 = \xi_0'^0$. Using the coupling time constructed in Theorem 32 and the stopping time theorem (h is bounded), we can write for any $s \geqslant 0$

$$h(\mathbf{e}_0) - h(\underline{\mathbf{e}}_0) = \mathbb{E}^{\epsilon_1}_{\mathbf{e}_0, \underline{\mathbf{e}}_0}\big[h(\mathbf{e}_{T+s}) - h(\underline{\mathbf{e}}_{T+s})\big].$$

The stopping time T was constructed to ensure that

$$\mathbf{e}_{T+s} = (g_{T+s}, \xi_{T+s}) = \Big(\big((y_{T+s}, x_{T+s}), g^1_{T+s}, g^2_{T+s}, g^3_{T+s}\big), \xi_{T+s}\Big)$$

and

$$\underline{\mathbf{e}}_{T+s} = (\underline{g}_{T+s}, \underline{\xi}_{T+s}) = \Big(\big((y_{T+s}, x_{T+s}), \underline{g}^1_{T+s}, \underline{g}^2_{T+s}, \underline{g}^3_{T+s}\big),\ \xi_{T+s} + C_1\mathbf{g}'^1 + c_2\mathbf{g}'^2 + c_3\mathbf{g}'^3\Big),$$

for some $(g^1_{T+s}, g^2_{T+s}, g^3_{T+s})$, and $(\underline{g}^1_{T+s}, \underline{g}^2_{T+s}, \underline{g}^3_{T+s})$ that need not be equal, but with $|c_2|, |c_3| \leqslant \varepsilon$. However there exists an isometry ρ of $\mathbb{R}^3 \subset \mathbb{R}^{1,3}$ such that

$$\underline{g}_{T+s}\,\rho = g_{T+s}.$$

Write $\underline{\mathbf{e}}_{T+s}\,\rho = (\underline{g}_{T+s}\,\rho, \underline{\xi}_{T+s})$. Since the function $h\Big(\big((y,x), g^1, g^2, g^3\big), \xi\Big)$ does not depend on g^1, g^2, g^3,

$$h(\underline{\mathbf{e}}_{T+s}\,\rho) = h(\underline{\mathbf{e}}_{T+s}).$$

As $\{y_r\}_{r\geqslant 0}$ diverges $\mathbb{P}^{\epsilon_1}$-almost surely to $+\infty$, we know from Lemma 37 that for s large enough, y_{T+s} is large enough to ensure that

$$\underline{\mathbf{e}}_{T+s}\rho \in \mathbf{e}_{T+s}\mathcal{V}.$$

Applying the bounded convergence theorem, it follows that

$$\big|h(\mathbf{e}_0) - h(\underline{\mathbf{e}}_0)\big| \leqslant \mathbb{E}^{\epsilon_1}_{\mathbf{e}_0,\underline{\mathbf{e}}_0}\Big[\lim_{s\to+\infty}\big|h(\mathbf{e}_{T+s}) - h(\underline{\mathbf{e}}_{T+s})\big|\Big] \leqslant \eta.$$

Since $\eta > 0$ is arbitrary, the result follows. ▷

5.2.4 Towards a second conditioning

Notation Given $\ell \in \mathbb{R}$, denote by τ_ℓ the translation $\xi \in \mathbb{R}^{1,3} \mapsto \xi + \ell\mathbf{g}'^0$.
We can see that bounded $L^{h^{\epsilon_1}h^{\epsilon_1}_\ell}$-harmonic functions depend only on y and ξ'^0 from the following observation.

Lemma 39 *h is a bounded $L^{h^{\epsilon_1}h^{\epsilon_1}_\ell}$-harmonic function iff $h\circ\tau_{\ell-\ell'}$ is a bounded $L^{h^{\epsilon_1}h^{\epsilon_1}_{\ell'}}$-harmonic function.*

So, given a bounded $L^{h^{\epsilon_1}h^{\epsilon_1}_\ell}$-harmonic function h, and $\ell' \leqslant \ell$,

$$L\Big(h^{\epsilon_1}h^{\epsilon_1}_{\ell'}\,\big(h\circ\tau_{\ell-\ell'}\big)\Big) = 0.$$

Thus the function

$$2^n \int_{\ell-2^{-n}}^{\ell} h^{\epsilon_1}_{\ell'}(.)(h\circ\tau_{\ell-\ell'})(.)d\ell'$$

is a bounded $L^{h^{\epsilon_1}}$-harmonic function,[18] and so it depends only on y and ξ'^0. Its limit $h^{\epsilon_1}_\ell h$ also depends only on y and ξ'^0; as $h^{\epsilon_1}_\ell$ only depends on y and ξ'^0, so does h.

[18] It is $\leqslant 2^n\,\|h\|_\infty\mathbb{P}^{\epsilon_1}_{.}\big(R^{\sigma_\infty}_\infty \in [\ell-2^{-n}, \ell]\big)$.

Corollary 40 *Let $\ell \in \mathbb{R}$. Bounded $L^{h^{\epsilon_1} h_\ell^{\epsilon_1}}$-harmonic functions only depend on y and ξ'^0.*

Notation Denote by $\mathbb{P}_u^{\epsilon_1,\ell}$ the law of the $L^{h^{\epsilon_1} h_\ell^{\epsilon_1}}$-diffusion (or $\left(h^{\epsilon_1} h_\ell^{\epsilon_1}\right)$-process), started from $u \in \mathbb{H} \times \mathbb{R}^{1,3}$.

Now, to show that bounded $L^{h^{\epsilon_1} h_\ell^{\epsilon_1}}$-harmonic functions are constant, we are going to study the $\left(h^{\epsilon_1} h_\ell^{\epsilon_1}\right)$-process. The two-dimensional process $\left\{(y_s, \xi_s'^0)\right\}$ happens to be a *diffusion*, under $\mathbb{P}_u^{\epsilon_1,\ell}$; this reduces the initial seven-dimensional problem to a two-dimensional one. We shall show in Propositions **46** and **47** that two independent trajectories of this process couple naturally; the conclusion will follow.

The constant $\ell \in \mathbb{R}$ is fixed in the next section.

5.3 Bounded $L^{h^{\epsilon_1} h_\ell^{\epsilon_1}}$-harmonic functions

5.3.1 Preliminary remarks

First, let us see what the $h_\ell^{\epsilon_1}$-transform $L^{h^{\epsilon_1} h_\ell^{\epsilon_1}}$ of $L^{h^{\epsilon_1}}$ looks like. It is defined on the open halfspace

$$\left\{(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}\,;\; \ell - \xi'^0 > 0\right\},$$

For our convenience we shall write

$$\alpha(\xi) = \ell - \xi'^0,$$

or simply α; so the operator $L^{h^{\epsilon_1} h_\ell^{\epsilon_1}}$ is defined on $\{\alpha > 0\}$.

The transform adds a drift[19]

$$2\frac{y^2}{2}\frac{\partial_y h_\ell^\sigma}{h_\ell^\sigma}\partial_y = y^2 \frac{G'(y\alpha) + y\alpha G''(y\alpha)}{yG'(y\alpha)}\partial_y = y\left(-2 + \frac{2}{y\alpha}\right)\partial_y,$$

to $L^{h^{\epsilon_1}}$. So we have for any $f \in \mathcal{C}_0^\infty\left(\{\alpha > 0\}\right)$,

$$L^{h^{\epsilon_1} h_\ell^\sigma} f = \frac{y^2}{2}(\partial_x^2 + \partial_y^2) f + \left(\frac{2}{\alpha(\xi)} - \frac{y}{2}\right)\partial_y f + \partial_\xi f(\dot{\xi}, \xi).\dot{\xi}.$$

The evolution of the $h^{\epsilon_1} h_\ell^{\epsilon_1}$-process is determined by the following stochastic differential system.

$$dy_s = y_s dw_s^y + y_s\left(\frac{2}{\alpha(\xi_s) y_s} - \frac{1}{2}\right) ds,$$
$$dx_s = y_s dw_s^x,$$

[19] We saw in equation (5.5) that $\frac{r^2}{2}G''(r) + \left(\frac{3r}{2} - 1\right)G'(r) = 0$.

$$
\begin{aligned}
d\xi_s'^0 &= \frac{ds}{y_s}, \\
d\xi_s'^1 &= \frac{|x_s|^2 + y_s^2 + 1}{2y_s} ds, \\
d\xi_s'^j &= \frac{x_s^{j-1}}{y_s} ds, \quad j = 2, \dots, d.
\end{aligned} \tag{5.17}
$$

It is defined until its explosion time, defined as the infimum of its exit time from all compacta and $\inf\{s > 0; \alpha(\xi_s) = 0\}$.

Notation We shall write $\alpha_s \equiv \alpha(\xi_s) = \ell - \xi_s'^0$.

As explained in the preceding paragraph, we are interested in the behaviour of the process (y, ξ'^0). Remark that since

$$\xi_s'^0 = \ell - \alpha_s,$$

we deduce from (5.17) that

$$
\begin{aligned}
dy_s &= y_s dw_s^y + y_s \left(\frac{2}{\alpha_s y_s} - \frac{1}{2} \right) ds, \\
d\alpha_s &= -\frac{ds}{y_s};
\end{aligned}
$$

so,

Fact 41 *The process $\{(y_s, \alpha_s)\}_{0 \leqslant s < S}$ is a diffusion under any $\mathbb{P}_u^{\epsilon_1, \ell}$, $u \in \mathbb{H} \times \mathbb{R}^{1,3}$.*

Notice that since $d\alpha_s = -\frac{ds}{y_s} < 0$, the process $\{\alpha\}_{0 \leqslant s < S}$ decreases.

Lemma 42 *We have*: $d(y_s\alpha_s) = (y_s\alpha_s) dw_s^y + \left(1 - \frac{y_s\alpha_s}{2}\right) ds.$

◁ Use Itô's formula.

$$
\begin{aligned}
d(y_s\alpha_s) &= y_s \frac{-ds}{y_s} + \alpha_s y_s \, dw_s^y + \alpha_s \left(\frac{2}{\alpha_s} - \frac{y_s}{2} \right) ds \\
&= (y_s\alpha_s) dw_s^y + \left(1 - \frac{y_s\alpha_s}{2}\right) ds.
\end{aligned}
$$

▷

Corollary 43 • *The process $\{y_s\alpha_s\}_{0 \leqslant s < S}$ is a positive recurrent diffusion on $\mathbb{R}_+^*$.*

• *The $\left(h^{\epsilon_1} h_\ell^{\epsilon_1}\right)$-process $\mathbb{P}_u^{\epsilon_1, \ell}$-almost surely does not explode, for every starting point $u \in \mathbb{H} \times \mathbb{R}^{1,3}$.*

The density[20] of the invariant probability of the diffusion $\{y_s\alpha_s\}_{s\geqslant 0}$ is proportional to $t^{-3}e^{-\frac{2}{t}}$.[21] The equation governing the evolution of y_s is integrable. Set

$$z_s = e^{-w_s^y - 2\int_0^s \frac{dr}{y_r\alpha_r} + s}.$$

Using the fact that

$$dz_s = -z_s dw_s^y + \left(\frac{3}{2} - \frac{2}{y_s\alpha_s}\right) z_s ds,$$

and that

$$d\langle y_s, z_s\rangle = -y_s z_s\, ds,$$

one finds

$$d(y_s z_s) = 0,$$

that is

$$y_s = y\, e^{w_s^y + 2\int_0^s \frac{dr}{y_r\alpha_r} - s}. \tag{5.18}$$

As we know the invariant probability of the positive recurrent diffusion $\{y_s\alpha_s\}_{s\geqslant 0}$, the ergodic theorem gives us an almost sure equivalent of the integral

$$\int_0^s \frac{du}{y_u\alpha_u} = \frac{\int_0^\infty t^{-4}e^{-\frac{2}{t}}\, dt}{\int_0^\infty t^{-3}e^{-\frac{2}{t}}\, dt} s + o(s)$$

An integration by parts gives $\frac{\int_0^\infty t^{-4}e^{-\frac{2}{t}}\, dt}{\int_0^\infty t^{-3}e^{-\frac{2}{t}}\, dt} = 1$, so that the law of iterated logarithm and formula (5.18) provide a precise estimate of y_s.

Proposition 44 *We have almost surely*: $\log(y_s) = s + o(s)$.

As expected,

Lemma 45 $\{\alpha_s\}_{s\geqslant 0}$ *almost surely decreases to* 0 *as* $s \to +\infty$.

$\lhd$ As $\{\alpha_s\}_{s\geqslant 0}$ decreases, the event $\big\{\{\alpha_s\}_{s\geqslant 0}$ does not tends to $0\big\}$ can be decribed as $\bigcup_{n\geqslant 1}\{\forall s \geqslant 0,\ \alpha_s \geqslant \frac{1}{n}\}$. But y_s diverging to $+\infty$ one has $y_s\alpha_s \to +\infty$ on the event $\{\forall s \geqslant 0,\ \alpha_s \geqslant \frac{1}{n}\}$. The diffusion $\{y_s\alpha_s\}_{s\geqslant 0}$ being

[20] With respect to Lebesgue measure on $\mathbb{R}_+^*$.

[21] This density $m(t)$ must satisfy the equation $\big(\frac{t^2}{2}m(t)\big)'' = \big((1-\frac{t}{2})m(t)\big)'$. If we set $v(t) = \frac{t^2}{2}m(t)$, this function must satisfy the equation $v''(t) = \big(\{\frac{2}{t^2} - \frac{1}{t}\}v(t)\big)'$, *i.e.*, up to an additive constant $v'(t) = \left\{\frac{2}{t^2} - \frac{1}{t}\right\} v(t)$. This gives $v(t) = t^{-1}e^{-\frac{2}{t}}$, and $m(t) = t^{-3}e^{-\frac{2}{t}}$, ignoring the constant.

positive recurrent each event $\{\forall\, s \geqslant 0,\ \alpha_s \geqslant \frac{1}{n}\}$ need to be of zero probability, so $\{\alpha_s\}_{s\geqslant 0}$ happens to decrease to 0 almost surely. $\triangleright$

To investigate the system, we shall take $\left(\frac{1}{y_s\alpha_s}, \alpha_s\right)$ as coordinates rather than (y_s, α_s). Set $b_s \equiv \frac{1}{y_s\alpha_s}$. We have

$$\begin{aligned} d\left(\frac{1}{y_s\alpha_s}\right) \equiv db_s &= -b_s dw_s^y + b_s\left(\frac{3}{2} - b_s\right) ds, \\ d\alpha_s &= -\alpha_s b_s ds, \end{aligned} \tag{5.19}$$

or

$$\begin{aligned} b_s &= b_0 - \int_0^s b_r dw_r^y + \int_0^s b_r\left(\frac{3}{2} - b_r\right) dr \\ \alpha_s &= \alpha_0\, e^{-\int_0^s b_r dr}. \end{aligned} \tag{5.20}$$

The diffusion b is positive recurrent.

5.3.2 An automatic coupling

In this paragraph,

(a) we show that two independent copies of (b, α), started from different points, meet with probability 1. So, two independent copies of (y, α) started from different points meet with probability 1.
(b) This implies that $L^{h^{\epsilon_1} h_\ell^{\epsilon_1}}$ has no non-constant bounded harmonic functions only depending on y and α (or y and ξ'^0).

(a) For $(b_0, \alpha_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, denote by $\mathbb{P}_{(b_0,\alpha_0)}$ the law of the diffusion (b, α) started from (b_0, α_0), defined on the canonical space $\Omega \equiv \mathcal{C}(\mathbb{R}^{\geqslant 0}, \mathbb{R}^2)$. For $\omega \in \Omega$ and $R > 0$, set

$$T_R(\omega) \equiv \inf\{s > 0; \log\alpha_s(\omega) \leqslant -R\}.$$

As $\log\alpha_s = \log\alpha_0 - \int_0^s b_r\, dr$, and b_r is continuous, > 0, T_R is (almost surely) a strictly increasing continuous function of R.

Proposition 46 *The process $\{b_{T_R}\}_{R\geqslant -\log\alpha_0}$ is a recurrent diffusion on $\mathbb{R}_+^*$.*

$\triangleleft$ On the one hand, $\big\{-\log(\alpha_s)\big\}_{s\geqslant 0}$ being an additive functional of $\{b_s\}_{s\geqslant 0}$, it is well known that the time transform of b by the inverse of $-\log(\alpha)$ remains a diffusion. On the other hand, since b is positive recurrent, one can invoke the ergodic theorem and find some constants $0 < c < C < +\infty$, such that one has almost surely $c\, s \leqslant -\log\alpha_s \leqslant C\, s$, from a random time $s_0(\omega)$ onwards. So, for R large enough ($\geqslant \frac{s_0}{c}$),

$$\frac{R}{C} \leqslant T_R \leqslant \frac{R}{c}.$$

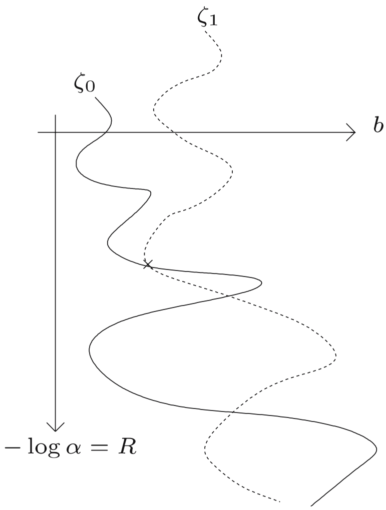


Fig. 4 Coupling of two independent trajectories of (b, α)

Then, the recurrence of $\{b_s\}_{s\geqslant 0}$ implies that of $\{b_{T_R}\}_{R\geqslant -\log\alpha_0}$. $\triangleright$

Note that the trajectory of (b_s, α_s) is the same as that of $\{b_{T_R}\}_{R\geqslant\alpha_0}$ in (b, R) coordinates, as is illustrated in Fig. 4 (two trajectories of this process are drawn). It remains to use the following elementary result to prove point **a)**.

Proposition 47 *Let* $\{\mathbb{P}_x\}_{x\in\mathbb{R}_+^*}$ *be the family of laws associated with a recurrent diffusion on* $\mathbb{R}_+^*$*. Let* $x_0 \neq x_1 \in \mathbb{R}_+^*$ *and* $\mathbb{P}_{x_0,x_1}$ *be the law of a couple* $(\mathfrak{x}, \mathfrak{x}')$ *of independent diffusions, with laws* $\{\mathbb{P}_x\}_{x\in\mathbb{R}_+^*}$*, started from* x_0 *and* x_1*, respectively. Then,* $\mathbb{P}_{x_0,x_1}$*-almost surely,* $\inf\{s > 0;\, \mathfrak{x}_s = \mathfrak{x}'_s\} < +\infty$.

$\triangleleft$ Map the state space $\mathbb{R}_+^*$ on $\mathbb{R}$ by the scale function to get two independent recurrent diffusions $\underline{\mathfrak{x}}$ and $\underline{\mathfrak{x}}'$ on $\mathbb{R}$, in natural scale. These are independent continuous local martingales with brackets increasing to $+\infty$ as time goes to $+\infty$. So the local martingale $\underline{\mathfrak{x}} - \underline{\mathfrak{x}}'$, whose bracket

$$\langle \underline{\mathfrak{x}} - \underline{\mathfrak{x}}' \rangle = \langle \underline{\mathfrak{x}} \rangle + \langle \underline{\mathfrak{x}}' \rangle \to +\infty,$$

has the trajectories of a Brownian motion; in particular, it hits 0 in a finite time. Since the scale function is injective, $\underline{\mathfrak{x}}$ and $\underline{\mathfrak{x}}'$ coincide at that time. $\triangleright$

The preceding two propositions show that *two independent copies of* (y, α)*, started from different points, almost surely meet at a positive finite time.*[22]

(b) Let us see why this fact implies that the infinitesimal generator of (b, α) has no non-constant bounded harmonic functions.

Let $\zeta_0 \neq \zeta_1$ be two points of $\mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\mathbb{P}_{\zeta_0}, \mathbb{P}_{\zeta_1}$, be the laws of the diffusion (b, α) started from ζ_0 and ζ_1 respectively. Let $\mathfrak{x}$ and $\mathfrak{x}'$, $\Omega \times \Omega \to \Omega$ be the first and

[22] Actually, they meet at arbitrarily large times.

second projection, respectively. The system

$$\left(\Omega\times\Omega, \mathcal{F}\otimes\mathcal{F}, \left\{\sigma\left((\mathfrak{x}_s, \mathfrak{x}'_s); s\leqslant t\right)\right\}_{t\geqslant 0}, \mathbb{P}_{\zeta_0,\zeta_1}\equiv\mathbb{P}_{\zeta_0}\otimes\mathbb{P}_{\zeta_1}\right)$$

describes the evolution of two independent copies of (b, α), started from ζ_0 and ζ_1.

Set

$$T_0(\omega,\omega')\equiv\inf\{s>0; \mathfrak{x}_s(\omega,\omega')\in\mathfrak{x}'_{[0,+\infty[}(\omega,\omega')\}$$

and

$$T_1(\omega,\omega')\equiv\inf\{s>0; \mathfrak{x}'_s(\omega,\omega')\in\mathfrak{x}_{[0,+\infty[}(\omega,\omega')\}.$$

These random times are not stopping times *with respect to the* σ*-algebra* $\{\sigma((\mathfrak{x}_s, \mathfrak{x}'_s); s\leqslant t)\}_{t\geqslant 0}$. Yet, point (a) tells us that they are finite $\mathbb{P}_{\zeta_0,\zeta_1}$-almost surely. As a consequence, the sets

$$\Omega_1\equiv\{\omega'\in\Omega\ ;\ T_0(\omega,\omega')<+\infty,\ \mathbb{P}_{\zeta_0}(d\omega)-a.s.\}$$

and

$$\Omega_0\equiv\{\omega\in\Omega\ ;\ T_1(\omega,\omega')<+\infty,\ \mathbb{P}_{\zeta_1}(d\omega')-a.s.\}$$

verify

$$\mathbb{P}_{\zeta_1}(\Omega_1)=\mathbb{P}_{\zeta_0}(\Omega_0)=1.$$

Now, given $(\omega,\omega')\in\Omega_0\times\Omega_1$, the trajectories of (b_s,α_s) and (b'_s,α'_s) in $\mathbb{R}\times\mathbb{R}^*_+$ are the same as those of b_{T_R} and b'_{T_R}, in (b, R) coordinates, as illustrated in Fig. 4. Setting

$$R_0=\inf\left\{R\ ;\ b_{T_R}\in b'_{T_{[-\log(\alpha'_0),\infty[}}\right\},$$

and

$$R_1=\inf\{R\ ;\ b'_{T_R}\in b_{T_{[-\log(\alpha_0),\infty[}}\},$$

one hase trivially

$$b'_{T_{R_1}}=b_{T_{R_0}};$$

back to the processes $\mathfrak{x}$ and $\mathfrak{x}'$, it means that one has $\mathbb{P}_{\zeta_0,\zeta_1}$-almost surely

$$\mathfrak{x}_{T_0(\omega,\omega')}(\omega,\omega')=\mathfrak{x}'_{T_1(\omega,\omega')}(\omega,\omega').$$

Proposition 48 *Any bounded function h, harmonic with respect to the infinitesimal generator of the diffusion* (y, α) *is constant.*

$\lhd$ Let $\omega' \in \Omega_1$. The application $\omega \in \Omega \mapsto T_0(\omega, \omega')$ is a $\{\sigma(\mathfrak{x}_s; s \leqslant t)\}_{t \geqslant 0}$ stopping time, $\mathbb{P}_{x_0}(d\omega)$-almost surely finite, and $\{h(\mathfrak{x}_t(\omega, \omega'))\}_{t \geqslant 0}$ is a $\{\{\sigma(\mathfrak{x}_s\,;\, s \leqslant t)\}_{t \geqslant 0}, \mathbb{P}_{x_0}(d\omega)\}$ bounded martingale. The stopping time theorem applies.

$$h(\zeta_0) = \int h\big(\mathfrak{x}_{T_0(\omega,\omega')}(\omega, \omega')\big)\mathbb{P}_{\zeta_0}(d\omega);$$

integrating with respect to $\mathbb{P}_{\zeta_1}(d\omega')$, one gets

$$h(\zeta_0) = \int h\big(\mathfrak{x}_{T_0(\omega,\omega')}(\omega, \omega')\big)\mathbb{P}_{\zeta_0,\zeta_1}(d\omega, d\omega') \tag{5.21}$$

In the same way, one can show that

$$h(\zeta_1) = \int h\big(\mathfrak{x}_{T_1(\omega,\omega')}(\omega, \omega')\big)\mathbb{P}_{\zeta_0,\zeta_1}(d\omega, d\omega'). \tag{5.22}$$

The equation $h(\zeta_0) = h(\zeta_1)$ now comes from the fact that $\mathfrak{x}_{T_0(\omega,\omega')}(\omega, \omega')$ and $\mathfrak{x}_{T_1(\omega,\omega')}(\omega, \omega')$ are $\mathbb{P}_{\zeta_0,\zeta_1}(d\omega, d\omega')$ almost surely equal. $\rhd$

Since any bounded $L^{h^{\epsilon_1}h_\ell^{\epsilon_1}}$-harmonic function depends only on y and ξ'^0, that is on y and α, any such function is harmonic with respect to the infinitesimal generator of the diffusion (y, α); so it is constant.

This proves that the L-harmonic functions $h^{\epsilon_1}h_\ell^{\epsilon_1}$ are minimal.

• Remember we fixed some $\sigma \in \mathbb{S}^2$ in the introduction of Sect. **5**, and chose the basis $\{\epsilon_0, \ldots, \epsilon_3\}$ of $\mathbb{R}^{1,3}$ in such a way that

$$\epsilon_0 = \varepsilon_0, \quad \epsilon_1 = \sigma.$$

Denote by h^σ and h_ℓ^σ the functions h^{ϵ_1} and $h_\ell^{\epsilon_1}$. As $\sigma \in \mathbb{S}^2$ and $\ell \in \mathbb{R}$ were arbitrary, Proposition **48** justifies point *3* of Theorem **18**: "The functions $h^\sigma h_\ell^\sigma$, $\sigma \in \mathbb{S}^2$, $\ell \in \mathbb{R}$, are L-harmonic minimal functions".

Write $\mathbb{P}^\sigma_{\dot\xi,\xi}$ instead of $\mathbb{P}^{\epsilon_1}_{\dot\xi,\xi}$, and

$$R_\infty^\sigma = \lim_{s \to +\infty} q(\xi_s, \varepsilon_0 + \sigma)$$

the $\mathbb{P}^\sigma_{\dot\xi,\xi}$-almost sure preceding limit.

One obtains a coordinate free expression of the function h_ℓ^σ, whose formula was given in (5.7), remarking that if $\dot\xi \in \mathbb{H}$ has $\{\epsilon_0, \ldots, \epsilon_3\}$-halfspace coordinates (y, x),

$$\frac{1}{y} = q(\dot\xi, \epsilon_0 + \epsilon_1) = q(\dot\xi, \varepsilon_0 + \sigma),$$

and $\xi'^0 = q(\xi, \varepsilon_0 + \sigma)$. So,

$$h^\sigma_\ell(\dot{\xi}, \xi) = \frac{1}{q(\dot{\xi}, \varepsilon_0+\sigma)} \left(\frac{q(\dot{\xi}, \varepsilon_0+\sigma)}{\ell - q(\xi, \varepsilon_0+\sigma)} \right)^3 \exp\left(- \frac{2q(\dot{\xi}, \varepsilon_0+\sigma)}{\ell - q(\xi, \varepsilon_0+\sigma)} \right) \mathbf{1}_{\ell > q(\xi, \varepsilon_0+\sigma)} \tag{5.23}$$

The following classical decomposition[23] asserts that $\mathbb{P}^\sigma_{\dot{\xi},\xi}$ is the law of the relativistic diffusion $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s \geqslant 0}$, conditioned on the event $\{\sigma_\infty = \sigma\}$: for any measurable event $A \subset \Omega$ and any measurable $B \subset \mathbb{S}^2$,

$$\mathbb{P}_{\dot{\xi},\xi}(A, \sigma_\infty \in B) = \int_B \mathbb{P}^\sigma_{\dot{\xi},\xi}(A)\, h^\sigma(\dot{\xi}, \xi)\, d\sigma, \quad (\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}.$$

One obtains the $\mathbb{P}_{\dot{\xi},\xi}$-almost sure existence of the limits σ_∞ and $R^{\sigma_\infty}_\infty$, and the determination of the law of the pair $(\sigma_\infty, R^{\sigma_\infty}_\infty)$ as a consequence of this identity. Theorem **18** is now entirely proved.

6 Comments

Asymptotic behaviour of the relativistic diffusion From a formal point of view, the information on the asymptotic behaviour of $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s \geqslant 0}$ is not contained in the invariant σ-algebra but in the tail σ-algebra

$$\tau\big(\{(\dot{\xi}_s, \xi_s)\}\big) = \bigcap_{t>0} \sigma\big((\dot{\xi}_s, \xi_s)\,;\ s \geqslant t\big).$$

The invariant σ-algebra is a subalgebra of $\tau\big(\{(\dot{\xi}_s, \xi_s)\}\big)$. These two σ-algebras are generally distinct (see [20], for instance).

In the same way as the analytic counterpart of the invariant σ-algebra of the diffusion $\{(\dot{\xi}_s, \xi_s)\}_{s \geqslant 0}$ is the Poisson boundary of L, it is a classical result that the analytical counterpart of its tail σ-algebra is the Poisson boundary of the operator $L + \partial_s$ on $\left(\mathbb{H} \times \mathbb{R}^{1,3}\right) \times \mathbb{R}^*_+$. Using Harnack inequality (Theorem **16**), and Derrienic's $0-2$ law as in [1] (Cor. 3.2, p.32), one can show that any bounded $(L + \partial_s)$-harmonic function on $\left(\mathbb{H} \times \mathbb{R}^{1,3}\right) \times \mathbb{R}^*_+$ does not depend on s; so, is L-harmonic. This means that the invariant σ-algebra of $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s \geqslant 0}$ and the tail σ-algebra of $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s \geqslant 0}$ coincide, up to $\mathbb{P}_{\dot{\xi},\xi}$-null sets, under any probability $\mathbb{P}_{\dot{\xi},\xi}$, $(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}$. So, Theorem **1** gives a complete description of the asymptotic behaviour of the relativistic diffusion.[24]

A geometrical description of the Poisson boundary of L In so far as $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s \geqslant 0}$ is the only diffusion on $\mathbb{H} \times \mathbb{R}^{1,3}$ whose law is invariant under the (natural) action of the affine isometries on $\mathbb{H} \times \mathbb{R}^{1,3}$ (leaving $\mathbb{H}$ fixed), it is natural to wonder to what

[23] See [22], Sect. 7.2, Theorem 2.2.

[24] The details of this paragraph can be found in the forthcoming article [4], in the Appendix.

extent the asymptotic behaviour of the relativistic diffusion reflects the geometry of Minkowski spacetime. Is the space $\mathbb{H} \times \mathbb{R}^{1,3}$ naturally endowed with a boundary $\partial\left(\mathbb{H} \times \mathbb{R}^{1,3}\right)$ such that

- the trajectories $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s \geqslant 0}$ converge $\mathbb{P}_{\dot{\xi},\xi}$-almost surely towards a point $u_\infty \in \partial\left(\mathbb{H} \times \mathbb{R}^{1,3}\right)$,
- the tail σ-algebra of $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s \geqslant 0}$ and $\sigma(u_\infty)$ coincide up to $\mathbb{P}_{\dot{\xi},\xi}$-null sets,

whatever $(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}$?

One sees such a situation for instance in the study of Brownian motion on a Cartan-Hadamard manifold, with a bounded sectional curvature less than or equal to some constant $-\varepsilon < 0$. Such an n-dimensional manifold $\mathbb{V}$ has a natural compactification.

Given a point basis, polar coordinates are well defined on $\mathbb{V}$. One identifies two paths $\{\gamma_t\}_{t \geqslant 0}$ and $\{\gamma'_t\}_{t \geqslant 0}$, leaving every compact, and viewed in polar coordinates $\left\{(\rho_t, \sigma_t)\right\}_{t \geqslant 0}$, $\left\{(\rho'_t, \sigma'_t)\right\}_{t \geqslant 0}$, if the directions σ_t and σ'_t of γ_t and γ'_t converge to the same point $\sigma_\infty \in \mathbb{S}^{n-1}$. This equivalence relation enables one to add a boundary $\partial\mathbb{V}$ to $\mathbb{V}$, homeomorphic to $\mathbb{S}^{n-1}$. A point of $\partial\mathbb{V}$ corresponds to $\sigma \in \mathbb{S}^{n-1}$ if it is in the same class as the geodesic $\{(t, \sigma)\}_{t \geqslant 0}$.

Every geodesic happens to converge towards some point of the boundary; so the added sphere $\mathbb{S}^{n-1}$ really represents all the possible behaviours of a geodesic at the infinity. In this context, Brownian motion $\{w_s\}_{s \geqslant 0}$ on $\mathbb{V}$ also almost surely converges towards some point $\sigma_\infty \in \mathbb{S}^{n-1}$.[25] Anderson proved in [2] that $\sigma(\sigma_\infty)$ and the invariant σ-algebra of w coincide up to $\mathbb{P}_{w_0}$-null sets, whatever the initial condition w_0 is (actually, he proves a much stronger result). In that sense, Brownian motion eventually behaves as a geodesic.

The situation is in some sense analogous in our Lorentzian framework, but the structure of the boundary is linked to the causal structure of $\mathbb{R}^{1,3}$.

Definition/Proposition • *A* $\mathcal{C}^1$ *path* $\gamma : [0, +\infty[\to \mathbb{R}^{1,3}$ *is said to be* causal *if* $q(\dot{\gamma}_s) \geqslant 0$, *for any* $s \geqslant 0$.

- *The* causal past of a point $\xi \in \mathbb{R}^{1,3}$ *is the set* $J^-(\xi) = \left\{\zeta \in \mathbb{R}^{1,3}\,;\, q(\xi - \zeta) \geqslant 0\right\}$ *of points* $\zeta \in \mathbb{R}^{1,3}$ *such that there exists a causal* $(\mathcal{C}^1)$ *path* γ *from* ζ *to* ξ.

Causal paths represent the motion in spacetime of objects moving at a speed less than or equal to the speed of light. Remarking that

- two points ξ and ξ' of $\mathbb{R}^{1,3}$ are equal if, and only if, they have the same causal past,
- for a causal path $\{\gamma_t\}_{t \geqslant 0}$, the sets $J^-(\gamma_t)$ increase:

$$\text{if } s \leqslant t, \quad J^-(\gamma_s) \subset J^-(\gamma_t),$$

it seems reasonable to define the following equivalence relation.

Definition 49 Two $(\mathcal{C}^1)$ causal paths $\{\gamma_t\}_{t \geqslant 0}$ and $\{\gamma'_t\}_{t \geqslant 0}$, leaving every compact, are said to be equivalent if

$$\bigcup_{t>0} J^-(\gamma_t) = \bigcup_{t>0} J^-(\gamma'_t).$$

[25] See [23], for instance.

One calls *causal boundary* the space of equivalent classes of causal paths leaving every compact. So, one identifies two points at the infinity if they have the same past. Such a construction in a general Lorentzian setting goes back to Penrose [17]. One can show that the causal boundary of $\mathbb{R}^{1,3}$ can be identified with a cylinder $\mathbb{S}^2 \times [-\infty, +\infty]$, where one identifies $\mathbb{S}^2 \times \{-\infty\}$ and $\mathbb{S}^2 \times \{+\infty\}$ to a single point. Denote by $\mathbf{C}$ this boundary and by $\mathbf{p}$ the point $\mathbb{S}^2 \times \{+\infty\}$. The set $\mathbf{C}\backslash\{\mathbf{p}\}$ can be naturally identified with the equivalent class of lightlike geodesics under the preceding equivalence relation **49**.[26]

The following theorem gives a geometrical, intrinsic version of Theorem **1**, analogue to what happens in a Riemannian framework, and shows that the relevant geometrical structure is the causal structure of spacetime.

Theorem 50 [3, Sect. 3.1] *Let* $(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,3}$.

1. *The process* $\{\xi_s\}_{s\geqslant 0}$ *converge* $\mathbb{P}_{\dot{\xi},\xi}$*-almost surely towards some random point* $\xi_\infty \in \mathbf{C}\backslash\{\mathbf{p}\}$. *In that sense,* $\{\xi_s\}_{s\geqslant 0}$ *eventually behaves as a lightlike geodesic.*
2. *The invariant* σ*-algebra of* $\left\{(\dot{\xi}_s, \xi_s)\right\}_{s\geqslant 0}$ *and* $\sigma(\xi_\infty)$ *coincide up to* $\mathbb{P}_{\dot{\xi},\xi}$*-null sets.*

Conclusion

If the description of the Poisson boundary of L given in Theorem **18** is new, the result in itself is not new.

Recall $\widetilde{L}$ is the infinitesimal generator of the diffusion $\left\{(g_s, \xi_s)\right\}_{s\geqslant 0}$ on $\mathcal{G}$, constructed in Sect. **3.2**. One deduces from the left invariance of the vector fields generating this diffusion that the sequence $\left\{(g_n, \xi_n)\right\}_{n\geqslant 0}$ is a right random walk on $\mathcal{G}$. One can show that the Poisson boundary of $\widetilde{L}$ is equal to the Poisson boundary of the random walk.

The problem of the determination of the Poisson boundary of some random walk on some locally compact group, with countable basis, was completely solved by Raugi, in [24], under some moment condition on the jump law of the random walk.[27] The study of $\left\{(g_n, \xi_n)\right\}_{n\geqslant 0}$ falls within this framework. Raugi's description of its Poisson boundary involves algebraic decompositions of the group $\mathcal{G}$.

One can check that his description coincides with that of L-harmonic functions, made in Theorem **18**. As a by side, one obtains that each bounded $\widetilde{L}$-harmonic function is an L-harmonic function.[28]

The relativistic diffusion studied in this article admits a natural extension on Lorentzian manifolds, introduced by Franchi and Le Jan in their article [16]. The article contains a deep study of the asymptotic behaviour of the relativistic diffusion in the general Schwarzschild spacetime. Though its Poisson boundary was not determined, its seems possible that it could be described as the causal boundary of Schwarzschild's space. As no algebraic calculus can be done in non-homogeneous

[26] More information on the subject in [17]. This simple case is treated in [3], Sect. 3.1.

[27] These are difficult results.

[28] Details on this paragraph can be found in [4].

spaces, we hope that the coupling scheme used in this article may help to attack the problem in the general framework of Lorentzian manifolds.

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