Probab. Theory Relat. Fields (2008) 141:247–281
DOI 10.1007/s00440-007-0085-y

Martingale transforms and L^p-norm estimates of Riesz transforms on complete Riemannian manifolds

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Received: 21 November 2006 / Revised: 4 June 2007 / Published online: 17 July 2007

Abstract Under the condition that the Bakry–Emery Ricci curvature is bounded from below, we prove a probabilistic representation formula of the Riesz transforms associated with a symmetric diffusion operator on a complete Riemannian manifold. Using the Burkholder sharp L^p-inequality for martingale transforms, we obtain an explicit and dimension-free upper bound of the L^p-norm of the Riesz transforms on such complete Riemannian manifolds for all $1 < p < \infty$. In the Euclidean and the Gaussian cases, our upper bound is asymptotically sharp when $p \to 1$ and when $p \to \infty$.

Mathematics Subject Classification (2000) Primary: 53C21 · 58J65; Secondary: 58J40 · 60J65

Keywords Riesz transforms · Bakry–Emery Ricci curvature · Martingale transforms · Burkholder sharp L^p-inequality for martingale subordination

1 Introduction

The purpose of this paper is to use martingale transforms to obtain an explicit and dimension-free upper bound for the L^p-norm of the Riesz transforms on complete

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Dedicated to my daughter Yun-Xuan.

Research partially supported by a Delegation in CNRS at the University of Paris-Sud during the 2005–2006 academic year.

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Riemannian manifolds with suitable curvature bound conditions. Before going to describe problems and results on Riemannian manifolds, we would like to review some historical backgrounds and recall some known results on Euclidean spaces.

In 1927, Riesz [50] proved that the Hilbert transform on the real line, defined by the principal value of the singular integral

$$Hf(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy,$$

or formally

$$H = \frac{d}{dx}\left(-\frac{d^2}{dx^2}\right)^{-1/2},$$

is bounded in $L^p(\mathbb{R}, dx)$ for all $p > 1$. This result has been considered as one of the most important discoveries in analysis of the last century. To extend it from $\mathbb{R}$ to $\mathbb{R}^n$, Calderon and Zygmund [15] developed the theory of singular integrals, in which one of the most basic examples is the Riesz transforms on $\mathbb{R}^n$, defined by the principal value of the singular integrals

$$R_j f(x) = \frac{\Gamma((n+1)/2)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}} dy, \quad j = 1, \dots, n,$$

or formally

$$R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}, \quad j = 1, \dots, n,$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator on $\mathbb{R}^n$. The vector Riesz transform on $\mathbb{R}^n$ is defined by

$$R_0(\Delta) := (R_1, \dots, R_n) = \nabla(-\Delta)^{-1/2},$$

where ∇ is the gradient operator on $\mathbb{R}^n$. It is well known (see Stein [55]) that the Riesz transforms R_j are weak (1, 1) and are bounded in $L^p(\mathbb{R}^n, dx)$ for all $p > 1$. To see the significant applications of the Riesz transforms in elliptic and parabolic PDEs, SDEs and in the study of Navier–Stokes equations, we refer the reader to [24,36,37,56].

In recent years, there has been considerable interest in finding the exact value or obtaining a good estimate of the L^p-norm of the Riesz transforms. One of the motivations for such study can be seen in Donaldson and Sullivan [22] and in Iwaniec and Martin [29,30], where it has been pointed out that the knowledge of the exact value or a good estimate of the L^p-norm of the Riesz transforms on $\mathbb{R}^n$ will lead important applications in the study of quasi-conformal mappings and related nonlinear geometric PDEs as well as in the L^p-Hodge decomposition theory.

In 1972, Pichorides [46] proved that the L^p-norm of the Hilbert transform is given by

$$\|H\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad \forall\, p > 1.$$

Here and throughout of this paper, we denote

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

In [30], Iwaniec and Martin proved that the L^p-norm of the Riesz transforms R_j is given by

$$\|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad j = 1, \ldots, n. \tag{1}$$

In [14], Bañuelos and Wang gave an alternative proof of (1) by using the Gundy–Varopoulos probabilistic representation formula of the Riesz transforms and the Burkholder sharp L^p-inequality for martingale transforms. Moreover, Bañuelos and Wang [14] proved that for all $p > 1$ the L^p-norm of the vector Riesz transform

$$R_0(\Delta) = (R_1, \ldots, R_n) = \nabla(-\Delta)^{-1/2}$$

has an explicit and dimension-free upper bound

$$\|R_0(\Delta)\|_{p,p} \le 2(p^* - 1). \tag{2}$$

Similar problems have been studied for the Riesz transforms associated with the Ornstein–Uhlenbeck operators which plays a fundamental role in the Malliavin calculus on the Wiener space (see [41]). Let L be the Ornstein–Uhlenbeck operator on the n-dimensional Gaussian space $(\mathbb{R}^n, \gamma_n)$ or on the infinite dimensional Wiener space $(W(\mathbb{R}^n), \mu)$, where γ_n is the standard Gaussian measure on $\mathbb{R}^n$, i.e., $d\gamma_n(x) = (2\pi)^{-n/2} e^{-\frac{\|x\|^2}{2}} dx$, and μ is the standard Wiener measure (i.e., the law of standard Brownian motion on $\mathbb{R}^n$) on the Wiener space $W(\mathbb{R}^n) = C([0, 1], \mathbb{R}^n)$. More precisely, the Orsntein–Uhlenbeck operator on $(\mathbb{R}^n, \gamma_n)$ can be given by

$$L = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j}, \tag{3}$$

and the Ornstein–Uhlenbeck operator on the Wiener space $(W(\mathbb{R}^n), \mu)$ can be formally given by the similar formula by taking $n = \infty$ in (3). In [44], Meyer introduced a family of Riesz transforms associated with the Ornstein–Uhlenbeck operator L by

$$R_a(L) = \nabla(a - L)^{-1/2},$$

where a is a non-negative constant, ∇ is the gradient operator on $\mathbb{R}^n$ or the Malliavin gradient operator on the Wiener space $W(\mathbb{R}^n)$, and proved that for all $p > 1$ and all $a \geq 0$ the Riesz transforms $R_a(L)$ are bounded in L^p with respect to the Gaussian measure γ_n or the Wiener measure μ. In [47], Pisier gave an alternative proof of Meyer's result by using the L^p-boundedness of the Hilbert transform and proved that for all $p > 1$, the L^p-norm of the Riesz transform $R_0(L) := \nabla(-L)^{-1/2}$ associated with the Ornstein–Uhlenbeck operator L satisfies

$$\|R_0(L)^{-1/2}\|_{p,p} \leq K_p, \tag{4}$$

where

$$K_p = O(p) \text{ as } p \to \infty, \quad \text{and} \quad K_p = O((p-1)^{-3/2}) \text{ as } p \to 1.$$

By extending the Gundy–Varopoulos formula of the Riesz transforms to S^n and using the Burkholder sharp L^p-inequality for martingale transforms, Arcozzi [2] proved that the Riesz transform $R_0(\Delta_{S^n}) := \nabla(-\Delta_{S^n})^{-1/2}$ on the n-dimensional unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ satisfies

$$\|R_0(\Delta_{S^n})\|_{p,p} \leq 2(p^* - 1), \quad \forall\, p > 1, \tag{5}$$

where Δ_{S^n} denotes the Laplace–Beltrami operator on S^n. Taking the Poincaré limit from $S^n(\sqrt{n})$ to the infinite dimensional Wiener space $W(\mathbb{R}^n)$ and using (5), Arcozzi [2] obtained that

$$\|R_0(L)\|_{p,p} \leq 2(p^* - 1), \quad \forall\, p > 1. \tag{6}$$

We now turn to the study of the Riesz transforms on complete Riemannian manifolds. Since Stein [54] introduced in 1970 the Riesz transforms on compact Lie groups and initiated the approach of using the Littlewood–Paley inequalities to prove the L^p-boundedness of the Riesz transforms, many people have tried to establish the L^p-boundedness of the Riesz transforms on various geometric objects. In 1983, Strichartz [53] introduced the notion of the Riesz transforms on complete non-compact Riemannian manifolds and raised the problem whether one can establish the L^p-boundedness of the Riesz transforms on a class of complete non-compact Riemannian manifolds. Since then, some geometric and analytic conditions on complete non-compact Riemannian manifolds have been found out for an affirmative answer to Strichartz's problem. For these, we refer the reader to Strichartz [53] for non-compact symmetric Riemannian manifolds of rank one, to Lohoué [38] for Cartan–Hadamard manifolds on which the Riemannian curvature and its first and second derivatives are bounded, to Bakry [4–7] for complete Riemannian manifolds on which the Ricci curvature or the so-called Bakry–Emery Ricci curvature is bounded from below, to Chen [16] and Li [32] for the weak (1, 1)-property of the Riesz transform on complete Riemannian manifolds with non-negative Ricci curvature, to Coulhon and Duong [18,19] and Auscher et al. [1] for complete Riemannian manifolds satisfying the doubling volume property, the Faber–Krahn inequalities and some additional heat kernel regularity conditions, and to the author [34,33] for complete Riemannian manifolds on which the negative part of Ricci curvature satisfies some gaugeability conditions.

Let (M, g) be a complete Riemannian manifold, $n = \dim M$, Δ the non-positive Laplace–Beltrami operator, ∇ the gradient operator, and ν the Riemannian volume measure on (M, g), i.e., $d\nu(x) = \sqrt{\det g(x)}dx$. Let μ be a weighted volume measure on M defined by $d\mu(x) = e^{-\phi(x)}d\nu(x)$, where $\phi \in C^2(M)$. Then for all $f, g \in C_0^\infty(M)$, we have

$$\int_M (\nabla f, \nabla g)d\mu = -\int_M fLgd\mu = -\int_M gLfd\mu,$$

where L is the weighted Laplacian with respect to μ on M and can be given by

$$L = \Delta - \nabla\phi \cdot \nabla.$$

Following Bakry and Emery [8], see also Bakry [5,7], the Ricci curvature associated with L is defined by

$$Ric(L) := Ric + \nabla^2\phi.$$

Here $\nabla^2\phi$ denotes the Hessian of ϕ with respect to the Levi–Civita connection on (M, g). According to [39,33], we call $Ric(L)$ the Bakry–Emery Ricci curvature of L (or μ) on (M, g).

Suppose that there exists a non-negative constant a such that

$$Ric(L) \geq -a.$$

Using a martingale approach to the Littlewood–Paley inequalities, Bakry [5] proved that for any $p > 1$, there exists a universal constant C_p which is independent of $n = \dim M$ and a, such that for all $f \in C_0^\infty(M)$,

$$\|R_a(L)f\|_{L^p(\mu)} \leq C_p\|f\|_{L^p(\mu)}, \tag{7}$$

where

$$R_a(L) := \nabla(a - L)^{-1/2} \tag{8}$$

is the Riesz transform associated with the symmetric diffusion operator L and the constant a. Equivalently, for all $p > 1$, the L^p-norm of the Riesz transform $R_a(L)$ with respect to the measure μ, i.e.,

$$\|R_a(L)\|_{p,p} := \sup_{f \neq 0} \frac{\|R_a(L)f\|_{L^p(\mu)}}{\|f\|_{L^p(\mu)}},$$

has a dimension-free upper bound which depends only on p

$$\|R_a(L)\|_{p,p} \leq C_p.$$

The dimension-free phenomenon for the upper bound of the L^p-norm of the Riesz transforms is a very interesting property. Taking $M = \mathbb{R}^n$ and $\phi(x) = \frac{\|x\|^2}{2} + \frac{n}{2}\log 2\pi$, we see that $\mu = \gamma_n$, and $L = \Delta - x \cdot \nabla$ is the Ornstein–Uhlenbeck operator on the Gaussian space $(\mathbb{R}^n, \gamma_n)$. In this case, it is well known that $Ric(L) = Id$. By Bakry [4,5], the L^p-norm of the Riesz transform $R_0(L)$ associated with the Ornstein–Uhlenbeck operator L on the Gaussian space $(\mathbb{R}^n, \gamma_n)$ is bounded from above by a universal constant which is dimension-free. This leads one to recapture Meyer's result on the L^p-boundedness of the Riesz transform associated with the Ornstein–Uhlenbeck operator on the infinite dimensional Wiener space.

It is natural to ask the problem what is the explicit p-dependence of the L^p-norm of the Riesz transforms on complete Riemannian manifolds with suitable curvature bound conditions. In particular, it is very interesting to study the following problem suggested by Le Jan in a private discussion in July 2001.

Problem 1.1 *What is the asymptotic behavior* (*when* $p \to 1$ *and when* $p \to \infty$, *respectively*) *of the* L^p*-norm of the Riesz transform* $\|R_0(L)\|_{p,p}$ (*respectively,* $\|R_a(L)\|_{p,p}$) *under the curvature condition* $Ric(L) \geq 0$ (*respectively,* $Ric(L) \geq -a$ *for a constant* $a > 0$)*?*

To help the reader to see the importance of the sharp estimate of the L^p-norm of the Riesz transforms, let us mention three applications.

(1) For symmetric diffusion operators on complete Riemannian manifolds with a positive lower bound of the Bakry–Emery Ricci curvature, we can prove the following L^p-Poincaré inequality (see Sect. 5):

Theorem 1.2 *Let* (M, g) *be a complete Riemannian manifold,* $\phi \in C^2(M)$. *Suppose that there exists a constant* $\rho > 0$ *such that*

$$Ric(L) = Ric + \nabla^2\phi \geq \rho.$$

Then for all $p > 1$ *and for all* $f \in W^{1,p}(M, \mu) := \{f \in L^p(\mu) : |\nabla f| \in L^p(\mu)\}$, *we have*

$$\|f - \mu(f)\|_p \leq \frac{\|R_0(L)\|_{q,q}}{\sqrt{\rho}}\|\nabla f\|_p, \tag{9}$$

where $q = \frac{p}{p-1}$, $\mu(f) = \int_M f d\mu$, *and* $L = \Delta - \nabla\phi \cdot \nabla$.

Note that, on any complete Riemannian manifold, integration by parts yields $\|R_0(L)\|_{2,2} = 1$. For symmetric diffusion operators with a positive lower bound of the Bakry–Emery Ricci curvature, Theorem 1.2 is a natural extension of the well known Bakry–Emery criterion (see [8]) for the L^2-Poincaré inequality to the L^p-Poincaré inequality for all $p > 1$.

(2) The sharp estimate of the L^p-norm of the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ will lead the following inequalities with optimal constants (see Sect. 5):

Theorem 1.3 *Let $p > 1$. Suppose that the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(M, \mu)$. Then for all $f \in W^{1,p}(M, \mu)$ we have*

$$\|R_a(L)\|_{p,p}^{-1}\|\nabla f\|_p \leq \|\sqrt{a-L}f\|_p \leq \sqrt{a}\|f\|_p + \|R_a(L)\|_{q,q}\|\nabla f\|_p, \quad (10)$$

where $q = \frac{p}{p-1}$, and $W^{1,p}(M, \mu) := \{f \in L^p(\mu) : |\nabla f| \in L^p(\mu)\}$. In the case where $a = 0$, we require that $f \in (\mathrm{Ker}L)^{\perp}$.

(3) Similarly to the case where $M = \mathbb{R}^n$, see [22,29,30,36,37], we believe that an explicit and good estimate of the L^p-norm of the Riesz transforms on Riemannian manifolds will lead some important applications in the study of quasi-conformal mappings and related geometric nonlinear PDEs as well as in the study of Navier–Stokes equations on Riemannian manifolds with suitable curvature bound conditions.

We are now in a position to state the main result of this paper.

Theorem 1.4 *Let (M, g) be a complete Riemannian manifold with a Riemannian metric g. Let $\phi \in C^2(M)$, $L = \Delta - \nabla\phi \cdot \nabla$, and $\mu(dx) = e^{-\phi(x)}\sqrt{\mathrm{det} g(x)}dx$.*

(1) *Suppose that*

$$Ric(L) = Ric + \nabla^2\phi \geq 0.$$

Then for any $p > 1$, we have

$$\|R_0(L)\|_{p,p} \leq 2(p^* - 1). \quad (11)$$

(2) *Suppose that there exists a constant $a > 0$ such that*

$$Ric(L) = Ric + \nabla^2\phi \geq -a.$$

Then for any $p > 1$, we have

$$\|R_a(L)\|_{p,p} \leq 2(p^* - 1)(1 + 4\|T_1\|_p), \quad (12)$$

where

$$T_1 := \inf\{t > 0 : \|B_t\| = 1\}$$

is the first hitting time of the standard 3-dimensional Brownian motion B_t to the unit sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$.

Theorem 1.4 provides us with an explicit and dimension-free upper bound of the L^p-norm for the Riesz transforms associated with symmetric diffusion operators on complete Riemannian manifolds whose Bakry–Emery Ricci curvature is bounded from below. In particular, the following corollary extends Bañuelos and Wang's upper bound (2) to the Riesz transform $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ associated with the Laplace–Beltrami operator on any complete Riemannian manifold with non-negative Ricci curvature.

Corollary 1.5 *Let M be a complete Riemannian manifold with non-negative Ricci curvature. Then for any $p > 1$, the L^p-norm of the Riesz transform $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ with respect to the volume measure ν satisfies*

$$\|R_0(\Delta)\|_{p,p} \leq 2(p^* - 1).$$

Due to the dimension-free phenomenon of the L^p-norm upper bound estimate of the Riesz transforms, Theorem 1.4 allows us to extend Arcozzi's estimate (6), which holds for Meyer's Riesz transforms associated with the Ornstein–Uhlenbeck operator on the classical Wiener space, to the Riesz transforms associated with a generalized Ornstein–Uhlenbeck operator on an infinite dimensional abstract Wiener space. More precisely, we have the following result.

Corollary 1.6 *Let H be a real separable Hilbert space which is densely embedded in a real separable Banach space W, $A : H \to H$ be a self-adjoint positive linear operator, μ be the Gaussian measure on W with the covariance A. Let $L = \Delta - Ax \cdot \nabla$ be the generalized Ornstein–Uhlenbeck operator on the abstract Wiener space (W, H, μ) in the sense of Gross. Then $Ric(L) = A$, and for any $p > 1$, the L^p-norm of the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ satisfies*

$$\|R_0(L)\|_{p,p} \leq 2(p^* - 1).$$

Theorem 1.4 together with Corollaries 1.5 and 1.6 give us some partial answers to Le Jan's problem concerning the asymptotically sharp upper bound of the Riesz transforms associated with symmetric diffusion operators on complete Riemannian manifolds with suitable curvature bound conditions. Using the known results in [30,14,2] and in Larsson-Cohn [31], we will see in Sect. 6 that at least in the Euclidean case and in the Gaussian case, the upper bound of the form $O(p^* - 1)$ for the L^p-norm of the Riesz transforms $\nabla(-\Delta)^{-1/2}$ and $\nabla(-L)^{-1/2}$ (where L is the Ornstein–Uhlenbeck operator) is asymptotically sharp when $p \to 1$ and when $p \to \infty$. In general, we would like to pose the following conjecture.

Conjecture 1.7 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that $Ric(L) = Ric + \nabla^2\phi \geq 0$. Then there exists a constant $c > 0$ such that for all $p > 1$, we have*

$$c(p^* - 1)(1 + o(1)) \leq \|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1). \tag{13}$$

In particular, on any complete Riemannian manifold M with non-negative Ricci curvature, for all $p > 1$, we have

$$c(p^* - 1)(1 + o(1)) \leq \|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1). \tag{14}$$

To prove Theorem 5.2, we develop a probabilistic approach using martingale transforms in the study of the Riesz transforms and related problems on complete Riemannian manifolds. This probabilistic approach is quite different from the well known martingale approach to the Littlewood–Paley inequalities initiated by Meyer [42–44] and developed by Bakry [4–7] among others (cf. [57,58,51,34]). We will first prove a probabilistic representation formula for the Riesz transforms on complete Riemannian manifolds (see Theorem 3.2 below) and then use the Burkholder sharp L^p-inequality for martingale subordination. We have been inspired by the earlier work due to Gundy and Varopoulos [27], Gundy and Silverstein [28], Gundy [25,26], Song [52], Bañuelos and Wang [14], and Arcozzi [2] for the probabilistic representation formulas and the L^p-norm estimates of the Riesz transforms on $\mathbb{R}^n$, S^n and on Wiener spaces. To save the length of this section which has already been very long, we will review the work of [27,28,25,26,52,14,2] in Sect. 3. We would like to point out that, when we deal with the Riesz transforms on complete Riemannian manifolds, we do need some new ideas and some new arguments. In [35], we will extend this approach to obtain an explicit and dimension-free upper bound for the Riesz transforms associated with the Hodge Laplacian and the Witten Laplacian on k-forms on a complete Riemannian manifold with suitable curvature conditions.

This paper is organized as follows. In Sect. 2 we recall the probabilistic representation formulas of the heat semigroups and the Poisson semigroups generated by the Witten Laplacian on one-forms. In Sect. 3 we prove the probabilistic representation formula for the Riesz transforms associated with symmetric diffusion operators on complete Riemannian manifolds with suitable curvature bound conditions. In Sect. 4 we prove Theorem 1.4. In Sect. 5, we prove Theorems 1.2 and 1.3. In Sect. 6, we give some remarks and use the results in [30,14,2,31] to give some new examples which support Conjecture 1.7.

2 Heat semigroups and Poisson semigroups on one-forms

In this section, we recall the probabilistic representation formulas for the heat semigroup and the Poisson semigroup generated by the Witten Laplacian on one-forms. The results in this section have been well known in the literature.

2.1 Probabilistic representation of heat semigroup on one-forms

Let $d_k : \Lambda^k(T^*M) \to \Lambda^{k+1}(T^*M)$ be the exterior differential operator acting on differential k-forms on M. Standard argument using integration by parts formula shows that the $L^2(\mu)$-adjoint of d_k, denoted by $d^*_{k,\phi} : \Lambda^{k+1}(T^*M) \to \Lambda^k(T^*M)$, is given by

$$d^*_{k,\phi} = d^* + i_{\nabla\phi},$$

where $i_{\nabla\phi}$ denotes the inner multiplication by $\nabla\phi$ on Λ^{k+1}.

We now define the Witten Laplacian on k-forms as follows:

$$\Box_{k,\phi} := d_{k-1}d^*_{k-1,\phi} + d^*_{k,\phi}d_k.$$

Denote $\Box_\phi := \Box_{1,\phi}$. Then the following commutation formula holds

$$d(-L)f = \Box_\phi df, \ \ \forall\, f \in C^\infty(M).$$

Note that $L = \Delta - \nabla\phi \cdot \nabla$ is a non-positive operator in $L^2(\mu)$, while $\Box_\phi$ is non-negative.

Let ∇ be the Levi–Civita connection on M. Let

$$\Delta_\phi = \mathrm{Tr}\nabla^2 - \nabla_{\nabla\phi}$$

be the weighted Laplace–Beltrami operator acting on one-forms. Then we have the generalized Bochner–Weitzenböck formula

$$\Box_\phi = -\Delta_\phi + Ric(L), \tag{15}$$

Let X_t be the L-diffusion process starting at $X_0 = x$ on M. Let $M_t \in \mathrm{End}(T_xM, T_{X_t}M)$ be the solution of the following covariant ordinary differential equation

$$\frac{\nabla}{\partial t}(M_t v) = -Ric(L)(X_t)(M_t v), \quad \forall\, v \in T_xM, \tag{16}$$

with the initial condition $M_0 = Id_{T_xM}$. Here $\frac{\nabla}{\partial t}$ denotes the Levi–Civita covariant derivative operator along $\{X_t, t \geq 0\}$. Then, using the generalized Bochner–Weitzenböck formula (15) and the Feynman–Kac formula, we have the following probabilistic representation formula for the heat semigroup $e^{-t\Box_\phi}$ on one-forms: for all $\omega \in C_0^\infty(M, \Lambda^1(T^*M))$,

$$e^{-t\Box_\phi}\omega(x) = E_x\left[M_t^*\omega(X_t)\right], \quad \forall\, x \in M, t \geq 0. \tag{17}$$

This kind of probabilistic representation formulas for the heat semigroup on one-forms goes back to Malliavin [40] (in the case where $\phi \equiv 0$) and has been systematically developed by Elworthy et al. [23].

2.2 Probabilistic representation of Poisson semigroups on one-forms

Using the Bochner subordination formula, the Poisson semigroup $e^{-t\sqrt{a-L}}$ on functions and the Poisson semigroup $e^{-t\sqrt{a+\Box_\phi}}$ on one-forms can be defined as follows:

for all $f \in C_0^\infty(M)$, $\omega \in C_0^\infty(M, \Lambda^1(T^*M))$, and for all $(x, t) \in M \times \mathbb{R}^+$,

$$e^{-t\sqrt{a-L}} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}(a-L)} f(x) e^{-u} u^{-1/2} du,$$

$$e^{-t\sqrt{a+\Box_\phi}} \omega(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}(a+\Box_\phi)} \omega(x) e^{-u} u^{-1/2} du.$$

We can also give a probabilistic representation formula for the Poisson semigroup $e^{-t\sqrt{a-L}}$ and the Poisson semigroup $e^{-t\sqrt{a+\Box_\phi}}$. To this end, let B_t be the standard Brownian motion on $\mathbb{R}$ starting from $B_0 = y > 0$. Define

$$\tau_y = \inf\{t > 0 : B_t = 0\}.$$

Then it is well known that for all $\lambda > 0$, we have

$$E_y \left[e^{-\lambda \tau_y} \right] = e^{-y\sqrt{\lambda}}.$$

By this and the spectral decomposition, we have the following probabilistic representation formula for the Poisson semigroup on functions

$$e^{-y\sqrt{a-L}} f(x) = E_y \left[e^{-(a-L)\tau_y} f(x) \right] = E_y \left[e^{-a\tau_y} E_x [f(X_{\tau_y})] \right].$$

That is

$$e^{-y\sqrt{a-L}} f(x) = E_{(x,y)} \left[e^{-a\tau_y} f(X_{\tau_y}) \right].$$

Similarly, we have the following probabilistic representation formula for the Poisson semigroup $e^{-y\sqrt{a+\Box_\phi}}$ on one-forms: for all $\omega \in C_0^\infty(M, \Lambda^1(T^*M))$,

$$e^{-y\sqrt{a+\Box_\phi}} \omega(x) = E_y \left[e^{-(a+\Box_\phi)\tau_y} \omega(x) \right] = E_y \left[e^{-a\tau_y} E_x [M^*_{\tau_y} \omega(X_{\tau_y})] \right].$$

That is

$$e^{-y\sqrt{a+\Box_\phi}} \omega(x) = E_{(x,y)} \left[e^{-a\tau_y} M^*_{\tau_y} \omega(X_{\tau_y}) \right].$$

3 Probabilistic representation of Riesz transforms

In the literature, the first probabilistic representation formula of the Riesz transform is due to Gundy and Varopoulos [27]. In [28], Gundy and Silverstein gave an alternative proof of the Gundy–Varopoulos representation formula using time reversal argument. See also Gundy [26], Bass [9] and Dellacherie et al. [20]. We now follow Gundy [26] to explain the Gundy-Varopoulos representation formula for the Riesz transforms on $\mathbb{R}^n$. Let X_t be the Brownian motion on $\mathbb{R}^n$ starting from the Lebesgue measure dx. Let B_t be a 1D Brownian motion starting from $y > 0$ with $E[B_t^2] = 2t$. That is, the generator of B_t is $\frac{d^2}{dy^2}$ instead of $\frac{1}{2}\frac{d^2}{dy^2}$. Set

$$\tau = \inf\{t > 0 : B_t = 0\}.$$

Let f be a suitable nice function defined on $\mathbb{R}^n$, $Q(f)(x, y) = e^{-y\sqrt{-\Delta}} f(x)$ be the Poisson integral of f, i.e., the harmonic extension of f on $\mathbb{R}^n \times \mathbb{R}^+$. In [26], Gundy proved that

$$\frac{1}{2} f(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau \frac{\partial}{\partial y} Q(f)(X_s, B_s) dB_s \middle| X_\tau = x \right]. \tag{18}$$

Replacing f by $R_j f = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2} f$ and using $\frac{\partial}{\partial y} Q(R_j f) = -\sqrt{-\Delta} Q(R_j f) = -\frac{\partial}{\partial x_j} Q(f)$, the above Gundy formula implies

$$-\frac{1}{2} R_j f(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau \frac{\partial}{\partial x_j} Q(f)(X_s, B_s) dB_s \middle| X_\tau = x \right].$$

Let $A_j = (a_{ik})$ be the $(n + 1) \times (n + 1)$ matrix with $a_{ik} = 0$ unless $i = n + 1$ and $k = j$, and $a_{(n+1)j} = 1$. Then we get the Gundy–Varopoulos formula

$$-\frac{1}{2} R_j f(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau A_j \nabla Q(f)(X_s, B_s)(dX_s, dB_s) \middle| X_\tau = x \right]. \tag{19}$$

In the literature, (X_t, B_t) is called the background radiation on $\mathbb{R}^n \times \mathbb{R}^+$, and an expression like $\int_0^t A\nabla Q(f)(X_s, B_s)d(X_s, dB_s)$ is called the martingale transform of the martingale $\int_0^t \nabla Q(f)(X_s, B_s)(dX_s, dB_s)$ by the matrix A. Thus, the Gundy–Varopoulos formula (19) represents the Riesz transform $R_j f$ as the terminal conditional expectation of a martingale transform with respect to the background radiation on $\mathbb{R}^n \times \mathbb{R}^+$. Based on (19) and the Burkholder sharp L^p-inequality for martingale transforms, Bañuelos and Wang [14] proved that $\|R_0(\Delta)\|_{p,p} \leq 2(p^* - 1)$ for all $p > 1$.

In [25,26], Gundy obtained a similar probabilistic representation formula for the Riesz transforms associated with the Ornstein–Uhlenbeck operator on finite dimensional Gaussian space and the infinite dimensional Wiener space. Using this representation formula and the Burkholder inequality for stochastic integrals, Gundy [26] gave a new proof of the L^p-boundedness of Meyer's Riesz transforms on the Wiener space.

Using the technique of *grossissement de filtration*, Song [52] extended Gundy's formula (18) to a wide class of real separable Banach spaces E equipped with a so-called C-semigroup. Moreover, Song [52] proved that the gradient of a nice scalar function f on such a Banach space E can be represented as the terminal conditional expectation of a martingale transform of the Poisson integral of f composed with the space-time Brownian motion on $E \times \mathbb{R}$ (See Lemma 5 in [52]). This will certainly imply a probabilistic representation formula for the Riesz transforms associated with the generator of the C-semigroups on Banach spaces. For technical reasons, we will not recall Song's formula in details (for which we need to introduce some new definitions and new notation).

In [2], Arcozzi extended the Gundy–Varopoulos representation formula for the Riesz transforms on $\mathbb{R}^n$ to the Riesz transforms on compact Lie groups and to the Riesz transforms on S^n. Using a similar approach as in [14], it was proved in [2] that $\|R_0(\Delta_{S^n})\|_{p,p} \le 2(p^*-1)$. Taking the Poincaré limit, Arcozzi obtained that the Riesz transform $R_0(L)$ associated with the Ornstein–Uhlenbeck operator on the Wiener space satisfies $\|R_0(L)\|_{p,p} \le 2(p^*-1)$ for all $p > 1$.

Inspired by the earlier work due to Gundy and Varopoulos [27], Gundy and Silverstein [28], Gundy [25,26], Song [52] and Arcozzi [2], and using some new ideas and new arguments in the setting of complete Riemannian manifolds, we will prove a probabilistic representation formula for the Riesz transforms associated with a symmetric diffusion operator on a complete Riemannian manifold. To state our main result of this section (Theorem 3.2), we need some notations and some preliminary results.

3.1 Background radiation processes

Let M be a complete Riemannian manifold, $L = \Delta - \nabla\phi\cdot\nabla$ be the weighted Laplacian with respect to the weighted volume measure $d\mu(x) = e^{-\phi(x)}dv(x)$ on M. Suppose that there exists a non-negative constant a such that

$$Ric(L) = Ric + \nabla^2\phi \ge -a.$$

By [3], see also [33], it is well known that the heat semigroup generated by L is conservative.

Let X_t be the diffusion process on M whose infinitesimal generator is L and whose initial measure is μ. By Itô's theory for diffusion processes on Riemannian manifolds, we have

$$dX_t = U_t dW_t - \nabla\phi(X_t)dt,$$

where W_t is the Brownian motion on $\mathbb{R}^n$, $U_t \in \mathrm{End}(T_xM, T_{X_t}M)$ denotes the stochastic parallel transport along $\{X_s, 0 \le s \le t\}$.

Let B_t be a 1D Brownian motion starting from $y > 0$ with $E[B_t^2] = 2t$. That is, the generator of B_t is $\frac{d^2}{dy^2}$ instead of $\frac{1}{2}\frac{d^2}{dy^2}$. Set

$$\tau = \inf\{t > 0 : B_t = 0\}.$$

Following Meyer [43], Gundy and Varopoulos [27] and Gundy [25,26], we introduce the so-called *background radiation process* $Z_t := (X_t, B_t)$ on $M \times \mathbb{R}^+$. In fact, $\{Z_t, t \in [0, \tau]\}$ is a diffusion process on $M \times \mathbb{R}^+$ whose infinitesimal generator is $L + \frac{d^2}{dy^2}$ and whose initial distribution is $\mu \otimes \delta_y$ supported on the hypersurface $M \times \{y\}$ at time $t = 0$. The process $\{Z_t, t \in [0, \tau]\}$ terminates at time $t = \tau$ upon hitting the boundary $M \times \{0\}$. Let $P_{(x,y)}$ be the probability law of $Z_t = (X_t, B_t)$ starting at $(x, y) \in M \times \mathbb{R}^+$. We define the measures $\{P_y, y > 0\}$ on the path space $C([0, \infty), M \times \mathbb{R})$ as

$$P_y(Z_t \in \mathcal{B}) = \int_{\mathbb{R}^n} P_{(x,y)}(Z_t \in \mathcal{B}) d\mu(x), \tag{20}$$

for all Borel sets $\mathcal{B} \subset M \times \mathbb{R}^+$. Let E_y be the expectation corresponding to P_y.

The following result is essentially due to Meyer [43]. See also Bañuelos and Lindeman [13].

Proposition 3.1 *Suppose that there exists a non-negative constant a such that*

$$Ric(L) = Ric + \nabla^2\phi \geq -a.$$

Then, for all non-negative measurable functions f on M, we have

$$E_y[f(X_\tau)] = \int_M f(x) d\mu(x). \tag{21}$$

Moreover, for all non-negative measurable functions F or for all measurable F such that $F(x, \eta)\eta \in L^1(\lambda(dx) \otimes d\eta)$, we have

$$E_y\left[\int_0^\tau F(Z_t)dt\right] = 2\int_0^\infty \int_M F(x, \eta)(y \wedge \eta) d\mu(x) d\eta. \tag{22}$$

Proof Let $\nu_\tau(dt) = P_{(0,y)}(\tau = t)dt$ be the law of τ given by (23). Let $p_t(x, y)$ be the heat kernel of the heat semigroup generated by L. Using the independence of X_t and B_t, and by the Fubini theorem, we have

$$\begin{aligned} E_y[f(X_\tau)] &= \int_M \int_M \int_0^\infty f(z) p_t(x, z) d\mu(z) d\mu(x) d\nu(t) \\ &= \int_M f(z) \int_0^\infty \int_M p_t(x, z) d\mu(x) d\nu_\tau(t) d\mu(z). \end{aligned}$$

Since the heat semigroup generated by L is conservative, we have $\int_M p_t(x,z)d\mu(x) = 1$ for all $x \in M$ and all $t > 0$. Hence

$$E_y[f(X_\tau)] = \int_M \int_0^\infty f(z)d\nu_\tau(t)d\mu(z) = \int_M f(x)d\mu(x).$$

This proves (21). To prove (22), let us first consider the case $F(x,y) = f(x)g(y)$, where $f \in C_b(M)$ and $g \in C_b(\mathbb{R}^+)$. Since X_t and B_t are independent and τ depends only on B_t,

$$\begin{aligned}
E_{(x,y)}\left[\int_0^\tau F(Z_s)ds\right] &= E_{(x,y)}\left[\int_{\mathbb{R}^+}\int_0^t f(X_s)g(B_s)dsd\nu_\tau(t)\right] \\
&= E_y\left[\int_{\mathbb{R}^+}\int_0^t E_x f(X_s)g(B_s)dsd\nu_\tau(t)\right] \\
&= E_y\left[\int_{\mathbb{R}^+}\int_0^t P_s f(x)g(B_s)dsd\nu_\tau(t)\right].
\end{aligned}$$

Integrating in x with respect to μ on M, and using the Fubini formula, we have

$$\begin{aligned}
E_y\left[\int_0^\tau F(Z_s)ds\right] &= \int_M E_{(x,y)}\left[\int_{\mathbb{R}^+}\int_0^t f(X_s)g(B_s)dsd\nu_\tau(t)\right]d\mu(x) \\
&= \int_M E_y\left[\int_{\mathbb{R}^+}\int_0^t P_s f(x)g(B_s)dsd\nu_\tau(t)\right]d\mu(x) \\
&= E_y\left[\int_{\mathbb{R}^+}\int_0^t \left(\int_M P_s f(x)d\mu(x)\right) g(B_s)dsd\nu_\tau(t)\right].
\end{aligned}$$

As P_s is symmetric with respect to μ, we have

$$\int_M P_s f(x)d\mu(x) = \int_M f(x)d\mu(x).$$

This yields that

$$E_y\left[\int_0^\tau F(Z_s)ds\right]=\int_M f(x)d\mu(x)E_y\left[\int_{\mathbb{R}^+}\int_0^t g(B_s)dsd\nu_\tau(t)\right]$$
$$=\int_M f(x)d\mu(x)E_y\left[\int_0^\tau g(B_s)ds\right].$$

Let $G_{\mathbb{R}^+}(y,\xi)$ be the Green function of $\frac{d^2}{dy^2}$ on $\mathbb{R}^+$. Then

$$E_y\left[\int_0^\tau g(B_s)ds\right]=\int_{\mathbb{R}^+} g(\xi)G_{\mathbb{R}^+}(y,\xi)d\xi.$$

Therefore

$$E_y\left[\int_0^\tau F(Z_s)ds\right]=\int_M f(x)d\mu(x)\int_{\mathbb{R}^+} g(\xi)G_{\mathbb{R}^+}(y,\xi)d\xi.$$

Notice that

$$G_{\mathbb{R}^+}(y,\xi)=2(y\wedge\xi).$$

Hence

$$E_y\left[\int_0^\tau F(Z_s)ds\right]=2\int_0^\infty\int_M f(x)dxg(\xi)(y\wedge\xi)d\xi.$$

This proves that (22) holds for $F(x,y)=f(x)g(y)$. Using the monotone class theorem, one can prove that (22) holds in the general case where F is a bounded measurable function on $M\times\mathbb{R}^+$ or when $F(x,\eta)\eta\in L^1(\lambda(dx)\otimes d\eta)$. □

3.2 Killed Brownian motion on the half line

In p. 185 of [43], Meyer described the duality between the killed Brownian motion on the half line $\overline{\mathbb{R}^+}=[0,\infty]$ and the 3D Bessel processes as follows: *D'une manière intuitive, on peut donc dire que le retourné du processus de Bessel issu de* λ_0 *est le "mouvement brownien venant de l'infini et tué en* 0*"*. Here, $\lambda_0=\lambda(dx)\otimes\delta_0$ denotes the Lebesgue measure on the hyperplane $\mathbb{R}^n\times\{0\}$. That is to say, the time reversal process of Brownian motion on $[0,\infty]$ starting from infinity and killed at 0 is the 3D Bessel process starting from 0. See also Sect. 31 Chap. III (p. 301) in Rogers and Williams [48].

More precisely, let B_t be the 1D Brownian motion starting at $y \in \mathbb{R}^+$, $\tau_y = \inf\{t > 0 : B_t = 0\}$, and set

$$\widetilde{B}_t = B_{\tau_y - t}, \quad t \in [0, \tau_y].$$

Let

$$L_y = \sup\{t > 0 : \widetilde{B}_t = y\}$$

be the last exit time of $\{\widetilde{B}_t\}$ from y. Then, it is well known, see [43,20,48,49], that L_y has the same law as τ_y, i.e.,

$$P\left(L_y \in dz\right) = P\left(\tau_y \in dz\right) = \frac{y}{\sqrt{2\pi}} z^{-3/2} e^{-y^2/2z} dz. \tag{23}$$

Moreover, $\{\widetilde{B}_t, t \in [0, L_y]\}$ is the conditional 3D Bessel process starting at 0 and with terminal value y conditioning at L_y, i.e., $\widetilde{B}_{L_y} = y$. Note that $\lim\limits_{y\to\infty} \tau_y = \lim\limits_{y\to\infty} L_y = \infty$. This leads us to identify $\widetilde{B}_t = B_{\tau_y - t}$ with the standard 3D Bessel process starting from 0 when taking the initial position of Brownian motion at infinity, i.e., when $B_0 = y \to \infty$.

3.3 The representation formula of Riesz transform

We are now in a position to state the main result of this section.

Theorem 3.2 *Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that for some non-negative constant $a \in \mathbb{R}^+$,*

$$Ric(L) := Ric + \nabla^2\phi \geq -a.$$

Then, for any $f \in C_0^\infty(M)$ (when $a = 0$ and $\mu(M) < \infty$, we require that $\int_M f d\mu = 0$), we have

$$-\frac{1}{2} R_a(L) f(x) = \lim_{y\to+\infty} E_y \left[\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} d Q_a(f)(X_s, B_s) dB_s \mid X_\tau = x \right], \tag{24}$$

where $Q_a(f)(x, y) = e^{-y\sqrt{a-L}} f(x)$, X_t denotes the L-diffusion process on M with initial distribution μ, B_t denotes the Brownian motion on $\mathbb{R}$ starting from $y > 0$ with $E[B_t^2] = 2t$, $\tau = \inf\{t > 0 : B_t = 0\}$, $M_t \in \mathrm{End}(T^*_{X_0}M, T^*_{X_t}M)$ is the solution to the following covariant differential equation

$$\frac{\nabla}{\partial t}(M_t v) = -Ric(L)(X_t)(M_t v), \quad \forall\, v \in T^*_{X_0}M, \tag{25}$$

with the initial condition $M_0 = Id_{T_{X_0}M}$, *where* $\frac{\nabla}{\partial t} := U_t \frac{\partial}{\partial t} U_t^{-1}$ *denotes the covariant ordinary derivative with respect to the Levi–Civita connection on* M *along the trajectory of the* L*-diffusion process* X_t, $U_t \in \mathrm{End}(T^*_{X_0}M, T^*_{X_t}M)$ *is the stochastic parallel transport along* $\{X_s, 0 \le s \le t\}$.

We will prove Theorem 3.2 in Subsect. 3.4. Below we give some examples.

Example 3.3 Taking $M = \mathbb{R}^n$ and $\phi = 0$ in Theorem 3.2, we have $M_t = Id$ for all $t \ge 0$. Hence

$$-\frac{1}{2}\nabla(-\Delta)^{-1/2} f(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau \nabla u(X_s, B_s) dB_s \mid X_\tau = x \right]. \tag{26}$$

Since $\nabla(-\Delta)^{-1/2} = (R_1, \ldots, R_n)$, where $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$, formula (26) allows us to recapture the Gundy–Varopoulos probabilistic representation formula (19) for the Riesz transforms $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$.

Example 3.4 Taking $M = \mathbb{R}^n$ and $\phi(x) = \frac{\|x\|^2}{2} + \frac{n}{2}\log(2\pi)$ in Theorem 3.2, we have $\mu = \gamma_n$ and $M_t = e^{-t} Id$ for all $t \ge 0$. This gives us the Gundy probabilistic representation formula for the Riesz transform associated with the Ornstein–Uhlenbeck operator $L = \Delta - x \cdot \nabla$ on $(\mathbb{R}^n, \gamma_n)$ (see Gundy [25,26] and Song [52]):

$$-\frac{1}{2}\nabla(a-L)^{-1/2} f(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{(a+1)(s-\tau)} \nabla e^{-B_s\sqrt{a-L}} f(X_s) dB_s \mid X_\tau = x \right].$$

In particular

$$-\frac{1}{2}\nabla(-L)^{-1/2} f(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{(s-\tau)} \nabla e^{-B_s\sqrt{-L}} f(X_s) dB_s \mid X_\tau = x \right].$$

Note that the dimension $n = dim\mathbb{R}^n$ does not play any role in the above formulas. Hence, these formulas also hold for Meyer's Riesz transforms associated with the Ornstein–Uhlenbeck operator on the infinite dimensional Wiener space.

Example 3.5 Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $k \in \mathbb{R}$ such that

$$Ric(L) = Ric + \nabla^2\phi = -k.$$

Then $M_t = e^{kt} Id$. Hence, for all $a \ge k \vee 0$, we have

$$-\frac{1}{2}\nabla(a-L)^{-1/2} f(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{(a-k)(s-\tau)} \nabla e^{-B_s\sqrt{a-L}} f(X_s) dB_s \mid X_\tau = x \right].$$

Example 3.6 Taking $M = S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ with the standard Riemannian metric, we have $Ric = n - 1$. Hence

$$
\begin{aligned}
&-\frac{1}{2}\nabla(a - \Delta_{S^n})^{-1/2} f(x) \\
&\quad = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{(a+n-1)(s-\tau)} \nabla e^{-B_s\sqrt{a-\Delta_{S^n}}} f(X_s) dB_s \mid X_\tau = x \right].
\end{aligned}
$$

This recaptures the probabilistic representation formula of the Riesz transform on S^n, see Arcozzi [2]. Moreover, taking $M = S^n(\sqrt{n-1}) = \{x \in \mathbb{R}^{n+1} : \|x\| = \sqrt{n-1}\}$ with the standard Riemannian metric, we have $Ric = 1$. Hence

$$
\begin{aligned}
&-\frac{1}{2}\nabla(a - \Delta_{S^n(\sqrt{n-1})})^{-1/2} f(x) \\
&\quad = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{(a+1)(s-\tau)} \nabla e^{-B_s\sqrt{a-\Delta_{S^n(\sqrt{n-1})}}} f(X_s) dB_s \mid X_\tau = x \right].
\end{aligned}
$$

Taking the Poincaré limit, the above formula leads us to recapture Gundy's probabilistic representation formula for Meyer's Riesz transforms associated with the Ornstein–Uhlenbeck operator L on the Wiener space

$$
\begin{aligned}
&-\frac{1}{2}\nabla(a - L)^{-1/2} f(x) \\
&\quad = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{(a+1)(s-\tau)} \nabla e^{-B_s\sqrt{a-L}} f(X_s) dB_s \mid X_\tau = x \right],
\end{aligned}
$$

where X_t denotes the Ornstein–Uhlenbeck process on the Wiener space $W(\mathbb{R}^n) = C([0,1], \mathbb{R}^n)$.

3.4 A probabilistic representation formula on one-forms

In this subsection we prove a probabilistic representation formula on one-forms which will be used in the proof of Theorem 3.2.

Lemma 3.7 *Let $\eta \in C_0^\infty(M, \Lambda^1(T^*M))$, and $\eta_a(x, y) := e^{-y\sqrt{a+\Box_\phi}}\eta(x)$. Then*

$$
\begin{aligned}
\eta(X_\tau) &= e^{a\tau} M_\tau^{*,-1} \eta_a(X_0, B_0) \\
&\quad + \int_0^\tau e^{a(\tau-s)} M_\tau^{*,-1} M_s^* \left(\nabla, \frac{\partial}{\partial y}\right) \eta_a(X_s, B_s)(U_s dW_s, dB_s).
\end{aligned}
$$

Proof By the covariant Itô formula on Riemannian manifolds, cf. [23,45], and using (25), we have

$$\begin{aligned}
&\frac{\nabla}{\partial t}(e^{-at}M_t^*\eta_a(X_t, B_t))\\
&\quad= -e^{-at}M_t^*\,(a + Ric(L)(X_t))\,\eta_a(X_t, B_t)\\
&\qquad+e^{-at}M_t^*\left(\nabla, \frac{\partial}{\partial y}\right)\eta_a(X_t, B_t)(dX_t, dB_t)\\
&\qquad+e^{-at}M_t^*\nabla^2\eta_a(X_t, B_t)\langle dX_t, dX_t\rangle + e^{-at}M_t^*\frac{\partial^2}{\partial y^2}\eta_a(X_t, B_t)dt\\
&\quad= -e^{-at}M_t^*\left(a + Ric(L)(X_t) + \nabla_{\nabla\phi(X_t)}\right)\eta_a(X_t, B_t)dt\\
&\qquad+e^{-at}M_t^*\left(\nabla, \frac{\partial}{\partial y}\right)\eta_a(X_t, B_t)(U_t dW_t, dB_t)\\
&\qquad+e^{-at}M_t^*\left(\Delta + \frac{\partial^2}{\partial y^2}\right)\eta_a(X_t, B_t)dt\\
&\quad= e^{-at}M_t^*\left(\nabla, \frac{\partial}{\partial y}\right)\eta_a(X_t, B_t)(U_t dW_t, dB_t),
\end{aligned}$$

where in the last step we have used the generalized Bochner–Weitzenböck formula

$$\Box_\phi = -\Delta + \nabla_{\nabla_\phi} + Ric(L).$$

and

$$\frac{\partial^2}{\partial y^2}\omega_a(x, y) = (a + \Box_\phi)\omega(x, y).$$

Hence

$$\frac{\nabla}{\partial t}(e^{-at}M_t^*\eta_a(X_t, B_t)) = e^{-at}M_t^*\left(\nabla, \frac{\partial}{\partial y}\right)\eta_a(X_t, Y_t)(U_t dW_t, dY_t).$$

Integrating from $t = 0$ to $t = \tau$, we complete the proof of Lemma 3.7. $\Box$

The following probabilistic representation formula on one-forms is a natural extension of Gundy's formula (18) and Song's formula (Lemma 5 in [52]) mentioned in the beginning of this section.

Theorem 3.8 *Let $\omega \in C_0^\infty(M, \Lambda^1(T^*M))$, and $\omega_a(x, y) := e^{-y\sqrt{a+\Box_\phi}}\omega(x)$. Then*

$$\frac{1}{2}\omega(x) = \lim_{y\to+\infty} E_y\left[\int_0^\tau e^{a(s-\tau)}M_\tau M_s^{-1}\frac{\partial}{\partial y}\omega_a(X_s, B_s)dB_s \mid X_\tau = x\right]. \tag{27}$$

Proof Let $Z_t = (X_t, B_t)$, $\eta \in C_0^\infty(M, \Lambda^1(T^*M))$. By Lemma 3.7, we have

$$\begin{aligned}
&\int_M \left(E_y \left[\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s \mid X_\tau = x \right],\ \eta(x) \right) d\mu(x) \\
&\quad = E_y \left[\left(\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s,\ \eta(X_\tau) \right) \right] \\
&\quad = I_1(y) + I_2(y),
\end{aligned}$$

where

$$\begin{aligned}
I_1(y) &= E_y \left[\left(\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s, e^{a\tau} M_\tau^{*,-1} \eta_a(Z_0) \right) \right], \\
I_2(y) &= E_y \left[\left(\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s, \right.\right. \\
&\qquad \left.\left. \int_0^\tau e^{a(\tau-s)} M_\tau^{*,-1} M_s^* (\nabla, \partial_y) \eta_a(Z_s)(U_s dW_s, dB_s) \right) \right].
\end{aligned}$$

By the martingale property of the Itô stochastic integral $\int_0^t e^{as} M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s$, we have

$$I_1(y) = E_y \left[\left(E \left[\int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s \middle| Z_0 \right], \eta_a(Z_0) \right) \right] = 0.$$

On the other hand, using the Itô L^2-isometry identity, we have

$$\begin{aligned}
I_2(y) &= E_y \left[\left(\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s, \right.\right. \\
&\qquad \left.\left. \int_0^\tau e^{a(\tau-s)} M_\tau^{*,-1} M_s^* (\nabla, \partial_y) \eta_a(Z_s)(U_s dW_s, dB_s) \right) \right] \\
&= E_y \left[\int_0^\tau \left(e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s), e^{a(\tau-s)} M_\tau^{*,-1} M_s^* \frac{\partial}{\partial y} \eta_a(Z_s) \right) ds \right] \\
&= E_y \left[\int_0^\tau \left(\frac{\partial}{\partial y} \omega_a(Z_s),\ \frac{\partial}{\partial y} \eta_a(Z_s) \right) ds \right].
\end{aligned}$$

The Green function of the background radiation process is given by $2(y \wedge z)$. Thus

$$E_y\left[\int_0^\tau \left(\frac{\partial}{\partial y}\omega_a(Z_s),\ \frac{\partial}{\partial y}\eta_a(Z_s)\right) ds\right]$$
$$= 2\int_M \int_0^\infty (y \wedge z)\left(\frac{\partial}{\partial z}\omega_a(x,z), \frac{\partial}{\partial z}\eta_a(x,z)\right) dz d\mu(x).$$

Using the spectral decomposition, we can prove the Littlewood–Paley identity

$$\lim_{y\to\infty}\int_M \int_0^\infty (y \wedge z)\left(\frac{\partial}{\partial z}\omega_a(x,z), \frac{\partial}{\partial z}\eta_a(x,z)\right) dz d\mu(x) = \frac{1}{4}\int_M (\omega(x), \eta(x))d\mu(x).$$

Therefore

$$\langle \omega, \eta\rangle_{L^2(\mu)}$$
$$= 2\lim_{y\to\infty}\int_M \left(E_y\left[\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1}\frac{\partial}{\partial y}\omega_a(Z_s)\cdot dB_s \mid X_\tau = x\right], \eta(x)\right) d\mu(x).$$

The proof of Theorem 3.8 is completed. □

3.5 Proof of Theorem 3.2

We now prove Theorem 3.2. Using the commutation formula $d(-L) = \square_\phi d$, we have

$$\frac{\partial}{\partial y}e^{-y\sqrt{a+\square_\phi}}(d(a-L)^{-1/2}f) = -(a+\square_\phi)^{1/2}e^{-y\sqrt{a+\square_\phi}}(d(a-L)^{-1/2}f)$$
$$= -d(a-L)^{1/2}e^{-t\sqrt{a-L}}(a-L)^{-1/2}f$$
$$= -de^{-t\sqrt{a-L}}f = -dQ_a(f)(\cdot, t).$$

Applying Theorem 3.8 to $\omega = d(a-L)^{-1/2}f$, we obtain

$$-\frac{1}{2}R_a(L)f(x) = \lim_{y\to\infty} E_y\left[\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} dQ_a(f)(X_s, B_s)dB_s \mid X_\tau = x\right].$$

The proof of Theorem 3.2 is completed. □

4 Proof of Theorem 1.4

We are now in a position to give a probabilistic proof of Theorem 1.4. The proof is inspired by the ones of Bañuelos and Wang [14] and Arcozzi [2] for the L^p-norm estimates of the Riesz transforms on $\mathbb{R}^n$ and on S^n. We have also benefited from Gundy [26] and Donati-Martin and Yor [21]. It would be very interesting if one can find an analytic proof of Theorem 1.4.

For all $1 < p < \infty$, the conditional expectation $E[\cdot \mid X_\tau = x]$ is contractive in L^p. Thus

$$
\begin{aligned}
&\|R_a(L)f\|_p^p \\
&= 2^p \int_M \left| E_y \left[\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} dQ_a(f)(X_s, B_s) dB_s \mid X_\tau = x \right] \right|^p d\mu(x) \\
&\le 2^p \int_M E_y \left[\left| \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} dQ_a(f)(X_s, B_s) dB_s \right|^p \mid X_\tau = x \right] d\mu(x) \\
&= 2^p E_y \left[\left| \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} dQ_a(f)(X_s, B_s) dB_s \right|^p \right].
\end{aligned}
$$

Let

$$
I := \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} dQ_a(f)(X_s, B_s) dB_s.
$$

Let $e_1, \ldots, e_n$ be a normal orthonormal basis at $T_{X_s}M$ such that $\nabla_{e_i} e_j(X_s) = 0$ for all $i, j = 1, \ldots, n$. Let $e_1^*, \ldots, e_n^*$ be the dual basis of $e_1, \ldots, e_n$. Then

$$
dQ_a(f)(X_s, B_s) = \sum_{j=1}^n \nabla_{e_j} Q_a(f)(X_s, B_s) e_j^*.
$$

Let $\widetilde{e}_i = U_\tau U_s^{-1} e_i$, and $\widetilde{e}_i^* = U_\tau U_s^{-1} e_i^*$. Then $\widetilde{e}_1, \ldots, \widetilde{e}_n$ is a normal orthonormal basis at $T_{X_\tau}M$, and its dual basis is $\widetilde{e}_1^*, \ldots, \widetilde{e}_n^*$. Let (A_{ij}) be the matrix representation of $A = M_\tau M_s^{-1} \in \mathrm{End}(T_{X_s}M, T_{X_\tau}M)$ in the orthonormal basis $e_1, \ldots, e_n$ and $\widetilde{e}_1, \ldots, \widetilde{e}_n$. We have

$$
\begin{aligned}
I &= \int_0^\tau e^{a(s-\tau)} \sum_{j=1}^n \nabla_{e_j} Q_a(f)(X_s, B_s) M_\tau M_s^{-1} e_j^* dB_s \\
&= \int_0^\tau e^{a(s-\tau)} \sum_{i,j=1}^n \nabla_{e_j} Q_a(f)(X_s, B_s) A_{ij} \widetilde{e}_i^* dB_s.
\end{aligned}
$$

Note that, at the point X_s, we have

$$\sum_{i,j=1}^{n} \nabla_{e_j} Q_a(f)(X_s, B_s) A_{ij} \widetilde{e}_i^* dB_s$$
$$= \sum_{i=1}^{n} \widetilde{e}_i^* \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{in} & 0 \end{pmatrix} \begin{pmatrix} \nabla_{e_1} Q_a(f) \\ \cdots \\ \nabla_{e_j} Q_a(f) \\ \cdots \\ \nabla_{e_n} Q_a(f) \\ \partial_y Q_a(f) \end{pmatrix} \cdot \begin{pmatrix} U_s dW_s^1 \\ \cdots \\ U_s dW_s^j \\ \cdots \\ U_s dW_s^n \\ dB_s \end{pmatrix}$$
$$= \sum_{i=1}^{n} \widetilde{e}_i^* A_i (\nabla, \partial_y) Q_a(f)(X_s, B_s) \cdot (U_s dW_s, dB_s),$$

where A_i denotes the $(n+1) \times (n+1)$-matrix between $\widetilde{e}_i^*$ and $(\nabla Q_a(f), \partial_y Q_a(f))$ in the first equality. Therefore, the Itô stochastic integral I can be rewritten as a martingale transform with respect to the space-time Brownian motion

$$I = \int_0^{\tau} e^{a(s-\tau)} \sum_{i=1}^{n} \widetilde{e}_i^* A_i (\nabla, \partial_y) Q_a(f)(X_s, B_s) \cdot (U_s dW_s, dB_s).$$

By Burkholder's sharp L^p-inequality for martingale transforms [10–12], see also Bañuelos and Wang [14], we have

$$\|R_a(L) f\|_p \leq 2(p^* - 1) \sup_{s \in [0,\tau]} e^{a(s-\tau)} \left\| \sum_{i=1}^{n} \widetilde{e}_i^* A_i \right\|_{sp}$$
$$\times \left\| \int_0^{\tau} (\nabla, \partial_y) Q_a(f)(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p, \tag{28}$$

where $\left\| \sum_{i=1}^n \widetilde{e}_i^* A_i \right\|_{sp}$ denotes the spectral norm of $\sum_{i=1}^n \widetilde{e}_i^* A_i \in \mathrm{End}(T_{X_s} M, T_{X_\tau} M)$, which is defined by

$$\left\| \sum_{i=1}^{n} \widetilde{e}_i^* A_i \right\|_{sp}^2 := \sup_{v \in T_{X_s} M, v \neq 0} \frac{\sum_{i=1}^n \sum_{j=1}^n |A_{ij} v_j|^2}{\sum_{j=1}^n |v_j|^2},$$

where $v = \sum_{j=1}^n v_j e_j$. By the matrix representation formula of $A = M_\tau M_s^{-1}$ in the orthonormal basis $e_1, \ldots, e_n$ and $\widetilde{e}_1, \ldots, \widetilde{e}_n$, we can easily verify that

$$\left\| \sum_{i=1}^{n} \widetilde{e}_i^* A_i \right\|_{sp} = \sup_{v \in T_{X_s} M, \|v\|=1} \|M_\tau M_s^{-1} v\| = \|M_\tau M_s^{-1}\|,$$

where $\|M_\tau M_s^{-1}\|$ is the operator norm of $M_\tau M_s^{-1} \in \mathrm{End}(T_{X_s}M, T_{X_\tau}M)$. Now, by (25), under the condition $Ric(L) = Ric + \mathrm{Hess}\,\phi \geq -a$, for all $v \in T_{X_s}M$, we have

$$\frac{d}{ds}\|M_\tau M_s^{-1}v\|^2 = Ric(L)(M_\tau M_s^{-1}v, M_\tau M_s^{-1}v) \geq -a\|M_\tau M_s^{-1}v\|^2.$$

Moreover, at time $s = \tau$, we have $M_\tau M_\tau^{-1} = Id$. The Gronwall inequality yields

$$\|M_\tau M_s^{-1}v\|^2 \leq e^{a(\tau-s)}\|v\|^2, \quad \forall s \in [0, \tau].$$

Hence

$$\sup_{s\in[0,\tau]} e^{a(s-\tau)} \left\| \sum_{i=1}^n \widetilde{e}_i^* A_i \right\|_{sp} \leq 1. \tag{29}$$

On the other hand, $Q_a f(x, y) = e^{-y\sqrt{a-L}} f(x)$ satisfies the Poisson equation

$$(\partial^2/\partial y^2 + L)Q_a f = aQ_a f.$$

By Itô's formula, we have

$$\begin{aligned}
&\int_0^\tau (\nabla, \partial_y)Q_a(f)(X_s, B_s) \cdot (U_s dW_s, dB_s) \\
&= Q_a(f)(X_\tau, B_\tau) - Q_a(f)(X_0, B_0) - a\int_0^\tau Q_a f(X_s, B_s)ds \\
&= f(X_\tau) - Q_a(f)(x, y) - a\int_0^\tau Q_a f(X_s, B_s)ds.
\end{aligned}$$

Notice that, for $f \in C_0^\infty(M)$ (with the additional condition $\mu(f) = 0$ if $\mu(M) < +\infty$ and $a = 0$),

$$Q_a(f)(x, +\infty) = \lim_{y\to+\infty} e^{-y\sqrt{a-L}} f(x) = 0.$$

Combining this with (28) and (29), we obtain

$$\|R_a(L)f\|_p \leq 2(p^*-1)\left(E\left|f(X_\tau)-a\int_0^\tau Q_af(X_s,B_s)ds\right|^p\right)^{1/p}$$
$$\leq 2(p^*-1)\left(\|f(X_\tau)\|_p + a\left\|\int_0^\tau Q_af(X_s,B_s)ds\right\|_p\right)$$
$$= 2(p^*-1)\left(\|f\|_p + a\left\|\int_0^\tau Q_af(X_s,B_s)ds\right\|_p\right).$$

Similar to Gundy [26] (p. 41), by the independence of B and X, and using Minkowski's inequality, we have

$$\left\|\int_0^\tau Q_af(X_s,B_s)ds\right\|_p^p = E_B\left[E_X\left(\left|\int_0^\tau Q_a(f)(X_s,B_s)ds\right|^p \Bigg| B\right)\right]$$
$$\leq E_B\left[\left\{\int_0^\infty \left(E_X\left[|Q_a(f)(X_s,B_s)|^p \big| B\right]\right)^{\frac{1}{p}} ds\right\}^p\right]$$
$$\leq E_B\left[\left(\int_0^\infty e^{-B_s\sqrt{a}}ds\right)^p\right]\|f\|_p^p,$$

where in the last step we have used the fact that for all $1 < p < \infty$ and all $f \in L^p(\mu)$,

$$\|Q_a(f)(\cdot, y)\|_p \leq e^{-y\sqrt{a}}\|f\|_p$$

which can be easily proved by using the Bochner subordination and the L^p-contractivity of e^{tL}. Hence

$$\|R_a(L)f\|_p \leq 2(p^*-1)\left(1 + a\left\|\int_0^\infty e^{-B_s\sqrt{a}}ds\right\|_p\right)\|f\|_p.$$

Thus, when $Ric(L) \geq 0$, we have proved that

$$\|R_0(L)f\|_p \leq 2(p^*-1)\|f\|_p.$$

Now suppose that $Ric(L) \geq -a$, where $a > 0$. Let $\widetilde{B}_s = B_{\tau_y - s}$ be the time reversal process of B_s. By [43,20,48,49], see Sect. 3.2, when $y \to \infty$, we have $\tau_y \to \infty$ and we can regard $\widetilde{B}_s$ as the 3D Bessel process starting from 0. Hence, for a standard 1D Brownian motion β_s, we have

$$d\widetilde{B}_s = d\beta_s + \frac{ds}{\widetilde{B}_s}.$$

This yields

$$d(\sqrt{a}\widetilde{B}_s) = d\beta_{as} + \frac{d(as)}{\sqrt{a}\widetilde{B}_s}.$$

Therefore

$$\sqrt{a}\widetilde{B}_s = \widetilde{B}_{as} \ \text{ in law.}$$

That is to say, when $Ric(L) \geq -a$ with $a > 0$, we have proved that

$$\|R_a(L)f\|_p \leq 2(p^* - 1)\left(1 + \left\|\int_0^\infty e^{-\widetilde{B}_s} ds\right\|_p\right)\|f\|_p.$$

Let

$$\xi := \int_0^\infty e^{-y_s} ds,$$

where $y_s = \widetilde{B}_s$ is the 3D Bessel process starting at $y_0 = 0$. By Donati-Martin and Yor (see Table 1 in [21], p. 1052), if we let T_1 be the first hitting time of y_s at 1, i.e.,

$$T_1 := \inf\{s > 0 : y_s = 1\}.$$

then T_1 has the same law with the random variable

$$\theta := \int_0^\infty e^{-2y_s} ds.$$

Now $2y_s = y_{4s}$ in law. Hence

$$\theta = \int_0^\infty e^{-2y_s} ds = \int_0^\infty e^{-y_{4s}} ds = \frac{1}{4}\int_0^\infty e^{-y_s} ds = \frac{\xi}{4} \ \text{ in law.}$$

That is to say, $\xi = 4T_1$ in law. The proof of Theorem 1.4 is completed. □

5 Two applications

In this section we give two applications of the main results of this paper. In Sect. 5.1 we prove the L^p-Poincaré inequality for a symmetric diffusion operator on a complete Riemannian manifold on which the Bakry–Emery Ricci curvature is bounded below by a strictly positive constant. In Sect. 5.2 we prove the inequalities of equivalence of Sobolev norms with optimal constants. We would like to emphasis that the results of this section remain valid for any estimate of the Riesz transforms.

5.1 The L^p-Poincaré inequality

It is well known that if M is a compact Riemannian manifold then the following Poincaré inequality holds: for any $p \in [1, n)$, there exists a constant C_p such that for all $f \in C^1(M)$,

$$\|f - \overline{f}\|_p \leq C_p \|\nabla f\|_p, \tag{30}$$

where $\overline{f} := \frac{1}{\mathrm{Vol}(M)} \int_M f(x) d\nu(x)$. In this subsection we study the problem of the explicit dependence of the constant C_p on p and on the bound of the Ricci curvature. The answer to this problem is the following result.

Theorem 5.1 *Let (M, g) be a compact Riemannian manifold, ν be the normalized volume measure. Suppose that there exists a positive constant $\rho > 0$ such that*

$$Ric \geq \rho.$$

Then for any $p > 1$ and for all $f \in W^{1,p}(M, \nu)$ we have

$$\|f - \nu(f)\|_p \leq \frac{\|R_0(\Delta)\|_{q,q}}{\sqrt{\rho}} \|\nabla f\|_p,$$

where $q = \frac{p}{p-1}$, $\nu(f) := \int_M f(x) d\nu(x)$, and $W^{1,p}(M, \nu) := \{f \in L^p(\nu) : |\nabla f| \in L^p(\nu)\}$.

Theorem 5.1 is a special case of the following general result.

Theorem 5.2 *Let (M, g) be a complete Riemannian manifold, $\mu(dx) = e^{-\phi(x)} d\nu(x)$ be a probability measure. Suppose that $\phi \in C^2(M)$ and there exists a positive constant $\rho > 0$ such that*

$$Ric(L) = Ric + \nabla^2 \phi \geq \rho.$$

Then for any $p > 1$ and for all $f \in W^{1,p}(M, \mu)$ we have

$$\|f - \mu(f)\|_p \leq \frac{\|R_0(L)\|_{q,q}}{\sqrt{\rho}} \|\nabla f\|_p,$$

where $q = \frac{p}{p-1}$, $\mu(f) := \int_M f(x) d\mu(x)$, and $W^{1,p}(M, \mu) := \{f \in L^p(\mu) : |\nabla f| \in L^p(\mu)\}$.

For symmetric diffusion operators with a positive lower bound of the Bakry–Emery Ricci curvature, Theorem 5.2 is a natural extension of the well known Bakry–Emery criterion for the L^2-Poincaré inequality (see [8]) to the L^p-Poincaré inequality for all $p > 1$. To prove Theorem 5.2, we need the some preliminary results.

Lemma 5.3 *(cf. [40,4,5,23]) Under the same conditions as in Theorem 5.2, for any $\omega \in C_0^1(M, \Lambda^1(T^*M))$, we have*

$$\left| e^{-t\Box_\phi}\omega(x) \right| \leq e^{-\rho t} e^{tL} |\omega(x)|, \quad \forall\, t \geq 0,\ x \in M.$$

Proposition 5.4 *Under the same conditions as in Theorem 5.2, for any $p > 1$ and any $\omega \in C_0^1(M, \Lambda^1(T^*M))$, we have*

$$\left\| \Box_\phi^{-1/2}\omega \right\|_p \leq \frac{\|\omega\|_p}{\sqrt{\rho}},$$

Proof By the Bochner subordination we have

$$\Box_\phi^{-1/2}\omega = \pi^{-1/2}\int_0^\infty e^{-t\Box_\phi}\omega \frac{dt}{\sqrt{t}}.$$

The Minkowski inequality and Lemma 5.3 imply

$$\begin{aligned}
\|\Box_\phi^{-1/2}\omega\|_p &= \pi^{-1/2}\left\| \int_0^\infty e^{-t\Box_\phi}\omega \frac{dt}{\sqrt{t}} \right\|_p \\
&\leq \pi^{-1/2}\int_0^\infty \left\| e^{t\Box_\phi}\omega \right\|_p \frac{dt}{\sqrt{t}} \\
&\leq \pi^{-1/2}\int_0^\infty e^{-\rho t}\|\omega\|_p \frac{dt}{\sqrt{t}} \\
&= \frac{\|\omega\|_p}{\sqrt{\rho}}.
\end{aligned}$$

□

Proof of Theorem 5.2 Replacing f by $f - \mu(f)$ and by the density argument, we need only prove Theorem 5.2 for all $f \in C_0^1(M)$ satisfying $\mu(f) = 0$. In this case, we have

$$f = R_0(L)^*\Box_\phi^{-1/2} df, \tag{31}$$

where $R_0(L)^*$ denotes the $L^2(\mu)$-adjoint of $R_0(L)$. To see (31), using $-L = d_\phi^* d$ and $d(-L) = \Box_\phi d$, we have

$$
\begin{aligned}
R_0(L)^* \Box_\phi^{-1/2} df &= \left(d(-L)^{-1/2}\right)^* d(-L)^{-1/2} f \\
&= (-L)^{-1/2} d_\phi^* d(-L)^{-1/2} f \\
&= (-L)^{-1/2}(-L)(-L)^{-1/2} f \\
&= f.
\end{aligned}
$$

By (31), Proposition 5.4, and using

$$\|R_0(L)^*\|_{p,p} = \|R_0(L)\|_{q,q},$$

we obtain

$$
\begin{aligned}
\|f\|_p &= \left\| R_0(L)^* \Box_\phi^{-1/2} df \right\|_p \\
&\le \|R_0(L)^*\|_{p,p} \|\Box_\phi^{-/2}\|_{p,p} \|df\|_p \\
&\le \frac{\|R_0(L)\|_{q,q}}{\sqrt{\rho}} \|\nabla f\|_p.
\end{aligned}
$$

This completes the proof of Theorem 5.2. □

5.2 Equivalence of Sobolev norms

As another application of Theorem 1.4, we have the following equivalence of Sobolev norms on M. Notice that the constants appeared here are optimal and dimension-free.

Theorem 5.5 *Under the same conditions as in Theorem 1.4, for all $p > 1$, we have*

$$
\begin{aligned}
\|\nabla(a-L)^{-1/2}\|_{p,p}^{-1} \|\nabla f\|_p &\le \|\sqrt{a-L} f\|_p \\
&\le \sqrt{a}\|f\|_p + \|\nabla(a-L)^{-1/2}\|_{q,q} \|\nabla f\|_p, \quad (32)
\end{aligned}
$$

where $q = \frac{p}{p-1}$. In the case where $a = 0$, we require that $f \in (\mathrm{Ker} L)^\perp$.

Proof We only consider the case $a > 0$. By the L^p-boundedness of the Riesz transform $\nabla(a-L)^{-1/2}$, we have

$$\|\nabla(a-L)^{-1/2} f\|_p \le \|\nabla(a-L)^{-1/2}\|_{p,p} \|f\|_p.$$

This implies the left hand side inequality in (32). Below we prove the right side one.

By Hölder's inequality, we have

$$\|f\|_p = \sup_{\|g\|_q = 1} \int_M \langle f, g \rangle d\mu.$$

For all $f, g \in C_0^\infty(M)$, integration by parts yields

$$\begin{aligned}\int_M \langle f, g\rangle d\mu &= \int_M \langle (a-L)^{-1/2}(a+d_\phi^* d)(a-L)^{-1/2}g, f\rangle d\mu \\ &= a\int_M (a-L)^{-1/2}g(a-L)^{-1/2}f d\mu \\ &\quad + \int_M \langle d(a-L)^{-1/2}g, d(a-L)^{-1/2}f\rangle d\mu \\ &\le a\|(a-L)^{-1/2}g\|_q\|(a-L)^{-1/2}f\|_p \\ &\quad + \|d(a-L)^{-1/2}\|_{q,q}\|g\|_q\|d(a-L)^{-1/2}f\|_p.\end{aligned}$$

Using the Bochner subordination formula and the L^q-contractivity of the heat semi-group $P_t = e^{tL}$, we have

$$\|(a-L)^{-1/2}g\|_q \le \frac{\|g\|_q}{\sqrt{a}}, \quad \forall\, q > 1.$$

Thus

$$\|f\|_p \le \sqrt{a}\|(a-L)^{-1/2}f\|_p + \|d(a-L)^{-1/2}\|_{q,q}\|d(a-L)^{-1/2}f\|_p.$$

This yields

$$\|\sqrt{a-L}f\|_p \le \sqrt{a}\|f\|_p + \|d(a-L)^{-1/2}\|_{q,q}\|df\|_p.$$

The proof of Theorem 5.5 is completed. □

6 Remarks and conjecture

In this section we give some remarks and pose a conjecture.

Remark 6.1 On any complete Riemannian manifold, using integration by parts, we have $\|R_0(\Delta)\|_{2,2} = \|R_0(L)\|_{2,2} = 1$, and for all $a > 0$, $\|R_a(\Delta)\|_{2,2} \le 1$ and $\|R_a(L)\|_{2,2} \le 1$. Hence, even if M is a complete Riemannian manifold with non-negative Ricci curvature, or if L is a symmetric diffusion operator on a complete Riemannian manifold whose Bakry–Emery Ricci curvature $Ric(L)$ is non-negative, the upper bound $2(p^* - 1)$ for the L^p-norm of the Riesz transforms $R_0(\Delta)$ and $R_0(L)$ is not the best. In the case where $M = \mathbb{R}^n$ and $p > 2$, Iwaniec and Martin [30] proved that

$$\|R_0(\Delta_{\mathbb{R}^n})\|_{p,p} \le \sqrt{\pi}\cot\left(\frac{\pi}{2p}\right).$$

This estimate is slightly better than Bañuelos and Wang's upper bound $2^*(p-1)$ when $p \to \infty$. Let us remind that it is still an open problem to find the exact value of $\|R_0(\Delta)\|_{p,p}$ even in the case where $(M, \mu) = (\mathbb{R}^n, dx)$ or $(M, \mu) = (S^n, \nu)$ when $n \geq 2$ and $p \neq 2$.

Remark 6.2 In [31], Larsson-Cohn proved that, for the Riesz transform associated with the Ornstein–Uhlenbeck operator on the Gaussian space $(\mathbb{R}^n, \gamma_n)$ or on the infinite dimensional Wiener space $(W(\mathbb{R}^n), \mu)$, when $p \to 1$,

$$\frac{2}{\pi}\frac{1}{p-1}(1+o(1)) \leq \|R_0(L)\|_{p,p} \leq \frac{2}{p-1},$$

and when $p \to \infty$,

$$\frac{1}{\pi}p(1+o(1)) \leq \|R_0(L)\|_{p,p} \leq \sqrt{\frac{2e}{\pi}}p(1+o(1)).$$

From these and by Iwaniec and Martin [30], Bañuelos and Wang [14] and Arcozzi [2], we see that at least in the case where $(M, \mu) = (\mathbb{R}^n, dx)$, $(M, \mu) = (\mathbb{R}^n, \gamma_n)$, or $(M, \mu) = (S^n, \nu)$, and in the case where L is the Ornstein–Uhlenbeck operator on the Wiener space, the upper bound of the form $O(p^* - 1)$ for the L^p-norm of the Riesz transforms $R_0(\Delta)$ or $R_0(L)$ is asymptotically sharp when $p \to 1$ and when $p \to \infty$.

In fact, in the case where $M = \mathbb{R}^n$, for all $p > 1$, we have $\|R_0(\Delta_{\mathbb{R}^n})\|_{p,p} \geq \|R_j\|_{p,p} = \cot(\frac{\pi}{2p^*})$. Moreover, $\|R_0(\Delta_{\mathbb{R}^n})\|_{p,p} \leq \frac{2}{p-1}$ for all $1 < p < 2$, and $\|R_0(\Delta_{\mathbb{R}^n})\|_{p,p} \leq \sqrt{\pi}\cot(\frac{\pi}{2p})$ for all $p > 2$. Therefore, when $p \to 1$, we have

$$\frac{2}{\pi}\frac{1}{p-1}(1+o(1)) \leq \|R_0(\Delta_{\mathbb{R}^n})\|_{p,p} \leq \frac{2}{p-1},$$

and when $p \to \infty$,

$$\frac{2}{\pi}p(1+o(1)) \leq \|R_0(\Delta_{\mathbb{R}^n})\|_{p,p} \leq \frac{2}{\sqrt{\pi}}p(1+o(1)).$$

In the case where $M = S^n$, from Arcozzi [2], we have

$$\cot\left(\frac{\pi}{2p^*}\right) \leq \|R_0(\Delta_{S^n})\|_{p,p} \leq 2(p^* - 1).$$

This also implies that, when $p \to 1$ and when $p \to \infty$,

$$\frac{2}{\pi}(p^* - 1)(1+o(1)) \leq \|R_0(\Delta_{S^n})\|_{p,p} \leq 2(p^* - 1).$$

Stimulated by the results in [30,14,2,31], we would like to pose the following conjecture.

Conjecture 6.3 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that $Ric(L) = Ric + \nabla^2\phi \geq 0$. Then there exists a constant $c > 0$ such that for all $p > 1$, we have*

$$c(p^* - 1)(1 + o(1)) \leq \|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1). \tag{33}$$

In particular, on any complete Riemannian manifold M with non-negative Ricci curvature, for all $p > 1$, we have

$$c(p^* - 1)(1 + o(1)) \leq \|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1). \tag{34}$$

In the case where M is a universal covering of a compact Riemannian manifold with non-negative Ricci curvature, the Cheeger–Gromoll splitting theorem says that M is isometric to the Riemannian product manifold $\mathbb{R}^k \times N$ with the product metric $ds_M^2 = ds_{\mathbb{R}^k}^2 + ds_N^2$, $k \geq 1$, where N is an $(n-k)$-dimensional compact Riemannian manifold with non-negative Ricci curvature. In this case, we can prove that the L^p-norm of the Riesz transform $\nabla(-\Delta_M)^{-1/2}$ satisfies (34). Furthermore, assuming that M is a complete Riemannian manifold with non-negative Ricci curvature and contains a line, then the Cheeger–Gromoll splitting theorem says that M is isometric to the Riemannian product manifold $\mathbb{R}^k \times N$ with product metric, where N is an $(n-k)$-dimensional complete Riemannian manifold which does not contain any line. Let $\phi(x, y) = \phi_1(x) + \phi_2(y)$, where $\phi_1(x) = 0$, or $\phi_1(x) = \frac{\|x\|^2}{2} + \frac{n}{2}\log(2\pi)$, $\phi_2 \in C^2(N)$, $(x, y) \in M = \mathbb{R}^k \times N$. Suppose that $Ric_N + \nabla^2\phi_2 \geq \rho$ for some constant $\rho > 0$, or more generally $Ric_N + \nabla^2\phi_2 \geq 0$ and $\mu_2(N) < \infty$, where $d\mu_2(y) = e^{-\phi_2(y)}\nu_N(dy)$. Then we can prove that the L^p-norm of the Riesz transform $\nabla(-L)^{-1/2}$ satisfies (33). Similarly, if M is isometric to the Riemannian product $S^k \times N$, $k \geq 1$, and $\phi(x, y) = \phi_1(x) + \phi_2(y)$, where $\phi_1(x) = 0$, $Ric_N + \nabla^2\phi_2 \geq 0$ and $\mu_2(N) < \infty$, the above conjecture is true. Due to the limit of the space of the paper, we leave the proofs of these results to the reader.

Finally, let us mention that it is well known that the Riesz transforms are not bounded in L^p for $p = 1, \infty$. In the case where M is a complete Riemannian manifold with non-negative Ricci curvature, it has been proved that the Riesz transform $\nabla(-\Delta_M)^{-1/2}$ is weak $(1, 1)$ and is bounded from H^1 to L^1, see [16,32,6,17,18]. However, it is still an open problem whether the weak $(1, 1)$-norm and the H^1–L^1 norm of the Riesz transform $\nabla(-\Delta_M)^{-1/2}$ are independent of $n = dim M$ (even in the case where $M = \mathbb{R}^n$). Moreover, it is also a long time open problem whether Meyer's Riesz transform on the Wiener space is weak $(1, 1)$ and is bounded from H^1 to L^1.

Acknowledgments The author would like to thank Professors D. Bakry, J.-M. Bismut, T. Coulhon, C. Donati-Martin, M. Ledoux, Y. Le Jan, N. Lohoué, P. Malliavin, S. Song and M. Yor for valuable discussions during many years. Part of this work was done when the author was supported by a grant of Délégation au CNRS at Le Laboratoire de Mathématiques d'Orsay de l'Université Paris-Sud. The author wishes to thank CNRS for the support and Le Laboratoire de Mathématiques d'Orsay for a very nice hospitality. Finally, the author would like to thank the associate editor for his interest and an anonymous referee for his careful reading and valuable comments on the submitted paper.

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