

Infinite volume limit of the Abelian sandpile model in dimensions $d \geq 3$

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Abstract We study the Abelian sandpile model on \mathbb{Z}^d . In $d \geq 3$ we prove existence of the infinite volume addition operator, almost surely with respect to the infinite volume limit μ of the uniform measures on recurrent configurations. We prove the existence of a Markov process with stationary measure μ , and study ergodic properties of this process. The main techniques we use are a connection between the statistics of waves and uniform two-component spanning trees and results on the uniform spanning forest measure on \mathbb{Z}^d .

Keywords Abelian sandpile model · Wave · Addition operator · Uniform spanning tree · Two-component spanning tree · Loop-erased random walk · Tail triviality

Mathematics Subject Classification (2000) 60K35 · 82C22

1 Introduction

The Abelian sandpile model (ASM), introduced by Bak et al. [2], has been studied extensively in the physics literature, mainly because of its remarkable “self-organized” critical state. Many exact results were obtained by Dhar using the group structure of the addition operators acting on recurrent configurations introduced in [4], see

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for example [6,5] for reviews. The relation between recurrent configurations and spanning trees, introduced by Majumdar and Dhar [22], has been used by Priezzhev to compute the stationary height probabilities of the two-dimensional model in the thermodynamic limit [25]. Later on, Ivashkevich, Ktitarev and Priezzhev introduced the concept of “waves” to study the avalanche statistics, and made a connection between two-component spanning trees and waves [9,10]. In [26] this connection was used to argue that the critical dimension of the ASM is $d = 4$.

From the mathematical point of view, one is interested in the thermodynamic limit, both for the stationary measures and for the dynamics. Recently, in [1] the connection between recurrent configurations and spanning trees, combined with results of Pemantle [24] on existence and uniqueness of the uniform spanning forest measure on \mathbb{Z}^d , has led to the result that the uniform measures μ_V on recurrent configurations in finite volume have a unique thermodynamic (weak) limit μ . (Note: for $d > 4$ the limit was only established for regular volumes such as cubes centered at the origin. The extension to arbitrary V is given in the appendix of this paper.) In [18] the existence of a unique limit μ was proved for an infinite tree, and a Markov process generated by Poissonian additions to recurrent configurations was constructed.

A natural continuation of [1] is therefore to investigate the dynamics defined on μ -typical configurations. The first question here is to study the addition operators. We prove that at least in $d \geq 3$, the addition operators a_x , $x \in \mathbb{Z}^d$ can be defined on μ -typical configurations. This turns out to be a rather simple consequence of the transience of the simple random walk, and we obtain that the avalanche resulting from adding a particle at a given site will be locally finite μ -almost surely (all sites topple finitely many times).

Next, in order to construct a stationary process from the infinite volume addition operators, it is crucial that the measure μ is invariant under a_x . We show that this is the case if the avalanche triggered by adding a particle at x is μ -almost surely finite (only finitely many topplings occur). In order to establish almost sure finiteness of avalanches, we first prove that the statistics of waves has a bounded density with respect to the uniform two-component spanning tree. The final step then is to show that the component of the uniform two-component spanning tree corresponding to the wave is almost surely finite in the infinite volume limit when $d \geq 3$. We deduce this from known results on the uniform spanning forest [3,24]. The case $d = 2$ remains an important open question.

Given existence of a_x , and stationarity of μ under its action, we can apply the formalism developed in [18] to construct a stationary process which is informally described as follows. Starting from a μ -typical configuration η , at each site $x \in \mathbb{Z}^d$ grains are added on the event times of a Poisson process N_t^x with mean $\varphi(x)$, where $\varphi(x)$ satisfies the condition

$$\sum_{x \in \mathbb{Z}^d} \varphi(x) G(0, x) < \infty,$$

with G the Green function of simple random walk in \mathbb{Z}^d . The condition ensures that the number of topplings at 0 caused by additions at all sites has finite expectation at any time $t > 0$.

In this paper we further study the ergodic properties of the infinite volume process. We show that tail triviality of the measure μ implies ergodicity of the process. We prove that μ has trivial tail in any dimension $d \geq 2$. For $2 \leq d \leq 4$ this is a rather straightforward consequence of the fact that the edge configuration of the uniform spanning tree is a (non-local) coding of the height-configuration, that is, from the spanning tree in *infinite volume* one can reconstruct the infinite height configuration almost surely. This is not the case in $d > 4$ where we need a separate argument.

Our paper is organized as follows. We start with some notation and definitions, recalling some basic facts about the ASM. In Sects. 3 and 4 we prove existence of the addition operator a_x when $d \geq 3$, and show invariance of the measure μ , assuming finiteness of avalanches. In Sect. 5 we prove existence of inverse addition operators. Sects. 6–8 are devoted to establishing finiteness of avalanches in dimensions $d \geq 3$. In Sect. 6 we make the precise link between avalanches and waves, in Sect. 7 we prove that all waves are finite if the uniform two-component spanning tree has almost surely a finite component. In Sect. 8 we prove the required finiteness of the component in dimensions $d \geq 3$. Finally, in Sects. 9 and 10 we discuss tail triviality of the stationary measure, and correspondingly, ergodicity of the stationary process.

The review papers [12,21,27] explain many points that are presented in less detail here, and may be useful for the reader.

2 Notation and definitions

We consider the Abelian sandpile model, as introduced in [2] and generalized by Dhar [4]. One starts from a toppling matrix Δ_{xy} , indexed by sites in \mathbb{Z}^d . In this paper Δ will always be the degree minus the adjacency matrix (in other words, minus the discrete lattice Laplacian):

$$\Delta_{xy} = \begin{cases} 2d & \text{if } x = y, \\ -1 & \text{if } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

A height configuration is a map $\eta : \mathbb{Z}^d \rightarrow \mathbb{N} = \{1, 2, \dots\}$, and a stable height configuration is such that $\eta(x) \leq \Delta_{xx}$ for all $x \in \mathbb{Z}^d$. A site where $\eta(x) > \Delta_{xx}$ is called an unstable site.

All stable configurations are collected in the set $\Omega = \{1, 2, \dots, 2d\}^{\mathbb{Z}^d}$. We endow Ω with the product topology. For $V \subseteq \mathbb{Z}^d$, $\Omega_V = \{1, 2, \dots, 2d\}^V$ denotes the stable configurations in volume V . If $\eta \in \Omega$ and $W \subseteq \mathbb{Z}^d$, then η_W denotes the restriction of η to the subset W . We also use η_W for the restriction of $\eta \in \Omega_V$ to a subset $W \subseteq V$. Given $\eta \in \Omega_V$, $\xi \in \Omega_{V^c}$, we let $\eta_V \xi_{V^c}$ denote the configuration that agrees with η in V and with ξ in V^c . We define the matrix Δ_V as the finite volume analogon of Δ , indexed now by the sites in V . That is, $(\Delta_V)_{xy} = \Delta_{xy}$, $x, y \in V$. Depending on the context, we sometimes interpret Δ_V as a matrix indexed by \mathbb{Z}^d . In that case $(\Delta_V)_{xy} = \Delta_{xy} I[x \in V] I[y \in V]$, where $I[\cdot]$ denotes an indicator function.

The toppling of a site x in finite volume V is defined on configurations $\eta : V \rightarrow \mathbb{N}$ by

$$T_x(\eta)(y) = \eta(y) - (\Delta_V)_{xy} \tag{2.1}$$

A toppling is called legal if the toppled site was unstable, otherwise it is called illegal. The stabilization of an unstable configuration is defined to be the stable result of a sequence of legal topplings, i.e.,

$$\mathcal{S}_V(\eta) = T_{x_n} \circ T_{x_{n-1}} \circ \dots \circ T_{x_1}(\eta), \tag{2.2}$$

where all topplings are legal and such that $\mathcal{S}_V(\eta)$ is stable. That $\mathcal{S}_V(\eta)$ is well-defined follows from [4, 23], see also [7]. If η is stable, then we define $\mathcal{S}_V(\eta) = \eta$. The addition operators are then defined by

$$a_{x,V}\eta = \mathcal{S}_V(\eta + \delta_x), \tag{2.3}$$

where $\delta_x(y)$ is 1 for $y = x$ and 0 otherwise. As long as we are in finite volume, $a_{x,V}$ is well-defined and $a_{x,V}a_{y,V} = a_{y,V}a_{x,V}$ (Abelian property).

The dynamics of the finite volume ASM is described as follows: at each discrete time step choose at random a site X according to a probability measure $p(x) > 0$, $x \in V$, and apply $a_{X,V}$ to the configuration. After time n , the configuration is $\prod_{i=1}^n a_{X_i,V} \dots a_{X_n,V} \eta$ where X_1, \dots, X_n is an i.i.d. sample from p . This gives a Markov chain with transition operator

$$Pf(\eta) = \sum_{x \in V} p(x) f(a_{x,V}\eta) \tag{2.4}$$

A function $f : \Omega \rightarrow \mathbb{R}$ is called local, if it only depends on finitely many coordinates, that is, there exists finite $V \subseteq \mathbb{Z}^d$, and $g : \Omega_V \rightarrow \mathbb{R}$ such that $f(\eta) = g(\eta_V)$, $\eta \in \Omega$. Local functions are dense in the space of continuous functions on Ω with respect to uniform convergence.

Given a function $F(V)$ defined for all sufficiently large finite volumes in \mathbb{Z}^d , and taking values in a metric space with metric ρ , we say that $\lim_V F(V) = a$, if for all $\varepsilon > 0$ there exists finite W , such that $\rho(F(V), a) < \varepsilon$ whenever $V \supseteq W$. For a probability measure ν , \mathbb{E}_ν will denote expectation with respect to ν . The boundary of V is defined by $\partial V = \{y \in V : y \text{ has a neighbour in } V^c\}$, while its exterior boundary is defined by $\partial_e V = \{y \in V^c : y \text{ has a neighbour in } V\}$.

2.1 Recurrent configurations

A stable configuration $\eta \in \Omega_V$ is called recurrent, if it is recurrent in the Markov chain defined in Sect. 2. Equivalently, η is recurrent, if for any $x \in V$ there exists $n = n_{x,\eta}$ such that $a_{x,V}^n \eta = \eta$. We denote by \mathcal{R}_V the set of recurrent configurations. The addition operators $a_{x,V}$ restricted to \mathcal{R}_V have well-defined inverses $a_{x,V}^{-1}$, and therefore

form an Abelian group under composition. From this fact one easily concludes that the uniform measure μ_V on \mathcal{R}_V is the unique invariant measure of the Markov chain. One can compute the number of recurrent configurations:

$$|\mathcal{R}_V| = \det(\Delta_V), \tag{2.5}$$

see [4]. Another important identity of [4] is the following. Denote by $N_V(x, y, \eta)$ the number of legal topplings at y needed to obtain $a_x \eta$ from $\eta + \delta_x$. Then the expectation of N_V satisfies

$$\mathbb{E}_{\mu_V}(N_V(x, y, \eta)) = G_V(x, y) \stackrel{\text{def}}{=} (\Delta_V^{-1})_{xy} \quad (\text{Dhar's formula}). \tag{2.6}$$

From this and the Markov inequality, one also obtains $G_V(x, y)$ as an estimate of the μ_V -probability that a site y has to be toppled if one adds at x . We also note that for our specific choice of Δ , G_V is $(2d)^{-1}$ times the Green function of simple random walk in V killed upon exiting V .

Recurrent configurations are characterized by the so-called burning algorithm [4]. A configuration η is recurrent if and only if it does not contain a so-called forbidden sub-configuration, that is, a subset $W \subseteq V$ such that for all $x \in W$:

$$\eta(x) \leq - \sum_{y \in W \setminus \{x\}} \Delta_{xy}. \tag{2.7}$$

From this explicit characterization, one easily infers a consistency property: if $\eta \in \mathcal{R}_V$ and $W \subseteq V$ then $\eta_W \in \mathcal{R}_W$. This suggests a natural definition of ‘‘recurrent configurations in infinite volume’’: we say that $\eta \in \Omega$ is recurrent, if its restriction to any finite V is. We denote this set by \mathcal{R} :

$$\mathcal{R} = \{\eta \in \Omega : \eta_V \in \mathcal{R}_V \text{ for all finite } V \subseteq \mathbb{Z}^d\}.$$

2.2 Infinite volume: basic questions and results

In studying infinite volume limits of the ASM, the following questions can be addressed. In this (non-exhaustive) list, any question can be asked only after a positive answer to all previous questions.

1. Do the measures μ_V weakly converge to a measure μ ? Does μ concentrate on the set \mathcal{R} ?
2. Is the addition operator a_x defined on μ -a.e. configuration $\eta \in \mathcal{R}$, and does it leave μ invariant? Does the Abelian property still hold in infinite volume?
3. Can one define a natural Markov process on \mathcal{R} with stationary distribution μ ?
4. Does the stationary Markov process of question 3 have good ergodic properties?

Question 1 is easily solved for the one-dimensional lattice \mathbb{Z} , however, μ is trivial, concentrating on the single configuration that is identically 2. Hence no further questions on our list are relevant in that case. See [20] for a result on convergence to

equilibrium in this case. For an infinite regular tree, the first three questions have been answered affirmatively and the fourth question remained open [18]. For dissipative models, that is when $\Delta_{xx} > 2d$, all four questions are affirmatively answered when Δ_{xx} is sufficiently large [19].

For \mathbb{Z}^d , question 1 is positively answered in any dimension $d \geq 2$, using a correspondence between spanning trees and recurrent configurations and properties of the uniform spanning forest on \mathbb{Z}^d [1]. The limiting measure μ is translation invariant. The proof of convergence in [1] in the case $d > 4$ is restricted to regular volumes, such as a sequence of cubes centered at the origin. In the appendix, we prove convergence along an arbitrary sequence of volumes using a random walk result [13].

In this paper we study questions 2, 3 and 4 for \mathbb{Z}^d , $d \geq 3$, and all questions are affirmatively answered.

The main problem is to prove that avalanches are almost surely finite. This is done by a decomposition of avalanches into a sequence of waves (cf. [10, 11]), and studying the almost sure finiteness of the waves. The latter can be achieved by a two-component spanning tree representation of waves, as introduced in [10, 11]. We then study the uniform two-component spanning tree in infinite volume and prove that the component containing the origin is almost surely finite. This turns out to be sufficient to ensure finiteness of all waves.

3 Existence of the addition operator

In this section we show convergence of the finite volume addition operators to an infinite volume addition operator when $d \geq 3$. This turns out to be easy, but in order to make appropriate use of this infinite volume addition operator, we need to establish that μ is invariant under its action, and for the latter we need to show that avalanches are finite μ -a.s.

Let $a_{x,V}$ denote the addition operator acting on Ω_V . We define a corresponding operator acting on Ω using the finite (V)-volume rule, that is, grains falling out of V disappear. More precisely, for $\eta \in \Omega$ and $V \ni x$, we define (with slight abuse of notation)

$$a_{x,V}\eta = (a_{x,V}\eta_V)\eta_V^c. \tag{3.1}$$

Given $\eta \in \Omega$, call $N_V(x, y, \eta)$ the number of topplings caused at y by addition at x in η , using the finite (V)-volume rule. Then

$$\eta + \delta_x - \Delta_V N_V(x, \cdot, \eta) = a_{x,V}\eta, \quad \eta \in \Omega, x \in V \tag{3.2}$$

where Δ_V is indexed by \mathbb{Z}^d .

We start with the following simple lemma:

Lemma 3.3 *$N_V(x, y, \eta)$ is a non-decreasing function of V and depends on η only through η_V .*

Proof Let $V \subseteq W$. Suppose we add a grain at x in configuration η . We perform topplings inside V until inside V the configuration is stable, using the finite (W)-volume rule. The result of this procedure is a configuration $(a_{x,V} \eta_V) \xi_{V^c}$, where possibly $\xi_{V^c \cap W}$ is not stable. In that case we perform all the necessary topplings still needed to stabilize $(a_{x,V} \eta_V) \xi_{V^c \cap W}$ inside W , using the finite (W)-volume rule. This can only cause potential extra topplings at any site y inside V .

From Lemma 3.3 and by monotone convergence:

$$\mathbb{E}_\mu(\sup_V N_V(x, y, \eta)) = \lim_V \mathbb{E}_\mu(N_V(x, y, \eta)). \tag{3.4}$$

By weak convergence of μ_V to μ , and by Dhar’s formula (2.6):

$$\begin{aligned} \lim_V \mathbb{E}_\mu(N_V(x, y, \eta)) &= \lim_V \lim_{W \supseteq V} \mathbb{E}_{\mu_W}(N_V(x, y, \eta)) \\ &\leq \lim_V \lim_{W \supseteq V} \mathbb{E}_{\mu_W}(N_W(x, y, \eta)) \\ &= \lim_W G_W(x, y) = G(x, y), \end{aligned} \tag{3.5}$$

where $G(x, y) = \Delta_{xy}^{-1}$ is $(2d)^{-1}$ times the Green function of simple random walk in \mathbb{Z}^d . In the last step we used that $d \geq 3$, otherwise $G_W(x, y)$ diverges as $W \uparrow \mathbb{Z}^d$. This proves that for all $x, y \in \mathbb{Z}^d$, $N(x, y, \eta) = \sup_V N_V(x, y, \eta)$ is μ -a.s. finite. Hence

$$\mu\left(\forall x, y \in \mathbb{Z}^d : N(x, y, \eta) < \infty\right) = 1. \tag{3.6}$$

Therefore, on the event in (3.6), we can define

$$a_x \eta = \eta + \delta_x - \Delta N(x, \cdot, \eta). \tag{3.7}$$

It is easy to see that $a_x \eta$ is stable. This is because $a_x \eta(y)$ is already determined by the number of topplings at y and its neighbours, and this is the same as it was in a large enough finite volume V . By similar reasons, we also get

$$a_x \eta = \lim_V a_{x,V} \eta, \quad \mu\text{-a.s.}, \tag{3.8}$$

where $a_{x,V}$ is defined in (3.1).

From its definition, one sees that a_x is well behaved with respect to translations. Let θ_x denote the shift on configuration, that is, $\theta_x \eta(y) = \eta(y - x)$. Then

$$a_x = \theta_x \circ a_0 \circ \theta_{-x}, \tag{3.9}$$

whenever either side is defined.

Note that with the above definition of a_x , there can be infinite avalanches. However, if the volume increases, it cannot happen that the number of topplings at a fixed site diverges, and that is the only problem for defining a_x (a problem which may arise

in $d = 2$). More precisely, an infinite avalanche that leaves eventually every finite region does not pose a problem for defining the addition operator. However, as we will see later on, infinite avalanches do cause problems in defining a stationary process, at least with our current methods. Hence we need extra arguments to show that the total number of topplings is finite μ -a.s.

We define the avalanche cluster caused by addition at x to be the set

$$C_x(\eta) = \{y \in \mathbb{Z}^d : N(x, y, \eta) > 0\} \tag{3.10}$$

We say that the avalanche at x is finite in η if $C_x(\eta)$ is a finite set. We say that μ has the finite avalanche property, if for all $x \in \mathbb{Z}^d$, $\mu(|C_x| < \infty) = 1$.

In Sects. 6–8, we prove the following theorem:

Theorem 3.11 *Assume $d \geq 3$. Then μ has the finite avalanche property, that is, $\mu(|C_x| < \infty) = 1$ for all $x \in \mathbb{Z}^d$.*

In Sect. 4, we show that Theorem 3.11 has the following consequence.

Proposition 3.12 *Assume $d \geq 3$. Then μ is invariant under the action of a_x , $x \in \mathbb{Z}^d$, that is, for any μ -integrable function f and for any $x \in \mathbb{Z}^d$,*

$$\int f(a_x \eta) d\mu = \int f(\eta) d\mu. \tag{3.13}$$

Before moving on to the proofs of Theorem 3.11 and Proposition 3.12, we prove some of their easy consequences.

Integrating (3.7) over μ and using Proposition 3.12, we easily obtain the following infinite volume analogue of Dhar’s formula.

Proposition 3.14 *Assume $d \geq 3$. Then*

$$\mathbb{E}_\mu(N(x, y, \eta)) = G(x, y) \tag{3.15}$$

At this point, we cannot compose different a_x , since a_x is only defined almost surely.

Proposition 3.16 *Assume $d \geq 3$. There exists a μ -measure one set $\Omega' \subseteq \mathcal{R}$ with the following properties.*

- (i) *For any $\eta \in \Omega'$ and $x \in \mathbb{Z}^d$, there exists finite $V_x(\eta) \subseteq \mathbb{Z}^d$, such that for all $W \supseteq V_x(\eta)$*

$$a_x \eta = a_{x,W} \eta.$$

- (ii) *For any $\eta \in \Omega'$ and $x \in \mathbb{Z}^d$ we have $a_x \eta \in \Omega'$.*

Consequently, for any $\eta \in \Omega'$, and any $x_1, \dots, x_n \in \mathbb{Z}^d$, $a_{x_n} a_{x_{n-1}} \dots a_{x_1} \eta$ is well-defined and all avalanches involved are finite.

Proof Define

$$\Omega_0 = \{\eta \in \mathcal{R} : |\mathcal{C}_x(\eta)| < \infty \text{ for all } x \in \mathbb{Z}^d\}.$$

By Theorem 3.11 and since μ is concentrated on \mathcal{R} , we have $\mu(\Omega_0) = 1$. Property (i) in the proposition is satisfied for all $\eta \in \Omega_0$. For (ii), we need to find a subset of Ω_0 invariant under all the a_x 's. For $n \geq 1$, define inductively the sets

$$\Omega_n = \Omega_{n-1} \cap \bigcap_{x \in \mathbb{Z}^d} a_x^{-1}(\Omega_{n-1}),$$

where a_x^{-1} here denotes inverse image (not to be confused with the inverse operator defined later). Since the a_x are measure preserving, it follows by induction that $\mu(\Omega_n) = 1$ for all n . Also, a_x maps Ω_n into Ω_{n-1} . Therefore, $\Omega' = \bigcap_{n \geq 0} \Omega_n$ satisfies both properties stated.

The following proposition shows that the Abelian property holds in infinite volume.

Proposition 3.17 *Assume $d \geq 3$. Then*

$$a_x a_y \eta = a_y a_x \eta, \quad \eta \in \Omega'. \tag{3.18}$$

Proof By Proposition 3.16, for $\eta \in \Omega'$ and for $W \supseteq V_y(\eta) \cup V_x(a_y \eta) \cup V_x(\eta) \cup V_y(a_x \eta)$ we have

$$a_x a_y \eta = a_{x,W} a_{y,W} \eta = a_{y,W} a_{x,W} \eta = a_y a_x \eta. \tag{3.19}$$

4 Invariance of μ under a_x

In this section, we show that μ is invariant under the addition operators, if it has the finite avalanche property, that is, we show that Theorem 3.11 implies Proposition 3.12.

Proof of Proposition 3.12 (assuming theorem 3.11) It is enough to prove the claim for f a local function. In that case, we have

$$\begin{aligned} \int f(a_x \eta) d\mu &= \int f(a_{x,V} \eta) d\mu + \epsilon_1(V, f) \\ &= \int f(a_{x,V} \eta) d\mu_W + \epsilon_1(V, f) + \epsilon_2(V, W, f) \\ &= \int f(a_{x,W} \eta) d\mu_W + \epsilon_1(V, f) + \epsilon_2(V, W, f) + \epsilon_3(V, W, f) \end{aligned}$$

Here ϵ_1 and ϵ_2 can be made arbitrarily small by (3.8) and by weak convergence. We also have

$$|\epsilon_3(V, W, f)| \leq 2 \|f\|_\infty \mu_W(a_{x,W} f \neq a_{x,V} f).$$

Next, by invariance of μ_W under the action of $a_{x,W}$,

$$\int f(a_{x,W}\eta)d\mu_W = \int fd\mu_W = \int fd\mu + \epsilon_4(W, f). \tag{4.1}$$

Here, by weak convergence, ϵ_4 can be made arbitrarily small. Therefore, combining the estimates, we conclude

$$\left| \int f(a_x\eta)d\mu - \int f(\eta)d\mu \right| \leq C \limsup_V \limsup_{W \supseteq V} \mu_W(a_{x,W}f \neq a_{x,V}f). \tag{4.2}$$

Define the avalanche cluster in volume W by

$$C_{x,W}(\eta) = \{y \in W : N_W(x, y, \eta) > 0\}, \quad \eta \in \Omega.$$

Let D_f denote the dependence set of the local function f . On the event $C_{x,W}(\eta) \cap \partial V = \emptyset$ we have $a_{x,V}\eta = a_{x,W}\eta$. Hence, provided $D_f \subseteq V$, we have

$$\mu_W(a_{x,W}f \neq a_{x,V}f) \leq \mu_W(C_{x,W} \cap \partial V \neq \emptyset).$$

The event on the right hand side is a cylinder event (only depends on heights in V). Therefore, the right hand side approaches $\mu(C_x \cap \partial V \neq \emptyset)$, as $W \uparrow \mathbb{Z}^d$. By Theorem 3.11,

$$\lim_V \mu(C_x \cap \partial V \neq \emptyset) = \mu(|C_x| = \infty) = 0,$$

which completes the proof.

5 Inverse addition operators

In this section we prove that a_x has an inverse defined μ -a.s., provided μ has the finite avalanche property. In fact, we show that a_x^{-1} is the limit of finite volume inverses. Let $a_{x,V}^{-1}$ denote the inverse of $a_{x,V}$ acting on \mathcal{R}_V . We define a corresponding operator acting on \mathcal{R} , by

$$a_{x,V}^{-1}\eta = (a_{x,V}^{-1}\eta_V)\eta_{V^c}, \quad \eta \in \mathcal{R}.$$

This is well-defined, since $\eta_V \in \mathcal{R}_V$.

Recall that under the finite avalanche property, Proposition 3.16 provided a set Ω' of recurrent configurations such that for any $\eta \in \Omega'$ and every $x \in \mathbb{Z}^d$, there exists a finite set $V_x(\eta)$ such that $a_x\eta = a_{x,V_x(\eta)}\eta$.

Proposition 5.1 *Assume $d \geq 3$. There exists a μ -measure one set $\Omega'' \subseteq \Omega'$ with the following properties.*

- (i) For any $\eta \in \Omega''$ and $x \in \mathbb{Z}^d$ there exists finite $\bar{V} = \bar{V}_x(\eta)$ such that $a_{x,W}^{-1}\eta = a_{x,\bar{V}}^{-1}\eta$ for all $W \supseteq \bar{V}$.
- (ii) If we define $a_x^{-1}\eta = a_{x,\bar{V}_x(\eta)}^{-1}\eta$, then $a_x^{-1}a_x\eta = a_x a_x^{-1}\eta = \eta$ for $\eta \in \Omega''$.
- (iii) As operators on $L_2(\mu)$, $a_x^* = a_x^{-1}$, that is, the a_x are unitary operators.

Proof We first prove that

$$\lim_{\bar{V}} \mu \left(\exists W \supseteq \bar{V} : a_{x,W}^{-1}\eta \neq a_{x,\bar{V}}^{-1}\eta \right) = 0. \tag{5.2}$$

We have

$$\begin{aligned} & \mu \left(\exists W \supseteq \bar{V} : a_{x,W}^{-1}\eta \neq a_{x,\bar{V}}^{-1}\eta \right) \\ &= \mu \left(\exists W \supseteq \bar{V} : a_{x,W}^{-1}a_x\eta \neq a_{x,\bar{V}}^{-1}a_x\eta \right) \\ &= \mu \left(\exists W \supseteq \bar{V} : a_{x,W}^{-1}a_x\eta \neq a_{x,\bar{V}}^{-1}a_x\eta \text{ and } \forall W \supseteq \bar{V}, a_{x,W}\eta = a_{x,\bar{V}}\eta \right) \\ &\quad + \epsilon_{\bar{V}} \\ &= \epsilon_{\bar{V}}. \end{aligned} \tag{5.3}$$

Here we used the invariance of μ under a_x in the first step. The last step follows because if $a_x\eta = a_{x,W}\eta = a_{x,\bar{V}}\eta$, then

$$a_{x,W}^{-1}a_x\eta = a_{x,W}^{-1}a_{x,W}\eta = \eta = a_{x,\bar{V}}^{-1}a_{x,\bar{V}}\eta = a_{x,\bar{V}}^{-1}a_x\eta. \tag{5.4}$$

As for $\epsilon_{\bar{V}}$ we have

$$\epsilon_{\bar{V}} \leq \mu \left(\exists W \supseteq \bar{V} : a_{x,W}\eta \neq a_{x,\bar{V}}\eta \right) \tag{5.5}$$

which converges to zero as $\bar{V} \uparrow \mathbb{Z}^d$, by the finite avalanche property.

Since the events in (5.2) are decreasing in \bar{V} , we get that property (i) of the proposition is satisfied for μ -a.e. η . Let us define a_x^{-1} on this set by setting $a_x^{-1}\eta = a_{x,\bar{V}_x(\eta)}^{-1}\eta$. Then (5.4) shows that a_x has an inverse on a full measure set. By standard arguments, similar to the one in Proposition 3.16, we can shrink the set Ω' appropriately to a set Ω'' of full measure such that Ω'' is invariant under a_x and a_x^{-1} for all $x \in \mathbb{Z}^d$. Then (i) and (ii) will hold for Ω'' .

The last statement of the proposition is an obvious consequence of the first two.

The above proposition proves that as operators on $L_2(\mu)$, the a_x generate an (Abelian) unitary group, which we denote by G .

6 Waves and avalanches

The goal of Sects. 6, 7 and 8 is to prove Theorem 3.11, saying that μ has the finite avalanche property.

We will decompose avalanches into so-called waves, that correspond to carrying out topplings in a special order. We prove that almost surely, there is a finite number of waves, and that all waves are almost surely finite. Without loss of generality, we assume that the particle is added at $x = 0$, the origin. Since the site where we add will remain fixed throughout Sects. 6, 7 and 8, henceforth we drop indices referring to x from our notation, and simply write $\mathcal{C} = \mathcal{C}(\eta)$ for the avalanche cluster $\mathcal{C}_0(\eta)$.

We start by recalling the definition of a wave from [10, 11]. Consider a finite volume $W \ni 0$, and add a grain at site 0 in a stable configuration. If the site becomes unstable, then topple it once and topple all other sites that become unstable, except 0. It is easy to see that in this procedure a site can topple at most once. The set of toppled sites is called the first wave. Next, if 0 has to be toppled again, we start a second wave, that is we topple 0 once again, and topple all other sites that become unstable. We continue as long as 0 needs to be toppled.

We define $\alpha_W(\eta)$ to be the number of waves caused by addition at 0 in the volume W . By definition, α_W is the number of topplings at 0, that is $\alpha_W(\eta) = N_W(0, 0, \eta)$. For fixed W , let $\mathcal{C}_W(\eta)$ denote the avalanche cluster in volume W . We decompose \mathcal{C}_W as

$$\mathcal{C}_W(\eta) = \bigcup_{i=1}^{\alpha_W(\eta)} \Xi_W^i(\eta), \tag{6.1}$$

where $\Xi_W^i(\eta)$ is the i th wave in W caused by addition at 0.

We can define waves in infinite volume as we defined the toppling numbers and avalanches in Sect. 3, by monotonicity in the volume. More precisely, the definition is as follows. Fix $\eta \in \Omega$, and assume that 0 is unstable in $\eta + \delta_0$. By the argument of Lemma 3.3 in Sect. 3, $\Xi_W^1(\eta)$ is non-decreasing in W , and therefore we can define $\Xi^1(\eta) = \cup_W \Xi_W^1(\eta)$. Let η_1 denote the configuration obtained by toppling every site in $\Xi^1(\eta)$ once, that is $\eta_1 = \lim_W \left[\prod_{x \in \Xi_W^1(\eta)} T_x \right] \eta$. Note that carrying out the first wave in the unstable configuration $\eta + \delta_0$ results in exactly $\eta_1 + \delta_0$. By the definition of a wave, all sites are stable in η_1 . If 0 is unstable in $\eta_1 + \delta_0$, we consider $\Xi_W^1(\eta_1)$. Since this is again nondecreasing in W , we can define $\Xi^2(\eta) = \cup_W \Xi_W^1(\eta_1)$. This is the second wave in infinite volume. Let $\eta_2 = \lim_W \left[\prod_{x \in \Xi_W^1(\eta_1)} T_x \right] \eta_1$. Then $\eta_2 + \delta_0$ is the result of carrying out the first two waves in infinite volume on $\eta + \delta_0$. Note that if $|\Xi^1(\eta)| < \infty$, we have $\Xi_W^1(\eta_1) = \Xi_W^2(\eta)$ for all large W , and consequently, $\Xi^2(\eta) = \lim_W \Xi_W^2(\eta)$. We similarly define inductively $\Xi^i(\eta) = \cup_W \Xi_W^1(\eta_{i-1})$ and $\eta_i = \lim_W \left[\prod_{x \in \Xi_W^1(\eta_{i-1})} T_x \right] \eta_{i-1}$. Under the assumption $|\Xi^j(\eta)| < \infty, 1 \leq j < i$, we also have $\Xi^i(\eta) = \lim_W \Xi_W^i(\eta)$. For convenience, we define Ξ_W^i or Ξ^i as the empty set, whenever such waves do not exist.

The easy part in proving finiteness of avalanches is to show that the number of waves is finite. Since $\alpha_W(\eta)$ is non-decreasing in W , it has a pointwise limit $\alpha(\eta)$, and as in (3.5),

$$\mathbb{E}_\mu(\alpha) \leq G(0, 0) < \infty. \tag{6.2}$$

This implies $\alpha < \infty$ μ -a.s.

In order to prove that $\mathcal{C}(\eta)$ is finite μ -a.s., we show, by induction on i , that all sets $\Xi^i(\eta)$ are finite μ -a.s. We base the proof on the following proposition, proved in Sects. 7 and 8.

Proposition 6.3 *Assume $d \geq 3$. For $i \geq 1$ we have*

$$\lim_V \limsup_{W \supseteq V} \mu_W(\Xi_W^i \not\subseteq V) = 0. \tag{6.4}$$

Proof of Theorem 3.11 (assuming Proposition 6.3) We prove by induction on i that $\mu(|\Xi^i| < \infty) = 1, i \geq 1$. To start the induction, note that $\{\Xi^1 \subseteq V\}$ is a local event, hence weak convergence of μ_W to μ and Proposition 6.3 with $i = 1$ imply that $\mu(|\Xi^1| < \infty) = 1$. Assume now that $\mu(|\Xi^j| < \infty) = 1, 1 \leq j < i$. Then

$$\mu(\Xi^i \not\subseteq V) \leq \mu(\Xi^i \not\subseteq V, \Xi^j \subseteq V', 1 \leq j < i) + \mu(\Xi^j \not\subseteq V' \text{ for some } 1 \leq j < i). \tag{6.5}$$

By the induction hypothesis, the second term on the right hand side can be made arbitrarily small by choosing V' large. For fixed V' , the event in the first term is a local event (only depends on sites in $V' \cup \partial_e V' \cup V \cup \partial_e V$). Therefore, the first term on the right hand side of (6.5) equals

$$\lim_W \mu_W(\Xi_W^i \not\subseteq V, \Xi_W^j \subseteq V', 1 \leq j < i) \leq \limsup_W \mu_W(\Xi_W^i \not\subseteq V). \tag{6.6}$$

Here the right hand side goes to 0 as $V \uparrow \mathbb{Z}^d$, by Proposition 6.3, showing that $\mu(|\Xi^i| < \infty) = 1$.

On the event $\{\alpha < \infty\} \cap \{|\Xi^i| < \infty, i \geq 1\}$, we can pass to the limit in (6.1) and obtain the decomposition

$$\mathcal{C}(\eta) = \bigcup_{i=1}^{\alpha(\eta)} \Xi^i(\eta). \tag{6.7}$$

It follows that $\mu(|\mathcal{C}| < \infty) = 1$, which completes the proof of Theorem 3.11.

7 Finiteness of waves

In this section we prove Proposition 6.3 saying that waves are finite. The proof is based on a correspondence with two-component spanning trees due to Ivashkevich et al. [10], which is recalled below. The correspondence allows us to use some results on the uniform spanning forest that are stated separately as Proposition 7.11 below. The argument is completed with a proof of Proposition 7.11 in Sect. 8.

We now describe the representation of waves as two-component spanning trees from [9, 10]. Consider a configuration $\eta_W \in \mathcal{R}_W$ with $\eta_W(0) = 2d$, and suppose we add a particle at 0. Consider the first wave, which is entirely determined by the

recurrent configuration $\eta_{W \setminus \{0\}} \in \mathcal{R}_{W \setminus \{0\}}$. The result of the first wave on $W \setminus \{0\}$ is given by

$$S_W^1(\eta) = \left(\prod_{j \sim 0} a_{j, W \setminus \{0\}} \right) \eta_{W \setminus \{0\}} \in \mathcal{R}_{W \setminus \{0\}}. \tag{7.1}$$

We associate to any $\xi \in \mathcal{R}_{W \setminus \{0\}}$ a tree $\mathcal{T}_W(\xi)$. The tree will represent a wave starting at 0 in ξ . For the definition of the tree, we use Majumdar and Dhar’s tree construction [22].

Denote by \widehat{W} the graph obtained from \mathbb{Z}^d by identifying all sites in $\mathbb{Z}^d \setminus (W \setminus \{0\})$ to a single site $\delta_{\widehat{W}}$ (removing loops). By [22], there is a one-to-one map between recurrent configurations $\xi \in \mathcal{R}_{W \setminus \{0\}}$ and spanning trees of \widehat{W} . The correspondence is given by following the spread of an avalanche started at $\delta_{\widehat{W}}$. Initially, each neighbour of $\delta_{\widehat{W}}$ receives a number of grains equal to the number of edges connecting it to $\delta_{\widehat{W}}$, which results in every site in $W \setminus \{0\}$ toppling exactly once. The spanning tree records the sequence in which topplings have occurred. There is some flexibility in how to carry out the topplings (and hence in the correspondence with spanning trees), and here we make a specific choice in accordance with [10]. Namely, we first transfer grains from $\delta_{\widehat{W}}$ only to the neighbours of 0, and carry out all possible topplings. We call this the first phase. When we apply the process to $\xi = \eta_{W \setminus \{0\}}$, the set of sites that topple in the first phase is precisely $\Xi_W^1(\eta) \setminus \{0\}$. Next we transfer grains from $\delta_{\widehat{W}}$ to the boundary sites of W , which will cause topplings at all sites that were not in the wave; this is the second phase.

The two phases can alternatively be described via the burning algorithm of Dhar [4], which in the above context looks as follows. For convenience, let \widetilde{W} denote the graph obtained by identifying all sites in $\mathbb{Z}^d \setminus W$ to a single site δ_W . That is, \widetilde{W} can be obtained from \widehat{W} by identifying 0 and δ_W , and calling it $\delta_{\widetilde{W}}$. We start with all sites of \widetilde{W} declared unburnt. At step 0 we burn 0 (the origin). At step t ,

$$\text{burn all sites } y \text{ for which } \xi(y) > \text{current number of unburnt neighbours of } y. \tag{7.2}$$

The process stops at some step $T = T(\xi)$. The sites that burn up to time T is precisely the sites toppling in the first phase. We continue by burning δ_W in step $T + 1$, and then repeating (7.2) as long as there are unburnt sites.

Following Majumdar and Dhar’s construction [22], we connect with an edge each $y \in W \setminus \{0\}$ burnt at time t to a unique neighbour y' (called the parent of y) burnt at time $t - 1$. This defines a spanning subgraph of \widetilde{W} with two tree components, having roots 0 and δ_W . We denote by $\mathcal{T}_W(\xi)$ the tree component having root 0. Since $\mathcal{T}_W(\xi)$ does not contain the vertex δ_W , it can be identified with a subgraph of \mathbb{Z}^d , and we will do so in what follows. (Identifying 0 and δ_W merges the two trees into a spanning tree of \widehat{W} , yielding the usual spanning tree representation of ξ .)

With slight abuse of language, we refer to the two-component spanning subgraph as a two-component spanning tree. By observations made earlier, when $\xi = \eta_{W \setminus \{0\}}$, the vertex set of $\mathcal{T}_W(\eta_{W \setminus \{0\}})$ is the first wave $\Xi_W^1(\eta)$.

We can generalize the above construction to further waves as follows. We define

$$S_W^k(\eta) = \left(\prod_{j \sim 0} a_{j, W \setminus \{0\}} \right)^k \eta_{W \setminus \{0\}} \in \mathcal{R}_{W \setminus \{0\}}, \quad k \geq 1, \eta \in \mathcal{R}_W. \tag{7.3}$$

If there exists a k th wave, then its result on $W \setminus \{0\}$ is given by (7.3). Applying the above constructions to $\xi = S_W^{k-1}(\eta)$, we obtain that the k th wave (if there is one) is represented by $\mathcal{T}_W(S_W^{k-1}(\eta))$.

If now $\xi \in \mathcal{R}_{W \setminus \{0\}}$ is distributed according to $\mu_{W \setminus \{0\}}$, then $\mathcal{T}_W(\xi)$ is a random subtree of \mathbb{Z}^d . We will prove that this random tree has a weak limit \mathcal{T} , which is almost surely finite. But first let us show that this is actually sufficient for finiteness of all waves.

Consider the first wave, and let $W \supseteq V$. By construction, $\Xi_W^1(\eta)$ is precisely the vertex set of $\mathcal{T}_W(\eta_{W \setminus \{0\}})$, therefore

$$\mu_W(\Xi_W^1(\eta) \not\subseteq V) = \mu_W(\mathcal{T}_W(\eta_{W \setminus \{0\}}) \not\subseteq V). \tag{7.4}$$

Here the right hand side is determined by the distribution of $\eta_{W \setminus \{0\}}$ under μ_W . This is different from the law of $\eta_{W \setminus \{0\}}$ under $\mu_{W \setminus \{0\}}$, which is simply the uniform measure on $\mathcal{R}_{W \setminus \{0\}}$. It is the latter that we can get information about using the correspondence to spanning trees. Indeed, under $\mu_{W \setminus \{0\}}$, the spanning tree corresponding to $\eta_{W \setminus \{0\}}$ is uniformly distributed on the set of spanning trees of \tilde{W} . In order to translate our results back to μ_W , we show that the former distribution has a bounded density with respect to the latter. This will be a consequence of the following lemma.

Lemma 7.5 *Assume $d \geq 3$. There is a constant $C(d) > 0$ such that*

$$\sup_{V \subseteq \mathbb{Z}^d} \frac{|\mathcal{R}_{V \setminus \{0\}}|}{|\mathcal{R}_V|} \leq C(d), \tag{7.6}$$

where the supremum is over finite sets.

Proof By Dhar’s formula (2.5),

$$|\mathcal{R}_{V \setminus \{0\}}| = \det(\Delta_{V \setminus \{0\}}) = \det(\Delta'_V)$$

where Δ'_V denotes the matrix indexed by sites $y \in V$ and defined by $(\Delta'_V)_{yz} = (\Delta_{V \setminus \{0\}})_{yz}$ for $y, z \in V \setminus \{0\}$, and $(\Delta'_V)_{0z} = (\Delta'_V)_{z0} = \delta_0(z)$. We have

$$\Delta_V + P = \Delta'_V$$

where P is a matrix for which $P_{yz} = 0$ unless $y, z \in N = \{u \in \mathbb{Z}^d : |u| \leq 1\}$. Moreover, $\max_{y, z \in V} |P_{yz}| \leq 2d - 1$. Hence

$$\frac{|\mathcal{R}_{V \setminus \{0\}}|}{|\mathcal{R}_V|} = \frac{\det(\Delta_V + P)}{\det(\Delta_V)} = \det(I + G_V P).$$

We have $(G_V P)_{yz} = 0$ unless $z \in N$. Therefore

$$\det(I + G_V P) = \det((I + G_V P)_N). \tag{7.7}$$

By transience of the simple random walk in $d \geq 3$, we have $\sup_V \sup_{y,z} G_V(y, z) \leq G(0, 0) < \infty$, and therefore the determinant of the matrix $(I + G_V P)_N$ in (7.7) is bounded by a constant depending on d .

We note that an alternative proof of Lemma 7.5 can be given based on the following idea. Consider the graph \bar{W} obtained by adding an extra edge e between 0 and δ_W in \tilde{W} . Then the ratio in (7.6) can be expressed in terms of the probability that a uniformly chosen spanning tree of \bar{W} contains e . By standard spanning tree results [3, Theorem 4.1], the latter is the same as the probability that a random walk in \bar{W} started at 0 first hits δ_W through e .

We continue with the bounded density argument. For any configuration $\xi \in \mathcal{R}_{W \setminus \{0\}}$ we have

$$\mu_W(\eta_{W \setminus \{0\}} = \xi) = \frac{1}{|\mathcal{R}_W|} |\{k \in \{1, \dots, 2d\} : (k)_0 \xi_{W \setminus \{0\}} \in \mathcal{R}_W\}|. \tag{7.8}$$

Therefore,

$$\frac{\mu_W(\eta_{W \setminus \{0\}} = \xi)}{\mu_{W \setminus \{0\}}(\eta_{W \setminus \{0\}} = \xi)} \leq \frac{|\mathcal{R}_{W \setminus \{0\}}|}{|\mathcal{R}_W|} 2d \leq C, \tag{7.9}$$

where, by (7.6), $C > 0$ does not depend on ξ or on W . From this estimate, it follows that

$$\frac{\mu_W(\mathcal{T}_W(\eta_{W \setminus \{0\}}) \not\subseteq V)}{\mu_{W \setminus \{0\}}(\mathcal{T}_W(\eta_{W \setminus \{0\}}) \not\subseteq V)} \leq C. \tag{7.10}$$

For a more convenient notation, we let $\nu_W^{(0)}$ denote the probability measure assigning equal mass to each spanning tree of \widehat{W} , or alternatively, to each two-component spanning tree of \tilde{W} . We can view $\nu_W^{(0)}$ as a measure on $\{0, 1\}^{\mathbb{E}^d}$ in a natural way, where \mathbb{E}^d is the set of edges of \mathbb{Z}^d . By the Majumdar–Dhar correspondence [22], $\nu_W^{(0)}$ corresponds with the measure $\mu_{W \setminus \{0\}}$, and the law of \mathcal{T}_W under $\mu_{W \setminus \{0\}}$ is that of the component of 0 under $\nu_W^{(0)}$. We keep the notation \mathcal{T}_W when referring to $\nu_W^{(0)}$.

We are ready to present the proof of Proposition 6.3 based on the proposition below, whose proof is given in Sect. 8.

- Proposition 7.11** (i) For any $d \geq 1$, the limit $\lim_W \nu_W^{(0)} = \nu^{(0)}$ exists.
 (ii) Assume $d \geq 3$. Denote the component of 0 under $\nu^{(0)}$ by \mathcal{T} . Then $\nu^{(0)}(|\mathcal{T}| < \infty) = 1$.

Proof of Proposition 6.3 (assuming Proposition 7.11) By Proposition 7.11 (i), we have

$$\lim_{W \geq V} \mu_{W \setminus \{0\}}(\mathcal{T}_W(\eta_{W \setminus \{0\}}) \not\subseteq V) = \lim_{W \geq V} \nu_W^{(0)}(\mathcal{T}_W \not\subseteq V) = \nu^{(0)}(\mathcal{T} \not\subseteq V). \tag{7.12}$$

By Proposition 7.11 (ii), the right hand side of (7.12) goes to zero as $V \uparrow \mathbb{Z}^d$, and together with (7.10) and (7.4), we obtain the $i = 1$ case of (6.4).

Finiteness of the other waves follows similarly. For $k \geq 2$ we have by (7.10)

$$\begin{aligned} \mu_W(\Xi_W^k(\eta) \not\subseteq V) &\leq \mu_W(\mathcal{T}_W(S_W^{k-1}\eta) \not\subseteq V) \\ &\leq C\mu_{W \setminus \{0\}}(\mathcal{T}_W(S_W^{k-1}\eta) \not\subseteq V) \\ &= C\mu_{W \setminus \{0\}}(\mathcal{T}_W(\eta_{W \setminus \{0\}}) \not\subseteq V), \end{aligned} \tag{7.13}$$

where the last step follows by invariance of $\mu_{W \setminus \{0\}}$ under $\prod_{j \sim 0} a_{j, W \setminus \{0\}}$. We have already seen in (7.12) that the right hand side of (7.13) goes to zero as $W, V \uparrow \mathbb{Z}^d$, which completes the proof of Proposition 6.3.

8 Finiteness of two-component spanning trees

In this section, we complete the arguments for finiteness of avalanches by proving Proposition 7.11, which amounts to showing that the weak limit of \mathcal{T}_W as $W \uparrow \mathbb{Z}^d$ is almost surely finite. For this, we briefly review below some results on the uniform spanning forest; see [3, 17] for more background.

The main statement of Proposition 7.11 is part (ii). It can be deduced from a well-known theorem, namely that all trees in the uniform spanning forest in \mathbb{Z}^d have a single end. This fact has been known for more general graphs than \mathbb{Z}^d , see [3, Theorem 10.1]. Russell Lyons informed us (private communication) that he, Ben Morris and Oded Schramm have independently proved a more general version of statement (ii) in the context of giving a new and more widely applicable proof of the single end theorem; see [16] and [17].

For finite $W \subseteq \mathbb{Z}^d$, let ν_W denote the probability measure assigning equal weight to each spanning tree of \widehat{W} . ν_W is known as the uniform spanning tree measure in W with wired boundary conditions (UST). We use the algorithm below, due to Wilson [29], to analyze random samples from $\nu_W^{(0)}$ and ν_W .

Let G be a finite connected graph. By simple random walk on G we mean the random walk which at each step jumps to a random neighbour, chosen uniformly. For a path $\pi = [\pi_0, \dots, \pi_m]$ on G , define the loop-erasure of π , denoted $\text{LE}(\pi)$, as the path obtained by erasing loops chronologically from π .

Wilson's algorithm. Pick a vertex $r \in G$, called the root. Enumerate the vertices of G as x_1, \dots, x_k . Let $(S_n^{(i)})_{n \geq 1}, 1 \leq i \leq k$ be independent simple random walks started at x_1, \dots, x_k , respectively. Let

$$T^{(1)} = \min\{n \geq 0 : S_n^{(1)} = r\},$$

and set

$$\gamma^{(1)} = \text{LE}(S^{(1)}[0, T^{(1)}]).$$

Now recursively define $T^{(i)}, \gamma^{(i)}, i = 2, \dots, k$ as follows. Let

$$T^{(i)} = \min\{n \geq 0 : S_n^{(i)} \in \cup_{1 \leq j < i} \gamma^{(j)}\},$$

and

$$\gamma^{(i)} = \text{LE}(S^{(i)}[0, T^{(i)}]).$$

(If $x_i \in \cup_{1 \leq j < i} \gamma^{(j)}$, then $\gamma^{(i)}$ is the single point x_i .) Let $T = \cup_{1 \leq i \leq k} \gamma^{(i)}$. Then T is a spanning tree of G and is uniformly distributed [29].

Applying the algorithm with $G = \tilde{W}$ and root δ_W gives a sample from ν_W . Similarly, applying the method with $G = \hat{W}$ and root $\delta_{\hat{W}}$ we get a sample from $\nu_W^{(0)}$. It will be convenient to think of the latter construction also taking place in \tilde{W} , via the one-to-one correspondence between the edges of \hat{W} and \tilde{W} . Note that under this correspondence, a path in \hat{W} that does not use $\delta_{\hat{W}}$ as an internal vertex, maps to a path in \tilde{W} . Hence the two-component spanning tree in \tilde{W} can be built from loop-erased random walks by regarding $\{0, \delta_W\}$ as the ‘‘root’’. In other words, the walks attach either to a piece growing from 0, or to a piece growing from δ_W , and these two growing pieces yield the two components.

One can extend Wilson’s algorithm to infinite graphs G if random walk on G is transient [3]. In this case, one chooses the root to be ‘‘at infinity’’, and note that loop-erasure makes sense for infinite paths that visit each site finitely many times.

The measures ν_W can be realized on the same sample space, $\{0, 1\}^{\mathbb{Z}^d}$, as $\nu_W^{(0)}$ introduced earlier. It is well known that ν_W has a weak limit ν as $W \uparrow \mathbb{Z}^d$, called the (wired) uniform spanning forest (USF) on \mathbb{Z}^d [3,24]. When $d \geq 3$, the USF can be constructed directly by Wilson’s method in \mathbb{Z}^d , rooted at infinity [3, Theorem 5.1].

We write ω for the random set of edges present under ν , that is, we identify $\omega \in \{0, 1\}^{\mathbb{Z}^d}$ with the set of edges e for which $\omega(e) = 1$. This allows us to view ω as a (random) subgraph of \mathbb{Z}^d . We say that an infinite tree T has one end, if there are no two disjoint infinite paths in T . It is known that

$$\begin{aligned} \nu(\text{all components of } \omega \text{ are infinite trees}) &= 1, \\ \nu(\text{each component of } \omega \text{ has one end}) &= 1; \end{aligned}$$

see [3,16].

Proof of Proposition 7.11 Denote the random set of edges present in the two-component spanning tree of \tilde{W} by ω_W . Let W_n be an increasing sequence of finite volumes exhausting \mathbb{Z}^d . If B is a finite set of edges, [3, Corollary 4.3] implies that $\nu_{W_n}^{(0)}(B \subseteq \omega_{W_n})$ is increasing in n . This is sufficient to imply the weak convergence $\lim_W \nu_W^{(0)} = \nu^{(0)}$, and the limiting spanning forest ω is uniquely determined by the conditions

$$\nu^{(0)}(B \subseteq \omega) = \lim_{n \rightarrow \infty} \nu_{W_n}^{(0)}(B \subseteq \omega_{W_n}),$$

as B varies over finite edge-sets (see the discussion in [3, Sect. 5]). This proves part (i) of the proposition.

For part (ii), assume $d \geq 3$. The configuration under $\nu^{(0)}$ can be constructed by Wilson’s method directly, by [3, Theorem 5.1]. Since here 0 is part of the boundary, the simple random walks in this construction are either killed when they hit the component growing from 0, or they attach to a component growing from infinity.

Assume now that $\nu^{(0)}(|\mathcal{T}| = \infty) = c_1 > 0$, and we reach a contradiction. We consider the construction of the configuration under $\nu^{(0)}$ via Wilson’s algorithm. Suppose that the first random walk, call it $S^{(1)}$, starts from $x \neq 0$. Write $x \leftrightarrow y$ to denote that x and y are in the same component. Then we have

$$\nu^{(0)}(x \not\leftrightarrow 0) = \Pr(S^{(1)} \text{ does not hit } 0) = 1 - \frac{G(x, 0)}{G(0, 0)} \rightarrow 1 \text{ as } |x| \rightarrow \infty.$$

In particular, there exists an $x \in \mathbb{Z}^d$, such that

$$\nu^{(0)}(|\mathcal{T}| = \infty, x \not\leftrightarrow 0) \geq c_1/2. \tag{8.1}$$

Fix such an x . Let $B(x, n)$ denote the box of radius n centered at x . Fix n_0 such that $0 \in B(x, n_0)$, and 0 is not a boundary point of $B(x, n_0)$. By inclusion of events, (8.1) implies

$$\nu^{(0)}(0 \leftrightarrow \partial B(x, n), x \not\leftrightarrow 0) \geq c_1/2 \tag{8.2}$$

for all $n \geq n_0$. For fixed $n \geq n_0$, let $y_1 = x$, and let y_2, \dots, y_K be an enumeration of the sites of $\partial B(x, n)$. We use Wilson algorithm with this enumeration of sites. Let $S^{(i)}$ and $T^{(i)}$ denote the i th random walk and the corresponding hitting time determined by the algorithm. We use these random walks to analyze the configuration under both $\nu^{(0)}$ and ν .

The event on the left hand side of (8.2) can be recast as

$$\{T^{(1)} = \infty, \exists 2 \leq j \leq K \text{ such that } T^{(j)} < \infty, S_{T^{(j)}}^{(j)} = 0\}, \tag{8.3}$$

and hence this event has probability at least $c_1/2$. On the above event, there is a first index $N, 2 \leq N \leq K$, such that the walk $S^{(N)}$ hits $B(x, n_0)$ at some random time σ , where $\sigma < T^{(N)}$. Let A be the latter event. Since A contains the event in (8.3), A also has probability at least $c_1/2$. Let $p = p(x, n_0)$ denote the minimum over $z \in \partial B(x, n_0)$ of the probability that a random walk started at z hits x before 0 without exiting $B(x, n_0)$. Clearly, $p > 0$, and is independent of n .

Let B denote the subevent of A on which after time σ , the walk $S^{(N)}$ hits the loop-erasure of $S^{(1)}$ before hitting 0 (and without exiting $B(x, n_0)$). We have $\Pr(B | A) \geq p$. Now we regard the random walks as generating ν . By the definition of N , on the event $A \cap B$, the hitting times $T^{(1)}, \dots, T^{(N)}$, have the same values as in the construction for $\nu^{(0)}$, since the walks do not hit 0. Moreover, on $A \cap B$, the tree containing x has two disjoint paths from $\partial B(x, n_0)$ to $\partial B(x, n)$: one is part of the infinite path generated by $S^{(1)}$, the other part of the path generated by $S^{(N)}$. Therefore, the probability of the

existence of two such paths is at least $p(c_1/2)$, for all $n \geq n_0$. However, this probability should go to zero as $n \rightarrow \infty$, because under ν , each tree has one end almost surely. This is a contradiction, proving part (ii) of the proposition.

9 Tail triviality of μ

In this section we study ergodic properties of μ . In Sect. 10 we are going to use the $d \geq 3$ part of the following theorem.

Theorem 9.1 *The measure μ is tail trivial for any $d \geq 2$.*

Our proof of Theorem 9.1 is divided into two parts. The argument in the case $2 \leq d \leq 4$ is quite simple, and is given in Sect. 9.1. The case $d > 4$ is quite involved, and is given in Sects. 9.2 and 9.3.

9.1 The case $2 \leq d \leq 4$

Proof of Theorem 9.1 [Case $2 \leq d \leq 4$] The proof is based on the fact that the uniform spanning forest measure ν is tail trivial [3, Theorem 8.3]. Let $\mathcal{X} \subseteq \{0, 1\}^{\mathbb{Z}^d}$ denote the set of spanning trees of \mathbb{Z}^d with one end. Recall the uniform spanning forest measure ν from Sect. 8. It was shown by Pemantle [24] that when $2 \leq d \leq 4$, the measure ν is concentrated on \mathcal{X} . We can regard any $\omega \in \mathcal{X}$ as a tree “rooted at infinity”, that is, we call x an ancestor of y if and only if x lies on the unique path from y to infinity.

It is shown in [1] that there is a mapping $\psi : \mathcal{X} \rightarrow \Omega$ such that μ is the image of ν under ψ . Moreover, ψ has the following property. Let $T_x = T_x(\omega)$ denote the tree consisting of all ancestors of x and its $2d$ neighbours in ω . In other words, T_x is the union of the paths leading from x and its neighbours to infinity. It follows from the results in [1] that $\eta_x = (\psi(\omega))_x$ is a function of T_x alone.

Assume that $f(\eta)$ is a bounded tail measurable function. Then for any n , f is a function of $\{\eta_x : \|x\|_\infty \geq n\}$ only. This means that $f(\eta) = f(\psi(\omega)) = g(\omega)$ is a function of the family $\{T_x(\omega) : \|x\|_\infty \geq n\}$. Let $\mathcal{F}_k = \sigma(\omega_e : e \cap [-k, k]^d = \emptyset)$. For $1 \leq k < n$ consider the event

$$E_{n,k} = \bigcap_{x:\|x\|_\infty \geq n} \{T_x \cap [-k, k]^d = \emptyset\}.$$

Observe that $E_{n,k} \in \mathcal{F}_k$, and $gI[E_{n,k}]$ is \mathcal{F}_k -measurable. Using that ω has a single end ν -a.s., it is not hard to check that for any $k \geq 1$

$$\lim_{n \rightarrow \infty} \nu(E_{n,k}) = 1.$$

Letting $n \rightarrow \infty$, this implies that there is an \mathcal{F}_k -measurable function \hat{g}_k , such that $g = \hat{g}_k \nu$ -a.s. Since this holds for any $k \geq 1$, tail triviality of ν implies that g is constant

ν -a.s. Letting c denote the constant, this implies

$$\mu(f(\eta) = c) = \nu(f(\psi(\omega)) = c) = 1,$$

which completes the proof in the case $2 \leq d \leq 4$. □

9.2 Coding of the sandpile in the case $d > 4$

The simple proof in Sect. 9.1 does not work when $d > 4$, due to the fact that the coding of the sandpile configuration by the USF breaks down. Nevertheless, it turns out that a coding is possible if we add extra randomness to the USF, namely, a random ordering of its components. Due to the presence of this random ordering, however, we have not been able to deduce tail triviality of μ directly from tail triviality of ν , and we need a separate argument.

We start by recalling results from [1]. Let \mathcal{X} denote the set of spanning forests of \mathbb{Z}^d with infinitely many components, where each component is infinite and has a single end. The USF measure ν is concentrated on \mathcal{X} [3]. Given $x \in \mathbb{Z}^d$ and $\omega \in \mathcal{X}$, let $T_x^{(1)}(\omega), \dots, T_x^{(k)}(\omega)$ denote the trees consisting of all ancestors of x and its $2d$ neighbours in ω . Here $k = k_x(\omega) \geq 1$. Each $T_x^{(i)}$ is a union of infinite paths starting at x or a neighbour of x , and has a unique vertex $v_x^{(i)}$ where these paths “first meet”. In other words, $v_x^{(i)}$ is the first vertex that is common to all of the paths. Let $F_x^{(i)}(\omega)$ denote the finite tree consisting of all descendants of $v_x^{(i)}$ in $T_x^{(i)}(\omega)$. Let \mathcal{F} denote the collection of finite rooted trees in \mathbb{Z}^d . Let Σ_l denote the set of permutations of the symbols $\{1, \dots, l\}$.

The sandpile height at x is a function of $\{F_x^{(i)}(\omega), v_x^{(i)}(\omega)\}_{i=1}^k$ and a random $\sigma_x \in \Sigma_k$, in the following sense.

Lemma 9.2 *There are functions $\psi_l : \mathcal{F}^l \times \Sigma_l, l = 1, 2, \dots$ such that if σ_x is a uniform random element of Σ_k , given ω , then*

$$\eta_x = \psi_{k_x}((F_x^{(1)}, v_x^{(1)}), \dots, (F_x^{(k)}, v_x^{(k)}), \sigma_x) \tag{9.3}$$

has the distribution of the height variable at x under μ .

Proof This follows from the proofs of Lemma 3 and Theorem 1 in [1].

Remark 9.4 Here it is convenient to think of σ_x as a random ordering of those components of ω that contain at least one neighbour of x . Then one can also view η_x as a function of $\{T_x^{(i)}\}_{i=1}^k$ and σ_x .

Next we turn to a description of the joint distribution of $\{\eta_x\}_{x \in A_0}$ for finite $A_0 \subseteq \mathbb{Z}^d$. Let $A = A_0 \cup \partial_e A_0$. Let $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(K)}, K = K_A(\omega)$, denote the components of the USF intersecting A . Each $\mathcal{C}^{(i)}$ contains a unique vertex $v_A^{(i)}$ where the paths from $A \cap \mathcal{C}^{(i)}$ to infinity first meet. Let $F_A^{(i)}$ denote the finite tree consisting in the portion of these paths up to $v_A^{(i)}$. In other words, $F_A^{(i)}$ is the union of the paths from $A \cap \mathcal{C}^{(i)}$

to $v_A^{(i)}$. Each rooted tree $(F_x^{(j)}, v_x^{(j)})$, $x \in A_0$, $1 \leq j \leq k_x$ is a subtree of some $F_A^{(i)}$, $1 \leq i \leq K$ and the former are determined by the latter. Let $\sigma_A \in \Sigma_K$ be uniformly distributed, given ω . For each $x \in A_0$, σ_A induces a permutation in Σ_{k_x} , by restriction. Then the lemma below follows from the results in [1].

Lemma 9.5 *The height configuration in A_0 is a function of $\{(F_A^{(i)}, v_A^{(i)})\}_{i=1}^K$ and σ_K . Moreover, the joint distribution of $\{\sigma_x\}_{x \in A_0}$ is the one induced by σ_A . \square*

From the above, we obtain the following description of μ in terms of the USF and a random ordering of its components. Let $\omega \in \mathcal{X}$ be distributed according to ν . Given ω , we define a random partial ordering $<_\omega$ on \mathbb{Z}^d in the following way. Let $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots$ be an enumeration of the components of ω , and let U_1, U_2, \dots be i.i.d. random variables, given ω , having the uniform distribution on $[0, 1]$. Define the random partial order $<_\omega$ depending on ω and $\{U_i\}_{i \geq 1}$ by letting $x <_\omega y$ if and only if $x \in \mathcal{C}^{(i)}$, $y \in \mathcal{C}^{(j)}$ and $U_i < U_j$. Even though $<_\omega$ is defined for sites, it is simply an ordering of the components of ω . The distribution of $<_\omega$ is in fact uniquely characterized by the property that it induces the uniform permutation on any finite set of components, and one could define it by this property, without reference to the U 's. This in turn shows that the distribution is independent of the ordering $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots$ initially chosen.

Let $\mathcal{Q} = \{0, 1\}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ denote the space of binary relations (where for $q \in \mathcal{Q}$ we interpret $q(x, y) = 1$ as $x < y$, and $q(x, y) = 0$ otherwise). We denote the joint law of $(\omega, <)$ on $\mathcal{X} \times \mathcal{Q}$ by $\tilde{\nu}$. From the couple $(\omega, <)$, we can recover the random permutations σ_x as follows. If $v_x^{(1)}, \dots, v_x^{(k)}$ are as defined earlier, then

$$(\sigma_x(1), \dots, \sigma_x(k)) = (j_1, \dots, j_k) \text{ if and only if } v_x^{(j_1)} <_\omega \dots <_\omega v_x^{(j_k)}. \tag{9.6}$$

The discussion above, and Lemma 9.5 easily implies the following lemma.

Lemma 9.7 *Suppose that $(\omega, <_\omega)$ has distribution $\tilde{\nu}$. Let η_x be given by (9.3), where σ_x is defined by (9.6). Then $\{\eta_x\}_{x \in \mathbb{Z}^d}$ has distribution μ . In particular, there is a $\tilde{\nu}$ -a.s. defined function $\psi : \mathcal{X} \times \mathcal{Q} \rightarrow \Omega$ such that μ is the image of $\tilde{\nu}$ under ψ . \square*

Before we start the argument proper, we need to recall some further terminology from [1]. Given finite rooted trees $(\bar{F}, \bar{v}) = (F_i, v_i)_{i=1}^K$ and a finite set of sites A , define the events

$$D(\bar{v}) = \{v_1, \dots, v_K \text{ are in distinct components of } \omega\},$$

$$B(\bar{F}, \bar{v}) = D(\bar{v}) \cap \{F_A^{(i)} = F_i, v_A^{(i)} = v_i \text{ for } 1 \leq i \leq K\},$$

We also need versions of these events for finite $\Lambda \subseteq \mathbb{Z}^d$. The wired UST ω_Λ in volume Λ can be viewed as the union of one or more components ($x, y \in \Lambda$ are in the same component if they are connected without using the special vertex δ_Λ). Let $\mathcal{C}_\Lambda^{(1)}, \dots, \mathcal{C}_\Lambda^{(K)}$ be the list of components intersecting A . We define $v_{A,\Lambda}^{(i)}$ and $F_{A,\Lambda}^{(i)}$

analogously to the infinite volume case, this time using the components $C_\Lambda^{(i)}$. Now we define

$$D_\Lambda(\bar{v}) = \{v_1, \dots, v_K \text{ are in distinct components of } \omega_\Lambda\},$$

$$B_\Lambda(\bar{F}, \bar{v}) = D_\Lambda(\bar{v}) \cap \{F_{A,\Lambda}^{(i)} = F_i, v_{A,\Lambda}^{(i)} = v_i \text{ for } 1 \leq i \leq K\}.$$

9.3 Proof in the case $d > 4$

9.3.1 Outline of the proof

Recall that tail triviality is equivalent to the following [8, Proposition 7.9]. For any cylinder event E' and $\varepsilon > 0$ there exists n such that (with $V_n = [-n, n]^d \cap \mathbb{Z}^d$) for any event $R' \in \mathcal{F}_{V_n^c}$ we have

$$|\mu(E' \cap R') - \mu(E')\mu(R')| \leq \varepsilon. \tag{9.8}$$

Let $E = \psi^{-1}(E')$ and $R = \psi^{-1}(R')$, where ψ is as in Lemma 9.7. Suppose that E' depends on the sites in the finite set A_0 , and put $A = A_0 \cup \partial_e A_0$.

For the proof, we want to show that E and R “decouple”, if n is sufficiently large. We try to achieve this by showing that they can be approximated by events that depend on portions of ω that are spatially separated. The main difficulty is that dependence between E and R also exists due to the ordering $<$, and it requires work to show that the dependence on $<$ also decouples. Below we give a rough outline of strategy for this.

By Lemma 9.5, the occurrence or not of the event E is determined by a collection of finite tree subgraphs $(F_A^{(i)}, v_A^{(i)})$ of ω , and an ordering of these trees. We get an approximation of the event E , if we consider the contribution of only those configurations for which $F_A^{(i)} \subseteq V_r, 1 \leq i \leq K$ for some large r . Suppose that we have also approximated R by an event that depends on the restriction of ω to V_m^c , where $r < m < n$. Condition on the restriction of ω to V_m^c , and also on the restriction of $<$ to this portion of ω . In particular, for any $w_1, w_2 \in \partial V_m$, our conditioning specifies whether $w_1 < w_2$ or not.

For fixed $1 \leq i < j \leq K$, let π_i and π_j denote the paths in ω from $v_A^{(i)}$ and $v_A^{(j)}$, respectively, to ∂V_m . Suppose π_i and π_j end in vertices $w(i)$ and $w(j)$, respectively. The conditional probability of $\{v_A^{(i)} < v_A^{(j)}\}$ is determined by the conditional probability of $w(i) < w(j)$. If $r \ll m$, then due to fluctuations in the behaviour of the paths π_i and π_j , the conditional probability of the events $\{w(i) < w(j)\}$ and $\{w(j) < w(i)\}$ will be approximately equal, and we obtain the desired decoupling.

In the next section, we specify suitable approximations of E and R .

9.3.2 Approximating E and R

We first have a closer look at the event E . We define

$$S(\bar{F}, \bar{v}, \sigma) = B(\bar{F}, \bar{v}) \cap \{v_{\sigma(1)} < \dots < v_{\sigma(K)}\},$$

$$\mathcal{G}_E = \{(\bar{F}, \bar{v}, \sigma) : S(\bar{F}, \bar{v}, \sigma) \subseteq E\},$$

$$\mathcal{G}_E(r) = \{(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E : F_i \subseteq V_r \text{ for } 1 \leq i \leq K\}.$$

The event E is a disjoint union of $S(\bar{F}, \bar{v}, \sigma)$ over $(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E$. By Lemma 9.7, we have

$$\mu(E') = \tilde{\nu}(E) = \sum_{(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E} \frac{1}{K!} \nu(B(\bar{F}, \bar{v})). \tag{9.9}$$

We also define an analogue of S in a finite volume Λ . Assume that the relation \prec_∂ is prescribed on the exterior boundary of Λ . For any realization of the wired UST ω_Λ there is a unique extension of \prec_∂ into Λ , denoted \prec_Λ , where $x \prec_\Lambda y$ if and only if they are connected (in ω_Λ) to boundary vertices $w(x)$ and $w(y)$ satisfying $w(x) \prec_\partial w(y)$. Using this extension, we define

$$S_\Lambda(\bar{F}, \bar{v}, \sigma) = B_\Lambda(\bar{F}, \bar{v}) \cap \{v_{\sigma(1)} \prec_\Lambda \dots \prec_\Lambda v_{\sigma(K)}\}.$$

We let $\tilde{\nu}_{\Lambda, \prec_\partial}$ denote the law of $(\omega_\Lambda, \prec_\Lambda)$ with boundary condition \prec_∂ .
 Introduce

$$G = G(r) = \{F_A^{(i)} \subseteq V_r \text{ for } 1 \leq i \leq K\},$$

where we assume that $A_0 \subseteq V_r \subseteq V_n$. Now $E \cap G$ is a disjoint union of the events $S(\bar{F}, \bar{v}, \sigma)$ over $(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E(r)$. Since A is fixed, there exists $r_0(\varepsilon)$ such that for $r \geq r_0(\varepsilon)$ we have $\nu(G(r)^c) \leq \varepsilon$. The event $E \cap G(r)$ will serve as an approximation for E .

Turning to R , we define

$$\mathcal{H} = \mathcal{H}_n = \bigcup_{x \in V_n^c} \bigcup_{i=1}^{k_x} \text{vertex set of } T_x^{(i)}$$

$$\mathcal{D} = \mathcal{D}_n = \mathbb{Z}^d \setminus \mathcal{H}_n.$$

The occurrence of R is determined by the collection of edges joining vertices in \mathcal{H} together with the restriction of \prec to \mathcal{H} . We also introduce for $r < m < n$ and $V_m \subseteq \Lambda \subseteq V_n$ the events

$$F = F(n, m) = \{\mathcal{H}_n \cap V_m = \emptyset\} \quad \text{and} \quad F_\Lambda = F_\Lambda(n, m) = \{\mathcal{D}_n = \Lambda\}.$$

Here F is the event that the portion of ω determining the sandpile configuration in V_n^c does not intersect V_m . The event $R \cap F$ will serve as an approximation of R , as mentioned in Sect. 9.3.1. However, we will further decompose F as the disjoint union $F = \bigcup_\Lambda F_\Lambda$. The reason is that the conditional law of ν inside \mathcal{D} , given F_Λ is simple: it is ν_Λ (see Sect. 9.3.3 below).

The value of m will be chosen large with respect to r . It is easy to see that there exists $n_0(m, \varepsilon)$, such that if $n \geq n_0(m, \varepsilon)$ then $v(F^c) \leq \varepsilon$. This is because $F(n, m)$ is monotone increasing in n , and $\bigcap_{n=m+1}^\infty F(n, m)^c = \emptyset$, since each component of the USF has a single end.

For technical reasons, we will in fact need a further subevent of F , on which, given the configuration in \mathcal{H} , with high conditional probability:

$$\text{for all } x, y \in V_r \quad x \leftrightarrow y \text{ implies } x \leftrightarrow y \text{ inside } V_m.$$

Let

$$J = \{\forall x, y \in V_r : \text{if } x \leftrightarrow y \text{ then } x \leftrightarrow y \text{ inside } V_m\}.$$

There exists $m_0(r, \varepsilon)$, such that if $m \geq m_0(r, \varepsilon)$, then $v(J^c) \leq \varepsilon \varepsilon_1$, where we have set $\varepsilon_1 = \varepsilon_1(r) = \varepsilon/|\mathcal{G}_E(r)|$. Define the event

$$J_0 = F \cap \{v(J^c \mid \omega_{\mathcal{H}_n}) \leq \varepsilon_1\},$$

where $\omega_{\mathcal{H}_n}$ denotes the configuration on the set of edges touching \mathcal{H}_n . By Markov's inequality,

$$v(J_0^c) \leq v(F^c) + v(F \cap \{v(J^c \mid \omega_{\mathcal{H}_n}) \geq \varepsilon_1\}) \leq \varepsilon + \frac{v(J^c)}{\varepsilon_1} \leq 2\varepsilon.$$

Summarizing the above, if $r \geq r_0$, $m \geq m_0(r, \varepsilon)$ and $n \geq n_0(m, \varepsilon)$, we have

$$|\mu(E' \cap R') - \tilde{v}(E \cap G \cap R \cap J_0)| \leq 3\varepsilon. \tag{9.10}$$

In the next section, we obtain a decomposition of the event $E \cap G \cap R \cap J_0$, that allows us to analyze it via Wilson's method.

9.3.3 Decomposition of $E \cap G \cap R \cap J_0$

We are going to regard the edges of ω being directed towards infinity. By the definition of \mathcal{H} , there are no directed edges from \mathcal{H} to \mathcal{D} . Therefore, given the restriction of ω to \mathcal{H} , the conditional law of ω in \mathcal{D} is that of the wired uniform spanning tree in \mathcal{D} , that is ν_D . One can see this by using Wilson's method rooted at infinity to first generate \mathcal{H} and the configuration on \mathcal{H} , and then the configuration in \mathcal{D} .

Note that the event F_Λ only depends on the portion of ω outside Λ . We want to rewrite the second term on the left hand side of (9.10) by conditioning on F_Λ , the portion of ω outside Λ , and the restriction of \prec to $\mathbb{Z}^d \setminus \Lambda$. By the previous paragraph, the conditional distribution of (ω, \prec) inside Λ is given by $\tilde{v}_{\Lambda, \prec_\partial}$, where \prec_∂ is determined by the conditioning.

The above implies

$$\tilde{v}(E \cap G \cap R \cap J_0) = \sum_{V_m \subseteq \Lambda \subseteq V_n R \cap J_0 \cap F_\Lambda} \int \tilde{v}_{\Lambda, <_\partial}(E \cap G) d\tilde{v}. \tag{9.11}$$

Since the integration in (9.11) is over a subset of J_0 , in what follows, we assume that the boundary condition $<_\partial$ is compatible with the event J_0 , in the sense that it arises from a configuration belonging to J_0 . The expression $\tilde{v}_{\Lambda, <_\partial}(E \cap G)$ can be further decomposed as follows:

$$\tilde{v}_{\Lambda, <_\partial}(E \cap G) = \sum_{(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E(r)} \tilde{v}_{\Lambda, <_\partial}(S_\Lambda(\bar{F}, \bar{v}, \sigma)). \tag{9.12}$$

In the remainder of the proof our aim is to show that the summand in (9.12) is close to $v(B(\bar{F}, \bar{v}))/K!$, uniformly in Λ and the boundary condition, if m is large enough. In the next section, we formulate precisely the statement we need as Lemma 9.13, and prove the theorem given Lemma 9.13. Finally, in Sect. 9.3.5, we complete the argument by proving Lemma 9.13.

9.3.4 Decoupling lemma and proof of theorem

Lemma 9.13 *There exists a universal constant C and $m_1(r, \varepsilon)$, such that for any $(\bar{F}, \bar{v}, \sigma) \in \mathcal{G}_E(r)$, $m \geq \max\{m_0(r, \varepsilon), m_1(r, \varepsilon)\}$, $V_m \subseteq \Lambda$ and any boundary condition $<_\partial$ compatible with J_0 , we have*

$$\left| \tilde{v}_{\Lambda, <_\partial}(S_\Lambda(\bar{F}, \bar{v}, \sigma)) - \frac{v(B(\bar{F}, \bar{v}))}{K!} \right| \leq C\varepsilon_1 = C \frac{\varepsilon}{|\mathcal{G}_E(r)|}. \tag{9.14}$$

Proof of Theorem 9.1 [Case $d > 4$] assuming Lemma 9.13 Given $\varepsilon > 0$ let $r \geq r_0(\varepsilon)$, $m \geq \max\{m_0(r, \varepsilon), m_1(r, \varepsilon)\}$ and $n \geq n_0(m, \varepsilon)$. Then the estimate in (9.14), formula (9.9) and (9.12) imply

$$|\tilde{v}_{\Lambda, <_\partial}(E \cap G) - \tilde{v}(E \cap G)| \leq C\varepsilon.$$

Substituting this into (9.11), and performing the integral and the sum, we get

$$|\tilde{v}(E \cap G \cap R \cap J_0) - \tilde{v}(E \cap G)\tilde{v}(R \cap J_0)| \leq C\varepsilon.$$

Due to $r \geq r_0(\varepsilon)$ and $n \geq n_0(m, \varepsilon)$, we have $\tilde{v}(G^c) \leq \varepsilon$ and $\tilde{v}(J_0) \leq 2\varepsilon$, which yields

$$|\mu(E' \cap R') - \mu(E')\mu(R')| \leq C'\varepsilon,$$

with a universal constant C' . This proves Theorem 9.1 in the case $d > 4$. □

9.3.5 Proof of decoupling lemma

We prove Lemma 9.13 by analyzing the event $B_\Lambda(\bar{F}, \bar{v})$ in terms of Wilson’s algorithm. The proof is similar to the proof of Lemma 3 in [1], however it does not seem possible to use that result directly. Before starting the proof proper, we introduce some notation.

Fix $(F_i, v_i)_{i=1}^K$ and $\sigma \in \Sigma_K$. Let $A = \{y_1, \dots, y_{|A|}\}$. We apply Wilson’s method to generate part of the wired UST in Λ with the following enumeration of sites:

$$v_1, \dots, v_K, \quad y_1, \dots, y_{|A|}.$$

Let $S^{(i)}, i = 1, \dots, K$ be independent simple random walks started at v_i . Let $\gamma_\Lambda^{(i)}$ denote the loop-erasure of $S^{(i)}$ up to its first exit time from Λ . We define a random walk event C_Λ whose occurrence will be equivalent to the occurrence of $B_\Lambda(\bar{F}, \bar{v})$, by Wilson’s method. Since the event $D_\Lambda(\bar{v})$ has to occur, we require that for $i = 1, \dots, K$, $S^{(i)}$ upto its first exit time be disjoint from $\cup_{1 \leq j < i} \gamma_\Lambda^{(j)}$. In addition, the fact that $B_\Lambda(\bar{F}, \bar{v})$ has to occur, gives conditions on the paths starting at $y_1, \dots, y_{|A|}$, namely, these paths have to realize the events $(F_A^{(i)}, v_A^{(i)}) = (F_i, v_i)$, given the paths $\{\gamma_\Lambda^{(i)}\}_{i=1}^K$. These implicit conditions define C_Λ . More precisely, the loop-erased walk η_1 started at y_1 has to equal the path in $\cup_i F_i$ from y_1 to $\{v_1, \dots, v_K\}$. The loop-erased walk η_2 started at y_2 has to equal the path in $\cup_i F_i$ from y_2 to $\{v_1, \dots, v_K\} \cup \eta_1$, and so on.

We write \Pr for probabilities associated with random walk events, and we couple the constructions in different volumes by using the same infinite random walks $S^{(i)}$. We also define the random walk event C , corresponding to $B(\bar{F}, \bar{v})$, analogously to the finite volume case.

Proof of Lemma 9.13 Let $W_\Lambda^{(i)}$ denote the first vertex $S^{(i)}$ visits in $\mathbb{Z}^d \setminus \Lambda$. Then we have

$$\tilde{v}_{\Lambda, < \vartheta}(S_\Lambda(\bar{F}, \bar{v}, \sigma)) = \Pr(C_\Lambda, W_\Lambda^{(\sigma(1))} < \dots < W_\Lambda^{(\sigma(K))}). \tag{9.15}$$

For $r < l < m$ we consider the event C_{V_l} , and write C_l for short. It is not hard to see that $\lim_\Lambda I[C_\Lambda] = I[C]$, Pr-a.s., which implies that for l large enough, $\Pr(C_l \Delta C) \leq \varepsilon_1$. (Here Δ denotes symmetric difference.) Hence the difference between the right hand side of (9.15) and

$$\Pr(C_l, W_\Lambda^{(\sigma(1))} < \dots < W_\Lambda^{(\sigma(K))}) \tag{9.16}$$

is at most $2\varepsilon_1$. Recall that $\Lambda \supseteq V_m$, and $m > l$. By conditioning on the first exit points from V_l , (9.16) can be written as

$$\Pr(C_l) \Pr(W_\Lambda^{(\sigma(1))} < \dots < W_\Lambda^{(\sigma(K))} \mid W_l^{(1)}, \dots, W_l^{(K)}). \tag{9.17}$$

The first factor here differs from $\Pr(C) = \nu(B(\bar{F}, \bar{v}))$ by at most ε_1 . If m is large with respect to l , the value of the second factor is essentially independent of σ . This

is because the distributions of $W_\Lambda^{(i)}$ and $W_\Lambda^{(j)}$ given $W_l^{(i)}$ and $W_l^{(j)}$ (respectively), can be made arbitrarily close in total variation distance. This implies that the difference between (9.17) and

$$\Pr(C) \Pr(W_\Lambda^{(1)} < \dots < W_\Lambda^{(K)} \mid W_l^{(1)}, \dots, W_l^{(K)})$$

is at most ε_1 , if m is large enough, uniformly in Λ .

Observe that if $W_\Lambda^{(i)} \leftrightarrow W_\Lambda^{(j)}$ for some $1 \leq i < j \leq K$, then the event J^c occurs. Since the boundary condition $<_\partial$ is compatible with J_0 , we have

$$\Pr(C_\Lambda, W_\Lambda^{(i)} \leftrightarrow W_\Lambda^{(j)} \text{ for some } 1 \leq i < j \leq K) \leq \tilde{v}_{\Lambda, <_\partial}(J^c) \leq \varepsilon_1. \tag{9.18}$$

It follows that for some universal constant C , if m is large enough

$$|\Pr(C_\Lambda, W_\Lambda^{(\sigma(1))} < \dots < W_\Lambda^{(\sigma(K))}) - \Pr(C)/K!| \leq C\varepsilon_1.$$

This proves the lemma. □

10 Ergodicity of the stationary process

Arrived at this point, we can apply the results in [18], and we obtain the following.

Theorem 10.1 *Let $\varphi : \mathbb{Z}^d \rightarrow (0, \infty)$ be an addition rate such that*

$$\sum_{x \in \mathbb{Z}^d} \varphi(x)G(0, x) < \infty. \tag{10.2}$$

Then the following hold.

1. *The closure of the operator on $L_2(\mu)$ defined on local functions by*

$$L_\varphi f = \sum_{x \in \mathbb{Z}^d} \varphi(x)(a_x - I)f \tag{10.3}$$

is the generator of a stationary Markov process $\{\eta_t : t \geq 0\}$.

2. *Let $N_t^\varphi(x)$ denote Poisson processes with rate $\varphi(x)$ that are independent (for different x). The limit*

$$\eta_t = \lim_{V \uparrow \mathbb{Z}^d} \left[\prod_{x \in V} a_x^{N_t^\varphi(x)} \right] \eta \tag{10.4}$$

exists a.s. with respect to the product of the Poisson process measures on N_t^φ with the stationary measure μ on the $\eta \in \Omega$. Moreover, η_t is a cadlag version of the process with generator L_φ .

Let $\{\eta_t : t \geq 0\}$ be the stationary process with generator $L_\varphi = \sum_x \varphi(x)(a_x - I)$. We recall that a process is called ergodic if every (time-)shift invariant measurable set has measure zero or one. For a Markov process, this is equivalent to the following: if $S_t f = f$ for all $t > 0$, then f is constant μ -a.s. This in turn is equivalent to the statement that $Lf = 0$ implies f is constant μ -a.s. The tail σ -field on Ω is defined as usual:

$$\mathcal{F}_\infty = \bigcap_{n \in \mathbb{N}} \sigma\{\eta(x) : |x| \geq n\} \tag{10.5}$$

A function f is tail measurable if its value does not change by changing the configuration in a finite number of sites, that is, if

$$f(\eta) = f(\xi_V \eta_{V^c})$$

for every ξ and $V \subseteq \mathbb{Z}^d$ finite.

Theorem 10.6 *The stationary process of Theorem 10.1 is mixing.*

Proof Recall that G denotes the group generated by the unitary operators a_x on $L_2(\mu)$. Consider the following statements.

1. The process $\{\eta_t : t \geq 0\}$ is ergodic.
2. The process $\{\eta_t : t \geq 0\}$ is mixing.
3. Any G -invariant function is μ -a.s. constant.
4. μ is tail trivial.

Then we have the following implications: 1, 2 and 3 are equivalent and 4 implies 3. This will complete the proof, because 4 holds by Theorem 9.1.

It is easy to see that on $L_2(\mu)$,

$$L^* = \sum_{x \in \mathbb{Z}^d} \varphi(x)(a_x^{-1} - I). \tag{10.7}$$

Hence L and L^* commute, that is, L is a normal operator. The equivalence of 1 and 2 then follows from [28, Lemmas 6 and 7] and an adaptation to continuous time. To see the equivalence of 1 and 3: invariance of μ under a_x and a_x^{-1} implies

$$\begin{aligned} \langle Lf | f \rangle &= -\frac{1}{2} \sum_{x \in \mathbb{Z}^d} \varphi(x) \int (a_x f - f)^2 d\mu, \\ &= \langle L^* f | f \rangle \\ &= -\frac{1}{2} \sum_{x \in \mathbb{Z}^d} \varphi(x) \int (a_x^{-1} f - f)^2 d\mu. \end{aligned} \tag{10.8}$$

Hence $Lf = 0$ is equivalent to f being invariant under all a_x and a_x^{-1} , and thus under the action of G . Finally, to prove the implication $4 \Rightarrow 3$, we will show that

a function invariant under the action of G is tail measurable. Suppose $f : \Omega \rightarrow \mathbb{R}$, and $f = a_x f = a_x^{-1} f$ μ -a.s. for all $x \in \mathbb{Z}^d$. There exists a full measure G -invariant subset Ω_0 so that the restriction of f to Ω_0 is G -invariant. If η and ζ are elements of Ω_0 and differ in a finite number of coordinates, then

$$\zeta = \prod_{x \in \mathbb{Z}^d} a_x^{\zeta(x) - \eta(x)} \eta \tag{10.9}$$

and hence $f(\eta) = f(\zeta)$. This implies that f is μ -a.s. equal to a tail measurable function.

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Appendix

In this section we show how to extend the argument of [1] in the case $d > 4$ and prove $\lim_{\Lambda} \mu_{\Lambda} = \mu$. Using the notation of Sect. 9.2, let

$$\begin{aligned} X_{\Lambda,i} &= \text{dist}_{\Lambda}(v_i, \delta_{\Lambda}), \quad i = 1, \dots, K, \\ Y_{\Lambda} &= \max_{1 \leq i < j \leq K} |X_{\Lambda,i} - X_{\Lambda,j}|. \end{aligned}$$

where dist_{Λ} denotes graph distance in the uniform spanning tree ω_{Λ} . We define the random permutation σ_K^* by the requirement:

$$\sigma_K^* = \sigma \quad \text{if and only if} \quad X_{\Lambda,\sigma(1)} \leq \dots \leq X_{\Lambda,\sigma(K)},$$

where we take a fixed but otherwise arbitrary rule to settle ties. Let $K(\bar{F}) = \max_{1 \leq i \leq K} \text{diam}(F_i)$. The required extension follows once we show the following analogues of [1, Eqs. (18) and (19)].

$$\lim_{\Lambda} \mu_{\Lambda} (B_{\Lambda}(\bar{F}, \bar{v}), Y_{\Lambda} \leq K(\bar{F})) = 0, \tag{A.1}$$

and

$$\lim_{\Lambda} \mu_{\Lambda} (B_{\Lambda}(\bar{F}, \bar{v}), \sigma_K^* = \sigma, Y_{\Lambda} > K(\bar{F})) = \frac{1}{K!} \mu (B(\bar{F}, \bar{v})). \tag{A.2}$$

Most of the argument in [1] does apply to general volumes, and here we detail only those points where differences arise. We use the notation introduced in Sect. 9.3.5 for Wilson’s algorithm.

We start with the proof of (A.1). Let $x, y \in \mathbb{Z}^d$ be fixed, and let $S^{(1)}$ and $S^{(2)}$ be independent simple random walks starting at x and y , respectively. Let $T_{\Lambda}^{(1)}$ and $T_{\Lambda}^{(2)}$

be the first exit times from Λ for these random walks. The required extension of (A.1) follows from an extension of (27)[1], which in turn follows from the statement

$$\lim_{\delta \rightarrow 0} \limsup_{\Lambda} \Pr \left(1 - \delta \leq \frac{T_{\Lambda}^{(1)}}{T_{\Lambda}^{(2)}} \leq 1 + \delta \right) = 0. \tag{A.3}$$

Statement (A.3) is proved in [13].

For the extension of (A.2), we recall from Sect. 9.2 the events $B_{\Lambda}(\bar{F}, \bar{v})$ and $B(\bar{F}, \bar{v})$ defined for a collection $(F_i, v_i)_{i=1}^K$. Let $S^{(i)}, i = 1, \dots, K$ be independent random walks started at v_i , respectively. Let $T_{\Lambda}^{(i)}$ be the exit time of $S^{(i)}$ from Λ . Also recall the random walk events C_{Λ} and C , and that C_m and $T_m^{(i)}$ are short for C_{Λ} and $T_{\Lambda}^{(i)}$ when $\Lambda = [-m, m]^d \cap \mathbb{Z}^d$. By the arguments in [1], the required extension of (A.2) follows, once we show an extension of (32)[1], namely that for any permutation $\sigma \in \Sigma_K$

$$\lim_{m \rightarrow \infty} \lim_{\Lambda} \Pr \left(C_m, T_{\Lambda}^{\sigma(1)} < \dots < T_{\Lambda}^{\sigma(K)} \right) = \Pr(C) \frac{1}{K!}. \tag{A.4}$$

Observe that C_m and the collection $\tilde{T}_{\Lambda, m}^{(i)} = T_{\Lambda}^{(i)} - T_m^{(i)}, i = 1, \dots, K$ are conditionally independent, given $\{S^{(i)}(T_m^{(i)})\}_{i=1}^K$. Therefore, using (A.3), the left hand side of (A.4) equals

$$\lim_{m \rightarrow \infty} \lim_{\Lambda} \Pr(C_m) \Pr \left(\tilde{T}_{\Lambda, m}^{\sigma(1)} < \dots < \tilde{T}_{\Lambda, m}^{\sigma(K)} \right). \tag{A.5}$$

The second probability approaches $1/K!$ for any fixed m , and hence the limit in (A.5) equals $\Pr(C)/K!$. This completes the proof of the required extension of (A.2).

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