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BSDE with quadratic growth and unbounded terminal value

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Abstract. In this paper, we study the existence of solution to BSDE with quadratic growth and unbounded terminal value. The main idea consists in using a localization procedure together with a priori bounds.

1. Introduction

In this paper we are concerned with real valued backward stochastic differential equations – BSDEs for short in the remaining –

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s \cdot dB_s, \qquad 0 \leq t \leq T$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Such equations have been extensively studied since the first paper of E. Pardoux and S. Peng [8]. The full list of contributions is too long to give and we will only quote results in our framework.

Our setting is mainly the following : the generator, namely the function f, is of quadratic growth in the variable z and the terminal condition, the random variable ξ, will not be bounded. BSDEs with quadratic growth have been first studied by M. Kobylanski in her PhD (see [4, 5]) and then by J.-P. Lepeltier and J. San Martin in [6] and more recently in [7]. We should point out that BSDEs with quadratic growth in the variable z have found applications in control and finance, see, e.g., J.-M. Bismut [1], N. El Karoui, R. Rouge [2], Y. Hu, P. Imkeller, M. Müller [3], ...

All the general results on BSDEs with quadratic growth require that the terminal condition ξ is a bounded random variable. The boundedness of the terminal condition appears, from the point of view of the applications, to be restrictive and, moreover, from a theoretical point of view, is not necessary to obtain a solution. Indeed, let us consider the following well known equation :

$$Y_t = \xi + \frac{1}{2}\int_t^T |Z_s|^2\, ds - \int_t^T Z_s\, dB_s, \qquad 0 \leq t \leq T\ ;$$

the change of variables $P_t = e^{Y_t}$, $Q_t = e^{Y_t} Z_t$, leads to the equation

$$P_t = e^{\xi} - \int_t^T Q_s\, dB_s$$

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which has a solution as soon as e^{ξ} is integrable. Actually, it can be shown, since $\{e^{Y_t}\}_{t\in[0,T]}$ is a supermartingale, that the integrability of e^{ξ} is also a necessary condition to obtain a solution for this BSDE. We refer to [7] for details. In this last paper, the result obtained for quadratic BSDEs require ξ to be bounded except for this example.

In this particular case we see that the existence of exponential moments of the terminal condition is sufficient to construct a solution to our BSDE. Our paper will be focused on the theoretical study of these BSDEs but with unbounded terminal value with only exponential moments. To fill the gap between boundedness and existence of exponential moments, we will use an approach based upon a localization procedure together with a priori bounds. Let us quickly explain how it works with a simple example.

Let $f : \mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ be a continuous function and ξ be a nonnegative terminal condition such that

$$|f(y,z)| \leq \frac{1}{2}|z|^2, \qquad \mathbb{E}\left[e^{\xi}\right] < \infty,$$

and let us try to construct a solution to the BSDE

$$Y_t = \xi + \int_t^T f(Y_s, Z_s)\,ds - \int_t^T Z_s \cdot dB_s, \quad 0 \leq t \leq T.$$

As mentioned before, BSDEs with quadratic growth in the variable z can be solved when the terminal solution is bounded. That is why we introduce (Y^n, Z^n) as the minimal solution to the BSDE

$$Y_t^n = \xi \wedge n + \int_t^T f(Y_s^n, Z_s^n)\,ds - \int_t^T Z_s^n \cdot dB_s,$$

and of course we want to pass to the limit when $n \to \infty$ in this equation. The process Y^n is known to be bounded but the estimate depends on $\|\xi \wedge n\|_\infty$ and thus is far from being useful when ξ is not bounded. The first step of our approach consists in finding an estimation for Y^n independent of n. In this example, we can use the explicit formula mentioned before to show that

$$0 \leq -\ln\mathbb{E}\left(e^{-(\xi\wedge n)} \mid \mathcal{F}_t\right) \leq Y_t^n \leq \ln\mathbb{E}\left(e^{\xi\wedge n} \mid \mathcal{F}_t\right) \leq \ln\mathbb{E}\left(e^{\xi} \mid \mathcal{F}_t\right).$$

With these inequalities in hands, we introduce the stopping time

$$\tau_k = \inf\left\{t \in [0,T] : \ln\mathbb{E}\left(e^{\xi} \mid \mathcal{F}_t\right) \geq k\right\} \wedge T$$

and instead of working on the time interval $[0,T]$ we will restrict ourselves to $[0,\tau_k]$ by considering the BSDE

$$Y_{t\wedge\tau_k}^n = Y_{\tau_k}^n + \int_{t\wedge\tau_k}^{T\wedge\tau_k} f\left(Y_s^n, Z_s^n\right)ds - \int_{t\wedge\tau_k}^{T\wedge\tau_k} Z_s^n \cdot dB_s, \qquad 0 \leq t \leq T.$$

By construction, we have $\sup_n \sup_t \left\|Y_{t\wedge\tau_k}^n\right\|_\infty \leq k$. This last property together with the fact that the sequence $(Y^n)_{n\geq 1}$ is nondecreasing allows us, with the help of a result of Kobylanski, to pass to the limit when $n \to \infty$, k being fixed and then to send k to infinity to get a solution.

The rest of the paper is organized as follows. Next section is devoted to the notations we use during this text. In Section 3, we claim our main result that we prove in Section 4. Finally the last section is devoted to some additional results on BSDEs with quadratic growth in z.

2. Notations

For the remaining of the paper, let us fix a nonnegative real number $T > 0$. First of all, $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmented natural filtration of B which satisfies the usual conditions. In this paper, we will always use this filtration. The sigma-field of predictable subsets of $[0, T] \times \Omega$ is denoted $\mathcal{P}$.

As mentioned in the introduction, a BSDE is an equation of the following type

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s \cdot dB_s, \qquad 0 \leq t \leq T. \tag{1}$$

The function f is called the generator and ξ the terminal condition. Let us recall that a generator is a random function $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ which is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ and a terminal condition is simply a real $\mathcal{F}_T$–measurable random variable.

By a solution to the BSDE (1) we mean a pair $(Y, Z) = \{(Y_t, Z_t)\}_{t\in[0,T]}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^d$ such that $\mathbb{P}$–a.s., $t \longmapsto Y_t$ is continuous, $t \longmapsto Z_t$ belongs to $\mathrm{L}^2(0, T)$, $t \longmapsto f(t, Y_t, Z_t)$ belongs to $\mathrm{L}^1(0, T)$ and $\mathbb{P}$–a.s.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s \cdot dB_s, \qquad 0 \leq t \leq T.$$

We will use the notation BSDE(ξ, f) to say that we consider the BSDE whose generator is f and whose terminal condition is ξ; $\big(Y^f(\xi), Z^f(\xi)\big)$ means a solution to the BSDE(ξ, f). A solution $\big(Y^f(\xi), Z^f(\xi)\big)$ is said to be minimal if $\mathbb{P}$-a.s., for each $t \in [0, T]$, $Y_t^f(\xi) \leq Y_t^g(\zeta)$ whenever $\mathbb{P}$–a.s. $\xi \leq \zeta$ and $f(t, y, z) \leq g(t, y, z)$ for all (t, y, z). $\big(Y^f(\xi), Z^f(\xi)\big)$ is said to be minimal in some space $\mathcal{B}$ if it belongs to this space and the previous property holds true as soon as $(Y^g(\zeta), Z^g(\zeta)) \in \mathcal{B}$.

For any real $p > 0$, $\mathcal{S}^p$ denotes the set of real-valued, adapted and càdlàg processes $\{Y_t\}_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \mathbb{E}\left[\sup\nolimits_{t\in[0,T]} |Y_t|^p\right]^{1\wedge 1/p} < +\infty.$$

If $p \geq 1$, $\|\cdot\|_{\mathcal{S}^p}$ is a norm on $\mathcal{S}^p$ and if $p \in (0, 1)$, $(X, X') \longmapsto \left\|X - X'\right\|_{\mathcal{S}^p}$ defines a distance on $\mathcal{S}^p$. Under this metric, $\mathcal{S}^p$ is complete. M^p denotes the set of (equivalent classes of) predictable processes $\{Z_t\}_{t\in[0,T]}$ with values in $\mathbb{R}^d$ such that

$$\|Z\|_{\mathrm{M}^p} := \mathbb{E}\left[\left(\int_0^T |Z_s|^2\, ds\right)^{p/2}\right]^{1\wedge 1/p} < +\infty.$$

For $p \geq 1$, $\mathrm{M}^p(\mathbb{R}^n)$ is a Banach space endowed with this norm and for $p \in (0, 1)$, M^p is a complete metric space with the resulting distance. We set $\mathcal{S} = \cup_{p>1}\mathcal{S}^p$, $\mathrm{M} = \cup_{p>1}\mathrm{M}^p$ and $\mathcal{S}^\infty$ stands for the set of predictable bounded processes.

3. Quadratic BSDEs

In this section, we consider BSDE(ξ, f) when the generator f has a linear growth in y and a quadratic growth in z. We denote (H1) the assumption: there exist $\alpha \geq 0$, $\beta \geq 0$ and $\gamma > 0$ such that $\mathbb{P}$–a.s.

$$\begin{aligned} &\forall t \in [0, T], \qquad (y, z) \longmapsto f(t, y, z) \text{ is continuous,} \\ &\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, \qquad |f(t, y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2. \end{aligned} \tag{H1}$$

Concerning the terminal condition ξ, we will assume that

$$\mathbb{E}\left[e^{\gamma e^{\beta T}|\xi|}\right] < +\infty. \tag{H2}$$

We will use also a stronger assumption on the integrability of ξ namely

$$\exists \lambda > \gamma e^{\beta T}, \qquad \mathbb{E}\left[e^{\lambda|\xi|}\right] < +\infty. \tag{H3}$$

It is clear that we can assume without loss of generality that $\alpha \geq \beta/\gamma$.

As we explained in the introduction, our method relies heavily on a priori estimate. To obtain such estimations, we will use the change of variable $P_t = e^{\gamma Y_t}$, $Q_t = \gamma e^{\gamma Y_t} Z_t$; if (Y, Z) is a solution to the BSDE(ξ, f), (P, Q) solves the BSDE

$$P_t = e^{\gamma\xi} + \int_t^T F(s, P_s, Q_s)\, ds - \int_t^T Q_s \cdot dB_s, \quad 0 \leq t \leq T,$$

with the function F defined by

$$F(s, p, q) = \mathbf{1}_{p>0}\left(\gamma p\, f\left(s, \frac{\ln p}{\gamma}, \frac{q}{\gamma p}\right) - \frac{1}{2}\frac{|q|^2}{p}\right). \tag{2}$$

In view of the growth of the generator f, we have $F(s, p, q) \leq \mathbf{1}_{p>0}\, p(\alpha\gamma + \beta|\ln p|)$. For notational convenience, we denote by H the function

$$\forall p \in \mathbb{R}, \qquad H(p) = p\,(\alpha\gamma + \beta \ln p)\,\mathbf{1}_{[1,+\infty)}(p) + \gamma\alpha\mathbf{1}_{(-\infty,1)}(p).$$

It is straightforward to check that, since $\alpha \geq \beta/\gamma$, H is convex and locally Lipschitz continuous and that, for any real $p > 0$, $p\,(\alpha\gamma + \beta|\ln p|) \leq H(p)$. Thus we deduce the inequality

$$\forall s \in [0, T],\ \forall p \in \mathbb{R},\ \forall q \in \mathbb{R}^d, \qquad F(s, p, q) \leq H(p). \tag{3}$$

To get an upper bound for Y_t, the idea is to compare more or less P_t with $\phi_t(\xi)$ where, for any real z, $\{\phi_t(z)\}_{0\leq t\leq T}$ stands for the solution to the integral equation

$$\phi_t = e^{\gamma z} + \int_t^T H(\phi_s)\, ds, \qquad 0 \leq t \leq T. \tag{4}$$

Using the convexity of H, we will able to prove that

$$P_t \le \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right), \qquad Y_t \le \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right).$$

Before proving this result rigorously, let us recall that the integral equation (4) can be solved easily. Indeed, we have, for any $z \ge 0$,

$$\phi_t(z) = \exp\left(\gamma\alpha \frac{e^{\beta(T-t)} - 1}{\beta}\right) \exp\left(z\gamma e^{\beta(T-t)}\right), \quad \text{if } \beta > 0,$$

and $\phi_t(z) = e^{\gamma\alpha(T-t)} e^{\gamma z}$ if $\beta = 0$. Let us consider the case where $z < 0$. If $e^{\gamma z} + T\gamma\alpha \le 1$ then the solution is

$$\phi_t = e^{\gamma z} + \gamma\alpha(T - t)$$

and otherwise there exists $0 < S < T$ such that $e^{\gamma z} + \gamma\alpha(T - S) = 1$ and

$$\phi_t = \left[e^{\gamma z} + \gamma\alpha(T - t)\right] \mathbf{1}_{(S,T]}(t) + \exp\left(\gamma\alpha \frac{e^{\beta(S-t)} - 1}{\beta}\right) \mathbf{1}_{[0,S]}(t).$$

It is plain to check that $t \mapsto \phi_t(z)$ is decreasing and that $z \mapsto \phi_t(z)$ is increasing and continuous.

Lemma 1. *Let the assumption* (H1) *hold and let ξ be a bounded $\mathcal{F}_T$–measurable random variable. If (Y, Z) is a solution to the BSDE(ξ, f) in $\mathcal{S}^\infty \times \mathrm{M}^2$ then*

$$-\frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(-\xi) \mid \mathcal{F}_t\right) \le Y_t \le \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right).$$

Proof. Let us set $\Phi_t = \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right)$. We have

$$\begin{aligned} \Phi_t &= \mathbb{E}\left(e^{\gamma\xi} + \int_t^T H(\phi_s(\xi))\, ds \,\middle|\, \mathcal{F}_t\right) \\ &= \mathbb{E}\left(e^{\gamma\xi} + \int_t^T \mathbb{E}\left(H(\phi_s(\xi)) \mid \mathcal{F}_s\right) ds \,\middle|\, \mathcal{F}_t\right). \end{aligned}$$

Thus writing the bounded Brownian martingale

$$\begin{aligned} &\mathbb{E}\left(e^{\gamma\xi} + \int_0^T \mathbb{E}\left(H(\phi_s(\xi)) \mid \mathcal{F}_s\right) ds \,\middle|\, \mathcal{F}_t\right) \\ &\quad = \mathbb{E}\left[e^{\gamma\xi} + \int_0^T \mathbb{E}\left(H(\phi_s(\xi)) \mid \mathcal{F}_s\right) ds\right] + \int_0^t \Psi_s \cdot dB_s \end{aligned}$$

(Φ, Ψ) solves the BSDE

$$\Phi_t = e^{\gamma\xi} + \int_t^T \mathbb{E}\left(H(\phi_s(\xi)) \mid \mathcal{F}_s\right) ds - \int_t^T \Psi_s \cdot dB_s.$$

On the other hand, if $(Y, Z) \in \mathcal{S}^\infty \times \mathrm{M}^2$ is a solution of (1), setting as before $P_t = e^{\gamma Y_t}$, $Q_t = \gamma e^{\gamma Y_t} Z_t$, we have

$$P_t = e^{\gamma\xi} + \int_t^T F(s, P_s, Q_s)\, ds - \int_t^T Q_s \cdot dB_s,$$

with F defined by (2). It follows that

$$\Phi_t - P_t = \int_t^T (H(\Phi_s) - H(P_s))\, ds + \int_t^T R_s\, ds - \int_t^T (\Psi_s - Q_s) \cdot dB_s$$

where, in view of the inequality (3) and since H is convex,

$$R_s = \mathbb{E}\left(H(\phi_s(\xi)) \mid \mathcal{F}_s\right) - H\left(\mathbb{E}\left(\phi_s(\xi) \mid \mathcal{F}_s\right)\right) + H(P_s) - F(s, P_s, Q_s)$$

is a nonnegative process. The function H is only locally Lipschitz but since Φ and P are bounded we can apply the comparison theorem to get $P_t \le \Phi_t$ and $Y_t \le \frac{1}{\gamma} \ln \Phi_t$.

Finally, since the function $-f(t, -y, -z)$ still satisfies the assumption (H1), we get also the inequality $-Y_t \le \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(-\xi) \mid \mathcal{F}_t\right)$. □

We are now in position to prove that under the assumptions described before the BSDE (1) has at least a solution.

Theorem 2. *Let the assumptions* (H1) *and* (H2) *hold. Then the BSDE* (1) *has at least a solution* (Y, Z) *such that :*

$$-\frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(-\xi) \mid \mathcal{F}_t\right) \le Y_t \le \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right). \tag{5}$$

If moreover, (H3) *holds, then Z belongs to* M^2.

Proof of the last part of Theorem 2. If (Y, Z) is a solution to the BSDE (1) such that the inequalities (5) hold, then

$$|Y_t| \le \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_0(|\xi|) \mid \mathcal{F}_t\right)$$

and, under the assumption (H3), we deduce that, for some $p > 1$,

$$\mathbb{E}\left[\sup\nolimits_{t\in[0,T]} e^{p\gamma|Y_t|}\right] < +\infty.$$

For $n \ge 1$, let τ_n be the following stopping time

$$\tau_n = \inf\left\{t \ge 0 : \int_0^t e^{2\gamma|Y_s|}|Z_s|^2\, ds \ge n\right\} \wedge T,$$

and let us consider the function from $\mathbb{R}_+$ into itself defined by

$$u(x) = \frac{1}{\gamma^2}\left(e^{\gamma x} - 1 - \gamma x\right).$$

$x \longmapsto u(|x|)$ is $\mathcal{C}^2$ and we have from Itô's formula, with the notation $\operatorname{sgn}(x) = -\mathbf{1}_{x\leq 0} + \mathbf{1}_{x>0}$,

$$\begin{aligned} u(|Y_0|) = u(|Y_{t\wedge\tau_n}|) \\ &+ \int_0^{t\wedge\tau_n} \left(u'(|Y_s|)\operatorname{sgn}(Y_s) f(s, Y_s, Z_s) - \frac{1}{2} u''(|Y_s|)|Z_s|^2 \right) ds \\ &- \int_0^{t\wedge\tau_n} u'(|Y_s|)\operatorname{sgn}(Y_s) Z_s \cdot dB_s. \end{aligned}$$

It follows from (H1) since $u'(x) \geq 0$ for $x \geq 0$ that

$$\begin{aligned} u(|Y_0|) \leq u(|Y_{t\wedge\tau_n}|) &+ \int_0^{t\wedge\tau_n} u'(|Y_s|)\,(\alpha + \beta|Y_s|)\, ds \\ &- \int_0^{t\wedge\tau_n} u'(|Y_s|)\operatorname{sgn}(Y_s) Z_s \cdot dB_s \\ &- \frac{1}{2}\int_0^{t\wedge\tau_n} \left(u''(|Y_s|) - \gamma\, u'(|Y_s|)\right) |Z_s|^2 ds. \end{aligned}$$

Moreover, we have $(u'' - \gamma u')(x) = 1$ for $x \geq 0$ and, taking expectation of the previous inequality, we get

$$\frac{1}{2}\mathbb{E}\left[\int_0^{T\wedge\tau_n} |Z_s|^2\, ds\right] \leq \mathbb{E}\left[\frac{1}{\gamma^2} \sup_{t\in[0,T]} e^{\gamma|Y_t|} + \frac{1}{\gamma}\int_0^T e^{\gamma|Y_s|}\,(\alpha + \beta|Y_s|)\, ds\right]$$

Fatou's lemma together with the fact that $e^{\gamma|Y_t|} \in \mathcal{S}^p$ gives the result. □

4. Proof of Theorem 2

Let us first construct a solution to the BSDE (1) in the case where ξ is nonnegative.

For each $n \in \mathbb{N}^*$, we set $\xi^n = \xi \wedge n$. Then it is known from [5, Theorem 2.3] that the BSDE

$$Y_t^n = \xi^n + \int_t^T f\left(s, Y_s^n, Z_s^n\right) ds - \int_t^T Z_s^n \cdot dB_s, \qquad 0 \leq t \leq T$$

has a minimal solution (Y^n, Z^n) in $\mathcal{S}^\infty \times \mathrm{M}^2$. Lemma 1 implies the inequalities

$$-\frac{1}{\gamma}\ln \mathbb{E}\left(\phi_t\left(-\xi^n\right) \mid \mathcal{F}_t\right) \leq Y_t^n \leq \frac{1}{\gamma}\ln \mathbb{E}\left(\phi_t\left(\xi^n\right) \mid \mathcal{F}_t\right).$$

Since we consider only minimal solutions, we have,

$$\forall t \in [0, T], \qquad Y_t^n \leq Y_t^{n+1}.$$

We define $Y = \sup_{n\geq 1} Y^n$.

Since $0 \leq \phi_t(\xi^n) \leq \phi_0(|\xi|)$ and $0 \leq \phi_t(-\xi^n) \leq \phi_0(|\xi|)$, we deduce from the dominated convergence theorem, noting that the random variable $\phi_0(|\xi|)$ is integrable by (H2), that

$$-\frac{1}{\gamma}\ln \mathbb{E}\left(\phi_t\left(-\xi\right) \mid \mathcal{F}_t\right) \leq Y_t \leq \frac{1}{\gamma}\ln \mathbb{E}\left(\phi_t\left(\xi\right) \mid \mathcal{F}_t\right).$$

In particular, we have $\lim_{t\to T} Y_t = \xi = Y_T$. Indeed, for each $S < T$,

$$\limsup_{t\to T} Y_t \leq \limsup_{t\to T} \frac{1}{\gamma}\ln \mathbb{E}\left(\phi_t\left(\xi\right) \mid \mathcal{F}_t\right) \leq \lim_{t\to T} \frac{1}{\gamma}\ln \mathbb{E}\left(\phi_S\left(\xi\right) \mid \mathcal{F}_t\right) = \frac{1}{\gamma}\ln \phi_S(\xi),$$

and $\lim_{S\to T} \frac{1}{\gamma}\ln \phi_S(\xi) = \xi$. We can do the same for $\liminf$.

Let us introduce the following stopping time :

$$\tau_k = \inf\left\{t \in [0,T] : \frac{1}{\gamma}\ln \mathbb{E}\left(\phi_0\left(|\xi|\right) \mid \mathcal{F}_t\right) \geq k\right\} \wedge T.$$

Then $(Y_k^n, Z_k^n) := (Y^n_{t\wedge\tau_k}, Z^n_t \mathbf{1}_{t\leq\tau_k})$ satisfies the following BSDE

$$Y_k^n = \xi_k^n + \int_t^T \mathbf{1}_{s\leq\tau_k} f\left(s, Y_k^n(s), Z_k^n(s)\right) ds - \int_t^T Z_k^n(s)\cdot dB_s,$$

where of course $\xi_k^n = Y_k^n(T) = Y^n_{\tau_k}$.

We are going to pass to the limit when n tends to $+\infty$ for k fixed in this last equation. The key point is that Y_k^n is increasing in n and remains bounded by k. At this stage, let us mention a mere generalization of Proposition 2.4 in [5].

Lemma 3 ([5]). *Let $(\xi_n)_{n\geq 1}$ be a sequence of $\mathcal{F}_T$–measurable bounded random variables and $(f_n)_{n\geq 1}$ be a sequence of generators which are continuous with respect to (y,z).*

We assume that $(\xi_n)_{n\geq 1}$ converges $\mathbb{P}$–a.s. to ξ, that $(f_n)_{n\geq 1}$ converges locally uniformly in (y,z) to the generator f, and also that

1. $\sup_{n\geq 1}\|\xi_n\|_\infty < +\infty$ *;*
2. $\sup_{n\geq 1}|f_n(t,y,z)|$ *satisfies the inequality in* (H1).

If for each $n \geq 1$, the BSDE(ξ_n, f_n) has a solution in $\mathcal{S}^\infty \times \mathrm{M}^2$, such that $\left(Y^{f_n}(\xi_n)\right)_{n\geq 1}$ is nondecreasing (respectively nonincreasing), then $\mathbb{P}$–a.s. $\left(Y_t^{f_n}(\xi_n)\right)_{n\geq 1}$ converges uniformly on $[0,T]$ to $Y_t = \sup_{n\geq 1} Y_t^{f_n}(\xi_n)$ (respectively $Y_t = \inf_{n\geq 1} Y_t^{f_n}(\xi_n)$), $\left(Z^{f_n}(\xi_n)\right)_{n\geq 1}$ converges to some Z in M^2 and (Y,Z) is a solution to BSDE(ξ, f) in $\mathcal{S}^\infty \times \mathrm{M}^2$.

Proof. It follows from Lemma 1 that there exists $r > 0$ such that, $\mathbb{P}$–a.s.

$$\forall n \geq 1, \quad \forall t \in [0,T], \qquad \left|Y_t^{f_n}(\xi_n)\right| \leq r.$$

Let us consider the continuous function $\rho(x) = xr/\max(r, |x|)$. Since $\rho(x) = x$ for $|x| \leq r$, $\left(Y^{f_n}(\xi_n), Z^{f_n}(\xi_n)\right)$ solves the BSDE(ξ_n, g_n) where $g_n(t, y, z) = f_n(t, \rho(y), z)$. Obviously, we have, for each $n \geq 1$,

$$|g_n(t, y, z)| \leq \alpha + \beta r + \frac{\gamma}{2}|z|^2,$$

and thus we can apply the result of Kobylanski. □

Setting $Y_k(t) = \sup_n Y_k^n(t)$, it follows from the previous lemma that there exists a process $Z_k \in \mathrm{M}^2$ such that $\lim_n Z_k^n = Z_k$ in M^2 and (Y_k, Z_k) solves the BSDE

$$Y_k(t) = \xi_k + \int_t^T \mathbf{1}_{s \leq \tau_k} f(s, Y_k(s), Z_k(s))\, ds - \int_t^T Z_k(s) \cdot dB_s, \tag{6}$$

where $\xi_k = \sup_n Y_{\tau_k}^n$. But $\tau_k \leq \tau_{k+1}$, and thus we get, coming back to the definition of Y_k, Z_k and Y,

$$Y_{t \wedge \tau_k} = Y_{k+1}(t \wedge \tau_k) = Y_k(t), \qquad Z_{k+1}(t)\, \mathbf{1}_{t \leq \tau_k} = Z_k(t).$$

As $\tau_k \to T$ and the Y_k's are continuous processes we deduce in particular that Y is continuous on $[0, T)$. On the other hand, as mentioned before $\lim_{t \to T} Y_t = \xi$ and Y_T is equal to ξ by construction. Thus Y is a continuous process on the closed interval $[0, T]$.

Then we define Z on $(0, T)$ by setting :

$$Z_t = Z_k(t), \quad \text{if } t \in (0, \tau_k).$$

From (6), (Y, Z) satisfies:

$$Y_{t \wedge \tau_k} = Y_{\tau_k} + \int_{t \wedge \tau_k}^{\tau_k} f(s, Y_s, Z_s) ds - \int_{t \wedge \tau_k}^{\tau_k} Z_s \cdot dB_s. \tag{7}$$

Finally, we have

$$\begin{aligned}
\mathbb{P}(\int_0^T |Z_s|^2\, ds = \infty) &= \mathbb{P}\left(\int_0^T |Z_s|^2\, ds = \infty, \tau_k = T\right) \\
&\quad + \mathbb{P}\left(\int_0^T |Z_s|^2\, ds = \infty, \tau_k < T\right) \\
&\leq \mathbb{P}\left(\int_0^{\tau_k} |Z_k(s)|^2\, ds = \infty\right) + \mathbb{P}(\tau_k < T),
\end{aligned}$$

and we deduce that, $\mathbb{P}$–a.s.

$$\int_0^T |Z_s|^2\, ds < \infty.$$

By sending k to infinity in (7), we deduce that (Y, Z) is a solution of (1).

Let us explain quickly how to extend this construction to the general case. Let us fix $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$ and set $\xi^{n,p} = \xi^+ \wedge n - \xi^- \wedge p$. Let us consider, $(Y^{n,p}, Z^{n,p})$ the minimal bounded solution to the BSDE

$$Y_t^{n,p} = \xi^{n,p} + \int_t^T f\left(s, Y_s^{n,p}, Z_s^{n,p}\right) ds - \int_t^T Z_s^{n,p} \cdot dB_s, \qquad 0 \le t \le T$$

which satisfies

$$-\frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t\left(-\xi^{n,p}\right) \mid \mathcal{F}_t\right) \le Y_t^{n,p} \le \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t\left(\xi^{n,p}\right) \mid \mathcal{F}_t\right).$$

We have,

$$\forall t \in [0,T], \qquad Y_t^{n,p+1} \le Y_t^{n,p} \le Y_t^{n+1,p},$$

and we define $Y^p = \sup_{n\ge 1} Y^{n,p}$ so that $Y_t^{p+1} \le Y_t^p$ and $Y_t = \inf_{p\ge 1} Y_t^p$.

By the dominated convergence theorem, we have

$$-\frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t\left(-\xi\right) \mid \mathcal{F}_t\right) \le Y_t \le \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t\left(\xi\right) \mid \mathcal{F}_t\right),$$

and in particular, we have $\lim_{t\to T} Y_t = \xi = Y_T$. $(Y_{t\wedge\tau_k}^{n,p}, Z_t^{n,p}\mathbf{1}_{t\le\tau_k})$ solves the BSDE

$$Y_{t\wedge\tau_k}^{n,p} = Y_{\tau_k}^{n,p} + \int_t^T \mathbf{1}_{s\le\tau_k} f\left(s, Y_s^{n,p}, Z_s^{n,p}\right) ds - \int_t^T Z_s^{n,p}\mathbf{1}_{s\le\tau_k} \cdot dB_s.$$

But, once again $Y_{t\wedge\tau_k}^{n,p}$ is increasing in n and decreasing in p and remains bounded by k. Arguing as before, setting $Y_k(t) = \inf_p \sup_n Y_{t\wedge\tau_k}^{n,p}$, there exists a process Z_k such that $\lim_p \lim_n Z^{n,p}(s)\mathbf{1}_{s\le\tau_k} = Z_k(s)$ and (Y_k, Z_k) still solves the BSDE (6). The rest of the proof is unchanged.

5. Additional results on quadratic BSDEs

5.1. Minimal solution

In this section, we give some complements on BSDEs with quadratic growth in z.

Proposition 4. *Let* (H1) *hold and assume moreover that there exists $r \ge 0$ such that $\mathbb{P}$–a.s.*

$$f(t,y,z) \ge -r\left(1 + |y| + |z|\right).$$

Let us assume also that (H3) *holds for ξ^+ and that, for some $p > 1$, $\xi^- \in \mathrm{L}^p$.*

Then BSDE(ξ, f) has a minimal solution in $\mathcal{S}$.

Proof. Without loss of generality, let us assume that r is an integer. For each integer $n \geq r$, let us consider the function

$$f_n(t, y, z) = \inf \left\{ f(t, p, q) + n|p - y| + n|q - z| : (p, q) \in \mathbb{Q}^{1+d} \right\}.$$

Then f_n is well defined and it is globally Lipschitz continuous with constant n. Moreover $(f_n)_{n \geq r}$ is increasing and converges pointwise to f. Dini's theorem implies that the convergence is also uniform on compact sets. We have also, for all $n \geq r$,

$$-r(1 + |y| + |z|) \leq f_n(t, y, z) \leq f(t, y, z)$$

Let (Y^n, Z^n) be the unique solution in $\mathcal{S}^p \times \mathrm{M}^{\mathrm{p}}$ to BSDE(ξ, f_n). It follows from the classical comparison theorem that

$$Y_t^r \leq Y_t^n \leq Y_t^{n+1}.$$

Let us prove that $Y_t^n \leq \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right)$. To do this let us recall that, since f_n is Lipschitz, $Y_t^n = \lim_{m \to +\infty} Y_t^{f_n}(\xi_m)$ where $\xi_m = \xi\, \mathbf{1}_{|\xi| \leq m}$. Moreover $\mathbb{E}\left(\phi_t(\xi_m) \mid \mathcal{F}_t\right) \longrightarrow \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right)$ a.s. since $\sup_{m \geq 1} |\phi_t(\xi_m)| \leq \phi_0(\xi^+)$ which is integrable. Thus we have only to prove that $Y_t^{f_n}(\xi_m) \leq \frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(\xi_m) \mid \mathcal{F}_t\right)$. We keep the notations of the beginning of Section 4. (Φ, Ψ) is solution to the BSDE

$$\Phi_t = e^{\gamma \xi_m} + \int_t^T H(\Phi_s)\, ds + \int_t^T \Gamma_s\, ds - \int_t^T \Psi_s \cdot dB_s,$$

where $\Gamma_s = \mathbb{E}\left(H(\phi_s(\xi)) \mid \mathcal{F}_s\right) - H(\Phi_s)$ is a nonnegative process since H is convex.

It follows by setting $U_t = \frac{1}{\gamma} \ln \Phi_t$, $V_t = \frac{\Psi_t}{\gamma \Phi_t}$ that (U, V) solves the BSDE

$$U_t = \xi_m + \int_t^T g(s, U_s, V_s)\, ds - \int_t^T V_s \cdot dB_s$$

where we have set $g(s, u, v) = (\alpha + \beta u)\mathbf{1}_{u \geq 0} + \alpha e^{\gamma |u|} \mathbf{1}_{u < 0} + \frac{\gamma}{2}|v|^2 + C_s$ with $C_s = \frac{1}{\gamma} e^{-\gamma U_s} \Gamma_s$. Since the process C is still nonnegative, we have the inequalities

$$f_n(t, u, v) \leq f(t, u, v) \leq g(t, u, v)$$

taking into account the fact that $\alpha\gamma \geq \beta$. Since f_n is Lipschitz continuous and $\left(Y^{f_n}(\xi_m) - U\right)^+$ belongs to $\mathcal{S}$, we can apply the extended comparison theorem (see Proposition 5) to get, for each $m \geq 1$, $Y_t^{f_n}(\xi_m) \leq U_t$ and thus the inequality we want to obtain.

We set $Y = \sup_{n \geq r} Y^n$ and, for $k \geq 1$,

$$\tau_k = \inf \left\{ t \in [0, T] : \max \left(\frac{1}{\gamma} \ln \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right), -Y_t^r \right) \geq k \right\} \wedge T.$$

Arguing as in the proof of Theorem 2, we construct a process Z such that (Y, Z) solves BSDE(ξ, f).

Let us show that this solution is minimal in $\mathcal{S}$. Let (Y', Z') be a solution to the BSDE(ξ', f') where $\xi \le \xi'$ and $f \le f'$. It is enough to check that $Y^n \le Y'$ to prove that $Y \le Y'$. But this is a direct consequence of Proposition 5. □

To be complete, let us claim and prove the extended comparison theorem that we used in the proof of the previous result.

Proposition 5. *Let (Y, Z) be a solution to BSDE(ξ, f) and (Y', Z') be a solution to BSDE(ξ', f'). We assume that $\xi \le \xi'$ and that f satisfies, for some constants μ and λ, $\mathbb{P}$–a.s.*

$$(y - y') \cdot \big(f(t, y, z) - f(t, y', z)\big) \le \mu |y - y'|^2;$$
$$\big|f(t, y, z) - f(t, y, z')\big| \le \lambda |z - z'|.$$

If $\big(Y - Y'\big)^+$ belongs to $\mathcal{S}$, then $\mathbb{P}$–a.s. $Y_t \le Y'_t$.

Proof. Let us fix $n \in \mathbb{N}^*$ and denote τ_n the stopping time

$$\tau_n = \inf\left\{t \in [0, T] : \int_0^t \left(|Z_s|^2 + \left|Z'_s\right|^2\right) ds \ge n\right\} \wedge T.$$

Tanaka's formula leads to the equation, setting $U_t = Y_t - Y'_t$, $V_t = Z_t - Z'_t$,

$$\begin{aligned} e^{\mu(t\wedge\tau_n)} U^+_{t\wedge\tau_n} \le e^{\mu\tau_n} U^+_{\tau_n} &- \int_{t\wedge\tau_n}^{\tau_n} e^{\mu s} \mathbf{1}_{U_s>0} V_s \cdot dB_s \\ &+ \int_{t\wedge\tau_n}^{\tau_n} e^{\mu s} \left\{\mathbf{1}_{U_s>0} \left(f(s, Y_s, Z_s) - f'\left(s, Y'_s, Z'_s\right)\right) - \mu U^+_s\right\} ds. \end{aligned} \tag{8}$$

First of all, we write

$$\begin{aligned} f(s, Y_s, Z_s) - f'\left(s, Y'_s, Z'_s\right) = {} & f(s, Y_s, Z_s) - f\left(s, Y'_s, Z_s\right) + f\left(s, Y'_s, Z_s\right) \\ & - f'\left(s, Y'_s, Z'_s\right) \end{aligned}$$

and we deduce, using the monotonicity of f in y that

$$\begin{aligned} \mathbf{1}_{U_s>0} \left(f(s, Y_s, Z_s) - f'\left(s, Y'_s, Z'_s\right)\right) - \mu U^+_s \le {} & \mathbf{1}_{U_s>0}\big(f\left(s, Y'_s, Z_s\right) \\ & - f'\left(s, Y'_s, Z'_s\right)\big). \end{aligned}$$

But $f\left(s, Y'_s, Z'_s\right) - f'\left(s, Y'_s, Z'_s\right)$ is nonpositive so that

$$\begin{aligned} \mathbf{1}_{U_s>0} \left(f(s, Y_s, Z_s) - f'\left(s, Y'_s, Z'_s\right)\right) - \mu U^+_s \le {} & \mathbf{1}_{U_s>0}\big(f\left(s, Y'_s, Z_s\right) \\ & - f\left(s, Y'_s, Z'_s\right)\big). \end{aligned}$$

Finally we set

$$\beta_s = \frac{\left(f\left(s, Y'_s, Z_s\right) - f\left(s, Y'_s, Z'_s\right)\right) V_s}{|V_s|^2}$$

which is a process bounded by λ. Coming back to (8), we obtain the following inequality

$$e^{\mu(t\wedge\tau_n)}U^+_{t\wedge\tau_n} \le e^{\mu\tau_n}U^+_{\tau_n} + \int_{t\wedge\tau_n}^{\tau_n} e^{\mu s}\mathbf{1}_{U_s>0}\beta_s\cdot V_s\,ds - \int_{t\wedge\tau_n}^{\tau_n} e^{\mu s}\mathbf{1}_{U_s>0}V_s\cdot dB_s$$

By Girsanov's theorem, we deduce that

$$\mathbb{E}^*\left[e^{\mu(t\wedge\tau_n)}U^+_{t\wedge\tau_n}\right] \le \mathbb{E}^*\left[e^{\mu\tau_n}U^+_{\tau_n}\right],$$

where $\mathbb{P}^*$ is the probability measure on $(\Omega,\mathcal{F}_T)$ whose density with respect to $\mathbb{P}$ is

$$D_T = \exp\left\{\int_0^T \beta_s\cdot dB_s - \frac{1}{2}\int_0^T |\beta_s|^2 ds\right\};$$

it is worth noting that, since β is a bounded process, D_T has moments of all order. Since we know that U^+ belongs to $\mathcal{S}$, we can easily send n to infinity to get

$$\mathbb{E}^*\left[e^{\mu t}U^+_t\right] \le 0.$$

Thus $U_t \le 0$ $\mathbb{P}^*$–a.s. and since $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F}_T)$, $Y_t \le Y'_t$ $\mathbb{P}$–a.s.. □

5.2. One extension

In this paragraph, we explain how we can extend our results to a more general setting allowing a superlinear growth of the generator in the variable y as in the work [6].

Let $h:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be a nondecreasing convex $\mathcal{C}^1$ function with $h(0)>0$ such that

$$\int_0^{+\infty}\frac{du}{h(u)} = +\infty.$$

We denote (H1') the assumption: there exists $\gamma>0$ such that $\mathbb{P}$–a.s.

$$\begin{aligned} &\forall t\in[0,T], \qquad (y,z)\longmapsto f(t,y,z) \text{ is continuous},\\ &\forall (t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d, \qquad |f(t,y,z)|\le h(|y|)+\frac{\gamma}{2}|z|^2, \\ &\sup_{y>0} e^{-\gamma y}h(y) < +\infty. \end{aligned} \tag{H1'}$$

Let us point out that the previous setting, namely the linear growth condition, corresponds to $h(y)=\alpha+\beta y$ but we can also have a superlinear growth in y; for instance, we can take $h(y)=\alpha\,(y+e)\ln(y+e)$.

Before giving our integrability condition for the terminal value ξ, let us explain what is the first modification we have to do. We consider only the case where h is not constant. According to the third point of (H1'), let us denote by $c=\sup_{p\in(0,1)}\gamma p h\left(-\frac{\ln p}{\gamma}\right)$ and let us define

$$p_0 = \inf\left\{p\ge 1 : \gamma p h\left(\frac{\ln p}{\gamma}\right)\ge c\right\}.$$

We define finally

$$H(p) = \gamma p h\left(\frac{\ln p}{\gamma}\right)\mathbf{1}_{p\geq p_0} + c\mathbf{1}_{p<p_0}.$$

Then H is convex and we have the following result.

Lemma 6. *Let $z \in \mathbb{R}$. The integral equation*

$$\phi_t = e^{\gamma z} + \int_t^T H(\phi_s)\,ds, \qquad 0 \leq t \leq T,$$

has a unique continuous solution $\{\phi_t(z)\}_{0\leq t\leq T}$ which is decreasing. Moreover, for each $t \in [0, T]$, the map $z \longmapsto \phi_t(z)$ is increasing and continuous.

Proof. ϕ_t is solution if and only if $u_t = \ln \phi_t/\gamma$ is a solution of the differential equation

$$u_t' = -\theta(u_t), \quad 0 \leq t \leq T, \quad u_T = z \geq 0,$$

where $\theta(x) = h(x)\mathbf{1}_{x\geq \frac{\ln p_0}{\gamma}} + \frac{c}{\gamma}e^{-\gamma x}\mathbf{1}_{x<\frac{\ln p_0}{\gamma}}$. Let us consider the function Θ defined by

$$\Theta(x) = \int_{-\infty}^{x} \frac{1}{\theta(u)}\,du, \quad x \in \mathbb{R}.$$

Since θ is positive, Θ is an increasing bijection from $\mathbb{R}$ onto $(0, \infty)$ of class $\mathcal{C}^1$. It's plain to check that the unique solution to the previous differential equation is $\Theta^{-1}(T - t + \Theta(z))$ since for any solution we have $\Theta(u_t)' = -1$. Thus

$$\phi_t = e^{\gamma\Theta^{-1}(T-t+\Theta(z))}$$

and the proof of the lemma is complete. □

We are now in position to give our second assumption.

$$\phi_0(|\xi|) \text{ is integrable.} \tag{H2'}$$

Exactly as in the linear case, we can prove the following existence result that generalizes Theorem 2.

Theorem 7. *Let assumptions* (H1') *and* (H2') *hold. Then the BSDE* (1) *has at least a solution (Y, Z) such that :*

$$-\frac{1}{\gamma}\ln \mathbb{E}\left(\phi_t(-\xi) \mid \mathcal{F}_t\right) \leq Y_t \leq \frac{1}{\gamma}\ln \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right).$$

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