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Invariant measures and disintegrations with applications to Palm and related kernels

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Abstract Consider a locally compact group G acting measurably on some spaces S and T. We prove a general representation of G-invariant measures on S and the existence of invariant disintegrations of jointly invariant measures on $S \times T$. The results are applied to Palm and related kernels associated with a stationary random pair (ξ, η), where ξ is a random measure on S and η is a random element in T.

Keywords Invariant measures and kernels · Disintegration · Skew factorization · Absolute continuity · Stationary random measures · Palm, Campbell, and supporting measures · Shift coupling · Gibbs and Papangelou kernels

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1 Introduction

Kernels—in the sense of measurable functions into a measure space—clearly abound in probability theory, appearing as conditional distributions, Markov transition functions, potential kernels, random measures, point processes, Palm measures, or Gibbs and Papangelou kernels. When the underlying distribution is invariant under a group of transformations, one expects those kernels to be invariant too, in an appropriate sense. In view of their non-uniqueness, it

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becomes an important and non-trivial problem to establish the existence of invariant versions.

Keeping such applications in mind, we prove in Sect. 2 a general representation of invariant measures on an abstract space S, when the underlying group G is locally compact and its action on S is proper but not necessarily transitive. The invariance problem for kernels is discussed in Sect. 3. Here we consider a locally compact group G, acting measurably on two abstract spaces S and T, along with a kernel μ from S to T. Typically, μ arises as a disintegration kernel of a jointly invariant measure M on $S \times T$, and we need to find a suitably invariant version of μ. This also involves the construction of an invariant supporting measure on S. Two totally different approaches are considered here, based on skew factorization and regularization (often called perfection), respectively.

The rest of the paper deals with some probabilistic applications. Thus, the basic invariance theorems for Palm, Gibbs, and Papangelou kernels appear in Sect. 5. Section 6 contains some characterizations of Palm measures and a related coupling theorem. In Sect. 7, we consider the dual problem of finding invariant representations of stationary random measures, including those in terms of stationary random measures on the underlying group. The preliminary Sect. 4 introduces the various kinds of Palm kernels and explains their connection to ordinary conditioning. Prompted by some remarks of a referee, we emphasize that all major results in this paper are (believed to be) entirely new, not just technical improvements of old results, unless something else is said to the contrary.

We turn to some historical and bibliographical comments. Invariant measures form a classical subject, covered by numerous textbooks and monographs, such as Hewitt and Ross [9]. Some basic probabilistic aspects are explored by Dynkin [4], and some recent developments are given by Schindler [29]. Elementary introductions appear in [12,27]. Invariant disintegrations, along with some applications to stochastic geometry, are discussed in [17].

The study of Palm measures goes back to the pioneering work of Palm [23], Khinchin [16], Kaplan [15], Ryll-Nardzewski [28], Slivnyak [30], Matthes [18], and Mecke [21]. Originally devised as a tool in queuing theory, as clarified by the modern expositions in [1,6], their importance for applications is not restricted to point processes, as decisively demonstrated by some recent work on superprocesses (Dawson [3]) and regenerative processes (Kallenberg [13]). The Papangelou and Gibbs kernels were discovered and explored by Papangelou [25,26] and Kallenberg [10,11], in work motivated by applications to stochastic geometry and statistical mechanics (cf. [20]).

Basic facts about random measures may be gathered from [11]. A comprehensive and diverse introduction to Palm measures is given in [2]. Further information on this rich subject appears in [11,19,33]. Elementary introductions are offered in [12,22].

We conclude with some remarks on notation. For measures μ, measurable functions f, and measurable subsets B on a common space S, we define $\mu f = \int f d\mu$ and $\mu[f;B] = \int_B f d\mu$. When $f \geq 0$, we write $f \cdot \mu$ for the measure $\nu \ll \mu$ with μ-density f, where $\ll$ denotes absolute continuity. The relation $\mu \sim \nu$

means that both $\mu \ll \nu$ and $\nu \ll \mu$. The restriction $1_B \cdot \mu$ of μ to B is often written as $1_B\mu$. All random elements are defined on an abstract probability space $(\Omega, \mathcal{A}, P)$, and we write $E\xi = \int \xi dP$ and $E[\xi; A] = \int_A \xi dP$ as before. We always assume Ω to be rich enough to support any randomization variables we may need. Probability distributions are written as $\mathcal{L}(\xi) = P\{\xi \in \cdot\}$, and $\xi \overset{d}{=} \eta$ means that $\mathcal{L}(\xi) = \mathcal{L}(\eta)$.

2 Invariant measures

Throughout this paper, we assume G to be an lcsc (locally compact, second countable Hausdorff) group with left and right Haar measures λ and $\tilde{\lambda}$ and with modular function Δ. When G acts on a space S, we define the associated *shifts* θ_r and *projections* π_s by $\theta_r s = \pi_s r = rs$, $r \in G$, $s \in S$. The sets $\pi_s G$ are called *orbits*, and the action is said to be *transitive* if $\pi_s G = S$ for all s. If even S is a topological space, then the action is said to be *topologically proper* if each projection π_s is continuous and such that $\pi_s^{-1} K$ is compact in G for every compact set $K \subset S$. The following result suggests a non-topological notion of properness. Here and below $\mathbb{N} = \{1, 2, \dots\}$.

Lemma 2.1 *For G acting measurably on S, these conditions are equivalent*:

(i) *There exists a measurable function $g > 0$ on S such that $\lambda(g \circ \pi_s) < \infty$ for all $s \in S$.*
(ii) *There exists a measurable partition $B_1, B_2, \dots$ of S such that $\lambda(\pi_s^{-1} B_k) < \infty$ for all $k \in \mathbb{N}$ and $s \in S$.*

Proof Suppose that (i) holds for some measurable function $g > 0$. Define

$$B_k = g^{-1}[k^{-1}, (k-1)^{-1}), \quad k \in \mathbb{N},$$

where $0^{-1} = \infty$. By Fubini's theorem, we get for any $s \in S$,

$$\sum\nolimits_k k^{-1} \lambda(\pi_s^{-1} B_k) = \lambda \sum\nolimits_k k^{-1} 1_{B_k} \circ \pi_s \leq \lambda(g \circ \pi_s) < \infty,$$

which shows that the sets $B_1, B_2, \dots$ satisfy (ii).

Conversely, assuming (ii) to hold for some sets $B_1, B_2, \dots$, we write

$$\begin{array}{lll} b_k(s) = (\lambda \circ \pi_s^{-1}) B_k, & s \in S, & k \in \mathbb{N}, \\ m(s) = \min\{k \in \mathbb{N};\, b_k(s) > 0\}, & s \in S, & \\ A_k = \{s \in S;\, m(s) = k\}, & k \in \mathbb{N}. & \end{array}$$

The A_k form a measurable partition of S, since $\sum_{k \geq 1} b_k(s) = \lambda G > 0$. Hence, the function

$$g(s) = \sum\nolimits_k 1_{A_k}(s) \sum\nolimits_n 2^{-n} 1_{B_n}(s) \frac{b_k(s)}{b_k(s) + b_n(s)}, \quad s \in S,$$

is measurable and satisfies $0 < g(s) < \infty$ for all $s \in S$. Since also

$$b_k(rs) = \int \lambda(dp)\, 1_{B_k}(prs) = \Delta(r) \int \lambda(dp)\, 1_{B_k}(ps) = \Delta(r)\, b_k(s),$$

and hence $1_{A_k}(rs) = 1_{A_k}(s)$, we have

$$\begin{aligned}
(\lambda \circ \pi_s^{-1})g &= \sum\nolimits_{k,n} 2^{-n} \int \lambda(dr)\, 1_{A_k \cap B_n}(rs)\, \frac{b_k(rs)}{b_k(rs) + b_n(rs)} \\
&= \sum\nolimits_{k,n} 2^{-n}\, 1_{A_k}(s)\, \frac{b_k(s)}{b_k(s) + b_n(s)} \int \lambda(dr)\, 1_{B_n}(rs) \\
&\le \sum\nolimits_k 1_{A_k}(s)\, b_k(s) \sum\nolimits_n 2^{-n} \\
&= b_{m(s)}(s) = (\lambda \circ \pi_s^{-1}) B_{m(s)} < \infty,
\end{aligned}$$

which shows that even (i) is fulfilled. □

When the conditions of Lemma 2.1 are satisfied, we say that G acts *properly* on S. This is trivially true when G is compact. If even S is lcsc and the action of G is continuous and topologically proper, then (ii) is clearly fulfilled for any partition of S into relatively compact, measurable subsets $B_1, B_2, \ldots$. Thus, the present notion of properness is then implied by the earlier, topological version.

When G acts measurably on $(S, \mathcal{S})$, we write $\mathcal{I}_S$ for the *G-invariant σ-field* in S, consisting of all sets $B \in \mathcal{S}$ such that $\theta_r^{-1} B = B$ for every $r \in G$. A measure μ on S is said to be *G-invariant* if $\mu \circ \theta_r^{-1} = \mu$ for all $r \in G$ and *s-finite* if $\mu_n \uparrow \mu$ for some bounded measures μ_n on S. Note that Fubini's theorem, usually proved for σ-finite measures, remains valid for s-finite ones.

Lemma 2.2 *Let G act measurably on S, and let μ and ν be s-finite, G-invariant measures on S. Then $\mu \ll \nu$ iff the same relation holds on $\mathcal{I}_S$. This holds automatically when G acts transitively on S and $\nu \neq 0$.*

Proof Assume that $\mu \ll \nu$ on $\mathcal{I}_S$, and fix any $B \in \mathcal{S}$ with $\nu B = 0$. Define $f(s) = \int 1_B(rs)\, \tilde{\lambda}(dr)$ and $A = \{s \in S; f(s) > 0\}$, and note that f and A are $\mathcal{S}$-measurable by Fubini's theorem. For any $r \in G$ we have $f(rs) = f(s)$ by the right invariance of $\tilde{\lambda}$, and therefore $\theta_r^{-1} A = A$, which shows that A is G-invariant and hence belongs to $\mathcal{I}_S$. Next, we may use Fubini's theorem for s-finite measures and the G-invariance of ν to get $\nu f = \tilde{\lambda}(G)\, \nu(B) = 0$, which implies $f = 0$ a.e. ν. This gives $\nu A = 0$, and so by hypothesis $\mu A = 0$, which implies $\mu f = 0$. Finally, by the s-finite version of Fubini's theorem and the G-invariance of μ, we have $\tilde{\lambda}(G)\, \mu(B) = \mu f = 0$, which implies $\mu B = 0$ since $\tilde{\lambda} \neq 0$. Thus, the relation $\mu \ll \nu$ extends to $\mathcal{S}$. The last assertion is obvious since $\mathcal{I}_S = \{\emptyset, S\}$ in the transitive case. □

We also need the following invariant version of the Radon–Nikodym theorem. The result is a special case of Theorem 3.5 (for T a singleton set), whose proof is independent of all results in this section.

Lemma 2.3 *Let G act measurably on S, and let $\mu \ll \nu$ be σ-finite and G-invariant measures on S. Then $\mu = f \cdot \nu$ for some measurable and G-invariant function $f \geq 0$ on S.*

Writing M for the convex set of G-invariant measures in $\mathcal{M}_S$, the class of σ-finite measures on S, we say that $\nu \in M$ is *extreme* if the relation $\nu = \nu' + \nu''$ with $\nu', \nu'' \in M$ implies that both ν' and ν'' are proportional to ν. We also say that ν is *ergodic* if $\nu A \wedge \nu A^c = 0$ for every $A \in \mathcal{I}_S$. When G acts properly on S, we define a kernel φ on S by

$$\varphi_s = \frac{\lambda \circ \pi_s^{-1}}{\lambda(g \circ \pi_s)}, \quad s \in S, \tag{1}$$

where g is such as in Lemma 2.1. The following result gives a unique representation of G-invariant measures on S.

Theorem 2.4 *Let G act properly on a Borel space S, fix g as in Lemma 2.1, and define a kernel φ by (1). Then a σ-finite measure ν on S is G-invariant iff $\nu = \int m\, \mu(dm)$ for some measure μ on the range $\varphi(S) = \{\varphi_s;\ s \in S\}$, in which case μ is unique and given by $\mu = (g \cdot \nu) \circ \varphi^{-1}$. The measures φ_s are either singular or equal, and ν is ergodic or extreme iff it is proportional to some φ_s, which is always true when G acts transitively on S.*

Though invariant measures form a classical subject covered by a vast literature (for different aspects, see [4,9,14,17,29]), the present result seems to be new. Only the transitive case is straightforward (cf. [27, p. 384] or [12, p. 41]). In the general case, our key idea is to use the kernel φ to provide a measurable labeling of the orbits.

Proof The measures φ_s are σ-finite since $\varphi_s g = 1$ for all $s \in S$, and the left-invariance of λ yields the G-invariance

$$\varphi_s \circ \theta_r^{-1} = \varphi_s, \quad r \in G,\ s \in S. \tag{2}$$

Noting that $\lambda \circ \pi_{rs}^{-1} = \Delta(r)\, \lambda \circ \pi_s^{-1}$ for all r and s, we see that also

$$\varphi_{rs} = \varphi_s, \quad r \in G,\ s \in S. \tag{3}$$

The mapping $s \mapsto \varphi_s$ is measurable by Fubini's theorem, and the diagonal D in $\mathcal{M}_S^2$ is measurable since $\mathcal{M}_S$ inherits the Borel property from S. Hence, the graph $\Phi^{-1}D = \{(s, \varphi_s);\ s \in S\}$ is measurable in $S \times \mathcal{M}_S$, where $\Phi(s,m) = (\varphi_s, m)$ on $S \times \mathcal{M}_S$. Finally, the projection $\varphi(S)$ of $\Phi^{-1}D$ onto $\mathcal{M}_S$ is universally measurable since S is Borel.

For any G-invariant measure ν on S, we define a measure $\tilde{\nu}$ on S by

$$\tilde{\nu} = \int g(s)\, \nu(ds)\, \varphi_s = \int m\, \mu(dm), \tag{4}$$

where $\mu = (g \cdot \nu) \circ \varphi^{-1}$. Note that $\tilde{\nu}$ is G-invariant by (2) and also σ-finite since $\tilde{\nu}g = \nu g < \infty$. If $A \in \mathcal{I}_S$ with $\tilde{\nu}A = 0$, we have

$$0 = \tilde{\nu}A = \lambda G \int_A \frac{g(s)\,\nu(ds)}{\lambda(g \circ \pi_s)}, \tag{5}$$

and we get $\nu A = 0$ since $g > 0$. This gives $\nu \ll \tilde{\nu}$ on $\mathcal{I}_S$, which extends to $\mathcal{S}$ by Lemma 2.2. Hence, Lemma 2.3 yields $\nu = h \cdot \tilde{\nu}$ for some measurable and G-invariant function $h \geq 0$ on S.

To identify h, we note that $\varphi_s(hf) = h(s)\,\varphi_s f$ for any $s \in S$ and measurable function $f \geq 0$, and so

$$\nu f = \tilde{\nu}(hf) = \int g(s)\,h(s)\,\nu(ds)\,\varphi_s f.$$

Applying this to the functions

$$f_\pm(s) = g(s)\,1\{(h(s) - 1)_\pm > 0\}, \quad s \in S,$$

where $1\{\cdot\} = 1_{\{\cdot\}}$, and using the G-invariance of h, we obtain

$$\int \nu(ds)\,g(s)\,(h(s) - 1)\,1\{(h(s) - 1)_\pm > 0\} = 0,$$

which yields $\nu(g|h - 1|) = 0$ and hence $h = 1$ a.e. ν. This gives $\nu = \tilde{\nu}$, and the representation (4) remains valid for ν. In the transitive case, (3) shows that φ_s is independent of s, and (4) reduces to $\nu = (\nu g)\,\varphi_s$ for all $s \in S$.

For measurable $M \subset \mathcal{M}_S$, the set $A = \varphi^{-1}M \subset S$ is G-invariant by (3), and so $\varphi_s[g; A] = 1_A(s) = 1_M(\varphi_s)$ for all $s \in S$. Assuming ν to be G-invariant with a representation $\int m\,\mu(dm)$, where μ is restricted to $\varphi(S)$, we get

$$\nu[g; A] = \int m[g; A]\,\mu(dm) = \mu M. \tag{6}$$

In particular, μ is uniquely determined by ν.

For any $s \in S$, consider a σ-finite and G-invariant measure $\nu \ll \varphi_s$. By Lemma 2.3 we have $\nu = h \cdot \varphi_s$ for some measurable and G-invariant function $h \geq 0$ on S, and so $\nu = h(s)\,\varphi_s$, which shows that ν is proportional to φ_s. In particular, if $\varphi_s = \nu + \nu'$ for some G-invariant measures ν and ν', then ν and ν' are both proportional to φ_s, which means that φ_s is extreme.

For any $s, t \in S$, consider the Lebesgue decomposition $\varphi_t = \nu + \nu'$ of φ_t with respect to φ_s, so that $\nu \ll \varphi_s$ and $\nu' \perp \varphi_s$. Then (2) yields for any $r \in G$

$$\varphi_t = \varphi_t \circ \theta_r^{-1} = \nu \circ \theta_r^{-1} + \nu' \circ \theta_r^{-1},$$

and also

$$\nu \circ \theta_r^{-1} \ll \varphi_s \circ \theta_r^{-1} = \varphi_s,$$
$$\nu' \circ \theta_r^{-1} \perp \varphi_s \circ \theta_r^{-1} = \varphi_s.$$

The uniqueness of the decomposition gives $\nu \circ \theta_r^{-1} = \nu$ and $\nu' \circ \theta_r^{-1} = \nu'$, which means that ν and ν' are again G-invariant. Since φ_t is extreme, we conclude that ν and ν' are both proportional to φ_t, and since $\nu \perp \nu'$, we have either $\nu = 0$ or $\nu' = 0$. Here $\nu = 0$ implies $\varphi_t \perp \varphi_s$. If instead $\nu' = 0$, we get $\varphi_t \ll \varphi_s$, and it follows as before that $\varphi_t = c\varphi_s$ for some constant $c \geq 0$. Noting that $1 = \varphi_t g = c\varphi_s g = c$, we obtain $\varphi_t = \varphi_s$. This shows that φ_s and φ_t are either singular or equal.

Now consider any extreme, G-invariant measure ν on S, represented as in (4) with $\mu = (g \cdot \nu) \circ \varphi^{-1}$. If μ is non-degenerate, we may choose a measurable set $M \subset \mathcal{M}_S$ such that $\mu M \wedge \mu M^c > 0$. Writing $A = \varphi^{-1}M$, we see from (6) that $\nu[g; A] \wedge \nu[g; A^c] > 0$, which implies $\nu A \wedge \nu A^c > 0$ since $g > 0$. The restrictions $1_A \cdot \nu$ and $1_{A^c} \cdot \nu$ are G-invariant since this is true for both A and ν. Since they are also non-zero and mutually singular, this contradicts the extremality of ν. We conclude that μ is degenerate and hence supported by a single measure φ_s, which means that ν is proportional to φ_s.

For any $A \in \mathcal{I}_S$, we have as in (5)

$$\varphi_s A \wedge \varphi_s A^c = \frac{\lambda G}{\lambda(g \circ \pi_s)} (1_A(s) \wedge 1_{A^c}(s)) = 0, \quad s \in S,$$

which shows that each φ_s is ergodic. Conversely, suppose that $\nu = \int m\, \mu(dm)$ is G-invariant and ergodic. If μ is not degenerate, we may choose $M \subset \mathcal{M}_S$ to be measurable with $\mu M \wedge \mu M^c > 0$. As before, we see that $A = \varphi^{-1}M$ is G-invariant with $\nu A \wedge \nu A^c > 0$, which contradicts the ergodicity of ν. Hence, μ is degenerate, which means that ν is proportional to some φ_s. □

Similar ideas can be used to construct σ-finite, invariant measures.

Proposition 2.5 *Let G act properly on S. Then for any s-finite, G-invariant measure μ on S, there exists a σ-finite, G-invariant measure $\nu \sim \mu$ on S, and ν is unique up to a normalization when the group action is transitive.*

Proof Since μ is s-finite, there exist some bounded measures $\mu_n \uparrow \mu$ on S, and we may define a bounded measure ρ on S by

$$\rho = \sum\nolimits_n 2^{-n} \frac{\mu_n}{\mu_n S \vee 1}.$$

For g as in Lemma 2.1 and the φ_s as in (1), we define $\nu = \int \rho(ds)\, \varphi_s$. Then ν is σ-finite since $\nu g = \rho S < \infty$, and by (2) it is also G-invariant.

For any $A \in \mathcal{S}$ we have $\mu A = 0$ iff $\mu_n A = 0$ for all n, which is equivalent to $\rho A = 0$. This shows that $\rho \sim \mu$. We also have $\nu \sim \rho$ on $\mathcal{I}_S$ since

$$\nu A = \lambda G \int_A \frac{\rho(ds)}{\lambda(g \circ \pi_s)}, \quad A \in \mathcal{I}_S.$$

Hence, $\nu \sim \mu$ on $\mathcal{I}_S$, which extends to $\mathcal{S}$ by Lemma 2.2. In the transitive case, (3) shows that φ_s is independent of s, and the last assertion follows as in Theorem 2.4. □

Finally, Lemma 2.3 yields a short proof for the uniqueness of invariant measures in the transitive case (cf. [12, p. 41]):

Corollary 2.6 *Let G act measurably and transitively on S. Then any two σ-finite, G-invariant measures on S agree up to a normalization.*

Proof Let ν_1 and ν_2 be σ-finite and G-invariant measures on S. Then $\nu_1 \ll \nu_1 + \nu_2 \equiv \nu$, and so by Lemma 2.3 we have $\nu_1 = h \cdot \nu$ for some G-invariant, measurable function $h \geq 0$ on S. Since the group action is transitive, h is just a constant $c \geq 0$, and we get $(1-c)\nu_1 = c\nu_2$, which means that ν_1 and ν_2 are proportional. □

3 Invariant kernels and disintegrations

Given two measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$, a *kernel* from S to T is defined as a function $\mu \geq 0$ on $S \times \mathcal{T}$ such that $\mu(s, B)$ is a σ-finite measure in B for fixed s and a measurable function of s for fixed B. (For basic properties of kernels and their products, see [12, pp. 20f.]) A *disintegration* (along S) of a measure M on $S \times T$ is a representation $M = \nu \otimes \mu$, where ν is a σ-finite measure on S and μ is a kernel from S to T. We begin with the basic existence result.

Lemma 3.1 *Let M be a σ-finite measure on $S \times T$, where T is Borel. Then $M = \nu \otimes \mu$ for some σ-finite measure ν on S and kernel μ from S to T. Here $\mu_s T < \infty$ for $s \in S$ a.e. ν iff $M(\cdot \times T)$ is σ-finite.*

Proof Choose a measurable function $f > 0$ on $S \times T$ with $Mf = 1$, and apply the existence theorem for conditional distributions (cf. [12, p. 107]) to the probability measure $f \cdot M$. We may choose $\nu = M(\cdot \times T)$, equivalent to $\mu_s T = 1$ a.e., iff $M(\cdot \times T)$ is σ-finite. □

Now suppose that G acts measurably on S and T, and let M be a measure on $S \times T$ with disintegration $\nu \otimes \mu$. Then for any $r \in G$ we note that $M \circ \theta_r^{-1} = \tilde{\nu} \otimes \tilde{\mu}$, where $\tilde{\nu} = \nu \circ \theta_r^{-1}$ and $\tilde{\mu}_{rs} = \mu_s \circ \theta_r^{-1}$. In the special case where M is jointly G-invariant, we get $\nu \otimes \mu = \tilde{\nu} \otimes \tilde{\mu}$, which suggests that we look for disintegrations $M = \nu \otimes \mu$ satisfying $\nu = \tilde{\nu}$ and $\mu = \tilde{\mu}$ for all $r \in G$. Thus, we want ν and μ to be *G-invariant*, where the latter invariance is defined by

$$\mu_{rs} = \mu_s \circ \theta_r^{-1}, \quad r \in G,\ s \in S. \tag{7}$$

We consider two methods of constructing invariant disintegrations. The easiest case is when S contains the group G as a factor.

Theorem 3.2 *Let G act measurably on S and T, where T is Borel, and consider a σ-finite, jointly G-invariant measure M on $G \times S \times T$. Then $M = \lambda \otimes \nu \otimes \mu$ for some G-invariant kernels ν from G to S and μ from $G \times S$ to T, given by*

$$\nu_r = \hat{\nu} \circ \theta_r^{-1}, \quad \mu_{r,s} = \hat{\mu}_{r^{-1}s} \circ \theta_r^{-1}, \qquad r \in G, \ s \in S,$$

for some measure $\hat{\nu}$ on S and kernel $\hat{\mu}$ from S to T. Here the measure $\hat{\nu} \otimes \hat{\mu}$ on $S \times T$ is unique.

Note that $\lambda \otimes \nu$ is again invariant, which makes $(\lambda \otimes \nu) \otimes \mu$ an invariant disintegration of M. Our proof begins with a simple factorization.

Lemma 3.3 *Let the σ-finite measure μ on $G \times S$ be left-invariant under shifts in G only. Then $\mu = \lambda \otimes \nu$ for a unique, σ-finite measure ν on S.*

Here the difficulty is to show that μ remains σ-finite on the class of measurable rectangles (cf. [2, pp. 461–462]).

Proof Fix any nonempty, open, relatively compact set $B \subset G$, and put

$$b(r) = \int 1_B(pr)\, \lambda(dp), \quad r \in G.$$

Since all right translates of B have the same properties, we get $0 < b(r) < \infty$ for $r \in G$. By the σ-finiteness of μ, we may next choose a measurable function $h > 0$ on $G \times S$ such that the function $b(r)h(r, s)$ is μ-integrable. Define

$$\hat{h}(s) = \int h(r^{-1}, s)\, \lambda(dr), \quad s \in S,$$
$$\hat{\mu}_A = \int_A \hat{h}(s)\, \mu(\cdot \times ds), \quad A \in \mathcal{S}.$$

Using the left invariance of λ and μ and Fubini's theorem, we get

$$\hat{\mu}_S B = \iint b(r)\, h(r, s)\, \mu(dr\, ds) < \infty.$$

Noting that each $\hat{\mu}_A$ is again left-invariant, we get $\hat{\mu}_A(rB) < \infty$ for every $r \in G$, and since every compact set in G is covered by finitely many translates rB, we conclude that the measures $\hat{\mu}_A$ are Radon. By the uniqueness of Haar measure up to a normalization, we obtain $\hat{\mu}_A = \rho_A \lambda$ for some constants ρ_A. Since $\hat{\mu}_A B$ is a bounded measure in A, the same thing is true for ρ_A, and we get $\hat{h} \cdot \mu = \lambda \otimes \rho$, which implies $\mu = \lambda \otimes \nu$ with $\nu = \hat{h}^{-1} \cdot \rho$. $\square$

The last lemma leads easily to the following *skew factorization*.

Lemma 3.4 *Let G act measurably on S, and consider a σ-finite, jointly G-invariant measure μ on $G \times S$. Then $\mu = (\lambda \otimes \nu) \circ \vartheta^{-1}$ for a unique, σ-finite measure ν on S, where*

$$\vartheta(r,s) = (r, rs), \quad r \in G,\ s \in S.$$

Proof Define the shifts θ_r and θ'_r on $G \times S$ by

$$\theta_r(p,s) = (rp, rs), \quad \theta'_r(p,s) = (rp, s), \quad p, r \in G,\ s \in S,$$

and note that $\vartheta^{-1} \circ \theta_r = \theta'_r \circ \vartheta^{-1}$ for all $r \in G$. Since μ is jointly G-invariant, we get

$$\mu \circ \vartheta \circ \theta_r'^{-1} = \mu \circ \theta_r^{-1} \circ \vartheta = \mu \circ \vartheta,$$

where $\vartheta = (\vartheta^{-1})^{-1}$. Hence, $\mu \circ \vartheta$ is invariant under the shifts θ'_r, and Lemma 3.3 yields the factorization $\mu \circ \vartheta = \lambda \otimes \nu$. Now apply ϑ^{-1} to both sides. □

Proof of Theorem 3.2 By Lemma 3.4 we have $M = (\lambda \otimes \rho) \circ \vartheta^{-1}$ for some σ-finite measure ρ on $S \times T$, where the mapping ϑ on $G \times S \times T$ is given by $\vartheta(r,s,t) = (r, rs, rt)$. Introducing a further disintegration $\rho = \hat{\nu} \otimes \hat{\mu}$ in terms of a measure $\hat{\nu}$ on S and a kernel $\hat{\mu}$ from S to T, we get $M = (\lambda \otimes \hat{\nu} \otimes \hat{\mu}) \circ \vartheta^{-1}$. Now define the kernels μ and ν by (19), and note that they are both G-invariant, in the sense that $\nu_r \circ \theta_p^{-1} = \nu_{pr}$ and $\mu_{r,s} \circ \theta_p^{-1} = \mu_{pr,ps}$ for all $p, r \in G$ and $s \in S$. To prove the required disintegration, we may write

$$\begin{aligned}(\lambda \otimes \nu \otimes \mu)f &= \int \lambda(dr) \int \nu_r(ds) \int \mu_{r,s}(dt) f(r,s,t) \\ &= \int \lambda(dr) \int \hat{\nu}(ds) \int \hat{\mu}_s(dt) f(r, rs, rt) \\ &= ((\lambda \otimes \hat{\nu} \otimes \hat{\mu}) \circ \vartheta^{-1})f = Mf,\end{aligned}$$

for any measurable function $f \geq 0$, where the iterated integrals should be read from right to left. The uniqueness of $\hat{\nu} \otimes \hat{\mu}$ follows from the same computation together with the uniqueness in Lemma 3.4. □

If an invariant supporting measure is already known, we can construct an associated invariant disintegration kernel by a suitable regularization.

Theorem 3.5 *Let G act measurably on S and T, where T is Borel. Consider some σ-finite, jointly G-invariant measures ν on S and M on $S \times T$ such that $M(\cdot \times T) \ll \nu$. Then $M = \nu \otimes \mu$ for a G-invariant kernel μ from S to T, which is unique when G acts transitively on S.*

Proof Since M and ν are σ-finite with $M(\cdot \times T) \ll \nu$ and T is Borel, there exists a kernel μ from S to T such that $M = \nu \otimes \mu$. Using the invariance of ν and M, we get

$$\mu_{rs} = \mu_s \circ \theta_r^{-1}, \quad s \in S \text{ a.e. } \nu, \;\; r \in G, \tag{8}$$

which may also be written as $\mu_s = \mu_{rs} \circ \theta_r$ for the same s and r. Now fix a measurable function $g \geq 0$ on G with $\tilde{\lambda} g = 1$. Since the mapping $(r,s) \mapsto \mu_{rs} \circ \theta_r$ on $G \times S$ is product measurable (cf. [12, p. 21]), we may define a kernel $\hat{\mu}$ from S to T by

$$\hat{\mu}_s = \int (g \cdot \tilde{\lambda})(dr)\, (\mu_{rs} \circ \theta_r), \quad s \in S.$$

Using (8) and Fubini's theorem gives

$$\mu_s = \mu_{rs} \circ \theta_r, \quad r \in G \text{ a.e. } \tilde{\lambda}, \;\; s \in S \text{ a.e. } \nu, \tag{9}$$

and so $\hat{\mu}_s = \mu_s$ a.e. ν, which shows that even $M = \nu \otimes \hat{\mu}$.

For any $r \in G$ and $s \in S$, we have

$$\hat{\mu}_{rs} \circ \theta_r = \int (g \cdot \tilde{\lambda})(dp)\, (\mu_{prs} \circ \theta_{pr}) = \iint (g \cdot \tilde{\lambda})^2 (dp\, dq)\, (\mu_{prs} \circ \theta_{pr}).$$

Using the right invariance of $\tilde{\lambda}$, we get for $r \in G$ and $s \in S$ and for any measurable function $f \geq 0$ on T,

$$\begin{aligned} |(\hat{\mu}_s - \hat{\mu}_{rs} \circ \theta_r) f| &= \left| \iint (g \cdot \tilde{\lambda})^2 (dp\, dq)\, (\mu_{ps} \circ \theta_p - \mu_{qrs} \circ \theta_{qr}) f \right| \\ &\leq \iint g(p)\, g(qr^{-1})\, \tilde{\lambda}^2 (dp\, dq)\, |(\mu_{ps} \circ \theta_p - \mu_{qs} \circ \theta_q) f|. \end{aligned}$$

Writing

$$A = \{ s \in S;\; \mu_{ps} \circ \theta_p = \mu_{qs} \circ \theta_q,\; (p,q) \in G^2 \text{ a.e. } \tilde{\lambda}^2 \},$$

we conclude that

$$\hat{\mu}_s = \hat{\mu}_{rs} \circ \theta_r, \quad r \in G, \;\; s \in A. \tag{10}$$

The set A is measurable by Fubini's theorem. Fixing any $r \in G$ and $s \in A$ and using the right invariance of $\tilde{\lambda}$ and the identity $\theta_{pr} = \theta_p \circ \theta_r$, we get

$$\tilde{\lambda}^2 \{ (p,q);\; \mu_{prs} \circ \theta_p \neq \mu_{qrs} \circ \theta_q \} = \tilde{\lambda}^2 \{ (p,q);\; \mu_{ps} \circ \theta_p \neq \mu_{qs} \circ \theta_q \} = 0,$$

and so $rs \in A$, which proves the invariance $\theta_r^{-1}A = A$. Finally, (9) yields $\nu A^c = 0$. Writing

$$\hat{\mu}'_s = 1_A(s)\, \hat{\mu}_s, \quad s \in S, \tag{11}$$

we get $\hat{\mu}'_s = \hat{\mu}_s = \mu_s$ a.e. ν, and so even $M = \nu \otimes \hat{\mu}'$. Using (10), (11), and the invariance of A, we note that $\hat{\mu}'_s = \hat{\mu}'_{rs} \circ \theta_r$ holds identically.

To prove the asserted uniqueness, suppose that $M = \nu \otimes \mu = \nu \otimes \mu'$ for some G-invariant kernels μ and μ'. Then $\mu_s = \mu'_s$ for $s \in S$ a.e. ν. Fixing any $s \in S$ with this property and using the invariance of μ and μ', we get $\mu_{rs} = \mu'_{rs}$ for any $r \in G$. In the transitive case, any element $t \in S$ equals rs for some $r \in G$, and it follows that $\mu = \mu'$. □

Additional regularity conditions may be needed to construct ν.

Corollary 3.6 *Let G act measurably on S and T, where T is Borel and the action on S is proper, and consider a σ-finite, jointly G-invariant measure M on $S \times T$. Then $M = \nu \otimes \mu$ for some σ-finite, G-invariant measure ν on S and G-invariant kernel μ from S to T. If the action on S is even transitive, then μ and ν are unique up to reciprocal normalizations.*

Proof The projection $M(\cdot \times T)$ is clearly s-finite and G-invariant, and so by Proposition 2.5 there exists a σ-finite, G-invariant measure $\nu \sim M(\cdot \times T)$. Since M is σ-finite and jointly G-invariant, Theorem 3.5 yields $M = \nu \otimes \mu$ for some G-invariant kernel μ from S to T. If the action of G on S is even transitive, then the same results show that ν and μ are unique up to reciprocal normalizations. □

Under suitable conditions, we can go even further and represent a G-invariant kernel μ from S to T in terms of an invariant kernel $\eta \circ \mu$ from S to G, where η is an invariant kernel from $\mathcal{M}_T$ to G and $(\eta \circ \mu)_s = \eta(\mu_s, \cdot)$.

Theorem 3.7 *Let G act measurably on S and T, where T is Borel and the action on T is proper and transitive, and fix an $a \in T$. Then for any σ-finite, G-invariant measure ν on S and G-invariant kernel μ from S to T, there exists a G-invariant kernel η from $\mathcal{M}_T$ to G such that*

$$\mu_s = (\eta \circ \mu)_s \circ \pi_a^{-1}, \quad s \in S \textit{ a.e. } \nu. \tag{12}$$

Proof Since ν and μ are both G-invariant and G acts properly and transitively on T, we see as in Proposition 2.5 that μ has a supporting measure $\rho = \lambda \circ \pi_a^{-1}$. Then Theorem 3.5 yields a G-invariant kernel ζ from T to $\mathcal{M}_T$ such that

$$\int \nu(ds) \int \mu_s(dt) f(t, \mu_s) = \int \rho(dt) \int \zeta_t(dm) f(t, m), \tag{13}$$

for any measurable function $f \geq 0$. Next, we may define an invariant kernel $\tilde{\zeta}$ from G to $\mathcal{M}_T$ by $\tilde{\zeta}_r = \zeta_{ra}$, $r \in G$. Letting $A \subset \mathcal{M}_T$ be measurable and

G-invariant with $\nu\{s;\, \mu_s \in A\} = 0$, we get by (13) and the G-invariance of ζ

$$0 = \int \nu(ds)\, \mu_s(T)\, 1_A(\mu_s) = \int \rho(dt)\, \zeta_t(A) = \lambda(G)\, \zeta_a(A),$$

and so $\tilde{\zeta}_r A = \zeta_{ra} A = \zeta_a A = 0$ for all $r \in G$. Hence, $(\lambda \otimes \tilde{\zeta})(G \times \cdot) \ll \nu\{s;\, \mu_s \in \cdot\}$ on the G-invariant σ-field in $\mathcal{M}_T$, which extends by Lemma 2.2 to the entire σ-field in $\mathcal{M}_T$, since both sides are G-invariant because of the invariance of λ, ν, $\tilde{\zeta}$, and μ. By another application of Theorem 3.5, there exists a G-invariant kernel η from $\mathcal{M}_T$ to G satisfying

$$\int \lambda(dr) \int \tilde{\zeta}_r(dm)\, f(r,m) = \int \nu(ds) \int \eta_{\mu_s}(dr)\, f(r,\mu_s). \tag{14}$$

Using (13), (14), and the definitions of $\tilde{\zeta}$ and ρ, we get for any measurable function $f \geq 0$ on $T \times \mathcal{M}_T$

$$\begin{aligned}\int \nu(ds) \int (\eta_{\mu_s} \circ \pi_a^{-1})(dt)\, f(t,\mu_s) &= \int \lambda(dr) \int \zeta_{ra}(dm)\, f(ra,m)\\ &= \int \nu(ds) \int \mu_s(dt)\, f(t,\mu_s),\end{aligned}$$

and (12) follows since f was arbitrary. □

4 Palm and Gibbs kernels

A *random measure* on a measurable space $(S, \mathcal{S})$ is defined as a kernel ξ from the basic probability space $(\Omega, \mathcal{A}, P)$ to S. We always assume ξ to be *uniformly σ-finite*, in the sense that there exist a measurable partition $B_1, B_2, \ldots$ of S such that $\xi B_k < \infty$ a.s. for all k. Note that ξ may also be regarded as a random element in $\mathcal{M}_S$, the space of σ-finite measures on S, endowed with the σ-field generated by all evaluation maps $\mu \mapsto \mu B$, $B \in \mathcal{S}$. When ξ is restricted to the the subspace $\mathcal{N}_S$ of integer-valued measures, it is called a *point process* on S. Assuming S to be Borel, we may then write ξ as a finite or countable sum of random unit masses δ_{τ_k}, and we say that ξ is *simple* if the τ_k are a.s. distinct, so that a.s. $\xi\{s\} = 0$ or 1 for all s.

Given a random measure ξ on S and a random element η in a space $(T, \mathcal{T})$, we define the associated *Campbell measure* C on $S \times T$ by the integrals $Cf = E \int f(s,\eta)\xi(ds)$ for measurable $f \geq 0$. If C is σ-finite and T is Borel, then Lemma 3.1 yields a disintegration $C = \nu \otimes Q$ into a σ-finite measure ν on S and a kernel Q from S to T. Assuming (as we may) that $\nu \sim E\xi$, we call ν a *supporting measure* for ξ and Q the associated *Palm kernel*. The *Palm measures* Q_s are bounded for $s \in S$ a.e. ν iff the *intensity measure* $E\xi$ itself is σ-finite, in which case we may choose $\nu = E\xi$ and let the Q_s be probability measures on T, then called the *Palm distributions* of η with respect to ξ.

Multivariate Palm measures are obtained by taking $\xi = \bigotimes_k \xi_k$ for some random measures ξ_k on S_k, $k \le n$, so that Q becomes a kernel from $S = \bigotimes_k S_k$ to T. When ξ is a point process, the Palm measures of ξ with respect to itself satisfy $Q_s\{\mu;\, \mu\{s\} = 0\} = 0$ for almost every s, and we may introduce the *reduced Palm measures* $Q'_s = Q_s\{\mu;\, \mu - \delta_s \in \cdot\}$. The kernel Q' and its higher-dimensional counterparts may also be obtained directly, by disintegration of the *reduced Campbell measures* C'_n on $S^n \times \mathcal{N}_S$, given by

$$C'_n f = E \int \cdots \int f\Big(s_1, \ldots, s_n,\, \xi - \sum\nolimits_k \delta_{s_k}\Big)\, \xi^{(n)}(ds_1 \cdots ds_n),$$

where the *factorial measures* $\xi^{(n)}$ are given recursively by $\xi^{(1)} = \xi$ and

$$\xi^{(n)} f = \int \xi(ds_1) \int \cdots \int f(s_1, \ldots, s_n)\, (\xi - \delta_{s_1})^{(n-1)}(ds_2 \cdots ds_n).$$

When ξ is simple, $\xi^{(n)}$ is just the restriction of the product measure ξ^n to the non-diagonal part of S^n.

Owing to the a.e. symmetry of $Q'_{s_1,\ldots,s_n}$ in the subscripts s_k, we may identify the vectors $s = (s_1, \ldots, s_n)$ with the measures $m = \sum_k \delta_{s_k}$ and write Q'_m instead of Q'_s. Similarly, C'_n may be regarded as a measure on $\mathcal{N}_S^2$. Defining the *compound Campbell measure* C_∞ by $C_\infty f = \sum_{\mu \le \xi} f(\mu,\, \xi - \mu)$ for measurable $f \ge 0$, with summation over $\hat{\mathcal{N}}_S = \{\mu \in \mathcal{N}_S;\, \mu S < \infty\}$, we have $C_\infty = \sum_n C'_n/n!$ (cf. [11, p. 123]), which shows that the reduced Palm kernels of different orders are obtainable by disintegration of C_∞ with respect to the first component. A disintegration with respect to the second component yields the *Gibbs kernel* (see below), and for a.s. bounded ξ, the two kernels will essentially agree up to a normalization.

The previous definitions are justified by the following descriptions in terms of ordinary conditioning. Here $k^{(n)} = k!/(k-n)!$ for $k \ge n \ge 0$.

Proposition 4.1 *Consider a random measure ξ on a Borel space S, and fix a set $B \in \mathcal{S}$ with $\xi B < \infty$ a.s.*

(i) *Let η be a random element in T such that the Campbell measure of (ξ, η) is σ-finite. Consider a random element τ in S with $\tau \notin B$ when $\xi B = 0$, and such that $P[\tau \in \cdot \mid \xi, \eta] = 1_B \xi/\xi B$ a.s. on $\{\xi B > 0\}$. Then ξ has supporting measure $\nu = \mathcal{L}(\tau)$ on B and associated Palm measures Q_s of η, given for $s \in B$ a.e. ν by $Q_\tau = E[\,\xi B;\, \eta \in \cdot \mid \tau]$ a.s. on $\{\tau \in B\}$.*

(ii) *When ξ is a point process, fix any $n \in \mathbb{N}$, and consider a random element β in S^n with $\beta \notin B^n$ when $\xi B < n$, and such that $P[\beta \in \cdot \mid \xi] = (1_B \xi)^{(n)}/(\xi B)^{(n)}$ a.s. on $\{\xi B \ge n\}$. Then $\xi^{(n)}$ has supporting measure $\nu'_n = \mathcal{L}(\beta)$ on B^n and associated n-th order reduced Palm measures Q'_s, given for $s \in B^n$ a.e. ν'_n by $Q'_\beta = E[\,(\xi B)^{(n)};\, \xi - \mu_\beta \in \cdot \mid \beta]$ a.s. on $\{\beta \in B^n\}$.*

(iii) *When ξ is a point process, let ζ be a point process on S such that $P[\zeta \in \cdot \mid \xi] = 2^{-\xi B} \sum_{\mu \le 1_B \xi} \delta_\mu$ a.s. Then ξ has compound Campbell measure $\nu \otimes Q'$ on $\hat{\mathcal{N}}_B \times \mathcal{N}_S$, where $\nu = \mathcal{L}(\zeta)$ and $Q'_\zeta = E[\,2^{\xi B};\, \xi - \zeta \in \cdot \mid \zeta]$ a.s.*

In particular, (i) yields for any measurable function $f \geq 0$ on $S \times T$

$$\int f(s,t)\, Q_s(dt) = E[f(\tau,\eta)\, \xi B \,|\, \tau \in ds], \quad s \in B \text{ a.e. } \nu,$$

(cf. [12, p. 108]). If $\xi = \delta_\sigma$ for some random element σ in S, then ξ has supporting measure $E\xi = \mathcal{L}(\sigma)$, and the associated Palm distributions Q_s are regular conditional distributions $P[\eta \in \cdot \,|\, \sigma \in ds]$.

Proof (i) On B we have $\nu = \mathcal{L}(\tau) = E[\xi/\xi B;\, \xi B > 0]$, which is clearly a supporting measure of ξ. Letting $f \geq 0$ be measurable on $S \times T$ and using the disintegration theorem twice, we get

$$E\int_B f(s,\eta)\, \xi(ds) = Ef(\tau,\eta)\, \xi B = \int_B \nu(ds) \int f(s,t)\, Q_s(dt).$$

(ii) The supporting property of ν'_n may be verified as before. Letting $f \geq 0$ be measurable and supported by $B^n \times \mathcal{N}_S$, we get by repeated use of the disintegration theorem

$$\begin{aligned} E\int f(s,\xi-\mu_s)\, \xi^{(n)}(ds) &= E\,(\xi B)^{(n)} f(\beta,\xi-\mu_\beta) \\ &= \int \nu'_n(ds) \int f(s,\mu)\, Q'_s(d\mu). \end{aligned}$$

(iii) Using the definitions of ν, Q', and ζ, and applying the disintegration theorem twice, we get for any measurable function $f \geq 0$ on $\mathcal{N}_S^2$

$$\int \nu(d\mu) \int Q'_\mu(dm) f(\mu,m) = E\, 2^{\xi B} f(\zeta,\xi-\zeta) = E \sum_{\mu \leq 1_B \xi} f(\mu,\xi-\mu),$$

as required. □

We need to examine when the various intensity and Campbell measures are σ-finite or s-finite.

Lemma 4.2 *Consider a random measure or point process ξ on S and a random element η in T. Then*

(i) *the intensity measures $E\xi^n$ and $E\xi^{(n)}$ are s-finite*;
(ii) *the Campbell measures C_n of the pairs (ξ^n,η) are s-finite, and when ξ is η-measurable they are even σ-finite*;
(iii) *the reduced and compound Campbell measures C'_n and C_∞ of ξ are σ-finite.*

Proof (i) By the σ-finiteness of ξ, we may choose a measurable partition $B_1, B_2, \ldots$ of S such that $\xi B_k < \infty$ a.s. for all $k \in \mathbb{N}$. Introducing the random measures $\xi_n = \sum_{k \le n}(1_{B_k}\xi)\, 1\{\xi B_k \le n\}$, $n \in \mathbb{N}$, we note that $E\xi_n S \le n^2$ for all n and $\xi_n \uparrow \xi$. Since $E\xi_n \uparrow E\xi$ by monotone convergence, we conclude that $E\xi$ is s-finite. Applying this result to the random measures ξ^n and $\xi^{(n)} \le \xi^n$, we see that even $E\xi^n$ and $E\xi^{(n)}$ are s-finite.

(ii) The first assertion follows from part (i), since $C_n(\cdot \times T) = E\xi^n$ for all n. Now let ξ be η-measurable. Introduce a measurable partition $B_1, B_2, \ldots$ of S such that $\xi B_n < \infty$ a.s. for all $n \in \mathbb{N}$. For every n we may next choose a measurable partition $A_{n1}, A_{n2}, \ldots$ of T such that $\xi B_n \le k$ a.s. on $\{\eta \in A_{nk}\}$ for all $k \in \mathbb{N}$. Then the sets $B_n \times A_{nk}$ form a countable partition of $S \times T$ satisfying $C_1(B_n \times A_{nk}) \le k$ for all $n, k \in \mathbb{N}$, which shows that C_1 is σ-finite. Applying this result to ξ^n yields the corresponding result for C_n.

(iii) Let the sets $B_n \uparrow S$ be measurable and such that $\xi B_n < \infty$ a.s. for all n. For any $n, k \in \mathbb{N}$, define $A_{nk} = \{\mu;\ \mu B_n \le k\}$ and $D_n = \{\mu;\ \mu B_n^c = 0\}$, and note that the sets $D_n \times A_{nk}$ form a countable partition of $\hat{\mathcal{N}}_S \times \mathcal{N}_S$. Since

$$C_\infty(D_n \times A_{nk}) = E\Big[2^{\xi B_n};\ \xi B_n \le k\Big] \le 2^k < \infty,$$

we see that C_∞ is σ-finite. Hence, we may choose a measurable function $f > 0$ on $\hat{\mathcal{N}}_S \times \mathcal{N}_S$ such that $C_\infty f < \infty$. Putting $g_n(s;\ \mu) = f(\sum_k \delta_{s_k};\ \mu)$ for $s = (s_1, \ldots, s_n)$ and using the formula $C_\infty = \sum_n C'_n/n!$, we obtain

$$C'_n g = E\int f\Big(\sum\nolimits_k \delta_{s_k},\ \xi - \sum\nolimits_k \delta_{s_k}\Big)\, \xi^{(n)}(ds) < \infty,$$

which shows that even C'_n is σ-finite. □

The definitions of *Gibbs* and *Papangelou kernels* Γ and Γ_1, associated with a point process ξ on S, are based on and justified by the following result, implicit in [11, pp. 121ff], and [14, p. 460]. To avoid obscuring complications, we restrict our attention to simple point processes. Given a measure μ on $S \times T$, we define the reflected version $\tilde{\mu}$ on $T \times S$ by $\tilde{\mu} f = \mu \tilde{f}$, where $\tilde{f}(s,t) = f(t,s)$.

Theorem 4.3 *Let ξ be a simple point process on a Borel space S with compound and reduced Campbell measures C_∞ and C'_1. Then there exist some maximal kernels Γ on $\mathcal{N}_S$ and Γ_1 from $\mathcal{N}_S$ to S such that*

$$\mathcal{L}(\xi) \otimes \Gamma \le \tilde{C}_\infty, \quad \mathcal{L}(\xi) \otimes \Gamma_1 \le \tilde{C}'_1. \tag{15}$$

The random measures $\gamma = \Gamma \circ \xi$ on $\mathcal{N}_S$ and $\gamma_1 = \Gamma_1 \circ \xi$ on S are given by

$$\gamma\{\mu;\ \mu \in A,\ \mu B^c = 0\} = \frac{P[1_B\xi \in A \mid 1_{B^c}\xi]}{P[\xi B = 0 \mid 1_{B^c}\xi]}, \tag{16}$$

$$\gamma_1 A = \frac{P[\xi A = \xi B = 1 \mid 1_{B^c}\xi]}{P[\xi B = 0 \mid 1_{B^c}\xi]}, \quad A \subset B, \tag{17}$$

a.s. on $\{\xi B = 0\}$ *for any* $B \in \mathcal{S}$*, along with the conditions*

$$\gamma\{\mu;\, \mu(\operatorname{supp}\xi) > 0\} = \gamma_1(\operatorname{supp}\xi) = 0 \qquad a.s.$$

Here the *maximality* of Γ means that, if Γ' is any other kernel on $\mathcal{N}_S$ satisfying (15), then $\Gamma'(\xi,\cdot) \le \Gamma(\xi,\cdot)$ a.s. The kernels Γ and Γ_1 are related by

$$\Gamma_1(\mu, B) = \Gamma(\mu, \{\nu;\, \nu B = \nu S = 1\}), \quad B \in \mathcal{S}. \tag{18}$$

The next result, implicit in Kallenberg [11, pp. 128ff] and [14, p. 460] gives necessary and sufficient conditions for equality in (15). Condition (iii) below is often denoted by (Σ).

Lemma 4.4 *For* ξ *as in Theorem* 4.3*, these conditions are equivalent*:

(i) $C_\infty(\hat{\mathcal{N}}_S \times \cdot) \ll \mathcal{L}(\xi)$,
(ii) $C_1'(S \times \cdot) \ll \mathcal{L}(\xi)$,
(iii) $P[\xi B = 0 \,|\, 1_{B^c}\xi] > 0$*, a.s. on* $\{\xi B = 1\}$ *for all* $B \in \mathcal{S}$*.*

In that case, (iii) *remains true on* $\{\xi B < \infty\}$*.*

5 Stationarity and invariance

We turn to the case when ξ and η are *jointly G-stationary*, in the sense that $\theta_r(\xi,\eta) \overset{d}{=} (\xi,\eta)$ for all $r \in G$, where $\theta_r\mu = \mu \circ \theta_r^{-1}$.

Lemma 5.1 *Let G act measurably on S and T. Then for any jointly G-stationary random measure* ξ *on S and random element* η *in T, the Campbell measure of* (ξ,η) *is jointly G-invariant on* $S \times T$*. The corresponding statement holds for the reduced and compound Campbell measures of a point process* ξ *on S.*

Proof For any $r \in G$ and measurable $f \ge 0$, we note that

$$(C \circ \theta_r^{-1})f = E\int f(s, r\eta)\,(\theta_r\xi)(ds),$$
$$(C_1' \circ \theta_r^{-1})f = E\int f(\theta_r\xi - \delta_s)\,(\theta_r\xi)(ds),$$
$$(C_\infty \circ \theta_r^{-1})f = E\sum\nolimits_\mu f(\mu,\, \theta_r\xi - \mu),$$

and similarly for general C_n and C_n'. □

The results of the previous sections yield invariant versions of the Palm and supporting measures. First we consider the *factorization* approach, where the underlying idea goes back to Matthes [18] for stationary point processes on $\mathbb{R}$, to Mecke [21] for random measures on locally compact Abelian groups, and to Tortrat [34] for non-Abelian groups.

Corollary 5.2 *Let G act measurably on S and T, where T is Borel. Consider a random measure ξ on $G \times S$ and a random element η in T, such that (ξ, η) is G-stationary with a σ-finite Campbell measure. Then ξ has a G-invariant supporting measure $\lambda \otimes \nu$ and associated G-invariant Palm measures $Q_{r,s}$ of η, where the kernels ν and Q are given, for any $r \in G$ and $s \in S$, by*

$$\nu_r = \hat{\nu} \circ \theta_r^{-1}, \quad Q_{r,s} = \hat{Q}_{r^{-1}s} \circ \theta_r^{-1}, \tag{19}$$

in terms of some measure $\hat{\nu}$ on S and kernel $\hat{Q}$ from S to T. The measure $\hat{\nu} \otimes \hat{Q}$ on $S \times T$ is then unique.

Proof Use Theorem 3.2 and Lemma 5.1. □

The situation of the last corollary arises naturally in the context of multivariate Palm measures. Indeed, the n-th order Palm measures with respect to some jointly stationary random measures $\xi_1, \ldots, \xi_n$ on G may be regarded as univariate Palm measures with respect to the stationary random measure $\xi = \bigotimes_k \xi_k$ on $G \times S$, where $S = G^{n-1}$ with the group action on S defined componentwise. This leads to invariant versions, as in (19), of the supporting measure and associated Palm kernel.

More symmetric representations are available in special cases, such as when $G = \mathbb{R}$. Here let H_d denote the hyperplane $\{x \in \mathbb{R}^d; \bar{x} = 0\}$, where $\bar{x} = d^{-1} \sum_k x_k$, and put $x' = x - \bar{x}\mathbf{1}$ with $\mathbf{1} = (1, \ldots, 1)$, so that $x' \in H_d$. Also write $Lx = (\bar{x}, x')$ and $\varphi = L^{-1}$, and let λ be Lebesgue measure on $\mathbb{R}$.

Proposition 5.3 *Let $\mathbb{R}$ act measurably on a Borel space T, and consider some random measures $\xi_1, \ldots, \xi_d$ on $\mathbb{R}$ and a random element η in T, all jointly stationary with a σ-finite Campbell measure C. Put $\xi = \bigotimes_k \xi_k$. Then (ξ, η) has the invariant Palm and supporting measures*

$$\nu = (\lambda \otimes \hat{\nu}) \circ \varphi^{-1}; \qquad Q_x = \hat{Q}_{x'} \circ \theta_{\bar{x}}^{-1}, \quad x \in \mathbb{R}^d, \tag{20}$$

for some measure $\hat{\nu}$ on H_d and kernel $\hat{Q}$ from H_d to T.

Proof Noting that $L \circ \theta_r = \theta'_r \circ L$ for all $r \in \mathbb{R}$ with $\theta'_r(p, s) = (p + r, s)$, we may check that $(\xi \circ L^{-1}, \eta)$ is stationary under shifts in $\mathbb{R}$ and T alone. Hence, Corollary 5.2 shows that $\xi \circ L^{-1}$ has a supporting measure $\tilde{\nu} = \lambda \otimes \hat{\nu}$ and associated Palm measures

$$\tilde{Q}_{r,s} = \hat{Q}_s \circ \theta_r^{-1}, \quad r \in \mathbb{R},\ s \in H_d,$$

for some σ-finite measure $\hat{\nu}$ on H_d and a kernel $\hat{Q}$ from H_d to T. Using the relation $L \circ \theta_r = \theta'_r \circ L$ and its inversion $\theta_r \circ \varphi = \varphi \circ \theta'_r$, we may verify that the measure ν and kernel Q in (20) are invariant. Letting $g \geq 0$ be a measurable function on $\mathbb{R} \times H_d \times T$ and using the definitions of ν, $\tilde{\nu}$, Q, $\hat{Q}$, $\tilde{Q}$, and L, we obtain

$$E\int g(Lx,\eta)\,\xi(dx) = \iint \tilde{\nu}(dr\,ds)\int \tilde{Q}_{r,s}(dt)\,g(r,s,t)$$
$$= \int \nu(dx)\int Q_x(dt)\,g(Lx,t).$$

Now substitute $g(r,s,t) = f(\varphi(r,s),t)$ for any measurable function $f \geq 0$ on $\mathbb{R}^d \times T$ to get $Cf = (\nu \otimes Q)f$, as required. □

We turn to the *regularization* approach, which goes back to Ryll-Nardzewski (1961) for point processes on $\mathbb{R}$, with an extension to homogeneous spaces due to Papangelou (1974a).

Corollary 5.4 *Let G act measurably on S and T, where T is Borel. Consider a random measure ξ on S and a random element η in T such that (ξ,η) is G-stationary with a σ-finite Campbell measure C. Then for any G-invariant supporting measure ν of ξ, the associated Palm kernel of η has a G-invariant version Q, which is unique when G acts transitively on S.*

Proof Note that $C(\cdot\times T) \ll \nu$ by the supporting property of ν, and use Theorem 3.5 and Lemma 5.1. □

The last result relies on the existence of an invariant supporting measure:

Corollary 5.5 *Let G act measurably on S, and consider a G-stationary random measure ξ on S. Then ξ has a G-invariant supporting measure ν, whenever $E\xi$ is σ-finite or G acts properly on S. In general, ν is unique up to a normalization when G acts transitively on S, and a σ-finite, G-invariant measure ν on S is a supporting measure of ξ iff $\nu \sim E\xi$ on $\mathcal{I}_S$.*

Proof If $E\xi$ is σ-finite, we may choose $\nu = E\xi$. In general, $E\xi$ is s-finite by Lemma 4.2, and so ν exists by Proposition 2.5 when G acts properly on S. If instead G acts transitively on S, then the uniqueness holds by Corollary 2.6. The final assertion holds by Lemma 2.2, since ν is supporting iff $\nu \sim E\xi$. □

We turn to the invariance of reduced Palm measures.

Proposition 5.6 *Let G act properly on S, and let ξ be a G-stationary point process on S. Then the reduced Palm kernel Q' from $\hat{\mathcal{N}}_S$ to $\mathcal{N}_S$ and the associated supporting measure ν on $\hat{\mathcal{N}}_S$ have G-invariant versions.*

Proof Since G acts properly on S, there exists a measurable function $g > 0$ on S such that $\lambda(g \circ \pi_s) < \infty$ for every $s \in S$. Define $\tilde{g}(\mu) = \mu g$ for $\mu \in \hat{\mathcal{N}}_S$, and note that $0 < \tilde{g} < \infty$ on $\hat{\mathcal{N}}'_S$. For any $\mu \in \hat{\mathcal{N}}_S$ we have

$$\lambda(\tilde{g}\circ\pi_\mu) = \int \lambda(dr)\int \mu(ds)\,g(rs) = \int \mu(ds)\,\lambda(g\circ\pi_s) < \infty,$$

since μ has finite support. Thus, the induced group action on $\hat{\mathcal{N}}'_S = \hat{\mathcal{N}}_S \setminus \{0\}$ is proper. Now use Corollary 3.6 and Lemmas 4.2 and 5.1. □

Condition (Σ) of Lemma 4.4 simplifies in the stationary case:

Lemma 5.7 *Let G act measurably on S. Then conditions* (i) *and* (ii) *of Lemma 4.4 are equivalent to the same conditions on $\mathcal{I}_{\mathcal{N}_S}$.*

Proof The measures C_1' and C_∞ are σ-finite by Lemma 4.2 and jointly G-invariant by Lemma 5.1. Hence, their projections $C_1'(S \times \cdot)$ and $C_\infty(\hat{\mathcal{N}}_S \times \cdot)$ are s-finite and G-invariant. The assertion now follows by Lemma 2.2. □

Let us now consider the invariance of Gibbs and Papangelou kernels.

Theorem 5.8 *Let G act measurably on a Borel space S, and consider a G-stationary, simple point process ξ on S. Then the associated Gibbs and Papangelou kernels Γ and Γ_1 have G-invariant versions, and the triple (ξ, γ, γ_1) is G-stationary, where $\gamma = \Gamma \circ \xi$ and $\gamma_1 = \Gamma_1 \circ \xi$.*

Proof The compound Campbell measure C_∞ is jointly G-invariant by Lemma 5.1. For any $r \in G$ and measurable $f \geq 0$ on $\mathcal{N}_S^2$, we may apply (15) to $f \circ \theta_r$ and use the G-invariance of C_∞ and G-stationarity of ξ to get

$$\begin{aligned} C_\infty f = C_\infty(f \circ \theta_r) &\geq E \int f(\theta_r \mu, \theta_r \xi)\, \Gamma(\xi, d\mu) \\ &= E \int f(\mu, \xi)\, \Gamma(\theta_r^{-1} \xi,\, \theta_r^{-1}(d\mu)). \end{aligned}$$

Hence, the maximality of Γ yields $\Gamma(\theta_r^{-1}\xi, \cdot) \circ \theta_r^{-1} \leq \Gamma(\xi, \cdot)$ a.s. for all $r \in G$. Replacing r by r^{-1} and using the G-stationarity of ξ gives the reverse inequality, and so in fact $\Gamma(\theta_r \xi, \cdot) = \Gamma(\xi, \cdot) \circ \theta_r^{-1}$ a.s. for each $r \in G$. In particular, the measures $\mathcal{L}(\xi)$ and $M = \mathcal{L}(\xi) \otimes \Gamma$ are both G-invariant, and so by Theorem 3.5 we may choose a version of Γ satisfying

$$\Gamma(\theta_r \mu, \cdot) = \Gamma(\mu, \cdot) \circ \theta_r^{-1}, \quad \mu \in \mathcal{N}_S, \quad r \in G. \tag{21}$$

To prove the corresponding relation for Γ_1, we may apply (18) and (21) for any $r \in G$ and $B \in \mathcal{S}$ to get

$$\Gamma_1(\theta_r \mu, B) = \Gamma(\mu,\, \theta_r^{-1}\{\nu;\, \nu B = \nu S = 1\}) = \Gamma_1(\mu, \theta_r^{-1} B),$$

which shows that $\Gamma_1(\theta_r \mu, \cdot) = \Gamma_1(\mu, \cdot) \circ \theta_r^{-1}$ for all $\mu \in \mathcal{N}_S$ and $r \in G$. Finally, we may use the G-stationarity of ξ and the G-invariance of Γ and Γ_1 to obtain

$$\theta_r(\xi, \gamma, \gamma_1) = (\theta_r \xi,\, \Gamma(\theta_r \xi, \cdot),\, \Gamma_1(\theta_r \xi, \cdot)) \overset{d}{=} (\xi, \gamma, \gamma_1),$$

which proves the joint G-stationarity of ξ, γ, and γ_1. □

6 Characterization and shift coupling

We begin with a characterization of Campbell measures.

Lemma 6.1 *Let G act measurably on S and T, where S is Borel. Consider some σ-finite, jointly G-invariant measures C on $S \times T$ and ν on T, where $\nu T = 1$. Then these conditions are equivalent*:

(i) *C is the Campbell measure of a G-stationary pair (ξ, η) with $\mathcal{L}(\eta) = \nu$,*
(ii) *$C(S \times \cdot) \ll \nu$ on $\mathcal{I}_T$.*

Here (ii) *extends to $\mathcal{T}$, and in* (i) *we may choose $\xi = \mu \circ \eta$ for a G-invariant kernel μ from T to S, in which case μ is a.s. unique and $\tilde{C} = \nu \otimes \mu$.*

Proof Clearly (i) implies (ii). Now assume (ii). Then Theorem 3.5 yields $\tilde{C} = \nu \otimes \mu$ for some G-invariant kernel μ from T to S. Letting $\mathcal{L}(\eta) = \nu$ and putting $\xi = \mu \circ \eta$, we get for measurable $f \geq 0$

$$E \int \xi(ds) f(s, \eta) = (\nu \otimes \mu)\tilde{f} = Cf, \tag{22}$$

which shows that (ξ, η) has Campbell measure C. The stationarity of (ξ, η) is clear from the invariance of ν and μ. Thus, (ii) implies (i). If (i) holds with $\xi = \mu \circ \eta$, we see as in (22) that $\tilde{C} = \nu \otimes \mu$, which yields the asserted ν-a.s. uniqueness of μ. Finally, (ii) extends to $\mathcal{T}$ by Lemma 2.2, since ν and $C(S \times \cdot)$ are both G-invariant and s-finite by Lemma 4.2. □

Next we extend a coupling result of Thorisson [31,33] (cf. [7] and [12, p. 209]). Say that G acts *injectively* on S if all projections $\pi_s : G \to S$ are injective.

Theorem 6.2 *Let G act measurably on some Borel spaces S and T. Consider some jointly G-stationary random measure ξ on S and G-ergodic random element η in T, where $E\xi$ is σ-finite with associated G-invariant Palm kernel Q. Then under each of these conditions:*

(i) *G acts transitively on S,*
(ii) *G is compact,*

we have $Q_s = \mathcal{L}(\gamma_s \eta)$, $s \in S$ a.e. $E\xi$, for some G-valued process γ on S. If G acts injectively on S, we can arrange that $\gamma_{rs} = r\gamma_s$ for all $r \in G$ and $s \in S$.

Proof First assume (i), and fix any $A \in \mathcal{I}_T$. Using the invariance of Q, the transitivity on S, and the ergodicity of η, we see that $Q_s A$ is independent of s and that $P\{\eta \in A\}$ equals 0 or 1. Letting $B \in \mathcal{S}$ and $s \in S$, we get

$$(E\xi B)\, Q_s A = C(B \times A) = E[\xi B;\, \eta \in A] = (E\xi B)\, P\{\eta \in A\},$$

which implies $Q_s = \mathcal{L}(\eta)$ on $\mathcal{I}_T$ since A and B were arbitrary. Hence, the coupling theorem of Thorisson [32,33] (cf. [12, p. 198]) yields $Q_s = \mathcal{L}(\gamma_s \eta)$ for some random elements γ_s in G.

Next assume (ii). Here we may assume $\lambda G = 1$ and take $g \equiv 1$ in Lemma 2.1, so that $\varphi_s = \lambda \circ \pi_s^{-1}$ for all s in S or T. Since η is stationary and ergodic, Theorem 2.4 yields $\mathcal{L}(\eta) = \varphi_{t_0}$ for some $t_0 \in T$. Writing $A = \varphi^{-1}\{\varphi_{t_0}\}$, we see from the same theorem (or from (6)) that

$$C(S \times A^c) = E[\xi S;\ \eta \in A^c] \ll P\{\eta \in A^c\} = \varphi_{t_0} A^c = 0.$$

Applying Theorem 2.4 to the σ-finite, invariant measure $E\xi$ on S gives

$$0 = C(S \times A^c) = \int E\xi(ds) \int \lambda(dr)\, Q_{rs} A^c.$$

Since the integrals $\int \lambda(dr)\, Q_{rs}$ are invariant probability measures on T, Theorem 2.4 yields $\int \lambda(dr)\, Q_{rs} = \varphi_{t_0}$ for $s \in S$ a.e. $E\xi$. By the invariance of Q we obtain $Q_s = \varphi_{t_0} = \mathcal{L}(\eta)$ on $\mathcal{I}_T$ for the same $s \in S$, and again the assertion follows from Thorisson's theorem.

For every orbit O in S, we may fix a point $s \in O$ and choose a random element γ_s in G with $\mathcal{L}(\gamma_s \eta) = Q_s$. If the maps π_s are injective, any other point in O has a unique representation rs, and we may define $\gamma_{rs} = r\gamma_s$. By the invariance of Q we get for measurable $f \geq 0$

$$Ef(\gamma_{rs}\eta) = Ef(r\gamma_s\eta) = Q_s(f \circ \theta_r) = Q_{rs} f,$$

which shows that $\mathcal{L}(\gamma_{rs}\eta) = Q_{rs}$ for all r. Also note that the defining relation holds identically since $\gamma_{prs} = p\gamma_{rs}$ for all $p, r \in G$. □

We proceed to characterize the Palm measures Q_s of a random element η with given distribution ν.

Theorem 6.3 *Let G act measurably on some Borel spaces S and T. Fix some σ-finite, G-invariant measures ρ on S and ν on T with $\nu T = 1$, along with a G-invariant kernel Q from S to T. Then* (ii) *implies* (i)*, where*

- (i) *Q is the Palm kernel for ρ of a G-stationary pair (ξ, η) with $\mathcal{L}(\eta) = \nu$,*
- (ii) *$Q_s \ll \nu$ on $\mathcal{I}_T$, $s \in S$ a.e. ρ,*

and equivalence holds under each of these conditions:

- (iii) *G acts transitively on S or T,*
- (iv) *G acts properly on S, and ρ is ergodic,*
- (v) *G is compact, ν is ergodic, and $E\xi$ is σ-finite.*

In (i) *we may choose $\xi = \mu \circ \eta$ for some G-invariant kernel μ from T to S, and* (iii) *allows us to replace* (ii) *by the same condition for a fixed $s \in S$.*

For $S = G = \mathbb{R}$ and $T = \Omega$, a related but more elaborate statement is proved by Getoor [8, p. 110]. Typically, the absolute continuity in (ii) fails on $\mathcal{T}$.

Proof The measure $C = \rho \otimes Q$ on $S \times T$ is clearly σ-finite and jointly G-invariant. Assuming (ii), we have $C(S \times \cdot) \ll \nu$ on $\mathcal{I}_T$. Then by Lemma 6.1, C is the Campbell measure of a G-stationary pair (ξ, η), where $\mathcal{L}(\eta) = \nu$ and $\xi = \mu \circ \eta$ for some G-invariant kernel μ from T to S. Hence, Q is the associated Palm kernel for the measure ρ. Putting $B = \{s \in S;\ Q_s \neq 0\}$, we may replace ρ by the supporting measure $\rho' = 1_B \rho$, which is again G-invariant by the invariance of Q. This proves (i).

Now assume (i). Then $C(S \times \cdot) \ll \nu$ on $\mathcal{I}_T$ by Lemma 6.1. The G-invariance of Q yields $Q_{rs} = Q_s$ on $\mathcal{I}_T$ for all $r \in G$ and $s \in S$. If G acts transitively on S, we obtain $Q_s = Q_{s'}$ on $\mathcal{I}_T$ for all $s, s' \in S$, which implies $C = \rho \otimes Q_s$ on $\mathcal{S} \times \mathcal{I}_T$ for every s. Hence, the stated absolute continuity yields $Q_s \ll \nu$ on $\mathcal{I}_T$ for all s. If instead G acts transitively on T, then $\mathcal{I}_T = \{\emptyset, T\}$, and $Q_s \ll \nu$ holds trivially. This shows that (i) and (iii) together imply (ii) for any fixed s.

Next, assume (i) and (iv). Then Theorem 2.4 yields $\rho = (\rho g)\varphi_s$ for some $s \in S$. By (3) we have

$$A \equiv \{s \in S;\ \rho = (\rho g)\, \varphi_s\} = \varphi^{-1}\{\rho/\rho g\} \in \mathcal{I}_S.$$

Using (1) and the G-invariance of Q, we get for any $s \in A$

$$C(S \times \cdot) = \frac{(\rho g)\, \lambda G}{\lambda(g \circ \pi_s)}\, Q_s \ \text{ on } \mathcal{I}_T, \quad s \in A,$$

and so by (i) and Lemma 6.1 we see that (ii) holds on A. To extend (ii) to A^c, fix any $s \in A$, and use (1) and the invariance of A to get

$$\rho A^c = (\rho g)\, \varphi_s A^c = \frac{(\rho g)\, \lambda G}{\lambda(g \circ \pi_s)}\, 1_{A^c}(s) = 0.$$

Finally, assume (i) and (v). Then $Q_s T < \infty$, $s \in S$ a.e. ρ, and Theorem 6.2 yields $Q_s = (Q_s T)\nu$ on $\mathcal{I}_T$, $s \in S$ a.e. ρ, which implies (ii). □

7 Invariant representations

Interchanging the roles of S and T yields invariant representations of stationary random measures. Such representations are usually taken to be part of the definition (as in [22, p. 309]). An exception is Getoor [18, pp. 108, 112], who (referring to [5]) derives perfection results when $S = G = \mathbb{R}$ and $T = \Omega$.

Theorem 7.1 *Let G act measurably on S and T, where S is Borel, and consider a random measure ξ on S and some random elements γ in G and η in T such that (ξ, γ, η) is G-stationary with a σ-finite Campbell measure. Then*

$$E[\xi \mid \gamma, \eta] = \mu \circ (\gamma, \eta) \quad a.s., \tag{23}$$

for some G-invariant kernel μ *from* $G \times T$ *to* S *of the form*

$$\mu_{r,t} = \hat{\mu}_{r^{-1}t} \circ \theta_r^{-1}, \quad r \in G, \ t \in T, \tag{24}$$

where $\hat{\mu}$ *is a kernel from* T *to* S*. Here* $\hat{\mu}$ *is unique a.e.* $\mathcal{L}(\gamma^{-1}\eta)$*.*

In particular, we get $\xi = \mu \circ (\gamma, \eta)$ a.s. when ξ is (γ, η)-measurable.

Proof Note that $\lambda = \mathcal{L}(\gamma)$ is a left and right Haar measure on G. By Lemmas 3.4 and 5.1, the Campbell measure C satisfies $\tilde{C} = (\lambda \otimes \hat{C}) \circ \vartheta^{-1}$ for some σ-finite measure $\hat{C}$ on $T \times S$, where $\vartheta(r, t, s) = (r, rs, rt)$ for $r \in G$, $s \in S$, and $t \in T$. Writing $\hat{\nu} = \mathcal{L}(\gamma^{-1}\eta)$, we get on $\mathcal{T}$

$$\hat{C}(\cdot \times S) = E[\xi S; \gamma^{-1}\eta \in \cdot] \ll P\{\gamma^{-1}\eta \in \cdot\} = \hat{\nu},$$

which allows the further disintegration $\tilde{C} = (\lambda \otimes \hat{\nu} \otimes \hat{\mu}) \circ \vartheta^{-1}$ in terms of a kernel $\hat{\mu}$ from T to S. Now define a kernel ν from G to T by $\nu_r = \hat{\nu} \circ \theta_r^{-1}$ for $r \in G$. Using the definitions of ν and $\hat{\nu}$, the substitution rule for integrals, Fubini's theorem, the right invariance and normalization of λ, and the G-stationarity of (γ, η), we may easily verify that $\lambda \otimes \nu = \mathcal{L}(\gamma, \eta)$.

Next define a kernel μ from $G \times T$ to S by (24), and note that μ is G-invariant. Expressing μ and ν in terms of $\hat{\mu}$ and $\hat{\nu}$ and using the definition of ϑ, we may check that $\tilde{C} = \lambda \otimes \nu \otimes \mu$. Hence, by the definition of C, we obtain for any $B \in \mathcal{S}$ and measurable $f \geq 0$ on $G \times T$

$$E\, E[\xi B \mid \gamma, \eta] f(\gamma, \eta) = E\,(\xi B) f(\gamma, \eta) = E\, \mu(\gamma, \eta, B) f(\gamma, \eta),$$

and (23) follows since S is Borel and B and f are arbitrary.

To prove the asserted uniqueness, consider another kernel $\hat{\mu}'$ from T to S, such that the generated kernel μ' from $G \times T$ to S, defined as in (24), satisfies (23). Applying (23) to both μ and μ' yields $\mu(\gamma, \eta, \cdot) = \mu'(\gamma, \eta, \cdot)$ a.s., and so by (24) for both μ and μ' we have $\hat{\mu}(\gamma^{-1}\eta, \cdot) = \hat{\mu}'(\gamma^{-1}\eta, \cdot)$ a.s., which means that $\hat{\mu} = \hat{\mu}'$ a.s. $\mathcal{L}(\gamma^{-1}\eta)$. □

We give another representation of this type, valid under different conditions and reducing to $\xi = \mu \circ \eta$ a.s. when ξ is η-measurable.

Theorem 7.2 *Let* G *act measurably on* S *and* T*, where* S *is Borel. Consider a random element* η *in* T *and a random measure* ξ *on* S *such that* (ξ, η) *is* G*-stationary with a* σ*-finite Campbell measure. Then*

$$E[\xi \mid \eta] = \mu \circ \eta \quad a.s. \tag{25}$$

for some a.s. unique, G-invariant kernel μ *from* T *to* S*. The uniqueness holds identically when* G *acts transitively on* T*.*

Proof The Campbell measure C is jointly G-invariant by Lemma 5.1, and the measure $\nu = \mathcal{L}(\eta)$ is G-invariant by stationarity and satisfies $C(S \times \cdot) \ll \nu$. Hence, Theorem 3.5 yields a disintegration $\tilde{C} = \nu \otimes \mu$ in terms of some G-invariant kernel μ from T to S. Relation (25) now follows, as before, by the definition of C. The a.s. uniqueness of μ is clear from (25), and it holds identically in the transitive case by the G-invariance of μ. □

Under stronger assumptions, we may represent a G-stationary random measure ξ on S in terms of a stationary random measure $\mu \circ \xi$ on G.

Theorem 7.3 *Let G act properly and transitively on a Borel space S, and fix an $a \in S$. Then for any G-stationary random measure ξ on S, there exists a G-invariant kernel μ from $\mathcal{M}_S$ to G such that*

$$\xi = (\mu \circ \xi) \circ \pi_a^{-1} \quad a.s.$$

Proof Using the properness and transitivity of the group action, we see from Proposition 2.5 and Lemma 4.2 that ξ has supporting measure $\nu = \lambda \circ \pi_a^{-1}$. By Theorem 5.4 we may choose a G-invariant version Q of the associated Palm kernel of ξ with respect to itself. Now define a G-invariant kernel $\hat{Q}$ from G to $\mathcal{M}_S$ by $\hat{Q}_r = Q_{ra}$ for all $r \in G$. Applying Theorem 6.3 with $T = \mathcal{M}_S$, first to Q and then to $\hat{Q}$, we see that the latter is the Palm kernel with respect to λ of a pair $(\mu \circ \xi, \xi)$, where μ is a G-invariant kernel from $\mathcal{M}_S$ to G.

Next define a kernel $\hat{\mu}$ from $\mathcal{M}_S$ to S by $\hat{\mu}_m = \mu_m \circ \pi_a^{-1}$ for all $m \in \mathcal{M}_S$. Using the G-invariance of μ and the fact that θ_r and π_a commute on G for every $r \in G$, we may check that $\hat{\mu}$ is again G-invariant. By the definitions of $\hat{\mu}$, μ, $\hat{Q}$, and ν, we get for any measurable function $f \geq 0$ on $S \times \mathcal{M}_S$,

$$\begin{aligned} E \int (\hat{\mu} \circ \xi)(ds) f(s, \xi) &= \int \lambda(dr) \int \hat{Q}_r(dm) f(ra, m) \\ &= \int \nu(ds) \int Q_s(dm) f(s, m), \end{aligned}$$

which shows that Q is also the Palm kernel of $(\hat{\mu} \circ \xi, \xi)$ with respect to ν. Hence, the uniqueness in Lemma 6.1 yields $\xi = \hat{\mu} \circ \xi = (\mu \circ \xi) \circ \pi_a^{-1}$ a.s. □

The following dual result follows immediately from Theorem 3.7.

Corollary 7.4 *Let G act measurably on S and T, where T is Borel and the action on T is proper and transitive, and fix any $a \in T$. Consider a random measure ξ on S and a random element η in T, such that (ξ, η) has the G-invariant supporting measure ν and Palm kernel Q. Then there exists a G-invariant kernel μ from $\mathcal{M}_T$ to G such that*

$$Q_s = (\mu \circ Q)_s \circ \pi_a^{-1}, \quad s \in S \ a.e. \ \nu.$$

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