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The chaotic-representation property for a class of normal martingales

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Abstract Suppose $Z = (Z_t)_{t \geqslant 0}$ is a normal martingale which satisfies the structure equation

$$\mathrm{d}[Z]_t = (\alpha(t) + \beta(t) Z_{t-})\, \mathrm{d}Z_t + \mathrm{d}t.$$

By adapting and extending techniques due to Parthasarathy and to Kurtz, it is shown that, if α is locally bounded and β has values in the interval $[-2, 0]$, the process Z is unique in law, possesses the chaotic-representation property and is strongly Markovian (in an appropriate sense). If also β is bounded away from the endpoints 0 and 2 on every compact subinterval of $[0, \infty[$ then Z is shown to have locally bounded trajectories, a variation on a result of Russo and Vallois.

Keywords Azéma martingale · Chaotic-representation property · Normal martingale · Predictable-representation property · Structure equation

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1 Introduction

A local martingale Z has the predictable representation property (henceforth PRP) if for any local martingale M there exists a predictable, Z-integrable process H such that

$$M_t = M_0 + \int_0^t H_s \, dZ_s \quad \forall t \geqslant 0.$$

This concept is of considerable intrinsic interest; as is well known, it is equivalent (when the initial filtration is trivial) to the law of Z being extremal [23]. The PRP is also important for many applications, in areas such as filtering, control theory and mathematical finance; the ideas in [4] (which concerns the former topics) and [1,7] (which concern the latter) may all be applied to the martingales discussed below, for example.

A strictly stronger notion [9] than this is the chaotic-representation property (henceforth CRP). Suppose Z is a normal martingale, i.e., Z and $t \mapsto Z_t^2 - t$ are both martingales, and recall that the iterated stochastic integrals

$$\int_{\{0 \leqslant t_1 < \cdots < t_n\}} f(t_1, \ldots, t_n) \, dZ_{t_1} \cdots dZ_{t_n}$$

are well defined for all $n \geqslant 1$ and deterministic, square-integrable functions f. If these integrals, together with the constant functions, are dense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ (where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space and $\mathcal{F}$ is generated by Z) then Z has the CRP. It is simple to verify that this implies the PRP and so, if the CRP holds, there exists a predictable process Φ such that

$$d[Z]_t = \Phi_t \, dZ_t + dt. \tag{1}$$

This is known as the *structure equation* for Z. The following question now presents itself: given a normal martingale Z which satisfies (1), does it have the CRP?

If Φ is deterministic then Dermoune [6] and Émery [8] have shown that the CRP holds and Z has independent increments; conversely if Z is a martingale with independent increments which satisfies (1) then the process Φ may be taken to be deterministic [22].

The next simplest case is when Φ is *affine*: $\Phi_t = \alpha(t) + \beta(t) Z_{t-}$ for all $t \geqslant 0$, where α and β are real valued. Émery [8] proved that if $\alpha \equiv a$ and $\beta \equiv b$ for constants a and b then any martingale Z which satisfies (1) is unique in law and, if $b \in [-2, 0]$, has the CRP. Russo and Vallois [20] considered the case where α and β are locally bounded and they established boundedness (which implies the CRP) if $\alpha \equiv 0$, $\beta(t) \in [-2, 0]$ and $\int_0^t |\beta(s)|^{-1} \, ds < \infty$ for all $t \geqslant 0$.

In this article, it is demonstrated that uniqueness in law and the CRP hold whenever α is locally bounded and $\beta(t) \in [-2, 0]$ for all $t \geqslant 0$. This is established by extending a comparison argument of Parthasarathy [13], to show that certain vectors are analytic for certain associated multiplication operators, and then by using an idea of Kurtz [11], which allows the CRP to be deduced from the self-adjointness of these operators.

In fact, a stronger result is established, by letting $\mathcal{F}_0$, the initial σ-algebra for Z, be non-trivial and working with structure equations where α and β are $L^\infty(\mathcal{F}_0)$ valued. This allows proof of (an appropriate version of) the strong Markov property for these martingales (called *Azéma martingales*, extending terminology due to Émery).

A variant of the Russo-Vallois result [20, Proposition 4.4] is also obtained; the requirement that $\alpha \equiv 0$ is removed but a stronger condition is imposed on β, namely that

$$\sup\{|1 + \beta(s)| : s \in [0, t]\} < 1 \quad \forall\, t \geqslant 0,$$

i.e., β is strictly bounded away from the endpoints of the interval $[-2, 0]$ on any compact subinterval of $[0, \infty[$.

The working below takes place in the Guichardet interpretation of Boson Fock space which, as Meyer observed, serves as a universal space for the investigation of normal martingales and is therefore a completely classical object. Although many of the ideas leading to the results herein came from quantum stochastic calculus, this article makes (almost) no explicit use of these techniques and may be read by any probabilist. (The sole exception is the proof of Proposition 29, for which a purely classical demonstration seems to be lacking.)

Section 2 describes Guichardet space and the chaotic-representation theorem of Kurtz; some examples of Azéma martingales are given in Sect. 3. The main results are in Sect. 4, together with two conjectures, and Sect. 5 is concerned with the strong Markov property.

1.1 Notation and conventions

The expression $\mathbf{1}_P$ has the value 1 if the proposition P is true and 0 if it is false. The symbol := is read as 'is defined to equal'; the set $\mathbb{R}_+ := [0, \infty[$, $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$,

$$\mathsf{P} := \{\sigma \subseteq \mathbb{R}_+ : |\sigma| < \infty\} \quad \text{and} \quad \mathsf{P}_n := \{\sigma \subseteq \mathbb{R}_+ : |\sigma| = n\} \qquad \forall\, n \in \mathbb{Z}_+,$$

where $|A|$ denotes the cardinality of the set A. Singleton sets are identified with their elements, so $s \in \mathsf{P}_1$ for all $s \in \mathbb{R}_+$. If $s, t \in \mathbb{R}_+$ then $s \wedge t := \min\{s, t\}$, $\sigma_{(s,t]} := \sigma \cap]s, t]$ and $\sigma_{t]} := \sigma \cap [0, t]$. The quantity 0^0 has the value 1, as has any empty product; an empty sum has the value 0. The L^p spaces considered herein are complex in general, with the notation $L^p(\cdot; \mathbb{R})$ distinguishing the real versions.

2 Preliminaries

Definition 1 Let $Z = (Z_t)_{t \geqslant 0}$ be a *normal martingale* defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., a martingale with càdlàg paths such that $(Z_t^2 - t)_{t \geqslant 0}$ is also a martingale, both with respect to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ which is right continuous and such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets; it is assumed throughout that the σ-algebra $\mathcal{F}$ is generated by Z. For all $n \in \mathbb{N}$ the linear map

$$I_n \colon L^2\bigl(\Delta^n; L^2(\mathcal{F}_0)\bigr) \to L^2(\mathcal{F}); \quad f \mapsto \int\limits_{\Delta^n} f(t_1, \ldots, t_n)\, \mathrm{d}Z_{t_1} \cdots \mathrm{d}Z_{t_n}$$

is a well-defined isometry, where $\Delta^n := \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n : t_1 < \cdots < t_n\}$ and $I_n(f)$ is obtained by extending the obvious definition when

$$f(t_1, \ldots, t_n)(\omega) = \mathbf{1}_{t_1 \in [a_1, b_1[} \cdots \mathbf{1}_{t_n \in [a_n, b_n[} f_0(\omega) \quad \forall\, t_1, \ldots, t_n \in \Delta^n, \ \ \omega \in \Omega,$$

where $a_1 \leqslant b_1 \leqslant \cdots \leqslant a_n \leqslant b_n$ and $f_0 \in L^2(\mathcal{F}_0)$ (cf. [5, XXI.1]). Let $\Xi_0 := L^2(\mathcal{F}_0)$, let $I_0 \colon \Xi_0 \to L^2(\mathcal{F})$ be the inclusion map and let

$$\Xi_n := \bigl\{ I_n(f) : f \in L^2\bigl(\Delta^n; L^2(\mathcal{F}_0)\bigr) \bigr\} \ \forall\, n \in \mathbb{N} \quad \text{and} \quad \Xi := \bigoplus_{n=0}^{\infty} \Xi_n;$$

the *chaos space* Ξ is a closed subspace (indeed, an $L^\infty(\mathcal{F}_0)$ submodule) of $L^2(\mathcal{F})$ (since Ξ_m is orthogonal to Ξ_n if $m \neq n$). If $\Xi = L^2(\mathcal{F})$ then Z has the *chaotic-representation property* (henceforth, CRP) *conditional on* $\mathcal{F}_0$ (or the *CRP at time 0*, in Émery's terminology [9]). If $\mathcal{F}_0$ is trivial then Z has the CRP in the usual sense.

Notation 2 Recall Guichardet's interpretation of Φ, the Boson Fock space over $L^2(\mathbb{R}_+)$: $\Phi = L^2(\mathsf{P})$, where P is the class of all finite subsets of $\mathbb{R}_+$ and

$$\|f\|^2 = \int\limits_{\mathsf{P}} |f(\sigma)|^2 \, \mathrm{d}\sigma = |f(\emptyset)|^2 + \sum_{n=1}^{\infty} \int\limits_{\Delta^n} |f(t_1, \ldots, t_n)|^2 \, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

for all $f \in \Phi$. (Each element of P_n may be regarded as a point in $\mathbb{R}^n$ with increasing coordinates, and P_n inherits its measurable structure from this; a set $A \subseteq \mathsf{P}$ is measurable if and only if $\iota_n(A \cap \mathsf{P}_n)$ is measurable for all $n \geqslant 1$, where the mapping $\iota_n \colon \mathsf{P}_n \to \mathbb{R}^n$; $\{t_1 < \cdots < t_n\} \mapsto (t_1, \ldots, t_n)$.)

Let $\widetilde{\Phi} := L^2\bigl(\mathsf{P}; L^2(\mathcal{F}_0)\bigr)$ and note that

$$U \colon \Xi \to \widetilde{\Phi}; \quad (UF)(\emptyset) = f_0, \quad (UF)(\sigma) = (f_{|\sigma|} \circ \iota_{|\sigma|})(\sigma) \quad \forall\, \sigma \in \mathsf{P} \setminus \mathsf{P}_0, \quad (2)$$

where $F = \sum_{n=0}^{\infty} I_n(f_n)$, is an isometric isomorphism.

The *exponential vector* in Φ corresponding to $u \in L^2(\mathbb{R}_+)$ is the function $\pi_u \colon \mathsf{P} \to \mathbb{C};\ \pi_u(\sigma) = \prod_{s\in\sigma} u(s)$ and $U^{-1}\pi_u = \mathcal{E}(u)$, the *Doléans exponential*:

$$\mathcal{E}(u) = 1 + \int_0^\infty u(t)\mathbb{E}[\mathcal{E}(u)|\mathcal{F}_t]\,\mathrm{d}Z_t = 1 + \sum_{n=1}^\infty I_n(u^{\otimes n}),$$

where a predictable version is taken of $t \mapsto \mathbb{E}[\mathcal{E}(u)|\mathcal{F}_t]$ and the function

$$u^{\otimes n} \colon (t_1, \dots, t_n) \mapsto u(t_1)\cdots u(t_n) \in L^2(\Delta^n)$$

for all $u \in L^2(\mathbb{R}_+)$. The linear span of the exponential vectors corresponding to bounded functions with compact support is denoted $\mathcal{E}_{00}$ and is dense in Φ.

Definition 3 Let Z be a normal martingale and, for all $t \geqslant 0$, let

$$\widehat{Z}_t \colon D(\widehat{Z}_t) := \{F \in \Xi : (Z_t - Z_0)F \in \Xi\} \subseteq \Xi \to \Xi;\ F \mapsto (Z_t - Z_0)F.$$

Note that $\widehat{Z}_t$ is symmetric and closed. (If $(F_n)_{n\geqslant 1} \subseteq D(\widehat{Z}_t)$ is such that $F_n \to F$ and $\widehat{Z}_t F_n \to G$ then, passing to a subsequence, $F_{n_k} \to F$ and $(Z_t - Z_0)F_{n_k} \to G$ almost everywhere, so $(Z_t - Z_0)F = G$. Hence $F \in D(\widehat{Z}_t)$ and $\widehat{Z}_t F_n \to \widehat{Z}_t F$, as required.)

Lemma 4 *If $(M, \mathcal{M}, \mu)$ is a finite measure space and $F \colon M \to \mathbb{R}$ is $\mathcal{M}$ measurable then the operator*

$$\{G \in L^2(\mathcal{M}) : FG \in L^2(\mathcal{M})\} \subseteq L^2(\mathcal{M}) \to L^2(\mathcal{M});\ G \mapsto FG$$

is self adjoint, and is bounded if and only if $\operatorname{ess\,sup}\{|F(m)| : m \in M\} < \infty$.

Proof This follows from [17, Proposition VIII.3.1] and, for the final statement, the spectral-radius formula [17, Theorem VI.6]. □

The following theorem is due to Kurtz [11, Théorème 8.1] (for trivial $\mathcal{F}_0$ and $Z_0 = 0$; the extension is straightforward).

Theorem 5 (Kurtz) *The martingale Z has the CRP conditional on $\mathcal{F}_0$ if and only if $\widehat{Z}_t$, as given in Definition 3, is self adjoint for all $t \geqslant 0$.*

Definition 6 The notation $F \colon \mathbb{R}_+ \to L^2(\mathcal{G})$ (where $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$) means that the process $F : \mathbb{R}_+ \times \Omega \to \mathbb{C}$ is measurable with respect to the product σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}$, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel σ-algebra on $\mathbb{R}_+$, and $F(t) : \Omega \to \mathbb{C};\ \omega \mapsto F(t,\omega)$ is such that $\mathbb{E}[|F(t)|^2] < \infty$ for all $t \geqslant 0$.

Definition 7 The normal martingale Z is an *Azéma martingale* if there exist processes $A, B \colon \mathbb{R}_+ \to L^2(\mathcal{F}_0; \mathbb{R})$ such that the following *structure equation* is satisfied:

$$[Z]_t = \int_0^t (A(s) + B(s)Z_{s-})\,\mathrm{d}Z_s + t \quad \forall\, t \geqslant 0, \tag{3}$$

where $[Z] = ([Z]_t)_{t \geqslant 0}$ is the quadratic variation of Z; the shorthand notation

$$\mathrm{d}[Z]_t = (A(t) + B(t)Z_{t-})\,\mathrm{d}Z_t + \mathrm{d}t$$

is also used. Of most interest are structure equations of the form

$$\mathrm{d}[Z]_t = (\alpha(t) + \beta(t)Z_{t-})\,\mathrm{d}Z_t + \mathrm{d}t, \tag{4}$$

where $\alpha, \beta \colon \mathbb{R}_+ \to \mathbb{R}$ are Borel measurable.

3 Examples

Let Z be an Azéma martingale with $Z_0 = 0$ which satisfies (4).

Example 8 If $\beta \equiv 0$ and α is constant then either $\alpha \equiv 0$, in which case Z is Brownian motion (a theorem due to Lévy [16, Theorem II.38]), or $\alpha \equiv a \neq 0$, so Z is a compensated Poisson process of the form

$$Z_t = a(N_{t/a^2} - t/a^2) \quad \forall\, t \geqslant 0,$$

where N is a Poisson process with intensity 1 and unit jumps [8, p. 69]. These processes are well known to have the CRP.

Example 9 If $\beta \equiv 0$ and α is any Borel-measurable function then Z has independent increments and may be realised as follows: if W is a standard Brownian motion and P an independent Poisson point process on $\mathbb{R}_+$ with intensity $\mathbf{1}_{\alpha(t)\neq 0}\,\mathrm{d}t/\alpha(t)^2$ then

$$Z_t = \int_0^t \mathbf{1}_{\alpha(s)=0}\,\mathrm{d}W_s + M_t \quad \forall\, t \geqslant 0,$$

where M is a purely discontinuous martingale with jumps at the points of P such that $\Delta M_t \in \{0, \alpha(t)\}$ for all $t \geqslant 0$. This explicit construction is due to Émery [8, Proposition 4], who has demonstrated that uniqueness in law and the CRP hold in this case. (See also work of Dermoune [6] and Utzet [22].)

Remark 10 If Z has independent increments then, by [22, p. 409, Commentaires du Séminaire], the process $(\alpha(t) + \beta(t)Z_{t-})_{t \geqslant 0}$ is equal almost everywhere (with respect to the product of Lebesgue measure on $\mathbb{R}_+$ and $\mathbb{P}$) to some deterministic process. From this, it is a straightforward exercise to show that $\beta = 0$ almost everywhere on $\mathbb{R}_+$.

Example 11 If $\alpha \equiv a$ and $\beta \equiv b$ then Z is an Azéma martingale of the type studied by Émery [8, Sect. (e)]; existence and uniqueness in law holds for all $a, b \in \mathbb{R}$, and if $b \in [-2, 0]$ then Z has the CRP. There are two important examples (as well as those given above) with explicit descriptions: if $a = 0$ and $b = -2$ then Z is the *parabolic martingale*, such that $Z_t^2 = t$ for all $t \geqslant 0$; if $a = 0$ and $b = -1$ then Z is the *first Azéma martingale*, which may be realised by taking a standard Brownian motion W and setting

$$Z_t = \operatorname{sign}(W_t)\sqrt{2(t - G_t)} \quad \forall\, t \geqslant 0,$$

where $\operatorname{sign}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x<0}$ for all $x \in \mathbb{R}$ and $G_t := \sup\{s \in [0,t] : B_s = 0\}$.

Example 12 If $\alpha(t) = 1 - t$ for all $t \geqslant 0$ and $\beta \equiv -1$ then Z is the classical martingale associated to the monotone Poisson process [2]; this is unique in law and has the CRP. The process $Y = (Y_t := Z_t + t)_{t\geqslant 0}$ has many similarities to the first Azéma martingale: it is determined by the level set $\mathcal{U} := \{t \geqslant 0 : Y_t = 1\}$ (which is almost surely non-empty, compact, without isolated points, of zero Lebesgue measure and of Hausdorff dimension no more than 1/2) together with choices either to increase or to decrease after each time in $\mathcal{U}$. The sample paths of this process have the explicit form

$$Y_t = -W_\bullet\big(-\exp(-1 - t + G_t)\big) \quad \forall\, t \geqslant 0,$$

where $G_t := \sup\{s \in [0,t] : Y_s = 1\} \in \{-\infty\} \cup\,]0,t]$ and $W_\bullet$ is one of the two branches of the Lambert W function which take real values. (Recall that W is the many-valued inverse to the complex function $z \mapsto ze^z$.) More information on this process may be found in [3].

Example 13 If $\alpha \equiv 0$ and $\beta \colon \mathbb{R}_+ \to [-2,0]$ is Borel measurable and such that $\int_0^t |\beta(s)|^{-1}\,\mathrm{d}s < \infty$ for all $t \geqslant 0$ then Z has locally bounded trajectories and so has the CRP; this is a result of Russo and Vallois [20, Proposition 4.4].

Example 14 Taviot [21, Théorème 4.0.2] has proved an existence theorem which gives a solution to (4) if α and β are *càglàd*, i.e., left continuous and with right limits everywhere.

4 Results

When $\mathcal{F}_0$ is trivial, the following result may be derived from the chaotic Kabanov formula of Privault et al. [15, Theorem 1].

Lemma 15 *Let Z be an Azéma martingale such that $Z_0 \in L^\infty(\mathcal{F}_0)$ which satisfies (3) and suppose $t \geqslant 0$ is such that A and B are* uniformly bounded on $[0,t]$*, i.e.,*

$$\|A\|_{\infty,t} := \operatorname{ess\,sup}\{|A(s,\omega)| : s \in [0,t],\ \omega \in \Omega\} < \infty \quad \textit{and} \quad \|B\|_{\infty,t} < \infty$$

(where the essential supremum is with respect to the product of Lebesgue measure on $[0,t]$ and $\mathbb{P}$). If $n \in \mathbb{N}$ and $f \in L^2\big(\Delta^n; L^2(\mathcal{F}_0)\big)$ then

$$(Z_t - Z_0) I_n(f) = I_{n-1}(f_t^-) + I_n(f_t^\circ) + I_{n+1}(f_t^+) \in \Xi, \tag{5}$$

where $t_0 := 0$, $\tilde{A}(s) := A(s) + Z_0 B(s)$, $B_{t]}(s) := \mathbf{1}_{s\in[0,t]} B(s)$,

$$f_t^-(t_1,\ldots,t_{n-1}) := \sum_{k=1}^{n-1} \int_{t_{k-1}\wedge t}^{t_k\wedge t} \prod_{l=k}^{n-1} \bigl(1 + B_{t]}(t_l)\bigr) f(t_1,\ldots,t_{k-1},s,t_k,\ldots,t_{n-1})\,\mathrm{d}s$$

$$+ \int_{t_{n-1}\wedge t}^{t} f(t_1,\ldots,t_{n-1},s)\,\mathrm{d}s,$$

$$f_t^\circ(t_1,\ldots,t_n) := \sum_{k=1}^{n} \mathbf{1}_{t_k\in[0,t]} \tilde{A}(t_k) \prod_{l=k+1}^{n} \bigl(1 + B_{t]}(t_l)\bigr) f(t_1,\ldots,t_n),$$

$$f_t^+(t_1,\ldots,t_{n+1}) := \sum_{k=1}^{n+1} \mathbf{1}_{t_k\in[0,t]} \prod_{l=k+1}^{n+1} \bigl(1 + B_{t]}(t_l)\bigr) f(t_1,\ldots,\widehat{t_k},\ldots,t_{n+1})$$

and $(t_1,\ldots,\widehat{t_k},\ldots,t_{n+1})$ *is the n-tuple obtained by removing* t_k *from the* $n+1$*-tuple* $(t_1,\ldots,t_{n+1})$.

Proof First, observe that $f_t^- \in L^2\bigl(\Delta^{n-1}; L^2(\mathcal{F}_0)\bigr)$, $f_t^\circ \in L^2\bigl(\Delta^n; L^2(\mathcal{F}_0)\bigr)$ and $f_t^+ \in L^2\bigl(\Delta^{n+1}; L^2(\mathcal{F}_0)\bigr)$, with

$$\|f_t^-\| \leqslant n(1 + \|B\|_{\infty,t})^{n-1} t^{1/2} \|f\|, \tag{6}$$

$$\|f_t^\circ\| \leqslant n\|\tilde{A}\|_{\infty,t}(1 + \|B\|_{\infty,t})^{n-1} \|f\| \tag{7}$$

$$\text{and} \quad \|f_t^+\| \leqslant (n+1)(1 + \|B\|_{\infty,t})^{n-1} t^{1/2} \|f\|. \tag{8}$$

Next, let $I_n(f) = \int_0^\infty G_s\,\mathrm{d}Z_s$, i.e., G is a predictable version of

$$t_n \mapsto \int_0^{t_n} \cdots \int_0^{t_2} f(t_1,\ldots,t_n)\,\mathrm{d}Z_{t_1}\cdots\mathrm{d}Z_{t_{n-1}} = I_{n-1}(f(\cdot,t_n)_{t_n]}),$$

where $g_{t]}(t_1,\ldots,t_k) := \mathbf{1}_{t_k\in[0,t]} g(t_1,\ldots,t_k)$ for all $g \in L^2(\Delta^k)$, or $G = f$ if $n = 1$. By the integration-by-parts formula for semimartingales,

$$(Z_t - Z_0) I_n(f) = \int_0^t \int_0^s G_r\,\mathrm{d}Z_r\,\mathrm{d}Z_s + \int_0^\infty (Z_{(s\wedge t)-} - Z_0) G_s\,\mathrm{d}Z_s + \int_0^t G_s\,\mathrm{d}[Z]_s$$

$$= \int_0^t \int_0^s G_r\,\mathrm{d}Z_r\,\mathrm{d}Z_s + \int_0^t \tilde{A}(s) G_s\,\mathrm{d}Z_s + \int_0^t G_s\,\mathrm{d}s$$

$$+ \int_0^\infty \bigl(1 + B_{t]}(s)\bigr)(Z_{(s\wedge t)-} - Z_0) G_s\,\mathrm{d}Z_s;$$

since $(Z_{r-})_{r\geqslant 0}$ is predictable, the process $s \mapsto (1+B_{t]}(s))Z_{(s\wedge t)-}G_s$ is a predictable version of $s \mapsto (1+B_{t]}(s))Z_{s\wedge t}G_s$. Thus, if $n=1$ then

$$(Z_t - Z_0)I_1(f) = \int_0^t f(s)\,\mathrm{d}s + \int_{\Delta^1} \mathbf{1}_{t_1\in[0,t]}\tilde{A}(t_1)f(t_1)\,\mathrm{d}Z_{t_1} + \int_{\Delta^2} \big(\mathbf{1}_{t_2\in[0,t]}f(t_1) + \mathbf{1}_{t_1\in[0,t]}\big(1+B_{t]}(t_2)\big)f(t_2)\big)\,\mathrm{d}Z_{t_1}\,\mathrm{d}Z_{t_2},$$

as required. Now suppose that (5) holds as claimed for some $n \geqslant 1$; from the above,

$$\begin{aligned}(Z_t - Z_0)I_{n+1}(f) = &\int_{\Delta^{n+1}} \mathbf{1}_{t_{n+1}\in[0,t]}f(t_1,\ldots,t_{n+1})\,\mathrm{d}Z_{t_1}\cdots\mathrm{d}Z_{t_n}\,\mathrm{d}t_{n+1}\\ &+ \int_{\Delta^{n+1}} \mathbf{1}_{t_{n+1}\in[0,t]}\tilde{A}(t_{n+1})f(t_1,\ldots,t_{n+1})\,\mathrm{d}Z_{t_1}\cdots\mathrm{d}Z_{t_{n+1}}\\ &+ \int_{\Delta^{n+2}} \mathbf{1}_{t_{n+2}\in[0,t]}f(t_1,\ldots,t_{n+1})\,\mathrm{d}Z_{t_1}\cdots\mathrm{d}Z_{t_{n+2}}\\ &+ \int_0^\infty \big(1+B_{t]}(s)\big)(Z_{(s\wedge t)-} - Z_0)I_n(f(\cdot,s)_{s]})\,\mathrm{d}Z_s\end{aligned}$$

and this final term is the sum of three integrals:

$$\begin{aligned}&\int_0^\infty \big(1+B_{t]}(s)\big)I_{n-1}(f^-_{s\wedge t}(\cdot,s)_{s]})\,\mathrm{d}Z_s\\ &\quad= \int_0^\infty \int_{\Delta^{n-1}} \sum_{k=1}^{n-1} \int_{t_{k-1}\wedge(s\wedge t)}^{t_k\wedge(s\wedge t)} \prod_{l=k}^{n-1}\big(1+B_{s\wedge t]}(t_l)\big)\big(1+B_{t]}(s)\big)\\ &\qquad\times \mathbf{1}_{t_{n-1}\in[0,s]}f(t_1,\ldots,t_{k-1},r,t_k,\ldots,t_{n-1},s)\,\mathrm{d}r\,\mathrm{d}Z_{t_1}\cdots\mathrm{d}Z_{t_{n-1}}\,\mathrm{d}Z_s\\ &\qquad+ \int_0^\infty \int_{\Delta^{n-1}} \int_{t_{n-1}\wedge(s\wedge t)}^{s\wedge t} \big(1+B_{t]}(s)\big)\\ &\qquad\times \mathbf{1}_{r\in[0,s]}f(t_1,\ldots,t_{n-1},r,s)\,\mathrm{d}r\,\mathrm{d}Z_{t_1}\cdots\mathrm{d}Z_{t_{n-1}}\,\mathrm{d}Z_s\\ &\quad= \int_{\Delta^n} \sum_{k=1}^{n} \int_{t_{k-1}\wedge t}^{t_k\wedge t} \prod_{l=k}^{n}\big(1+B_{t]}(t_l)\big)f(t_1,\ldots,t_{k-1},r,t_k,\ldots,t_n)\,\mathrm{d}r\,\mathrm{d}Z_{t_1}\cdots\mathrm{d}Z_{t_n},\end{aligned}$$

$$
\begin{aligned}
&\int_0^\infty \bigl(1 + B_{t]}(s)\bigr) I_n(f^\circ_{s\wedge t}(\cdot, s)_{s]})\,\mathrm{d}Z_s \\
&\quad = \int_0^\infty \int_{\Delta^n} \sum_{k=1}^{n} \mathbf{1}_{t_k \in [0,s\wedge t]} \tilde{A}(t_k) \prod_{l=k+1}^{n} \bigl(1 + B_{s\wedge t]}(t_l)\bigr)\bigl(1 + B_{t]}(s)\bigr) \\
&\qquad \times \mathbf{1}_{t_n \in [0,s]} f(t_1, \ldots, t_n, s)\,\mathrm{d}Z_{t_1} \cdots \mathrm{d}Z_{t_n}\,\mathrm{d}Z_s \\
&\quad = \int_{\Delta^{n+1}} \sum_{k=1}^{n} \mathbf{1}_{t_k \in [0,t]} \tilde{A}(t_k) \prod_{l=k+1}^{n+1} \bigl(1 + B_{t]}(t_l)\bigr) f(t_1, \ldots, t_{n+1})\,\mathrm{d}Z_{t_1} \cdots \mathrm{d}Z_{t_{n+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\infty \bigl(1 + B_{t]}(s)\bigr) I_{n+1}(f^+_{s\wedge t}(\cdot, s)_{s]})\,\mathrm{d}Z_s \\
&\quad = \int_0^\infty \int_{\Delta^{n+1}} \sum_{k=1}^{n+1} \mathbf{1}_{t_k \in [0,s\wedge t]} \prod_{l=k+1}^{n+1} \bigl(1 + B_{s\wedge t]}(t_l)\bigr)\bigl(1 + B_{t]}(s)\bigr) \\
&\qquad \times \mathbf{1}_{t_{n+1} \in [0,s]} f(t_1, \ldots, \widehat{t_k}, \ldots, t_{n+1}, s)\,\mathrm{d}Z_{t_1} \cdots \mathrm{d}Z_{t_{n+1}}\,\mathrm{d}Z_s \\
&\quad = \int_{\Delta^{n+2}} \sum_{k=1}^{n+1} \mathbf{1}_{t_k \in [0,t]} \prod_{l=k+1}^{n+2} \bigl(1 + B_{t]}(t_l)\bigr) f(t_1, \ldots, \widehat{t_k}, \ldots, t_{n+2})\,\mathrm{d}Z_{t_1} \cdots \mathrm{d}Z_{t_{n+2}}.
\end{aligned}
$$

The result follows by induction. □

Definition 16 A function $f\colon \mathsf{P} \to \mathbb{C}$ is a *test vector* if it is measurable and there exist constants T, C, $M > 0$ such that $|f(\sigma)| \leqslant \mathbf{1}_{\sigma \subseteq [0,T]} CM^{|\sigma|}$ for all $\sigma \in \mathsf{P}$. The collection of all such functions forms a vector space, denoted $\mathcal{T}$, which contains $\mathcal{E}_{00}$ and is dense in Φ.

A *generalised test vector* is a measurable function $f\colon \mathsf{P} \to L^2(\mathcal{F}_0)$ for which there exist constants T, C, $M > 0$ such that $\|f(\sigma)\|_{L^\infty(\mathcal{F}_0)} \leqslant \mathbf{1}_{\sigma \subseteq [0,T]} CM^{|\sigma|}$ for all $\sigma \in \mathsf{P}$. The set $\widetilde{\mathcal{T}}$ of all generalised test vectors is a vector space which contains $\{\pi_u \otimes f_0 : \pi_u \in \mathcal{E}_{00},\ f_0 \in L^\infty(\mathcal{F}_0)\}$, where $(\pi_u \otimes f_0)(\sigma) := \omega \mapsto \pi_u(\sigma) f_0(\omega)$ for all $\sigma \in \mathsf{P}$, and is dense in $\widetilde{\Phi}$.

Theorem 17 *Suppose Z is an Azéma martingale with $Z_0 \in L^\infty(\mathcal{F}_0)$ which satisfies* (3) *and $t \geqslant 0$ is such that A and B are uniformly bounded on* $[0, t]$. *If U is the isomorphism* (2) *and $\widehat{Z}_t$ is as in Definition* 3 *then $U^{-1} f \in D(\widehat{Z}_t)$ for all $f \in \widetilde{\mathcal{T}}$ and, if $\sigma \in \mathsf{P}$,*

$$(U\widehat{Z}_t U^{-1}f)(\sigma) = \int_0^t \prod_{r\in\sigma_{(s,t]}} \bigl(1+B(r)\bigr) f(\sigma\cup s)\,\mathrm{d}s + \sum_{s\in\sigma_{t]}} \prod_{r\in\sigma_{(s,t]}} \bigl(1+B(r)\bigr) f(\sigma\setminus s)$$
$$+ \sum_{s\in\sigma_{t]}} \tilde{A}(s) \prod_{r\in\sigma_{(s,t]}} \bigl(1+B(r)\bigr) f(\sigma).$$

Proof If there exists $n \in \mathbb{Z}_+$ such that $f(\sigma) \equiv 0$ for all $\sigma \in \mathsf{P}\setminus\mathsf{P}_n$ then this claim is simply a translation of Lemma 15 (or is immediately verified, for the case $n = 0$). Furthermore, the estimates (6–8) imply that if $f \in \widetilde{\mathcal{T}}$ and $U^{-1}f = \sum_{n=0}^\infty I_n(f_n)$ then $\sum_{n=1}^\infty \|f^+_{n,t}\|^2 + \|f^\circ_{n,t}\|^2 + \|f^-_{n,t}\|^2$ is convergent, which gives the result. □

Lemma 18 *For $c > 0$ let $N = (N_t)_{t\geqslant 0}$ be a compensated Poisson process with jump size c and intensity c^{-2}, defined on the probability space $(\Omega_{\mathrm{P}}, \mathcal{F}_{\mathrm{P}}, \mathbb{P}_{\mathrm{P}})$, and suppose that $\mathcal{F}_{\mathrm{P}}$ is generated by N. There exists an isometric isomorphism $U_{\mathrm{P}}\colon L^2(\mathcal{F}_{\mathrm{P}}) \to \Phi$ such that the operator of multiplication by N_t in $L^2(\mathcal{F}_{\mathrm{P}})$ equals $U_{\mathrm{P}}^{-1}\widetilde{N}_t U_{\mathrm{P}}$ on $U_{\mathrm{P}}^{-1}(\mathcal{T})$, where the operator $\widetilde{N}_t$ acts in Φ so that*

$$(\widetilde{N}_t f)(\sigma) := \int_0^t f(\sigma\cup s)\,\mathrm{d}s + \sum_{s\in\sigma_{t]}} f(\sigma\setminus s) + c|\sigma_{t]}| f(\sigma) \quad \forall\, \sigma \in \mathsf{P}.$$

If $u \in L^2(\mathbb{R}_+)$ then $U_{\mathrm{P}}^{-1}\pi_u$ equals the stochastic exponential $\mathcal{E}(u)$ and

$$\mathcal{E}(u)\mathcal{E}(v) = \exp\left(\int_0^\infty u(s)v(s)\,\mathrm{d}s\right)\mathcal{E}(u+v+cuv) \tag{9}$$

for all u, $v \in L^2(\mathbb{R}_+)\cap L^4(\mathbb{R}_+)$.

Proof It is well known (cf. Example 8) that N satisfies the structure equation

$$\mathrm{d}[N]_t = c\,\mathrm{d}N_t + \mathrm{d}t$$

and has the CRP; the claims about U_{P} and $(\widetilde{N}_t)_{t\geqslant 0}$ thus follow immediately from Theorem 17. Yor's formula [16, Theorem II.37] implies the remark about the product of stochastic exponentials. □

Remark 19 The operator $\widetilde{N}_t$ of Lemma 18 extends to a self-adjoint operator in $\widetilde{\Phi}$ by ampliation with the identity; this operator (denoted in the same manner) acts in $\widetilde{\Phi}$ so that, if $\sigma \in \mathsf{P}$ and $\omega \in \Omega$,

$$(\widetilde{N}_t f)(\sigma)(\omega) = \int_0^t f(\sigma\cup s)(\omega)\,\mathrm{d}s + \sum_{s\in\sigma_{t]}} f(\sigma\setminus s)(\omega) + c|\sigma_{t]}| f(\sigma)(\omega).$$

Definition 20 For processes A, $B\colon \mathbb{R}_+ \to L^2(\mathcal{F}_0;\mathbb{R})$ and a random variable $Z_0 \in L^\infty(\mathcal{F}_0;\mathbb{R})$, let

$$\tilde{A}\colon \mathbb{R}_+ \to L^2(\mathcal{F}_0;\mathbb{R});\ t \mapsto A(t) + Z_0 B(t).$$

For all $t \geqslant 0$, define linear operators $\widetilde{X}_t$, $\widetilde{Y}_t$ and $\widetilde{Z}_t$ in $\widetilde{\Phi}$ by setting, for all $\sigma \in \mathsf{P}$,

$$(\widetilde{X}_t f)(\sigma) := \int_0^t \prod_{r\in\sigma_{(s,t]}} \big(1 + B(r)\big) f(\sigma \cup s)\,\mathrm{d}s + \sum_{s\in\sigma_{t]}} \prod_{r\in\sigma_{(s,t]}} \big(1 + B(r)\big) f(\sigma \setminus s),$$
$$(\widetilde{Y}_t f)(\sigma) := \sum_{s\in\sigma_{t]}} \tilde{A}(s) \prod_{r\in\sigma_{(s,t]}} \big(1 + B(r)\big) f(\sigma),$$

and $(\widetilde{Z}_t f)(\sigma) := (\widetilde{X}_t f)(\sigma) + (\widetilde{Y}_t f)(\sigma)$, with maximal domains

$$D(\widetilde{W}_t) := \left\{ f \in \widetilde{\Phi} : \int_{\mathsf{P}} \mathbb{E}\big[|(\widetilde{W}_t f)(\sigma)|^2\big]\,\mathrm{d}\sigma < \infty \right\} \quad \forall\, W \in \{X, Y, Z\};$$

note that $\widetilde{X}_t + \widetilde{Y}_t \subseteq \widetilde{Z}_t$.

Notation 21 The expression $A \equiv \alpha$ means that $\alpha\colon \mathbb{R}_+ \to \mathbb{R}$ is such that $A(t,\omega) = \alpha(t)$ for all $t \geqslant 0$ and $\omega \in \Omega$, i.e., for all $t \geqslant 0$ the function $A(t)$ is constant and equals $\alpha(t)$; the same applies, *mutatis mutandis*, to B.

Remark 22 The decomposition of $\widetilde{Z}_t$ as the sum of the operators $\widetilde{X}_t$ and $\widetilde{Y}_t$ is a generalisation of Hudson and Parthasarathy's method [10, Sect. 6] of obtaining the Poisson process (as the perturbation of quantum Brownian motion $Q = A + A^\dagger$ by addition of the gauge process Λ). (Here, A represents the annihilation process of quantum stochastic calculus and has nothing to do with the structure equation (3).)

Proposition 23 *If* $t \geqslant 0$ *is such that* B *is uniformly bounded on* $[0,t]$ *then* $\widetilde{\mathcal{T}}$ *is an* invariant subspace *for* $\widetilde{X}_t$*, i.e.,* $\widetilde{\mathcal{T}} \subseteq D(\widetilde{X}_t)$ *and* $\widetilde{X}_t(\widetilde{\mathcal{T}}) \subseteq \widetilde{\mathcal{T}}$*; if* A *is also uniformly bounded on* $[0,t]$ *then* $\widetilde{\mathcal{T}}$ *is an invariant subspace for both* $\widetilde{Y}_t$ *and* $\widetilde{Z}_t$ *as well.*

Proof For $f \in \widetilde{\mathcal{T}}$ let T, C, $M > 0$ satisfy $\|f(\sigma)\|_{L^\infty(\mathcal{F}_0)} \leqslant \mathbf{1}_{\sigma\subseteq[0,T]} C M^{|\sigma|}$ for all $\sigma \in \mathsf{P}$. If $c := \|\tilde{A}\|_{\infty,t}$ and $d := \|B\|_{\infty,t}$ then

$$\|(\widetilde{X}_t f)(\sigma)\|_{L^\infty(\mathcal{F}_0)}$$
$$\leqslant t(1+d)^{|\sigma|} C M^{|\sigma|+1} \mathbf{1}_{\sigma\subseteq[0,T]} + |\sigma|(1+d)^{|\sigma|} C M^{|\sigma|-1} \mathbf{1}_{\sigma\subseteq[0,\max\{t,T\}]}$$

and

$$\|(\widetilde{Y}_t f)(\sigma)\|_{L^\infty(\mathcal{F}_0)} \leqslant |\sigma| c (1+d)^{|\sigma|} C M^{|\sigma|} \mathbf{1}_{\sigma\subseteq[0,T]} \quad \forall\, \sigma \in \mathsf{P},$$

which gives the result as claimed. □

The proof of the following theorem is a generalisation of a technique used by Parthasarathy [13, Sect. 2].

Theorem 24 *Suppose $t \geqslant 0$ is such that A is uniformly bounded on $[0,t]$ and $B(s,\omega) \in [-2,0]$ for all $s \in [0,t]$ and $\omega \in \Omega$. Every vector of the form $\pi_u \otimes f_0$, where $\pi_u \in \mathcal{E}_{00}$ and $f_0 \in L^\infty(\mathcal{F}_0)$, is an analytic vector for $\widetilde{Z}_t$.*

Proof Letting $c := \|\tilde{A}\|_{\infty,t}$, if $\sigma \in \mathsf{P}$ and $\omega \in \Omega$ then

$$\begin{aligned}|\widetilde{Z}_t f(\sigma)(\omega)| &\leqslant \int_0^t |f(\sigma \cup s)(\omega)|\,\mathrm{d}s + \sum_{s\in\sigma_{t]}} |f(\sigma \setminus s)(\omega)| + c|\sigma_{t]}|\,|f(\sigma)(\omega)| \\ &= (\widetilde{N}_t|f|)(\sigma)(\omega),\end{aligned}$$

where $\widetilde{N}_t$ is defined in Remark 19. If $f \geqslant 0$ (i.e., $f(\sigma)(\omega) \geqslant 0$ for all $\sigma \in \mathsf{P}$ and $\omega \in \Omega$) then $\widetilde{N}_t f \geqslant 0$, so if $|\widetilde{Z}_t^{n-1} f| \leqslant \widetilde{N}_t^{n-1}|f|$ then

$$|\widetilde{Z}_t^n f| = |\widetilde{Z}_t(\widetilde{Z}_t^{n-1} f)| \leqslant \widetilde{N}_t|\widetilde{Z}_t^{n-1} f| \leqslant \widetilde{N}_t(\widetilde{N}_t^{n-1}|f|) = \widetilde{N}_t^n|f|,$$

hence induction yields the inequality $|\widetilde{Z}_t^n f| \leqslant \widetilde{N}_t^n |f|$ for all $n \in \mathbb{Z}_+$. It follows that if $\pi_u \in \mathcal{E}_{00}$ and $f_0 \in L^\infty(\mathcal{F}_0)$ then, by the Cauchy-Schwarz-Bunyakovskii inequality and Lemma 18,

$$\begin{aligned}\|\widetilde{Z}_t^n(\pi_u \otimes f_0)\|_{\widetilde{\Phi}}^2 &\leqslant \|\widetilde{N}_t^n \pi_{|u|} \otimes |f_0|\,\|_{\widetilde{\Phi}}^2 \\ &= \|(U_{\mathrm{P}}^{-1}\widetilde{N}_t U_{\mathrm{P}})^n U_{\mathrm{P}}^{-1}\pi_{|u|}\|_{L^2(\mathcal{F}_{\mathrm{P}})}^2 \|\,|f_0|\,\|_{L^2(\mathcal{F}_0)}^2 \\ &= \mathbb{E}_{\mathrm{P}}\big[|N_t^n \mathcal{E}(|u|)|^2\big]\mathbb{E}[|f_0|^2] \\ &\leqslant \mathbb{E}_{\mathrm{P}}\big[N_t^{4n}\big]^{1/2}\mathbb{E}_{\mathrm{P}}\big[\mathcal{E}(|u|)^4\big]^{1/2}\mathbb{E}\big[|f_0|^2\big].\end{aligned}$$

Thus $\pi_u \otimes f_0$ is an analytic vector for $\widetilde{Z}_t$ if $\mathcal{E}(|u|) \in L^4(\mathbb{P}_{\mathrm{P}})$ and the power series $\sum_{n=0}^\infty \mathbb{E}_{\mathrm{P}}\big[N_t^{4n}\big]^{1/4} z^n/n!$ has strictly positive radius of convergence. The first follows from (9):

$$\mathcal{E}(|u|)^2 = \exp(\|u\|^2)\mathcal{E}(2|u| + c|u|^2) \in L^2(\mathcal{F}_{\mathrm{P}}),$$

since $2|u| + c|u|^2 \in L^2(\mathbb{R}_+)$ if u is bounded and has compact support. For the second, as $c^{-1}(N_t + c^{-1}t)$ has a Poisson distribution with mean $c^{-2}t$,

$$\begin{aligned}&\mathbb{E}_{\mathrm{P}}\big[\big(c^{-1}(N_t + c^{-1}t)\big)^{4n}\big] \\ &\quad= \sum_{k=0}^\infty \frac{e^{-c^{-2}t}(c^{-2}t)^k k^{4n}}{k!} \\ &\quad\leqslant e^{-c^{-2}t}\sum_{k=0}^\infty \frac{t^k(k+1)\cdots(k+4n)}{c^{2k}k!} = e^{-c^{-2}t}\frac{\mathrm{d}^{4n}}{\mathrm{d}t^{4n}}\sum_{k=0}^\infty \frac{t^{k+4n}}{c^{2k}k!}\end{aligned}$$

$$
\begin{aligned}
&= e^{-c^{-2}t} \frac{\mathrm{d}^{4n}}{\mathrm{d}t^{4n}} \left(t^{4n} e^{c^{-2}t} \right) = e^{-c^{-2}t} \sum_{k=0}^{4n} \binom{4n}{k} \frac{\mathrm{d}^{4n-k}}{\mathrm{d}t^{4n-k}} t^{4n} \frac{\mathrm{d}^k}{\mathrm{d}t^k} e^{c^{-2}t} \\
&\leqslant (4n)! \sum_{k=0}^{4n} \binom{4n}{k} (c^{-2}t)^k = (4n)!(1 + c^{-2}t)^{4n}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathbb{E}_{\mathrm{P}}[N_t^{4n}]^{1/4} = \|N_t\|^n_{L^{4n}(\mathbb{P}_{\mathrm{P}})} &\leqslant (\|N_t + c^{-1}t\|_{L^{4n}(\mathbb{P}_{\mathrm{P}})} + \|c^{-1}t\|_{L^{4n}(\mathbb{P}_{\mathrm{P}})})^n \\
&\leqslant 2^n (\|N_t + c^{-1}t\|^n_{L^{4n}(\mathbb{P}_{\mathrm{P}})} + (c^{-1}t)^n) \\
&\leqslant (4n)!^{1/4} (2c + 2c^{-1}t)^n + (2c^{-1}t)^n
\end{aligned}
$$

and $\sum_{n=0}^{\infty} (4n)!^{1/4} (2c + 2c^{-1}t)^n z^n / n!$ has radius of convergence $(8c + 8c^{-1}t)^{-1}$, the result follows. □

Theorem 25 *Let Z be an Azéma martingale with $Z_0 \in L^\infty(\mathcal{F}_0)$ which satisfies* (3)*, where A is* locally uniformly bounded*, i.e., $\|A\|_{\infty,t} < \infty$ for all $t \geqslant 0$, and $B(t,\omega) \in [-2, 0]$ for all $t \geqslant 0$ and $\omega \in \Omega$. Conditional on $\mathcal{F}_0$, the process Z is unique in law and has the CRP.*

Proof Theorem 24 implies that $U^{-1}(\pi_u \otimes f_0)$ is an analytic vector for $\widehat{Z}_t$ whenever $t \geqslant 0$, where $\pi_u \in \mathcal{E}_{00}$ and $f_0 \in L^\infty(\mathcal{F}_0)$, since $\widetilde{Z}_t$ and $U^{-1}\widehat{Z}_t U$ agree on $\widetilde{\mathcal{T}}$, by Theorem 17, which is invariant under their action, by Proposition 23. Hence $\widehat{Z}_t$ is self adjoint, by Nelson's theorem on analytic vectors [18, Theorem X.39], and Theorem 5 gives the CRP conditional on $\mathcal{F}_0$. Furthermore, $\widehat{Z}_t$ is determined by $\tilde{A}$ and B, since it equals $U^{-1}\widetilde{Z}_t U$ on $U^{-1}\widetilde{\mathcal{T}}$, which is a core for $\widehat{Z}_t$: by the analytic-vector theorem,

$$
\begin{aligned}
&(\widehat{Z}_t|_{U^{-1}\widetilde{\mathcal{T}}})^* = \overline{\widehat{Z}_t|_{U^{-1}\widetilde{\mathcal{T}}}} \subseteq \overline{\widehat{Z}_t} = \widehat{Z}_t = \widehat{Z}_t^* \\
&\qquad\qquad \Longrightarrow \quad \widehat{Z}_t \supseteq \overline{\widehat{Z}_t|_{U^{-1}\widetilde{\mathcal{T}}}} = (\widehat{Z}_t|_{U^{-1}\widetilde{\mathcal{T}}})^{**} \supseteq \widehat{Z}_t^{**} = \widehat{Z}_t.
\end{aligned}
$$

Hence the characteristic function

$$
\begin{aligned}
(\lambda_1, \ldots, \lambda_n) \mapsto\ &\mathbb{E}\big[\exp\big(i(\lambda_1 Z_{t_1} + \cdots + \lambda_n Z_{t_n})\big)\big] \\
&= \exp\big(i(\lambda_1 + \cdots + \lambda_n)Z_0\big)\langle 1, \exp(i\lambda_1 \widehat{Z}_{t_1}) \cdots \exp(i\lambda_n \widehat{Z}_{t_n})1\rangle
\end{aligned}
$$

is determined by Z_0, A and B; thus Z is unique in law conditional on $\mathcal{F}_0$. □

Corollary 26 *An Azéma martingale Z which satisfies* (4) *is unique in law and has the CRP if Z_0 is sure, α is locally bounded and $\beta(t) \in [-2, 0]$ for all $t \geqslant 0$.*

Proof This follows from Theorem 25 by taking $\mathcal{F}_0$ to be trivial. □

Example 27 Parthasarathy demonstrated [14, Sect. 2] that if $b \in [-2,0[$ and $x \neq 0$ then there exists an Azéma martingale $X^{b,x}$ such that

$$X_0^{b,x} = x \quad \text{and} \quad \mathrm{d}[X^{b,x}]_t = bX_{t-}^{b,x}\,\mathrm{d}X_t^{b,x} + \mathrm{d}t$$

and proved that $X^{b,x}$ and $(xX^{b,1}_{t/x^2})_{t \geqslant 0}$ are identical in law; for $b = 0$ this is just the scaling property of Brownian motion [19, I.3.4]. Émery [8, Sect. (e)] noted that this identity is a consequence of uniqueness in law for solutions of such a structure equation with the prescribed initial condition, which holds for all $b \in \mathbb{R}$. The result established above implies that if X^x is an Azéma martingale such that

$$X_0^x = x \neq 0 \quad \text{and} \quad \mathrm{d}[X^x]_t = \beta(t)X_{t-}^x\,\mathrm{d}X_t^x + \mathrm{d}t,$$

where $\beta\colon \mathbb{R}_+ \to [-2,0]$ is Borel measurable and satisfies $\beta(t) = \beta(t/x^2)$ for almost every $t \geqslant 0$, then, given that X^1 also exists, X^x and $(xX^1_{t/x^2})_{t\geqslant 0}$ are identical in law. (Non-trivial examples of such β are readily found.)

Conjecture 28 If $B \equiv 0$ and $A \equiv \alpha$ then the operators $(\widetilde{Z}_t)_{t\geqslant 0}$ correspond (at least formally) to the process with independent increments described in Example 9, which has the CRP. As this holds whether or not α is locally bounded, it is conjectured that $\widetilde{Z}_t$ is self adjoint for all $t \geqslant 0$ and any processes $A, B\colon \mathbb{R}_+ \to L^2(\mathcal{F}_0;\mathbb{R})$ with $B(t,\omega) \in [-2,0]$ for all $t \geqslant 0$ and $\omega \in \Omega$.

Proposition 29 *If $t \geqslant 0$ is such that $B(s,\omega) = \beta(s) \in [-2,0]$ for all $s \in [0,t]$ and $\omega \in \Omega$, where $\int_0^t |\beta(s)|^{-1}\,\mathrm{d}s < \infty$, then $\widetilde{X}_t$ is bounded.*

Proof (Sketch) This is in imitation of a similar result given by Russo and Vallois [20, Proposition 4.4] (which itself follows an idea of Émery); however, their proof relies upon the existence of an Azéma martingale $X = (X_s)_{0\leqslant s\leqslant t}$ such that

$$\mathrm{d}[X]_s = \beta(s)X_{s-}\,\mathrm{d}X_s + \mathrm{d}s,$$

whereas the following demonstration uses only the operators $(\widetilde{X}_s)_{0\leqslant s\leqslant t}$.

These operators may be shown to satisfy the quantum stochastic differential equation

$$\mathrm{d}\widetilde{X}_s = \beta(s)\widetilde{X}_s\,\mathrm{d}\Lambda_s + \mathrm{d}A_s + \mathrm{d}A_s^\dagger$$

on $\mathcal{T}$, taking $\mathcal{F}_0$ to be trivial. (Here, similarly to Remark 22, $\mathrm{d}A_s$ relates to the quantum-stochastic annihilation process and has nothing to do with the coefficient function A.) It follows that $\mathrm{d}\widetilde{X}_s^2 = 2\widetilde{X}_s\,\mathrm{d}\widetilde{X}_s + \mathrm{d}[\widetilde{X}]_s$ on $\mathcal{T}$ as well, by the quantum Itô product formula, where $\mathrm{d}[\widetilde{X}]_s = \beta(s)\widetilde{X}_s\,\mathrm{d}\widetilde{X}_s + \mathrm{d}s$. Thus

$$\mathrm{d}\widetilde{X}_s^2 = (\beta(s)+2)\widetilde{X}_s(\beta(s)\widetilde{X}_s\,\mathrm{d}\Lambda_s + \mathrm{d}A_s + \mathrm{d}A_s^\dagger) + \mathrm{d}s$$

and

$$\begin{aligned}\|\widetilde{X}_t f\|^2 &= \langle f, \widetilde{X}_t^2 f\rangle \\ &= \int_0^t \frac{\beta(s)+2}{\beta(s)} \|\beta(s)\widetilde{X}_s \nabla_s f + f\|^2 - \frac{2}{\beta(s)} \|f\|^2 \, \mathrm{d}s \\ &\leqslant -\int_0^t \frac{2}{\beta(s)} \, \mathrm{d}s \|f\|^2\end{aligned}$$

for all $f \in \mathcal{E}_{00}$, where the definition of the gradient operator, $\nabla_s \pi_u := u(s)\pi_u$, is extended by linearity; the result follows. □

Proposition 30 *If A is uniformly bounded on $[0,t]$ and $\|1+B\|_{\infty,t} < 1$ for some $t \geqslant 0$ then $\widetilde{Y}_t$ is bounded.*

Proof If $c := \|\tilde{A}\|_{\infty,t}$ and $q := \|1+B\|_{\infty,t} < 1$ then

$$|\widetilde{Y}_t f(\sigma)(\omega)| \leqslant \sum_{s \in \sigma_{t]}} c q^{|\sigma_{(s,t]}|} |f(\sigma)(\omega)| \leqslant c(1-q)^{-1} |f(\sigma)(\omega)| \quad \forall\, \sigma \in \mathsf{P},$$

whence $\|\widetilde{Y}_t\| \leqslant c(1-q)^{-1}$. □

Corollary 31 *If Z is an Azéma martingale such that $Z_0 \in L^\infty(\mathcal{F}_0)$ which satisfies* (3)*, where A is locally uniformly bounded, $B \equiv \beta$ and*

$$\sup\{|1+\beta(s)| : s \in [0,t]\} < 1 \quad \forall\, t \geqslant 0,$$

then Z has locally bounded trajectories.

Proof This follows by combining the two previous propositions. □

Conjecture 32 Since $\widetilde{X}_t$ is bounded under weaker conditions than those required in Corollary 31, it is tempting to conjecture that $\widetilde{Y}_t$ is also, i.e., that if $B(s,\omega) = \beta(s) \in [-2,0]$ for all $s \in [0,t]$ and $\omega \in \Omega$ then uniform boundedness of A on $[0,t]$ and the existence of $\int_0^t |\beta(s)|^{-1} \, \mathrm{d}s$ are sufficient for $\widetilde{Y}_t$ (and so $\widetilde{Z}_t$) to be bounded. It is possible that an 'intrinsic' proof of Proposition 29 (i.e., one that relies directly upon the definition of $\widetilde{X}_t$, rather than its interpretation as part of a hypothetical Azéma martingale) would point the way to establishing such a result.

5 The strong Markov property

Definition 33 Let $X = (X_t)_{t \geqslant 0}$ be an $\mathbb{R}^d$-valued process with càdlàg trajectories which is adapted to the filtration $(\mathcal{F}_t)_{t \geqslant 0}$. If

$$\mathbb{E}[f(X_{t+T}) | \sigma(X_T)] = \mathbb{E}[f(X_{t+T}) | \mathcal{F}_T] \quad \forall\, t \geqslant 0 \tag{10}$$

for any bounded, Borel-measurable function $f\colon \mathbb{R}^d \to \mathbb{R}$ and any finite stopping time T then X has the *strong Markov property*.

Proposition 34 *If X satisfies (10) for any bounded, Borel-measurable function f and any* bounded *stopping time T then X has the strong Markov property.*

Proof Let T be a finite stopping time and let $T_n := T \wedge n$ for all $n \geqslant 1$. Lévy's upward convergence theorem [19, Theorem II.50.3] and the dominated-convergence theorem, together with (10), imply (after some working) that if f is a bounded, Borel-measurable function and $t \geqslant 0$ then

$$\mathbb{E}[f(X_{t+T})|\mathcal{F}_T] = \lim_{n\to\infty} \mathbb{E}[f(X_{t+T})|\sigma(X_{T_n})].$$

For all $n \geqslant 1$ there exists a bounded, Borel-measurable function $g_n\colon \mathbb{R}^d \to \mathbb{R}$ such that $\|g_n\|_\infty := \sup\{|g_n(x)| : x \in \mathbb{R}^d\} \leqslant \|f\|_\infty$ and $\mathbb{E}[f(X_{t+T})|\sigma(X_{T_n})] = g_n(X_{T_n})$; since

$$\mathbb{E}[|g_n(X_T) - g_n(X_{T_n})|] \leqslant 2\mathbb{P}(T > n)\|g_n\|_\infty \leqslant 2\mathbb{P}(T > n)\|f\|_\infty \to 0$$

as $n \to \infty$, $\mathbb{E}[f(X_{t+T})|\mathcal{F}_T] = \lim_{n\to\infty} g_n(X_T)$ is measurable with respect to $\sigma(X_T)$. Furthermore, it is now simple to check that

$$\mathbb{E}[\mathbb{E}[f(X_{t+T})|\mathcal{F}_T]h(X_T)] = \mathbb{E}[f(X_{t+T})h(X_T)]$$

for any bounded, Borel-measurable function $h\colon \mathbb{R}^d \to \mathbb{R}$; the result follows. □

Proposition 35 *If X is an $\mathbb{R}^d$-valued process as in Definition 33 then the $\mathbb{R}^{d+1}$-valued process $(X_t, t)_{t\geqslant 0}$ has the strong Markov property if and only if*

$$\mathbb{E}[f(X_{t+T})|\sigma(X_T, T)] = \mathbb{E}[f(X_{t+T})|\mathcal{F}_T] \quad \forall\, t \geqslant 0 \tag{11}$$

for any bounded, continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$ and any bounded stopping time T.

Proof One direction is clear; the converse follows from an application of the monotone-class theorem [16, Theorem I.8] and Proposition 34. □

Lemma 36 *If X is a normal martingale for the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ and T is a bounded stopping time then $Y = (Y_t := X_{t+T})_{t\geqslant 0}$ is a normal martingale for the filtration $(\mathcal{F}_{t+T})_{t\geqslant 0}$, and if $t \geqslant 0$ then*

$$\int_0^t F_{s+T}\,\mathrm{d}Y_s = \int_T^{t+T} F_s\,\mathrm{d}X_s \tag{12}$$

for any $(\mathcal{F}_s)_{s\geqslant 0}$-predictable process F such that $\mathbb{E}[\int_T^{t+T} F_s^2\,\mathrm{d}s] < \infty$.

Proof The optional-sampling theorem [16, Theorem I.17] shows that Y is normal; the rest may be obtained from results in [12, II.2]. □

Theorem 37 *If Z is an Azéma martingale which satisfies (4), with Z_0 sure, α locally bounded and $\beta(t) \in [-2,0]$ for all $t \geqslant 0$, then $(Z_t,t)_{t\geqslant 0}$ has the strong Markov property. In the special case when $\alpha \equiv \alpha(0)$ and $\beta \equiv \beta(0)$ then Z has the strong Markov property.*

Proof Let T be a bounded stopping time and let

$$T_n := T \wedge \inf\{t > 0 : |Z_t - Z_0| > n\} \quad \forall\, n \in \mathbb{N},$$

so that T_n is a bounded stopping with $Z_{T_n} \in L^\infty(\mathcal{F}_{T_n})$ and $T_n \uparrow T$ as $n \to \infty$; to see the first claim, note that $|Z_{T_n-}| \leqslant n + |Z_0|$ and

$$|\Delta Z_{T_n}| \leqslant |\alpha(T_n)| + |\beta(T_n)|\,|Z_{T_n-}| \leqslant \sup\{|\alpha(s)| : s \in [0,r]\} + 2n,$$

where $r \geqslant 0$ is such that $T \leqslant r$ surely, since $\Delta Z_t^2 = (\alpha(t) + \beta(t)Z_{t-})\Delta Z_t$ for all $t \geqslant 0$.

If $W = (W_t := Z_{t+T_n})_{t\geqslant 0}$ then, by Lemma 36, W is a normal martingale with respect to the filtration $(\mathcal{F}_{t+T_n})_{t\geqslant 0}$ and

$$\begin{aligned}[W]_t &= W_t^2 - W_0^2 - 2\int_0^t W_{s-}\,\mathrm{d}W_s = Z_{t+T_n}^2 - Z_{T_n}^2 - 2\int_{T_n}^{t+T_n} Z_{s-}\,\mathrm{d}Z_s \\ &= [Z]_{t+T_n} - [Z]_{T_n} = t + \int_{T_n}^{t+T_n} (\alpha(s) + \beta(s)Z_{s-})\,\mathrm{d}Z_s \\ &= t + \int_0^t (A(s) + B(s)W_{s-})\,\mathrm{d}W_s,\end{aligned}$$

where $A(t) := \alpha(t + T_n)$ and $B(t) := \beta(t + T_n)$ for all $t \geqslant 0$; note that

$$\mathbb{R}_+ \times \Omega \ni (t,\omega) \mapsto A(t)(\omega) \quad \text{and} \quad \mathbb{R}_+ \times \Omega \ni (t,\omega) \mapsto B(t)(\omega)$$

are measurable with respect to $\mathcal{B}(\mathbb{R}_+) \otimes \sigma(T_n)$ (and so $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_{T_n}$). If $t \geqslant 0$ then $\|A\|_{\infty,t} \leqslant \|\alpha\|_{\infty,t+r} < \infty$, where $r \geqslant 0$ is as above, and $B(t)(\omega) \in [-2,0]$ for all $t \geqslant 0$ and $\omega \in \Omega$.

Furthermore, W is also a normal martingale with respect to $(\mathcal{G}_t)_{t\geqslant 0}$, where $\mathcal{G}_t := \sigma(W_s : s \in [0,t]) \vee \sigma(T_n)$ for all $t \geqslant 0$, since $\mathcal{G}_t \subseteq \mathcal{F}_{t+T_n}$ for all $t \geqslant 0$ and therefore

$$\mathbb{E}[W_t|\mathcal{G}_s] = \mathbb{E}[\mathbb{E}[W_t|\mathcal{F}_{s+T_n}]|\mathcal{G}_s] = \mathbb{E}[W_s|\mathcal{G}_s] = W_s$$

and

$$\mathbb{E}[W_t^2 - t|\mathcal{G}_s] = \mathbb{E}[\mathbb{E}[W_t^2 - t|\mathcal{F}_{s+T_n}]|\mathcal{G}_s] = \mathbb{E}[W_s^2 - s|\mathcal{G}_s] = W_s^2 - s$$

if $0 \leqslant s \leqslant t$. As $A(t)$, $B(t) \in L^\infty(\mathcal{G}_0)$ for all $t \geqslant 0$, the uniqueness-in-law result contained in Theorem 25 implies that, for all $u \in \mathbb{R}$,

$$\mathbb{E}[e^{iuZ_{t+T_n}}|\mathcal{F}_{T_n}] = \mathbb{E}[e^{iuW_t}|\mathcal{F}_{T_n}] = \mathbb{E}[e^{iuW_t}|\mathcal{G}_0] = \mathbb{E}[e^{iuZ_{t+T_n}}|\sigma(Z_{T_n}, T_n)];$$

letting $n \to \infty$, the result follows:

$$\mathbb{E}[|\mathbb{E}[e^{iuZ_{t+T_n}}|\mathcal{F}_{T_n}] - \mathbb{E}[e^{iuZ_{t+T}}|\mathcal{F}_{T_n}]|] \leqslant \mathbb{E}[|e^{iuZ_{t+T_n}} - e^{iuZ_{t+T}}|] \to 0,$$

by the dominated-convergence theorem, and

$$\mathbb{E}[|\mathbb{E}[e^{iuZ_{t+T}}|\mathcal{F}_{T_n}] - \mathbb{E}[e^{iuZ_{t+T}}|\mathcal{F}_T]|] \to 0,$$

by Lévy's upward convergence theorem; the same working holds if $\mathcal{F}_{T_n}$ and $\mathcal{F}_T$ are replaced by $\sigma(T_n, Z_{T_n})$ and $\sigma(T, Z_T)$, respectively.

As for the final claim, in this case A and B do not depend on T_n, so it suffices to take $\mathcal{G}_t := \sigma(W_s : s \in [0, t])$ for all $t \geqslant 0$. □

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References

1. Amendinger, J.: Martingale representation theorems for initially enlarged filtrations. Stoch. Process. Appl. **89**, 101–116 (2000)
2. Belton, A.C.R.: The monotone Poisson process. In: Proceedings of the 25th Conference on Quantum Probability and Related Topics, Będlewo, 2004. Banach Center Publications, Warsaw (to appear)(2006)
3. Belton, A.C.R.: On the path structure of a particular Azéma martingale. University College, Cork (Preprint)(2006)
4. Brémaud, P., Yor, M.: Changes of filtrations and of probability measures. Z. Wahrscheinlichke-itstheor. Verwandte Geb. **45**, 269–295 (1978)
5. Dellacherie, C., Maisonneuve, B., Meyer, P.-A.: Probabilités et potentiel. Chapitres XVII–XXIV: processus de Markov (fin), compléments de calcul stochastique. Hermann, Paris (1992)
6. Dermoune, A.: Distributions sur l'espace de P. Lévy et calcul stochastique. Ann. Inst. Henri Poincaré, Probab. Stat. **26**, 101–119 (1990)
7. Dritschel, M., Protter, P.: Complete markets with discontinuous security price. Finance Stoch. **3**, 203–214 (1999)
8. Émery, M.: On the Azéma martingales. In: Azéma, J., Meyer, P.-A., Yor, M. (eds.) Sémin. Probab. XXIII. Lecture Notes on Mathematics, vol. 1372, pp. 66–87. Springer, Heidelberg (1989)
9. Émery, M.: On the chaotic representation property for martingales. In: Ibragimov, I.A., Zaitsev, A.Yu. (eds.) Probability Theory and Mathematical Statistics, St. Petersburg, 1993, pp. 155–166. Gordon and Breach, Amsterdam (1996)

10. Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolutions. Commun. Math. Phys. **93**, 301–323 (1984)
11. Kurtz, D.: Répresentation nucléaire des martingales d'Azéma. In: Azéma, J., Émery, M., Ledoux, M., Yor, M. (eds.) Sémin. Probab. XXXVI. Lecture Notes on Mathematics, vol. 1801, pp. 457–476. Springer, Heidelberg (2003)
12. Le Jan, Y.: Martingales et changement de temps. In: Dellacherie, C., Meyer, P.-A., Weil, M. (eds.) Sémin. Probab. XIII. Lecture Notes on Mathematics, vol. 721, pp. 385–399. Springer, Heidelberg (1979)
13. Parthasarathy, K.R.: Azéma martingales and quantum stochastic calculus. In: Bahadur, R.R. (ed.) Proceedings of the R.C. Bose Symposium on Probability, Statistics and Design of Experiments, pp. 551–569. Wiley Eastern, New Delhi (1990)
14. Parthasarathy, K.R.: Azéma martingales with drift. Probab. Math. Stat. **15**, 461–468 (1995)
15. Privault, N., Solé, J.L., Vives, J.: Chaotic Kabanov formula for the Azéma martingales. Bernoulli **6**, 633–651 (2000)
16. Protter, P.: Stochastic integration and differential equations. A new approach. Applications of Mathematics, vol. 21. Springer, Heidelberg (1990)
17. Reed, M., Simon, B.: Methods of modern mathematical physics. I: Functional analysis. Academic, New York (1972)
18. Reed, M., Simon, B.: Methods of modern mathematical physics. II: Fourier analysis, self-adjointness. Academic, New York (1975)
19. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes and martingales. vol. 1 Foundations, 2nd edn., Cambridge University Press, Cambridge (2000)
20. Russo, F., Vallois, P.: Product of two multiple stochastic integrals with respect to a normal martingale. Stoch. Process. Appl. **73**, 47–68 (1998)
21. Taviot, G.: Martingales et équations de structure : étude géométrique. Thèse, Université Louis Pasteur Strasbourg 1 (1999)
22. Utzet, F.: Les processus à accroissements indépendants et les équations de structure. In: Azéma, J., Meyer, P.-A., Yor, M. (eds.) Sémin. Probab. XXVI. Lecture Notes on Mathematics, vol. 1526, pp. 405–409. Springer, Heidelberg (1992)
23. Yor, M.: Sous-espaces denses dans L^1 ou H^1 et représentation des martingales (avec J. de Sam Lazaro pour l'appendice). In: Dellacherie, C., Meyer, P.-A., Weil, M. (eds.) Sémin. Probab. XII. Lecture Notes on Mathematics, vol. 649, pp. 265–309 (1978)