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Gaussian free fields for mathematicians

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Abstract The d-dimensional Gaussian free field (GFF), also called the (Euclidean bosonic) massless free field, is a d-dimensional-time analog of Brownian motion. Just as Brownian motion is the limit of the simple random walk (when time and space are appropriately scaled), the GFF is the limit of many incrementally varying random functions on d-dimensional grids. We present an overview of the GFF and some of the properties that are useful in light of recent connections between the GFF and the Schramm–Loewner evolution.

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1 Introduction

The d-dimensional *Gaussian free field* (GFF) is a natural d-dimensional-time analog of Brownian motion. Like Brownian motion, it is a simple random object of widespread application and great intrinsic beauty. It plays an important role in statistical physics and the theory of random surfaces, particularly in the case $d = 2$. It is also a starting point for many constructions in quantum field theory [11,12,15].

The main purpose of this paper is to provide a mostly self-contained mathematical introduction to the GFF for readers familiar with basic probability (Gaussian variables, σ-algebras, Brownian motion, etc.), but not necessarily versed in the language of quantum field theory or conformal field theory. We will review the classical continuum constructions (Dirichlet quadratic forms, abstract Wiener spaces, Gaussian Hilbert spaces, Schwinger functions, chaos decomposition, etc.) and assemble basic facts about discrete Gaussian free fields.

Several results from this paper are cited in a recent work by the author and Schramm, which studies contour lines of the discrete Gaussian free field and shows that their scaling limits are forms of the Schramm–Loewner evolution SLE_4 [23]. We also expect these facts to be cited in forthcoming work relating SLE_κ to the GFF for other values of κ.

Although [23] is a long and technical work, it contains an elementary twenty-page introduction with many additional references to the history of the contour line problem and many other pointers to the physics literature. To avoid duplicating this effort, we will not discuss SLE at any length here. We also will not discuss the Virasoro algebra or the use of the GFF in the Coulomb gas theory (topics discussed at length in several reference texts, including [5,6,17]), and we generally make no attempt to survey the physics literature here. Although this work is primarily a survey, we will also present without references several simple results (including the natural coupling of harmonic crystals with the GFF via finite elements and the coupling of the GFF and Brownian motion via "field exploration") that we have not found articulated in the literature.

Remark 1.1 In the physics literature, what we call the GFF is often called the *massless free field* or the *Euclidean bosonic massless free field*—or else introduced without a title as something like "the field whose action is the Dirichlet energy" or "the Gaussian field with point covariances given by Green's function."

2 Gaussian free fields

2.1 Standard Gaussians

Consider the space $H_s(D)$ of smooth, real-valued functions on $\mathbb{R}^d$ that are supported on a compact subset of a domain $D \subset \mathbb{R}^d$ (so that, in particular, their first derivatives are in $L^2(D)$). This space has a *Dirichlet inner product* defined by $(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2)\, dx$. Denote by $H(D)$ the Hilbert space completion of $H_s(D)$. (The space $H(D)$ is in fact a *Sobolev space*, sometimes written $\mathbb{H}_0^1(D)$ or $W_0^{1,2}(D)$ [1].) The quantity $(f, f)_\nabla$ is called the *Dirichlet energy* of f.

Let g be a bijective map from D to another domain D'. If g is a translation or an orthogonal rotation, then it is not hard to see that

$$\int_{D'} \nabla(f_1 \circ g^{-1}) \cdot \nabla(f_2 \circ g^{-1})\, \mathrm{d}x = \int_D (\nabla f_1 \cdot \nabla f_2)\, \mathrm{d}x.$$

If $g(x) = cx$ for a constant c, then

$$\int_{D'} \nabla(f_1 \circ g^{-1}) \cdot \nabla(f_2 \circ g^{-1})\, \mathrm{d}x = c^{d-2} \int_D (\nabla f_1 \cdot \nabla f_2)\, \mathrm{d}x.$$

In the special case $d = 2$, the equality holds without the c^{d-2} term. In fact, an elementary change of variables calculation implies that this equality holds for any conformal map g and any $f_1, f_2 \in H(D)$. (It is enough to verify this for $f_1, f_2 \in H_s(D)$.) In other words, the Dirichlet inner product is invariant under conformal transformations when $d = 2$. (This is one reason that the GFF is a useful tool in the study of conformally invariant random two dimensional fractals like SLE [23].)

When D is a geometric manifold without boundary (e.g., the unit torus $\mathbb{R}^d/\mathbb{Z}^d$), we define $H_s(D)$ to be the set of all *zero mean* smooth functions on D, and again we take $H(D)$ to be its completion to a Hilbert space with the Dirichlet inner product.

Note that by integration by parts, $(f_1, f_2)_\nabla = (f_1, -\Delta f_2)$, where Δ is the Dirichlet Laplacian operator and $(\cdot,\cdot)$ is the standard inner product for functions on D. Throughout this paper, we use the notation $(f_1, f_2)_\nabla := \int_D (\nabla f_1 \cdot \nabla f_2)\, dx$ and $(f_1, f_2) := \int_D (f_1 f_2)\, dx$ when the integrals clearly make sense (even if f_1 and f_2 do not necessarily belong to $H(D)$ and $L^2(D)$, respectively). We also write $\|f\| = (f, f)^{1/2}$ and $\|f\|_\nabla = (f, f)_\nabla^{1/2}$.

Given any finite-dimensional real vector space V with (positive definite) inner product $(\cdot,\cdot)$, denote by μ_V the probability measure $e^{-(v,v)/2} Z^{-1} dv$, where dv is Lebesgue measure on V and Z is a normalizing constant. The following is well known (and easy to prove) [19].

Proposition 2.1 *Let v be a Lebesgue measurable random variable on $V = \mathbb{R}^d$ with inner product $(\cdot,\cdot)$ as above. Then the following are equivalent:*

1. *v has law μ_V.*
2. *v has the same law as $\sum_{j=1}^d \alpha_j v_j$ where $v_1, \dots, v_d$ are a deterministic orthonormal basis for V and the α_j are i.i.d. Gaussian random variables with mean zero and variance one.*
3. *The characteristic function of v is given by*

$$\mathbb{E} \exp(i(v,t)) = \exp\left(-\frac{1}{2}\|t\|^2\right)$$

   for all $t \in \mathbb{R}^d$.
4. *For each fixed $w \in V$, the inner product (v,w) is a zero mean Gaussian random variable with variance (w,w).*

A random variable satisfying one of the equivalent items in Proposition 2.1 is called a *standard Gaussian random variable on V*. Roughly speaking, the GFF is a standard Gaussian random variable h on $H(D)$. Because $H(D)$ is infinite dimensional, some care is required to make this precise. One might naively try to define h as a random element of $H(D)$ whose projections onto finite dimensional subspaces of $H(D)$ are standard Gaussian random variables on those subspaces. However, it is easy see that this is impossible. (Expanded in terms of an orthonormal basis, the individual components of h would have to be i.i.d. Gaussians—and hence a.s. the sum of their squares would be infinite, implying $h \notin H(D)$.)

We will now review two commonly used (and closely related) ways to define standard Gaussian random variables on infinite dimensional Hilbert spaces: the abstract Wiener space approach and the Gaussian Hilbert space approach.

2.2 Abstract Wiener spaces

One way to construct a standard Gaussian random variable h on an infinite dimensional Hilbert space H, proposed by Gross in 1967, is to define h as a random element not of H but of a larger Banach space B containing H as a subspace [16]. To this end, [16] defines a norm $|\cdot|$ on H to be **measurable** if for each $\epsilon > 0$, there is a finite-dimensional subspace E_ϵ of H for which

$$E \perp E_\epsilon \implies \mu_E(\{x \in E : \ |x| > \epsilon\}) < \epsilon,$$

where μ_E is the standard Gaussian measure on E. In particular, we may cite the following proposition [16]. (Throughout this subsection, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm of H.)

Proposition 2.2 *If T is a* Hilbert Schmidt operator *on H (i.e.,*

$$\sum \|Tf_j\|^2 < \infty$$

for some orthonormal basis $\{f_j\}$ of H), then the norm $\|T \cdot\|$ is measurable.

Write B for the Banach space completion of H under the norm $|\cdot|$, B' for the space of continuous linear functionals on B, and $\mathcal{B}$ for the smallest σ-algebra in which the functionals in B' are measurable. Since each element of B' is a continuous linear functional on H, we may view B' as a subset of H. Thus $B' \subset H \subset B$. When $b \in B$ and $f \in B'$, we use the inner product notation (f, b) to denote the value of the functional f at b. (When $f \in H$, this is equal to the inner product of f and b in H.) Given any finite dimensional subspace E of B' with H-orthonormal basis $v_1, \dots, v_k$, the map $\phi_E : B \to E$ given by $\phi_E(b) = \sum(v_j, b)v_j$ is an extension to B of the orthogonal projection map from H to E. Let μ_E be the standard Gaussian measure on E. Gross proved the following:

Theorem 2.3 *If $|\cdot|$ is measurable, then there is a unique probability measure P on $(B, \mathcal{B})$ for which $P(\phi_E^{-1}S) = \mu_E(S)$ for each finite dimensional subspace E of B' and each Lebesgue measurable $S \subset E$.*

By Proposition 2.1, we can restate this as follows:

Theorem 2.4 *If $|\cdot|$ is measurable, then there is a unique probability measure P such that if h is a random variable with probability measure P then for any $f \in B'$, the random variable (h, f) is a one-dimensional Gaussian of zero mean and variance $(f, f)^2$.*

The triple (H, B, P) is called an *abstract Wiener space*. The example that motivated Gross's construction is the standard Wiener space, in which $H = H((0, 1))$, endowed with the Dirichlet inner product, $|\cdot|$ is the supremum norm, and B is the set of continuous functions on $[0, 1]$ that vanish on $\{0, 1\}$. Using the Hilbert space $H(D)$ (with Dirichlet inner product) we can now give a definition:

Definition 2.5 *Given a measurable norm $|\cdot|$ on $H(D)$ and B', $\mathcal{B}$, B as above, the* Gaussian free field determined by norm $|\cdot|$ *is the unique B-valued, $\mathcal{B}$-measurable random variable h with the property that for every fixed $f \in B'$, the random variable $(h, f)_\nabla$ is a Gaussian of variance $\|f\|_\nabla$. Equivalently, $h = \sum \alpha_j f_j$, where α_j are i.i.d. Gaussians of unit variance and zero mean and the f_j are elements of B which form an orthonormal basis for $H(D)$ — and the sum is defined within the space B. (It is not hard to see that the partial sums $\sum_{j=1}^m \alpha_j f_j$ converge almost surely in B [16].)*

Remark 2.6 We can analogously define the *complex Gaussian free field determined by norm* $|\cdot|$ by replacing H, B', and B with their complex analogs and writing $h = h_1 + ih_2$, where the h_1 and h_2 are independent real Gaussian free fields.

2.3 Choosing a measurable norm

We now construct one natural family of measurable norms on $H(D)$ using the eigenvalues of the Laplacian. Suppose that $\{e_j\}$ are eigenvectors of the Dirichlet Laplacian on D which form an orthonormal basis of $L^2(D)$ endowed with the usual inner product and have negative eigenvalues $\{\lambda_j\}$ (ordered to be non-increasing in j). Then an orthonormal basis for $H(D)$ is given by $f_j = (-\lambda_j)^{-1/2}e_j$, since integration by parts implies $(e_j, e_k)_\nabla = (e_j, -\Delta e_k) = 0$ whenever $j \neq k$ and $(f_j, f_j)_\nabla = ((-\lambda_j)^{-1/2}e_j, (-\lambda_j)^{1/2}e_j)_\nabla = 1$. (This choice of the f_j is *not* invariant under conformal transformations of D when $d = 2$.)

The reader may recall that by Weyl's formula, if $D \subset \mathbb{R}^d$ is bounded, then $j^{2/d}/(-\lambda_j)$ tends to a constant as $j \to \infty$. (References and much more precise estimates on the growth of λ_j are given in [22].) We define powers of the negative Dirichlet Laplacian by writing, for each $a \in \mathbb{R}$,

$$(-\Delta)^a \sum \beta_j e_j := \sum (-\lambda_j)^a \beta_j e_j,$$

a definition which makes sense even when a is not an integer. We then formally define $\mathcal{L}_a(D) := (-\Delta)^a L^2(D)$ to be the set of sums of the form $\sum \beta_j e_j$ for which $\sum \beta_j (-\lambda_j)^{-a} e_j \in L^2(D)$. (When $a < 0$, this sum $\sum \beta_j e_j$ may not converge in $L^2(D)$, but since $(-\lambda_j)^{-a}$ is polynomial in j, it always converges in the space of distributions on D; see Remark 2.8.)

Since integration by parts gives

$$(f, g)_\nabla = (f, (-\Delta)g) = \left((-\Delta)^{1/2} f, (-\Delta)^{1/2} g\right),$$

the map $(-\Delta)^{-1/2}$ gives a Hilbert space isomorphism from $L^2(D)$ (with the L^2 inner product) to $H(D)$ (with the Dirichlet inner product). Thus we may write $H(D) = \mathcal{L}_{-1/2}(D)$.

Similarly, for any $a \in \mathbb{R}$, we may view $\mathcal{L}_a(D)$ as a Hilbert space whose inner product $(\cdot,\cdot)_a$ is the pullback of the L^2 inner product, i.e., $(f, g)_a = ((-\Delta)^{-a} f, (-\Delta)^{-a} g)$. We abbreviate $\|f\|_a := ((-\Delta)^{-a} f, (-\Delta)^{-a} f)^{1/2}$ for the corresponding norm. An equivalent way to define $\mathcal{L}_a(D)$ is as the Hilbert space closure of $H_s(D)$ under this norm.

Proposition 2.7 *Suppose D is a bounded domain in $\mathbb{R}^d$. Then we have the following:*

1. *In the space of formal sums $\sum \beta_j e_j$ (or the space of distributions) we have $\mathcal{L}_a(D) \subset \mathcal{L}_b(D)$ whenever $a < b$.*
2. *$\|\cdot\|_b$ is a measurable norm on $\mathcal{L}_a(D)$ (where the latter has inner product $(\cdot,\cdot)_a$) whenever $a < b - d/4$.*
3. *When $f \in \mathcal{L}_{-a}(D)$, the functional $g \to (f, g)$ is continuous on $\mathcal{L}_a(D)$.*

Proof The first item is immediate since $\sum (-\lambda_j)^{-2a} |\beta_j|^2 < \infty$ implies $\sum (-\lambda_j)^{-2b} |\beta_j|^2 < \infty$. To prove the second, we first write $\|f\|_b = \|T_{b-a} f\|_a$ where $T_c := (-\Delta)^{-c}$. Let $\{g_j\}$ be an orthonormal basis for $\mathcal{L}_b$ under the inner product $(\cdot,\cdot)_b$. Then $\|f\|_b$ is a Hilbert Schmidt operator (and hence measurable by Proposition 2.2) provided that $\sum \|T_a g_j\|_b^2 = \sum (-\lambda_j)^{2a-2b} < \infty$. Weyl's formula implies that this holds provided that $\sum j^{2(2a-2b)/d} < \infty$, which in turn holds whenever

$2(2a-2b)/d < -1$, i.e., $a < b - d/4$. The final statement in the proposition is trivial since

$$(f, g) = ((-\Delta)^a f, (-\Delta)^a g)_a = (f, (-\Delta)^{2a} g)_a,$$

and $(-\Delta)^{2a} g \in \mathcal{L}_a$. □

Proposition 2.7 implies that, although we cannot construct the GFF h as a random element of $H(D)$, we can construct h as a random element of $B = \mathcal{L}_b(D)$, provided $b > \frac{d-2}{4}$, using the abstract Wiener space definition given in Sect. 2.2.

In particular, when $d = 1$, we may take $b = 0$ and define h as a random element of $L^2(D)$. When $d = 2$, we cannot define h as a random element of $L^2(D)$ (indeed, from the power series expansion, we expect the L^2 norm of h to be almost surely infinite), but we can define h as a random element of $B = \mathcal{L}_b(D)$ for any $b > 0$. In this case, we may view $(h, \cdot)$ as a random continuous linear functional on $\mathcal{L}_{-b}(D) \subset L^2(D)$ for any $b > 0$. In general, $\rho \to (h, \rho)$ is a random continuous linear functional on $\mathcal{L}_{-b}(D)$ whenever $b > \frac{d-2}{4}$.

Remark 2.8 Sometimes it is convenient to restrict attention to smooth, compactly supported test functions ρ. Following the usual definition, we say that h is a **distribution** if $(h, \cdot)$ is well defined as a functional on the space $H_s(D)$ of smooth compactly supported functions and this functional is continuous with respect to the topology of uniform convergence of all derivatives. If $\rho \in H_s(D)$, then $(-\Delta)^a \rho \in H_s(D) \subset L^2(D)$ for each positive integer a, and it follows that $\rho \in \mathcal{L}_a(D)$ for all a. If $h \in \mathcal{L}_b(D)$ for some b, then $(h, \cdot)$ is a continuous functional on $\mathcal{L}_c(D)$ for any negative integer $c < -b$. This implies that the restriction of $(h, \cdot)$ to $H_s(D)$ is continuous in the topology of uniform convergence of all derivatives (since uniform convergence of all derivatives in particular implies convergence in $\mathcal{L}_c(D)$), so h is also a distribution. Many texts (e.g., [15]) simply define the GFF to be the random distribution determined in this way. Since $H_s(D)$ is dense in each of the larger spaces $L_b(D)$, we don't lose any information by restricting $(h, \cdot)$ to smooth functions, since there is a unique way to extend $(h, \cdot)$ to a continuous function on the larger space.

Remark 2.9 Let ϕ be rotationally symmetric smooth positive bump function on $\mathbb{R}^d$ whose integral is 1 and which vanishes outside of the unit ball in $\mathbb{R}^d$. Let $f_{r,z}(x) = r^{-d}\phi((x-z)/r)$. This function is the density of a probability measure on the ball $B_r(z)$ of radius $r > 0$ centered at $z \in \mathbb{R}^d$. Let D_r be the set of pairs (r, z) for which $r > 0$ and $B_r(z) \subset D$. Then the map $(r, z) \to f_{r,z}$ is continuous from D_r to $\mathcal{L}_b(D)$ for any $b \in \mathbb{R}$. Hence if h is an instance of the GFF defined by one of the norms discussed above, then $\psi((r, z)) = (h, f_{r,z})$ is a random continuous function from D_r to $\mathbb{R}$. Similar arguments show that all the derivatives of ψ are continuous almost surely. Since the span of the $f_{r,z}$ is dense, h is almost surely determined by the random smooth function ψ.

2.4 Gaussian Hilbert spaces

The definition of the GFF in terms of abstract Wiener spaces has an aesthetic and practical drawback in that the choice of measurable norm is somewhat arbitrary, and it does not yield a description of the random variable $(h,f)_\nabla$ for general $f \in H(D)$. In this section we give a way to make $(h,\cdot)_\nabla$ well defined as a random variable for each $f \in H(D)$—accepting, of course, the fact that $f \to (h,f)_\nabla$ cannot be defined as a continuous functional.

Consider the probability space $(\Omega, \mathcal{F}, \mu)$ where Ω is the set of real sequences $\alpha = \{\alpha_j\}, j \geq 1$, $\mathcal{F}$ is the smallest σ-algebra in which the coordinate projections $\alpha \to \alpha_j$ are measurable, and μ is the probability measure in which the α_j are i.i.d. Gaussian variables of unit variance and zero mean.

In the previous section, we defined the Gaussian free field (GFF) to be the formal sum $h = \sum_{j=1}^{\infty} \alpha_j f_j$ (which converges in a larger space B), where the f_j are an ordered orthonormal basis for $H(D)$ and the α_j are i.i.d. Gaussians. Now, for any fixed $f \in H(D) = \sum \beta_j f_j$, the inner product $(h,f)_\nabla$ is a random variable that *can* be almost surely well defined as the limit of the partial sums $\sum_{j=1}^{k} \beta_j \alpha_j$. (It is important here that we fix the order of summation in advance, since the sequence $\beta_j \alpha_j$ is not necessarily a.s. absolutely summable.)

Now we have a formal definition:

Definition 2.10 *The* Gaussian free field derived from the ordered orthonormal basis $\{f_j\}$ *is the indexed collection $\mathcal{G}(D)$ of random variables $(h,f)_\nabla$ described above.*

A more abstract definition, which does not specifically reference a basis or an ordering, is as follows. First, we take the following definition from [19]:

Definition 2.11 *A* Gaussian linear space *is a real linear space of random variables, defined on an arbitrary probability space $(\Omega, \mathcal{F}, \mu)$, such that each variable in the space is a centered (i.e., mean zero) Gaussian. A* Gaussian Hilbert space *is a Gaussian linear space which is complete, i.e., a closed subspace of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$, consisting of centered Gaussian variables, which inherits the standard $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ inner product: $(X,Y) = \int XY d\mu$. We also assume that $\mathcal{F}$ is the smallest σ-algebra in which these random variables are measurable.*

Note that if $X_1, \ldots, X_n$ are any real random variables with the property that all linear combinations of the X_j are centered Gaussians, then the joint law of the X_j is completely determined by the covariances $\mathrm{Cov}[X_j, X_k] = \mathbb{E}(X_j X_k)$, and it is a linear transformation of the standard normal distribution. A similar statement holds for infinite collections of random variables [19]. Then we have:

Definition 2.12 *A* Gaussian free field *is any Gaussian Hilbert space $\mathcal{G}(D)$ of random variables denoted by "$(h,f)_\nabla$"—one variable for each $f \in H(D)$—that inherits the Dirichlet inner product structure of $H(D)$, i.e.,*

$$\mathbb{E}[(h,a)_\nabla (h,b)_\nabla] = (a,b)_\nabla.$$

In other words, the map from f to the random variable $(h,f)_\nabla$ is an inner product preserving map from $H(D)$ to $\mathcal{G}(D)$.

By the identity $(a,b) = \frac{1}{2}[(a+b,a+b) - (a,a) - (b,b)]$, this map is inner product preserving if and only if it is norm-preserving — i.e., the variance of $(h,f)_\nabla$ is $(f,f)_\nabla$ for each $f \in H(D)$ — and linear. Thus we have the following:

Proposition 2.13 *An $H(D)$-indexed linear space of random variables denoted $(h,f)_\nabla$ is a Gaussian free field if and only if the map from $f \in H(D)$ to the random variable $(h,f)_\nabla$ is linear and each $(h,f)_\nabla$ is a centered Gaussian with variance $(f,f)_\nabla$.*

Throughout the remainder of this text, we will adopt the Gaussian Hilbert space approach and view the variables $(h,f)_\nabla$ as being well defined for all $f \in H(D)$. Equivalently, we view (h,ρ) as being well defined for all $\rho \in (-\Delta)H(D) = \mathcal{L}_{1/2}$.

Remark 2.14 When ρ_1 and ρ_2 are in $H_s(D)$, the covariance of (h,ρ_1) and (h,ρ_2) can be written as $(-\Delta^{-1}\rho_1, -\Delta^{-1}\rho_2)_\nabla = (-\Delta^{-1}\rho_1, \rho_2)$. Since $-\Delta^{-1}\rho$ can be written using the Green's function kernel as

$$[-\Delta^{-1}\rho](x) = \int_D G(x,y)\rho(y)\mathrm{d}y,$$

we may also write:

$$\mathrm{Cov}[(h,\rho_1),(h,\rho_2)] = \int_{D\times D} \rho_1(x)\rho_2(y)G(x,y)\mathrm{d}x\,\mathrm{d}y$$

When $\rho_1 = \rho_2 = \rho$, the above expression has an interpretation in electrostatics as the *energy of assembly* of an electric charge density ρ (grounded at ∂D), and $\Delta^{-1}\rho$ is the *electrostatic potential* of that density. The energy of assembly of a density of charge is the amount of energy required to move charge into that configuration starting from a zero-energy configuration (in which the potential is everywhere zero). Thus the Laplacian $\mathfrak{p} = (-\Delta)h$ is, at least intuitively, a random electrostatic charge distribution in which the probability of $\mathfrak{p}$ is proportional to

$$\exp(-\text{energy of assembly of } \mathfrak{p}).$$

(See [23] for more references relevant to this interpretation.)

2.5 Simple examples

Let D be the unit torus $\mathbb{R}^d/\mathbb{Z}^d$. As before $H_s(D)$ is the set of smooth functions on D with zero mean and $H(D)$ is the Hilbert space closure of $H_s(D)$ using the

Dirichlet inner product. An orthonormal basis for the complex version of $H(D)$ is given by eigenvectors of the Laplacian, which have the form $f_k(x) = \frac{1}{2\pi|k|}\mathrm{e}^{2\pi i x\cdot k}$, for $k \in \mathbb{Z}^d\backslash\{0\}$. Thus, the complex GFF on D is a random distribution whose Fourier transform consists of i.i.d. complex Gaussians times $(2\pi|k|)^{-1}$.

If $d \geq 2$, then for any fixed $x \in D$ and any fixed ordering of the k's, the partial sums of $\sum_{j=1}^{k} \alpha_j f_j(x)$ diverge almost surely, since the variance of the partial sums are given by $(2\pi)^{-2}\sum|k|^{-2}$, and this sum diverges when $d \geq 2$.

When $d = 1$, the limit h can be defined a.s. at any given x and is a complex Gaussian. Since the above sum converges to $(2\pi)^{-2}2\zeta(2) = 1/12$, the real and imaginary components of $h(x)$ each have variance $1/12$. In fact, it is not hard to see that the difference $h(x) - h(0)$ can be written as $(h, f^x)_\nabla = (h, \delta_x - \delta_0)$ where $f^x = -\Delta^{-1}(\delta_x - \delta_0)$ is continuous and linear on $(0, x)$ and $(x, 1)$. By computing dot products of $(f^x, f^y)_\nabla$, the reader may verify that h has the same law as a multiple of the Brownian bridge on the circle, normalized by adding a constant so that it has zero mean. A similar argument shows that the one-dimensional GFF on an interval is a multiple of the Brownian bridge on that interval, and an even simpler argument shows that the one-dimensional GFF on $(0, \infty)$ is a Brownian motion. In the latter case, we may take $[-\Delta^{-1}\delta_x](y) = \min\{x, y\}$ and $G(x, y) = \min\{x, y\}$. The variance of $h(x) = (h, \delta_x)$ is $G(x, x) = x$.

Remark 2.15 The definition of the Dirichlet inner product, and hence the Gaussian free field, has an obvious analog any manifold on which the Dirichlet energy can be defined. In particular, since the Dirichlet inner product is conformally invariant when $d = 2$, the Dirichlet energy has a canonical definition for Riemann surfaces. There is also a "free boundary" version in which we replace $H_s(D)$ by the set of all smooth, mean zero functions on D with first derivatives in $L^2(D)$.

Remark 2.16 The GFF has a natural dynamic analog in which each $(h_t, f)_\nabla$ is the Ornstein-Uhlenbeck process with zero mean whose stationary distribution has variance $\|f\|_\nabla^2$. Thus, instead of taking $(h, f_j)_\nabla$ to be an i.i.d. sequence of random variables, we take $(h_t, f_j)_\nabla$ to be an i.i.d. sequence of Ornstein-Uhlenbeck processes parameterized by t.

2.6 Field averages and the Markov property

If $-\Delta a = \rho$ is constant on an open subset $D' \subset D$ and equal to zero outside of D' (i.e., a is harmonic outside of D'), then we can think of $(h, a)_\nabla = (h, \rho)$ as describing (up to a constant multiple) the mean value of h on D'. We will retain that interpretation when h is chosen from the Gaussian free field—i.e., we think of h as fluctuating so rapidly that it is not necessarily even well defined as a function, but the "average value of h on D'" is well-defined.

Since Hilbert spaces are self dual, if ρ is any probability measure on D for which $f \to \rho f := \int f \mathrm{d}\rho$ is a continuous linear functional on $H(D)$ (which is the case if and only if $\sum[\rho f_j]^2 < \infty$), then there is an f for which $\rho g = (f,g)_\nabla$ for all $g \in H(D)$, and we have $\rho = -\Delta f \in \Delta H(D)$.

For example, if $d = 2$ and ρ is the uniform measure on a line segment L in the interior of D, then ρh is well-defined. In this case, the reader may check that $f \to \rho f$ is continuous on $H_s(D)$. (The sums $\sum[\rho f_j]^2 < \infty$ can be computed explicitly when D is a rectangle; continuity then follows for domains that are subsets of that rectangle.)

Another important observation is that if H_1 and H_2 are any closed orthogonal subspaces of $H(D)$, then $(h,\cdot)_\nabla$ restricted to these two subspaces is independent. To be precise, denote by $\mathcal{F}_{H_j}$ the smallest σ-algebra in which $h \to (h,f)_\nabla$ is a measurable function for each $f \in H_j$. Then it is clear that $\mathcal{F}_{H_1}$ and $\mathcal{F}_{H_2}$ together generate $\mathcal{F}$, and moreover, μ is independent on these two subalgebras.

For example, given an open subset U of D, we can write $H_U(D)$ for the closure of the set of smooth functions that are supported in a compact subset of U. If $a \in H_U(D)$ and b is harmonic in U, then integration by parts implies $(a,b)_\nabla = (a,-\Delta b) = 0$. Thus $H_U(D)$ is orthogonal to the closed subspace $H_U^\perp(D)$ of functions that are harmonic on U.

Theorem 2.17 *The spaces $H_U(D)$ and $H_U^\perp(D)$ span $H(D)$.*

Proof To see this it is enough to show that if $f \in H_s(D)$, then f can be written as $a+b$, with $a \in H_U(D), b \in H_U^\perp(D)$. Roughly speaking, we would like to set b to be the unique continuous function which is equal to f outside of U and harmonic inside of U, and then write $a = f - b$. But in some cases—e.g., if U is the complement of a discrete set of points—there is no b with this property. We will give a slightly modified definition of b and show $b \in H_U(D)$ and $f - b \in H_U^\perp(D)$.

Let $b_\delta(x)$ be the expected value of f at the point at which a Brownian motion started at x first exits the set U_δ of points of distance more than δ from the complement of U. Then $a_\delta(x) = f - b_\delta(x)$ is supported on a compact subset of U and is clearly in $H_U(D)$. Since $H_U(D)$ is the closure of the union of the H_{U_δ}, the a_δ's—which are projections onto the increasing (as $\delta \to 0$) subspaces H_{U_δ}—converge to some function $a \in H_U(D)$. The b_δ thus must converge to some b, and $b \in H_U^\perp$ (since the limit of harmonic functions is harmonic), and $f = a + b$. □

For short, we will write $\mathcal{F}_U = \mathcal{F}_{H_U}$ and $\mathcal{F}_U^\perp = \mathcal{F}_{H_U^\perp}$. The σ-algebra $\mathcal{F}_U^\perp$ is one in which random variables of the form $(h,f)_\nabla = (h,-\Delta f)$ are measurable whenever Δf vanishes on U. Intuitively, it allows us to measure the "values" of h outside of U. On the other hand, $\mathcal{F}_U$ allows us to measure the "values" of h inside of U modulo the harmonic functions on U. The independence of the GFF on $\mathcal{F}_U$ and $\mathcal{F}_U^\perp$ can be interpreted as saying that *given* the values of h outside of U, the distribution of the values of h in U is a harmonic extension of the values of h on the boundary of U *plus* an independent GFF on U. This property of the GFF is called a *Markov property*. It holds, in particular, if $d = 2$ and the complement of U is a simple path in D; in this case, the $\mathcal{F}_U^\perp$-measurable functions measure the values of h along that path (or at least the average values of h along subintervals of that path).

If U is closed and $x \in D\backslash U$, then it is not hard to see that the projection $f_{x,U}$ of $-\Delta^{-1}\delta_x$ onto H_U has finite Dirichlet energy and that its Laplacian is supported on the boundary of $D\backslash U$. Although $h(x) = (h, \delta_x) = (h, -\Delta^{-1}\delta_x)_\nabla$ is not a well-defined random variable, we may still intuitively interpret $(h, f_{x,U})_\nabla$ as the "expected" value of $h(x)$ given the values of h in U. The reader may check that the function $(h, f_{x,U})_\nabla$ is almost surely harmonic in $D\backslash U$. (More precisely, since the event "$(h, f_{x,U})_\nabla$ is harmonic" is not in our σ-algebra, one shows that the function $(h, f_{x,U})_\nabla$ defined on dyadic rational points of $D\backslash U$ almost surely extends continuously to a harmonic function in all of $D\backslash U$; this is the same way one proves continuity of Brownian motion in, e.g., Chapter 7 of [7].) We interpret this function as the harmonic extension to $D\backslash U$ of the "values" of h on the boundary of U.

2.7 Field exploration: Brownian motion and the GFF

In this subsection, we describe a simple way of using a space-filling curve to give a linear correspondence between the GFF and Brownian motion. Roughly speaking, we "explore" the field h along a space-filling curve, and the Brownian motion goes up or down depending on whether the values we encounter are greater than or less than what we expect. Then each of the random variables $(h, f)_\nabla$ can be viewed as an appropriate stochastic integral of this Brownian motion. Although analogous constructions hold in higher dimensions, we will assume for simplicity that $d = 2$ and D is a simply connected bounded domain.

First, choose f_0 so that Δf_0 is a negative constant on D. Then let $\gamma : [0, 1] \to D$ be a continuous space-filling curve. For each t, denote by γ_t the compact set $\gamma([0, t])$, and let P_t be the projection onto the subspace $H^\perp_{D\backslash\gamma_t}$ of functions harmonic in $D\backslash\gamma_t$. We also require that γ remains continuous when it is parameterized in such a way that $\|P_t(f_0)\|^2_\nabla = t$ for all $t \in [0, \|f_0\|^2_\nabla]$. (This will be the case provided that γ_s is a proper subset of γ_t whenever $s < t$. In other words, although γ may intersect itself, it cannot spend an entire positive-length interval of time retracing points that have already been seen.)

By decomposing f_0 into its projection onto the complementary subspaces $\mathcal{F}^\perp_{D\backslash\gamma_t}$ and $\mathcal{F}_{D\backslash\gamma_t}$, we easily observe the following:

$$W(t) := \mathbb{E}((h, f_0)_\nabla | \mathcal{F}^\perp_{D\backslash\gamma_t}) = (h, P_t(f_0))_\nabla.$$

Clearly, W is a martingale, and each $W(t) - W(s)$ is Gaussian with variance $|s - t|$. Hence, W has the same law as a Brownian motion (in the smallest σ-algebra where each $W(t)$ is measurable).

Now, the reader may easily verify that the linear span of the functions $P_t(f_0)$, with $t \in [0, 1]$, is dense in $H(D)$. Thus, given the Brownian motion $W(t)$, it should be possible, almost surely, to determine $(h, f_j)_\nabla$ for each j. To this end, observe that for any other f, the value

$$W_f(t) := \mathbb{E}((h, f)_\nabla | \mathcal{F}^\perp_{D\backslash\gamma_t}) = (h, P_t(f))_\nabla$$

is also a martingale, and is Brownian motion when time is parameterized by $\|P_t(f)\|y_\nabla^2$. The question is, how are W_f and W related?

The answer would be obvious if we had $f = aP_s(f_0)$ for some fixed constants a and $0 < s < 1$. In this case, $W_f(t) = aW(\min\{s,t\})$. A similar result holds if f is any finite sum of such functions. We have now defined $W_f(t)$ for a dense linear space of functions $f \in H(D)$, and we may choose an orthonormal basis $\{f_j\}$ for $H(D)$ from among that space. Given an arbitrary $f = \sum \alpha_j f_j$ and any fixed t, we can take $W_f(t)$ to be the limit of the partial sums of $\sum \alpha_j W_{f_j}(t)$.

The above discussion gives a *linear* correspondence between the GFF and a Brownian motion. That correspondences of this sort should exist is not surprising, given that a Brownian motion can be interpreted as a Gaussian free field on the interval $I = (0, \|f_0\|_\nabla^2)$ that is only required to vanish at the left endpoint (i.e., $H(I)$ is the Dirichlet inner product Hilbert space completion of the set of smooth functions on I that vanish at zero). The map that sends the function $g^s(t) = \min\{s,t\}$ in $H(I)$ to $P_s(f_0) \in H(D)$ extends to a Hilbert space isomorphism between $H(D)$ and $H(I)$ and the correspondence between the GFF and Brownian motion is induced by this isomorphism.

2.8 Circle averages and thick points

Fix a domain $D \subset \mathbb{R}^2$ on which the GFF is defined. For $x \in D$ and $t \in \mathbb{R}$, write $B_x(t)$ for the mean value of the GFF on the circle of radius e^{-t} centered at x (provided t is large enough that the disc enclosed by this circle lies in D). The reader may verify that for each $x \in D$, the law of $B_x(t)$ is that of a multiple of a Brownian motion. If $x \neq y$, then the GFF Markov property implies that the Brownian motions $B_x(t)$ and $B_y(t)$ grow independently of one another once t is large enough so that $2e^{-t} < |x - y|$. Let

$$A_x(t) = \frac{\int_{s=t}^{\infty} B_x(s)e^{-s}\mathrm{d}s}{\int_{s=t}^{\infty} e^{-s}\mathrm{d}s} = \int_{s=t}^{\infty} B_x(s)e^{t-s}\mathrm{d}s$$

be the mean value of the GFF on the disc of radius e^{-t}.

One can generate various fractal subsets of D by considering the set of points x for which $B_x(t)$ or $A_x(t)$ are in some sense highly atypical. For example, one might consider the set of x for which $\limsup |B_x(t)| \leq C$ for a constant C, or more generally the set of x for which $\limsup |B_x(t) - f(t)| \leq C$ for a some function f.

In [18] the authors fix a constant $0 \leq a \leq 2$ and define a *thick point* as a point $x \in D$ for which $\lim A_x(t)/t = \sqrt{a}$. They prove that the Hausdorff dimension of the set of thick points of a GFF is almost surely equal to $2 - a$.

3 General results for Gaussian Hilbert spaces

By replacing the Dirichlet inner product with a different bilinear form, it is possible to construct different types of Gaussian Hilbert spaces, some of which play important roles in constructive quantum field theory [15]. Most of the results in the previous section depended heavily on the choice of inner product. In this section, we will back up and make some statements about the Gaussian free field that are largely independent of this choice.

3.1 Moments and Schwinger functions

The moments of the variables (h,ρ) may be computed explicitly. Suppose that $\rho_j = -\Delta f_j$ for $1 \le j \le k$. First, we know that $\mathbb{E}\left[(h,\rho)_\nabla\right] = 0$ and $\mathbb{E}\left[(h,\rho_1)(h,\rho_2)\right] = (f_1,f_2)_\nabla$. We can now cite the following from the first chapter of [19]:

Theorem 3.1 *We have*

$$\mathbb{E}[(h,\rho_1)\cdots(h,\rho_k)] = \sum_M \prod_{j=1}^{k/2} (f_{M_{j,1}}, f_{M_{j,2}})_\nabla$$

where M ranges over the set of all partitions $M = \{(M_{j,1}, M_{j,2})\}$ of $\{1,\dots,k\}$ into $k/2$ disjoint pairs. In particular, this value is zero whenever k is odd.

If each $(f_j,f_k)_\nabla$ is positive (which will be the case, e.g., if each ρ_j is a positive probability density function), we can interpret the above value as the partition function for a process that chooses a random perfect matching M of the complete graph on $1,\dots,k$ with probability proportional to

$$\prod_{j=1}^{k/2} (-\Delta^{-1}\rho_{M_{j,1}}, \rho_{M_{j,2}})$$

i.e., each edge (i,j) is weighted by $(f_i,f_j)_\nabla$. (This is a useful mnemonic, if nothing else.) These perfect matchings are simple examples of *Feynman diagrams* (see [19] or [15]). The moment computations above apply to general Gaussian Hilbert spaces when $(\cdot,\cdot)_\nabla$ is replaced by the appropriate covariance inner product.

Now, for any $x_1,\dots,x_k \in D$, one would also like to define the point moments $S_k(x_1,\dots,x_k) := \mathbb{E}\prod_{j=1}^k h(x_j)$. In the case of the GFF when $d \ge 2$, these $h(x_j)$ are not defined as random variables. If they *were* defined as random variables, then, writing $\rho = -\Delta f$, we would also expect that $\mathbb{E}(h,\rho) = \int_D S_1(x)\rho(x)\mathrm{d}x$ and more generally

$$\mathbb{E}\prod_{j=1}^k (h,\rho_j) = \int_{D^k} S_k(x_1,\dots,x_k)\prod \rho_j(x_j)\mathrm{d}x.$$

It turns out that for a broad class of Gaussian and non-Gaussian random fields that includes the GFF, there *do* exist functions (or at least distributions)

$S_k : D^k \to \mathbb{R}$, called *Schwinger functions*, for which the latter statement holds, at least when the ρ_j are smooth (Proposition 6.1.4 of [15]).

In the case of the GFF, the reader may verify that S_k is identically zero when k is odd and $S_2(x_1, x_2) = G(x_1, x_2)$. For even $k > 2$, S_k can be computed from S_2 using the expansion given in Theorem 3.1.

3.2 Wiener decompositions and Wick products

From Chapter 2 of [19], we cite the following:

Theorem 3.2 *The set of polynomials in the $(h, f_j)_\nabla$ is a dense subspace of $L^2(\Omega, \mathcal{F}, \mu)$.*

The space $L^2(\Omega, \mathcal{F}, \mu)$, endowed with the inner product $(X, Y) = \mathbb{E}(XY)$, can be viewed as the closure of the direct sum of Hilbert spaces $H^{:n:}$, each of which is the closure of the set of degree n polynomials in $(h, f_j)_\nabla$ that are orthogonal to all degree $n-1$ polynomials (in particular, $H^{:1:} = H(D)$). The decomposition of an element of $L^2(\Omega, \mathcal{F}, \mu)$ into these spaces is sometimes called the *Wiener chaos decomposition* (Wiener 1938, Itô 1951, Segal 1956; see Chapter 2 of [19]). See Chapters 2 and 3 of [19] for the explicit form of the projection operators onto each $H^{:n:}$, as well as a natural orthonormal basis for each $H^{:n:}$—defined in terms of an orthonormal basis of $H(D)$.

The above implies in particular that if we use the two-dimensional Gaussian free field to define conformally invariant random sets or random loop ensembles (e.g., via SLE constructions), then any L^2 function of these random objects (e.g., the number of loops encircling a given disc) can be expanded in terms of this orthonormal basis (although in practice this may be difficult to do explicitly).

Now, given any $\eta_1 \in H^{:m:}$ and $\eta_2 \in H^{:n:}$, the *Wick product* of η_1 and η_2 is the projection of $\eta_1\eta_2$ onto $H^{:m+n:}$. Explicit formulae for the Wick product are given in [19].

3.3 Other fields

The "massive" free fields (see Chapter 6 of [15]) may be defined as a collection of Gaussian random variables (h, ρ) with covariances given by

$$\mathrm{Cov}[(h, \rho_1), (h, \rho_2)] = ((-\Delta + m^2)^{-1}\rho_1, \rho_2),$$

where m is a real number called the **mass**. Here, we can either require the test functions ρ to be smooth (the random distribution interpretation) or let them belong to the Hilbert space completion of the smooth functions under the inner product $((-\Delta + m^2)^{-1}\rho_1, \rho_2)$ (the Gaussian Hilbert space interpretation).

Equivalently, we may consider random variables $(h, f)^m_\nabla := (h, f)_\nabla + m^2(h, f)$ where $f = (-\Delta + m^2)^{-1}\rho$. We then have

$$\mathrm{Cov}[(h, f_1)^m_\nabla, (h, f_2)^m_\nabla] = (f_1, f_2)^m_\nabla.$$

The GFF is the case $m = 0$. Most of the results in this paper have straightforward analogs in the case $m \neq 0$.

Among the other fields discussed in quantum field theory (see, e.g., Chapters 8, 10, and 11 of [15]) are probability measures that are absolutely continuous with respect to the massive or massless free fields and have Radon-Nikodym derivatives given by $e^{-V}Z^{-1}$, where V is a "potential" function on Ω and Z is an appropriate normalizing constant.

In the more interesting "interacting particle" settings, however, this V is undefined or infinite for μ-almost all points in Ω. This happens, for example, if $d \geq 2$ and we write $V(h) = \int_D P(h)$ where P is an even polynomial. In this case, we can define approximations V_n to V by writing $V_n(h) = V(h_n)$ where the h_n is a natural approximation to h (e.g., h_n could be the partial sum $\sum_{j=1}^n \alpha_j f_j$, or it could be one of the discrete lattice approximations in the subsequent section). We then seek to define a field which is the limit, in some sense, of (appropriately normalized versions of) the probability measures $e^{-V_n}Z_n^{-1}\mu$. When it exists, the limiting measure is in general not absolutely continuous with respect to μ. See [15] for a mathematically rigorous approach to constructing fields, including the so-called $P(\phi)$ **fields** (in particular, the celebrated ϕ^4 fields), in a way that uses the Gaussian fields as a starting point.

4 Harmonic crystals and discrete approximations of the GFF

4.1 Harmonic crystals and random walks

Let Λ be a finite graph with a positive weight function w on its edges. If ϕ_1 and ϕ_2 are functions on Λ, we denote their Dirichlet inner product by

$$(\phi_1, \phi_2)_\nabla = \sum_{e=(x,y)} w(e)(\phi_1(y) - \phi_1(x))(\phi_2(y) - \phi_2(x))$$

where the sum is over all edges $e = (x, y)$ of Λ. Now, fix a "boundary" $\partial\Lambda$, which, for now, we can take to be any non-empty subset of the vertices of Λ. Then the set $H(\Lambda)$ of real-valued functions on Λ whose values are fixed to be zero (or some other pre-determined set of boundary values) on $\partial\Lambda$ is a $|\Lambda| - |\partial\Lambda|$ dimensional Hilbert space under the Dirichlet inner product. This is also a finite dimensional Gaussian Hilbert space, where the probability density at ϕ is proportional to $e^{-\|\phi\|_\nabla^2/2}$.

When Λ is a large subset of a d-dimensional lattice graph L, and $\partial\Lambda$ is the set of vertices that border points of $L\backslash\Lambda$, the resulting "discretized random surface" model is commonly called the *discrete Gaussian free field* (DGFF), the *discrete massless free field*, or the *harmonic crystal*.

Next, consider a random walk in which each edge $e = (x, y)$ is activated by an independent exponential clock with intensity $w(e)$, at which point if the position is x, it switches to y, and vice versa.

Given ϕ and $x \in \Lambda\backslash\partial\Lambda$, write Y for the expected value of $\phi(x)$ at the first neighbor x of y hit by a random walk starting at y, i.e.,

$$Y = \frac{\sum_{e=(x,y)} w(e)\phi(x)}{\sum_{e=(x,y)} w(e)}.$$

When the weight function w is constant, Y is simply the average value of ϕ on the neighbors of x; when w is not constant, Y is an appropriate weighted average of these values.

The reader may verify the following one-point discrete Markov property: the random variable $\phi(y) - Y$ is independent of the values of $\phi(x)$ for $x \neq y$ and this random variable has mean zero and variance $1/\sum_{e=(x,y)} w(e)$.

This property implies that if we fix the values of ϕ on $\partial\Lambda$ (where these values are not necessarily equal to zero), then the expected value of any $\phi(x)$, for $x \in \Lambda$, will be the expected value of $\phi(x_h)$, where x_h is the first vertex on $\partial\Lambda$ that is hit by a random walk beginning at x.

We claim that it also implies that $\mathrm{Cov}[\phi(x), \phi(y)]$ is equal to the expected amount of time that a particle started at x will spend at y before hitting the boundary. (This function in x and y is also called the *discrete Green's function* on Λ.) To see this, first observe that when y is fixed, both sides are discrete harmonic (with respect to the random walk) in $\Lambda\backslash\{y\}$. It is then enough to compute the discrete Laplacian (with respect to the random walk) at y itself and observe that it is equal to $1/\sum_{e\ni y} w(e)$ for both sides: for the right hand side (the random walk interpretation), this follows from well-known properties of exponential clocks. For the left hand side, since Y and $\phi(y) - Y$ are independent and $\mathbb{E}(\phi(y) - Y)^2 = 1/\sum_{e=(x,y)} w(e)$, we have

$$\mathbb{E}\phi(y)(\phi(y) - Y) = \mathbb{E}(\phi(y) - Y)^2 + \mathbb{E}Y(\phi(y) - Y) = 1\Big/\sum_{e=(x,y)} w(e).$$

As an example, if Λ is $[-n,n]^d \subset \mathbb{Z}^d$ and $w = 1$, then it is not hard to see (from well-known properties of random walks) that the variance of $\phi(0)$ is asymptotically proportional to n if $d = 1$, $\log(n)$ if $d = 2$, and a constant if $d \geq 3$ (since the random walk is transient in the latter case).

See [2–4,13] and the references therein for these and many more results about harmonic crystals and various generalizations of the harmonic crystal. For an analog of the Green's function interpretation of variance that applies in the continuum case (known as the *Dynkin isomorphism theorem*), see [8–10].

4.2 Discrete approximations: triangular lattice

Suppose $d = 2$, and let L be the standard triangular lattice (the dual of the honeycomb lattice). Now, suppose we restrict the GFF on D to the σ-algebra $\mathcal{F}_{H_n}$ where $H_n(D)$ is the set of continuous functions that are affine on the each of triangles of $\frac{1}{n}L$ and that vanish on the boundary of D. Since $H_n(D)$ is a finite dimensional Hilbert space, it is self-dual, and hence a sample from the GFF determines an element h_n of $H_n(D)$, with probability proportional to $\exp[-\|h_n\|_\nabla^2]$. Observe also that h_n is determined by its values on the vertices of the triangular mesh, and that $\|h_n\|_\nabla^2$ is equal to $\frac{\sqrt{3}}{6}\sum |h_n(j) - h_n(i)|^2 + |h_n(k) - h_n(i)|^2 + |h_n(k) - h_n(j)|^2$ where the sum is over all triangles (i, j, k) in the mesh. (The area of each triangle is $\sqrt{3}/(4n^2)$ and the gradient squared is $\frac{2}{3}n^2[|h_n(j)-h_n(i)|^2+|h_n(k)-h_n(i)|^2+|h_n(k)-h_n(j)|^2]$.) Since each interior edge of D is contained in two triangles, this is also equal to $3^{-1/2}\sum |h_n(j) - h_n(i)|^2$, where the sum is taken over nearest neighbor pairs (i, j).

In other words, h_n is distributed like $3^{1/4}$ times the harmonic crystal (with unit weights) on the set of vertices of $\frac{1}{n}L \cap D$, where the boundary vertices are precisely those that lie on a triangle which is not completely contained in D.

It is not hard to see that the union of the spaces $H_n(D)$ is dense in $H(D)$, and that for any function $f \in H(D)$, f minus its projection onto $H_n(D)$ will tend to zero in n. Thus, the h_n are approximations of h in the sense that for any fixed $f \in H(D)$, we have $(h_n, f)_\nabla \to (h, f)_\nabla$ almost surely. For any fixed n, it easy to define contour lines of the continuous function h_n.

4.3 Discrete approximations: other lattices

Now, again, suppose that $d = 2$ but that we replace the standard lattice with any doubly periodic triangular lattice L' (i.e., a doubly periodic planar graph in which all faces are triangles). Once again, we can restrict the GFF to the functions that are linear on the triangles of $\frac{1}{n}L'$. In this case, the heights at the corners turn out to have the same law as a harmonic crystal on this triangular lattice graph in which the weight $w(e)$ corresponding to a given edge e is given by $w(e) = [\cot(\theta_1) + \cot(\theta_2)]/2$, where θ_1 and θ_2 are the angles opposite e in the two triangles that are incident to e. (Note, of course, that the weights are unchanged by constant rescaling.)

That is (as the reader may verify), the discrete Dirichlet energy

$$(h_n, h_n)_\nabla = \sum w((x, y))[h_n(y) - h_n(x)]^2,$$

where h_n is viewed as a function on the graph, is the same as the continuous Dirichlet energy $(h_n, h_n)_\nabla$, where h_n is extended piecewise linearly to all of D.

In particular, we can get the discrete GFF on a square grid by dividing each square into a pair of triangles (either direction). In this case, the diagonal edges have $w(e) = 0$, since both θ_1 and θ_2 are right angles. The vertical and horizontal edges have $w(e) = 1$, since θ_1 and θ_2 are equal to 45 degrees in this case. (Note that $w(e)$ can be negative if one or both of the θ_i exceeds 90 degrees. Earlier, we assumed that w was positive; however, since the quadratic form corresponding to such a w is still positive definite, the definition of the harmonic crystal still makes sense in this setting.)

Similar results hold when $d > 2$ if we replace triangles with d-dimensional simplices.

4.4 Computer simulations of harmonic crystals

One fortunate feature of the discrete Gaussian free fields is the ease with which they can be simulated on computers (see Fig. 1). Consider first the special case that Λ is an $m \times n$ torus grid graph. In this case, the discrete exponential functions form an orthogonal basis of the set of complex, mean-zero functions on Λ, so the elements in a discrete Fourier transform of a discrete GFF on Λ are independently distributed Gaussians with easily computed variances. For example, the Mathematica code shown below generates and plots an instance of the discrete GFF on an $m \times n$ torus.

```
ListPlot3D[ Re[Fourier[Table[
  (InverseErf[2 Random[]-1]+I InverseErf[2 Random[]-1])*
If[j+k==2,0,1/Sqrt[(Sin[(j-1)*Pi/m]^2+ Sin[(k-1)* Pi/n]^2)]],
{j,m}, k,n]]]]
```

The code generates a complex GFF and then plots its real part. To parse the code, note that the second line simply produces a complex Gaussian random variable in Mathematica. For each $1 \leq j \leq m$, $1 \leq k \leq n$, the third line gives one over the gradient norm of the discrete exponential $(x, y) \to \eta^{(j-1)x}\zeta^{(k-1)y}$ (where η and ζ are mth and nth roots of unity, respectively) on the $m \times n$ torus (unless $j = k = 1$, in which case it gives zero). The Fourier transform of the corresponding matrix is the Gaussian free field on the torus. Now suppose Λ is a simply connected induced subgraph of the torus with values h_0 assigned to its boundary; to sample a GFF on Λ with these boundary values, one may first sample an instance h of the GFF on the torus and then replace h with $h + \tilde{h}$ where $\tilde{h}$ is the discrete harmonic interpolation of the function $h_0 - h$ (defined on $\partial\Lambda$) to all of Λ. The most time-consuming part of this algorithm is computing the discrete harmonic interpolation (but it is not hard to compute an approximate interpolation).

4.5 Central limit theorems for random surfaces

Kenyon in [20] proved that random domino tiling height functions (in regions with certain kinds of boundary conditions) converge to the GFF as the mesh size gets finer. (To be precise, [20] studies random discrete height functions h_ϵ—for which the lattice spacing is ϵ—and shows that for smooth density functions ρ, (h_ϵ, ρ) converges in law to $(h, \rho) = (h, -\Delta^{-1}\rho)_\nabla$.) Also, [14,21] give a similar Gaussian free field convergence result for a class of discretized random surfaces known as *Ginzburg-Landau* $\nabla\phi$ *random surfaces* or *anharmonic crystals* and [14] shows further that certain time-varying versions of these processes converge to the dynamic GFF.

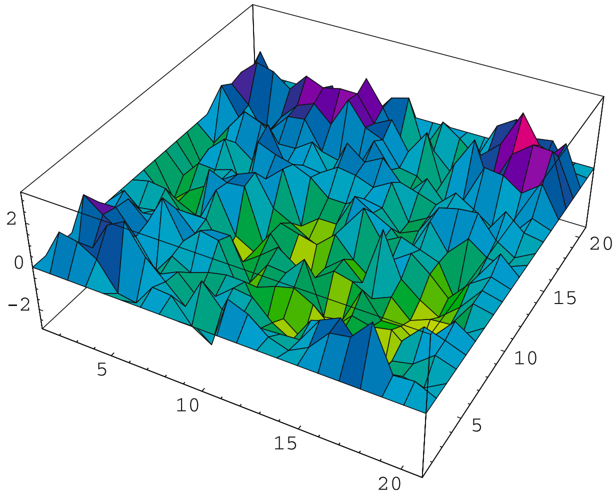

Fig. 1 Discrete Gaussian free field on 20 by 20 grid with zero boundary conditions

See [24] for more references on random surface models (assuming both discrete and continuous height values) with convex nearest neighbor potential functions and discussion of the division of the so-called *gradient Gibbs measures* into *smooth* phases (in which the variance of the height difference between two points is bounded, independently of the distance between those points) and *rough* phases (in which the variance of these height differences tends to infinity as the points get further apart). An important open question is whether every two-dimensional rough phase has a scaling limit given by a linear transformation of the GFF.

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