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Survival and coexistence in stochastic spatial Lotka–Volterra models

J. Theodore Cox · Edwin A. Perkins

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Abstract A spatially explicit, stochastic Lotka–Volterra model was introduced by Neuhauser and Pacala in Neuhauser and Pacala (Ann. Appl. Probab. **9**, 1226–1259, 1999). A low density limit theorem for this process was proved by the authors in Cox and Perkins (Ann. Probab. **33**, 904–947, 2005), showing that certain generalized rescaled Lotka–Volterra models converge to super-Brownian motion with drift. Here we use this convergence result to extend what is known about the parameter regions for the Lotka–Volterra process where (i) survival of one type holds, and (ii) coexistence holds.

Keywords Lotka–Volterra · Voter model · Super–Brownian motion

Mathematics Subject Classification (2000) Primary: 60K35 · 60G57; Secondary: 60F05 · 60J80

1 Introduction and statement of results

In [11], Neuhauser and Pacala introduced a stochastic spatial version of the Lotka–Volterra model for competition between two species. In [5], the authors

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J. T. Cox
Department of Mathematics, Syracuse University, Syracuse, 13244 NY, USA

E. A. Perkins (✉)
Department of Mathematics, The University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, BC V6T 1Z2, Canada
e-mail: perkins@math.ubc.ca

proved that in three or more dimensions these processes, suitably rescaled in time and space, converge to a super-Brownian motion with drift. In this paper the goal is to obtain information about survival and coexistence for the Lotka–Volterra models from corresponding information about the limiting super-Brownian motion. This methodology has been successfully applied before in similar settings, as in [8], where the long range contact process is treated. See [7] for a good reference to the general approach. We begin by defining the Lotka–Volterra process.

Following [11], we let $\xi_\cdot = \{\xi_t, t \geq 0\}$ denote a $\{0,1\}^{\mathbf{Z}^d}$-valued Feller process, with the interpretation $\xi_t(x) = i$ means there is a plant of species i ($i = 0$ or 1) at time t at site $x \in \mathbf{Z}^d$ (the d-dimensional integer lattice). When a plant dies it is immediately replaced, and the rate at which this happens and the type of the new plant incorporates both intraspecific and interspecific effects. To specify the dynamics precisely, we need a kernel $p(x), x \in \mathbf{Z}^d$, and nonnegative interaction parameters α_0, α_1. We suppose throughout that $p(x,y) = p(y-x)$ is an irreducible, symmetric random walk kernel on $\mathbf{Z}^d$, such that $p(0) = 0$ and $\sum_{x \in \mathbf{Z}^d} x^i x^j p(x) = \delta_{ij}\sigma^2 < \infty$. For $\xi \in \{0,1\}^{\mathbf{Z}^d}$ the densities $f_i = f_i(\xi) = f_i(x,\xi)$ are defined by

$$f_i(x,\xi) = \sum_{y \in \mathbf{Z}^d} p(y-x) 1\{\xi(y) = i\}, \quad i = 0, 1.$$

We define the Lotka–Volterra *rate function* $c(x,\xi)$ by

$$c(x,\xi) = c_1(x,\xi) 1\{\xi(x) = 0\} + c_0(x,\xi) 1\{\xi(x) = 1\}, \tag{1.1}$$

where

$$\begin{aligned} c_1(x,\xi) &= f_1(f_0 + \alpha_0 f_1)(x,\xi) = f_1(x,\xi) + (\alpha_0 - 1) f_1(x,\xi)^2, \\ c_0(x,\xi) &= f_0(f_1 + \alpha_1 f_0)(x,\xi) = f_0(x,\xi) + (\alpha_1 - 1) f_0(x,\xi)^2. \end{aligned} \tag{1.2}$$

Here $c_1(x,\xi)$ [respectively, $c_0(x,\xi)$] is the infinitesimal rate at which a 1 replaces a 0 [respectively, a 0 replaces a 1] at location x in state ξ. By a standard theorem [see Theorem B3 of [10] and Remark 2.5 below], $c(x,\xi)$ determines a unique, $\{0,1\}^{\mathbf{Z}^d}$-valued Markov process ξ_t. More precisely Corollary 2.4 and Proposition 2.1 show the above rates determine a unique $\{0,1\}^{\mathbf{Z}^d}$-valued Feller process through the generator described in Proposition 2.1(c) below. We will refer to this process as the LV(α_0, α_1) process and let P^α or $P^\alpha_{\xi_0}$ denote its law starting at ξ_0. Hence if $\xi(x) = 1$, $f_1(x,\xi) + \alpha_1 f_0(x,\xi)$ is the death rate of the type 1 plant at x and the other factor $f_0(x,\xi)$ is the probability that it is immediately colonized by a type 0 plant. Therefore α_1 represents the competitive intensity of a "neighbouring" type 0 on a type 1 and 1 is the corresponding intensity for a "neighbouring" type on its own type. Also p is here playing a dual role both as a dispersal and competition kernel. Similarly α_0 represents the competitive intensity of a "neighbouring" 1 on a type 0. If $\alpha_0 = \alpha_1 = 1$, the LV(α_0, α_1)

process reduces to the well-known voter model [see [9] and [10] for references]. Note that $\alpha = (1, 1)$ is a special turning point for the model since $\alpha_i < 1$ means each type fares better in the presence of the other type while $\alpha_i > 1$ means each type prefers to be surrounded by its own type. Those familiar with [11] will have noted we have set their additional fecundity parameter λ to be one.

For $A \subset \mathbf{Z}^d$ we will use ξ_t^A to denote the process with initial state ξ_0 given by $\xi_0(x) = 1$ iff $x \in A$, and will write ξ_t^0 for $\xi_t^{\{0\}}$. Also, it will be convenient to use the notation $|\xi| = \sum_{x\in\mathbf{Z}^d} \xi(x)$, $\xi \in \{0,1\}^{\mathbf{Z}^d}$.

The fundamental questions about ξ_t concern *survival* and *coexistence*, which we now define. For given $\alpha = (\alpha_0, \alpha_1)$:

(i) *Survival* occurs if $P^\alpha\left(|\xi_t^0| > 0 \text{ for all } t \geq 0\right) > 0$.

(ii) *1's take over* if there is survival and $P^\alpha(\xi_t(x) = 1 \mid |\xi_t| > 0) \to 1$ as $t \to \infty$ for all x.

(iii) *Coexistence* occurs if there is a stationary distribution ν for $\xi_.$ such that

$$\nu\left(\{\zeta : \sum_x \zeta(x) = \sum_x (1 - \zeta(x)) = \infty\}\right) = 1.$$

Questions of coexistence of types using related systems of sde's have also been studied by Blath, Etheridge and Meredith [1].

To discuss survival we first recall some basic facts and definitions concerning monotonicity and coupling from [9] and [10]. Let $c(x, \xi)$, $\tilde{c}(x, \xi)$ be two rate functions which satisfy (2.3) below. This is a technical condition, satisfied in the cases of interest to us, which by Theorem B3 of [10] implies these rates uniquely determine associated $\{0,1\}^{\mathbf{Z}^d}$-valued Feller processes, $\xi_.$ and $\tilde{\xi}_.$, respectively, through the appropriate spin-flip generator described in Proposition 2.1(c) below. Write $\tilde{\xi} \leq \xi$ if the inequality holds pointwise. Assume

$$c(x, \xi) \leq \tilde{c}(x, \tilde{\xi}) \text{ when } \tilde{\xi} \leq \xi \text{ and } \tilde{\xi}(x) = 1,$$

and

$$c(x, \xi) \geq \tilde{c}(x, \tilde{\xi}) \text{ when } \tilde{\xi} \leq \xi \text{ and } \xi(x) = 0.$$

Given initial conditions $\xi_0 \geq \tilde{\xi}_0$ in $\{0,1\}^{\mathbf{Z}^d}$ one can then construct both processes $\xi_.$ and $\tilde{\xi}_.$ with the corresponding initial states so that $\xi_t \geq \tilde{\xi}_t$ for all $t \geq 0$ a.s. (see Theorem III.1.5 of [9]). In this case we say ξ_t stochastically dominates $\tilde{\xi}_t$ and write $\xi_t \succeq \tilde{\xi}_t$. A special case occurs when $c = \tilde{c}$, in which case we say ξ is monotone or attractive (see Theorem III.2.2 of [9]).

It follows as a special case of Propositions 8.1 and 8.2 below (although the reader can easily carry out the required calculation directly now) that if $p_* = \inf\{p(x) : p(x) > 0\}$ and $\underline{\alpha} = 1 - (2 - p_*)^{-1} \in [\frac{1}{3}, \frac{1}{2}]$, then $\mathrm{LV}(\alpha_0, \alpha_1)$ is monotone for $\alpha_0 \wedge \alpha_1 \geq \underline{\alpha}$ and is stochastically increasing in $\alpha_0 \in [\underline{\alpha}, \infty)$ and decreasing in $\alpha_1 \in [\underline{\alpha}, \infty)$. In addition, Proposition 8.2 also implies

$$\begin{aligned} &\text{if } 0 \leq \alpha_0' \leq \alpha_0,\ 0 \leq \alpha_1 \leq \alpha_1', \quad \text{and either } \alpha_0 \wedge \alpha_1 \geq \underline{\alpha} \text{ or } \alpha_0' \wedge \alpha_1' \geq \underline{\alpha}, \\ &\quad \text{then } \mathrm{LV}(\alpha_0', \alpha_1') \leq \mathrm{LV}(\alpha_0, \alpha_1). \end{aligned} \tag{1.3}$$

The survival and extinction regions for the Lotka–Volterra models are defined as

$$S = \{(\alpha_0, \alpha_1) : P^{\alpha}(|\xi_t^0| > 0 \quad \text{for all } t > 0) > 0\}$$

and $E = S^c$, respectively. For $\alpha_0 \geq \underline{\alpha}$, let

$$h(\alpha_0) = \sup\{\alpha_1 : (\alpha_0, \alpha_1) \in S\} \in [0, \infty]\,,$$

where $(\sup \emptyset = 0)$. It follows from the above monotonicity results that h is non-decreasing on $\{\alpha_0 \geq \underline{\alpha} : h(\alpha_0) \geq \underline{\alpha}\}$, the region in the $\alpha_0 - \alpha_1$ plane to the left of the portion of graph(h) in $[\underline{\alpha}, \infty)^2$ is in E and the region below it is in S (see Fig. 1). Note that (1.3) is used in these last two assertions.

We recall now results of [11] concerning survival and coexistence. Corollary 1 of [11] states that 1's take over for α_0, α_1 satisfying

$$\alpha_1 < \begin{cases} 1 + \frac{1}{p_*}(\alpha_0 - 1) & \text{if } 1 - p_* \leq \alpha_0 \leq 1, \\ 1 + p_*(\alpha_0 - 1) & \text{if } \alpha_0 > 1. \end{cases} \tag{1.4}$$

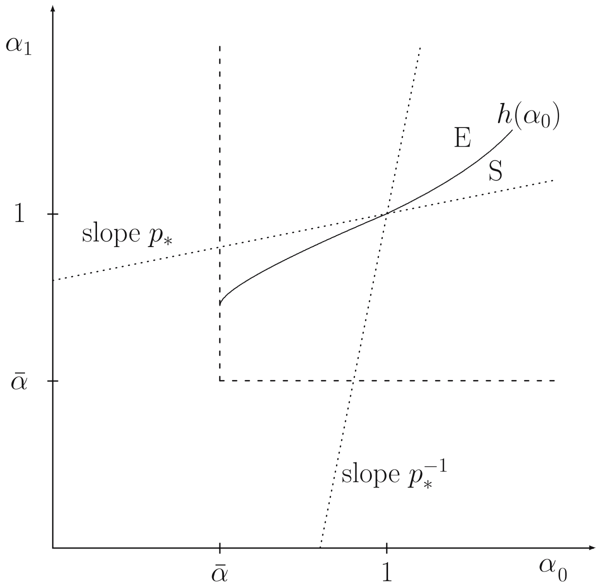

Fig. 1 Extinction and survival regions

Similarly, the 0's take over if

$$\alpha_0 < \begin{cases} 1 + \frac{1}{p_*}(\alpha_1 - 1) & \text{if } 1 - p_* \le \alpha_1 \le 1, \\ 1 + p_*(\alpha_1 - 1) & \text{if } \alpha_1 > 1. \end{cases} \tag{1.5}$$

Their neat proof relies on a stochastic comparison with some biased voter models. If $\alpha_0 \ge 1$ and $\tilde{\xi}, \xi \in \{0,1\}^{\mathbf{Z}^d}$ satisfy $\tilde{\xi} \le \xi$, then (the second inequality below is the only one requiring a moments thought)

$$c_1(x,\xi) = f_1(1 + (\alpha_0 - 1)f_1)(x,\xi) \ge f_1(1 + (\alpha_0 - 1)p_*)(x,\tilde{\xi}) \equiv \tilde{c}_1(x,\tilde{\xi}),$$

and

$$c_0(x,\xi) = f_0(f_1 + \alpha_1 f_0)(x,\xi) \le \alpha_1 f_0(x,\tilde{\xi}) \equiv \tilde{c}_0(x,\tilde{\xi}).$$

Therefore by the discussion above, $\tilde{c}(x,\tilde{\xi}) = \tilde{c}_1(x,\tilde{\xi})1(\tilde{\xi}(x) = 0) + \tilde{c}_0(x,\tilde{\xi})1(\tilde{\xi}(x) = 1)$ is the jump rate of a biased voter model $\tilde{\xi}_\cdot$ satisfying $\tilde{\xi}_\cdot \le \xi_\cdot$, where $\xi_\cdot$ is $\mathrm{LV}(\alpha_0,\alpha_1)$. If $\alpha_1 < 1 + (\alpha_0 - 1)p_*$, the 1's have a positive bias for $\tilde{\xi}$ and so the 1's will take over. In fact infinitely many 1's will drive out the 0's from any bounded set in finite time a.s. (see [2]). This implies the same conclusion for $\xi_\cdot$. A similar argument goes through for $\alpha_0 < 1$, hence giving (1.4). Then (1.15) follows by interchanging the roles of 0 and 1.

In terms of survival, (1.4) and (1.15) imply (see Fig. 1)

$$\begin{aligned} p_*(\alpha_0 - 1) \le h(\alpha_0) - 1 \le \tfrac{1}{p_*}(\alpha_0 - 1) \quad & \text{if } \alpha_0 \ge 1, \\ \tfrac{1}{p_*}(\alpha_0 - 1) \le h(\alpha_0) - 1 \le p_*(\alpha_0 - 1) \quad & \text{if } 1 - p_* \le \alpha_0 \le 1. \end{aligned} \tag{1.6}$$

Hence $h(1) = 1$, as indicated in Fig. 1. In fact we know $(1,1) \in E$ as the the voter model starting from a finite configuration will die out in finite time since $|\xi_t|$ is a non-negative martingale. Note that p_* is a highly unstable function of p and so one would not expect the above results to be sharp even locally near $(1,1)$.

Our first result gives more refined information on the behavior of $h(\alpha_0)$ for α_0 near 1 and will make use of an invariance principle established in [5] which we now state. Let $\{\hat{B}_t^x, x \in \mathbf{Z}^d\}$ be a coalescing random walk system: each $\hat{B}_t^x$ is a rate 1 random walk on $\mathbf{Z}^d$ with kernel p, with $\hat{B}_0^x = x$, the walks move independently until they collide, and then move together thereafter. For finite $A \subset \mathbf{Z}^d$ let $\tau(A) = \inf\{s : |\{\hat{B}_s^x, x \in A\}| = 1\}$ be the time at which the particles starting from A coalesce into a single particle, and write $\tau(a,b,\dots)$ when $A = \{a,b,\dots\}$. Let γ_e be the escape probability

$$\gamma_e = \sum_{e \in \mathbf{Z}^d} p(e) P(\tau(0,e) = \infty), \tag{1.7}$$

and also define the coalescing probabilities

$$\beta = \sum_{e,e' \in \mathbf{Z}^d} p(e)p(e')P(\tau(e,e') < \infty, \tau(0,e) = \tau(0,e') = \infty),$$
$$\delta = \sum_{e,e' \in \mathbf{Z}^d} p(e)p(e')P(\tau(0,e) = \tau(0,e') = \infty).$$

To describe β and δ, consider a collection of three coalescing random walks, two of which start with independent initial conditions with law $p(\cdot)$, and the third of which starts at the origin. Then β is the probability that the first two random walks coalesce but neither one of these walks ever meets the third random walk, and δ is this probability plus the probability that there is no coalescing of any two of the walks. (We will soon be assuming $d \geq 3$ so that these probabilities are non-zero.)

Consider now a sequence $\{\xi^N_\cdot, N = 1, 2, \ldots\}$ of Lotka–Volterra models on $\mathbf{Z}^d$ with kernel p and interaction parameters α_i^N satisfying:

$$|\xi_0^N| < \infty \quad \text{for all } N, \quad \text{and } \theta_i^N = N(\alpha_i^N - 1) \to \theta_i \in \mathbf{R} \quad \text{as } N \to \infty \quad \text{for } i = 0, 1. \tag{1.8}$$

Let $\mathcal{M}_F$ be the space of finite Borel measures on $\mathbf{R}^d$, endowed with the topology of vague convergence. Let $\mathbf{S_N} = \mathbf{Z}^d/\sqrt{N}$ and let $X^N_\cdot$ denote the $\mathcal{M}_F$-valued process defined by

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbf{S_N}} \xi_{Nt}^N(x\sqrt{N})\delta_x, \tag{1.9}$$

where δ_x is the unit point mass at x. We will use P_N to denote the law of $X^N_\cdot$ on $D(\mathbf{R}^+, \mathcal{M}_F)$. We make the following assumption about the initial states ξ_0^N:

$$X_0^N \to X_0 \quad \text{in } \mathcal{M}_F \quad \text{as } N \to \infty. \tag{1.10}$$

The following result is Theorem 1.2 of [5].

Theorem A *Assume $d \geq 3$. If the above assumptions hold, then $P_N \Rightarrow P_{X_0}^{2\gamma_e,\theta,\sigma^2}$ as $N \to \infty$, the law of super-Brownian motion started at X_0 with branching coefficient $2\gamma_e$, drift coefficient*

$$\theta = \theta_0\beta - \theta_1\delta, \tag{1.11}$$

and diffusion coefficient σ^2.

This limiting super-Brownian motion X is the unique $\mathcal{M}_F$-valued diffusion satisfying the following martingale problem, where $\mathcal{F}_t^X = \cap_{u>t}\sigma(X_s : s \leq u)$: for all infinitely differentiable bounded ϕ with bounded partial derivatives,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s\left(\frac{\sigma^2\Delta\phi}{2}\right) \mathrm{d}s - \theta\int_0^t X_s(\phi)\,\mathrm{d}s$$

is a continuous $(\mathcal{F}_t^X)$-martingale, with $M_0(\phi) = 0$ and predictable square function

$$\langle M(\phi)\rangle_t = \int_0^t X_s(2\gamma_e\phi^2)\,\mathrm{d}s.$$

We refer the reader to [12] for a general treatment of super-Brownian motion. For now, we only point out that if the drift θ of this super-Brownian motion $X_.$ is positive, and $X_0(\mathbf{R}^d) > 0$, then $X_.$ has positive probability of survival, meaning

$$P(X_t \neq 0 \quad \text{for all } t \geq 0) = 1 - \mathrm{e}^{-\theta X_0(\mathbf{R}^d)/\gamma_e} > 0 \tag{1.12}$$

(see Exercise II.5.3 of [12]). This suggests that $\mathrm{LV}(\alpha_0,\alpha_1)$ models with interaction rates sufficiently close to $(1,1)$ and satisfying $\beta(\alpha_0-1)-\delta(\alpha_1-1) > 0$ should survive. Our first result, Theorem 1 below, shows that this is indeed the case. As we will be using Theorem A and its refinement and generalizations, we will assume throughout that

the spatial dimension d is 3 or more.

The extension of Theorem A to the biologically important two-dimensional case will be given in [6]. The extension of the results in this paper to $d = 2$ is a topic of current research.

To state our result, we let

$$m_0 = \beta/\delta, \tag{1.13}$$

and observe that $m_0 < 1$, since (recall our earlier verbal description of β and δ) $\delta = \beta + \sum_{e,e'} p(e)p(e')P(\tau(0,e) = \tau(0,e') = \tau(e,e') = \infty) > \beta$ for $d \geq 3$. For $0 < \eta < m_0$ let S^η be the set of all $(\alpha_0,\alpha_1) \in [0,\infty)^2$, $(\alpha_0,\alpha_1) \neq (1,1)$, such that

$$\alpha_1 - 1 < \begin{cases} (m_0-\eta)(\alpha_0-1) & \text{if } \alpha_0 \geq 1, \\ (m_0+\eta)(\alpha_0-1) & \text{if } \alpha_0 < 1. \end{cases} \tag{1.14}$$

Theorem 1 *For $0 < \eta < m_0$ there exists $r(\eta) > 0$ such that survival holds for all $(\alpha_0,\alpha_1) \in S^\eta$ such that $|\alpha_0 - 1| < r(\eta)$.*

We may assume that $r(\eta)$ is non-decreasing without loss of generality. Taking the union over η in Theorem 1, we see that near $\alpha_0 = 1$, h is bounded below by a continuous function $\underline{h}$ which is differentiable at $\alpha_0 = 1$ and satisfies $\underline{h}(1) = 1$

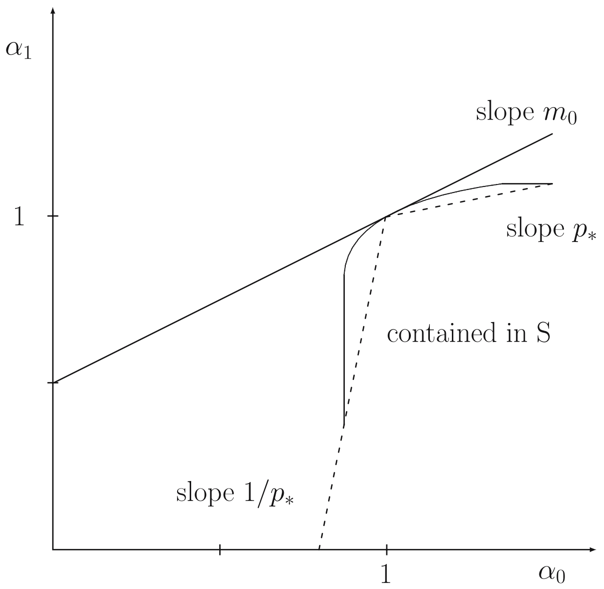

Fig. 2 Comparison of lower bounds on S

and $\underline{h}'(1) = m_0$ (see Fig. 2). Hence if $h'(1)$ exists it must be m_0. In a future work with Rick Durrett we will use different arguments to show this result is locally sharp for $\alpha_0 < 1$ and close to 1. In particular we will show that the left-hand derivative of h at 1 does equal m_0. It was already conjectured in [11] (in a slightly different form–see Conjecture 2 there) that $h(\alpha_0) = \alpha_0$ for $\alpha_0 \geq 1$, which would imply the right-hand derivative of h at $\alpha_0 = 1$ is 1. This discontinuity in the derivative at $\alpha_0 = 1$ can be thought of as a sudden increase in the survival region as α_0 passes below 1. As this is the regime in which 1's prefer to be surrounded by 0's, it allows for the survival of sparse fractal-like configurations of 1's which after rescaling are nicely modeled by the super-Brownian motion arising in Theorem A.

From Fig. 2 we see that Theorem 1 represents a significant increase on the known lower bound on S from that given by (1.6), at least near $(1, 1)$. The increase is most noticeable for $\alpha_0 < 1$ but is also significant for $\alpha_0 > 1$. To see this we now compare m_0 with p_* (note the crude inequalities in what follows and also that p_* will be 0 if p has infinite range):

$$\begin{aligned}\beta &= \sum_e \sum_{e'} p(e)p(e')P(\tau(e,e') < \infty, \tau(0,e) = \tau(0,e') = \infty)\\ &> \sum_e p(e)^2 P(\tau(0,e) = \infty)\\ &\geq p_* \sum_e p(e)P(\tau(0,e) = \infty)\end{aligned}$$

$$> p_* \sum_e \sum_{e'} p(e)p(e')P(\tau(0,e) = \tau(0,e') = \infty)$$
$$= p_*\delta.$$

Therefore we have $m_0 = \beta/\delta > p_*$. In the nearest-neighbour case, $p(x) = 2d^{-1}1(\|x\|_1 = 1)$ and simulations carried out by David Lubin give the following:

d	m_0	$p_* = 1/2d$
3	0.38	0.167
4	0.20	0.125
5	0.14	0.1
6	0.11	0.083

The proof of Theorem 1 uses the comparison with $2K$-dependent oriented percolation described in Chap. 4 of [7] to interchange the limits $N \to \infty$ and $t \to \infty$. Briefly, the idea is to construct the Lotka–Volterra process ξ_t and a super-critical oriented percolation process on the same space with the property that ξ_t "lies above" the percolation process, implying survival. Although this approach has become a standard tool, there are some subtleties in our implementation of the method. For example, we make use of some explicit upper bounds on the critical percolation probability for $2K$-dependent oriented percolation (see Remark 5.2). One byproduct of our proof of this result is the following.

Corollary 2 *Assume* (α_0, α_1) *is as in Theorem* 1 *for some* $0 < \eta < m_0$. *Then there is a* $p_0 = p_0(\alpha_0, \alpha_1) > 0$ *such that* $P(\xi_t^0(0) = 1) \geq p_0$ *for all* $t \geq 0$.

We also will use a modification of Theorem A (see Theorem C in Sect. 2 below) to derive the following quantitative version of Theorem 1.

Corollary 3 *For each* $0 < \eta < m_0$ *there are* $c_{1.15}(\eta), r(\eta) > 0$ *such that for all* $(\alpha_0, \alpha_1) \in S^\eta$ *with* $|\alpha_0 - 1| < r(\eta)$, *the* $LV(\alpha_0, \alpha_1)$ *process* ξ_t *satisfies*

$$P^\alpha(|\xi_t^0| > 0 \text{ for all } t \geq 0) \geq c_{1.15}(\eta)[|\alpha_0 - 1| + |\alpha_1 - 1| \wedge r(\eta)]. \quad (1.15)$$

A delicate aspect of these results is that one is getting non-trivial lower bounds on survival for (α_0, α_1) near $(1, 1)$, a point at which survival fails.

We turn now to the question of coexistence. As coexistence cannot occur if infinitely many 1's (or 0's) take over with probability one, (1.4) and (1.15) imply that the coexistence region

$$C = \{(\alpha_0, \alpha_1) : \text{ coexistence occurs for LV}(\alpha_0, \alpha_1)\}$$

satisfies

$$C \cap [0,1]^2 \subset \{(\alpha_0, \alpha_1) \in [0,1]^2 : \frac{1}{p_*}(\alpha_0 - 1) \leq \alpha_1 - 1 \leq p_*(\alpha_0 - 1)\}.$$

This result attracted considerable attention as it shows that a stochastic spatial model may reduce the parameter region for which coexistence holds from that in the corresponding "mean field" model. The latter is the natural ordinary differential equation model in which space is ignored (see (1.2) of [11] but with $\lambda = 1$ in that work), and coexistence occurs for all $(\alpha_0, \alpha_1) \in (0,1)^2$ as it is trivial to see there is a stable non-trivial equilibrium point in this parameter regime. It is of course natural to think that a spatial model would allow for an increased coexistence set, but the reason for the shrinkage is explained in [11]–in a spatial model for $\alpha_i < 1$, small colonies of 1's focus their positive affects on the *nearby* 0's which return the favour by driving them out. It is therefore natural to ask how much the coexistence region shrinks and our next result answers this query for (α_0, α_1) near $(1,1)$.

It is reasonable to suppose that coexistence might hold for parameter values for which both 0's and 1's survive. For $0 < \eta < m_0$ let C^η be the set of all $(\alpha_0, \alpha_1) \in [0,1]^2$ such that

$$\frac{1}{m_0 + \eta}(\alpha_0 - 1) \le \alpha_1 - 1 \le (m_0 - \eta)(\alpha_0 - 1).$$

Recall that $m_0 < 1$ so this sector is non-empty.

Theorem 4 *For* $0 < \eta < m_0$ *there exists* $r(\eta) > 0$ *such that coexistence holds for all* $(\alpha_0, \alpha_1) \in C^\eta$ *and* $1 - \alpha_0 < r(\eta)$.

Taking a union over η in Theorem 4 (again we may assume $r(\eta)$ is non-decreasing), we see that C includes a region $(1,1) \in C_0 \subset [0,1]^2$ such that

$$C_0 = \{(\alpha_0, \alpha_1) \in [0,1]^2 : f_0(\alpha_0) \le \alpha_1 \le f_1(\alpha_0)\},$$

where $f_0 \le f_1$ are continuous increasing functions such that $f_0(1) = f_1(1) = 1$, $f_0'(1) = 1/m_0$ and $f_1'(1) = m_0$ (see Fig. 3). We conjecture that these slopes are sharp.

Another consequence of Theorem 4 is that (for $d \ge 3$) coexistence holds on the diagonal $\alpha_0 = \alpha_1 = \alpha \le 1$ for α sufficiently close to 1. By Theorem of 1 of [11], coexistence also holds along the diagonal for α sufficiently close to 0 (a point where survival holds by the arguments in [11]). This gives some additional support, at least for $d \ge 3$, for the conjecture in [11] that coexistence occurs on the entire diagonal $0 \le \alpha \le 1$.

The proof of Theorem 4 allows us to say more about coexistence. Let $B(\ell) = [-\ell, \ell]^d \cap \mathbf{Z}^d$ and if $q \in [0,1]$, let $\{\xi_0^q(x) : x \in \mathbf{Z}^d\}$ be iid Bernoulli random variables with $P(\xi_0^q(x) = 1) = q$.

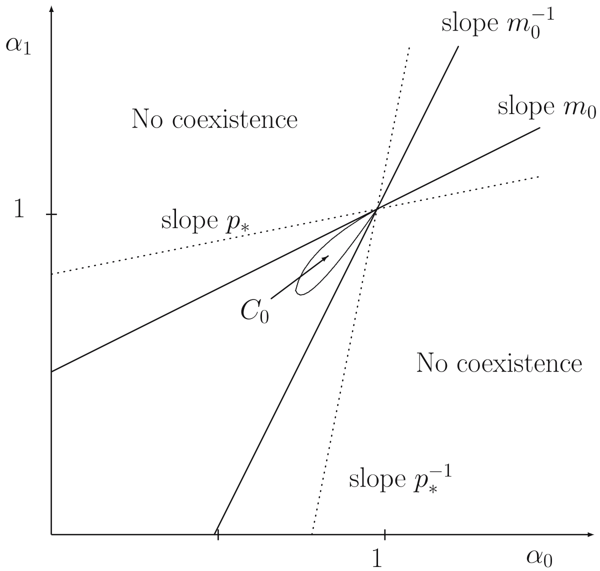

Fig. 3 Coexistence holds on C_0

Corollary 5 *Assume* (α_0, α_1) *satisfies the hypotheses of Theorem 4 for some* $0 < \eta < m_0$*, and* $\xi_0 = \xi_0^q$ *for some* $0 < q < 1$*. For any* $\varepsilon > 0$ *there are positive* ℓ_ε*,* t_ε *such that*

$$P^{\alpha}_{\xi_0}\left(\left(\sum_{x\in B(\ell_\varepsilon)} \xi_t(x)\right) \wedge \left(\sum_{x\in B(\ell_\varepsilon)} (1-\xi_t(x))\right) \geq \frac{1}{\varepsilon}\right) \geq 1-\varepsilon \quad \text{for all } t \geq t_\varepsilon.$$

Looking at (1.2), we can consider $\mathrm{LV}(\alpha_0, \alpha_1)$ as a particular quadratic perturbation of the voter model. All of the above results will be derived as special cases of results which apply to a large class of voter model perturbations including general polynomial perturbations. See Theorem 4.1 for the general version of Theorem 1 and Corollary 3, and Theorem 6.1 for the general version of Theorem 4. This general setting was introduced in [5]. For example, it allows one to extend the above class of Lotka–Volterra models to allow for different competition kernels for each type which may also be distinct from the dispersal kernel p. More specifically, let p^b and p^d be arbitrary kernels on $\mathbf{Z}^d$ such that $p^b(0) = p^d(0) = 0$ and define $f_i^b(x,\xi), f^d(x,\xi)$, for $i = 0, 1$ in the obvious way using these kernels. The spin-flip rates now become

$$\begin{aligned} c_1(x,\xi) &= f_1(f_0^b + \alpha_0 f_1^b)(x,\xi) = f_1 + (\alpha_0 - 1) f_1 f_1^b(x,\xi) \\ c_0(x,\xi) &= f_0(f_1^d + \alpha_1 f_0^d)(x,\xi) = f_0 + (\alpha_1 - 1) f_0 f_0^d(x,\xi) \\ c(x,\xi) &= c_1(x,\xi)1(\xi(x)=0) + c_0(x,\xi)1(\xi(x)=1). \end{aligned} \tag{1.16}$$

Our general perturbation results will apply if for some $C_{1.17} > 0$,

$$p^b(a) \wedge p^d(a) \le C_{1.17} p(a) \quad \text{for all } a \in \mathbf{Z}^d. \tag{1.17}$$

This condition is needed to ensure monotonicity near $(1,1)$.

Define

$$\beta' = \sum_{e,e' \in \mathbf{Z}^d} p(e) p^b(e') P(\tau(e,e') < \infty, \tau(0,e) = \tau(0,e') = \infty),$$
$$\delta' = \sum_{e,e' \in \mathbf{Z}^d} p(e) p^d(e') P(\tau(0,e) = \tau(0,e') = \infty).$$

and also β'' and δ'', which are β' and δ' with the roles of p^b and p^d reversed. Let $m_0' = \beta'/\delta'$ and $m_0'' = \beta''/\delta''$. Then the conclusion of Theorem 1 holds in this more general setting with m_0' in place of m_0 (see Theorem 8.3) and the conclusion of Theorem 4 holds with $\frac{1}{m_0''}$ and m_0' in place of $\frac{1}{m_0}$ and m_0, respectively (see Theorem 8.5). Here one should note that $\frac{1}{m_0''} < m_0'$ by an elementary argument [see (8.13)].

A second example we can treat is the (full) Neuhauser-Pacala model with their fecundity parameter λ. The rate functions in this case are

$$c_1(x,\xi) = \frac{\lambda f_1}{\lambda f_1 + f_0}(f_0 + \alpha_0 f_1), \quad c_0(x,\xi) = \frac{f_0}{\lambda f_1 + f_0}(f_1 + \alpha_1 f_0).$$

[In (1.2) $\lambda = 1$]. If $\alpha_0, \alpha_1, \lambda$ are all near 1 we can view these rates as defining a perturbation of the basic voter model, in this case a non-polynomial perturbation. Nevertheless, our results apply to this model (at least if p has finite range), and we can prove survival and coexistence in suitable values of $(\alpha_0, \alpha_1, \lambda)$ near (1,1,1). We do not include the details here.

The general voter model perturbations are introduced in Sect. 2 along with the corresponding generalization of Theorem A and a modification of this extension which is used to get the quantitative lower bound in Corollary 3. In Sect. 3 we establish a key comparison estimate (Lemma 3.2) which plays an important role in our comparison with oriented percolation. The generalized convergence theorem is used in Sect. 4 to prove the key propagation estimate which will make our underlying oriented percolation process super-critical. The general version of Theorem 1 is stated as well, along with part of the proof. Section 5 gives the oriented percolation construction and some standard consequences. The general co-existence results then follow easily in Sect. 6 and the general version of Corollary 3 is established in Sect. 7. Finally in Sect. 8 we use the general results to prove the analogues of the theorems stated above in the setting of distinct dispersal and competition kernels described above. The above theorems are then derived as special cases.

2 Construction and basic properties

We begin with a construction of $\{0,1\}^{\mathbf{Z}^d}$-valued Markov processes ξ_t, which start from initial states ξ_0 satisfying $|\xi_0| < \infty$. The construction, modelled after the one given in Chapter 2 of [7], is useful for coupling purposes.

Assume $c_i : \mathbf{Z}^d \times \{0,1\}^{\mathbf{Z}^d} \to [0,\infty), i = 0,1$ are bounded, measurable functions, and define c by $c(x,\xi) = c_1(x,\xi)1(\xi(x) = 0) + c_0(x,\xi)1(\xi(x) = 1)$. Assume there is a finite constant $C_{2.1}$ such that

$$\sum_x c_1(x,\xi) \le C_{2.1}|\xi| \quad \text{for all } \xi \in \{0,1\}^{\mathbf{Z}^d}. \tag{2.1}$$

For $A \subset \mathbf{Z}^d$ define $\xi|_A \in \{0,1\}^{\mathbf{Z}^d}$ by $\xi\big|_A(x) = \xi(x)$ for $x \in A$ and $\xi(x) = 0$ otherwise.

Let $\{N^{x,i}, x \in \mathbf{Z}^d, i = 0,1\}$ be independent Poisson point processes on $\mathbf{R}_+ \times \mathbf{R}_+$ with intensity $ds \times du$ (Lebesgue measure). $N^{x,i}$ will be used to switch the type at x to type i. For $s < t$ and $I' \subset \mathbf{R}^d$ let

$$\mathcal{G}([s,t] \times I') = \sigma(N^{x,0}(A), N^{x,1}(B), A, B \subset [s,t] \times \mathbf{R}^d, x \in I' \cap \mathbf{Z}^d), \tag{2.2}$$

and define $\mathcal{G}_t^{I'} = \mathcal{G}([0,t] \times I')$ and $\mathcal{G}_t = \mathcal{G}_t^{\mathbf{R}^d}$. In practice I' will be a large open box outside of which we will freeze the components of our particle system at 0. When translated in space, this will give us a sub-process with built-in independence for sufficiently spaced initial conditions to which we can apply known survival results for oriented percolation. The following result constructs our processes in terms of the Poisson processes $N^{x,i}$.

Proposition 2.1 *Let $\xi_0 : \mathbf{Z}^d \to \{0,1\}$ be random, independent of $\{N^{x,i} : x \in \mathbf{Z}^d, i = 0,1\}$, and satisfy $|\xi_0| < \infty$ a.s. Fix $I' \subset \mathbf{R}^d$ such that $\xi_0(x) = 0$ for all $x \notin I'$, and let $\mathcal{F}_t^{I'} = \sigma(\xi_0) \vee \mathcal{G}_t^{I'}$.*

(a) *There is a unique $\mathcal{F}_\cdot^{I'}$-adapted solution, $\xi_\cdot = \xi_\cdot[0,\xi_0,I']$ to*

$$\xi_t(x) = \begin{cases} \xi_0(x) + \int_0^t \int 1(\xi_{s-}(x)=0)1(u \le c_1(x,\xi_{s-}))N^{x,0}(\mathrm{d}s,\mathrm{d}u) \\ \quad - \int_0^t \int 1(\xi_{s-}(x)=1)1(u \le c_0(x,\xi_{s-}))N^{x,1}(\mathrm{d}s,\mathrm{d}u) \quad \forall t \ge 0, & \text{if } x \in I', \\ 0 \quad \forall t \ge 0, & \text{if } x \notin I'. \end{cases} \tag{SDE)(I'}$$

Moreover, $|\xi_t| < \infty$ for all $t \ge 0$ a.s.

(b) *Assume that $c(x,\xi)$ is monotone. Then*

   (i) *$\xi_t[0,\xi_0,I'] \le \xi_t[0,\xi_0,\mathbf{R}^d]$ for all $t \ge 0$ a.s.*

(ii) *Assume* $\tilde{\xi}_0$ *satisfies the same conditions as* ξ_0*, and let* $\tilde{\xi}_\cdot[0,\tilde{\xi}_0,I']$ *denote the corresponding solution to* $(SDE)(I')$*. If* $\xi_0 \le \tilde{\xi}_0$ *a.s., then* $\xi_t[0,\xi_0,I'] \le \tilde{\xi}_t[0,\tilde{\xi}_0,I']$ *for all* $t \ge 0$ *a.s.*

(c) *Assume also that* $c(x,\xi)$ *satisfies*

$$\sup_x \sum_u \sup_\xi |c(x,\xi) - c(x,\xi^u)| < \infty, \tag{2.3}$$

where $\xi^u(x) = 1(x \ne u)\xi(x) + 1(x = u)(1 - \xi(x))$*. Then* $\xi_\cdot[0,\xi_0,I']$ *is the unique* $\{0,1\}^{\mathbf{Z}^d}$*-valued Feller process with initial law given by that of* ξ_0 *and whose generator is the closure of*

$$\Omega f(\xi) = \begin{cases} \sum_{x\in\mathbf{Z}^d} c(x,\xi)(f(\xi^x) - f(\xi)) & \text{if } x \in I' \\ 0 & \text{if } x \notin I' \end{cases}$$

on the set of functions $f : \{0,1\}^{\mathbf{Z}^d} \to \mathbf{R}$ *depending on only finitely many coordinates.*

Proof (a) Let $T_0 = 0$ and

$$\Lambda_t = \sum_x \left[\int_0^t \int 1(u \le c_1(x,\xi_0)) N^{x,0}(\mathrm{d}s,\mathrm{d}u) + \int_0^t \int \xi_0(x) 1(u \le \|c_0\|_\infty) N^{x,1}(\mathrm{d}s,\mathrm{d}u) \right]. \tag{2.4}$$

Then Λ is a well-defined cadlag increasing process, since if we take the expected value, with respect to the Poisson processes, of the right side above, (2.1) implies

$$\int_0^t \sum_x c_1(x,\xi_0)\mathrm{d}s + \int_0^t \sum_x \xi_0(x)\|c_0\|_\infty ds \le \int_0^t (C_{2.1} + \|c_0\|_\infty)|\xi_0|\mathrm{d}s < \infty \quad \text{for all } t > 0 \text{ a.s.}$$

If T_1 is the first jump time of Λ. then the existence of a unique $\mathcal{F}_t^{I'}$-adapted solution to $(SDE)(I')$ (denoted $\xi_\cdot$) up to and including T_1 is clear (set it equal to 0 for $t > T_1$). Moreover it is easy to use (2.1) to check that $|\xi_{T_1}| < \infty$ a.s. This allows us to repeat the above argument with ξ_{T_1} in place of ξ_0 and $N^{x,i}((T_1, T_1 + t] \times A)$ in place of $N^{x,i}([0,t] \times A)$ to show the existence of a unique $\mathcal{F}_t^{I'}$-adapted solution, $\xi_\cdot$ to $(SDE)(I')$ up to and including T_2, the time of the second jump of

$$\Lambda_t = \sum_x \int_0^t \int 1(u \le c_1(x, \xi_{s-}))N^{x,0}(\mathrm{d}s, \mathrm{d}u)$$
$$+ \int_0^t \int \xi_{s-}(x) 1(u \le \|c_0\|_\infty) N^{x,1}(\mathrm{d}s, \mathrm{d}u).$$

Continuing in this way we may construct a unique $\mathcal{F}_t^{I'}$-adapted solution up until $T_\infty = \lim T_n$, where T_n is the nth jump time of Λ. It remains to show that $T_\infty = \infty$ a.s. and this follows as in Lemma 2.1 of [4] by bounding T_n by the nth jump time of a pure birth process.
(b) Define $\{T_n\}$ as in the proof of (a). Implicit in the above construction is the fact that the jump times of $\xi_\cdot[0, \xi_0, I']$ and $\xi_\cdot[0, \xi_0, \mathbf{R}^d]$ are included in the set $\{T_n\}$. Therefore to prove (i) it suffices to prove

$$\xi_{T_n}[0, \xi_0, I'](x) \le \xi_{T_n}[0, \xi_0, \mathbf{R}^d](x) \quad \text{a.s. for all } x \in \mathbf{Z}^d \quad \text{and} \quad n \in \mathbf{Z}_+. \tag{2.5}$$

We proceed by induction on n. As $n = 0$ is trivial, assume (2.3) holds for all x and all $k < n$. Now fix $x \in I'$.
Case 1 $\xi_{T_n-}[0, \xi_0, \mathbf{R}^d](x) = 1$ and $\xi_{T_n}[0, \xi_0, \mathbf{R}^d](x) = 0$: Here we must have that

$$N^{x,1}(\{T_n\} \times [0, c_0(x, \xi_{T_n-}[0, \xi_0, \mathbf{R}^d])]) = 1.$$

If $\xi_{T_n-}[0, \xi_0, I'](x) = 1$, then since $\xi_{T_n-}[0, \xi_0, I'] \le \xi_{T_n-}[0, \xi_0, \mathbf{R}^d]$ a.s. by induction, monotonicity implies $N^{x,1}(\{T_n\} \times [0, c_0(x, \xi_{T_n-}[0, \xi_0, I'])]) = 1$. Consequently, we must have $\xi_{T_n}[0, \xi_0, I'](x) = 0$ a.s. If $\xi_{T_n-}[0, \xi_0, I'](x) = 0$, then $\xi_{T_n}[0, \xi_0, I'](x) = 0$ a.s. because $N^{x,0}$ and $N^{x,1}$ have no common jump times a.s.
Case 2 $\xi_{T_n-}[0, \xi_0, I'](x) = 0$ and $\xi_{T_n}[0, \xi_0, I'](x) = 1$: Necessarily,

$$N^{x,0}(\{T_n\} \times [0, c_1(x, \xi_{T_n-}[0, \xi_0, I'])]) = 1.$$

If $\xi_{T_n-}[0, \xi_0, \mathbf{R}^d](x) = 0$, then on account of our induction hypothesis and monotonicity, it follows that $c_1(x, \xi_{T_n-}[0, \xi_0, \mathbf{R}^d]) \ge c_1(x, \xi_{T_n-}[0, \xi_0, I'])$. As in Case 1, we obtain the conclusion $\xi_{T_n}[0, \xi_0, \mathbf{R}^d](x) = 1$. If $\xi_{T_n-}[0, \xi_0, \mathbf{R}^d](x) = 1$ this conclusion is trivial as before.
Remaining cases: The conclusion $\xi_{T_n}[0, \xi_0, I'](x) \le \xi_{T_n}[0, \xi_0, \mathbf{R}^d](x)$ is trivial for these cases, and so the proof of (2.5) is complete, and (i) is proved.

The proof of (ii) is similar. We start with the first jump of $\Lambda + \tilde{\Lambda}$, where $\tilde{\Lambda}$ is defined as in (2.2), but with $\tilde{\xi}_0$ in place of ξ_0 and proceed inductively.
(c) It is easy to use the stochastic calculus for Poisson point processes to see that for functions f depending on only finitely many coordinates, $f(\xi_t) - f(\xi_0) - \int_0^t \Omega f(\xi_s) ds$ is an $\mathcal{F}_t^{\mathbf{R}^d}$-martingale. The result now follows from Theorem B3 of [10] and Theorem I.5.2 of [9]. □

We note here that the theorems quoted from [9] and [10] do not require finiteness of $|\xi_0|$.

We now apply the above construction to the *voter model perturbations* of [5] which generalize the Lotka–Volterra model. Let P_F be the set of finite subsets of $\mathbf{Z}^d$, and

$$\ell^1(P_F) = \left\{ \gamma : P_F \to \mathbf{R} : \|\gamma\|_1 = \sum_{A \in P_F} |\gamma(A)| < \infty \right\},$$

and for $(\beta,\delta) \in \ell^1(P_F)^2$, set $\|(\beta,\delta)\|_1 = \|\beta\|_1 + \|\delta\|_1$. For $A \in P_F$, put $\chi(A,x,\xi) = \prod_{e \in A} \xi(x+e)$. For $(x,\xi) \in \mathbf{Z}^d \times \{0,1\}^{\mathbf{Z}^d}$ and $(\beta,\delta) \in \ell^1(P_F)^2$, define

$$\begin{aligned} c_0^{\beta,\delta}(x,\xi) &\equiv c_0(x,\xi) = f_0(x,\xi) + \textstyle\sum_{A \in P_F} \delta(A)\chi(A,x,\xi)\,, \\ c_1^{\beta,\delta}(x,\xi) &\equiv c_1(x,\xi) = f_1(x,\xi) + \textstyle\sum_{A \in P_F} \beta(A)\chi(A,x,\xi)\,, \\ c^{\beta,\delta}(x,\xi) &\equiv c(x,\xi) \;\; = c_1(x,\xi)1(\xi(x)=0) + c_0(x,\xi)1(\xi(x)=1). \end{aligned} \tag{2.6}$$

This definition should be compared to the rates for the Lotka–Volterra model (1.2). In (1.2) we consider small $|\alpha_i - 1|$, making $\mathrm{LV}(\alpha_0,\alpha_1)$ a (quadratic) perturbation of the voter model. Later we will be assuming $\beta(A)$ and $\delta(A)$ are small and so the above can be viewed as (possibly infinite degree) polynomial perturbations of the voter model.

For $(\beta,\delta) \in \ell^1(P_F)^2$ we introduce the following conditions:

$$\text{There is an } n_1 \in \mathbf{N} \text{ such that } \beta(A) = \delta(A) = 0 \quad \text{if } \mathrm{card}(A) \equiv |A| > n_1. \tag{P1}$$

For all $(x,\xi) \in \mathbf{Z}^d \times \{0,1\}^{\mathbf{Z}^d}$,

$$c^{\beta,\delta}(x,\xi) \geq 0, \tag{P2}$$

and

$$p(x) + \sum_{\substack{A \in P_F \\ A \ni x}} \beta(A)\chi(A \setminus \{x\},0,\xi) \geq 0 \quad \text{and} \quad -p(x) + \sum_{\substack{A \in P_F \\ A \ni x}} \delta(A)\chi(A \setminus \{x\},0,\xi) \leq 0. \tag{P3}$$

There is a constant K_4 such that

$$\sum_{A \in P_F} \delta(A)\chi(A,0,\xi) \geq -K_4 f_0(0,\xi) \;\forall \xi \in \{0,1\}^{\mathbf{Z}^d} \quad \text{such that } \xi(0) = 1. \tag{P4}$$

$$\beta(\emptyset) = 0. \tag{P5}$$

If $S \subset \ell^1(P_F)^2$, we say (P) holds uniformly on S iff (P1)–(P5) hold for all $(\beta,\delta) \in S$ with n_1 and K_4 independent of the choice of $(\beta,\delta) \in S$.

Remark 2.2 (a) Note that the rates in (2.6) are translation invariant. That is, if $\tau_x\xi(y) = \xi(x+y)$, then $c^{\beta,\delta}(x,\xi) = c^{\beta,\delta}(0,\tau_x\xi)$.
(b) It is not difficult to see that (P3) is implied by the simpler condition: for all $x \in \mathbf{Z}^d$,

$$p(x) \geq -\beta(\{x\}) + \sum_{A:x\in A,|A|>1} \beta(A)^- \quad \text{and} \quad p(x) \geq \delta(\{x\}) + \sum_{A:x\in A,|A|>1} \delta(A)^+. \tag{P3$'$}$$

Here $\beta(A)^-$ and $\delta(A)^+$ are the negative part of $\beta(A)$ and positive part of $\delta(A)$, respectively.
(c) (P4) is used to make comparisons with a biased voter model in [5]. If $\{x : p(x) > 0\}$ is finite then (P4) follows from (P2) and $\delta \in \ell^1(P_F)$ (see Lemma 1.7 of [5]).
(d) The condition (P5) implies $c(x,\xi) = 0$ for $\xi \equiv 0$, so that $\xi \equiv 0$ is a trap. The condition that makes $\xi \equiv 1$ a trap is

$$\sum_{A\in P_F} \delta(A) = 0. \tag{P5$'$}$$

We will impose this condition in Theorem 6.1.
(e) As in [5], there is no loss in generality in assuming that $\beta(A) = \delta(A) = 0$ if $0 \in A$.
(f) In Sect. 8 we will show that for $\mathrm{LV}(\alpha_0,\alpha_1)$, we may write $c(x,\xi) = c^{\beta_\alpha,\delta_\alpha}(x,\xi)$ where (P) holds uniformly on $\{(\beta_\alpha,\delta_\alpha) : \alpha_0 \wedge \alpha_1 \geq \frac{1}{2}\}$ (see Propostion 8.1).

Condition (P3) will give monotonicity of the above spin-flip processes.

Proposition 2.3 *Assume $(\beta,\delta) \in \ell^1(P_F)^2$ satisfy (P1) and (P2). Then $c^{\beta,\delta}$ is monotone if and only if (P3) holds.*

Proof For $\xi \in \{0,1\}^{\mathbf{Z}^d}$ and $x \in \mathbf{Z}^d$, define $\xi_x \in \{0,1\}^{\mathbf{Z}^d}$ by $\xi_x(y) = \xi(y)$ if $y \neq x$ and $\xi_x(x) = 1$. We claim that $c^{\beta,\delta}$ is monotone if and only if:

$$\begin{aligned} &\text{for all } x \neq 0,\ c_0(0,\xi) - c_0(0,\xi_x) \geq 0 \text{ whenever } \xi(0) = 1 \\ &\quad \text{and } c_1(0,\xi) - c_1(0,\xi_x) \leq 0 \text{ whenever } \xi(0) = 0. \end{aligned} \tag{2.7}$$

Necessity of this condition is obvious. To prove sufficiency, let $\underline{\xi} \leq \xi$ satisfy $\underline{\xi}(0) = 1$ and let us prove that $c_0(0,\xi) \leq c_0(0,\underline{\xi})$. There is a sequence $\{\xi_n\}$ so that $\underline{\xi} = \xi_1 \leq \xi_n \uparrow \xi$ and $\xi_{n+1} = (\xi_n)_{x_n}$ for some x_n, for all n. (2.7) implies

$$c_0(0,\xi_n) = c_0(0,(\xi_{n-1})_{x_{n-1}}) \leq c_0(0,\xi_{n-1}) \leq \cdots \leq c_0(0,\underline{\xi}),$$

and so it suffices to show that $\lim_{n\to\infty} c_0(0,\xi_n) = c_0(0,\xi)$. This, however, is immediate by Dominated Convergence because $\delta \in \ell^1(P_F)$ and $\sum_x p(x) = 1 < \infty$. By translation invariance we may replace the location 0 with an arbitrary x in the above. Similar reasoning shows that $\xi(x) = 0$ implies $c_1(x,\underline{\xi}) \leq c_1(x,\xi)$, and the claim is proved. Finally, a simple calculation shows that (2.7) is equivalent to (P3) under (P1) and (P2). □

Corollary 2.4 *Assume $(\beta,\delta)\in\ell^1(P_F)^2$ satisfy conditions (P1)–(P3) and (P5). Then all the conclusions of Proposition 2.1 are valid for the rates $c^{\beta,\delta}(x,\xi)$.*

Proof The boundedness of $c_i^{\beta,\delta}$ is clear from $\beta,\delta\in\ell^1(P_F)$. Condition (2.1) follows easily from $\beta,\delta\in\ell^1(P_F)$ and (P5). Condition (P3) implies the monotonicity of the spin-flip system by Proposition 2.3. (P1), (P2) and $(\beta,\delta)\in\ell^1(P_F)^2$, easily imply that the rates $c^{\beta,\delta}$ satisfy (2.3). Hence, all parts of Proposition 2.1 apply. □

Remark 2.5 We note that as (2.3) holds, under the hypotheses of Corollary 2.4 we may apply Theorem B3 of [10] directly to see that the rates $c^{\beta,\delta}$ determine a unique $\{0,1\}^{\mathbf{Z}^d}$-valued Feller process satisfying the martingale problem in Theorem 2.1(c), which (by Proposition 2.1) we may construct via (SDE) if $|\xi_0|<\infty$. We call the associated process $\xi_\cdot$ a generalized voter model perturbation and let $P^{\beta,\delta}$ or $P^{\beta,\delta}_{\xi_0}$ denote its law on the space of cadlag $\{0,1\}^{\mathbf{Z}^d}$-valued paths. Survival and coexistence are defined in this setting, just as in Sect. 1. As noted in [5] [see (1.25) and (1.26)], the LV(α_0,α_1) is a particular generalized voter model perturbation.

Before proceeding further we state the analogue of Theorem A for these generalized voter model perturbations. Let (β_N,δ_N), $N\in\mathbf{N}$ be a sequence in $\ell^1(P_F)^2$ such that conditions (P1), (P2), (P4) and (P5) hold uniformly on $\{(\beta_N,\delta_N): N\in\mathbf{N}\}$, and suppose that for some $(\beta,\delta)\in\ell^1(P_F)^2$,

$$(\beta_N,\delta_N)\to(\beta,\delta) \text{ in } \ell^1(P_F)^2 \text{ as } N\to\infty.$$

Let ξ_t be the voter model perturbation process with rate function $c^{\frac{\beta_N}{N},\frac{\delta_N}{N}}$, suppressing dependence on N. We recall that $\mathbf{S_N}=\mathbf{Z}^d/\sqrt{N}$, set $\xi_t^N(x)=\xi_{Nt}(x\sqrt{N})$, $x\in\mathbf{S_N}$, and define the $\mathcal{M}_F$-valued process $X_\cdot^N$ by

$$X_t^N=\frac{1}{N}\sum_{x\in\mathbf{S_N}}\xi_t^N(x)\delta_x. \tag{2.8}$$

Let P_N denote the law of $X_\cdot^N$ on $D(\mathbf{R}^+,\mathcal{M}_F)$, and assume the initial states ξ_0^N satisfy (1.10). Also, we recall the coalescing random walks $\hat{B}_t^x$, the coalescing times $\tau(A)$, the escape probability γ_e given in (1.7), and define $\sigma(A)=P(\tau(A)<\infty)$, $A\in P_F$. The following result is Corollary 1.6 of [5].

Theorem B *Assume $d\geq 3$. If (1.10) holds, then $P_N\Rightarrow P^{2\gamma_e,\theta,\sigma^2}_{X_0}$ as $N\to\infty$, the law of super-Brownian motion started at X_0 with branching coefficient $2\gamma_e$, drift coefficient*

$$\theta = \sum_{A \in P_F} \Big[\beta(A)\sigma(A) - (\beta(A) + \delta(A))\sigma(A \cup \{0\}) \Big] \tag{2.9}$$

and diffusion coefficient σ^2.

We also will need a slight variant of Theorem B.

Theorem C *Assume* $d \geq 3$, *and let* $\{X_\cdot^{N,i} : i \leq N\}$ *be iid copies of* $X_\cdot^N$ *as in* (2.8) *but with* $X_0^{N,i} = \frac{1}{N}\delta_0$ *and let* P_N *be the law of* $\sum_{i=1}^N X_\cdot^{N,i}$ *on* $D(\mathbf{R}_+, \mathcal{M}_F)$. *Then* $P_N \Rightarrow P_{\delta_0}^{2\gamma_e,\theta,\sigma^2}$ *as* $N \to \infty$, *the law of super-Brownian motion started at* δ_0 *with branching coefficient* $2\gamma_e$, *drift coefficient* θ *given in* (2.9), *and diffusion coefficient* σ^2.

Remark There is nothing special about δ_0. One could assume $X_0^{N,i} = \frac{1}{N}\delta_{x_{N,i}}$, where $x_{N,i} \in \mathbf{S_N}$ for $i \leq M_N$ and $X_0^N = \sum_{i \leq M_N} X_0^{N,i}$ converges to $X_0 \in \mathcal{M}_F$. The same conclusion then holds where the limiting super-Brownian motion now starts at X_0.

Proof The proof of Theorem C involves only minor and obvious changes in the proof of Theorem B from [5]. We mention only a few points and use notation from [5].

As in [5] one may bound each $X_t^{N,i}(\mathbf{1})$ by $\bar{X}_t^{N,i}(\mathbf{1})$, where $\bar{X}_t^{N,i}(\phi) = \frac{1}{N}\sum_x \phi(x)\bar{\xi}_t^{N,i}(x)$ and $\{\bar{\xi}_t^{N,i} : i \leq N\}$ are appropriate independent rescaled biased voter models. Using Lemma 4.1 of [5] to bound the first and second moment of the biased voter model one sees that

$$\begin{aligned}
E\left(\left(\sum_i \bar{X}_t^{N,i}(\mathbf{1})\right)^2\right) &\leq \mathrm{Var}\left(\sum_i \bar{X}_t^{N,i}(\mathbf{1})\right) + \left[E\left(\sum_i \bar{X}_t^{N,i}(\mathbf{1})\right)\right]^2 \\
&\leq \sum_i E(\bar{X}_t^{N,i}(\mathbf{1})^2) + \mathrm{e}^{\bar{c}t}\left(\sum_i X_0^{N,i}(\mathbf{1})\right)^2 \\
&\leq C(T)(X_0^N(\mathbf{1})^2 + X_0^N(\mathbf{1}))
\end{aligned}$$

for all $t \leq T$. The above bound and the strong L^2 inequality for the submartingale $(\sum_i \bar{X}_t^{N,i}(\mathbf{1}))^2)$ gives

$$E\left(\sup_{t \leq T} X_t^N(\mathbf{1})^2\right) \leq C(T,K) \quad \text{for } \sup_N X_0^N(\mathbf{1}) \leq K. \tag{2.10}$$

This is the analogue of Proposition 3.3 in [5].

The key technical bound in [5] is Lemma 5.1 of that work. Although the term being bounded is nonlinear in $X_0^N(\mathbf{1})$, the proof only uses linear bounds which carry over to our setting without change. There is even some simplification in the bound on (the analogue of) $\eta^N_{3,1}$ in (5.17) of [5] as the term $X^N_0(\mathbf{1})^2$ may be replaced by $X^N_0(\mathbf{1})$. This is because each of the initial conditions $\xi^{N,i}_0$ only charges a single site. This then leads to the analogue of the main bound (5.4) with the smaller term J in place of J^2. The proof of the key Proposition 3.4 of [5] then goes through as before now using (2.10). The proof of tightness and identification of the limit points now involve only trivial modifications. □

3 Comparison estimates

Let $(\beta,\delta) \in \ell^1(P_F)^2$ satisfy (P1)–(P5) and let I' be a bounded open box in $\mathbf{R}^d$. For initial ξ_0 with $|\xi_0| < \infty$ we may apply Proposition 2.1 with $c = c^{\beta,\delta}$, obtaining the solution $\xi_\cdot[0,\xi_0,I']$ of (SDE)(I'), which we will also write as $\underline{\xi}_\cdot$, suppressing the dependence on I'. More generally, if $t_0 \geq 0$ and $\underline{\xi}_{t_0} \in \{0,1\}^{\mathbf{Z}^d}$ is $\mathcal{G}_{t_0}$-measurable such that $|\underline{\xi}_{t_0}| < \infty$ and $\underline{\xi}_{t_0}(y) = 0$ for $y \notin I'$, let $\underline{\xi}(t) \equiv \underline{\xi}[t_0,\underline{\xi}_{t_0},I'](t)$ be the unique solution of

$$\begin{aligned}\underline{\xi}_t(x) &= \underline{\xi}_{t_0}(x) + \int_{t_0}^{t}\int 1(\underline{\xi}_{s-}(x)=0)1(u \leq c_1(x,\underline{\xi}_{s-}))N^{x,0}(\mathrm{d}s,\mathrm{d}u)\\ &\quad - \int_{t_0}^{t}\int 1(\underline{\xi}_{s-}(x)=1)1(u \leq c_0(x,\underline{\xi}_{s-}))N^{x,1}(\mathrm{d}s,\mathrm{d}u), \quad t \geq t_0,\ x \in I'\\ \underline{\xi}_t(x) &= 0, \quad t \geq t_0, x \notin I'.\end{aligned} \tag{SDE(t_0,I')}$$

The existence and uniqueness of a $\sigma(\underline{\xi}_{t_0}) \vee \mathcal{G}([t_0,t] \times I')$-adapted solution to $(SDE)(t_0,I')$ follows by applying Proposition 2.1 with $\xi_0 = \underline{\xi}_{t_0}$ and the Poisson point processes $N^{x,i}([t_0,t_0+t] \times A)$ in place of $N^{x,i}([0,t] \times A)$. Proposition 2.1 (b) (in the above setting) implies that for any $t_0 \geq 0$, whenever $\underline{\xi}_{t_0} \leq \xi_{t_0}$ are both $\mathcal{G}_{t_0}$-measurable,

$$\underline{\xi}_t[t_0,\underline{\xi}_{t_0},I'] \leq \underline{\xi}_t[t_0,\xi_{t_0},\mathbf{R}^d] \quad \text{for all } t \geq t_0 \text{ a.s.} \tag{3.1}$$

Fix $T > 0$ and natural numbers $K > 2$, $L > 1$ and N, and define $I' = (-KL\sqrt{N}, KL\sqrt{N})^d$. These parameters will be chosen with care in the next section, but for now their particular values will not be important. $[0,T] \times I'$ will serve as the space-time sets for our oriented percolation events, defined in Sect. 5. Given a deterministic initial ξ_0 such that $\xi_0(x) = 0$ for $x \notin I'$ (and hence $|\xi_0| < \infty$), let $\underline{\xi}_t = \underline{\xi}_t[0,\xi_0,I']$ and $\xi_t = \underline{\xi}_t[0,\xi_0,\mathbf{R}^d]$ be as defined above, and note that ξ_t is the (full) generalized voter model process with law $P^{\beta,\delta}_{\xi_0}$. We define the rescaled processes ξ^N_t and $\underline{\xi}^N_t$

$$\xi^N_t(x) = \xi_{Nt}(x\sqrt{N}) \quad x \in \mathbf{S_N} \quad \text{and} \quad \underline{\xi}^N_t(x) = \underline{\xi}_{Nt}(x\sqrt{N}), \quad x \in \mathbf{S_N}, \tag{3.2}$$

and their associated measure-valued processes

$$X_t^N = \frac{1}{N}\sum_{x\in\mathbf{S_N}} \xi_t^N(x)\delta_x \quad \text{and} \quad \underline{X}_t^N = \frac{1}{N}\sum_{x\in\mathbf{S_N}} \underline{\xi}_t^N(x)\delta_x.$$

By Proposition 2.1,

$$\underline{\xi}_t^N \le \xi_t^N \quad \text{and} \quad \underline{X}_t^N(\psi) \le X_t^N(\psi) \text{ for all } t\ge 0 \text{ and nonnegative } \psi \text{ a.s.} \tag{3.3}$$

The task of this section is to obtain a useful estimate (Lemma 3.2 below) of the difference $E(X_t^N(\mathbf{1})) - E(\underline{X}_t^N(\mathbf{1}))$ in terms of β, δ, the random walk kernel p, and the parameters T, K, L, N. For $A \in P_F$, $x \in \mathbf{S_N}$ and $\xi \in \{0,1\}^{\mathbf{S_N}}$, let $\beta_N(A) = N\beta(A)$, $\delta_N(A) = N\delta(A)$, $p_N(x) = p(x\sqrt{N})$, and

$$f_i^N(x,\xi) = \sum_{y\in\mathbf{S_N}} p_N(y-x)1(\xi(y)=i), \quad i=0,1.$$

If $\phi \in C_b([0,T]\times\mathbf{S_N})$ and $\dot\phi(t,x) = \frac{\partial\phi}{\partial t}(t,x) \in C_b([0,T]\times\mathbf{S_N})$, define

$$\mathcal{A}_N(\phi_t)(x) = \sum_{y\in\mathbf{S_N}} Np_N(y-x)(\phi_t(y)-\phi_t(x)).$$

Let B_t^N denote the continuous time random walk with generator $\mathcal{A}_N$ and semigroup P_t^N.

In Lemma 3.2 below, our bound on the difference $E(X_t^N(\mathbf{1})) - E(\underline{X}_t^N(\mathbf{1}))$ includes terms of the form $P(\sup_{s\le T}|B_s^N| > (K-1)L/3)$ and also $P(\sup_{s\le T}|\hat B_s^N| > (K-1)L/3)$, where $\hat B_t^N$ is a random walk defined below. The B_t^N term comes from the voter part of the dynamics, and the $\hat B_t^N$ term comes from the (β_N, δ_N) part. Define a probability mass function on $\mathbf{S_N}$ by

$$\hat p_N(x) = \begin{cases} \sum_{A: x\in -A/\sqrt{N}} \frac{\beta_N^+(A)}{|A|\|\beta_N^+\|_1} & \text{if } \|\beta_N^+\|_1 > 0 \\ 1\{x=0\} & \text{if } \|\beta_N^+\|_1 = 0, \end{cases} \tag{3.4}$$

with associated mean operator $\hat P_N\phi(x) = \sum_y \hat p_N(y-x)\phi(y)$. (In the Lotka–Volterra case, $\hat p_N(x) = p(x)$ if $\alpha_0 > 1$ and $1\{x=0\}$ if $\alpha_0 \le 1$.) Let

$$\hat{\mathcal{A}}_N\phi(x) = \|\beta_N^+\|_1(\hat P_N\phi(x) - \phi(x))$$

be the generator of the continuous time random walk $\hat B^N$, which takes jumps at rate $\|\beta_N^+\|_1$ according to the kernel $\hat p_N$. Let $\tilde{\mathcal{A}}_N = \mathcal{A}_N + \hat{\mathcal{A}}_N$ and let $\tilde P_t^N$ be the semigroup associated with the generator $\tilde{\mathcal{A}}_N$. Therefore $\tilde P_t^N$ is the semigroup associated with the random walk $\tilde{B}_t^N \equiv B_t^N + \hat{B}_t^N$, where B_t^N and $\hat{B}_t^N$ are independent copies of the random walks introduced above. We use P_x, $\hat{P}_x$ and $\tilde{P}_x$ to denote the laws of these three random walks.

The $\mathcal{A}_N$ random walk arises from the spatial motion in the rescaled voter model dynamics. The $\tilde{\mathcal{A}}_N$ random walk arises from the spatial motion implicit in the positive $\beta(A)$ terms in the series expansion for $c_1^{\beta,\delta}$ in (2.6). More specifically, we will bound $\chi(A,x,\xi) \le \frac{1}{|A|}\sum_a \xi(x+a)$ and consider the creation of a 1 at x "due to $\xi(x+a)$" as including a migration from $x+a$ to x, which when rescaled leads to this second random walk. This interpretation leads to the following bound.

Lemma 3.1 *For* $\psi \ge 0$,

$$E(X_t^N(\psi)) \le \mathrm{e}^{\|(\beta_N^+,\delta_N^-)\|_1 t} X_0^N(\tilde{P}_t^N \psi).$$

Proof By Proposition 2.3 of [5], for $\phi, \dot{\phi} \in C_b([0,t] \times \mathbf{S_N})$ with $\phi \ge 0$, we have

$$X_t^N(\phi) = X_0^N(\phi_0) + D_t^{N,1}(\phi) + D_t^{N,2}(\phi) - D_t^{N,3}(\phi) + M_t^N(\phi)$$

where

$$D_t^{N,1}(\phi) = \int_0^t X_s^N(\dot{\phi}_s + \mathcal{A}^N \phi_s)\,\mathrm{d}s,$$

$$D_t^{N,2}(\phi) = \frac{1}{N}\int_0^t \sum_{x\in\mathbf{S_N}} \phi_s(x) \sum_{A\in P_F} \beta_N(A)(1-\xi_s^N(x))\chi_N(A,x,\xi_s^N)\,\mathrm{d}s,$$

$$D_t^{N,3}(\phi) = \frac{1}{N}\int_0^t \sum_{x\in\mathbf{S_N}} \phi_s(x) \sum_{A\in P_F} \delta_N(A)\xi_s^N(x)\chi_N(A,x,\xi_s^N)\,\mathrm{d}s,$$

and $M_t^N(\phi)$ is a square integrable $\mathcal{G}_t$-martingale starting at 0. (The filtration $\mathcal{G}_t$ is not in [5] but it is trivial to verify the martingale property with respect to this filtration.) Now

$$\begin{aligned} D_t^{N,2}(\phi) &\le \frac{1}{N}\int_0^t \sum_{x\in\mathbf{S_N}} \phi_s(x) \sum_{A\in P_F} \beta_N^+(A)\frac{1}{|A|} \sum_{a\in A/\sqrt{N}} \xi_s^N(x+a)\,\mathrm{d}s \\ &= \frac{1}{N}\int_0^t \sum_{x\in\mathbf{S_N}} \phi_s(x) \sum_{a\in\mathbf{S_N}} \|\beta_N^+\|_1 \hat{p}^N(a)\xi_s^N(x-a)\,\mathrm{d}s \end{aligned}$$

$$= \frac{1}{N}\int_0^t \|\beta_N^+\|_1 \sum_{y\in\mathbf{S_N}} \xi_s^N(y) \sum_{a\in\mathbf{S_N}} \hat{p}^N(a)\phi_s(y+a)\,\mathrm{d}s$$

$$= \|\beta_N^+\| \int_0^t X_s^N(\hat{P}^N(\phi_s))\,\mathrm{d}s.$$

A shorter computation yields

$$D_t^{N,3} \geq -\frac{1}{N}\int_0^t \sum_{x\in\mathbf{S_N}} \phi_s(x) \sum_{A\in P_F} \delta_N^-(A)\xi_s^N(x)\,\mathrm{d}s = -\|\delta_N^-\|_1 \int_0^t X_s^N(\phi_s)\,\mathrm{d}s.$$

We have therefore established

$$X_t^N(\phi_t) \leq X_0^N(\phi_0) + \int_0^t X_s^N(\dot{\phi}_s + \mathcal{A}^N\phi_s)\,\mathrm{d}s + \|\beta_N^+\|_1 \int_0^t X_s^N(\hat{P}^N(\phi_s))\,\mathrm{d}s$$
$$+\|\delta_N^-\|_1 \int_0^t X_s^N(\phi_s)\,\mathrm{d}s + M_t^N(\phi).$$

Consequently,

$$X_t^N(\phi_t) \leq X_0^N(\phi_0) + \int_0^t \left(X_s^N(\dot{\phi}_s + \tilde{A}^N\phi_s) + (\|\beta_N^+\|_1 + \|\delta_N^-\|_1)X_s^N(\phi_s)\right)\mathrm{d}s$$
$$+M_t^N(\phi). \tag{3.5}$$

Now set

$$\phi_s(x) = \tilde{P}_{t-s}^N\psi(x)\mathrm{e}^{(\|\beta_N^+\|_1+\|\delta_N^-\|_1)(t-s)},$$

where ψ is a bounded non-negative function on $\mathbf{S_N}$. Then $\dot{\phi}_s = -\tilde{A}^N\phi_s - (\|\beta_N^+\|_1 + \|\delta_N^-\|_1)\phi_s$, and since integrability of $\sup_{s\leq t} X_s^N(1)$ follows from Proposition 2.1 of [5], we get

$$EX_t^N(\psi) \leq X_0^N(\tilde{P}_t^N\psi)\mathrm{e}^{(\|\beta_N^+\|_1+\|\delta_N^-\|_1)t},$$

from which the result follows for bounded non-negative ψ. It then follows by monotone convergence for any non-negative ψ. □

Let $I_N'' = (-\sqrt{N}(K-1)L/3, \sqrt{N}(K-1)L/3)^d$, and recall from (P1) that $\beta(A) = \delta(A) = 0$ if $|A| > n_1$. Here is the Comparison Lemma.

Lemma 3.2 *If* $\|(\beta_N,\delta_N)\|_1 \vee 1 \le \kappa$ *and* $\xi_0^N = \underline{\xi}_0^N$ *is supported on* $I = [-L,L]^d$, *then for all* $T > 0$,

$$E[X_T^N(\mathbf{1}) - \underline{X}_T^N(\mathbf{1})] \le X_0^N(\mathbf{1})3\mathrm{e}^{5\kappa n_1 T}\Bigg(\sum_{A\subset(I_N'')^c}|\beta_N(A)| + \hat{P}_0(\sup_{s\le T}|\hat{B}_s^N| > (K-1)L/3) + P(\sup_{s\le T}|B_s^N| > (K-1)L/3)\Bigg). \quad (3.6)$$

Proof For $A\in P_F$, $x\in\mathbf{S_N}$ and $\xi^N\in\{0,1\}^{\mathbf{S_N}}$, let $\chi_N(A,x,\xi^N) = \prod_{a\in A}\xi^N(x+\frac{a}{\sqrt{N}})$. Consider $(SDE)(I')$ on the rescaled lattice $\mathbf{S_N}$ with

$$c_1(x,\xi^N) \equiv c_{N,1}^{\beta,\delta}(x,\xi^N) = Nf_1^N(x,\xi^N) + \sum_{A\in P_F}\beta_N(A)\chi_N(A,x,\xi^N)$$

and

$$c_0(x,\xi^N) \equiv c_{N,0}^{\beta,\delta}(x,\xi^N) = Nf_0^N(x,\xi^N) + \sum_{A\in P_F}\delta_N(A)\chi_N(A,x,\xi^N).$$

Let $\phi \in C_b^1(\mathbf{R}_+ \times \mathbf{S_N})$, and define $\underline{\phi}_s(x) = \phi_s(x)1\{x\in I\}$. Multiply (the rescaled) $(SDE(I'))$ by $\frac{1}{N}\underline{\phi}_s(x)$, integrate by parts, and sum over x to see

$$\begin{aligned}\underline{X}_t^N(\phi_t) &= \underline{X}_0^N(\phi_0) + \int_0^t \underline{X}_s^N(\dot{\phi}_s)\,\mathrm{d}s \\ &\quad + \int_0^t \frac{1}{N}\sum_x \underline{\phi}_s(x)(1-\underline{\xi}_s^N(x)) \\ &\qquad \times\left[Nf_1^N(x,\underline{\xi}_s^N) + \sum_A \beta_N(A)\chi_N(A,x,\underline{\xi}_s^N)\right]\mathrm{d}s \\ &\quad - \int_0^t \frac{1}{N}\sum_x \underline{\phi}_s(x)\underline{\xi}_s^N(x) \\ &\qquad \times\left[Nf_0^N(x,\underline{\xi}_s^N) + \sum_A \delta_N(A)\chi_N(A,x,\underline{\xi}_s^N)\right]\mathrm{d}s + \underline{M}_t^N(\phi),\end{aligned}$$

where $\underline{M}_t^N$ is a square integrable martingale. The absolute summability of all these terms and integrability of the resulting sums follow easily from $E(\sup_{s\le t}X_s^N(\mathbf{1})^k) < \infty$ for all $k,t>0$ (Proposition 2.1 of [5]), $\underline{X}_t^N(\mathbf{1}) \le X_t^N(\mathbf{1})$ (Proposition 2.1 above), $\beta_N,\delta_N \in \ell^1(P_F)$, and $\beta_N(\emptyset) = 0$. The same reasoning shows $\underline{M}_t^N(\phi)$ is a square integrable martingale and not just a local martingale (see the proof of Proposition 2.3 of [5]). Let $\underline{B}_t^N$ denote the random walk B_t^N killed when it exits I', with cemetary state Δ, and let $\underline{P}_t^N$ be the associated semigroup, with generator

$$\underline{\mathcal{A}}^N(\psi)(x) = \sum_y Np_N(y-x)[\underline{\psi}(y) - \underline{\psi}(x)]1\{x \in I'\}.$$

Here $\underline{\psi}(y) = \psi(y)1\{y \in I'\}$ as above. With this notation, summation by parts yields

$$\begin{aligned}
&\frac{1}{N}\sum_x \underline{\phi}_s(x)\left[\left(1-\underline{\xi}_s^N(x)\right)Nf_1^N(x,\underline{\xi}_s^N) - \underline{\xi}_s^N(x)Nf_0^N(x,\underline{\xi}_s^N)\right]\\
&\quad= \frac{1}{N}\sum_{x,y}\underline{\phi}_s(x)Np_N(y-x)\left[\underline{\xi}_s^N(y)\left(1-\underline{\xi}_s^N(x)\right) - \underline{\xi}_s^N(x)\left(1-\underline{\xi}_s^N(y)\right)\right]\\
&\quad= \frac{1}{N}\sum_{x,y}\underline{\xi}_s^N(x)Np_N(y-x)[\underline{\phi}_s(y) - \underline{\phi}_s(x)]I(x \in I')\\
&\quad= \underline{X}_s^N(\underline{\mathcal{A}}_N(\phi_s)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\underline{X}_t^N(\phi_t) = \underline{X}_0^N(\phi_0) &+ \int_0^t \underline{X}_s^N(\dot{\phi}_s + \underline{\mathcal{A}}^N(\phi_s))\,\mathrm{d}s\\
&+\int_0^t \frac{1}{N}\sum_x \underline{\phi}_s(x)(1-\underline{\xi}_s^N(x))\sum_A \beta_N(A)\chi_N(A,x,\underline{\xi}_s^N)\,\mathrm{d}s\\
&-\int_0^t \frac{1}{N}\sum_x \underline{\phi}_s(x)\underline{\xi}_s^N(x)\sum_A \delta_N(A)\chi_N(A,x,\underline{\xi}_s^N)\,\mathrm{d}s + \underline{M}_t^N(\phi).
\end{aligned}$$

We now set $\phi_s = \underline{P}_{t-s}^N\psi$ where $\psi \in C^b(\mathbf{S_N})$, $\psi \geq 0$, so that $\underline{\phi}_s = \phi_s$ and $\dot{\phi}_s = -\underline{\mathcal{A}}_N(\phi_s)$. This gives

$$\begin{aligned}
\underline{X}_t^N(\psi) = \underline{X}_0^N(\underline{P}_t^N\psi) &+ \int_0^t \frac{1}{N}\sum_x \underline{P}_{t-s}^N\psi(x)\\
&\times\Bigg[(1-\underline{\xi}_s^N(x))\sum_A \beta_N(A)\chi_N(A,x,\underline{\xi}_s^N)\\
&\quad-\sum_A \delta_N(A)\chi_N(A\cup\{0\},x,\underline{\xi}_s^N)\Bigg]\mathrm{d}s + \underline{M}_t^N(\underline{P}_{t-\cdot}^N\psi).
\end{aligned}$$

A similar representation, with the semigroup P_t^N, holds for $X_t^N(\psi)$ (take $I' = \mathbf{R}^d$ in the above). If we set $\psi \equiv 1$ in the above two representations and take the difference, we obtain

$$\begin{aligned}
E[X_t^N(\mathbf{1}) - \underline{X}_t^N(\mathbf{1})] = X_0^N(\mathbf{1} - \underline{P}_t^N\mathbf{1}) \\
&+E\biggl(\int_0^t \biggl[\frac{1}{N}\sum_x (1-\xi_s^N(x))\sum_A \beta_N(A)\chi_N(A,x,\xi_s^N) \\
&-\frac{1}{N}\sum_x \underline{P}_{t-s}^N\mathbf{1}(x)(1-\underline{\xi}_s^N(x))\sum_A \beta_N(A)\chi_N(A,x,\underline{\xi}_s^N)\biggr] \\
&-\biggl[\frac{1}{N}\sum_x\sum_A \delta_N(A)\chi_N(A\cup\{0\},x,\xi_s^N) \\
&-\frac{1}{N}\sum_x\sum_A \underline{P}_{t-s}^N\mathbf{1}(x)\delta_N(A)\chi_N(A\cup\{0\},x,\underline{\xi}_s^N)\biggr]\,\mathrm{d}s\biggr).
\end{aligned} \tag{3.7}$$

We would like to estimate the above using Gronwall's Lemma and random walk probabilities. To do this, let d_s denote the integrand on the right-hand side, and write $d_s = \sum_{i=1}^4 d_s^i$, where

$$\begin{aligned}
d_s^1 &= \frac{1}{N}\sum_x\sum_A \beta_N(A)[(1-\xi_s^N(x))\chi_N(A,x,\xi_s^N) - (1-\underline{\xi}_s^N(x))\chi_N(A,x,\underline{\xi}_s^N)], \\
d_s^2 &= \frac{1}{N}\sum_x\sum_A \delta_N(A)[\chi_N(A\cup\{0\},x,\underline{\xi}_s^N) - \chi_N(A\cup\{0\},x,\xi_s^N)], \\
d_s^3 &= \frac{1}{N}\sum_x (1-\underline{P}_{t-s}^N\mathbf{1}(x))\sum_{A\neq\emptyset}\beta_N(A)(1-\underline{\xi}_s^N(x))\chi_N(A,x,\underline{\xi}_s^N), \\
d_s^4 &= \frac{1}{N}\sum_x (\underline{P}_{t-s}^N\mathbf{1}(x)-1)\sum_A \delta_N(A)\chi_N(A\cup\{0\},x,\underline{\xi}_s^N).
\end{aligned}$$

To sum over $A \neq \emptyset$ in d_s^3 we have used (P5). The Gronwall term comes from d_s^1 and d_s^2 as follows. By an elementary inequality and the fact that $\underline{\xi}_s^N \le \xi_s^N$,

$$\begin{aligned}
&|(1-\xi_s^N(x))\chi_N(A,x,\xi_s^N) - (1-\underline{\xi}_s^N(x))\chi_N(A,x,\underline{\xi}_s^N)| \\
&\quad \le |(1-\xi_s^N(x)) - (1-\underline{\xi}_s^N(x))| + \sum_{a\in A}\left|\xi_s^N\left(x+\frac{a}{\sqrt{N}}\right) - \underline{\xi}_s^N\left(x+\frac{a}{\sqrt{N}}\right)\right| \\
&\quad = \sum_{a\in A\cup\{0\}}\left[\xi_s^N\left(x+\frac{a}{\sqrt{N}}\right) - \underline{\xi}_s^N\left(x+\frac{a}{\sqrt{N}}\right)\right].
\end{aligned}$$

The same bound holds for $|\chi_N(A \cup \{0\}), x, \xi_s^N) - \chi_N(A \cup \{0\}), x, \underline{\xi}_s^N)|$, and thus

$$\begin{aligned} |d_s^1| + |d_s^2| &\le \frac{2}{N} \sum_x \sum_A (|\beta_N(A)| + |\delta_N(A)|) \\ &\quad \times \sum_{a \in A \cup \{0\}} \left(\xi_s^N\left(x + \frac{a}{\sqrt{N}}\right) - \underline{\xi}_s^N\left(x + \frac{a}{\sqrt{N}}\right)\right) \qquad (3.8) \\ &\le 2(X_s^N(\mathbf{1}) - \underline{X}_s^N(\mathbf{1}))(n_1+1)\|(\beta_N, \delta_N)\|_1 \le 2\kappa(n_1+1)(X_s^N(\mathbf{1}) - \underline{X}_s^N(\mathbf{1})), \end{aligned}$$

since (recall (P1)), $\beta(A) = \delta(A) = 0$ if $|A| > n_1$.

Turning to d_s^3, for $\emptyset \ne A \in P_F$, choose $\bar{a} = \bar{a}(A) \in A$ with $|\bar{a}| = \max_{i \le d} |a_i|$ minimal. Then

$$\begin{aligned} E(|d_s^3|) &\le E\left(\frac{1}{N}\sum_x (1 - \underline{P}_{t-s}^N \mathbf{1}(x)) \sum_{A \ne \emptyset} |\beta_N(A)| \underline{\xi}_s^N\left(x + \frac{\bar{a}(A)}{\sqrt{N}}\right)\right) \\ &= \sum_{A \ne \emptyset} |\beta_N(A)| E\left(\frac{1}{N}\sum_y \underline{\xi}_s^N(y)\left(1 - \underline{P}_{t-s}^N \mathbf{1}\left(y - \frac{\bar{a}(A)}{\sqrt{N}}\right)\right)\right). \qquad (3.9) \end{aligned}$$

If $E_N(s, A)$ is the expectation appearing in the above summand, then Lemma 3.1 implies (use $supp(X_0^N) \subset [-L, L]$ in the third line)

$$\begin{aligned} E_N(s, A) &\le \exp\{s\|(\beta_N^+, \delta_N^-)\|_1\} \\ &\quad \times \int_I \tilde{P}_x\left(P_{\tilde{B}_s^N - (\bar{a}(A)/\sqrt{N})}(\underline{B}_{t-s}^N = \Delta)\right) X_0^N(\mathrm{d}x) \\ &= \exp\{s\|(\beta_N^+, \delta_N^-)\|_1\} \int_I \hat{E}_0 \\ &\quad \times E_x\left(P_{B_s^N + \hat{B}_s^N - (\bar{a}(A)/\sqrt{N})}(\exists u \le t - s, B_u^N \notin I')\right) X_0^N(\mathrm{d}x) \\ &\le \mathrm{e}^{\kappa s} X_0^N(\mathbf{1}) \left[\hat{P}_0\left(|\hat{B}_s^N| + \frac{|\bar{a}(A)|}{\sqrt{N}} \ge \frac{(K-1)2L}{3}\right)\right. \\ &\quad \left. + P_0\left(\sup_{u \le t} |B_u^N| \ge \frac{(K-1)L}{3}\right)\right] \\ &\le \mathrm{e}^{\kappa s} X_0^N(\mathbf{1}) \left[1\left\{A \subset \left(\left(\frac{-\sqrt{N}(K-1)L}{3}, \frac{\sqrt{N}(K-1)L}{3}\right)^d\right)^c\right\}\right. \\ &\quad \left. + \hat{P}_0\left(|\hat{B}_s^N| \ge \frac{(K-1)L}{3}\right) + P_0\left(\sup_{u \le t} |B_u^N| \le \frac{(K-1)L}{3}\right)\right]. \end{aligned}$$

In the last line we argue that if $A \cap (-\sqrt{N}(K-1)L/3, \sqrt{N}(K-1)L/3)^d \ne \emptyset$, then $|\bar{a}(A)|/\sqrt{N} \le (K-1)L/3$, and so $|\hat{B}_s^N| + |\bar{a}(A)|/\sqrt{N} \ge (K-1)2L/3$ implies $|\hat{B}_s^N| \ge (K-1)L/3$. Use the above in (3.9) to obtain

$$E(|d_s^3|) \le \mathrm{e}^{\kappa s} X_0^N(\mathbf{1}) \Bigg[\sum_{A \subset (I''_N)^c} |\beta_N(A)| + \|\beta_N\|_1 \hat{P}_0(|\hat{B}_s^N| \ge (K-1)L/3)$$
$$+ \|\beta_N\|_1 P_0(\sup_{u \le t} |B_u^N| \ge (K-1)L/3) \Bigg]$$
$$\le \kappa \mathrm{e}^{\kappa s} X_0^N(\mathbf{1}) \Bigg[\sum_{A \subset (I''_N)^c} |\beta_N(A)| + \hat{P}_0(|\hat{B}_s^N| \ge (K-1)L/3)$$
$$+ P_0(\sup_{u \le t} |B_u^N| \ge (K-1)L/3) \Bigg]. \tag{3.10}$$

A simpler argument shows

$$E(|d_s^4|) \le \mathrm{e}^{\kappa s} X_0^N(\mathbf{1}) \|\delta_N\|_1 \Bigg[\hat{P}_0(|\hat{B}_s^N| \ge (K-1)L/2)$$
$$+ P_0 \left(\sup_{u \le t} |B_u^N| \ge (K-1)L/2 \right) \Bigg]. \tag{3.11}$$

Finally, recalling $\mathrm{supp}(X_0^N) \subset I$, we have the easy estimate

$$X_0^N(\mathbf{1} - \underline{P}_t^N \mathbf{1}) \le X_0^N(\mathbf{1}) P_0(\sup_{u \le t} |B_u^N| \ge (K-1)L). \tag{3.12}$$

Note that $\int_0^t \kappa \mathrm{e}^{\kappa s}\, ds \le \mathrm{e}^{\kappa t}$ and so if

$$F_N(t) = \mathrm{e}^{\kappa t} X_0^N(\mathbf{1}) \Bigg[\sum_{A \subset (I''_N)^c} |\beta_N(A)| + 2\hat{P}_0(\sup_{s \le t} |\hat{B}_s^N| \ge (K-1)L/3)$$
$$+ 2P_0(\sup_{u \le t} |B_u^N| \ge (K-1)L/3) \Bigg] + X_0^N(\mathbf{1}) P_0(\sup_{s \le t} |\hat{B}_s^N| \ge (K-1)L/3),$$

then we may use (3.8), (3.10), (3.11) and (3.12) in (3.7) to conclude

$$E(X_t^N(\mathbf{1}) - \underline{X}_t^N(\mathbf{1})) \le 2(n_1+1)\kappa \int_0^t E(X_s^N(\mathbf{1}) - \underline{X}_s^N(\mathbf{1}))\, \mathrm{d}s + F_N(t).$$

Recall that $E(X_t^N(\mathbf{1})) < \infty$ by Proposition 2.1 of [5]. As F_N is non-decreasing, Gronwall's Lemma implies

$$
\begin{aligned}
E(X_T^N(\mathbf{1}) - \underline{X}_T^N(\mathbf{1})) &\le e^{2\kappa(n_1+1)T} F_N(T) \\
&\le e^{2\kappa(n_1+1)T} 3 e^{\kappa T} X_0^N(\mathbf{1}) \\
&\quad \times \left[\sum_{A \subset (I_N'')^c} |\beta_N(A)| + \hat{P}_0\left(\sup_{s \le T} |\hat{B}_s^N| > (K-1)L/3 \right) \right. \\
&\quad \left. + P\left(\sup_{s \le T} |B_s^N| > (K-1)L/3 \right) \right],
\end{aligned}
$$

and the result follows. □

4 Weak survival: propagation bounds

We continue to work with generalized voter model perturbations satisfying (P) with laws $P^{\beta,\delta}$. Here is the goal of the next two sections. We will show that Theorem 1 follows as a special case (see Sect. 8). Recall that for $A \subset \mathbf{Z}^d$, $\sigma(A) = P(\tau(A) < \infty)$ where $\tau(A) = \inf\{s : |\{\hat{B}_s^x : x \in A\}| = 1\}$ is the coalescing time of our system of coalescing random walks (see Sect. 1).

Theorem 4.1 *Assume $S \subset \{(\beta,\delta) \in \ell^1(P_F)^2 : \|(\beta,\delta)\|_1 \le 1\}$ is relatively compact and (P) holds uniformly on S. For $\eta > 0$, let*

$$
S_\eta = \left\{ (\beta,\delta) \in S : \sum_{A \in P_F} [\beta(A)\sigma(A) - (\beta(A) + \delta(A))\sigma(A \cup \{0\})] \ge \eta \right\}.
$$

Then there exists $r = r(\eta, S) \in (0,1)$ and $C_{4.1} = C_{4.1}(\eta, S) > 0$ such that for all $\frac{(\beta,\delta)}{\|(\beta,\delta)\|_1} \in S_\eta$ such that $0 < \|(\beta,\delta)\|_1 \le r$,

$$
P^{\beta,\delta}(|\xi_t^0| > 0 \quad \text{for all } t > 0) \ge C_{4.1} \|(\beta,\delta)\|_1. \tag{4.1}
$$

In particular, survival holds for such (β,δ).

The expression appearing in the definition of S_η is the drift of the limiting super-Brownian motion in Theorem B. Its positivity is necessary and sufficient for the possible survival of the limiting super-Brownian motion and so after an interchange of limits one sees the above survival conclusion.

We will first prove survival, and then use additional arguments to obtain the bound (4.1). The proof of survival depends on a construction of a supercritical oriented percolation process which “lies beneath ξ_t”. The occupied sites of this oriented percolation process will correspond to large blocks of large mass for ξ_t. To prove the supercriticality we must show those large blocks propagate with high probability. This is Proposition 4.2 below and is the goal of the present section. The oriented percolation process is then constructed in Sect. 5, where the proof of survival is given. The bound (4.1) is proved in Sect. 7.

Let $I = [-L, L]^d$, and for $z \in \mathbf{Z}, I_z = 2zLe_1 + I$ and $I'_z = 2zLe_1 + (-KL, KL)^d$, where e_1 is the unit vector in the x_1 direction. Also introduce $I_z^N = \sqrt{N} I_z$ and $I_z^{\prime,N} = \sqrt{N} I'_z$. The parameters L, K, N will be natural numbers whose values will be selected in the proof of the next Proposition, along with two other parameters $J \in \mathbf{N}$ and $T \in [1, \infty)$. Assume ξ_0 is a given initial condition such that $|\xi_0| < \infty$. We will assume (β, δ) is as in Theorem 4.1, $\xi_t = \xi_t[0, \xi_0, \mathbf{R}^d]$, $\underline{\xi}_t = \underline{\xi}_t[0, \xi_0, I']$, ξ_t^N, $\underline{\xi}_t^N$, and X_t^N, $\underline{X}_t^N$ are defined as in the previous section. For example, ξ has law $P_{\xi_0}^{\beta,\delta}$, $\xi_t^N(x) = \xi_{Nt}(x\sqrt{N})$ for $x \in \mathbf{S_N}$ and $X_t^N = \frac{1}{N} \sum_{x \in \mathbf{S_N}} \xi_t^N(x) \delta_x$. The dependence on (β, δ) is suppressed in this notation, but we will often use $P_{\xi_0}^{\beta,\delta}$ for emphasis.

Proposition 4.2 *Let $\eta \in (0, 1)$ and assume S and S_η are as in Theorem 4.1. There are*
$L, K, J \in \mathbf{N}$, $T \geq 1$, and $r \in (0, 1]$ depending on (η, S), such that if

$$0 < \|(\beta, \delta)\|_1 \leq r, \quad \frac{(\beta, \delta)}{\|(\beta, \delta)\|_1} \in S_\eta, \quad N = \left\lfloor \|(\beta, \delta)\|_1^{-1/2} \right\rfloor^2, \quad \text{and} \quad \gamma_K = 6^{-4(2K+1)^2},$$

then

$$X_0^N(I) = X_0^N(\mathbf{1}) \geq J \text{ implies } P^{\beta,\delta}(\underline{X}_T^N(I_1) \geq J \quad \text{and} \quad \underline{X}_T^N(I_{-1}) \geq J) \geq 1 - \gamma_K. \tag{4.2}$$

Proof Assume $0 < \|(\beta, \delta)\|_1$ and $\frac{(\beta,\delta)}{\|(\beta,\delta)\|_1} \in S_\eta$. Now define N as above and set $\beta_N(A) = N\beta(A)$ and $\delta_N(A) = N\delta(A)$. First assume $\|(\beta, \delta)\|_1 \leq r(\eta) \leq 1/16$. Then an elementary argument shows that

$$\|(\beta, \delta)\|_1^{-1} \geq N \geq \frac{1}{2} \|(\beta, \delta)\|_1^{-1}. \tag{4.3}$$

This implies

$$d_N \equiv \sum_A [\beta_N(A)\sigma(A) - (\beta_N(A) + \delta_N(A))\, \sigma(A \cup \{0\})] \geq N\|(\beta, \delta)\|_1 \eta \geq \frac{\eta}{2} \tag{4.4}$$

and

$$d_N \leq \|\beta_N\|_1 + \|\beta_N + \delta_N\|_1 \leq 2\|(\beta_N, \delta_N)\|_1 = 2N\|(\beta, \delta)\|_1 \leq 2. \tag{4.5}$$

To achieve (4.2) we want to choose our constants so that (with X_t denoting the appropriate limiting super-Brownian motion from Theorem B):

(1) $X_T(I_1)$ and $X_T(I_{-1})$ are large with high probability (Lemma 4.3 below).

(2) $X_T^N(I_1) \approx X_T(I_1)$ and $X_T^N(I_{-1}) \approx X_T(I_{-1})$ with high probability (proof of (4.13) below).
(3) $\underline{X}_T^N(I_1) \approx X_T^N(I_1)$ and $\underline{X}_T^N(I_{-1}) \approx X_T^N(I_{-1})$ with high probability ((4.14) below).

We start choosing our constants, beginning with a new constant $c = c(\sigma)$ taken large enough to satisfy

$$\exp\left(-\frac{c^2K^2}{37\sigma^2 d^2}\right) \le \frac{1}{100}6^{-4(2K+1)^2} \quad \forall K \ge 1. \tag{4.6}$$

(Recall that $p(x)$ has covariance matrix $\sigma^2 I$.) The reason for this somewhat peculiar choice will become clear later. As σ is a constant throughout this work we will drop all dependence on it in our notation.

Next, choose $T = T(\eta) \ge 1$ sufficiently large so that if B_t denotes Brownian motion in $\mathbf{R}^d$ with diffusion parameter σ^2, then

$$e^{\eta T/2} \inf_{|x|\le c}\{P_x(B_1 \in [c, 3c]^d)\} \ge 5. \tag{4.7}$$

By increasing T slightly we may also assume $L = L(\eta) \equiv c\sqrt{T}$ is in $\mathbf{N}$. We have chosen T large so that a supercritical super-Brownian motion with drift $d_0 \in [\eta/2, 2]$ will have a large amount of mass in both I_1 and I_{-1} at time T with high probability provided it begins with a large amount of mass in I. More precisely the following Lemma follows exactly as for Lemma 12.1(b) in [8] using a simple Chebychev argument. Note that by monotonicity in X_0 it suffices to consider initial states X_0 with support contained in I.

Lemma 4.3 *There is a constant $C_{4.8} = C_{4.8}(\eta, T)$ such that if $X_\cdot$ is a super-Brownian motion with branching rate $2\gamma_e$, diffusion rate σ^2, drift $d_0 \in [\frac{\eta}{2}, 2]$, and initial state X_0 satisfying $X_0(I) \ge 1$, then*

$$P(X_T(I_1) \vee X_T(I_{-1}) \le 4X_0(I)) \le C_{4.8}/X_0(I). \tag{4.8}$$

This will allow us to use Theorem B to infer that similar results will hold for our rescaled Lotka–Volterra models.

We complete our selection of constants as follows. Choose $K \ge K_0 = \max\{4, 1 + \frac{3d\sigma}{c}\}$ large enough so that

$$6de^{8n_1 T}e^{-c^2K^2/36d^2\sigma^2} \le \frac{1}{3}e^{-c^2K^2/37d^2\sigma^2}. \tag{4.9}$$

Note that K really depends only on (η, S) since this is the case for $T = T(\eta)$ and $n_1 = n_1(S)$. Lastly, choose $J \in \mathbf{N}$ large enough so that

$$\frac{C_{4.8}(\eta, T)}{J} < \frac{1}{3}e^{-c^2K^2/37d^2\sigma^2}. \tag{4.10}$$

Since all of $T, C_{4.8}, K$, depend only on η and S, the same is true of $J = J(\eta, S)$.

If B^N_t is as in Sect. 3, then $B^N_\cdot \Rightarrow B_\cdot$, d-dimensional Brownian motion with covariance matrix $\sigma^2 I$. Therefore the functional central limit theorem shows that there are constants $\epsilon_N = \epsilon_N(K, c, T)$ with $\lim_{N\to\infty} \epsilon_N = 0$ such that

$$\begin{aligned} P_0\left(\sup_{t\le T} |B^N_t| > \frac{(K-1)L}{3}\right) &\le P_0\left(\sup_{s\le T} |B_s| > \frac{(K-1)L}{3}\right) + \epsilon_N \\ &\le 4dP_0\left(B^1_T > \frac{(K-1)L}{3d}\right) + \epsilon_N \qquad (4.11) \\ &\le 4d\exp(-((K-1)L)^2/18d^2\sigma^2 T) + \epsilon_N \\ &\le 4d\exp(-K^2c^2/36d^2\sigma^2) + \epsilon_N. \end{aligned}$$

In the next to last line we used our lower bound on K and the bound $P(B_1 > y) \le \mathrm{e}^{-y^2/2\sigma^2}$ for $y \ge \sigma$, and in the last line we used $K \ge 4$ (which implies $((K-1)/K)^2 \ge 1/2$).

Assume that the initial condition X^N_0 satisfies $X^N_0(\mathbf{1}) = X^N_0(I_0)$. As we have $\|(\beta_N, \delta_N)\|_1 \le 1$ by (4.3), we may use (4.11) and Lemma 3.2 with $\kappa = 1$ to conclude that

$$\begin{aligned} E(X^N_T(\mathbf{1}) - \underline{X}^N_T(\mathbf{1})) \le 3X^N_0(\mathbf{1})\mathrm{e}^{5n_1 T}\Bigg[\left(\sum_{A\subset(I''_N)^c} |\beta_N(A)|\right) \\ + \hat{P}_0\left(\sup_{s\le T} |\hat{B}^N_s| \ge \frac{(K-1)L}{3}\right) + 4d\exp\left(\frac{-K^2c^2}{36d^2\sigma^2}\right) + \epsilon_N\Bigg]. \end{aligned} \qquad (4.12)$$

We give now the analogue of Lemma 4.3 for our rescaled Lotka–Volterra processes, in fact for the processes $\underline{X}^N$ with additional killing on the boundary. The proof relies on Theorem B.

Lemma 4.4 *Let $\eta \in (0,1)$ and assume S and S_η are as in Theorem 4.1. There exists $r = r(\eta, S) > 0$, such that if $0 < \|(\beta,\delta)\|_1 \le r$, $\frac{(\beta,\delta)}{\|(\beta,\delta)\|_1} \in S_\eta$, supp$(X^N_0) \subset I$ and $X^N_0(I) \ge J$, then*

$$P^{\beta,\delta}(X^N_T(I_1) \wedge X^N_T(I_{-1}) \le 4J) \le (2/3)\mathrm{e}^{-c^2K^2/37d^2\sigma^2} \qquad (4.13)$$

and for $\Delta^N_T = (X^N_T(I_1) - \underline{X}^N_T(I_1)) \vee (X^N_T(I_{-1}) - \underline{X}^N_T(I_{-1}))$,

$$P^{\beta,\delta}(\Delta^N_T > 2J) \le (4/3)\mathrm{e}^{-c^2K^2/37d^2\sigma^2}. \qquad (4.14)$$

Proof If (4.13) fails we may assume without loss of generality there is a sequence (β^m, δ^m) in $\ell^1(P_F)^2$ such that $0 < \|(\beta^m,\delta^m)\|_1 \to 0$ and $(\hat{\beta}^m, \hat{\delta}^m) = \frac{(\beta^m,\delta^m))}{\|(\beta^m,\delta^m)\|_1} \in S^\eta$

and a sequence of initial conditions $X_0^{N_m}$ such that $\text{supp}(X_0^{N_m}) \subset I, X_0^{N_m}(I) \geq J$, and

$$P^{\beta^m,\delta^m}(X_T^{N_m}(I_1) \leq 4J) > \frac{1}{3}\mathrm{e}^{-c^2K^2/37d^2\sigma^2} \quad \forall m \geq 1. \tag{4.15}$$

Here of course

$$N_m = \lfloor \|(\beta^m, \delta^m)\|_1^{-1/2} \rfloor^2 \to \infty.$$

The monotonicity of $P^{\beta^m,\delta^m}(X_\cdot^{N_m} \in \cdot)$ in the initial condition, given by Proposition 2.3 and elementary scaling, allows us to assume $X_0^{N_m}(I) \to J$ as $m \to \infty$. By considering a subsequence (recall that S is relatively compact), we may assume without loss of generality that $(\hat{\beta}^m, \hat{\delta}^m) \to (\beta, \delta)$ in the closed unit ball of $\ell^1(P_F)^2$, and $X_0^{N_m} \to X_0 \in \mathcal{M}_F$ with $X_0(\mathbf{R}^d) = X_0(I) = J$. The former implies that

$$(\beta_{N_m}^m, \delta_{N_m}^m) \equiv N_m \|(\beta^m, \delta^m)\|_1 (\hat{\beta}^m, \hat{\delta}^m) \to (\beta, \delta) \text{ in } \ell^1(P_F)^2. \tag{4.16}$$

It is now easy to use our hypothesis that (P) holds uniformly in S to conclude that the hypotheses of Theorem B are in force. For example, our condition (P4) (uniformly over S) and (4.3) (we may assume $\|(\beta^m, \delta^m)\|_1 \leq 1/16$) imply there is a $K_4 > 0$ such that for all m,

$$\begin{aligned}\sum_A \delta_{N_m}^m(A)\chi(A,0,\xi) &= N_m\|(\beta^m,\delta^m)\|_1 \sum_A \hat{\delta}^m(A)\chi(A,0,\xi) \\ &\geq -\frac{K_4}{2} f_0(0,\xi),\end{aligned}$$

which is precisely the hypothesis that (P4) holds uniformly on $\{(\beta_{N_m}^m, \delta_{N_m}^m) : m \in \mathbf{N}\}$, required in Theorem B. The other conditions of Theorem B are easier to verify.

In addition, the bounds (4.4) and (4.5) are valid for

$$d_m = \sum_A \left[\beta_{N_m}^m(A)\sigma(A) - (\beta_{N_m}^m(A) + \delta_{N_m}^m(A))\sigma(A \cup \{0\})\right],$$

and so

$$\theta = \sum_A \left[\beta(A)\sigma(A) - (\beta(A) + \delta(A))\sigma(A \cup \{0\})\right] \in \left[\frac{\eta}{2}, 2\right],$$

by the ℓ^1-convergence of $(\beta_{N_m}^m, \delta_{N_m}^m)$ to (β, δ) (see (4.16)). Theorem B shows that $X_\cdot^{N_m} \Rightarrow X_\cdot$ where $X_\cdot$ is super-Brownian motion with branching coefficient

$2\gamma_e$, drift coefficient θ and diffusion coefficient σ^2. Therefore, since $X_T(\partial I_1) = 0$ a.s., we may use this weak convergence, Lemma 4.3 and (4.10) to obtain

$$\limsup_{m\to\infty} P^{\beta^m,\delta^m}(X_T^{N_m}(I_1) \le 4J) \le P_{X_0}(X_T(I_1) \le 4J) \le \frac{C_{4.8}}{J} < \frac{1}{3}\mathrm{e}^{-c^2K^2/37d^2\sigma^2}.$$

This contradicts (4.15) and so proves (4.13).

If (4.14) fails we may suppose there exist sequences $(\beta^m, \delta^m), X_0^{N_m}$ as before but with (4.15) replaced by

$$P^{\beta^m,\delta^m}(X_T^{N_m}(I_1) - \underline{X}_T^{N_m}(I_1) > 2J) > (2/3)\mathrm{e}^{-c^2K^2/37d^2\sigma^2}. \tag{4.17}$$

A simple Chebyshev argument implies that the left side in (4.17) is bounded above by

$$\begin{aligned}&(2J)^{-1}E(X_T^{N_m}(\mathbf{1}) - \underline{X}_T^{N_m}(\mathbf{1}))\\ &\quad\le \frac{X_0^{N_m}(\mathbf{1})3\mathrm{e}^{5n_1T}}{2J}\left[\sum_{A\subset(I''_{N_m})^c}|\beta^m_{N_m}(A)| + \hat{P}_0\left(\sup_{s\le T}|\hat{B}_s^{N_m}| > \frac{(K-1)L}{3}\right)\right.\\ &\qquad\qquad\left. +\epsilon_{N_m} + 4d\exp\left(\frac{-K^2c^2}{36d^2\sigma^2}\right)\right].\end{aligned} \tag{4.18}$$

We have used (4.12) in the last line. Note that $\hat{B}^{N_m}$ is the random walk defined prior to Lemma 3.1 with $\beta^m_{N_m}$ in place of β_N. The fact that $\beta^m_{N_m} \to \beta$ in ℓ^1, implies

$$\lim_{m\to\infty}\sum_{A\subset(I''_{N_m})^c}|\beta^m_{N_m}(A)| = 0, \tag{4.19}$$

and also implies that $\sqrt{N_m}\hat{B}^{N_m}$ converges weakly to $\hat{B}$, where $\hat{B}_u \in \mathbf{Z}^d$ is a random walk starting at 0, taking steps at rate $\|\beta^+\|_1$ according to

$$\hat{p}(x) = \begin{cases}\sum_{x\in -A}\frac{\beta^+(A)/|A|}{\|\beta^+\|_1} & \text{if } \|\beta^+\|_1 > 0\\ 1(x=0) & \text{if } \|\beta^+\|_1 = 0.\end{cases}$$

This shows that

$$\lim_{m\to\infty}\hat{P}_0\left(\sup_{s\le T}|\hat{B}_s^{N_m}| > \frac{(K-1)L}{3}\right) = 0. \tag{4.20}$$

Use (4.19), (4.20) and the convergence $\lim_{m\to\infty}\frac{X_0^{N_m}(\mathbf{1})}{J}=1$ in (4.18), and conclude that

$$\limsup_{m\to\infty} P(X_T^{N_m}(I_1)-\underline{X}_T^{N_m}(I_1)\geq 2J)\leq \mathrm{e}^{5n_1T}6d\exp\left(\frac{-K^2c^2}{36d^2\sigma^2}\right)$$
$$\leq \frac{1}{3}\exp\left(\frac{-K^2c^2}{37d^2\sigma^2}\right),$$

the last by (4.9). The above contradicts (4.17) and so the proof of (4.14) is complete. □

We can now end this section with the

Proof of Proposition 4.2 By decreasing $r(\eta)$ in Lemma 4.4, if necessary, we may assume $r(\eta)<1/16$ (to ensure (4.3)). Assume $0<\|(\beta,\delta)\|_1<r(\eta)$, $\frac{(\beta,\delta)}{\|(\beta,\delta)\|_1}\in S_\eta$, and X_0^N is supported on I and has total mass at least J. By Lemma 4.4,

$$P^{\beta,\delta}(\underline{X}_T^N(I_1)\leq 2J)\leq P^{\beta,\delta}(X_T^N(I_1)\leq 4J)+P^{\beta,\delta}(X_T^N(I_1)-\underline{X}_T^N(I_1)>2J)$$
$$\leq 2\mathrm{e}^{-c^2K^2/37d^2\sigma^2}.$$

Consequently, using the same bound for I_{-1} and (4.6), we get

$$P^{\beta,\delta}(\underline{X}_T^N(I_1)\geq 2J\text{ and }\underline{X}_T^N(I_{-1})\geq 2J)\geq 1-4\exp(-K^2c^2/37d^2\sigma^2)$$
$$\geq 1-\frac{4}{100}6^{-4(2K+1)^2}$$
$$\geq 1-\gamma_K.$$

□

5 Oriented percolation construction

In this section the setting is as in Sect. 4. Hence ξ_t denotes a generalized voter perturbation with parameters (β,δ). We will often write P for $P_{\xi_0}^{\beta,\delta}$. We will use Theorem 4.3 of [7] to define a super-critical oriented percolation process which lies beneath ξ_t, but, as it will be convenient to have some detailed knowledge of the percolation process, we will give an explicit description of its construction.

We begin by assuming that the parameters (β,δ), T,L,K,J,N are fixed; we do not yet impose the assumptions of the last section. We recall the notation $I=[-L,L]^d$, $I_z=2zLe_1+I$ for $z\in\mathbf{Z}$, $I_z'=2zLe_1+(-KL,KL)^d$, $I_z^N=\sqrt{N}I_z$, and $I_z'^{,N}=\sqrt{N}I_z'$. Let $\mathcal{L}=\{(z,n)\in\mathbf{Z}\times\mathbf{Z}_+ : z+n\text{ is even}\}$, and let $\{\mathcal{B}(z,n),\ (z,n)\in\mathcal{L}\}$ be a collection of iid Bernoulli random variables, independent of the Poisson processes $N^{x,i}$, such that $P(\mathcal{B}(z,n)=1)=1-\gamma_K$, where γ_K is as in Proposition 4.2.

Let us fix $\xi_0 \in \{0,1\}^{\mathbf{Z}^d}$ such that $|\xi_0| < \infty$, and define $\xi_\cdot = \xi_\cdot[0,\xi_0,\mathbf{R}^d]$. Let $B \subset \{z \in 2\mathbf{Z} : \xi_0(I_z^N) \ge NJ\}$. We are ready now for the construction.

Step 1. Let $W_0^B = B$, and for $z \in W_0^B$, define

$$\underline{\xi}_t^{(z,0)} = \underline{\xi}_t[0,\xi_0\big|_{I_z^N}, I_z'^N], \quad t \ge 0,$$

the unique solution to $(SDE)(0,I_z'^N)$ (see the beginning of Sect. 3). By definition, $\underline{\xi}_0^{(z,0)}(I_z^N) \ge NJ$ for all $z \in W_0^B$, and it follows from Proposition 2.1(b) that $\underline{\xi}_t^{(z,0)} \le \xi_t$ for all $t \ge 0$ and $z \in W_0^B$.

Step 2. Suppose $n \ge 0$. Assume $\{w(z,k) : (z,k) \in \mathcal{L}, k < n\}$, $W_n^B \subset \{z \in \mathbf{Z} : (z,n) \in \mathcal{L}\}$ and $\{\underline{\xi}_t^{(z,n)}, t \ge nTN\}$ for all $z \in W_n^B$ have all been defined, and for all such z satisfy $\underline{\xi}_{nNT}^{(z,n)}(I_z^N) \ge NJ$ and

$$\underline{\xi}_t^{(z,n)} \le \xi_t \quad \text{for all } t \ge nNJ.$$

For $z \notin W_n^B$ put $w(z,n) = \mathcal{B}(z,n)$. For $z \in W_n^B$, define

$$w(z,n) = \begin{cases} 1 & \text{if } \underline{\xi}_{(n+1)NT}^{(z,n)}(I_{z-1}^N) \wedge \underline{\xi}_{(n+1)NT}^{(z,n)}(I_{z+1}^N) \ge NJ \\ 0 & \text{otherwise.} \end{cases}$$

Now define

$$W_{n+1}^B = \{z \in \mathbf{Z} \colon \exists y \in W_n^B, |y-z| = 1, w(y,n) = 1\}.$$

For $z \in W_{n+1}^B$, let $y = z-1$ if $z-1 \in W_n^B$, and otherwise $y = z+1 \in W_n^B$, and define

$$\underline{\xi}_t^{(z,n+1)} = \underline{\xi}_t[(n+1)TN, \underline{\xi}_{(n+1)TN}^{(y,n)}\big|_{I_z^N}, I_z'^N], \quad t \ge (n+1)TN. \tag{5.1}$$

Then, by construction, $\underline{\xi}_{(n+1)NT}^{(z,n+1)}(I_z^N) \ge NJ$, and by (3.1) and our induction hypothesis, we get $\underline{\xi}_t^{(z,n+1)} \le \xi_t$ for all $t \ge (n+1)NJ$. This verifies the induction hypotheses for $n+1$ and allows us to iterate this construction. The above induction has established

$$\text{for all } n \ge 0 \text{ and } z \in W_n^B,\ \xi_{nNT}(I_z^N) \ge \underline{\xi}_{nNT}^{(z,n)}(I_z^N) \ge NJ. \tag{5.2}$$

In fact one readily sees from the above construction that

$$\begin{aligned} &z \in W_n^B \text{ iff there exist } x_0,\dots,x_n \text{ such that } x_0 \in B, x_n = z, \text{ and for } 0 \le i < n, \\ &\quad |x_{i+1}-x_i| = 1 \text{ and } \underline{\xi}_{(i+1)NT}^{(x_i,i)}(I_{x_i-1}^N) \wedge \underline{\xi}_{(i+1)NT}^{(x_i,i)}(I_{x_i+1}^N) \ge NJ. \end{aligned} \tag{5.3}$$

Assume now that (β, δ) and N are as in Proposition 4.2 for some $\eta > 0$ and T, L, K, J are selected as in Lemma 4.4 and so satisfy (4.6), (4.7), (4.9) and (4.10). To relate the above construction to that in Theorem 4.3 of [7], introduce

$$H = \{\xi_0 \in \{0,1\}^{\mathbf{Z}^d} : |\xi_0| < \infty, \sum_{x \in I_0^N} \xi_0(x) \geq NJ\},$$

and for $\xi_0 \in H$ define the event

$$G_{\xi_0} = \{\underline{\xi}_{NT}[0, \xi_0|_{I_0^N}, I'^N_0](I_1^N) \geq NJ \text{ and } \underline{\xi}_{NT}[0, \xi_0|_{I_0^N}, I'^N_0](I_{-1}^N) \geq NJ\}.$$

Let $\xi_0 \in H$ and as usual, $\xi_t = \xi_t[0, \xi_0, \mathbf{R}]$ is the unique solution of $(SDE)(\mathbf{R})$. By Proposition 2.1,

$$G_{\xi_0} \text{ is } \mathcal{G}([0, NT] \times (-KL\sqrt{N}, KL\sqrt{N}))\text{-measurable.} \tag{5.4}$$

On G_{ξ_0}, $\xi_{NT} \in \tau_{2L\sqrt{N}}(H) \cap \tau_{-2L\sqrt{N}}(H)$ (recall that $\tau_x(\xi)(y) = \xi_0(x+y)$) because

$$\xi_{NT}(I_1^N) \wedge \xi_{NT}(I_{-1}^N) \geq \underline{\xi}_{NT}[0, \xi_0|_{I_0^N}, I_0^{',N}](I_1^N) \wedge \underline{\xi}_{NT}[0.\xi_0|_{I_0^N}, I_0^{',N}](I_{-1}^N) \geq NJ.$$

Finally Proposition 4.2 and our hypotheses on (β, δ) imply that $P(G_{\xi_0}) \geq 1 - \gamma_K$. We have just verified the Comparison Assumptions required to apply Theorem 4.3 of [7] and so the proof of that result gives the following:

Lemma 5.1 *For every $k > 0$ and $(z_i, n_i) \in \mathcal{L}$, $i = 0, \ldots, k$ such that $|z_i - z_j| > 2K$ whenever $n_i = n_j$ $(i \neq j)$,*

$$P(w(z_i, n_i) = 0 \quad \text{for all } 1 \leq i \leq k) \leq \gamma_K^k. \tag{5.5}$$

Some explanation is perhaps in order here. We have replaced the integers T, L, k_0 in [7] with NT, $\sqrt{N}L$ and K, respectively. In [7] it is assumed that $\xi_.$ is a finite range process which in our setting amounts to $p(\cdot)$, $\beta(\cdot)$ and $\delta(\cdot)$ having finite support. This hypothesis is only used in [7] to construct $\xi_.$ as a solution of (SDE) and establish (5.4). We have been able to derive this thanks to Proposition 2.1, which in turn relies on $|\xi_0| < \infty$. Once $\xi_.$ is constructed in this way the finite range assumption plays no further role in the proof of Theorem 4.3 in [7]. We may consider initial conditions such that $|\xi_0| = \infty$ (see Remark 2.5) but when applying the above comparison with oriented percolation will always cull our initial condition to a finite one.

The above lemma shows that, in the terminology of [7], we have constructed a $2K$-dependent oriented percolation process with density at least $1 - \gamma_K$. According to Theorem 4.1 of [7], this implies that

$$P(W_n^B \neq \emptyset \quad \text{for all } n \geq 0 | W_0^B) \geq 0.95 \text{ on } \{W_0^B \neq \emptyset\}. \tag{5.6}$$

It follows from (5.2) and (5.6) that

$$P_{\xi_0}(|\xi_t| > 0 \text{ all } t \geq 0) \geq .95 \quad \text{if } \xi_0\left(\sqrt{N}I_0\right) \geq NJ. \tag{5.7}$$

If $\xi_0(x) = 1$ only at $x = 0$, and again denoting the corresponding process ξ_t^0, a simple application of the Markov property at time NT shows that we may construct $\{W_n : n \geq 0\}$ as above such that

$$W_0 = \begin{cases} \{0\} & \text{on } \left\{\xi_{NT}^0\left(\sqrt{N}I_0\right) \geq NJ\right\} \\ \emptyset & \text{otherwise,} \end{cases} \tag{5.8}$$

$$z \in W_n \text{ implies } \xi_{(n+1)NT}^0\left(\sqrt{N}I_z\right) \geq NJ, \tag{5.9}$$

and so by (5.6),

$$P(|\xi_t^0| > 0 \text{ for all } t \geq 0) \geq .95\, P(\xi_{NT}^0(\sqrt{N}I_0) \geq NJ). \tag{5.10}$$

We will use this in the proof of (4.1) below to get our quantitative lower bound but for now note the trivial consequence of the above:

$$P(|\xi_t^0| > 0 \quad \text{for all } t \geq 0) > 0. \tag{5.11}$$

Thus, we have proved survival for (β, δ) which satisfy the assumptions of Theorem 4.1.

Remark 5.2 We have spelled out this argument in some detail because there seem to be some differences in the way we have applied the oriented percolation comparison argument than in other applications of this method with which we are familiar (e.g. that in [8]). The scaling parameter N is intertwined with the underlying parameters (β, δ) and so changing it leads to a change in the underlying probability. We cannot just fix a parameter value of interest and prove survival, because, for a given η, we must consider infinitely many parameter values simultaneously. In the end the limit theorem (Theorem 4.1) nicely looks after this issue.

Perhaps more significant is the fact that we have needed the asymptotic upper bound from [7] for the critical probability for $2K$-dependent oriented percolation as $K \to \infty$. This arises because we have only been willing (or able) to carry out a first moment argument in our Comparison Lemma (Lemma 3.2) to bound the effect of our killing mechanism (as opposed to a second moment argument as in Lemma 12.1(a) of [8]). The complex nature of the Lotka–Volterra (and voter model perturbations) makes higher moment calculations less desirable (although unfortunately some will have to be carried out in a future work where we will show our survival results are sharp, at least for the basic

Lotka–Volterra examples). On the other hand, using first moments means we cannot simply increase J to beat out whatever critical probability arises after the choice of K. Instead we have a horse race between the Gaussian tail in K arising in the bound given in Lemma 3.2 and the upper bound on p_{crit}, $1-\gamma_K$, from Theorem 4.1 of [7]. The choice of c in (4.6) is made to ensure that the right term wins thanks to our large choice of box I.

Here is a standard consequence of our supercritical oriented percolation construction.

Proposition 5.3 *Assume (β,δ) satisfies the hypotheses of Theorem 4.1 for some $\eta>0$. There is a $p_0=p_0(\beta,\delta)>0$ such that $P(\xi_t^0(0)=1)\ge p_0$ for all $t\ge 0$.*

Proof Let $\{W_n\}$ be the $2K$-dependent oriented percolation process in (5.9). Lemma 4.4 of [3], (5.8) and (5.6) imply there are $\ell>0$ and $n_0\in\mathbf{N}$ such that for $n\ge n_0$,

$$P(W_n\cap[-\ell,\ell]\neq\emptyset)\ge 0.9P(\xi_{NT}^0(\sqrt{N}I_0)\ge NJ)\equiv p_1(\beta,\delta)>0.$$

Note that ℓ will also depend only on (β,δ) as all our parameters K,N,T do. By (5.9), if $L'=(2\ell\sqrt{N}+1)L$, this implies

$$P^{\beta,\delta}(\xi^0_{(n+1)NT}([-L',L']\times[-\sqrt{N}L,\sqrt{N}L]^{d-1})>0)\ge p_1\quad\text{for } n\ge n_0. \tag{5.12}$$

Let

$$p_2(t)=\inf\{P_{\xi_0}(\xi_t(0)=1):\xi_0([-L',L']^d)\ge 1\},$$

and set $p_2=\inf\{p_2(t):t\in[NT,(N+1)T]\}$. We claim $p_2=p_2(\ell,N,L,T)=p_2(\beta,\delta)$ is strictly positive. By monotonicity we may assume in the first infimum, that ξ_0 has support in $[-L',L']^d$ and hence ranges over a finite set. This shows that $p_2(t)>0$ for each $t>0$. Let

$$\begin{aligned}p_3(t)=p_3(\beta,\delta)(t)=\inf\{P^{\beta,\delta}_{\xi_0}(&\xi_s\text{ has no death event at }x=0\\&\text{for times }s\text{ in }[0,t]):\xi_0(0)=1\}.\end{aligned}$$

By (2.6), $p_3(t)\ge \mathrm{e}^{-t(1+\|\delta\|_1)}>0$. Note that if $\xi^0_{NT}(0)=1$ and there is no death event at 0 for times in $[NT,(N+1)T]$, then $\xi_t(0)=1$ for all $t\in[NT,(N+1)T]$. The Markov property therefore shows that

$$p_2\ge p_2(NT)p_3(NT)>0. \tag{5.13}$$

Assume $t\ge(n_0+2)NT\equiv t_0$ and choose $n\ge n_0$ such that $t\in[(n+2)NT,(n+3)NT]$. Another application of the Markov property together with (5.12) and (5.13) show that

$$P^{\beta,\delta}(\xi_t^0(0)=1)\ge p_1p_2>0.$$

This gives the required bound for $t \geq t_0$. It then follows for all $t \geq 0$ upon noting that $p_3(t_0) > 0$. □

We finish this section with an estimate needed in Sect. 6.

Lemma 5.4 *Assume (β, δ) satisfies the assumptions of Theorem 4.1 for some $\eta > 0$. Let ξ_t^q be the corresponding voter model perturbation process, where $\xi_0^q(x), x \in \mathbf{Z}^d$ are iid Bernoulli random variables with $P(\xi_0^q(x) = 1) = q > 0$. For each $\varepsilon > 0$ and $k \in \mathbf{N}$ there exist finite t_0 and M such that if $t \geq t_0$, then*

$$P^{\beta,\delta}\left(\sum_{x\in[-M,M]^d} \xi_t^q(x) \geq k\right) \geq 1 - \varepsilon. \tag{5.14}$$

To prepare for the proof of this result, let $B^q = \{z \in 2\mathbf{Z} : \xi_0^q(I_z^N) \geq NJ\}$. By decreasing $r(\eta)$, and hence increasing N, in Theorem 4.1, we may assume without loss of generality that $N^{d/2}(2L)^d \geq 2NJ$ (recall $d \geq 3$). This implies that the iid events $\{z \in B^q\}, z \in 2\mathbf{Z}$ satisfy $P(z \in B^q) = p'(\beta, \delta, q) > 0$. For positive integers ℓ define $B^{q,\ell} = B^q \cap [-2\ell, 2\ell]$. According to Theorem A.3 and its proof on pages 194–195 of [7], and after a few misprints are corrected, for $n \geq n_1(K)$,

$$\begin{aligned} P(W_{2n}^{B^{q,\ell+n}} \cap [-2\ell, 2\ell] \neq \emptyset) &\geq (1 - (1-p')^{\sqrt{n}})(1 - 2^{-8\ell} - 2^{-4n}\gamma_K^{-2\sqrt{n}}) \\ &\geq (1 - (1-p')^{\sqrt{n}})(1 - 2^{-\ell} - 2^{-n}) \\ &\geq 1 - (1-p')^{\sqrt{n}} - 2^{-\ell} - 2^{-n}. \end{aligned}$$

Here we will carry out our oriented percolation construction with $\xi_0 = \xi_0^q|_{[-2M_0,2M_0]^d}$ for appropriately large values of M_0–large enough so that it will give the same initial condition for W_n, B, as it would without the truncation at $2M_0$ (see below). By translation invariance and monotonicity in B, if $n \geq n_1$, we get

$$P(W_{2n}^{B^{q,\ell+n+|z|}} \cap [z - 2\ell, z + 2\ell] \neq \emptyset) \geq 1 - (1-p')^{\sqrt{n}} - 2^{-\ell} - 2^{-n},$$

and so if $z_1, \ldots, z_k \in \mathbf{Z} \cap [-M', M']$, then again for $n \geq n_1$,

$$\begin{aligned} P(W_{2n}^{B^{q,\ell+n+M'}} &\cap [z_j - 2\ell, z_j + 2\ell] \neq \emptyset \quad \text{for } j = 1, \ldots, k) \\ &\geq 1 - k((1-p')^{\sqrt{n}} + 2^{-\ell} + 2^{-n}). \end{aligned} \tag{5.15}$$

Here our initial condition is as above with $M_0 = \sqrt{N}L[2(\ell + n + M') + 1]$, where M' is chosen below.

Proof Let $k \in \mathbf{N}, \varepsilon > 0$ and $q \in (0, 1]$ be fixed. Choose $n_0, \ell_0 \in \mathbf{N}$ so that $n_0 \geq n_1$ and the right-hand side of (5.15) is at least $1 - \varepsilon$ for $n \geq n_0$ and $\ell \geq \ell_0$. Choose $z_1, \ldots, z_k \in \mathbf{Z}$ so that

$$|z_i - z_j| \geq K + 3 + 2\ell_0 \quad \text{for } i \neq j, \text{ and for all } j \leq k, \; |z_j| \leq k[K + 3 + 2\ell_0] \equiv M'. \tag{5.16}$$

Then (5.15) implies that

$$P(W_{2n}^{B^{q,\ell+n+M'}} \cap [z_j - 2\ell_0, z_j + 2\ell_0] \neq \emptyset \quad \text{for } j = 1, \ldots, k) \geq 1 - \varepsilon \quad \text{for } n \geq n_0,$$

which by our definition of W_n^B trivially implies

$$\begin{aligned} P(W_n^{B^{q,\ell+n+M'}} &\cap [z_j - 2\ell_0 - 1, z_j + 2\ell_0 + 1] \neq \emptyset \quad \text{for } j = 1, \ldots, k) \\ &\geq 1 - \varepsilon \quad \text{for } n \geq 2n_0. \end{aligned} \tag{5.17}$$

Fix $n \geq 2n_0$ and then ω in the event on the left-hand side of (5.17). Write B for $B^{q,\ell+n+M'}$. Choose $y_j \in W_n^B \cap [z_j - 2\ell_0 - 1, z_j + 2\ell_0 + 1]$ for $j = 1, \ldots, k$. By (5.3) (with $i = n - 1$ in that result), for each $j = 1, \ldots, k$,

$$\underline{\xi}_{nNT}^{(y_j-1,n-1)}(I_{y_j}^N) \geq NJ \text{ or } \underline{\xi}_{nNT}^{(y_j+1,n-1)}(I_{y_j}^N) \geq NJ.$$

Recalling (5.1) (with $n - 1$ in place of $n + 1$), this implies

$$\begin{aligned} \underline{\xi}_t^{(y_j-1,n-1)}(I_{y_j-1}^{\prime N}) &\geq 1 \quad \text{for all } t \in [(n-1)NT, nNT] \\ \text{or } \underline{\xi}_t^{(y_j+1,n-1)}(I_{y_j+1}^{\prime N}) &\geq 1 \quad \text{for all } t \in [(n-1)NT, nNT]. \end{aligned} \tag{5.18}$$

This is because if for some $t \geq (n-1)NT$, $\underline{\xi}_t^{(y_j\pm1,n-1)}(I_{y_j\pm1}^{\prime N}) = 0$, then $\underline{\xi}_s^{(y_j\pm1,n-1)} = 0$ for all $s \geq t$. A bit of arithmetic using (5.16) shows that $\{I_{y_j-1}^{\prime N} \cup I_{y_j+1}^{\prime N} : j = 1, \ldots, k\}$ are disjoint sets. If ξ_t is our generalized voter perturbation with $\xi_0 = \xi_0^q|_\Gamma$, where

$$\Gamma = [-2(\ell_0 + n + M')L\sqrt{N} - L\sqrt{N}, 2(\ell_0 + n + M')L\sqrt{N} + L\sqrt{N}]^d,$$

then $\xi_t \geq \underline{\xi}_t^{(y_j\pm1,n-1)}$ for all $t \geq (n-1)NT$ by our inductive construction. Therefore (5.18) and the disjointness noted above imply

$$\xi_t(\cup_{j=1}^k I_{y_j-1}^{\prime N} \cup I_{y_j+1}^{\prime N}) \geq k \quad \text{for all } t \in [(n-1)NT, nNT].$$

If $M = (2M' + 4\ell_0 + K)L\sqrt{N}$, this shows

$$\xi_t([-M, M]^d) \geq k \quad \text{for all } t \in [(n-1)NT, nNT].$$

By the monotonicity of ξ this proves that for all $t \geq t_0 = (n_0 - 1)NT$,

$$P(\xi_t^q([-M, M]^d) \geq k) \geq 1 - \varepsilon,$$

as required. $\square$

6 Coexistence

In order to prove coexistence we apply our survival criteria to the voter model perturbation processes with the role of 0's and 1's reversed. For $\xi \in \{0,1\}^{\mathbf{Z}^d}$ define the flipped configuration $\tilde{\xi} \in \{0,1\}^{\mathbf{Z}^d}$ by $\tilde{\xi}(x) = 1 - \xi(x)$ for all $x \in \mathbf{Z}^d$. Consider $(\beta, \delta) \in \ell^1(P_F)^2$ satisfying (P), and let $c(x,\xi) = c^{\beta,\delta}(x,\xi)$ be the associated rate function given in (2.6). Let ξ_t be the voter model perturbation process determined by $c(x,\xi)$. The flipped process $\tilde{\xi}_t$ has rate function $\tilde{c}(x,\xi) = c(x,\tilde{\xi})$, and monotonicity for $\tilde{c}(x,\xi)$ follows easily from monotonicity for $c(x,\xi)$. Furthermore, $\tilde{\xi}_t$ is in fact a voter model perturbation with rate function $\tilde{c}(x,\xi) = c^{\tilde{\beta},\tilde{\delta}}(x,\xi)$, where

$$\tilde{\beta}(A) = (-1)^{|A|} \sum_{B \supset A} \delta(B), \quad \tilde{\delta}(A) = (-1)^{|A|} \sum_{B \supset A} \beta(B). \tag{6.1}$$

To see this, first note that it follows easily from (P1) that $\|(\tilde{\beta}, \tilde{\delta})\|_1 \leq 2^{n_1} \|(\beta, \delta)\|$, so $(\tilde{\beta}, \tilde{\delta}) \in \ell^1(P_F)^2$. Next, it is easy to check that for $A \in P_F$,

$$\chi(A, x, \tilde{\xi}) = \prod_{y \in A} (1 - \xi(x + y)) = \sum_{B \subset A} (-1)^{|B|} \chi(B, x, \xi). \tag{6.2}$$

If $\xi(x) = 0$, and hence $\tilde{\xi}(x) = 1$, then

$$\begin{aligned}
\tilde{c}(x,\xi) = c(x,\tilde{\xi}) &= f_0(x,\tilde{\xi}) + \sum_{A \in P_F} \delta(A) \chi(A, x, \tilde{\xi}) \\
&= f_1(x,\xi) + \sum_{A \in P_F} \delta(A) \sum_{B \subset A} (-1)^{|B|} \chi(B, x, \xi) \\
&= f_1(x,\xi) + \sum_{B \in P_F} \tilde{\beta}(B) \chi(B, x, \xi),
\end{aligned}$$

where we have used (6.2) in the second equality. A similar argument applies if $\xi(x) = 1$, and this shows that $\tilde{c} = c^{\tilde{\beta},\tilde{\delta}}$. Clearly $(\tilde{\beta}, \tilde{\delta})$ also satisfies (P1) with the same n_1.

We have established that if $(\beta, \delta) \in \ell^1(P_F)^2$ satisfies (P), then $(\tilde{\beta}, \tilde{\delta})$ is also in $\ell^1(P_F)^2$ and satisfies (P1), (P2) and (P3). Here recall from Proposition 2.3 that under (P1) and (P2), (P3) is equivalent to monotonicity. It is easy to check that if (β, δ) also satisfies

$$\text{there is a constant } K_4 \text{ such that } \sum_{A\in P_F} \beta(A)\chi(A,0,\xi) \geq -K_4 f_1(0,\xi) \quad \forall \xi \in \{0,1\}^{\mathbf{Z}^d} \text{ such that } \xi(0)=1, \tag{P4$'$}$$

and

$$\sum_{A\in P_F} \delta(A) = 0. \tag{P5$'$}$$

then $(\tilde{\beta}, \tilde{\delta})$ satisfies (P4) (with the same K_4 as in (P4)$'$) and (P5).

Recall $\{\xi_0^q(x) : x \in \mathbf{Z}^d\}$ are iid Bernoulli random variables with $P(\xi_0(x) = 1) = q$ and $\tilde{\xi}_t(x) = 1 - \xi_t(x)$.

Theorem 6.1 *Assume $C \subset \{(\beta,\delta) \in \ell^1(P_F)^2 : \|(\beta,\delta)\|_1 \leq 1\}$ is relatively compact and (P), (P4)$'$ and (P5)$'$ hold uniformly on C. For $\eta > 0$, let C_η be the set of $(\beta,\delta) \in C$ such that*

$$\sum_{A\in P_F} \Big[\beta(A)\sigma(A) - (\beta(A)+\delta(A))\sigma(A\cup\{0\})\Big] \geq \eta,$$

and

$$\sum_{A\in P_F} \Big[\tilde{\beta}(A)\sigma(A) - (\tilde{\beta}(A)+\tilde{\delta}(A))\sigma(A\cup\{0\})\Big] \geq \eta.$$

Then there is an $r = r(\eta, S) \in (0,1)$ such that coexistence holds for all (β,δ) such that $\frac{(\beta,\delta)}{\|(\beta,\delta)\|_1} \in C_\eta$ and $0 < \|(\beta,\delta)\|_1 < r$. Moreover in this case there is a translation invariant probability μ such that

$$\sum_x \xi(x) = \sum_x \tilde{\xi}(x) = \infty \ \mu - \text{a.s.}$$

Proof For any initial ξ_0, Theorem I.1.8 of [9] shows that we may find a sequence $t_n \to \infty$ such that $\frac{1}{t_n}\int_0^{t_n} \xi_t dt \Rightarrow \xi_\infty$ as $n \to \infty$, and the law, μ, of ξ_∞ is a stationary distribution for $\xi_\cdot$. Furthermore, if the law of ξ_0 is translation invariant, then so is μ. We apply this in the case that ξ_0 is ξ_0^q for some $0 < q < 1$. Lemma 5.4 easily implies that

$$\mu\left(\sum_x \xi(x) = \infty\right) = 1.$$

The symmetry of our hypotheses allow us to reverse the roles of 0's and 1's in the above argument by considering $\tilde{\xi}_t^q$. Then $\tilde{\xi}_0^q(x), x \in \mathbf{Z}^d$ are iid with $P(\tilde{\xi}_0(x) = 1) = 1-q$, and (take a further subsequence if necessary) $\frac{1}{t_n}\int_0^{t_n} \tilde{\xi}_t^q \mathrm{d}t \Rightarrow \tilde{\mu}$, where $\tilde{\mu}(\xi \in A) = \mu(\tilde{\xi} \in \mu)$. By symmetry the same hypotheses are now satisfied by

$\tilde{\xi}^q$ (with $1-q$ in place of q) and so as before we obtain $\tilde{\mu}(\sum_x \xi(x) = \infty) = 1$, or $\mu(\sum_x 1(\xi(x) = 0) = \infty) = 1$. □

Corollary 6.2 *Assume* $0 < q < 1$, $\frac{(\beta,\delta)}{\|(\beta,\delta)\|_1} \in C_\eta$ *where* $0 < \|(\beta,\delta)\|_1 < r(\eta, S)$ *and* $r(\eta, S)$ *is as in Theorem 6.1. Let* ξ_t^q *denote the voter model perturbation under* $P^{\beta,\delta}$ *with* ξ_0^q *equal to the iid Bernoulli* (q) *random field. For any* $\varepsilon > 0$ *there are* $\ell_\varepsilon, t_\varepsilon > 0$ *such that*

$$P^{\beta,\delta}\left(\xi_t^q([-\ell_\varepsilon, \ell_\varepsilon]^d) > \frac{1}{\varepsilon},\ \tilde{\xi}_t^q([-\ell_\varepsilon, \ell_\varepsilon]^d) > \frac{1}{\varepsilon}\right) \geq 1 - \varepsilon \quad \textit{for all } t \geq t_\varepsilon.$$

Proof This is immediate from Lemma 5.4 and symmetry (as in the previous argument). □

7 Proof of (4.1)

We use the notation from Sect. 4. For $\mu \in \mathcal{M}_F$ of the form

$$(1/N) \sum_{x \in \mathbf{S_N}} \zeta\left(x\sqrt{N}\right) \delta_x$$

for some $\zeta \in \{0,1\}^{\mathbf{Z}^d}$, we will write $P_\mu^{\beta,\delta}(X_t^N \in \cdot) \equiv P_\mu(X_t^N \in \cdot)$ to refer to the law of the rescaled empirical process $X_t^N = \frac{1}{N}\sum_{x \in \mathbf{S_N}} \xi_{Nt}(x\sqrt{N})\delta_x$ with $\xi_0 = \zeta$. Hence ξ_t will be a generalized voter perturbation.

In view of (5.10), the fact that T, J, and $I_0 = I$ depend only on η, and our definition of N (recall (4.3)), we only need show there is an $r(\eta, S) > 0$ so that for $(\hat{\beta}, \hat{\delta}) \equiv \frac{(\beta,\delta)}{\|(\beta,\delta)\|_1} \in S^\eta$ and $\|(\beta,\delta)\|_1 < r(\eta, S)$,

$$P^{\beta,\delta}_{\frac{1}{N}\delta_0}(X_T^N(I) \geq J) \geq C/N, \tag{7.1}$$

where $C > 0$ is allowed to depend on (η, S) and hence on T, J, and I. By taking $r(\eta, S) \leq 1/16$, as in the proof of (4.3), we may, and shall, assume (β, δ) satisfies (4.4) and (4.5). Recall that $I = I_0 = [-L, L]^d$ for some $L \in \mathbf{N}$.

We proceed to prove (7.1) by contradiction, and so assume there is a sequence (β_k, δ_k) such that each $(\hat{\beta}_k, \hat{\delta}_k) \in S_\eta$, $\|(\beta_k, \delta_k)\|_1 \to 0$ (hence also $N_k \to \infty$), and

$$N_k\, P^{\beta_k,\delta_k}_{\frac{1}{N_k}\delta_0}(X_T^{N_k}(I) \geq J) \to 0 \quad \text{as } k \to \infty. \tag{7.2}$$

Here, as in Proposition 4.2, $N_k = \lfloor \|(\beta_k, \delta_k)\|_1^{-1/2} \rfloor^2$. Furthermore, by the relative compactness of S we may assume without loss of generality that for some $(\beta, \delta) \in \ell^1(P_F)^2$, $(\hat{\beta}_k, \hat{\delta}_k) \to (\beta, \delta)$ in $\ell^1(P_F)^2$ as $k \to \infty$, and so $N_k(\beta_k, \delta_k) \to (\beta, \delta)$ in the same space. We claim, by taking a further subsequence if necessary, that for all $0 < \varepsilon < J$,

$$N_k P^{\beta_k,\delta_k}_{\frac{1}{N_k}\delta_0}(X^{N_k}_{T/2}(I) \geq \varepsilon) \to 0 \quad \text{as } k \to \infty. \tag{7.3}$$

We first show how (7.3) leads to a contradiction, and then return to the derivation of (7.3) from (7.2).

As in the proof of Lemma 4.4, one easily checks that $(\beta^k_{N_k}, \delta^k_{N_k}) = N_k(\beta_k, \delta_k)$ satisfies the hypotheses of Theorem C, using the fact that $(\hat{\beta}_k, \hat{\delta}_k) \in S_\eta$. Theorem C and the fact that $X_{T/2}(\partial I) = 0$ a.s. imply that if $\{X^{N,i}, i \leq N\}$ are as in Theorem C, then

$$\sum_{i=1}^{N_k} X^{N_k,i}_{T/2}(I) \Rightarrow X_{T/2}(I),$$

where X is the super Brownian motion starting at δ_0 in Theorem C. Given (7.3), it follows that

$$P\left(\max_{i \leq N_k} X^{N_k,i}_{T/2}(I) \geq \varepsilon\right) \to 0 \quad \text{as } k \to \infty.$$

These last two facts imply

$$\sum_{i=1}^{N_k} X^{N_k,i}_{T/2}(I) \wedge \varepsilon \Rightarrow X_{T/2}(I) \quad \text{as } k \to \infty. \tag{7.4}$$

Using the independence of the $X^{N,i}_{\cdot}$ one sees that

$$\begin{aligned} \mathrm{Var}\left(\sum_{i=1}^{N_k} X^{N_k,i}_{T/2}(I) \wedge \varepsilon\right) &= \sum_{i=1}^{N_k} \mathrm{Var}(X^{N_k,i}_{T/2}(I) \wedge \varepsilon) \\ &\leq \sum_{i=1}^{N_k} E_{\frac{1}{N_k}\delta_0}((X^{N_k,i}_{T/2}(I) \wedge \varepsilon)^2) \\ &\leq \varepsilon \sum_{i=1}^{N_k} E_{\frac{1}{N_k}\delta_0}(X^{N_k,i}_{T/2}(I)). \end{aligned} \tag{7.5}$$

An elementary argument using Proposition 2.3 of [5] with $\phi = 1$ and Gronwall's lemma gives

$$E_{\frac{1}{N_k}\delta_0}(X^{N_k,i}_{T/2}(\mathbf{1})) \leq \mathrm{e}^{c_0 T} \frac{1}{N_k},$$

where c_0 is a universal constant thanks to the uniform bound (4.5). Therefore (7.5) implies $\mathrm{Var}(\sum_{i=1}^{N_k} X^{N_k,i}_{T/2}(I) \wedge \varepsilon) \leq \mathrm{e}^{c_0 T}\varepsilon$. By (7.4), Fatou's lemma and Skorokhod's a.s. representation theorem, we get $\mathrm{Var}(X_{T/2}(I)) \leq \mathrm{e}^{c_0 T}\varepsilon$ and as ε is arbitrary we have proved $X_{T/2}(I)$ is a constant a.s. This contradicts the fact that it has a positive variance (eg. by Exercise II.5.2 of [12]) since it's initial measure (δ_0) is non-zero. Therefore, (7.2) cannot hold. □

We now prove (7.3). By Cantor diagonalization we may fix $\varepsilon \in (0, J)$. Also we may assume the probability on the left-hand side of (7.3) is positive for all but finitely many k, or the conclusion is trivial. In the following argument, we consider realizations $\xi_t^0 = \zeta \in \{0,1\}^{\mathbf{Z}^d}$ such that $\zeta(I_0^{N_k}) \geq N_k\varepsilon$. For such a ζ we can choose, using some lexicographical order, $F_k(\zeta) = \widehat{\zeta} \leq \zeta$ such that $N_k\varepsilon \leq \widehat{\zeta}(I_0^{N_k}) \leq N_k J$ and $\widehat{\zeta}((I_0^{N_k})^c) = 0$. More formally we define an appropriate

$$\begin{aligned} F_k : \{\zeta \in \{0,1\}^{\mathbf{Z}^d} : \zeta(I_0^{N_k}) \geq N_k\varepsilon\} \\ \to \{\zeta \in \{0,1\}^{\mathbf{Z}^d} : N_k J \geq \zeta(I_0^{N_k}) \geq N_k\varepsilon,\ \zeta(x) = 0\ \forall x \notin I_0^{N_k}\} \end{aligned}$$

such that $F_k(\zeta) \leq \zeta$ for all ζ in the domain of F_k. The monotonicity given by Proposition 2.3 and scaling implies

$$P_\zeta^{\beta_k,\delta_k}(\xi_{N_k T/2}(I_0^{N_k}) \geq N_k J) \geq P_{F_k(\zeta)}^{\beta_k,\delta_k}(\xi_{N_k T/2}(I_0^{N_k}) \geq N_k J).$$

This inequality and the Markov property imply that

$$\begin{aligned} &P_{\frac{1}{N_k}\delta_0}^{\beta_k,\delta_k}(X_{T/2}^{N_k}(I_0) \geq \varepsilon, X_T^{N_k}(I_0) \geq J) \\ &\geq \int P^{\beta_k,\delta_k}(\xi_{N_k T/2}^0 \in \mathrm{d}\zeta) 1(\zeta(I_0^{N_k}) \geq N_k\varepsilon) P_{F_k(\zeta)}^{\beta_k,\delta_k}(\xi_{N_k T/2}(I_0^{N_k}) \geq N_k J). \end{aligned}$$

If we now adopt the notation $\widehat{X}_t^{N_k} = (1/N_k)\sum_{x\in\mathbf{S_N}} F_k(\xi_{N_k t})(x\sqrt{N})\delta_x$, and define (recall the conditioning event below has positive probability or we are done)

$$\nu_k(\cdot) = P_{\frac{1}{N_k}\delta_0}(\widehat{X}_{T/2}^{N_k} \in \cdot \mid X_{T/2}^{N_k}(I_0) \geq \varepsilon),$$

then the previous inequality can be written as

$$P_{\frac{1}{N_k}\delta_0}^{\beta_k,\delta_k}(X_T^{N_k}(I) \geq J \mid X_{T/2}^{N_k}(I) \geq \varepsilon) \geq \int_{\mathcal{M}_F} \nu_k(\mathrm{d}\mu) P_\mu^{\beta_k,\delta_k}(X_{T/2}^{N_k}(I) \geq J). \quad (7.6)$$

By construction, ν_k is concentrated on

$$\mathcal{M}'_F = \{\mu \in \mathcal{M}_F : \varepsilon \leq \mu(I_0) \leq J \quad \text{and} \quad \mu(I^c) = 0\}.$$

Since $\mathcal{M}'_F$ is compact we may suppose, by taking a subsequence, that $\nu_k \Rightarrow \nu \in \mathcal{M}_F$.

Let $\phi : \mathbf{R}^d \to [0,1]$ be continuous and satisfy $1_I \geq \phi \geq 1_{\underline{I}}$, where $\underline{I} = [-L + .5, L - .5]^d$, and let $\psi : \mathbf{R} \to [0,1]$ be a continuous non-decreasing function satisfying $1_{[J,\infty)} \geq \psi \geq 1_{[J+1,\infty)}$. Observe that the right-side of (7.6) is bounded below by $\int_{\mathcal{M}_F} \nu_k(d\mu) E_\mu(\psi(X^{N_k}_{T/2}(\phi)))$. By Theorem B, which applies as in the previous part of the proof,

$$\int_{\mathcal{M}_F} \nu_k(\mathrm{d}\mu) E_\mu(\psi(X^{N_k}_{T/2}(\phi))) \to \int_{\mathcal{M}_F} \nu(\mathrm{d}\mu) E_\mu(\psi(X_{T/2}(\phi)))$$

as $k \to \infty$. We may therefore conclude that

$$\begin{aligned} p_0 &\equiv \inf\{E_\mu(\psi(X_{T/2}(\phi))) : \mu \in \mathcal{M}_F, J \geq \mu(I) \geq \varepsilon, \mu(I^c) = 0\} \\ &\leq \liminf_{k\to\infty} P^{\beta_k,\delta_k}_{\frac{1}{N_k}\delta_0}(X^{N_k}_T(I) \geq J | X^{N_k}_{T/2}(I) \geq \varepsilon). \end{aligned} \tag{7.7}$$

The inf defining p_0 is attained at some non-zero μ_0, as it is the minimum of a continuous function on a compact set of non-zero measures. If $p_0 = 0$, then $X_{T/2}(\underline{I}) \leq J + 1 P_{\mu_0}$−a.s., which is impossible as $X_{T/2}(\underline{I})$ is a non-constant, non-negative infinitely divisible random variable (see eg. the beginning of Sect. II.7 of [12]). Therefore, $p_0 > 0$, and so (7.7) and (7.2) imply the claim (7.3). This completes the proof of (7.1) and hence (4.1). □

8 Application to the Lotka–Volterra Models

In this section we apply our general perturbation results to derive the theorems in Sect. 1 on the stochastic Lotka–Volterra models. We in fact will work with the more general multikernel Lotka–Volterra models with rates given by (1.16) for some $\alpha_0, \alpha_1 \geq 0$ and probability kernels p^b, p^d on $\mathbf{Z}^d$ such that $p^b(0) = p^d(0) = 0$. One may easily check (see Corollary 1.10 of [5]) that these rates correspond to voter model perturbations (i.e. are as in (2.6)) with

$$\beta(A) \equiv \beta_{\alpha_0}(A) = (\alpha_0 - 1)\begin{cases} p(a)p^b(a), & A = \{a\} \\ (p(a)p^b(a') + p(a')p^b(a)), & A = \{a,a'\},\ a \neq a' \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta(A) \equiv \delta_{\alpha_1}(A) = (\alpha_1 - 1)\begin{cases} 1, & A = \emptyset \\ (p(a)p^d(a) - p(a) - p^d(a)), & A = \{a\} \\ (p(a)p^d(a') + p(a')p^d(a)), & A = \{a,a'\},\ a \neq a' \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $(\beta_{\alpha_0}, \delta_{\alpha_1}) \in \ell^1(P_F)^2$ and it is easy to see that

$$\begin{aligned} \|(\beta,\delta)\|_1 &= |\alpha_0 - 1| + 2|\alpha_1 - 1|(2 - \sum_a p(a)p_d(a)) \\ &\in [|\alpha_0 - 1| + 2|\alpha_1 - 1|, |\alpha_0 - 1| + 4|\alpha_1 - 1|]. \end{aligned} \tag{8.1}$$

Conditions (P5) and (P1) (with $n_1 = 2$) are immediate and (P2) is clear from the original definition of the rates in Sect. 1. Condition (P4), with $K_4 = 1$ is checked as in Sect. 1 of [5] where it is verified in the case $p^b = p^d = p$ and left as an exercise in general. Here it amounts to noting that $(\alpha_1 - 1)f_0 f_0^b \geq -f_0$.

Finally consider the monotonicity condition (P3). A bit of algebra, which is best left for the reader, shows that the stronger condition (P3)′ is equivalent to

$$\begin{aligned} &\alpha_0 \geq 1 - \left(1 + \frac{p^b(a)}{p(a)} - p^b(a)\right)^{-1} \quad \text{and} \\ &\alpha_1 \geq 1 - \left(1 + \frac{p^d(a)}{p(a)} - p^d(a)\right)^{-1} \quad \forall a \in \mathbf{Z}^d. \end{aligned} \tag{8.2}$$

Here it is understood that $\frac{0}{0} = 0$ and otherwise the usual rules apply for division by 0 and ∞. This condition is obvious if $\alpha_0, \alpha_1 \geq 1$. To allow $\alpha_1 < 1$ we assume there is a finite constant $C_{8.3}$ such that

$$p^b(a) \vee p^d(a) \leq C_{8.3} p(a) \quad \text{for all } a \in \mathbf{Z}^d. \tag{8.3}$$

Under this condition, (8.2) becomes

$$\alpha_0 \geq \underline{\alpha}_0 \quad \text{and} \quad \alpha_1 \geq \underline{\alpha}_1, \tag{8.4}$$

where

$$\underline{\alpha}_0 = 1 - \left[1 + \sup\left\{\frac{p^b(a)}{p(a)} - p^b(a) : p(a) > 0\right\}\right]^{-1},$$

and

$$\underline{\alpha}_1 = 1 - \left[1 + \sup\left\{\frac{p^d(a)}{p(a)} - p^d(a) : p(a) > 0\right\}\right]^{-1}$$

will satisfy [by (8.3)]

$$\underline{\alpha}_0 \vee \underline{\alpha}_1 \leq 1 - (1 + C_{8.3})^{-1} < 1. \tag{8.5}$$

We have now proved

Proposition 8.1 *Assume* (8.3). *Then* (P) *holds uniformly on*

$$S' = \{(\beta_{\alpha_0}, \delta_{\alpha_1}) : \alpha_0 \geq \underline{\alpha}_0, \alpha_1 \geq \underline{\alpha}_1\}.$$

As has already been noted (Corollary 2.4 and Proposition 2.1) there is a unique $\{0,1\}^{\mathbf{Z}^d}$-valued monotone Feller process ξ_t, whose generator is determined by these rates as in Proposition 2.1(c). We say ξ_t is $\mathrm{LV}(\alpha_0, \alpha_1, p^b, p^d)$. It will be convenient to slightly strengthen the monotonicity.

Proposition 8.2 *Assume* (8.3). *Let* $0 \leq \alpha_0' \leq \alpha_0$, $0 \leq \alpha_1 \leq \alpha_1'$, *and assume either* $\alpha_i \geq \underline{\alpha}_i$, $i = 0, 1$, *or* $\alpha_i' \geq \underline{\alpha}_i$, $i = 0, 1$. *If* ξ_t *is* $\mathrm{LV}(\alpha_0, \alpha_1, p^b, p^d)$ *and* ξ_t' *is* $\mathrm{LV}(\alpha_0', \alpha_1', p^b, p^d)$ *with* $\xi_0 \geq \xi_0'$, *then* ξ_t *stochastically dominates* ξ_t'.

Proof Let $c_i(x,\xi)$ and $c_i'(x,\xi)$ be the spin-flip rates of ξ_t and ξ_t', respectively. By Theorem III.1.5 of [9], it suffices to show that for $\xi' \leq \xi$,

$$c_1'(x,\xi') \leq c_1(x,\xi) \quad \text{if } \xi(x) = 0,$$

and

$$c_0'(x,\xi') \geq c_0(x,\xi) \quad \text{if } \xi'(x) = 1.$$

Assume without loss of generality that $\alpha_i' \geq \underline{\alpha}_i$ for $i = 0, 1$. Then ξ_t' is monotone by the previous discussion. Therefore by Theorem III.2.2 of [9], $c_1'(x,\xi') \leq c_1'(x,\xi)$ if $\xi(x) = 0$ and $c_0'(x,\xi') \geq c_0'(x,\xi)$ if $\xi'(x) = 1$. Hence it suffices to show

$$c_1'(x,\xi) \leq c_1(x,\xi) \quad \text{if } \xi(x) = 0,$$

and

$$c_0'(x,\xi) \geq c_0(x,\xi) \quad \text{if } \xi(x) = 1.$$

The formulae for c_1 and c_0 (i.e., (1.16)) show that c_1 and c_0 are non-decreasing functions of α_0 and α_1, respectively, and these last two inequalities are then immediate. □

Using the notation from Sect. 1, if $e, e' \in \mathbf{Z}^d - \{0\}$, define

$$p_1(e,e') = P(\tau(e,e') < \infty, \tau(0,e) = \tau(0,e') = \infty),$$

and

$$p_2(e,e') = P(\tau(0,e) = \tau(0,e') = \infty).$$

Introduce

$$\beta' = \sum_{e,e' \in \mathbf{Z}^d} p(e)p^b(e')p_1(e,e'), \quad \delta' = \sum_{e,e' \in \mathbf{Z}^d} p(e)p^d(e')p_2(e,e'),$$

and $m_0' = \frac{\beta'}{\delta'}$.

Notation If $0 < \eta < m_0'$, let

$$f_\eta(\alpha_0) = 1 + \begin{cases} (m_0' - \eta)(\alpha_0 - 1) & \text{if } \alpha_0 \geq 1 \\ (m_0' + \eta)(\alpha_0 - 1) & \text{if } \alpha_0 < 1, \end{cases}$$

and

$$\hat{S}_\eta = \{(\alpha_0, \alpha_1) \in [0,\infty)^2 : (\alpha_0, \alpha_1) \neq (1,1), \alpha_1 - 1 \leq f_\eta(\alpha_0)\}.$$

Theorem 8.3 *Let ξ_t be $\mathrm{LV}(\alpha_0, \alpha_1, p^b, p^d)$ under P^α and assume (8.3) holds. If $0 < \eta < m_0'$ there is an $r(\eta) > 0$ and $C_{8.6}(\eta) > 0$ such that if $|\alpha_0 - 1| \leq r(\eta)$ and $(\alpha_0, \alpha_1) \in \hat{S}_\eta$, then*

$$P^\alpha(|\xi_t^0| > 0 \quad \text{for all } t \geq 0) \geq C_{8.6}(\eta)[|\alpha_0 - 1| + (|\alpha_1 - 1| \wedge r(\eta))], \tag{8.6}$$

and in particular survival holds for such (α_0, α_1).

Proof We apply Theorem 4.1 with

$$S = \{(\beta_{\alpha_0}, \delta_{\alpha_1}) : \alpha_0 \geq \underline{\alpha}_0, \alpha_1 \geq \underline{\alpha}_1, |\alpha_0 - 1| + 4|\alpha_1 - 1| \leq 1\}.$$

S is the image in $\ell^1(P_F)^2$ of $\{(\alpha_0, \alpha_1) : \alpha_0 \geq \underline{\alpha}_0, \alpha_1 \geq \underline{\alpha}_1, |\alpha_0 - 1| + 4|\alpha_1 - 1| \leq 1\}$ under a continuous map, and hence is a compact subset of the unit ball in $\ell^1(P_F)^2$, the last inclusion by (8.1). Implicit in the proof of Corollary 1.10 of [5] is the fact that

$$\begin{aligned} \theta(\alpha) &= \sum_{A \in P_F} [\beta_{\alpha_0}(A)\sigma(A) - (\beta_{\alpha_0}(A) + \delta_{\alpha_1}(A))\sigma(A \cup \{0\})] \\ &= (\alpha_0 - 1)\beta' - (\alpha_1 - 1)\delta'. \end{aligned} \tag{8.7}$$

Let $\|\alpha\|$ denotes $\|(\beta_{\alpha_0}, \delta_{\alpha_1})\|_1$. Proposition 8.1 allows us to apply Theorem 4.1 and so conclude from (8.7) that for $\eta' > 0$ there exists $r'(\eta') \in (0,1)$ and $C_{8.8}(\eta') > 0$ such that

$$0 < \|\alpha\| \leq r'(\eta') \quad \text{and} \quad \frac{\theta(\alpha)}{\|\alpha\|} \geq \eta' \text{ imply } P^\alpha(|\xi_t^0| > 0 \; \forall t \geq 0) \geq C_{8.8}(\eta')\|\alpha\|. \tag{8.8}$$

Fix $0 < \eta < m'_0$. For $(\alpha_0, \alpha_1) \in \hat{S}_\eta$, $\alpha_0 \neq 1$, define $m = (\alpha_1 - 1)/(\alpha_0 - 1)$. Then $m \le m'_0 - \eta$ or $m \ge m'_0 + \eta$, according as $\alpha_0 > 1$ or $\alpha_0 < 1$, respectively, and by the upper bound on $\|\alpha\|$ in (8.1) and a bit of arithmetic,

$$\frac{\theta(\alpha)}{\|\alpha\|} \ge \delta' \mathrm{sgn}(\alpha_0 - 1) \frac{m'_0 - m}{1 + 4|m|}. \tag{8.9}$$

As a function of m, $(m'_0 - m)/(1 + 4|m|)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Since $\eta < m'_0$, this implies the right side above cannot be smaller than $\eta\delta'/(1 + 4m'_0)$. Also, if $\alpha_0 = 1$ and $(\alpha_0, \alpha_1) \in \hat{S}_\eta$, then $\alpha_1 < 1$ and $\theta(\alpha)/\|\alpha\| \ge \delta'/4 \ge \delta'\eta/(1 + 4m'_0)$. Therefore, for $0 < \eta < m'_0$, if we set $\eta' = \delta'\eta/(1 + 4m_0)$, $r_0(\eta) = r'(\eta')/8$ and $C_{8.10}(\eta) = C_{8.8}(\eta')$, we have (using (8.1)),

$$\begin{aligned}&(\alpha_0, \alpha_1) \in \hat{S}_\eta \quad \text{and} \quad |\alpha_0 - 1| + |\alpha_1 - 1| < 2r_0(\eta) \text{ implies} \\ &P^\alpha(|\xi^0_t| > 0\ \forall t \ge 0) \ge C_{8.10}(\eta)[|\alpha_0 - 1| + |\alpha_1 - 1|].\end{aligned} \tag{8.10}$$

By decreasing $r_0(\eta)$ we also may assume

$$1 - r_0(\eta) \ge \underline{\alpha}_0 \vee \underline{\alpha}_1. \tag{8.11}$$

Finally choose $r(\eta) > 0$ small enough so that $r(\eta) \le r_0(\eta)$, and

$$\begin{aligned}[1 - r(\eta), 1 + r(\eta)] \times [0, 1 - r_0(\eta)] &\subset \hat{S}_\eta \cap ([1 - r(\eta), 1 + r(\eta)] \times [0, \infty)) \\ &\subset [1 - r(\eta), 1 + r(\eta)] \times [0, 1 + r_0(\eta)).\end{aligned} \tag{8.12}$$

Assume $(\alpha_0, \alpha_1) \in \hat{S}_\eta$ and $|\alpha_0 - 1| \le r(\eta)$. If $|\alpha_1 - 1| < r_0(\eta)$, then the hypotheses of (8.10) hold and that result gives the desired conclusion. Assume next that $|\alpha_1 - 1| \ge r_0(\eta)$. The second inclusion in (8.12) implies $\alpha_1 \le 1 - r_0(\eta) \equiv \alpha'_1$ and we may apply Proposition 8.2 with $\alpha'_0 = \alpha_0$ because by our choice of $r(\eta)$ and (8.11), $\alpha'_0 = \alpha_0 \ge \underline{\alpha}_0$ and $\alpha'_1 \ge \underline{\alpha}_1$. The first inclusion in (8.12) shows that $(\alpha'_0, \alpha'_1) \in \hat{S}_\eta$ and so Proposition 8.2 and (8.10) imply that

$$\begin{aligned}P^\alpha(|\xi^0_t| > 0\ \forall t \ge 0) \ge P^{\alpha'}(|\xi^0_t| > 0\ \forall t \ge 0) &\ge C_{8.10}(\eta)[|\alpha'_0 - 1| + |\alpha'_1 - 1|] \\ &= C_{8.10}(\eta)[|\alpha'_0 - 1| + r_0(\eta)] \\ &\ge C_{8.10}(\eta)[|\alpha_0 - 1| + r(\eta) \wedge |\alpha_1 - 1|].\end{aligned}$$

This completes the proof in either case. □

As an immediate application of the inclusion established in the above argument, the above comparison argument and Proposition 5.3 we also obtain.

Corollary 8.4 *Let ξ_t be* $\mathrm{LV}(\alpha_0,\alpha_1,p^b,p^d)$ *under P^α and assume* (8.3) *holds. Let $r(\eta)>0$ be as in Theorem* 8.3 *and assume that for some $0<\eta<m_0'$, $(\alpha_0,\alpha_1)\in\hat{S}_\eta$ and $|\alpha_0-1|<r(\eta)$. Then there is a $p_0=p_0(\alpha_0,\alpha_1)>0$ such that $P^\alpha(\xi_t^0(0)=1)\ge p_0$ for all $t\ge 0$.*

Let

$$\beta''=\sum_{e\in\mathbf{Z}^d}\sum_{e'\in\mathbf{Z}^d}p(e)p^d(e')p_1(e,e') \quad\text{and}\quad \delta''=\sum_{e\in\mathbf{Z}^d}\sum_{e'\in\mathbf{Z}^d}p(e)p^b(e')p_2(e,e'),$$

and $m_0''=\frac{\beta''}{\delta''}$. Note here we have reversed the roles of p^b and p^d from the definitions of β' and δ'. The facts that $p_2(e,e')\ge p_1(e,e')$ with strict inequality for $e\ne e'$ and that $supp(p)$ contains at least two points (by symmetry and $p(0)=0$) implies

$$m_0'm_0''<1. \tag{8.13}$$

Define

$$C=\{(\beta_{\alpha_0},\delta_{\alpha_1}):\alpha_i\in[\underline{\alpha}_i,1],\ i=0,1,\ |\alpha_0-1|+4|\alpha_1-1|\le 1\},$$

and

$$\hat{C}_\eta=\{(\alpha_0,\alpha_1)\in[0,1]^2:(m_0''+\eta)^{-1}(\alpha_0-1)\le\alpha_1-1\le(m_0'-\eta)(\alpha_0-1)\}.$$

By (8.13) for $\eta>0$ small enough, $\hat{C}_\eta$ contains infinitely many points in every neighbourhood of $(1,1)$.

Theorem 8.5 *Let ξ_t be* $\mathrm{LV}(\alpha_0,\alpha_1,p^b,p^d)$ *under P^α and assume* (8.3) *holds. For each $0<\eta<m_0'$, there is an $r(\eta)>0$ so that coexistence holds for all $(\alpha_0,\alpha_1)\in\hat{C}_\eta$ so that $1-\alpha_0<r(\eta)$.*

Proof We apply Theorem 6.1 to the above set C which as in the proof of Theorem 8.3 is a compact subset of the unit ball in $\ell^1(P_F)^2$. We have

$$\tilde{c}_0(x,\xi)=c_1(x,\tilde{\xi})=f_0(x,\xi)+(\alpha_0-1)f_0(x,\xi)f_0^b(x,\xi),$$

and

$$\tilde{c}_1(x,\xi)=c_0(x,\tilde{\xi})=f_1(x,\xi)+(\alpha_1-1)f_1(x,\xi)f_1^d(x,\xi).$$

It is now easy to check (P4)$'$ holds with $K_4=1$, just as for (P4), and it is also trivial to check (P5)$'$. Hence, as before, C satisfies the hypotheses of Theorem 6.1. We may again easily check that if $\tilde{\beta}$ and $\tilde{\delta}$ are defined as in Sect. 6 using the current rates, then

$$\sum_A[\tilde{\beta}(A)\sigma(A)-(\tilde{\beta}(A)+\tilde{\delta}(A))\sigma(A\cup\{0\})]=(\alpha_1-1)\beta''-(\alpha_0-1)\delta''.$$

The result now follows from Theorem 6.1 by means of an easy computation similar to that in the proof of Theorem 8.3. In fact there is some simplification now as there is no need to use the comparison result (Proposition 8.2) since making $1-\alpha_0$ small for $(\alpha_0,\alpha_1)\in\hat{C}_\eta$ forces $|1-\alpha_1|=1-\alpha_1$ to be small (and hence also $\|(\beta_{\alpha_0},\delta_{\alpha_1})\|_1$ small.). Note also we have not had to exclude (1,1) from $\hat{C}_\eta$ since coexistence for the voter model in more than two dimensions is well-known (e.g. Corollary V.1.13 of [9]). □

An application of Corollary 6.2 in the above setting gives us the following result.

Corollary 8.6 *Let ξ_t^q be* $\mathrm{LV}(\alpha_0,\alpha_1,p^b,p^d)$ *under P^α with initial condition ξ_0^q for some $0<q<1$ and assume* (8.3) *holds. Let $r(\eta)>0$ be as in Theorem 8.5 and assume that for some $0<\eta<m_0'$, $(\alpha_0,\alpha_1)\in\hat{C}_\eta$ and $|\alpha_0-1|<r(\eta)$. For any $\varepsilon>0$ there are positive ℓ_ε, t_ε such that*

$$P^\alpha\left(\Big(\sum_{x\in B(\ell_\varepsilon)}\xi_t^q(x)\Big)\wedge\Big(\sum_{x\in B(\ell_\varepsilon)}(1-\xi_t^q(x))\Big)\geq\frac{1}{\varepsilon}\right)\geq 1-\varepsilon\quad \textit{for all } t\geq t_\varepsilon.$$

Proofs of Theorem 1 Corollary 2 Corollary 3 Theorem 4 and Corollary 5 We simply apply the above results in the setting where $p^b=p^d=p$. In this case (8.3) is trivial with $C_{8.3}=1$ and so (8.5) implies $\underline{\alpha}_i\geq 1/2$, and we may replace $\underline{\alpha}_i$ with $1/2$ in Propositions 8.1 and 8.2. We also have $m_0'=m_0''=m_0\in(p_*,1)$ (see Sect. 1). Theorem 1, Corollary 2, Corollary 3, Theorem 4 and Corollary 5 are therefore special cases of Theorem 8.3, Corollary 8.4, (8.6), Theorem 8.5, and Corollary 8.6, respectively. □

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References

1. Blath, J., Etheridge, A.M., Meredith, M.E.: Coexistence in locally regulated competing populations. Preprint (2004)
2. Bramson, M., Griffeath, D.: On the Williams–Bjerknes tumour growth model. I. Ann. Probab. **9**, 173–185 (1981)
3. Bramson, M., Neuhauser, C.: Coexistence for a catalytic surface reaction model. Ann. Appl. Probab. **7**, 565–614 (1997)
4. Cox, J.T., Durrett, R., Perkins, E.A.: Rescaled voter models converge to super-Brownian motion. Ann. Probab. **28**, 185–234 (2000)
5. Cox, J.T., Perkins, E.A.: Rescaled Lotka–Volterra models converge to super-Brownian motion. Ann. Probab. **33**, 904–947 (2005)
6. Cox, J.T., Perkins, E.A.: Renormalization of the two-dimensional Lotka–Volterra Model. Preprint (2006)
7. Durrett, R.: Ten lectures on particle systems. Ecole d'été de Probabilités de St. Flour, XXIII. Lecture Notes in Math **1608**, pp. 97–201. Springer, Berlin Heidelberg New York (1995)
8. Durrett, R., Perkins, E.: Rescaled contact processes converge to super-Brownian motion for $d\geq 2$. Probab. Theory Rel. Fields **114**, 309–399 (1998)
9. Liggett, T.L.: Interacting particle systems. Springer, Berlin Heidelberbg New York (1985)

10. Liggett, T.L.: Interacting stochastic systems: contact, voter and exclusion processes. Springer, Berlin Heidelberg New York (1999)
11. Neuhauser, C., Pacala, S.W.: An explicitly spatial version of the Lotka–Volterra model with interspecific competition. Ann. Appl. Probab. **9**, 1226–1259 (1999)
12. Perkins, E.: Measure-valued processes and interactions. Pages 125-324, in École d'Été de Probabilités de Saint Flour XXIX-1999, Lecture Notes in Mathematics, **1781**, pp. 97–201. Springer, Berlin Heidelberg New York (2002)