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A subdiffusive behaviour of recurrent random walk in random environment on a regular tree

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Abstract We are interested in the random walk in random environment on an infinite tree. Lyons and Pemantle (Ann. Probab. **20**, 125–136, 1992) give a precise recurrence/transience criterion. Our paper focuses on the almost sure asymptotic behaviours of a recurrent random walk (X_n) in random environment on a regular tree, which is closely related to Mandelbrot's (C. R. Acad. Sci. Paris **278**, 289–292, 1974) multiplicative cascade. We prove, under some general assumptions upon the distribution of the environment, the existence of a new exponent $\nu \in \left(0, \frac{1}{2}\right]$ such that $\max_{0\le i\le n} |X_i|$ behaves asymptotically like n^{ν}. The value of ν is explicitly formulated in terms of the distribution of the environment.

Keywords Random walk · Random environment · Tree · Mandelbrot's multiplicative cascade

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1 Introduction

Random walk in random environment (RWRE) is a fundamental object in the study of random phenomena in random media. RWRE on $\mathbb{Z}$ exhibits rich regimes in the transient case [6], as well as a slow logarithmic movement in the recurrent case [23]. On $\mathbb{Z}^d$ (for $d \geq 2$), the study of RWRE remains a big challenge to mathematicians [24,25]. The present paper focuses on RWRE on a regular rooted tree, which can be viewed as an infinite-dimensional RWRE. Our main result reveals a rich regime à la Kesten–Kozlov–Spitzer, but this time even in the recurrent case; it also strongly suggests the existence of a slow logarithmic regime à la Sinai.

Let $\mathbb{T}$ be a b-ary tree ($\mathrm{b} \geq 2$) rooted at e. For any vertex $x \in \mathbb{T}\backslash\{e\}$, let $\overleftarrow{x}$ denote the first vertex on the shortest path from x to the root e, and $|x|$ the number of edges on this path (notation: $|e| := 0$). Thus, each vertex $x \in \mathbb{T}\backslash\{e\}$ has one parent $\overleftarrow{x}$ and b children, whereas the root e has b children but no parent. We also write $\overleftarrow{\overleftarrow{x}}$ for the parent of $\overleftarrow{x}$ (for $x \in \mathbb{T}$ such that $|x| \geq 2$).

Let $\omega := (\omega(x,y),\, x, y \in \mathbb{T})$ be a family of non-negative random variables such that $\sum_{y\in\mathbb{T}} \omega(x,y) = 1$ for any $x \in \mathbb{T}$. Given a realization of ω, we define a Markov chain $X := (X_n,\, n \geq 0)$ on $\mathbb{T}$ by $X_0 = e$, and whose transition probabilities are

$$P_\omega(X_{n+1} = y \,|\, X_n = x) = \omega(x,y).$$

Let $\mathbf{P}$ denote the distribution of ω, and let $\mathbb{P}(\cdot) := \int P_\omega(\cdot)\mathbf{P}(\mathrm{d}\omega)$. The process X is a $\mathbb{T}$-valued RWRE. (By informally taking $\mathrm{b} = 1$, X would become a usual RWRE on the half-line $\mathbb{Z}_+$.)

We assume the existence of $\varepsilon_0 > 0$ such that $\omega(x,y) \geq \varepsilon_0$ if either $x = \overleftarrow{y}$ or $y = \overleftarrow{x}$, and $\omega(x,y) = 0$ otherwise; in words, (X_n) is a nearest-neighbour walk, satisfying a (uniform) ellipticity condition.

For general properties of tree-valued processes, we refer to Peres [19] and Lyons and Peres [12]. See also Duquesne and Le Gall [2] and Le Gall [7] for continuous random trees. For a list of motivations to study RWRE on a tree, see Pemantle and Peres [17], p. 106.

We define

$$A(x) := \frac{\omega(\overleftarrow{x}, x)}{\omega(\overleftarrow{x}, \overleftarrow{\overleftarrow{x}})}, \quad x \in \mathbb{T}, \;\; |x| \geq 2. \tag{1.1}$$

Following Lyons and Pemantle [11], we assume throughout the paper that $(\omega(x,\bullet))_{x\in\mathbb{T}\backslash\{e\}}$ is a family of i.i.d. *non-degenerate* random vectors and that $(A(x),\; x \in \mathbb{T},\; |x| \geq 2)$ are identically distributed.

Let A denote a generic random variable having the common distribution of $A(x)$ (for $|x| \geq 2$). Define

$$p := \inf_{t\in[0,1]} \mathbf{E}(A^t). \tag{1.2}$$

We recall a recurrence/transience criterion from Lyons and Pemantle [11] (Theorem 1 and Proposition 2).

Theorem A (Lyons and Pemantle [11]) *With $\mathbb{P}$-probability one, the walk (X_n) is recurrent or transient, according to whether $p \le \frac{1}{\mathfrak{b}}$ or $p > \frac{1}{\mathfrak{b}}$. It is, moreover, positive recurrent if $p < \frac{1}{\mathfrak{b}}$.*

We study the recurrent case $p \le \frac{1}{\mathfrak{b}}$ in this paper. Our first result, which is not deep, concerns the positive recurrent case $p < \frac{1}{\mathfrak{b}}$.

Theorem 1.1 *If $p < \frac{1}{\mathfrak{b}}$, then*

$$\lim_{n\to\infty} \frac{1}{\log n} \max_{0\le i\le n} |X_i| = \frac{1}{\log[1/(q\mathfrak{b})]}, \quad \mathbb{P}\text{-a.s.}, \tag{1.3}$$

where the constant q is defined in (2.1), *and lies in* $(0, \frac{1}{\mathfrak{b}})$ *when $p < \frac{1}{\mathfrak{b}}$.*

Despite the warning of Pemantle [16] (there are many papers proving results on trees as a somewhat unmotivated alternative …to Euclidean space), it seems to be of particular interest to study the more delicate situation $p = \frac{1}{\mathfrak{b}}$ that turns out to possess rich regimes. We prove that, similarly to the Kesten–Kozlov–Spitzer theorem for *transient* RWRE on the line, (X_n) enjoys, even in the recurrent case, an interesting subdiffusive behaviour.

To state our main result, we define

$$\kappa := \inf\left\{t > 1 : \ \mathbf{E}(A^t) = \frac{1}{\mathfrak{b}}\right\} \in (1, \infty], \quad (\inf \emptyset = \infty) \tag{1.4}$$

$$\psi(t) := \log \mathbf{E}\left(A^t\right), \quad t \ge 0. \tag{1.5}$$

We use the notation $a_n \approx b_n$ to denote $\lim_{n\to\infty} \frac{\log a_n}{\log b_n} = 1$.

Theorem 1.2 *If $p = \frac{1}{\mathfrak{b}}$ and if $\psi'(1) < 0$, then*

$$\max_{0\le i\le n} |X_i| \approx n^{\nu}, \quad \mathbb{P}\text{-a.s.}, \tag{1.6}$$

where $\nu = \nu(\kappa)$ is defined by

$$\nu := 1 - \frac{1}{\min\{\kappa, 2\}} = \begin{cases} (\kappa-1)/\kappa, & \text{if } \kappa \in (1,2], \\ 1/2 & \text{if } \kappa \in (2,\infty]. \end{cases} \tag{1.7}$$

For the value of κ, see Fig. 1. Under the assumptions $p = \frac{1}{\mathfrak{b}}$ and $\psi'(1) < 0$, the value of κ lies in $(2, \infty]$ if and only if $\mathbf{E}(A^2) < \frac{1}{\mathfrak{b}}$; and $\kappa = \infty$ if moreover ess sup$(A) \le 1$.

Remark (i) It is known [14] that if $p = \frac{1}{\mathfrak{b}}$ and $\psi'(1) < 0$, then for **P**-almost all environment ω, (X_n) is null recurrent.

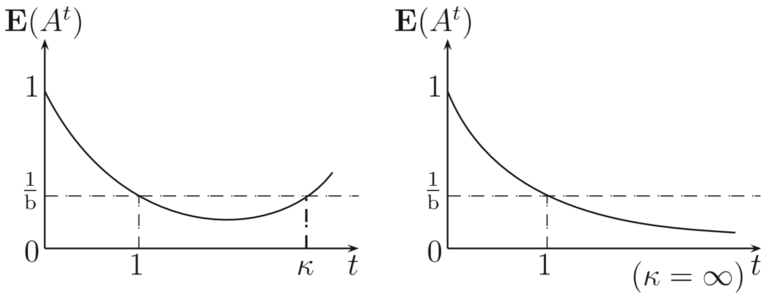

Fig. 1 The function $t \to \mathbf{E}(A^t)$ in the case $\psi'(1) < 0$ and $p = \frac{1}{\mathrm{b}}$

(ii) Since the walk is recurrent, $\max_{0\le i\le n} |X_i|$ cannot be replaced by $|X_n|$ in (1.3) and (1.6).

(iii) Theorem 1.2, which could be considered as a (weaker) analogue of the Kesten–Kozlov–Spitzer theorem, shows that tree-valued RWRE has even richer regimes than RWRE on $\mathbb{Z}$. In fact, recurrent RWRE on $\mathbb{Z}$ is of order of magnitude $(\log n)^2$, and has no n^a (for $0 < a < 1$) regime.

(iv) The case $\psi'(1) \ge 0$ leads to a phenomenon similar to Sinai's slow movement, and is studied in a forthcoming paper.

(v) The (uniform) ellipticity condition upon the environment can be weakened to some integrability condition.

(vi) The exponent ν is closely related to the (quenched) tail of excursions. More precisely, $P_\omega(H \ge n) \approx n^{-(1-\nu)/\nu}$, where H stands for the height of an excursion from the root to the root.

(vii) The non-degeneracy of the environment is necessary in Theorem 1.1. Theorem 1.2, on the other hand, holds even when the environment is degenerate, which corresponds to the case $A = \frac{1}{\mathrm{b}}$ and $\kappa = \infty$. In this case, $\max_{0\le i\le n} |X_i| \approx n^{1/2}$, $\mathbb{P}$-a.s., which is in agreement with Peres and Zeitouni [20].

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. In Sect. 3, we collect some elementary inequalities, which will be of frequent use later on. Theorem 1.2 is proved in Sect. 4, by means of a result (Proposition 4.2) concerning the solution of a recurrence equation which is closely related to Mandelbrot's multiplicative cascade. We prove Proposition 4.2 in Sect. 5.

Throughout the paper, c (possibly with a subscript) denotes a finite and positive constant; we write $c(\omega)$ instead of c when the value of c depends on the environment ω.

2 Proof of Theorem 1.1

We first introduce the constant q in the statement of Theorem 1.1, which is defined without the assumption $p < \frac{1}{\mathrm{b}}$. Let

$$\varrho(r) := \inf_{t\ge 0} \left\{ r^{-t}\, \mathbf{E}(A^t) \right\}, \quad r > 0.$$

Let $\underline{r} > 0$ be such that

$$\log \underline{r} = \mathbf{E}(\log A).$$

We mention that $\varrho(r) = 1$ for $r \in (0, \underline{r}]$, and that $\varrho(\cdot)$ is continuous and (strictly) decreasing on $[\underline{r}, \Theta)$ (where $\Theta := \operatorname{ess\,sup}(A) < \infty$), and $\varrho(\Theta) = \mathbf{P}(A = \Theta)$. Moreover, $\varrho(r) = 0$ for $r > \Theta$. See Chernoff [1]. We have $\underline{r} < \Theta$ since the environment is non-degenerate.

We define

$$\overline{r} := \inf\left\{r > 0 :\ \varrho(r) \le \frac{1}{\mathrm{b}}\right\}.$$

Clearly, $\underline{r} < \overline{r}$.

We define

$$q := \sup_{r \in [\underline{r}, \overline{r}]} r\varrho(r). \tag{2.1}$$

The following elementary lemma tells us that, instead of p, we can also use q in the recurrence/transience criterion of Lyons and Pemantle.

Lemma 2.1 *We have $q > \frac{1}{\mathrm{b}}$ (resp., $q = \frac{1}{\mathrm{b}}$, $q < \frac{1}{\mathrm{b}}$) if and only if $p > \frac{1}{\mathrm{b}}$ (resp., $p = \frac{1}{\mathrm{b}}$, $p < \frac{1}{\mathrm{b}}$).*

Proof By Lyons and Pemantle [11, p. 129], $p = \sup_{r \in (0,1]} r\varrho(r)$. Since $\varrho(r) = 1$ for $r \in (0, \underline{r}]$, there exists $\min\{\underline{r}, 1\} \le r^* \le 1$ such that $p = r^*\varrho(r^*)$.

(i) Assume $p < \frac{1}{\mathrm{b}}$. Then $\varrho(1) \le \sup_{r \in (0,1]} r\varrho(r) = p < \frac{1}{\mathrm{b}}$, which, by definition of $\overline{r}$, implies $\overline{r} < 1$. Therefore, $q \le p < \frac{1}{\mathrm{b}}$.

(ii) Assume $p \ge \frac{1}{\mathrm{b}}$. We have $\varrho(r^*) \ge p \ge \frac{1}{\mathrm{b}}$, which yields $r^* \le \overline{r}$. If $\underline{r} \le 1$, then $r^* \ge \underline{r}$, and thus $p = r^*\varrho(r^*) \le q$. If $\underline{r} > 1$, then $p = 1$, and thus $q \ge \underline{r}\,\varrho(\underline{r}) = \underline{r} > 1 = p$.
We have therefore proved that $p \ge \frac{1}{\mathrm{b}}$ implies $q \ge p$.
If moreover $p > \frac{1}{\mathrm{b}}$, then $q \ge p > \frac{1}{\mathrm{b}}$.

(iii) Assume $p = \frac{1}{\mathrm{b}}$. We already know from (ii) that $q \ge p$.
On the other hand, $\varrho(1) \le \sup_{r \in (0,1]} r\varrho(r) = p = \frac{1}{\mathrm{b}}$, implying $\overline{r} \le 1$. Thus $q \le p$.
As a consequence, $q = p = \frac{1}{\mathrm{b}}$. □

Having defined q, the next step in the proof of Theorem 1.1 is to compute invariant measures π for (X_n). We first introduce some notation on the tree. For any $m \ge 0$, let

$$\mathbb{T}_m := \{x \in \mathbb{T} :\ |x| = m\}.$$

For any $x \in \mathbb{T}$, let $\{x_i\}_{1 \le i \le \mathrm{b}}$ be the set of children of x.

If π is an invariant measure, then

$$\pi(x) = \frac{\omega(\overleftarrow{x}, x)}{\omega(x, \overleftarrow{x})}\pi(\overleftarrow{x}), \quad \forall x \in \mathbb{T}\backslash\{e\}.$$

By induction, this leads to [recalling A from (1.1)]: for $x \in \mathbb{T}_m$ $(m \geq 1)$,

$$\pi(x) = \frac{\pi(e)}{\omega(x, \overleftarrow{x})}\frac{\omega(e, x^{(1)})}{A(x^{(1)})}\exp\left(\sum_{z\in]\!]e,x]\!]}\log A(z)\right),$$

where $]\!]e,x]\!]$ denotes the shortest path $x^{(1)}, x^{(2)}, \ldots, x^{(m)} =: x$ from the root e (but excluded) to the vertex x. We note that $A(x^{(1)})$ cancels out in the above formula, which holds for *any* choice of $(A(e_i), 1 \leq i \leq \mathrm{b})$. We choose $(A(e_i), 1 \leq i \leq \mathrm{b})$ to be a random vector independent of $(\omega(x, y), |x| \geq 1, y \in \mathbb{T})$, and distributed as $(A(x_i), 1 \leq i \leq \mathrm{b})$, for any $x \in \mathbb{T}_m$ with $m \geq 1$.

By the ellipticity condition on the environment, we can take $\pi(e)$ to be sufficiently small so that for some $c_0 \in (0, 1]$,

$$c_0 \exp\left(\sum_{z\in]\!]e,x]\!]}\log A(z)\right) \leq \pi(x) \leq \exp\left(\sum_{z\in]\!]e,x]\!]}\log A(z)\right). \tag{2.2}$$

By Chebyshev's inequality, for any $r > \underline{r}$,

$$\max_{x\in\mathbb{T}_n}\mathbf{P}\left\{\pi(x) \geq r^n\right\} \leq \varrho(r)^n. \tag{2.3}$$

Since $\#\mathbb{T}_n = \mathrm{b}^n$, this gives $\mathbf{E}(\#\{x \in \mathbb{T}_n : \pi(x) \geq r^n\}) \leq \mathrm{b}^n\varrho(r)^n$. By Chebyshev's inequality and the Borel–Cantelli lemma, for any $r > \underline{r}$ and $\mathbf{P}$-almost surely for all large n,

$$\#\left\{x \in \mathbb{T}_n : \pi(x) \geq r^n\right\} \leq n^2\mathrm{b}^n\varrho(r)^n. \tag{2.4}$$

On the other hand, by (2.3),

$$\mathbf{P}\left\{\exists x \in \mathbb{T}_n : \pi(x) \geq r^n\right\} \leq \mathrm{b}^n\varrho(r)^n.$$

For $r > \overline{r}$, the expression on the right-hand side is summable in n. By the Borel–Cantelli lemma, for any $r > \overline{r}$ and $\mathbf{P}$-almost surely for all large n,

$$\max_{x\in\mathbb{T}_n}\pi(x) < r^n. \tag{2.5}$$

Proof of Theorem 1.1 (*upper bound*) Fix $\varepsilon > 0$ such that $q + 3\varepsilon < \frac{1}{\mathrm{b}}$.

We follow the strategy given in Liggett [8, p. 103], by introducing a positive recurrent birth-and-death chain $(\widetilde{X}_j, j \geq 0)$, starting from 0, with transition probability from i to $i+1$ (for $i \geq 1$) equal to

$$\frac{1}{\widetilde{\pi}(i)} \sum_{x \in \mathbb{T}_i} \pi(x)(1-\omega(x, \overleftarrow{x})),$$

where $\widetilde{\pi}(i) := \sum_{x \in \mathbb{T}_i} \pi(x)$. We note that $\widetilde{\pi}$ is a finite invariant measure for $(\widetilde{X}_j)$. Let

$$\tau_n := \inf\{i \geq 1 : X_i \in \mathbb{T}_n\}, \quad n \geq 0.$$

By Liggett [8, Theorem II.6.10], for any $n \geq 1$,

$$P_\omega(\tau_n < \tau_0) \leq \widetilde{P}_\omega(\widetilde{\tau}_n < \widetilde{\tau}_0),$$

where $\widetilde{P}_\omega(\widetilde{\tau}_n < \widetilde{\tau}_0)$ is the probability that $(\widetilde{X}_j)$ hits n before returning to 0. According to Hoel et al. [5, p. 32, Formula (61)],

$$\widetilde{P}_\omega(\widetilde{\tau}_n < \widetilde{\tau}_0) = c_1(\omega) \left(\sum_{i=0}^{n-1} \frac{1}{\sum_{x \in \mathbb{T}_i} \pi(x)(1-\omega(x, \overleftarrow{x}))} \right)^{-1},$$

where $c_1(\omega) \in (0, \infty)$ depends on ω. We arrive at the following estimate: for any $n \geq 1$,

$$P_\omega(\tau_n < \tau_0) \leq c_1(\omega) \left(\sum_{i=0}^{n-1} \frac{1}{\sum_{x \in \mathbb{T}_i} \pi(x)} \right)^{-1}. \tag{2.6}$$

We now estimate $\sum_{i=0}^{n-1} \frac{1}{\sum_{x \in \mathbb{T}_i} \pi(x)}$. For any fixed $0 = r_0 < \underline{r} < r_1 < \cdots < r_\ell = \overline{r} < r_{\ell+1}$,

$$\sum_{x \in \mathbb{T}_i} \pi(x) \leq \sum_{j=1}^{\ell+1} (r_j)^i \# \left\{ x \in \mathbb{T}_i : \pi(x) \geq (r_{j-1})^i \right\} + \sum_{x \in \mathbb{T}_i : \pi(x) \geq (r_{\ell+1})^i} \pi(x).$$

By (2.5), $\sum_{x \in \mathbb{T}_i : \pi(x) \geq (r_{\ell+1})^i} \pi(x) = 0$ **P**-almost surely for all large i. It follows from (2.4) that **P**-almost surely, for all large i,

$$\sum_{x \in \mathbb{T}_i} \pi(x) \leq (r_1)^i \mathrm{b}^i + \sum_{j=2}^{\ell+1} (r_j)^i i^2 \, \mathrm{b}^i \varrho(r_{j-1})^i.$$

Recall that $q = \sup_{r \in [\underline{r}, \overline{r}]} r \varrho(r) \geq \underline{r} \varrho(\underline{r}) = \underline{r}$. We choose $r_1 := \underline{r} + \varepsilon \leq q + \varepsilon$. We also choose the partition sufficiently fine so that $r_j \varrho(r_{j-1}) < q + \varepsilon$ for all

$2 \le j \le \ell + 1$. Thus, **P**-almost surely for all large i,

$$\sum_{x \in \mathbb{T}_i} \pi(x) \le (r_1)^i \mathrm{b}^i + \sum_{j=2}^{\ell+1} i^2\, \mathrm{b}^i (q+\varepsilon)^i = (r_1)^i \mathrm{b}^i + \ell\, i^2\, \mathrm{b}^i (q+\varepsilon)^i,$$

which implies (recall: $\mathrm{b}(q+\varepsilon) < 1$) that $\sum_{i=0}^{n-1} \frac{1}{\sum_{x\in\mathbb{T}_i} \pi(x)} \ge \frac{c_2}{n^2\, \mathrm{b}^n (q+\varepsilon)^n}$. Plugging this into (2.6) yields that, **P**-almost surely for all large n,

$$P_\omega(\tau_n < \tau_0) \le c_3(\omega)\, n^2\, \mathrm{b}^n (q+\varepsilon)^n \le [(q+2\varepsilon)\mathrm{b}]^n.$$

In particular, by writing $L(\tau_n) := \#\{1 \le i \le \tau_n : X_i = e\}$, we obtain

$$P_\omega\{L(\tau_n) \ge j\} = [P_\omega(\tau_n > \tau_0)]^j \ge \left\{1 - [(q+2\varepsilon)\mathrm{b}]^n\right\}^j,$$

which, by the Borel–Cantelli lemma, yields that, **P**-almost surely for all large n,

$$L(\tau_n) \ge \frac{1}{[(q+3\varepsilon)\mathrm{b}]^n}, \quad P_\omega\text{-a.s.}$$

Since $\{L(\tau_n) \ge j\} \subset \{\max_{0\le k \le 2j} |X_k| < n\}$, and since ε can be as close to 0 as possible, we obtain the upper bound in Theorem 1.1. □

Proof of Theorem 1.1 (*lower bound*) Assume $p < \frac{1}{\mathrm{b}}$. Recall that in this case, we have $\overline{r} < 1$. Let $\varepsilon > 0$ be small. Let $r \in (\underline{r}, \overline{r})$ be such that $\varrho(r) > \frac{1}{\mathrm{b}}\mathrm{e}^{\varepsilon}$ and that $r\varrho(r) \ge q\mathrm{e}^{-\varepsilon}$. Let L be a large integer with $\mathrm{b}^{-1/L} \ge \mathrm{e}^{-\varepsilon}$ and satisfying (2.7) below.

We start by constructing a Galton–Watson tree $\mathbb{G}$, which is a certain subtree of $\mathbb{T}$. The first generation of $\mathbb{G}$, denoted by $\mathbb{G}_1$ and defined below, consists of vertices $x \in \mathbb{T}_L$ satisfying a certain property. The second generation of $\mathbb{G}$ is formed by applying the same procedure to each element of $\mathbb{G}_1$, and so on. To be precise,

$$\mathbb{G}_1 = \mathbb{G}_1(L, r) := \left\{ x \in \mathbb{T}_L : \min_{z \in]\!]e, x]\!]} \prod_{y \in]\!]e, z]\!]} A(y) \ge r^L \right\},$$

where $]\!]e, x]\!]$ denotes as before the set of vertices (excluding e) lying on the shortest path relating e and x. More generally, if $\mathbb{G}_i$ denotes the ith generation of $\mathbb{G}$, then

$$\mathbb{G}_{n+1} := \bigcup_{u \in \mathbb{G}_n} \left\{ x \in \mathbb{T}_{(n+1)L} : \min_{z \in]\!]u, x]\!]} \prod_{y \in]\!]u, z]\!]} A(y) \ge r^L \right\}, \quad n = 1, 2, \ldots$$

We claim that it is possible to choose L sufficiently large such that

$$\mathbf{E}(\#\mathbb{G}_1) \geq \mathrm{e}^{-\varepsilon L}\mathrm{b}^L\varrho(r)^L. \tag{2.7}$$

Note that $\mathrm{e}^{-\varepsilon L}\mathrm{b}^L\varrho(r)^L > 1$, since $\varrho(r) > \frac{1}{\mathrm{b}}\mathrm{e}^{\varepsilon}$.

We admit (2.7) for the moment, which implies that $\mathbb{G}$ is super-critical. By theory of branching processes [4, p. 13], when n goes to infinity, $\frac{\#\mathbb{G}_{n/L}}{[\mathbf{E}(\#\mathbb{G}_1)]^{n/L}}$ converges almost surely (and in L^2) to a limit W with $\mathbf{P}(W > 0) > 0$. Therefore, on the event $\{W > 0\}$, for all large n,

$$\#(\mathbb{G}_{n/L}) \geq c_4(\omega)[\mathbf{E}(\#\mathbb{G}_1)]^{n/L}. \tag{2.8}$$

(For notational simplification, we only write our argument for the case when n is a multiple of L. It is clear that our final conclusion holds for all large n.)

Recall that according to the Dirichlet principle [3],

$$\begin{aligned} 2\pi(e)P_\omega\{\tau_n < \tau_0\} &= \inf_{h:\,h(e)=1,\,h(z)=0,\,\forall |z|\geq n} \sum_{x,y\in\mathbb{T}} \pi(x)\omega(x,y)(h(x)-h(y))^2 \\ &\geq c_5 \inf_{h:\,h(e)=1,\,h(z)=0,\,\forall z\in\mathbb{T}_n} \sum_{|x|<n}\sum_{y:\,x=\overleftarrow{y}} \pi(x)(h(x)-h(y))^2, \end{aligned} \tag{2.9}$$

the last inequality following from ellipticity condition on the environment. Clearly,

$$\begin{aligned} \sum_{|x|<n}\sum_{y:\,x=\overleftarrow{y}} \pi(x)(h(x)-h(y))^2 &= \sum_{i=0}^{(n/L)-1}\ \sum_{x:\,iL\leq|x|<(i+1)L}\ \sum_{y:\,x=\overleftarrow{y}} \pi(x)(h(x)-h(y))^2 \\ &:= \sum_{i=0}^{(n/L)-1} I_i, \end{aligned}$$

with obvious notation. For any i,

$$I_i \geq \mathrm{b}^{-L} \sum_{v\in\mathbb{G}_{i+1}}\ \sum_{x\in[\![v^\uparrow,v[\![}\ \sum_{y:\,x=\overleftarrow{y}} \pi(x)(h(x)-h(y))^2,$$

and for any $v \in \mathbb{G}_{i+1}$, denote by $v^\uparrow \in \mathbb{G}_i$ the unique element of $\mathbb{G}_i$ lying on the path $[\![e,v]\!]$ (in words, $v^\uparrow$ is the parent of v in the Galton–Watson tree $\mathbb{G}$), and the factor b^{-L} comes from the fact that each term $\pi(x)(h(x)-h(y))^2$ is counted at most b^L times in the sum on the right-hand side.

By (2.2), for $x \in [\![v^{\uparrow}, v[\![$, $\pi(x) \geq c_0 \prod_{u \in]\!]e,x]\!]} A(u)$, which, by the definition of $\mathbb{G}$, is at least $c_0\, r^{(i+1)L}$. Therefore,

$$\begin{aligned} I_i &\geq c_0\, \mathrm{b}^{-L} \sum_{v \in \mathbb{G}_{i+1}} \sum_{x \in [\![v^{\uparrow}, v[\![} \sum_{y: x = \overleftarrow{y}} r^{(i+1)L} (h(x) - h(y))^2 \\ &\geq c_0\, \mathrm{b}^{-L} r^{(i+1)L} \sum_{v \in \mathbb{G}_{i+1}} \sum_{y \in]\!]v^{\uparrow}, v]\!]} (h(\overleftarrow{y}) - h(y))^2. \end{aligned}$$

By the Cauchy–Schwarz inequality, $\sum_{y \in]\!]v^{\uparrow}, v]\!]} (h(\overleftarrow{y}) - h(y))^2 \geq \frac{1}{L} (h(v^{\uparrow}) - h(v))^2$. Accordingly,

$$I_i \geq c_0 \frac{\mathrm{b}^{-L} r^{(i+1)L}}{L} \sum_{v \in \mathbb{G}_{i+1}} (h(v^{\uparrow}) - h(v))^2,$$

which yields

$$\begin{aligned} \sum_{i=0}^{(n/L)-1} I_i &\geq c_0 \frac{\mathrm{b}^{-L}}{L} \sum_{i=0}^{(n/L)-1} r^{(i+1)L} \sum_{v \in \mathbb{G}_{i+1}} (h(v^{\uparrow}) - h(v))^2 \\ &\geq c_0 \frac{\mathrm{b}^{-L}}{L} \mathrm{b}^{-n/L} \sum_{v \in \mathbb{G}_{n/L}} \sum_{i=0}^{(n/L)-1} r^{(i+1)L} (h(v^{(i)}) - h(v^{(i+1)}))^2, \end{aligned}$$

where, $e =: v^{(0)}, v^{(1)}, v^{(2)}, \ldots, v^{(n/L)} := v$, is the shortest path (in $\mathbb{G}$) from e to v, and the factor $\mathrm{b}^{-n/L}$ results from the fact that each term $r^{(i+1)L}(h(v^{(i)}) - h(v^{(i+1)}))^2$ is counted at most $\mathrm{b}^{n/L}$ times in the sum on the right-hand side.

By the Cauchy–Schwarz inequality, for all $h : \mathbb{T} \to \mathbb{R}$ with $h(e) = 1$ and $h(z) = 0$ ($\forall z \in \mathbb{T}_n$), we have

$$\begin{aligned} &\sum_{i=0}^{(n/L)-1} r^{(i+1)L} (h(v^{(i)}) - h(v^{(i+1)}))^2 \\ &\quad \geq \frac{1}{\sum_{i=0}^{(n/L)-1} r^{-(i+1)L}} \left(\sum_{i=0}^{(n/L)-1} (h(v^{(i)}) - h(v^{(i+1)})) \right)^2 \\ &\quad = \frac{1}{\sum_{i=0}^{(n/L)-1} r^{-(i+1)L}} \geq c_6\, r^n. \end{aligned}$$

Therefore,

$$\sum_{i=0}^{(n/L)-1} I_i \geq c_0 c_6\, r^n\, \frac{\mathrm{b}^{-L}}{L} \mathrm{b}^{-n/L} \#(\mathbb{G}_{n/L})$$
$$\geq c_0 c_6 c_4(\omega)\, r^n\, \frac{\mathrm{b}^{-L}}{L} \mathrm{b}^{-n/L}\, [\mathbf{E}(\#\mathbb{G}_1)]^{n/L}\, \mathbf{1}_{\{W>0\}},$$

the last inequality following from (2.8). Plugging this into (2.9) yields that for all large n,

$$P_\omega\{\tau_n < \tau_0\} \geq c_7(\omega)\, r^n\, \frac{\mathrm{b}^{-L}}{L} \mathrm{b}^{-n/L}\, [\mathbf{E}(\#\mathbb{G}_1)]^{n/L}\, \mathbf{1}_{\{W>0\}}.$$

Recall from (2.7) that $\mathbf{E}(\#\mathbb{G}_1) \geq \mathrm{e}^{-\varepsilon L} \mathrm{b}^L \varrho(r)^L$. Therefore, on $\{W > 0\}$, for all large n, $P_\omega\{\tau_n < \tau_0\} \geq c_8(\omega)(\mathrm{e}^{-\varepsilon}\mathrm{b}^{-1/L}\mathrm{b} r \varrho(r))^n$, which is no smaller than $c_8(\omega)(\mathrm{e}^{-3\varepsilon} q\mathrm{b})^n$ (since $\mathrm{b}^{-1/L} \geq \mathrm{e}^{-\varepsilon}$ and $r\varrho(r) \geq q\mathrm{e}^{-\varepsilon}$ by assumption). Thus, by writing $L(\tau_n) := \#\{1 \leq i \leq n :\ X_i = e\}$ as before, we have, on $\{W > 0\}$,

$$P_\omega\{L(\tau_n) \geq j\} = [P_\omega(\tau_n > \tau_0)]^j \leq [1 - c_8(\omega)(\mathrm{e}^{-3\varepsilon} q\mathrm{b})^n]^j.$$

By the Borel–Cantelli lemma, for $\mathbf{P}$-almost all ω, on $\{W > 0\}$, we have, P_ω-almost surely for all large n, $L(\tau_n) \leq 1/(\mathrm{e}^{-4\varepsilon} q\mathrm{b})^n$, i.e.,

$$\max_{0 \leq k \leq \tau_0(\lfloor 1/(\mathrm{e}^{-4\varepsilon} q\mathrm{b})^n \rfloor)} |X_k| \geq n,$$

where $0 < \tau_0(1) < \tau_0(2) < \cdots$ are the successive return times to the root e by the walk (thus $\tau_0(1) = \tau_0$). Since the walk is positive recurrent, $\tau_0(\lfloor 1/(\mathrm{e}^{-4\varepsilon} q\mathrm{b})^n \rfloor) \sim \frac{1}{(\mathrm{e}^{-4\varepsilon} q\mathrm{b})^n} E_\omega[\tau_0]$ (for $n \to \infty$), P_ω-almost surely ($a_n \sim b_n$ meaning $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$). Therefore, for $\mathbf{P}$-almost all $\omega \in \{W > 0\}$,

$$\liminf_{n\to\infty} \frac{\max_{0\leq k\leq n} |X_k|}{\log n} \geq \frac{1}{\log[1/(q\mathrm{b})]}, \quad P_\omega\text{-a.s.} \tag{2.10}$$

We claim that modifying a finite number of transition probabilities does not change the value of $\liminf_{n\to\infty} \frac{\max_{0\leq k\leq n} |X_k|}{\log n}$. Indeed, for any $1 \leq i \leq \mathrm{b}$, let $(X_n^{(i)},\ n \geq 1)$ denote the restriction of the walk on the subtree $\mathbb{T}(e_i)$ of e_i, i.e., $(X_n^{(i)},\ n \geq 1)$ is the original walk by throwing away the excursions away from $\mathbb{T}(e_i)$. Analytically, $X_n^{(i)} = X_{\sigma_i(n)}$, where $\sigma_i(1) := \inf\{k > 0 : X_k \in \mathbb{T}(e_i)\}$ and $\sigma_i(n+1) := \inf\{k > \sigma_i(n) : X_k \in \mathbb{T}(e_i)\}$, $n \geq 1$. Then

$$\max_{0\leq k\leq \min_{1\leq i\leq \mathrm{b}} \sigma_i(n)} |X_k| \leq \max_{1\leq i\leq \mathrm{b}} \max_{0\leq k\leq n} |X_k^{(i)}| \leq \max_{0\leq k\leq \max_{1\leq i\leq \mathrm{b}} \sigma_i(n)} |X_k|.$$

By the law of large numbers and the positive recurrence of the walk, for any $1 \le i \le \mathrm{b}$, we have $\frac{\log \sigma_i(n)}{\log n} \to 1$, $\mathbb{P}$-a.s. Therefore,

$$\liminf_{n\to\infty} \frac{1}{\log n} \max_{0\le k\le n} |X_k| = \liminf_{n\to\infty} \frac{1}{\log n} \max_{1\le i\le \mathrm{b}} \max_{0\le k\le n} |X_k^{(i)}|, \quad \mathbb{P}\text{-a.s.}$$

In particular, $\mathbb{P}$-almost surely, $\liminf_{n\to\infty} \frac{\max_{0\le k\le n} |X_k|}{\log n}$ does not depend on the values of $\omega(e, e_i)$, $1 \le i \le \mathrm{b}$. A similar argument shows that modifying a finite number of transition probabilities does not change the value of $\liminf_{n\to\infty} \frac{\max_{0\le k\le n} |X_k|}{\log n}$, $\mathbb{P}$-almost surely. Recall from (2.7) that $\mathbf{P}\{W > 0\} > 0$. By Kolmogorov's zero–one law and (2.10), we obtain the lower bound in Theorem 1.1.

It remains to prove (2.7). Let $(A^{(i)})_{i\ge 1}$ be an i.i.d. sequence of random variables distributed as A. Clearly, for any $\delta \in (0, 1)$,

$$\begin{aligned}\mathbf{E}(\#\mathbb{G}_1) &= \mathrm{b}^L\, \mathbf{P}\left(\sum_{i=1}^{\ell} \log A^{(i)} \ge L \log r,\ \forall 1 \le \ell \le L \right) \\ &\ge \mathrm{b}^L\, \mathbf{P}\left((1-\delta) L \log r \ge \sum_{i=1}^{\ell} \log A^{(i)} \ge L \log r,\ \forall 1 \le \ell \le L \right).\end{aligned}$$

Let $t \ge 0$. We define a new probability $\mathbf{Q} = \mathbf{Q}_t$ by

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} := \frac{\mathrm{e}^{t \log A}}{\mathbf{E}(\mathrm{e}^{t \log A})} = \frac{A^t}{\mathbf{E}(A^t)}.$$

Then

$$\begin{aligned}\mathbf{E}(\#\mathbb{G}_1) &\ge \mathrm{b}^L\, \mathbf{E}_{\mathbf{Q}}\left[\frac{[\mathbf{E}(A^t)]^L}{\exp\{t \sum_{i=1}^{L} \log A^{(i)}\}} \mathbf{1}_{\{(1-\delta)L\log r \ge \sum_{i=1}^{\ell} \log A^{(i)} \ge L \log r, \forall 1\le \ell \le L\}} \right] \\ &\ge \mathrm{b}^L\, \frac{[\mathbf{E}(A^t)]^L}{r^{t(1-\delta)L}}\, \mathbf{Q}\left((1-\delta) L \log r \ge \sum_{i=1}^{\ell} \log A^{(i)} \ge L \log r,\ \forall 1 \le \ell \le L \right).\end{aligned}$$

To choose an optimal value of t, we fix $\widetilde{r} \in (r, \overline{r})$ with $\widetilde{r} < r^{1-\delta}$. Our choice of $t = t^*$ is such that $\varrho(\widetilde{r}) = \inf_{t\ge 0}\{\widetilde{r}^{-t}\mathbf{E}(A^t)\} = \widetilde{r}^{-t^*}\mathbf{E}(A^{t^*})$. With this choice, we have $\mathbf{E}_{\mathbf{Q}}(\log A) = \log \widetilde{r}$, so that $\mathbf{Q}\{(1-\delta)L\log r \ge \sum_{i=1}^{\ell} \log A^{(i)} \ge L \log r,\ \forall 1 \le \ell \le L\} \ge c_9$. Consequently,

$$\mathbf{E}(\#\mathbb{G}_1) \ge c_9\, \mathrm{b}^L\, \frac{[\mathbf{E}(A^{t^*})]^L}{r^{t^*(1-\delta)L}} = c_9\, \mathrm{b}^L\, \frac{[\widetilde{r}^{t^*} \varrho(\widetilde{r})]^L}{r^{t^*(1-\delta)L}} \ge c_9\, r^{\delta t^* L} \mathrm{b}^L \varrho(\widetilde{r})^L.$$

Since $\delta > 0$ can be as close to 0 as possible, the continuity of $\varrho(\cdot)$ on $[\underline{r}, \overline{r})$ yields (2.7), and thus completes the proof of Theorem 1.1. □

3 Some elementary inequalities

We collect some elementary inequalities in this section. They will be of use in the next sections, in the study of the null recurrence case.

Lemma 3.1 *Let $\xi \geq 0$ be a random variable.*

(i) *Assume that $\mathbb{E}(\xi^a) < \infty$ for some $a > 1$. Then for any $x \geq 0$,*

$$\frac{\mathbb{E}[(\frac{\xi}{x+\xi})^a]}{[\mathbb{E}(\frac{\xi}{x+\xi})]^a} \leq \frac{\mathbb{E}(\xi^a)}{[\mathbb{E}\xi]^a}. \tag{3.1}$$

(ii) *If $\mathbb{E}(\xi) < \infty$, then for any $0 \leq \lambda \leq 1$ and $t \geq 0$,*

$$\mathbb{E}\left\{\exp\left(-t\,\frac{(\lambda+\xi)/(1+\xi)}{\mathbb{E}[(\lambda+\xi)/(1+\xi)]}\right)\right\} \leq \mathbb{E}\left\{\exp\left(-t\,\frac{\xi}{\mathbb{E}(\xi)}\right)\right\}. \tag{3.2}$$

Remark When $a = 2$, (3.1) is a special case of Lemma 6.4 of Pemantle and Peres [18].

Proof of Lemma 3.1 We actually prove a very general result, stated as follows. Let $\varphi : (0,\infty) \to \mathbb{R}$ be a convex $\mathcal{C}^1$-function. Let $x_0 \in \mathbb{R}$ and let I be an open interval containing x_0. Assume that ξ takes values in a Borel set $J \subset \mathbb{R}$ (for the moment, we do not assume $\xi \geq 0$). Let $h : I \times J \to (0,\infty)$ and $\frac{\partial h}{\partial x} : I \times J \to \mathbb{R}$ be measurable functions such that

- $\mathbb{E}\{h(x_0,\xi)\} < \infty$ and $\mathbb{E}\{|\varphi(\frac{h(x_0,\xi)}{\mathbb{E}h(x_0,\xi)})|\} < \infty$;
- $\mathbb{E}[\sup_{x\in I}\{|\frac{\partial h}{\partial x}(x,\xi)| + |\varphi'(\frac{h(x,\xi)}{\mathbb{E}h(x,\xi)})|\,(\frac{|\frac{\partial h}{\partial x}(x,\xi)|}{\mathbb{E}\{h(x,\xi)\}} + \frac{h(x,\xi)}{[\mathbb{E}\{h(x,\xi)\}]^2}|\mathbb{E}\{\frac{\partial h}{\partial x}(x,\xi)\}|)\}] < \infty$;
- both $y \to h(x_0,y)$ and $y \to \frac{\partial}{\partial x}\log h(x,y)|_{x=x_0}$ are monotone on J.

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left\{\varphi\left(\frac{h(x,\xi)}{\mathbb{E}h(x,\xi)}\right)\right\}\Big|_{x=x_0} \geq 0, \quad \text{or} \quad \leq 0, \tag{3.3}$$

depending on whether $h(x_0,\cdot)$ and $\frac{\partial}{\partial x}\log h(x_0,\cdot)$ have the same monotonicity.

To prove (3.3), we observe that by the integrability assumptions,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left\{\varphi\left(\frac{h(x,\xi)}{\mathbb{E}h(x,\xi)}\right)\right\}\Big|_{x=x_0}\\ &\quad= \frac{1}{(\mathbb{E}h(x_0,\xi))^2}\,\mathbb{E}\left(\varphi'(h(x_0,\xi))\left[\frac{\partial h}{\partial x}(x_0,\xi)\mathbb{E}h(x_0,\xi) - h(x_0,\xi)\mathbb{E}\frac{\partial h}{\partial x}(x_0,\xi)\right]\right).\end{aligned}$$

Let $\widetilde{\xi}$ be an independent copy of ξ. The expectation expression $\mathbb{E}(\varphi'(h(x_0,\xi))[\cdots])$ on the right-hand side is

$$\begin{aligned}
&= \mathbb{E}\left(\varphi'(h(x_0,\xi))\left[\frac{\partial h}{\partial x}(x_0,\xi)h(x_0,\widetilde{\xi}) - h(x_0,\xi)\frac{\partial h}{\partial x}(x_0,\widetilde{\xi})\right]\right)\\
&= \frac{1}{2}\,\mathbb{E}\left(\left[\varphi'(h(x_0,\xi)) - \varphi'(h(x_0,\widetilde{\xi}))\right]\left[\frac{\partial h}{\partial x}(x_0,\xi)h(x_0,\widetilde{\xi}) - h(x_0,\xi)\frac{\partial h}{\partial x}(x_0,\widetilde{\xi})\right]\right)\\
&= \frac{1}{2}\,\mathbb{E}\left(h(x_0,\xi)h(x_0,\widetilde{\xi})\,\eta\right),
\end{aligned}$$

where

$$\eta := \left[\varphi'(h(x_0,\xi)) - \varphi'(h(x_0,\widetilde{\xi}))\right]\left[\frac{\partial \log h}{\partial x}(x_0,\xi) - \frac{\partial \log h}{\partial x}(x_0,\widetilde{\xi})\right].$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left\{\varphi\left(\frac{h(x,\xi)}{\mathbb{E}h(x,\xi)}\right)\right\}\Big|_{x=x_0} = \frac{1}{2(\mathbb{E}h(x_0,\xi))^2}\,\mathbb{E}\left(h(x_0,\xi)h(x_0,\widetilde{\xi})\,\eta\right).$$

Since $\eta \geq 0$ or ≤ 0 depending on whether $h(x_0,\cdot)$ and $\frac{\partial}{\partial x}\log h(x_0,\cdot)$ have the same monotonicity, this yields (3.3).

To prove (3.1) in Lemma 3.1, we take $x_0 \in (0,\infty)$, $J = \mathbb{R}_+$, I a finite open interval containing x_0 and away from 0, $\varphi(z) = z^a$, and $h(x,y) = \frac{y}{x+y}$, to see that the function $x \mapsto \frac{\mathbb{E}[(\frac{\xi}{x+\xi})^a]}{[\mathbb{E}(\frac{\xi}{x+\xi})]^a}$ is non-decreasing on $(0,\infty)$. By dominated convergence,

$$\lim_{x\to\infty}\frac{\mathbb{E}\big[\big(\frac{\xi}{x+\xi}\big)^a\big]}{\big[\mathbb{E}\big(\frac{\xi}{x+\xi}\big)\big]^a} = \lim_{x\to\infty}\frac{\mathbb{E}\big[\big(\frac{\xi}{1+\xi/x}\big)^a\big]}{\big[\mathbb{E}\big(\frac{\xi}{1+\xi/x}\big)\big]^a} = \frac{\mathbb{E}(\xi^a)}{[\mathbb{E}\xi]^a},$$

yielding (3.1).

The proof of (3.2) is similar. Indeed, applying (3.3) to the functions $\varphi(z) = \mathrm{e}^{-tz}$ and $h(x,y) = \frac{x+y}{1+y}$ with $x \in (0,1)$, we get that the function $x \mapsto \mathbb{E}\big\{\exp\big(-t\frac{(x+\xi)/(1+\xi)}{\mathbb{E}[(x+\xi)/(1+\xi)]}\big)\big\}$ is non-increasing on $(0,1)$; hence for $\lambda \in [0,\,1]$,

$$\mathbb{E}\left\{\exp\left(-t\frac{(\lambda+\xi)/(1+\xi)}{\mathbb{E}[(\lambda+\xi)/(1+\xi)]}\right)\right\} \leq \mathbb{E}\left\{\exp\left(-t\frac{\xi/(1+\xi)}{\mathbb{E}[\xi/(1+\xi)]}\right)\right\}.$$

On the other hand, we take $\varphi(z) = \mathrm{e}^{-tz}$ and $h(x,y) = \frac{y}{1+xy}$ (for $x \in (0,1)$) in (3.3) to see that $x \mapsto \mathbb{E}\big\{\exp\big(-t\frac{\xi/(1+x\xi)}{\mathbb{E}[\xi/(1+x\xi)]}\big)\big\}$ is non-increasing on $(0,1)$.

Therefore,

$$\mathbb{E}\left\{\exp\left(-t\frac{\xi/(1+\xi)}{\mathbb{E}[\xi/(1+\xi)]}\right)\right\} \le \mathbb{E}\left\{\exp\left(-t\,\frac{\xi}{\mathbb{E}(\xi)}\right)\right\},$$

which implies (3.2). □

The following lemma is due to Neveu [15]:

Lemma 3.2 *Let $\xi_1, \ldots, \xi_k$ be independent non-negative random variables such that for some $a \in [1, 2]$, $\mathbb{E}(\xi_i^a) < \infty (1 \le i \le k)$. Then*

$$\mathbb{E}\left[(\xi_1 + \cdots + \xi_k)^a\right] \le \sum_{k=1}^{k} \mathbb{E}(\xi_i^a) + \left(\sum_{i=1}^{k} \mathbb{E}\xi_i\right)^a.$$

The following inequality, borrowed from page 82 of Petrov [21], will be of frequent use.

Fact 3.3 *Let $\xi_1, \ldots, \xi_k$ be independent random variables. We assume that for any i, $\mathbb{E}(\xi_i) = 0$ and $\mathbb{E}(|\xi_i|^a) < \infty$, where $1 \le a \le 2$. Then*

$$\mathbb{E}\left(\left|\sum_{i=1}^{k} \xi_i\right|^a\right) \le 2\sum_{i=1}^{k} \mathbb{E}(|\xi_i|^a).$$

Lemma 3.4 *Fix $a > 1$. Let $(u_j)_{j\ge1}$ be a sequence of positive numbers, and let $(\lambda_j)_{j\ge1}$ be a sequence of non-negative numbers.*

(i) *If there exists some constant $c_{10} > 0$ such that for all $n \ge 2$,*

$$u_{j+1} \le \lambda_n + u_j - c_{10}\, u_j^a, \quad \forall 1 \le j \le n-1,$$

then we can find a constant $c_{11} > 0$ independent of n and $(\lambda_j)_{j\ge1}$, such that

$$u_n \le c_{11}\, (\lambda_n^{1/a} + n^{-1/(a-1)}), \quad \forall n \ge 1.$$

(ii) *Fix $K > 0$. Assume that $\lim_{j\to\infty} u_j = 0$ and that $\lambda_n \in [0, \frac{K}{n}]$ for all $n \ge 1$. If there exist $c_{12} > 0$ and $c_{13} > 0$ such that for all $n \ge 2$,*

$$u_{j+1} \ge \lambda_n + (1 - c_{12}\lambda_n)u_j - c_{13}\, u_j^a, \quad \forall 1 \le j \le n-1,$$

then for some $c_{14} > 0$ independent of n and $(\lambda_j)_{j\ge1}$ (c_{14} may depend on K),

$$u_n \ge c_{14}\, (\lambda_n^{1/a} + n^{-1/(a-1)}), \quad \forall n \ge 1.$$

Proof (i) Put $\ell = \ell(n) := \min\{n, \lambda_n^{-(a-1)/a}\}$. There are two possible situations.

First situation: there exists some $j_0 \in [n-\ell, n-1]$ such that $u_{j_0} \le (\frac{2}{c_{10}})^{1/a} \lambda_n^{1/a}$. Since $u_{j+1} \le \lambda_n + u_j$ for all $j \in [j_0, n-1]$, we have

$$u_n \le (n-j_0)\lambda_n + u_{j_0} \le \ell\lambda_n + \left(\frac{2}{c_{10}}\right)^{1/a} \lambda_n^{1/a} \le \left(1 + \left(\frac{2}{c_{10}}\right)^{1/a}\right) \lambda_n^{1/a},$$

which implies the desired upper bound.

Second situation: $u_j > (\frac{2}{c_{10}})^{1/a} \lambda_n^{1/a}$, $\forall j \in [n-\ell, n-1]$. Then $c_{10}\, u_j^a > 2\lambda_n$, which yields

$$u_{j+1} \le u_j - \frac{c_{10}}{2} u_j^a, \quad \forall j \in [n-\ell, n-1].$$

Since $a > 1$ and $(1-y)^{1-a} \ge 1 + (a-1)y$ (for $0 < y < 1$), this yields, for $j \in [n-\ell, n-1]$,

$$u_{j+1}^{1-a} \ge u_j^{1-a} \left(1 - \frac{c_{10}}{2} u_j^{a-1}\right)^{1-a} \ge u_j^{1-a} \left(1 + \frac{c_{10}}{2}(a-1)\, u_j^{a-1}\right) = u_j^{1-a} + \frac{c_{10}}{2}(a-1).$$

Therefore, $u_n^{1-a} \ge c_{15}\,\ell$ with $c_{15} := \frac{c_{10}}{2}(a-1)$. As a consequence, $u_n \le (c_{15}\,\ell)^{-1/(a-1)} \le (c_{15})^{-1/(a-1)}\,(n^{-1/(a-1)} + \lambda_n^{1/a})$, as desired.

(ii) Let us first prove:

$$u_n \ge c_{16}\, n^{-1/(a-1)}. \tag{3.4}$$

To this end, let n be large and define $v_j := u_j\,(1 - c_{12}\lambda_n)^{-j}$ for $1 \le j \le n$. Since $u_{j+1} \ge (1 - c_{12}\lambda_n)u_j - c_{13}u_j^a$ and $\lambda_n \le K/n$, we get

$$v_{j+1} \ge v_j - c_{13}(1 - c_{12}\lambda_n)^{(a-1)j-1}\, v_j^a \ge v_j - c_{17}\, v_j^a, \quad \forall\, 1 \le j \le n-1.$$

Since $u_j \to 0$, there exists some $j_0 > 0$ such that for all $n > j \ge j_0$, we have $c_{17}\, v_j^{a-1} < 1/2$, and

$$v_{j+1}^{1-a} \le v_j^{1-a} \left(1 - c_{17}\, v_j^{a-1}\right)^{1-a} \le v_j^{1-a} \left(1 + c_{18}\, v_j^{a-1}\right) = v_j^{1-a} + c_{18}.$$

It follows that $v_n^{1-a} \le c_{18}\,(n - j_0) + v_{j_0}^{1-a}$, which implies (3.4).

It remains to show that $u_n \ge c_{19}\, \lambda_n^{1/a}$. Consider a large n. The function $h(x) := \lambda_n + (1 - c_{12}\lambda_n)x - c_{13}x^a$ is increasing on $[0, c_{20}]$ for some fixed constant $c_{20} > 0$. Since $u_j \to 0$, there exists j_0 such that $u_j \le c_{20}$ for all $j \ge j_0$. We claim there exists $j \in [j_0, n-1]$ such that $u_j > (\frac{\lambda_n}{2c_{13}})^{1/a}$: otherwise, we would have $c_{13}\, u_j^a \le \frac{\lambda_n}{2} \le \lambda_n$ for all $j \in [j_0, n-1]$, and thus

$$u_{j+1} \ge (1 - c_{12}\,\lambda_n)u_j \ge \cdots \ge (1 - c_{12}\,\lambda_n)^{j-j_0}\, u_{j_0};$$

in particular, $u_n \geq (1 - c_{12}\,\lambda_n)^{n-j_0}\,u_{j_0}$ which would contradict the assumption $u_n \to 0$ (since $\lambda_n \leq K/n$).

Therefore, $u_j > (\frac{\lambda_n}{2c_{13}})^{1/a}$ for some $j \geq j_0$. By monotonicity of $h(\cdot)$ on $[0, c_{20}]$,

$$u_{j+1} \geq h(u_j) \geq h\left(\left(\frac{\lambda_n}{2c_{13}}\right)^{1/a}\right) \geq \left(\frac{\lambda_n}{2c_{13}}\right)^{1/a},$$

the last inequality being elementary. This leads to: $u_{j+2} \geq h(u_{j+1}) \geq h((\frac{\lambda_n}{2c_{13}})^{1/a}) \geq (\frac{\lambda_n}{2c_{13}})^{1/a}$. Iterating the procedure, we obtain: $u_n \geq (\frac{\lambda_n}{2c_{13}})^{1/a}$ for all $n > j_0$, which completes the proof of the Lemma. □

4 Proof of Theorem 1.2

Let $n \geq 2$, and let as before

$$\tau_n := \inf\{i \geq 1 : X_i \in \mathbb{T}_n\}.$$

We start with a characterization of the distribution of τ_n via its Laplace transform $\mathbb{E}(\mathrm{e}^{-\lambda\tau_n})$, for $\lambda \geq 0$. To state the result, we define $\alpha_{n,\lambda}(\cdot)$, $\beta_{n,\lambda}(\cdot)$ and $\gamma_n(\cdot)$ by $\alpha_{n,\lambda}(x) = \beta_{n,\lambda}(x) = 1$ and $\gamma_n(x) = 0$ (for $x \in \mathbb{T}_n$), and

$$\beta_{n,\lambda}(x) = \frac{(1 - \mathrm{e}^{-2\lambda}) + \sum_{i=1}^{\mathrm{b}} A(x_i)\beta_{n,\lambda}(x_i)}{1 + \sum_{i=1}^{\mathrm{b}} A(x_i)\beta_{n,\lambda}(x_i)}, \tag{4.1}$$

$$\alpha_{n,\lambda}(x) = \mathrm{e}^{-\lambda}\, \frac{\sum_{i=1}^{\mathrm{b}} A(x_i)\alpha_{n,\lambda}(x_i)}{1 + \sum_{i=1}^{\mathrm{b}} A(x_i)\beta_{n,\lambda}(x_i)}, \tag{4.2}$$

$$\gamma_n(x) = \frac{[1/\omega(x, \overleftarrow{x})] + \sum_{i=1}^{\mathrm{b}} A(x_i)\gamma_n(x_i)}{1 + \sum_{i=1}^{\mathrm{b}} A(x_i)\beta_n(x_i)}, \quad 1 \leq |x| < n, \tag{4.3}$$

where $\beta_n(\cdot) := \beta_{n,0}(\cdot)$, and for any $x \in \mathbb{T}$, $\{x_i\}_{1\leq i\leq \mathrm{b}}$ stands as before for the set of children of x.

Probabilistic interpretation: for $1 \leq |x| < n$, if $T_{\overleftarrow{x}} := \inf\{k \geq 0 : X_k = \overleftarrow{x}\}$, then $\alpha_{n,\lambda}(x) = E_\omega[\mathrm{e}^{-\lambda\tau_n}\mathbf{1}_{\{\tau_n < T_{\overleftarrow{x}}\}} \,|\, X_0 = x]$, $\beta_{n,\lambda}(x) = 1 - E_\omega[\mathrm{e}^{-\lambda(1+T_{\overleftarrow{x}})}\mathbf{1}_{\{\tau_n > T_{\overleftarrow{x}}\}} \,|\, X_0 = x]$, and $\gamma_n(x) = E_\omega[(\tau_n \wedge T_{\overleftarrow{x}}) \,|\, X_0 = x]$. We do not use these identities in the paper.

Proposition 4.1 *We have, for $n \geq 2$,*

$$E_\omega\left(\mathrm{e}^{-\lambda\tau_n}\right) = \mathrm{e}^{-\lambda}\, \frac{\sum_{i=1}^{\mathrm{b}} \omega(e, e_i)\alpha_{n,\lambda}(e_i)}{\sum_{i=1}^{\mathrm{b}} \omega(e, e_i)\beta_{n,\lambda}(e_i)}, \quad \forall\lambda \geq 0, \tag{4.4}$$

$$E_\omega(\tau_n) = \frac{1 + \sum_{i=1}^{\mathrm{b}} \omega(e, e_i)\gamma_n(e_i)}{\sum_{i=1}^{\mathrm{b}} \omega(e, e_i)\beta_n(e_i)}. \tag{4.5}$$

Proof Identity (4.5) can be found in Rozikov [22]. The proof of (4.4) is along similar lines; so we feel free to give an outline only. Let $g_{n,\lambda}(x) := E_\omega(\mathrm{e}^{-\lambda\tau_n}|X_0 = x)$. By the Markov property, $g_{n,\lambda}(x) = \mathrm{e}^{-\lambda}\sum_{i=1}^{\mathrm{b}}\omega(x,x_i)g_{n,\lambda}(x_i) + \mathrm{e}^{-\lambda}\omega(x,\overleftarrow{x})g_{n,\lambda}(\overleftarrow{x})$, for $|x| < n$. By induction on $|x|$ (such that $1 \le |x| \le n-1$), we obtain: $g_{n,\lambda}(x) = \mathrm{e}^{\lambda}(1-\beta_{n,\lambda}(x))g_{n,\lambda}(\overleftarrow{x}) + \alpha_{n,\lambda}(x)$, from which (4.4) follows. □

It turns out that $\beta_{n,\lambda}(\cdot)$ is closely related to Mandelbrot's multiplicative cascade [13]. Let

$$M_n := \sum_{x\in\mathbb{T}_n}\prod_{y\in]\!]e,x]\!]} A(y), \quad n \ge 1, \tag{4.6}$$

where $]\!]e, x]\!]$ denotes as before the shortest path relating e to x. We mention that $(A(e_i), 1 \le i \le \mathrm{b})$ is a random vector independent of $(\omega(x,y), |x| \ge 1, y \in \mathbb{T})$, and is distributed as $(A(x_i), 1 \le i \le \mathrm{b})$, for any $x \in \mathbb{T}_m$ with $m \ge 1$.

Let us recall some properties of (M_n) from Theorem 2.2 of Liu [9] and Theorem 2.5 of Liu [10]: under the conditions $p = \frac{1}{\mathrm{b}}$ and $\psi'(1) < 0$, (M_n) is a martingale, bounded in L^a for any $a \in [1,\kappa)$; in particular,

$$M_\infty := \lim_{n\to\infty} M_n \in (0,\infty), \tag{4.7}$$

exists $\mathbf{P}$-almost surely and in $L^a(\mathbf{P})$, and

$$\mathbf{E}\left(\mathrm{e}^{-sM_\infty}\right) \le \exp\left(-c_{21}\, s^{c_{22}}\right), \quad \forall s \ge 1; \tag{4.8}$$

furthermore, if $1 < \kappa < \infty$, then we also have

$$\frac{c_{23}}{x^\kappa} \le \mathbf{P}\left(M_\infty > x\right) \le \frac{c_{24}}{x^\kappa}, \quad x \ge 1. \tag{4.9}$$

We now summarize the asymptotic properties of $\beta_{n,\lambda}(\cdot)$ which will be needed later on.

Proposition 4.2 *Assume $p = \frac{1}{\mathrm{b}}$ and $\psi'(1) < 0$.*

(i) *For any $1 \le i \le \mathrm{b}$, $n \ge 2$, $t \ge 0$ and $\lambda \in [0, 1]$, we have*

$$\mathbf{E}\left\{\exp\left[-t\,\frac{\beta_{n,\lambda}(e_i)}{\mathbf{E}[\beta_{n,\lambda}(e_i)]}\right]\right\} \le \left\{\mathbf{E}\left(\mathrm{e}^{-t\,M_n/\Theta}\right)\right\}^{1/\mathrm{b}}, \tag{4.10}$$

where, as before, $\Theta := \operatorname{ess\,sup}(A) < \infty$.

(ii) *If $\kappa \in (2,\infty]$, then for any $1 \le i \le \mathrm{b}$ and all $n \ge 2$ and $\lambda \in [0, \frac{1}{n}]$,*

$$c_{25}\left(\sqrt{\lambda} + \frac{1}{n}\right) \le \mathbf{E}[\beta_{n,\lambda}(e_i)] \le c_{26}\left(\sqrt{\lambda} + \frac{1}{n}\right). \tag{4.11}$$

(iii) *If* $\kappa \in (1,2]$, *then for any* $1 \le i \le \mathrm{b}$, *when* $n \to \infty$ *and uniformly in* $\lambda \in [0, \frac{1}{n}]$,

$$\mathbf{E}[\beta_{n,\lambda}(e_i)] \approx \lambda^{1/\kappa} + \frac{1}{n^{1/(\kappa-1)}}, \tag{4.12}$$

where $a_n \approx b_n$ *denotes as before* $\lim_{n\to\infty} \frac{\log a_n}{\log b_n} = 1$.

The proof of Proposition 4.2 is postponed until Sect. 5. By admitting it for the moment, we are able to prove Theorem 1.2.

Proof of Theorem 1.2 Assume $p = \frac{1}{\mathrm{b}}$ and $\psi'(1) < 0$.

Let π be an invariant measure. By (2.2) and the definition of (M_n), $\sum_{x\in\mathbb{T}_n} \pi(x) \ge c_0\, M_n$. Therefore by (4.7), we have $\sum_{x\in\mathbb{T}} \pi(x) = \infty$, $\mathbf{P}$-a.s., implying that (X_n) is null recurrent.

We proceed to prove the lower bound in (1.6). By (4.3) and the ellipticity condition on the environment, $\gamma_n(x) \le \frac{1}{\omega(x,\overleftarrow{x})} + \sum_{i=1}^{\mathrm{b}} A(x_i)\gamma_n(x_i) \le c_{27} + \sum_{i=1}^{\mathrm{b}} A(x_i)\gamma_n(x_i)$. Iterating the argument yields

$$\gamma_n(e_i) \le c_{27}\left(1 + \sum_{j=2}^{n-1} M_j^{(e_i)}\right), \quad n \ge 3,$$

where

$$M_j^{(e_i)} := \sum_{x\in\mathbb{T}_j} \prod_{y\in]\!]e_i,x]\!]} A(y).$$

For future use, we also observe that

$$M_n = \sum_{i=1}^{\mathrm{b}} A(e_i)\, M_n^{(e_i)}, \quad n \ge 2. \tag{4.13}$$

Let $1 \le i \le \mathrm{b}$. Since $(M_j^{(e_i)}, j \ge 2)$ is distributed as $(M_{j-1}, j \ge 2)$, it follows from (4.7) that $M_j^{(e_i)}$ converges (when $j \to \infty$) almost surely, which implies $\gamma_n(e_i) \le c_{28}(\omega)\, n$. Plugging this into (4.5), we see that for all $n \ge 3$,

$$E_\omega(\tau_n) \le \frac{c_{29}(\omega)\, n}{\sum_{i=1}^{\mathrm{b}} \omega(e,e_i)\beta_n(e_i)} \le \frac{c_{30}(\omega)\, n}{\beta_n(e_1)}, \tag{4.14}$$

the last inequality following from the ellipticity assumption on the environment.

We now bound $\beta_n(e_1)$ from below (for large n). Let $1 \le i \le \mathrm{b}$. By (4.10), for $\lambda \in [0, 1]$ and $s \ge 0$,

$$\mathbf{E}\left\{\exp\left[-s\,\frac{\beta_{n,\lambda}(e_i)}{\mathbf{E}[\beta_{n,\lambda}(e_i)]}\right]\right\} \le \left\{\mathbf{E}\left(\mathrm{e}^{-s\,M_n/\Theta}\right)\right\}^{1/\mathrm{b}} \le \left\{\mathbf{E}\left(\mathrm{e}^{-s\,M_\infty/\Theta}\right)\right\}^{1/\mathrm{b}},$$

where, in the last inequality, we used the fact that (M_n) is a uniformly integrable martingale. Let $\varepsilon > 0$. Applying (4.8) to $s := n^{\varepsilon}$, we see that

$$\sum_n \mathbf{E}\left\{\exp\left[-n^{\varepsilon}\frac{\beta_{n,\lambda}(e_i)}{\mathbf{E}[\beta_{n,\lambda}(e_i)]}\right]\right\} < \infty. \tag{4.15}$$

In particular, $\sum_n \exp[-n^{\varepsilon}\frac{\beta_n(e_1)}{\mathbf{E}[\beta_n(e_1)]}]$ is $\mathbf{P}$-almost surely finite (by taking $\lambda = 0$; recalling that $\beta_n(\cdot) := \beta_{n,0}(\cdot)$). Thus, for $\mathbf{P}$-almost all ω and all sufficiently large n, $\beta_n(e_1) \geq n^{-\varepsilon}\,\mathbf{E}[\beta_n(e_1)]$. Going back to (4.14), we see that for $\mathbf{P}$-almost all ω and all sufficiently large n,

$$E_{\omega}(\tau_n) \leq \frac{c_{30}(\omega)\, n^{1+\varepsilon}}{\mathbf{E}[\beta_n(e_1)]}.$$

Let $m(n) := \lfloor \frac{n^{1+2\varepsilon}}{\mathbf{E}[\beta_n(e_1)]}\rfloor$. By Chebyshev's inequality, for $\mathbf{P}$-almost all ω and all sufficiently large n, $P_{\omega}(\tau_n \geq m(n)) \leq c_{31}(\omega)\, n^{-\varepsilon}$. Considering the subsequence $n_k := \lfloor k^{2/\varepsilon}\rfloor$, we see that $\sum_k P_{\omega}(\tau_{n_k} \geq m(n_k)) < \infty$, $\mathbf{P}$-a.s. By the Borel–Cantelli lemma, for $\mathbf{P}$-almost all ω and P_{ω}-almost all sufficiently large k, $\tau_{n_k} < m(n_k)$, which implies that for $n \in [n_{k-1}, n_k]$ and large k, we have $\tau_n < m(n_k) \leq \frac{n_k^{1+2\varepsilon}}{\mathbf{E}[\beta_{n_k}(e_1)]} \leq \frac{n^{1+3\varepsilon}}{\mathbf{E}[\beta_n(e_1)]}$ (the last inequality following from the estimate of $\mathbf{E}[\beta_n(e_1)]$ in Proposition 4.2). In view of Proposition 4.2, and since ε can be arbitrarily small, this gives the lower bound in (1.6) of Theorem 1.2.

To prove the upper bound, we note that $\alpha_{n,\lambda}(x) \leq \beta_n(x)$ for any $\lambda \geq 0$ and any $0 < |x| \leq n$ (this is easily checked by induction on $|x|$, or simply by looking at the probabilistic interpretation of $\alpha_{n,\lambda}(x)$ and $\beta_n(x)$). Thus, by (4.4), for any $\lambda \geq 0$,

$$E_{\omega}\left(\mathrm{e}^{-\lambda\tau_n}\right) \leq \frac{\sum_{i=1}^{\mathrm{b}} \omega(e, e_i)\beta_n(e_i)}{\sum_{i=1}^{\mathrm{b}} \omega(e, e_i)\beta_{n,\lambda}(e_i)} \leq \sum_{i=1}^{\mathrm{b}} \frac{\beta_n(e_i)}{\beta_{n,\lambda}(e_i)}.$$

We now fix $r \in (1, \frac{1}{\nu})$, where $\nu := 1 - \frac{1}{\min\{\kappa, 2\}}$ is defined in (1.7). It is possible to choose a small $\varepsilon > 0$ such that

$$\frac{1}{\kappa - 1} - \frac{r}{\kappa} > 3\varepsilon \quad \text{if } \kappa \in (1, 2], \qquad 1 - \frac{r}{2} > 3\varepsilon \quad \text{if } \kappa \in (2, \infty].$$

Let $\lambda = \lambda(n) := n^{-r}$. By (4.15), we have $\beta_{n,n^{-r}}(e_i) \geq n^{-\varepsilon}\,\mathbf{E}[\beta_{n,n^{-r}}(e_i)]$ for $\mathbf{P}$-almost all ω and all sufficiently large n, which yields

$$E_{\omega}\left(\mathrm{e}^{-n^{-r}\tau_n}\right) \leq n^{\varepsilon}\sum_{i=1}^{\mathrm{b}} \frac{\beta_n(e_i)}{\mathbf{E}[\beta_{n,n^{-r}}(e_i)]}.$$

It is easy to bound $\beta_n(e_i)$. For any given $x \in \mathbb{T}\backslash\{e\}$ with $|x| \leq n$, $n \mapsto \beta_n(x)$ is non-increasing (this is easily checked by induction on $|x|$). Chebyshev's inequality, together with the Borel–Cantelli lemma (applied to a subsequence, as we did in the proof of the lower bound) and the monotonicity of $n \mapsto \beta_n(e_i)$, readily yields $\beta_n(e_i) \le n^{\varepsilon}\, \mathbf{E}[\beta_n(e_i)]$ for almost all ω and all sufficiently large n. As a consequence, for $\mathbf{P}$-almost all ω and all sufficiently large n,

$$E_\omega\left(\mathrm{e}^{-n^{-r}\tau_n}\right) \le n^{2\varepsilon} \sum_{i=1}^{\mathrm{b}} \frac{\mathbf{E}[\beta_n(e_i)]}{\mathbf{E}[\beta_{n,n^{-r}}(e_i)]}.$$

By Proposition 4.2, this yields $E_\omega(\mathrm{e}^{-n^{-r}\tau_n}) \le n^{-\varepsilon}$ (for $\mathbf{P}$-almost all ω and all sufficiently large n; this is where we use $\frac{1}{\kappa-1} - \frac{r}{\kappa} > 3\varepsilon$ if $\kappa \in (1, 2]$, and $1 - \frac{r}{2} > 3\varepsilon$ if $\kappa \in (2, \infty]$). In particular, for $n_k := \lfloor k^{2/\varepsilon} \rfloor$, we have $\mathbf{P}$-almost surely, $E_\omega(\sum_k \mathrm{e}^{-n_k^{-r}\tau_{n_k}}) < \infty$, which implies that, $\mathbb{P}$-almost surely for all sufficiently large k, $\tau_{n_k} \ge n_k^r$. This implies that $\mathbb{P}$-almost surely for all sufficiently large n, $\tau_n \ge \frac{1}{2}\, n^r$. The upper bound in (1.6) of Theorem 1.2 follows. □

Proposition 4.2 is proved in Sect. 5.

5 Proof of Proposition 4.2

Let $\theta \in [0, 1]$. Let $(Z_{n,\theta})$ be a sequence of random variables, such that $Z_{1,\theta} \stackrel{\text{law}}{=} \sum_{i=1}^{\mathrm{b}} A_i$, where $(A_i, 1 \le i \le \mathrm{b})$ is distributed as $(A(x_i), 1 \le i \le \mathrm{b})$ (for any $x \in \mathbb{T}$), and that

$$Z_{j+1,\theta} \stackrel{\text{law}}{=} \sum_{i=1}^{\mathrm{b}} A_i \frac{\theta + Z_{j,\theta}^{(i)}}{1 + Z_{j,\theta}^{(i)}}, \quad \forall j \ge 1, \tag{5.1}$$

where $Z_{j,\theta}^{(i)}$ (for $1 \le i \le \mathrm{b}$) are independent copies of $Z_{j,\theta}$, and are independent of the random vector $(A_i, 1 \le i \le \mathrm{b})$.

Then, for any given $n \ge 1$ and $\lambda \ge 0$,

$$Z_{n,1-\mathrm{e}^{-2\lambda}} \stackrel{\text{law}}{=} \sum_{i=1}^{\mathrm{b}} A_i\, \beta_{n,\lambda}(e_i), \tag{5.2}$$

provided $(A_i, 1 \le i \le \mathrm{b})$ and $(\beta_{n,\lambda}(e_i), 1 \le i \le \mathrm{b})$ are independent.

Proposition 5.1 *Assume $p = \frac{1}{\mathrm{b}}$ and $\psi'(1) < 0$. Let κ be as in* (1.4). *For all $a \in (1, \kappa) \cap (1, 2]$, we have*

$$\sup_{\theta \in [0,1]} \sup_{j \ge 1} \frac{[\mathbf{E}(Z_{j,\theta})^a]}{(\mathbf{E}Z_{j,\theta})^a} < \infty.$$

Proof Let $a \in (1, 2]$. Conditioning on $A_1, \ldots, A_{\mathrm{b}}$, we can apply Lemma 3.2 to see that

$$\begin{aligned}
&\mathbf{E}\left[\left(\sum_{i=1}^{\mathrm{b}} A_i \frac{\theta + Z_{j,\theta}^{(i)}}{1 + Z_{j,\theta}^{(i)}}\right)^a \Big| A_1, \ldots, A_{\mathrm{b}}\right] \\
&\quad \le \sum_{i=1}^{\mathrm{b}} A_i^a \,\mathbf{E}\left[\left(\frac{\theta + Z_{j,\theta}}{1 + Z_{j,\theta}}\right)^a\right] + \left[\sum_{i=1}^{\mathrm{b}} A_i \,\mathbf{E}\left(\frac{\theta + Z_{j,\theta}}{1 + Z_{j,\theta}}\right)\right]^a \\
&\quad \le \sum_{i=1}^{\mathrm{b}} A_i^a \,\mathbf{E}\left[\left(\frac{\theta + Z_{j,\theta}}{1 + Z_{j,\theta}}\right)^a\right] + c_{32}\left[\mathbf{E}\left(\frac{\theta + Z_{j,\theta}}{1 + Z_{j,\theta}}\right)\right]^a,
\end{aligned}$$

where c_{32} depends on a, b and the bound of A (recalling that A is bounded away from 0 and infinity). Taking expectation on both sides, and in view of (5.1), we obtain

$$\mathbf{E}[(Z_{j+1,\theta})^a] \le \mathrm{b}\mathbf{E}(A^a)\mathbf{E}\left[\left(\frac{\theta + Z_{j,\theta}}{1 + Z_{j,\theta}}\right)^a\right] + c_{32}\left[\mathbf{E}\left(\frac{\theta + Z_{j,\theta}}{1 + Z_{j,\theta}}\right)\right]^a.$$

We divide by $(\mathbf{E}Z_{j+1,\theta})^a = \left[\mathbf{E}(\frac{\theta+Z_{j,\theta}}{1+Z_{j,\theta}})\right]^a$ on both sides, to see that

$$\frac{\mathbf{E}[(Z_{j+1,\theta})^a]}{(\mathbf{E}Z_{j+1,\theta})^a} \le \mathrm{b}\mathbf{E}(A^a)\frac{\mathbf{E}\left[\left(\frac{\theta+Z_{j,\theta}}{1+Z_{j,\theta}}\right)^a\right]}{\left[\mathbf{E}\left(\frac{\theta+Z_{j,\theta}}{1+Z_{j,\theta}}\right)\right]^a} + c_{32}.$$

Put $\xi = \theta + Z_{j,\theta}$. By (3.1), we have

$$\frac{\mathbf{E}\left[\left(\frac{\theta+Z_{j,\theta}}{1+Z_{j,\theta}}\right)^a\right]}{\left[\mathbf{E}\left(\frac{\theta+Z_{j,\theta}}{1+Z_{j,\theta}}\right)\right]^a} = \frac{\mathbf{E}\left[\left(\frac{\xi}{1-\theta+\xi}\right)^a\right]}{\left[\mathbf{E}\left(\frac{\xi}{1-\theta+\xi}\right)\right]^a} \le \frac{\mathbf{E}[\xi^a]}{[\mathbf{E}\xi]^a}.$$

Applying Lemma 3.2 to $k = 2$ yields that $\mathbf{E}[\xi^a] = \mathbf{E}[(\theta + Z_{j,\theta})^a] \le \theta^a + \mathbf{E}[(Z_{j,\theta})^a] + (\theta + \mathbf{E}(Z_{j,\theta}))^a$. It follows that $\frac{\mathbf{E}[\xi^a]}{[\mathbf{E}\xi]^a} \le \frac{\mathbf{E}[(Z_{j,\theta})^a]}{[\mathbf{E}Z_{j,\theta}]^a} + 2$, which implies that for $j \ge 1$,

$$\frac{\mathbf{E}[(Z_{j+1,\theta})^a]}{(\mathbf{E}Z_{j+1,\theta})^a} \le \mathrm{b}\mathbf{E}(A^a)\frac{\mathbf{E}[(Z_{j,\theta})^a]}{(\mathbf{E}Z_{j,\theta})^a} + (2\mathrm{b}\mathbf{E}(A^a) + c_{32}).$$

Thus, if $\mathrm{b}\mathbf{E}(A^a) < 1$ (which is the case if $1 < a < \kappa$), then

$$\sup_{j\ge 1} \frac{\mathbf{E}[(Z_{j,\theta})^a]}{(\mathbf{E}Z_{j,\theta})^a} < \infty,$$

uniformly in $\theta \in [0, 1]$. □

We now turn to the proof of Proposition 4.2. For the sake of clarity, the proofs of (4.10), (4.11) and (4.12) are presented in three distinct parts.

5.1 Proof of (4.10)

By (3.2) and (5.1), we have, for all $\theta \in [0, 1]$ and $j \geq 1$,

$$\mathbf{E}\left\{\exp\left(-t\,\frac{Z_{j+1,\theta}}{\mathbf{E}(Z_{j+1,\theta})}\right)\right\} \leq \mathbf{E}\left\{\exp\left(-t\sum_{i=1}^{\mathrm{b}} A_i \frac{Z_{j,\theta}^{(i)}}{\mathbf{E}(Z_{j,\theta}^{(i)})}\right)\right\}, \quad t \geq 0.$$

Let $f_j(t) := \mathbf{E}\left\{\exp\left(-t\frac{Z_{j,\theta}}{\mathbf{E}Z_{j,\theta}}\right)\right\}$ and $g_j(t) := \mathbf{E}(\mathrm{e}^{-tM_j})$ (for $j \geq 1$). We have

$$f_{j+1}(t) \leq \mathbf{E}\left(\prod_{i=1}^{\mathrm{b}} f_j(tA_i)\right), \quad j \geq 1.$$

On the other hand, by (4.13),

$$g_{j+1}(t) = \mathbf{E}\left\{\exp\left(-t\sum_{i=1}^{\mathrm{b}} A(e_i) M_{j+1}^{(e_i)}\right)\right\} = \mathbf{E}\left(\prod_{i=1}^{\mathrm{b}} g_j(tA_i)\right), \quad j \geq 1.$$

Since $f_1(\cdot) = g_1(\cdot)$, it follows by induction on j that for all $j \geq 1$, $f_j(t) \leq g_j(t)$; in particular, $f_n(t) \leq g_n(t)$. We take $\theta = 1 - \mathrm{e}^{-2\lambda}$. In view of (5.2), we have proved that

$$\mathbf{E}\left\{\exp\left(-t\sum_{i=1}^{\mathrm{b}} A(e_i)\frac{\beta_{n,\lambda}(e_i)}{\mathbf{E}[\beta_{n,\lambda}(e_i)]}\right)\right\} \leq \mathbf{E}\left\{\mathrm{e}^{-tM_n}\right\}, \tag{5.3}$$

which yields (4.10). □

Remark Let

$$\beta_{n,\lambda}(e) := \frac{(1 - \mathrm{e}^{-2\lambda}) + \sum_{i=1}^{\mathrm{b}} A(e_i)\beta_{n,\lambda}(e_i)}{1 + \sum_{i=1}^{\mathrm{b}} A(e_i)\beta_{n,\lambda}(e_i)}.$$

By (5.3) and (3.2), if $\mathbf{E}(A) = \frac{1}{\mathrm{b}}$, then for $\lambda \geq 0$, $n \geq 1$ and $t \geq 0$,

$$\mathbf{E}\left\{\exp\left(-t\frac{\beta_{n,\lambda}(e)}{\mathbf{E}[\beta_{n,\lambda}(e)]}\right)\right\} \leq \mathbf{E}\left\{\mathrm{e}^{-tM_n}\right\}.$$

5.2 Proof of (4.11)

Assume $p = \frac{1}{b}$ and $\psi'(1) < 0$. Since $Z_{j,\theta}$ is bounded uniformly in j, we have, by (5.1), for $1 \le j \le n-1$,

$$\begin{aligned}\mathbf{E}(Z_{j+1,\theta}) &= \mathbf{E}\left(\frac{\theta + Z_{j,\theta}}{1+Z_{j,\theta}}\right)\\ &\le \mathbf{E}\left[(\theta + Z_{j,\theta})(1 - c_{33}\, Z_{j,\theta})\right]\\ &\le \theta + \mathbf{E}(Z_{j,\theta}) - c_{33}\,\mathbf{E}\left[(Z_{j,\theta})^2\right]\\ &\le \theta + \mathbf{E}(Z_{j,\theta}) - c_{33}\,\left[\mathbf{E}Z_{j,\theta}\right]^2. \end{aligned} \tag{5.4}$$

By Lemma 3.4, we have, for any $K > 0$ and uniformly in $\theta \in [0, \frac{K}{n}]$,

$$\mathbf{E}(Z_{n,\theta}) \le c_{34}\left(\sqrt{\theta} + \frac{1}{n}\right) \le \frac{c_{35}}{\sqrt{n}}. \tag{5.5}$$

We mention that this holds for all $\kappa \in (1, \infty]$. In view of (5.2), this yields the upper bound in (4.11).

To prove the lower bound, we observe that

$$\mathbf{E}(Z_{j+1,\theta}) \ge \mathbf{E}\left[(\theta + Z_{j,\theta})(1 - Z_{j,\theta})\right] = \theta + (1-\theta)\mathbf{E}(Z_{j,\theta}) - \mathbf{E}\left[(Z_{j,\theta})^2\right]. \tag{5.6}$$

If furthermore $\kappa \in (2, \infty]$, then $\mathbf{E}[(Z_{j,\theta})^2] \le c_{36}\,(\mathbf{E}Z_{j,\theta})^2$ (see Proposition 5.1). Thus, for all $1 \le j \le n-1$,

$$\mathbf{E}(Z_{j+1,\theta}) \ge \theta + (1-\theta)\mathbf{E}(Z_{j,\theta}) - c_{36}\,(\mathbf{E}Z_{j,\theta})^2.$$

By (5.5), $\mathbf{E}(Z_{n,\theta}) \to 0$ uniformly in $\theta \in \left[0, \frac{K}{n}\right]$ (for any given $K > 0$). An application of (5.2) and Lemma 3.4 readily yields the lower bound in (4.11). □

5.3 Proof of (4.12)

We assume in this part $p = \frac{1}{b}$, $\psi'(1) < 0$ and $1 < \kappa \le 2$.

Let $\varepsilon > 0$ be small. Since $(Z_{j,\theta})$ is bounded, we have $\mathbf{E}[(Z_{j,\theta})^2] \le c_{37}\,\mathbf{E}[(Z_{j,\theta})^{\kappa-\varepsilon}]$, which, by Proposition 5.1, implies

$$\mathbf{E}\left[(Z_{j,\theta})^2\right] \le c_{38}\,\left(\mathbf{E}Z_{j,\theta}\right)^{\kappa-\varepsilon}. \tag{5.7}$$

Therefore, (5.6) yields that

$$\mathbf{E}(Z_{j+1,\theta}) \ge \theta + (1-\theta)\mathbf{E}(Z_{j,\theta}) - c_{38}\,(\mathbf{E}Z_{j,\theta})^{\kappa-\varepsilon}.$$

By (5.5), $\mathbf{E}(Z_{n,\theta}) \to 0$ uniformly in $\theta \in [0, \frac{K}{n}]$ (for any given $K > 0$). An application of Lemma 3.4 implies that for any $K > 0$,

$$\mathbf{E}(Z_{\ell,\theta}) \geq c_{14} \left(\theta^{1/(\kappa-\varepsilon)} + \frac{1}{\ell^{1/(\kappa-1-\varepsilon)}}\right), \quad \forall \theta \in [0, \frac{K}{n}], \ \forall 1 \leq \ell \leq n. \tag{5.8}$$

The lower bound in (4.12) follows from (5.2).

It remains to prove the upper bound. Define

$$Y_{j,\theta} := \frac{Z_{j,\theta}}{\mathbf{E}(Z_{j,\theta})}, \quad 1 \leq j \leq n.$$

We take $Z^{(x)}_{j-1,\theta}$ (for $x \in \mathbb{T}_1$) to be independent copies of $Z_{j-1,\theta}$, and independent of $(A(x),\ x \in \mathbb{T}_1)$. By (5.1), for $2 \leq j \leq n$,

$$\begin{aligned} Y_{j,\theta} &\overset{\text{law}}{=} \sum_{x\in\mathbb{T}_1} A(x) \frac{(\theta + Z^{(x)}_{j-1,\theta})/(1+Z^{(x)}_{j-1,\theta})}{\mathbf{E}\big[(\theta + Z^{(x)}_{j-1,\theta})/(1+Z^{(x)}_{j-1,\theta})\big]} \geq \sum_{x\in\mathbb{T}_1} A(x) \frac{Z^{(x)}_{j-1,\theta}/(1+Z^{(x)}_{j-1,\theta})}{\theta + \mathbf{E}[Z_{j-1,\theta}]} \\ &= \frac{\mathbf{E}[Z_{j-1,\theta}]}{\theta + \mathbf{E}[Z_{j-1,\theta}]} \sum_{x\in\mathbb{T}_1} A(x) Y^{(x)}_{j-1,\theta} \\ &\quad - \frac{\mathbf{E}[Z_{j-1,\theta}]}{\theta + \mathbf{E}[Z_{j-1,\theta}]} \sum_{x\in\mathbb{T}_1} A(x) \frac{(Z^{(x)}_{j-1,\theta})^2/\mathbf{E}(Z_{j-1,\theta})}{1+Z^{(x)}_{j-1,\theta}} \\ &\geq \sum_{x\in\mathbb{T}_1} A(x) Y^{(x)}_{j-1,\theta} - \Delta_{j-1,\theta}, \end{aligned}$$

where

$$\begin{aligned} Y^{(x)}_{j-1,\theta} &:= \frac{Z^{(x)}_{j-1,\theta}}{\mathbf{E}(Z_{j-1,\theta})}, \\ \Delta_{j-1,\theta} &:= \frac{\theta}{\theta + \mathbf{E}[Z_{j-1,\theta}]} \sum_{x\in\mathbb{T}_1} A(x) Y^{(x)}_{j-1,\theta} + \sum_{x\in\mathbb{T}_1} A(x) \frac{(Z^{(x)}_{j-1,\theta})^2}{\mathbf{E}(Z_{j-1,\theta})}. \end{aligned}$$

By (5.7), $\mathbf{E}\left[\frac{(Z^{(i)}_{j-1,\theta})^2}{\mathbf{E}(Z_{j-1,\theta})}\right] \leq c_{38}\,(\mathbf{E}Z_{j-1,\theta})^{\kappa-1-\varepsilon}$. On the other hand, by (5.8), $\mathbf{E}(Z_{j-1,\theta}) \geq c_{14}\,\theta^{1/(\kappa-\varepsilon)}$ for $2 \leq j \leq n$, and thus $\frac{\theta}{\theta+\mathbf{E}[Z_{j-1,\theta}]} \leq c_{39}\,(\mathbf{E}Z_{j-1,\theta})^{\kappa-1-\varepsilon}$. As a consequence, $\mathbf{E}(\Delta_{j-1,\theta}) \leq c_{40}\,(\mathbf{E}Z_{j-1,\theta})^{\kappa-1-\varepsilon}$.

If we write $\xi \overset{\text{st.}}{\geq} \eta$ to denote that ξ is stochastically greater than or equal to η, then we have proved that $Y_{j,\theta} \overset{\text{st.}}{\geq} \sum_{x\in\mathbb{T}_1}^{\text{b}} A(x) Y^{(x)}_{j-1,\theta} - \Delta_{j-1,\theta}$. Applying the same argument to each of $(Y^{(x)}_{j-1,\theta},\ x \in \mathbb{T}_1)$, we see that, for $3 \leq j \leq n$,

$$Y_{j,\theta} \overset{\text{st.}}{\geq} \sum_{u\in\mathbb{T}_1} A(u) \sum_{v\in\mathbb{T}_2:\, u=\overleftarrow{v}} A(v) Y^{(v)}_{j-2,\theta} - \left(\Delta_{j-1,\theta} + \sum_{u\in\mathbb{T}_1} A(u)\Delta^{(u)}_{j-2,\theta} \right),$$

where $Y^{(v)}_{j-2,\theta}$ (for $v \in \mathbb{T}_2$) are independent copies of $Y_{j-2,\theta}$, and are independent of $(A(w),\, w \in \mathbb{T}_1 \cup \mathbb{T}_2)$, and $(\Delta^{(u)}_{j-2,\theta},\, u \in \mathbb{T}_1)$ are independent of $(A(u),\, u \in \mathbb{T}_1)$ and are such that $\mathbb{E}[\Delta^{(u)}_{j-2,\theta}] \leq c_{40}\, (\mathbf{E}Z_{j-2,\theta})^{\kappa-1-\varepsilon}$.

By induction, we arrive at: for $j > m \geq 1$,

$$Y_{j,\theta} \overset{\text{st.}}{\geq} \sum_{x\in\mathbb{T}_m} \left(\prod_{y\in]\!] e,x]\!]} A(y) \right) Y^{(x)}_{j-m,\theta} - \Lambda_{j,m,\theta}, \tag{5.9}$$

where $Y^{(x)}_{j-m,\theta}$ (for $x \in \mathbb{T}_m$) are independent copies of $Y_{j-m,\theta}$, and are independent of the random vector $(A(w),\, 1 \leq |w| \leq m)$, and $\mathbf{E}(\Lambda_{j,m,\theta}) \leq c_{40} \sum_{\ell=1}^{m} (\mathbf{E}Z_{j-\ell,\theta})^{\kappa-1-\varepsilon}$.

Since $\mathbf{E}(Z_{i,\theta}) = \mathbf{E}\big(\frac{\theta+Z_{i-1,\theta}}{1+Z_{i-1,\theta}}\big) \geq \mathbf{E}(Z_{i-1,\theta}) - \mathbf{E}[(Z_{i-1,\theta})^2] \geq \mathbf{E}(Z_{i-1,\theta}) - c_{38}\, [\mathbf{E}Z_{i-1,\theta}]^{\kappa-\varepsilon}$ [by (5.7)], we have, for all $j \in (j_0, n]$ (with a large but fixed integer j_0) and $1 \leq \ell \leq j - j_0$,

$$\begin{aligned} \mathbf{E}(Z_{j,\theta}) &\geq \mathbf{E}(Z_{j-\ell,\theta}) \prod_{i=1}^{\ell} \left\{ 1 - c_{38}\, [\mathbf{E}Z_{j-i,\theta}]^{\kappa-1-\varepsilon} \right\} \\ &\geq \mathbf{E}(Z_{j-\ell,\theta}) \prod_{i=1}^{\ell} \left\{ 1 - c_{41}\, (j-i)^{-(\kappa-1-\varepsilon)/2} \right\}, \end{aligned}$$

the last inequality being a consequence of (5.5). Thus, for $j \in (j_0, n]$ and $1 \leq \ell \leq j^{(\kappa-1-\varepsilon)/2}$, $\mathbf{E}(Z_{j,\theta}) \geq c_{42}\, \mathbf{E}(Z_{j-\ell,\theta})$, which implies that for all $m \leq j^{(\kappa-1-\varepsilon)/2}$, $\mathbf{E}(\Lambda_{j,m,\theta}) \leq c_{43}\, m(\mathbf{E}Z_{j,\theta})^{\kappa-1-\varepsilon}$. By Chebyshev's inequality, for $j \in (j_0, n]$, $m \leq j^{(\kappa-1-\varepsilon)/2}$ and $r > 0$,

$$\mathbf{P}\left\{ \Lambda_{j,m,\theta} > \varepsilon r \right\} \leq \frac{c_{43}\, m(\mathbf{E}Z_{j,\theta})^{\kappa-1-\varepsilon}}{\varepsilon r}. \tag{5.10}$$

Let us go back to (5.9), and study the behaviour of $\sum_{x\in\mathbb{T}_m} (\prod_{y\in]\!] e,x]\!]} A(y)) Y^{(x)}_{j-m,\theta}$. Let $M^{(x)}$ (for $x \in \mathbb{T}_m$) be independent copies of M_∞ and independent of all other random variables. Since $\mathbf{E}(Y^{(x)}_{j-m,\theta}) = \mathbf{E}(M^{(x)}) = 1$, we have, by Fact 3.3, for any $a \in (1, \kappa)$,

$$\mathbf{E}\left\{\left|\sum_{x\in\mathbb{T}_m}\left(\prod_{y\in]\!]e,x]\!]} A(y)\right)(Y^{(x)}_{j-m,\theta}-M^{(x)})\right|^a\right\}$$
$$\le 2\mathbf{E}\left\{\sum_{x\in\mathbb{T}_m}\left(\prod_{y\in]\!]e,x]\!]} A(y)^a\right)\mathbf{E}\left(|Y^{(x)}_{j-m,\theta}-M^{(x)}|^a\right)\right\}.$$

By Proposition 5.1 and the fact that (M_n) is a martingale bounded in L^a, we have $\mathbf{E}(|Y^{(x)}_{j-m,\theta}-M^{(x)}|^a)\le c_{44}$. Thus,

$$\mathbf{E}\left\{\left|\sum_{x\in\mathbb{T}_m}\left(\prod_{y\in]\!]e,x]\!]} A(y)\right)\left(Y^{(x)}_{j-m,\theta}-M^{(x)}\right)\right|^a\right\}\le 2c_{44}\mathbf{E}\left\{\sum_{x\in\mathbb{T}_m}\prod_{y\in]\!]e,x]\!]} A(y)^a\right\}$$
$$=2c_{44}\,\mathrm{b}^m\,[\mathbf{E}(A^a)]^m.$$

By Chebyshev's inequality,

$$\mathbf{P}\left\{\left|\sum_{x\in\mathbb{T}_m}\left(\prod_{y\in]\!]e,x]\!]} A(y)\right)\left(Y^{(x)}_{j-m,\theta}-M^{(x)}\right)\right|>\varepsilon r\right\}\le\frac{2c_{44}\,\mathrm{b}^m[\mathbf{E}(A^a)]^m}{\varepsilon^a r^a}. \tag{5.11}$$

Clearly, $\sum_{x\in\mathbb{T}_m}(\prod_{y\in]\!]e,x]\!]}A(y))M^{(x)}$ is distributed as M_∞. We can thus plug (5.11) and (5.10) into (5.9), to see that for $j\in[j_0,n]$, $m\le j^{(\kappa-1-\varepsilon)/2}$ and $r>0$,

$$\mathbf{P}\left\{Y_{j,\theta}>(1-2\varepsilon)r\right\}\ge\mathbf{P}\left\{M_\infty>r\right\}-\frac{c_{43}\,m(\mathbf{E}Z_{j,\theta})^{\kappa-1-\varepsilon}}{\varepsilon r}$$
$$-\frac{2c_{44}\,\mathrm{b}^m[\mathbf{E}(A^a)]^m}{\varepsilon^a r^a}. \tag{5.12}$$

We choose $m:=\lfloor j^\varepsilon\rfloor$. Since $a\in(1,\kappa)$, we have $\mathrm{b}\mathbf{E}(A^a)<1$, so that $\mathrm{b}^m[\mathbf{E}(A^a)]^m\le\exp(-j^{\varepsilon/2})$ for all large j. We choose $r=\frac{1}{(\mathbf{E}Z_{j,\theta})^{1-\delta}}$, with $\delta:=\frac{4\kappa\varepsilon}{\kappa-1}$. In view of (4.9), we obtain: for $j\in[j_0,n]$,

$$\mathbf{P}\left\{Y_{j,\theta}>\frac{1-2\varepsilon}{(\mathbf{E}Z_{j,\theta})^{1-\delta}}\right\}\ge c_{23}\,(\mathbf{E}Z_{j,\theta})^{(1-\delta)\kappa}-\frac{c_{43}}{\varepsilon}\,j^\varepsilon\,(\mathbf{E}Z_{j,\theta})^{\kappa-\varepsilon-\delta}$$
$$-\frac{2c_{44}\,(\mathbf{E}Z_{j,\theta})^{(1-\delta)a}}{\varepsilon^a\exp(j^{\varepsilon/2})}.$$

Since $c_{14}/j^{1/(\kappa-1-\varepsilon)}\le\mathbf{E}(Z_{j,\theta})\le c_{35}/j^{1/2}$ [see (5.8) and (5.5), respectively], we can pick up sufficiently small ε, so that for $j\in[j_0,n]$,

$$\mathbf{P}\left\{Y_{j,\theta}>\frac{1-2\varepsilon}{(\mathbf{E}Z_{j,\theta})^{1-\delta}}\right\}\ge\frac{c_{23}}{2}\,(\mathbf{E}Z_{j,\theta})^{(1-\delta)\kappa}.$$

Recall that by definition, $Y_{j,\theta} = \frac{Z_{j,\theta}}{\mathbf{E}(Z_{j,\theta})}$. Therefore, for $j \in [j_0, n]$,

$$\mathbf{E}[(Z_{j,\theta})^2] \geq [\mathbf{E}Z_{j,\theta}]^2 \frac{(1-2\varepsilon)^2}{(\mathbf{E}Z_{j,\theta})^{2(1-\delta)}} \mathbf{P}\left\{ Y_{j,\theta} > \frac{1-2\varepsilon}{(\mathbf{E}Z_{j,\theta})^{1-\delta}} \right\} \geq c_{45}\, (\mathbf{E}Z_{j,\theta})^{\kappa+(2-\kappa)\delta}.$$

Of course, the inequality holds trivially for $0 \leq j < j_0$ (with possibly a different value of the constant c_{45}). Plugging this into (5.4), we see that for $1 \leq j \leq n-1$,

$$\mathbf{E}(Z_{j+1,\theta}) \leq \theta + \mathbf{E}(Z_{j,\theta}) - c_{46}\, (\mathbf{E}Z_{j,\theta})^{\kappa+(2-\kappa)\delta}.$$

By Lemma 3.4, this yields $\mathbf{E}(Z_{n,\theta}) \leq c_{47} \left\{ \theta^{1/[\kappa+(2-\kappa)\delta]} + n^{-1/[\kappa-1+(2-\kappa)\delta]} \right\}$. An application of (5.2) implies the desired upper bound in (4.12). □

Remark A close inspection on our argument shows that under the assumptions $p = \frac{1}{\mathrm{b}}$ and $\psi'(1) < 0$, we have, for any $1 \leq i \leq \mathrm{b}$ and uniformly in $\lambda \in \left[0, \frac{1}{n}\right]$,

$$\left(\frac{\alpha_{n,\lambda}(e_i)}{\mathbf{E}[\alpha_{n,\lambda}(e_i)]}, \frac{\beta_{n,\lambda}(e_i)}{\mathbf{E}[\beta_{n,\lambda}(e_i)]}, \frac{\gamma_n(e_i)}{\mathbf{E}[\gamma_n(e_i)]} \right) \xrightarrow{\text{law}} (M_\infty, M_\infty, M_\infty),$$

where "$\xrightarrow{\text{law}}$" stands for convergence in distribution, and M_∞ is the random variable defined in (4.7).

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