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Nick Dungey

Time regularity for random walks on locally compact groups

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Abstract. Let G be a compactly generated, locally compact group, and let T be the operator of convolution with a probability measure μ on G. Our main results give sufficient conditions on μ for the operator T to be analytic in $L^p(G)$, $1 < p < \infty$, where analyticity means that one has an estimate of form $\|(I - T)T^n\| \le cn^{-1}$ for all $n = 1, 2, \ldots$ in L^p operator norm. Counterexamples show that analyticity may not hold if some of the conditions are not satisfied.

1. Introduction and statement of results

Let $\mathcal{L}(X)$ be the space of bounded linear operators in a Banach space X. An operator $S \in \mathcal{L}(X)$ is said to be *analytic* (cf. [7, 4]) if there exists $c > 0$ with $\|(I - S)S^n\| \le cn^{-1}$ for all $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. This notion is an analogue, for the discrete time semigroup $(S^n)_{n\in\mathbb{N}}$, of the usual notion of analyticity for a continuous time semigroup $(\mathrm{e}^{tH})_{t\ge 0}$ which corresponds to an estimate $\|H\mathrm{e}^{tH}\| \le ct^{-1}$, $t > 0$. Analyticity is a time regularity property which is highly useful for study of the semigroup (S^n), especially in cases where S is not self-adjoint in Hilbert space so that the spectral theorem is not available.

The aim of this paper is to obtain very general conditions under which the Markov operators for random walks on groups are analytic in L^p spaces. More precisely, we consider a locally compact, compactly generated group G and a $\mu \in \mathbb{P}(G)$ where $\mathbb{P}(G)$ denotes the set of regular Borel probability measures on G. We are interested in analyticity of the (right-invariant) Markov operator $T = T_\mu$ defined by the convolution

$$(Tf)(h) = (\mu * f)(h) = \int_G d\mu(g)\, f(g^{-1}h) \tag{1}$$

for $h \in G$, $f \in L^p := L^p(G; dg)$, where dg denotes a fixed left invariant Haar measure on G. Most of our results deal with L^2, though they usually extend to

N. Dungey: School of Mathematics, The University of New South Wales, Sydney 2052, Australia. e-mail: `dungey@maths.unsw.edu.au`

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L^p when $1 < p < \infty$. For some related studies of analytic Markov operators in specific situations, see [4, 1, 2, 10, 11]. In particular, the present paper extends results of [11] for locally compact groups.

Our results suggest that much of the theory of self-adjoint Markov operators on groups could be extended to non-self-adjoint operators, and there may be possible future applications of our results in this direction.

To state our main results we fix some notation (for background material see [18, 12]). Since G is compactly generated, we may fix an open, relatively compact neighborhood U of the identity $e \in G$ which is symmetric ($U = U^{-1}$) and generates G. The standard modulus $\rho = \rho_U : G \to \mathbb{N}$ is then defined by

$$\rho(g) = \inf\{n \in \mathbb{N} : g \in U^n\}, \quad g \in G,$$

where $U^n := \{g_1 \cdots g_n : g_1, \ldots, g_n \in U\}$. A probability measure $\mu \in \mathbb{P}(G)$ is said to be *adapted* if the closed subgroup generated by its support $\mathrm{supp}(\mu)$ equals G, and is said to be *spread out* if there exists $n \in \mathbb{N}$ such that $\mu^{(n)}$ is not singular with respect to Haar measure dg, where $\mu^{(n)} := \mu * \cdots * \mu$ denotes the n-th convolution power of μ. For ν_1, ν_2 Borel measures on G, the notation $\nu_1 \geq \nu_2$ will mean that $\nu_1 - \nu_2$ is a positive measure. One sees that $\mu \in \mathbb{P}(G)$ is spread out if and only if there exist $n \in \mathbb{N}$, a constant $c > 0$ and a non-empty open set $V \subseteq G$ such that $\mu^{(n)} \geq c\chi_V$. (Here, χ_V denotes the characteristic function of $V \subseteq G$ or, more precisely, the measure $\chi_V(g)dg$.)

The following concept of *centeredness* (compare [1, 2, 17, 11]) plays a crucial role in our results. Consider the canonical homomorphism $\pi_0 : G \to G/G_0$ where $G_0 := \overline{[G, G]}$ is the closure in G of the commutator subgroup $[G, G]$. Now G/G_0 is a compactly generated locally compact abelian group, so that by a standard theorem (see [13, Theorem II.9.8]) it is isomorphic with $\mathbb{Z}^{q_1} \times \mathbb{R}^{q_2} \times M$ for some $q_1, q_2 \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and some compact abelian group M. Define the closed normal subgroup $G_1 := \pi_0^{-1}(\{0\} \times M)$ of G, and consider the canonical homomorphism $\pi : G \to G/G_1 \cong \mathbb{Z}^{q_1} \times \mathbb{R}^{q_2}$ with components $\pi^{(i)} : G \to \mathbb{R}$, $i \in \{1, \ldots, q_1 + q_2\}$. One says that $\mu \in \mathbb{P}(G)$ is *centered* if

$$\int_G d\mu(g)\, \pi^{(i)}(g) = 0$$

for all $i \in \{1, \ldots, q_1 + q_2\}$ where the integrals are assumed to be absolutely convergent.

Finally, δ_g denotes the probability measure concentrated at $g \in G$, and for $\mu \in \mathbb{P}(G)$ the involute $\mu^* \in \mathbb{P}(G)$ is defined by $\mu^*(A) = \mu(A^{-1})$ for Borel sets $A \subseteq G$. We say μ is *symmetric* if $\mu = \mu^*$; any symmetric μ which satisfies $\int d\mu(g)\, |\pi^{(i)}(g)| < \infty$, $i \in \{1, \ldots, q_1 + q_2\}$, is centered, but centered measures are not symmetric in general.

Our basic tool to study analyticity is the following abstract characterization due to Nevanlinna (see [15, Theorem 4.5.4], [16, Theorem 2.1], and [4, 5]). Recall that an operator $S \in \mathcal{L}(X)$ is said to be *power-bounded* if $\sup_{n \in \mathbb{N}} \|S^n\| < \infty$.

Theorem 1.1 *Let X be a complex Banach space and let $S \in \mathcal{L}(X)$. Put $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Then S is analytic and power-bounded if and only if the spectrum of S is a subset of $\mathbb{D} \cup \{1\}$ and the semigroup $(e^{-t(I-S)})_{t\geq 0}$ is bounded analytic.*

We shall denote by $\sigma_{L^p}(S)$ the L^p spectrum of any operator $S \in \mathcal{L}(L^p)$.

The main theorem of this paper is the following (unless otherwise stated, in the rest of the paper T will always denote the Markov operator (1) associated with $\mu \in \mathbb{P}(G)$).

Theorem 1.2 *Let $\mu \in \mathbb{P}(G)$ be centered, adapted, spread out, and such that $\int_G d\mu(g)\, \rho(g)^2 < \infty$. Suppose $\sigma_{L^2}(T) \subseteq \mathbb{D} \cup \{1\}$. Then T is analytic in L^2.*

The conclusion of Theorem 1.2 was obtained in [11] for a smaller class of centered adapted measures (essentially, the compactly supported measures with a bounded density with respect to Haar measure). See also [1, 2] for estimates implying analyticity in the particular case of groups of polynomial volume growth.

We obtain Theorem 1.2 as an immediately corollary of Theorem 1.1 and the following result.

Theorem 1.3 *Let $\mu \in \mathbb{P}(G)$ be centered, adapted, spread out, and such that $\int_G d\mu(g)\, \rho(g)^2 < \infty$. Then the semigroup $(e^{-t(I-T)})_{t\geq 0}$ is bounded analytic in L^2.*

Consider the left translation operators $L(g)$ and the difference operators ∂_g defined by $(L(g)f)(h) = f(g^{-1}h)$, $(\partial_g f)(h) = ((L(g) - I)f)(h) = f(g^{-1}h) - f(h)$ for all $g, h \in G$ and $f : G \to \mathbb{C}$. For the proof of Theorem 1.3, an essential ingredient is the "Dirichlet norm" Γ_2 defined by

$$\Gamma_2(f) := \left(\int_U \mathrm{d}u\, \|\partial_u f\|_2^2 \right)^{1/2} \tag{2}$$

for all $f \in L^2$. In fact, the proof depends on comparisons of $\Gamma_2(f)^2$ with the real and imaginary parts of the quadratic form $f \mapsto ((I - T)f, f)$, $f \in L^2$. Note that $I - T$ may be called the "discrete Laplacian" associated with $\mu \in \mathbb{P}(G)$ and can be written in the form

$$(I - T)f = -\int_G \mathrm{d}\mu(g)\, \partial_g f \tag{3}$$

for any $f \in L^p$.

In Theorems 1.2 and 1.3, the "finite second moment" hypothesis that $\int \mathrm{d}\mu\, \rho^2 < \infty$ can be slightly weakened to a condition on the "antisymmetric" part of μ; for details, see the end of Sect. 3. The weakened condition holds, for example, for arbitrary symmetric probability measures.

The next result, a corollary of Theorem 1.2, provides an interesting L^2 spatial regularity estimate. We write $\|\cdot\|_{p\to q}$ to denote the norm of a bounded linear operator from L^p to L^q.

Corollary 1.4 *Assume the hypotheses of Theorem 1.2. Then there exists $c > 0$ such that*

$$\|\partial_g T^n\|_{2\to 2} \le c\rho(g)n^{-1/2}$$

for all $n \in \mathbb{N}$ and $g \in G$.

Theorems 1.3 and 1.2 have analogues in L^p for $1 < p < \infty$, as follows.

Corollary 1.5 *Under the hypotheses of Theorem 1.3, then the semigroup $(\mathrm{e}^{-t(I-T)})_{t\ge 0}$ is bounded analytic in L^p for each $p \in (1,\infty)$.*

Moreover, under the hypotheses of Theorem 1.2, T is analytic in L^p for each $p \in (1,\infty)$.

Corollary 1.5 follows from Theorems 1.3 and 1.2 by interpolation methods. Indeed, $(\mathrm{e}^{-t(I-T)})_{t\ge 0}$ is a contraction semigroup in L^p for all $p \in [1,\infty]$, that is, $\|\mathrm{e}^{-t(I-T)}\|_{p\to p} \le 1$, and if it is bounded analytic in L^2 then a standard application of the Stein interpolation theorem (compare [9, Theorem 1.4.2]) shows that it is bounded analytic in L^p for $p \in (1,\infty)$. Similarly, since T is a contraction in L^p for all $p \in [1,\infty]$, the second statement of Corollary 1.5 follows by an interpolation theorem of Blunck [5] for analytic operators. Blunck's theorem states that if $S \in \mathcal{L}(L^{p_1}) \cap \mathcal{L}(L^{p_2})$ is power-bounded in L^{p_1} and L^{p_2}, and analytic in L^{p_1}, then it is analytic in L^p for all p strictly between p_1 and p_2.

None of the hypotheses "centered", "spread out" or "adapted" can be omitted from Theorems 1.2 and 1.3. In Sect. 5 we shall give counterexamples, and also obtain the following negative result for non-centered measures on amenable groups.

Theorem 1.6 *Let G be amenable and $\mu \in \mathbb{P}(G)$ such that $\int \mathrm{d}\mu(g)\, |\pi^{(i)}(g)| < \infty$ for all $i \in \{1,\dots,q_1+q_2\}$ and μ is not centered. Then the semigroup $(e^{-t(I-T)})_{t\ge 0}$ is not bounded analytic in L^2, hence T is not analytic in L^2. If in addition $\int d\mu(g)\, |\pi^{(i)}(g)|^2 < \infty$ for all i, then there exists $c > 0$ such that*

$$\|(I-T)e^{-t(I-T)}\|_{2\to 2} \ge ct^{-1/2}$$

for all $t \ge 1$.

The situation for non-amenable groups is as follows. If G is not amenable and $\mu \in \mathbb{P}(G)$ is any adapted probability measure, then it is known (see, for example, [3]) that the spectral radius $r(T) = \lim_{n\to\infty} \|T^n\|_{2\to 2}^{1/n}$ is strictly less than 1; one then has for some $c, \omega > 0$ an estimate $\|T^n\|_{2\to 2} \le c\mathrm{e}^{-\omega n}$, $n \in \mathbb{N}$, and it follows trivially that T is analytic in L^2. For this reason, the L^2 results of this paper are chiefly of interest for amenable groups.

Our final result is the following simple sufficient criterion for the spectral condition of Theorem 1.1, which generalizes criteria of [10, 11].

Theorem 1.7 *Let $\mu \in \mathbb{P}(G)$. If there exist $\nu \in \mathbb{P}(G)$ and a constant $\alpha \in (0,1)$ such that $\mu \ge \alpha(\nu^* * \nu)$, then $\sigma_{L^2}(T) \subseteq \mathbb{D} \cup \{1\}$.*

In particular, this result applies whenever μ satisfies either: (i) $\mu \ge \alpha\delta_e$, or (ii) $\mu \ge \varepsilon\chi_W$ for some $\varepsilon > 0$ and some neighborhood W of e in G.

The paper is organized as follows. Since the proof of Theorem 1.7 is relatively short, we give it first in Sect. 2. The proof of Theorem 1.3 is contained in Sects. 3 and 4. This proof depends crucially on a certain "Taylor expansion" estimate on G (see (4)) which was established in [11]. In order to keep the present paper largely self-contained, we give a proof of this estimate in an Appendix (Sect. 6) in the special case where G is discrete. Finally, in Sect. 5 we prove Theorem 1.6 and discuss counterexamples.

2. Proof of theorem 1.7

Fix $\mu, \nu \in \mathbb{P}(G)$ and $\alpha \in (0, 1)$ as in the statement of Theorem 1.7, and write $Tf := \mu * f$, $f \in L^p$. We also set $\overline{\nu} := \nu^* * \nu \in \mathbb{P}(G)$ and let $T'f := \alpha(\overline{\nu} * f)$. Then $\|T'\|_{2\to 2} \leq \alpha$. Note that

$$(T - T')f = (\mu - \alpha\overline{\nu}) * f$$

where $\mu - \alpha\overline{\nu}$ is a positive measure by hypothesis, and $(\mu - \alpha\overline{\nu})(G) = 1 - \alpha$. It follows that $\|T - T'\|_{2\to 2} \leq 1 - \alpha$. Next observe that $(T'f, f) = \alpha(\nu * f, \nu * f) \geq 0$, so that T' is a non-negative self-adjoint operator in L^2.

For any $A \subseteq \mathbb{C}$, $z \in \mathbb{C}$, let $\mathrm{d}(z, A) = \inf\{|z - a| : a \in A\}$ denote the distance between z and A. Whenever $\|f\|_2 = 1$, we have $(T'f, f) \in [0, \alpha]$ and

$$(Tf, f) = ((T - T')f, f) + (T'f, f) \in \Lambda_\alpha$$

where Λ_α is the set defined by $\Lambda_\alpha = \{z \in \mathbb{C} : d(z, [0, \alpha]) \leq 1 - \alpha\}$. By a standard Hilbert space result (see [14, Corollary V.3.3]), $\sigma_{L^2}(T)$ is contained in the closure in $\mathbb{C}$ of the set $\{(Tf, f) : \|f\|_2 = 1\}$. We conclude that $\sigma_{L^2}(T) \subseteq \Lambda_\alpha \subseteq \mathbb{D} \cup \{1\}$ where the last inclusion used that $\alpha \in (0, 1)$. This proves the first statement of the theorem.

In case μ satisfies $\mu \geq \alpha\delta_e$, then since $\delta_e^* * \delta_e = \delta_e$ we may apply the first statement of the theorem with $\nu = \delta_e$.

Alternatively, if $\mu \geq \varepsilon\chi_W$ where W is a neighborhood of e and $\varepsilon > 0$, then choose a compact neighborhood V of e with $V = V^{-1}$ and $VV \subseteq W$ and consider $\nu := |V|^{-1}\chi_V dg \in \mathbb{P}(G)$ where $|V| = dg(V) > 0$. It is then easy to see that $\mu \geq \varepsilon\chi_W \geq \alpha(\nu^* * \nu)$ for some sufficiently small $\alpha \in (0, 1)$, and again the first statement of the theorem applies.

3. Proof of theorem 1.3

The proof of Theorem 1.3 is based on the following well known criterion for analyticity of semigroups in Hilbert space (see for example [14, Theorem IX.1.24]).

Proposition 3.1 *Let $\mathcal{H}$ be a complex Hilbert space, let $S \in \mathcal{L}(\mathcal{H})$ and suppose there exists $c > 0$ such that*

$$|\operatorname{Im}(Sf, f)| \leq c \operatorname{Re}(Sf, f)$$

for all $f \in \mathcal{H}$. Then $(e^{-tS})_{t\geq 0}$ is a bounded analytic semigroup in $\mathcal{H}$; in fact, there exist constants $\theta_1 \in (0, \pi/2)$, $c_1 > 0$, such that $\|e^{-zS}\| \leq 1$ whenever $z \in \mathbb{C}$ with $|\arg(z)| < \theta_1$, and $\|Se^{-tS}\| \leq c_1 t^{-1}$ for all $t > 0$.

Theorem 1.3 will follow immediately from Proposition 3.1 together with the following two propositions.

Proposition 3.2 *Let $\mu \in \mathbb{P}(G)$ be centered and satisfy $\int_G d\mu(g)\,\rho(g)^2 < \infty$. Then there is $c > 0$ such that $|((I - T)f, f)| \le c\Gamma_2(f)^2$ for all $f \in L^2$.*

Proposition 3.3 *Let $\mu \in \mathbb{P}(G)$ be adapted and spread out. Then there exists $c > 0$ such that $\mathrm{Re}((I - T)f, f) \ge c^{-1}\Gamma_2(f)^2$ for all $f \in L^2$.*

In the rest of this section, we give the proof of Proposition 3.2. The proof of Proposition 3.3 will be given in Sect. 4 below (and that will complete the proof of Theorem 1.3).

The next theorem is the crucial tool in the proof of Proposition 3.2. It may be thought of as a type of second order Taylor expansion for arbitrary locally compact, compactly generated groups.

To state it, consider the homomorphism $\pi : G \to G/G_1$ defined as in Sect. 1 and identify $G/G_1 = \mathbb{Z}^{q_1} \times \mathbb{R}^{q_2} \subseteq \mathbb{R}^q$, where $q := q_1 + q_2$, so that $\pi(g) = (\pi^{(1)}(g), \ldots, \pi^{(q)}(g))$ for all $g \in G$. Let $e_1, \ldots, e_q$ be the standard basis of $\mathbb{R}^q$ (e_j has a 1 in the j-th position and zeroes elsewhere). As shown in [11, Sect. 9], we can find elements $z_1, \ldots, z_q \in G$ such that $\pi(z_j) = e_j$ and such that $z_{q_1+1}, \ldots, z_q$ are contained in one-parameter subgroups of G; more precisely, the latter condition means that there exist continuous homomorphisms $\theta_1, \ldots, \theta_{q_2}$ of $\mathbb{R}$ into G such that $z_{q_1+j} = \theta_j(1)$ and $\pi(\theta_j(t)) = te_{q_1+j}$ for all $t \in \mathbb{R}$ and $j \in \{1, \ldots, q_2\}$.

Theorem 3.4 *Let $z_1, \ldots, z_q$ be as above. There exists a $c > 0$ such that*

$$\left| \left(\left(\partial_g - \sum_{i=1}^{q} \pi^{(i)}(g)\partial_{z_i} \right) f_1, f_2 \right) \right| \le c\rho(g)^2\Gamma_2(f_1)\Gamma_2(f_2)$$

for all $g \in G$ and $f_1, f_2 \in L^2$.

Proof In [11, Sect. 9] this estimate is proved over compact subsets of G: that is, given any relatively compact $K \subseteq G$ one has

$$\left| \left(\left(\partial_g - \sum_{i=1}^{q} \pi^{(i)}(g)\partial_{z_i} \right) f_1, f_2 \right) \right| \le c(K)\Gamma_2(f_1)\Gamma_2(f_2) \tag{4}$$

for all $g \in K$, $f_1, f_2 \in L^2$, where $c(K) > 0$ is a constant depending on K. (See the Appendix, Sect. 6 below, for a simpler proof of (4) when G is discrete.) To extend this result we need the standard estimate (cf. [18, Proposition VII.3.2]) that

$$\|\partial_g f\|_2 \le c_U\rho(g)\Gamma_2(f) \tag{5}$$

for all $f \in L^2$, where c_U is a constant depending only on U, G and the choice of Haar measure dg.

Given any $g \in G$, let us set $n = \rho(g) \in \mathbb{N}$ and write $g = g_1 \cdots g_n$ where $g_1, \ldots, g_n \in U$. In what follows, c, c' will denote positive constants independent of g and of $f_1, f_2 \in L^2$. A direct computation yields the general identity

$$\partial_{g_1 g_2 \cdots g_n} = \sum_{j=1}^{n} \partial_{g_j} + \sum_{k<j} \partial_{g_k} L(g_{k+1} \cdots g_{j-1}) \partial_{g_j} \tag{6}$$

where the second sum is over all $j, k \in \{1, \ldots, n\}$ satisfying $k < j$, and where we set $L(g_{k+1} \cdots g_{j-1}) := I$ in case $j = k + 1$. But

$$\begin{aligned} |(\partial_{g_k} L(h) \partial_{g_j} f_1, f_2)| &= |(L(h) \partial_{g_j} f_1, \partial_{g_k^{-1}} f_2)| \\ &\leq \|\partial_{g_j} f_1\|_2 \|\partial_{g_k^{-1}} f_2\|_2 \leq c \Gamma_2(f_1) \Gamma_2(f_2) \end{aligned}$$

for all $j, k \in \{1, \ldots, n\}$ and $h \in G$, where the last step used (5) with g_j, g_k^{-1} replacing g. We deduce that

$$\left| \left(\left(\partial_g - \sum_{j=1}^{n} \partial_{g_j} \right) f_1, f_2 \right) \right| \leq c' n^2 \Gamma_2(f_1) \Gamma_2(f_2).$$

Then, since $\pi^{(i)}(g) = \sum_{j=1}^{n} \pi^{(i)}(g_j)$ and applying (4) with $K = U$, we obtain

$$\begin{aligned} &\left| \left(\left(\partial_g - \sum_{i=1}^{q} \pi^{(i)}(g) \partial_{z_i} \right) f_1, f_2 \right) \right| \\ &\quad \leq \left| \left(\left(\partial_g - \sum_{j=1}^{n} \partial_{g_j} \right) f_1, f_2 \right) \right| + \sum_{j=1}^{n} \left| \left(\left(\partial_{g_j} - \sum_{i=1}^{q} \pi^{(i)}(g_j) \partial_{z_i} \right) f_1, f_2 \right) \right| \\ &\quad \leq (cn^2 + cn) \Gamma_2(f_1) \Gamma_2(f_2) \leq c' \rho(g)^2 \Gamma_2(f_1) \Gamma_2(f_2) \end{aligned}$$

because $n = \rho(g) \leq \rho(g)^2$. □

Proof of Proposition 3.2 From (3) and since μ is assumed centered we have

$$((I - T) f, f) = - \int_G \mathrm{d}\mu(g) \left(\left[\partial_g - \sum_{i=1}^{q} \pi^{(i)}(g) \partial_{z_i} \right] f, f \right),$$

and by Theorem 3.4 then $|((I - T) f, f)| \leq c \int_G \mathrm{d}\mu(g)\, \rho(g)^2 \Gamma_2(f)^2 \leq c' \Gamma_2(f)^2$ for all $f \in L^2$. □

Remark The second moment hypothesis $\int \mathrm{d}\mu(g)\, \rho(g)^2 < \infty$ in Theorems 1.2 and 1.3 can be slightly weakened, in the following way.

Given $\mu \in \mathbb{P}(G)$ define the symmetric and antisymmetric parts $\mu_{\mathrm{s}} := 2^{-1}(\mu + \mu^*) \in \mathbb{P}(G)$ and $\mu_{\mathrm{as}} := 2^{-1}(\mu - \mu^*)$, so that $\mu = \mu_{\mathrm{s}} + \mu_{\mathrm{as}}$. Note that μ_{as} is a finite signed Borel measure, whose variation measure $|\mu_{\mathrm{as}}|$ satisfies $|\mu_{\mathrm{as}}| \leq \mu_{\mathrm{s}}$.

We claim that the hypothesis $\int \mathrm{d}\mu(g)\,\rho(g)^2 < \infty$ in the cited Theorems can be weakened to

$$\int_G \mathrm{d}|\mu_{as}|(g)\,\rho(g)^2 < \infty. \tag{7}$$

To see this, observe that for centered μ, writing $i = (-1)^{1/2}$ we have

$$\begin{aligned}\mathrm{Im}((I-T)f, f) &= (2i)^{-1}\left\{(I-T)f, f) - ((I-T^*)f, f)\right\}\\ &= i\int_G \mathrm{d}\mu_{as}(g)\,(\partial_g f, f)\\ &= i\int_G \mathrm{d}\mu_{as}(g)\left(\left[\partial_g - \sum_{j=1}^{q}\pi^{(j)}(g)\partial_{z_j}\right]f, f\right).\end{aligned}$$

Assuming (7) we obtain $|\,\mathrm{Im}((I-T)f, f)| \le c\Gamma_2(f)^2$, $f \in L^2$. This variation of the estimate of Proposition 3.2 leads, as before, to the conclusion of Theorem 1.3.

4. Further proofs

In this section we prove Proposition 3.3 and then Corollary 1.4.

We begin with a simple lemma whose proof is included for the reader's convenience.

Lemma 4.1 *Let $\mu, \nu \in \mathbb{P}(G)$ and $g_0, h_0 \in G$. Suppose that $g_0 \in \mathrm{supp}(\mu)$ and that $\nu \ge c\chi_V$ for some $c > 0$ and some relatively compact open set V with $h_0 \in V$. Then there exists $c' > 0$ and a relatively compact open set W such that $g_0h_0 \in W$ and $\mu * \nu \ge c'\chi_W$.*

Proof Note that $(\mu * \chi_V)(g) = \mu(gV^{-1})$ for all $g \in G$. By continuity of the group multiplication, we can choose relatively compact open sets W, W' with $g_0h_0 \in W$, $g_0 \in W'$ and $(W')^{-1}W \subseteq V$. It follows that $gV^{-1} \supseteq W'$ for all $g \in W$, and setting $\varepsilon := \mu(W') > 0$ we have

$$(\mu * \chi_V)(g) = \mu(gV^{-1}) \ge \varepsilon$$

for all $g \in W$. Then $\mu * \nu \ge c(\mu * \chi_V) \ge c\varepsilon\chi_W$. □

We are ready to prove Proposition 3.3. Given $\mu \in \mathbb{P}(G)$ which is adapted and spread out, let $\overline{\mu} := 2^{-1}\delta_e + 4^{-1}(\mu + \mu^*) \in \mathbb{P}(G)$ and consider the corresponding Markov operators $Tf := \mu * f$ and $\overline{T}f := \overline{\mu} * f$. Clearly $\overline{\mu}$ is adapted, spread out, symmetric, and $\overline{\mu} \ge 2^{-1}\delta_e$, and an easy calculation gives

$$((I-\overline{T})f, f) = 4^{-1}((I-T)f, f) + 4^{-1}((f, (I-T)f) = 2^{-1}\,\mathrm{Re}((I-T)f, f)$$

for all $f \in L^2$. Therefore by replacing μ by $\overline{\mu}$, without loss of generality we will assume in the rest of the proof that μ is symmetric and $\mu \ge 2^{-1}\delta_e$.

Let S be the set of all $g \in G$ for which there exist $n \in \mathbb{N}$, $c > 0$ and a relatively compact open neighborhood V of g satisfying $\mu^{(n)} \geq c\chi_V$. Then S is non-empty since μ is spread out; moreover, it is easy to check that S is an open, hence closed, subgroup of G (that $S = S^{-1}$ follows from the assumption that μ is symmetric). One also sees that $\operatorname{supp}(\mu) \subseteq S$, by applying Lemma 4.1 with $h_0 = e$. Since μ is adapted and S is a closed subgroup of G, we must have $S = G$.

We can therefore cover the compact set $\overline{U}$ with a finite collection of open, relatively compact sets V_k such that inequalities of form $\mu^{(n_k)} \geq c_k\chi_{V_k}$ hold for each k, with some $n_k \in \mathbb{N}$, $c_k > 0$. But $\mu \geq 2^{-1}\delta_e$ implies that $\mu^{(n+m)} \geq 2^{-m}\mu^{(n)}$ for all $m, n \in \mathbb{N}$. Hence, setting $N = \max_k n_k \in \mathbb{N}$ we have

$$\mu^{(N)} \geq c\chi_{\overline{U}}.$$

Since $\mu^{(N)}$ is symmetric, a standard calculation then yields

$$((I - T^N)f, f) = 2^{-1}\int_G \mathrm{d}\mu^{(N)}(g)\, \|\partial_g f\|_2^2 \geq c' \int_U \mathrm{d}g\, \|\partial_g f\|_2^2 = c'\Gamma_2(f)^2$$

for $f \in L^2$. But since T is a self-adjoint contraction in L^2, the spectral theorem easily shows that $((I-T^N)f, f) \leq c''((I-T)f, f)$ for all $f \in L^2$. Proposition 3.3 now follows.

Proof of Corollary 1.4 For all $f \in L^2$, applying (5), Proposition 3.3 and then Theorem 1.2 gives

$$\begin{aligned}\|\partial_g T^n f\|_2^2 &\leq c\rho(g)^2\Gamma_2(T^n f)^2\\ &\leq c'\rho(g)^2\operatorname{Re}((I-T)T^n f, T^n f) \leq c''\rho(g)^2 n^{-1}\|f\|_2^2,\end{aligned}$$

which implies the Corollary. □

5. Proof of theorem 1.6 and final remarks

To prove Theorem 1.6, we first examine the simplest case where $G = \mathbb{R}$ and $\mu \in \mathbb{P}(\mathbb{R})$. The L^2 spectral decomposition of the Markov operator T then comes from the Fourier transform $f \mapsto \widehat{f}$ which maps $L^2(\mathbb{R})$ unitarily onto itself. Explicitly,

$$(\widehat{Tf})(\xi) = \widehat{\mu}(\xi)\widehat{f}(\xi)$$

for all $f \in L^2 = L^2(\mathbb{R})$, $\xi \in \mathbb{R}$, where $\widehat{\mu} \in C(\mathbb{R})$ is the Fourier transform of μ given by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} \mathrm{d}\mu(x)\, \mathrm{e}^{-ix\xi}.$$

The spectrum $\sigma_{L^2}(T)$ is the set $\overline{\{\widehat{\mu}(\xi) : \xi \in \mathbb{R}\}}$, where in general $\overline{A}$ denotes the closure of a set $A \subseteq \mathbb{C}$.

Lemma 5.1 *For $\mu \in \mathbb{P}(\mathbb{R})$ as above, the semigroup $(\mathrm{e}^{-t(I-T)})_{t\geq 0}$ is bounded analytic in $L^2(\mathbb{R})$ if and only if there exists a $c > 0$ such that*

$$|\operatorname{Im}(1-\widehat{\mu}(\xi))| = |\operatorname{Im}\widehat{\mu}(\xi)| \leq c\operatorname{Re}(1-\widehat{\mu}(\xi)) \tag{8}$$

for all $\xi \in \mathbb{R}$.

Proof If (8) holds then $|\operatorname{Im}((I-T)f, f)| \leq c\operatorname{Re}((I-T)f, f)$ for all $f \in L^2$, hence $(\mathrm{e}^{-t(I-T)})$ is bounded analytic by Proposition 3.1.

Conversely, suppose that $(\mathrm{e}^{-t(I-T)})$ is bounded analytic. By standard semigroup theory (see [8, Theorem 2.33]) the spectrum $\sigma_{L^2}(I-T)$ must then be contained in a closed sector $\Theta(\omega) := \{z \in \mathbb{C} : |\arg(z)| \leq \omega\} \cup \{0\}$ for some $\omega \in (0, \pi/2)$. But $\sigma_{L^2}(I-T) = \overline{\{1-\widehat{\mu}(\xi) : \xi \in \mathbb{R}\}}$. Hence $|\arg(1-\widehat{\mu}(\xi))| \leq \omega$, which implies (8). □

Proof of Theorem 1.6 First suppose $G = \mathbb{R}$ and consider $\mu \in \mathbb{P}(\mathbb{R})$ with $\int_{\mathbb{R}} \mathrm{d}\mu(x)\,|x| < \infty$ and which is non-centered, that is, $a := \int_{\mathbb{R}} \mathrm{d}\mu(x)\,x \neq 0$. Differentiating $\widehat{\mu}(\xi) = \int_{\mathbb{R}} \mathrm{d}\mu(x)\,\mathrm{e}^{-ix\xi}$ with respect to ξ shows that $\widehat{\mu} \in C^1(\mathbb{R})$ and $\widehat{\mu}'(0) = -ia$. Since $\widehat{\mu}(0) = 1$ we have

$$\widehat{\mu}(\xi) = 1 - ia\xi + o(|\xi|)$$

where $o(|\xi|)$ denotes a function of ξ such that $\lim_{\xi\to 0} o(|\xi|)/|\xi| = 0$. Since $a \neq 0$, for some small $\varepsilon > 0$ we have

$$|1-\widehat{\mu}(\xi)| \geq \varepsilon|\xi|$$

for all $\xi \in [-\varepsilon, \varepsilon]$, while $\operatorname{Re}(1-\widehat{\mu}(\xi)) = o(|\xi|)$. Thus (8) cannot hold, and by Lemma 5.1, $(e^{-t(I-T)})$ is not bounded analytic in $L^2(\mathbb{R})$.

If in addition $\int \mathrm{d}\mu(x)\,x^2 < \infty$, then $\widehat{\mu} \in C^2(\mathbb{R})$ and we have a Taylor expansion

$$\widehat{\mu}(\xi) = 1 - ia\xi + O(\xi^2)$$

where $O(\xi^2)$ denotes a function satisfying $|O(\xi^2)| \leq c\xi^2$ for all ξ sufficiently close to 0. Then for some $\varepsilon > 0$ we have inequalities $|1-\widehat{\mu}(\xi)| \geq \varepsilon|\xi|$ and $\operatorname{Re}(1-\widehat{\mu}(\xi)) \leq \varepsilon^{-1}\xi^2$ for all $\xi \in [-\varepsilon, \varepsilon]$. We find that

$$\begin{aligned}\|(I-T)\mathrm{e}^{-t(I-T)}\|_{2\to 2} &= \sup_{\xi\in\mathbb{R}} |(1-\widehat{\mu}(\xi))\mathrm{e}^{-t(1-\widehat{\mu}(\xi))}| \\ &\geq \sup_{\xi\in[-\varepsilon,\varepsilon]} \varepsilon|\xi|\exp(-t\varepsilon^{-1}\xi^2) \geq c't^{-1/2}\end{aligned}$$

for all $t \geq 1$, where the last step followed by choosing $\xi = \varepsilon t^{-1/2}$. This proves Theorem 1.6 for the case $G = \mathbb{R}$.

Next, in case G is the discrete group $\mathbb{Z}$ of integers, we can prove Theorem 1.6 with a very similar analysis using the $\mathbb{Z}$-Fourier transform defined by $\widehat{f}(\xi) := \sum_{n\in\mathbb{Z}} f(n)\mathrm{e}^{-i\xi n}$, $\xi \in [-\pi, \pi]$, for $f \in L^2(\mathbb{Z})$. We leave the details to the reader.

Finally, for general G and a non-centered μ as in the hypothesis of the theorem, we may fix a $j \in \{1, \ldots, q_1 + q_2\}$ such that $\int_G \mathrm{d}\mu(g)\,\pi^{(j)}(g) \neq 0$. The image measure $\mu' := \pi^{(j)}(\mu)$ is then a non-centered probability measure on the group $G' := \pi^{(j)}(G)$ and G' is either $\mathbb{Z}$ or $\mathbb{R}$. We observe that $\int_{G'} \mathrm{d}\mu'(y)\,|y|^n = \int_G \mathrm{d}\mu(g)\,|\pi^{(j)}(g)|^n$ for any $n \in \mathbb{N}$.

Let $Sf := \mu' * f$ for $f \in L^2(G')$. A standard transference theorem for convolution operators on amenable locally compact groups (see [6, Theorem 2.4]) implies that

$$\|(I-S)\mathrm{e}^{-t(I-S)}\|_{2\to 2} \le \|(I-T)\mathrm{e}^{-t(I-T)}\|_{2\to 2}$$

(here, the operator norms are taken with respect to $L^2(G')$ and $L^2(G)$ respectively). Then Theorem 1.6 on G follows easily from the cases $\mathbb{R}$ or $\mathbb{Z}$ already considered. $\square$

In the rest of this section, we give two examples demonstrating that the hypotheses “spread out” or “adapted” cannot be omitted from Theorems 1.3 and 1.2.

Example We exhibit a probability measure which is centered and adapted but not spread out, such that the corresponding semigroup $(\mathrm{e}^{-t(I-T)})$ is not bounded analytic.

Let $G = \mathbb{R}$, fix an irrational number $\beta > 0$ and consider the finitely supported singular measure

$$\mu := \frac{\beta}{1+\beta}\delta_{-1} + \frac{1}{1+\beta}\delta_{\beta}.$$

Then $\mu \in \mathbb{P}(\mathbb{R})$ is centered and adapted but not spread out. Setting $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, we claim that

$$\sigma_{L^2}(T) = \overline{\{\widehat{\mu}(\xi) : \xi \in \mathbb{R}\}} = \{\beta(1+\beta)^{-1}z + (1+\beta)^{-1}w : z, w \in \mathbb{T}\}. \quad (9)$$

Taking $z = w$ in (9), it follows that $\sigma_{L^2}(T)$ contains $\mathbb{T}$ and that (8) fails; therefore, $(\mathrm{e}^{-t(I-T)})$ is not bounded analytic in $L^2(\mathbb{R})$.

To obtain the second equality of (9), just observe that

$$\widehat{\mu}(\xi) = \beta(1+\beta)^{-1}\mathrm{e}^{i\xi} + (1+\beta)^{-1}\mathrm{e}^{-i\beta\xi},$$

and use the fact that for any $z, w \in \mathbb{T}$ we can find a sequence $(\xi_k)_{k=1}^{\infty}$ of reals with $\lim_{k\to\infty} \mathrm{e}^{i\xi_k} = z$, $\lim_{k\to\infty} \mathrm{e}^{-i\beta\xi_k} = w$.

Finally, let us remark that if we put $\mu' := 2^{-1}(\mu + \delta_0) \in \mathbb{P}(\mathbb{R})$ and consider the associated Markov operator $T' = 2^{-1}(T + I)$, then we obtain an example where $(\mathrm{e}^{-t(I-T')})$ is not bounded analytic in $L^2(\mathbb{R})$ and in addition $\sigma_{L^2}(T') \subseteq \mathbb{D} \cup \{1\}$.

Example In this example we describe a μ which is centered and spread out but not adapted, such that $(e^{-t(I-T)})$ is not bounded analytic.

Let G be the semidirect product of $\mathbb{Z}$ with $\mathbb{Z}_2 = \{0, 1\} \cong \mathbb{Z}/(2\mathbb{Z})$, with respect to the action $\gamma : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ defined by

$$\gamma(0)m = m, \quad \gamma(1)m = -m$$

for all $m \in \mathbb{Z}$. Then $G = \{(m, z) : m \in \mathbb{Z}, z \in \mathbb{Z}_2\}$ is a discrete solvable group with product given by $(m, z)(m', z') = (m + \gamma(z)m', z +_{\mathbb{Z}_2} z')$. Set $\mathbb{Z}' := \{(m, 0) :$

$m \in \mathbb{Z}\}$, so that $\mathbb{Z}' \cong \mathbb{Z}$ is a subgroup of index 2 in G. An easy calculation shows that

$$[G, G] = \{(2m, 0) : m \in \mathbb{Z}\};$$

therefore $G/[G, G]$ is finite and $G_1 = G$. Hence every element of $\mathbb{P}(G)$ is centered in G.

Now fix a $\mu \in \mathbb{P}(\mathbb{Z}')$ which is finitely supported and non-centered in $\mathbb{Z}'$, that is, $\sum_{m\in\mathbb{Z}} m\mu((m, 0)) \neq 0$. We regard μ as a non-adapted element of $\mathbb{P}(G)$. We can also identify $L^2(\mathbb{Z}')$ with the subspace of $L^2(G)$ consisting of functions with support contained in $\mathbb{Z}'$. Since $T \in \mathcal{L}(L^2(G))$ maps $L^2(\mathbb{Z}')$ into itself and since μ is not centered in $\mathbb{Z}'$, Theorem 1.6 implies that for some $c > 0$,

$$\sup\{\|(I - T)\mathrm{e}^{-t(I-T)} f\|_2 : f \in L^2(\mathbb{Z}'),\ \|f\|_2 = 1\} \geq ct^{-1/2}$$

for all $t \geq 1$. In particular, $(\mathrm{e}^{-t(I-T)})$ is not bounded analytic in $L^2(G)$.

Remark The common element of both of the above examples is that μ, though centered in G, is not centered in some subgroup $H \subseteq G$ containing the support of μ. In the first example, this remark holds with $H = \{m + n\beta : m, n \in \mathbb{Z}\} \subseteq \mathbb{R}$, where H is given the discrete topology making it isomorphic with $\mathbb{Z}^2$.

By developing this observation, one might be able to obtain criteria for analyticity for some classes of non-spread out singular measures (for example, finitely supported measures on connected groups), but we will not pursue that here.

6. Appendix

The aim of this appendix is to give a straightforward proof of inequality (4) in the special case of a discrete group. We follow essentially [10]. (On general groups, topological difficulties occur - since, for example, $[G, G]$ need not be closed in G - and the proof of (4) is more elaborate: see [11].)

Let G be a finitely generated discrete group. Since G/G_1 is discrete we can then identify $G/G_1 = \mathbb{Z}^q$ for some $q \in \mathbb{N}_0$. Consider the homomorphism $\pi : G \to G/G_1 = \mathbb{Z}^q$ and fix elements $z_1, \ldots, z_q \in G$ such that $\pi(z_j) = e_j$ for $j \in \{1, \ldots, q\}$.

Let $\mathcal{D}_2$ denote the linear subspace of $\mathcal{L}(L^2)$ consisting of all finite linear combinations of the operators $L(h)\partial_{g_1}\partial_{g_2}$ for any $h, g_1, g_2 \in G$. The identity $L(h)\partial_{g_1}\partial_{g_2} = \partial_{hg_1h^{-1}}L(h)\partial_{g_2}$ implies that $\mathcal{D}_2$ is also equal to the linear space spanned by all operators $\partial_{g_1}L(h)\partial_{g_2}$ for $h, g_1, g_2 \in G$.

Since the compact subsets of G are just the finite subsets, we obtain (4) as an immediate consequence of the following two results.

Lemma 6.1 *For any $W \in \mathcal{D}_2$, there exists a $c(W) > 0$ such that*

$$|(Wf_1, f_2)| \leq c(W)\Gamma_2(f_1)\Gamma_2(f_2)$$

for all $f_1, f_2 \in L^2$.

Proposition 6.2 *One has $\partial_g - \sum_{i=1}^{q} \pi^{(i)}(g)\partial_{z_i} \in \mathcal{D}_2$ for all $g \in G$.*

Proof of Lemma 6.1. For any $h, g_1, g_2 \in G$, we have

$$\begin{aligned} |(\partial_{g_1} L(h)\partial_{g_2} f_1, f_2)| &= |(L(h)\partial_{g_2} f_1, \partial_{g_1^{-1}} f_2)| \\ &\le \|\partial_{g_2} f_1\|_2 \|\partial_{g_1^{-1}} f_2\|_2 \\ &\le c_U^2 \rho(g_2)\rho(g_1)\Gamma_2(f_1)\Gamma_2(f_2) \end{aligned}$$

where the last step used (5). The lemma follows by linearity. □

For the proof of Proposition 6.2 we need a lemma.

Lemma 6.3 *One has $\partial_g \in \mathcal{D}_2$ for all $g \in G_1$.*

Proof of Lemma 6.3 The identity (6) implies that

$$\partial_{g_1 \ldots g_n} - \sum_{j=1}^{n} \partial_{g_j} \in \mathcal{D}_2 \tag{10}$$

for all $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in G$. In particular, setting $n = 2$, $g_2 = g_1^{-1}$ one sees that $\partial_g + \partial_{g^{-1}} \in \mathcal{D}_2$, $g \in G$, and combining this with (10) shows that

$$\partial_{g^k} - k\partial_g \in \mathcal{D}_2 \tag{11}$$

for all $g \in G$ and $k \in \mathbb{Z}$. For the commutator $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$, we see from (10) that $\partial_{[g_1,g_2]} - (\partial_{g_1} + \partial_{g_1^{-1}} + \partial_{g_2} + \partial_{g_2^{-1}}) \in \mathcal{D}_2$ and then by (11) that $\partial_{[g_1,g_2]} \in \mathcal{D}_2$, for all $g_1, g_2 \in G$. Since the group $[G, G]$ is generated by all elements $[g_1, g_2]$, from (10) it then follows that $\partial_g \in \mathcal{D}_2$ for all $g \in [G, G]$.

Next, the group $G_1/[G, G] \cong M$ is both compact and discrete, hence finite. For any $g \in G_1$, we may therefore find $m \in \mathbb{N}$ such that $g^m \in [G, G]$ (one may actually take m to be the cardinality of M). Since $\partial_{g^m} \in \mathcal{D}_2$, from (11) we deduce that $\partial_g \in \mathcal{D}_2$, as desired. □

Proof of Proposition 6.2 Take an arbitrary $g \in G$. Set $t_j := \pi^{(j)}(g) \in \mathbb{Z}$ for $j \in \{1, \ldots, q\}$, and define $g' \in G$ by $g = z_1^{t_1} \cdots z_q^{t_q} g'$. Use of (10) and (11) shows that

$$\partial_g - \sum_{j=1}^{q} t_j \partial_{z_j} - \partial_{g'} \in \mathcal{D}_2.$$

Since $\pi(g) = (t_1, \ldots, t_q) = \pi(z_1^{t_1} \cdots z_q^{t_q})$, we have $\pi(g') = 0$, or in other words, $g' \in G_1$. Thus $\partial_{g'} \in \mathcal{D}_2$ by Lemma 6.3, and it follows that $\partial_g - \sum_j t_j \partial_{z_j} \in \mathcal{D}_2$ as desired. □

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