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Valleys and the maximum local time for random walk in random environment

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Abstract. Let $\xi(n, x)$ be the local time at x for a recurrent one-dimensional random walk in random environment after n steps, and consider the maximum $\xi^*(n) = \max_x \xi(n, x)$. It is known that $\limsup_n \xi^*(n)/n$ is a positive constant a.s. We prove that $\liminf_n (\log\log\log n)\, \xi^*(n)/n$ is a positive constant a.s.; this answers a question of P. Révész [5]. The proof is based on an analysis of the *valleys* in the environment, defined as the potential wells of record depth. In particular, we show that almost surely, at any time n large enough, the random walker has spent almost all of its lifetime in the two deepest valleys of the environment it has encountered. We also prove a uniform exponential tail bound for the ratio of the expected total occupation time of a valley and the expected local time at its bottom.

1. Introduction

Let $\omega = (\omega_x)_{x\in\mathbb{Z}_+}$ be a collection of i.i.d. random variables taking values in (0, 1). We will denote the distribution of ω by P. For each ω, we define the random walk in random environment (RWRE) $(X_n)_{n=0,1,2,...}$ as the Markov chain taking values in $\mathbb{Z}_+$ with $X_0 = 0$ and transition probabilities $P_\omega(X_{n+1} = 1|X_n = 0) = 1$, $P_\omega(X_{n+1} = x + 1|X_n = x) = \omega_x = 1 - P_\omega(X_{n+1} = x - 1|X_n = x)$ for $x > 0$. For fixed ω, we denote the distribution of the Markov chain $(X_0, X_1, \dots)$ with P_ω. As usual, we denote by $\mathbb{P}$ the joint distribution of $(\omega, (X_n))$. Throughout the paper, we make the following assumptions on the distribution of the environment ω. Let $\rho_i := (1 - \omega_i)/\omega_i$, $i = 1, 2, \dots$

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$$E(\log \rho_1) = \int \log \rho_1(\omega)P(\mathrm{d}\omega) = 0, \tag{1.1}$$

$$\mathrm{Var}(\log \rho_1) > 0; \tag{1.2}$$

there is $\delta \in (0, 1)$ such that

$$P(\delta \le \omega_1 \le 1 - \delta) = 1. \tag{1.3}$$

Assumption (1.1) guarantees that for P-almost all ω, the Markov chain is recurrent; (1.2) excludes the deterministic case of a simple random walk on the positive integers, and (1.3) is a technical assumption which could possibly be relaxed but is used extensively. Usually, one defines in the same way the RWRE on the integer axis, but for the questions we will consider, there is no difference between the two models, so we restrict attention to the RWRE on the positive integers for simplicity. A key property of recurrent RWRE is its strong localization: under our assumptions, Sinai [9] showed that that $X_n/(\log n)^2$ converges in distribution. A lot more is known about this model; we refer to the survey by Zeitouni [10] for limit theorems, large deviations results, and for further references.

Let $\xi(n, x) := |\{0 \le j \le n : X_j = x\}|$ denote the local time of the RWRE in x at time n and $\xi^*(n) := \sup_{x \in \mathbb{Z}_+} \xi(n, x)$ the maximal local time at time n. It was proved in [3] that for each non-decreasing function φ, $\limsup_{n\to\infty} \xi^*(n)/\varphi(n)$ and $\liminf_{n\to\infty} \xi^*(n)/\varphi(n)$ are $\mathbb{P}$-almost surely (possibly degenerate) constants. For the lim sup behavior of $\xi^*(n)$, it was shown in [5] and [7] that

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n} > 0 \quad \mathbb{P}\text{-a.s.}$$

(Clearly this lim sup is at most 1/2.) In his book, Révész [5] raised the problem of determining the lim inf behavior of $\xi^*(n)$. Our main result is the following.

Theorem 1.1. *There exists a constant* $c \in (0, \infty)$ *such that*

$$\liminf_{n\to\infty} \frac{\xi^*(n)}{n/\log\log\log n} = c, \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

In particular, (1.4) disproves the conjecture on page 303 of Révész [5]. We will shortly give a heuristic argument which explains why the three logarithms appear.

The *potential* corresponding to the RWRE is $V(x) = \sum_{i=1}^{x} \log \rho_i$, $x \in \mathbb{Z}_+$. As is well known, the potential governs the behavior of the RWRE in several senses, e.g.

- In an excursion starting from any site b, the logarithm of the expected number of visits to a site x before returning to b is roughly the potential difference $V(b) - V(x)$, see (3.5).
- Starting from the origin, the logarithm of the expected time to reach a site x is roughly $\max_{y \le x} V(y)$; see (3.1) for an upper bound, and observe that a similar lower bound follows from (3.5).

The proof of Theorem 1.1 is based on an analysis of the *valleys* in the potential, which is of independent interest. By a "valley" we mean a potential well of record depth; see Section 2.1 for a precise definition.

We will partition the environment into valleys, and show that at any time n, the particle performing RWRE has almost surely spent almost all of its lifetime in the two deepest valleys it has encountered. This almost sure localization theorem (Theorem 3.4) can be considered as the second main result of the paper. Furthermore, we define in (4.1) the *effective width* of a valley as the ratio of the expected total occupation time of the valley and the expected local time at its bottom, and prove a uniform exponential tail bound (4.28) for the effective width of valleys. The reason for the term "effective width" is that most of the occupation time in a valley is spent at sites where the potential is within an additive constant from its minimum in the valley; the number of these sites is the effective width, up to a multiplicative constant.

Theorem 1.1 is then established as follows:

Due to scaling properties of the potential, the depths of successive valleys grow at a geometric rate, whence the distance between bottoms of successive valleys also exhibit geometric growth, resulting with $O(\log R)$ valleys in a large interval $[0, R]$. By time n the random walker reaches a distance of order $(\log n)^2$ from the origin, thus visiting an order of $\log\log n$ valleys. The exponential tail bounds on effective widths imply that a.s., for all k, the kth valley encountered has effective width at most $O(\log k)$; conversely, a.s. for infinitely many k the effective width is at least $c \log k$. Hence, a.s. the maximal effective width of valleys seen by the walker up to time n is at most of order $\log\log\log n$, and up to a constant factor, this effective width is realized infinitely often.

The paper is organized as follows. In Section 2, we introduce the notion of valleys, and describe some scaling properties of such valleys. Section 3 is devoted to the study of the behavior of the RWRE within the valleys. We first give some background on hitting times and excursions. We then compare the occupation time of different valleys and prove that the RWRE spends most of its time in the last two visited valleys: Theorem 3.4 is the main result of this section. In Section 4, we compare the occupation time of valleys with the local time in sites. Our main tool here is to average over excursions of the RWRE. This comparison motivates our definition of the "effective width" of the valleys, whose asymptotic growth is studied in the second part of Section 4. Similarly to Section 2 this part does not concern the random walk, but only the environment. Finally, Theorem 1.1 is proved in Section 5.

2. Valleys

Recall that the potential V is a function of the environment, defined as follows:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i, & x = 1, 2, \dots, \\ 0, & x = 0. \end{cases}$$

Note that V is itself a sum of i.i.d. random variables, which are bounded by $C := |\log \delta - \log(1-\delta)|$, see (1.3). For fixed ω, P_ω is a reversible Markov chain, hence an electrical network in the sense of [2]. The conductance of the bonds is

$$C_{(x,x+1)} = \mathrm{e}^{-V(x)}, \quad x = 0, 1, 2, \dots \tag{2.1}$$

and the reversible measure μ (which is unique up to multiplication by a constant), is given by

$$\mu(x) = \begin{cases} \mathrm{e}^{-V(x)} + \mathrm{e}^{-V(x-1)}, & x = 1, 2, \dots, \\ 1, & x = 0. \end{cases} \tag{2.2}$$

For background on reversible Markov chains, we refer to [2].

2.1. Definition of valleys

Fix a constant $K_0 > 0$. We set $\theta_0 := 0$ and

$$\begin{aligned} \eta_0 &:= \inf\left\{i > 0 : \ V(i) - \min_{0\le j\le i} V(j) \ge K_0\right\}, \\ b_0 &:= \sup\left\{i < \eta_0 : \ V(i) = \min_{0\le j\le \eta_0} V(j)\right\}. \end{aligned}$$

We now define, for $k \ge 1$, inductively:

$$\begin{aligned} \theta_k &:= \inf\left\{i > \eta_{k-1} : \ V(i) \le V(b_{k-1})\right\}, \\ H^+_{k-1} &:= \max_{\eta_{k-1}\le j\le \theta_k} V(j) - V(b_{k-1}), \\ \eta_k &:= \inf\left\{i > \theta_k : \ V(i) - \min_{0\le j\le i} V(j) \ge H^+_{k-1}\right\}, \\ b_k &:= \sup\left\{i < \eta_k : \ V(i) = \min_{\theta_k\le j\le \eta_k} V(j)\right\}, \\ H^-_k &:= \max_{\eta_{k-1}\le j\le \theta_k} V(j) - V(b_k). \end{aligned}$$

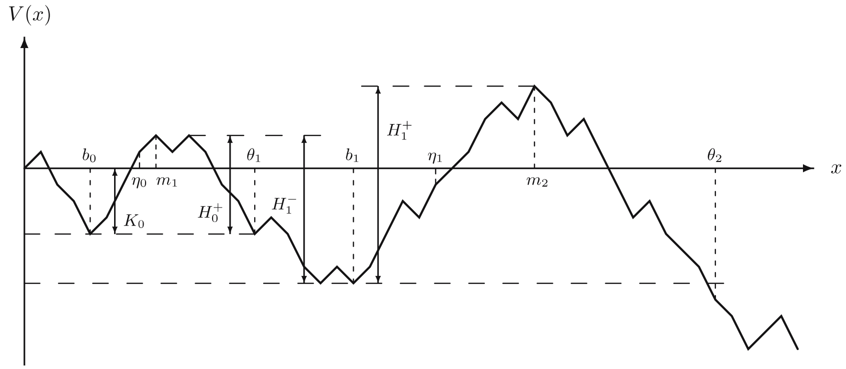
Let now

$$m_k := \inf\left\{i > \eta_{k-1} : \ V(i) = \max_{\eta_{k-1}\le j\le \theta_k} V(j)\right\}. \tag{2.3}$$

The piece $(V(i),\ m_k \le i < m_{k+1})$ is the k-th valley, H^-_k the left height of this valley, and H^+_k the right height. We call

$$H_k := \min\left\{H^-_k,\ H^+_k\right\},$$

the height of the k-th valley. Also, b_k is called the bottom of the k-th valley.



Remark.

1. In words, m_k is the beginning of the k-th valley. Note that $(\theta_k)_{k\geq 0}$ and $(\eta_k)_{k\geq 0}$ are sequences of stopping times (with respect to the natural filtration of the potential V), whereas $(b_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ are not.
2. Our definition of valleys is not exactly the standard definition of valleys in the sense of Sinai [9]. However, it follows from our definition that almost surely the heights $(H_k,\ k \geq 1)$ are increasing.
3. Here is a (very) rough description of the asymptotic behavior of the RWRE. When k is large, the time needed for the RWRE to exit from the kth valley is of order e^{H_k} [see (3.1) and Lemma 3.2]; and since H_k is of order e^k (Lemma 2.1), we have: $n \approx \mathrm{e}^{H_{N_n}}$, where N_n is the number of valleys visited by the RWRE in the first n steps. This leads to: $H_{N_n} \approx \log n$. On the other hand, V being the partial sum process of i.i.d. bounded mean-zero random variables, $H_k \approx x_k^{1/2}$ for any site x_k in the kth valley. Therefore, x_{N_n} is of order $(\log n)^2$; i.e., the maximal distance to the origin of the RWRE in the first n steps is of order $(\log n)^2$. In fact, a famous result of Sinai [9] says that $X_n/(\log n)^2$ converges in distribution (under $\mathbb{P}$) to a non-degenerate limit.

2.2. *Heights of valleys*

We now consider the asymptotic growth of the heights of the valleys.

Lemma 2.1. *We have, P-almost surely,*

$$\log H_k \sim \log H_k^+ \sim \log H_k^- \sim k, \qquad k \to \infty. \tag{2.4}$$

Proof. Assume for a moment that V is a Brownian motion. Then, the strong Markov property at θ_k and scaling properties imply that $(H_{k-1}^+/H_k^+,\ k \geq 2)$ is a sequence of i.i.d. random variables with common uniform distribution on $(0, 1)$. In particular, $E(\log(H_k^+/H_{k-1}^+)) = 1$. More precisely, since $(\eta_k, k \geq 1)$ is a sequence of stopping times, the random variables H_{k-1}^+/H_k^+, $k \geq 2$, are independent, and the probability of the event $\{H_k^+ \geq (1 + c)H_{k-1}^+\}$ is the probability that a standard

Brownian motion hits c before hitting -1. By the law of large numbers, P-almost surely,

$$\log H_k^+ = \log H_1^+ + \sum_{i=2}^{k} \log \frac{H_i^+}{H_{i-1}^+} \sim k, \quad k \to \infty. \tag{2.5}$$

Further, the strong Markov property at the stopping time θ_k implies that $(H_k^- - H_{k-1}^+)/H_{k-1}^+$ is an exponential random variable with mean 1. More precisely, the conditional distribution of $(H_k^- - H_{k-1}^+)/H_{k-1}^+$, given $H_{k-1}^+ = a$, is the distribution of $|\inf_{t<\sigma(a)} B_t| \cdot a^{-1}$, where (B_t) is a standard Brownian motion and $\sigma(a) := \inf\{s : B_s - \inf_{u<s} B_u = a\}$. By scaling, this distribution does not depend on a, hence equals the distribution of $|\inf_{t<\sigma(1)} B_t|$. Lévy's identity tells us that $(B_t - \inf_{s<t} B_s, |\inf_{s<t} B_s|)$ has the same distribution as $(|B_t|, L_t)$ where (L_t) is the local time of (B_t) at 0 (cf. [6, Theorem VI.2.3]). Therefore, $|\inf_{t<\sigma(1)} B_t|$ has the same distribution as L_τ, where $\tau := \inf\{t : |B_t| = 1\}$, and the distribution of L_τ is known to be exponential with mean 1 (for example, see Formula 3.3.2, page 213 of [1]). Using the Borel–Cantelli lemma, we see that P-almost surely,

$$\log \left(\frac{H_k^-}{H_{k-1}^+} \right) = O(\log \log k), \quad k \to \infty, \tag{2.6}$$

and thus (2.5) yields $\log H_k^- \sim k$, P-almost surely. This would prove the lemma if V was a Brownian motion.

In our case, V is the partial sum process associated with a sequence of i.i.d. bounded mean-zero random variables, so we have to be more careful. Let $k \geq 1$. We look at the random walk $(V(i + \theta_k) - V(\theta_k),\ i \geq 0)$, which is independent of $(V(i),\ i \leq \theta_k)$ (thus of H_{k-1}^- and H_{k-1}^+). This random walk can be embedded into a Brownian motion, say $(B_k(t),\ t \geq 0)$, in the sense of Skorokhod embedding, making $V(i + \theta_k) - V(\theta_k) = B_k(t_i),\ i \geq 0$, a random sequence of points on the path of $t \mapsto B_k(t)$, such that the maximum of the height differences $|B_k(t) - B_k(t_i)|$ for $t \in [t_i, t_{i+1}]$ is at most C. For any $r > 0$, let

$$\sigma_k(r) := \inf \left\{ t > 0 : \ B_k(t) - \inf_{s \in [0,t]} B_k(s) = r \right\},$$

and

$$\widetilde{H}_k^-(r) := r + \left| \inf_{0 \leq t \leq \sigma_k(r)} B_k(t) \right|.$$

Note that with $V(b_{k-1}) - C \leq V(\theta_k) \leq V(b_{k-1})$, given $H_{k-1}^+ = a > 0$, we have that

$$a + V(\theta_k) - V(b_k) \leq H_k^- \leq a + C + V(\theta_k) - V(b_k).$$

Further, $V(b_k)-V(\theta_k)$ is the minimum of $B_k(t_i)$ for those i such that $t_i \in [0, t_{\eta_k-\theta_k}]$, and since the Brownian increments between $B_k(t_i)$ are of height at most C, we have that

$$V(b_k) - V(\theta_k) - C \leq \inf_{0\leq t\leq t_{\eta_k-\theta_k}} B_k(t) \leq V(b_k) - V(\theta_k).$$

We thus conclude that if $H_{k-1}^+ = a > 2C$, then

$$\widetilde{H}_k^-(a-2C) \leq H_k^- \leq \widetilde{H}_k^-(a+2C). \tag{2.7}$$

More precisely, by the time $\sigma_k(a+2C)$ the Brownian motion made an increment of $a+2C$ over its minimal value and by the time $\sigma_k(a-2C)$ it made an increment of $a-2C$ over its minimal value. Since $\eta_k - \theta_k$ corresponds to the first value of i where $B_k(t_i)$ makes an increment of at least a from its minimum, and the Brownian increments between the points $B_k(t_i)$ are at most of height C, a fortiori,

$$\sigma_k(a-2C) \leq t_{\eta_k-\theta_k} \leq \sigma_k(a+2C),$$

which by the monotonicity of $u \mapsto \inf_{0\leq t\leq u} B_k(t)$ yields the inequality (2.7).

Similarly, we embed the random walk $(V(j+\eta_k)-V(\eta_k),\ j\geq 0)$ as a random sequence of points $W_k(s_j)$ on the path of an independent Brownian motion denoted $(W_k(s),\ s\geq 0)$, such that the maximum of the height differences $|W_k(s)-W_k(s_j)|$ for $s\in[s_j, s_{j+1}]$ is at most C, and without loss of generality, we assume that we are still working on the same probability space. Note that $V(\eta_k)-V(b_k)$ is within distance C of H_{k-1}^+ and that

$$H_k^+ = \max_{0\leq j\leq \theta_{k+1}-\eta_k} W_k(s_j) + V(\eta_k) - V(b_k),$$

where $\theta_{k+1}-\eta_k$ corresponds to the first value of j such that $W_k(s_j) \leq V(b_k) - V(\eta_k)$. Therefore, by a similar line of reasoning as before, given $H_{k-1}^+ = a > 2C$, we have that

$$S_k(-(a-2C)) \leq s_{\theta_{k+1}-\eta_k} \leq S_k(-(a+2C)),$$

where $S_k(r) := \inf\{s\geq 0 :\ W_k(s) = r\}$. Consequently, then also

$$\widetilde{H}_k^+(a-2C) \leq H_k^+ \leq \widetilde{H}_k^+(a+2C), \tag{2.8}$$

where for any $r>0$,

$$\widetilde{H}_k^+(r) := r + \sup_{0\leq s\leq S_k(-r)} W_k(s).$$

Recall that $H_k^+ = V(m_{k+1}) - V(b_k)$, is non-decreasing, and further

$$H_k^+ - H_{k-1}^+ \geq V(b_{k-1}) - V(b_k) \geq V(\theta_k) - V(b_k),$$

which is non-negative, and dominates the law of the negative part of $\log\rho_0$. Thus, by (1.2) we see that $H_k^+ \to \infty$, P-almost surely. Fixing $\varepsilon > 0$, we thus have that

P-almost surely, $\varepsilon H^+_{k-1} \geq 2C$ for all k large enough, in which case we have from (2.7) and (2.8) that

$$\widetilde{H}^{\pm}_k((1-\varepsilon)H^+_{k-1}) \leq H^{\pm}_k \leq \widetilde{H}^{\pm}_k((1+\varepsilon)H^+_{k-1})\,. \tag{2.9}$$

Without loss of generality we take the Brownian motions $B_k(\cdot), W_k(\cdot), k = 1, 2, \ldots,$ to be independent, and consequently, so are $\widetilde{H}^{\pm}_k(\cdot)$. Further, by the scaling properties of the Brownian motion, the law of $r^{-1}\widetilde{H}^{\pm}_k(r)$ is independent of $r > 0$ and k, resulting with i.i.d. random variables

$$Z^{\pm}_k := \frac{\widetilde{H}^{\pm}_k(uH^+_{k-1})}{uH^+_{k-1}}\,,$$

whose law is independent of $u > 0$. As we have already seen, $-1 + Z^-_k$ has the exponential distribution of mean 1 (being the same as $|\inf_{t<\sigma(1)} B_t|$) while $1/Z^+_k$ has the uniform law on (0, 1). Consequently, $E(\log\, Z^+_1) = 1$ and

$$k^{-1}\sum_{i=1}^{k} \log\, Z^+_i \to 1$$

P-almost surely. Since (2.9) holds for all but finitely many values of k and $\log(1\pm\varepsilon)$ can be arbitrarily small, it follows that also

$$k^{-1}\sum_{i=2}^{k} \log \frac{H^+_i}{H^+_{i-1}} \to 1\,,$$

P-almost surely. That is, $\log H^+_k \sim k$, P-almost surely.

A Borel–Cantelli argument as in the proof of (2.6), using (2.9), easily implies that $\log(H^-_k/H^+_{k-1}) = O(\log\log k)$, P-almost surely. Thus $\log\, H^-_k \sim k$, P-almost surely. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $\varepsilon > 0$. We have, P-almost surely for all sufficiently large k,*

$$H_k - H^+_{k-1} \geq (H^+_{k-1})^{1-\varepsilon}. \tag{2.10}$$

Proof. Observe that

$$P\left(\frac{H_k}{H^+_{k-1}}<1+\mathrm{e}^{-\varepsilon k/2}\right) \leq P\left(\frac{H^+_k}{H^+_{k-1}}<1+\mathrm{e}^{-\varepsilon k/2}\right) + P\left(\frac{H^-_k}{H^+_{k-1}}<1+\mathrm{e}^{-\varepsilon k/2}\right).$$

The distributions of H^+_k/H^+_{k-1} and H^-_k/H^+_{k-1} have already been mentioned in the case of a Brownian potential V: H^+_{k-1}/H^+_k is uniformly distributed on (0, 1), whereas $(H^-_k - H^+_{k-1})/H^+_{k-1}$ is an exponential random variable with mean 1. Therefore, $\sum_k P\big(H^+_k/H^+_{k-1}<1+\mathrm{e}^{-\varepsilon k/2}\big)<\infty$ and $\sum_k P\big(H^-_k/H^+_{k-1}<1+\mathrm{e}^{-\varepsilon k/2}\big) < \infty$. As a consequence, $\sum_k P\left(\frac{H_k}{H^+_{k-1}} < 1+\mathrm{e}^{-\varepsilon k/2}\right) < \infty$.

For our partial sum potential, we can easily use (2.9) to see that $\sum_k P\,\big(H_k/H^+_{k-1} < 1+\mathrm{e}^{-\varepsilon k/2}\big) < \infty$ still holds. By the Borel–Cantelli lemma, P-almost surely for k large enough, $H_k - H^+_{k-1} \geq H^+_{k-1}\mathrm{e}^{-\varepsilon k/2}$. This yields (2.10), as we know from Lemma 2.1 that $\log H^+_{k-1} \sim k$, P-almost surely. □

2.3. Other facts about valleys

Throughout the paper, we will subsequently use some asymptotic properties of the valleys. First, note that

$$K_0 + \sum_{i=1}^{k} (H_i^- + H_i^+) \ge \max_{0 \le x,\, y \le m_{k+1}} |V(x) - V(y)| \ge \max_{x \in [0,\, m_{k+1}]} |V(x)| \ge \frac{1}{2} H_k^- .$$

Hence, with $m_k \to \infty$, applying Chung's law of the iterated logarithm for the potential V, we have for each $\varepsilon \in (0, 1/4)$, that P-almost surely for all sufficiently large k,

$$K_0 + \sum_{i=1}^{k} (H_i^- + H_i^+) \ge m_{k+1}^{(1-0.5\varepsilon)/2} \ge (H_k^-)^{1-\varepsilon}. \tag{2.11}$$

In view of Lemma 2.1 the first inequality in (2.11) implies that P-almost surely,

$$m_k \le b_k \le m_{k+1} \le H_k^{2+\varepsilon}, \tag{2.12}$$

for all sufficiently large k. Further, by the same reasoning we have that P-almost surely,

$$\log \log m_k \sim \log k, \quad \text{for } k \to \infty. \tag{2.13}$$

We will also make use of the following: for each $\varepsilon \in (0, 1)$, we have P-almost surely for all k large enough,

$$\max_{m_k \le y \le z < b_k} (V(z) - V(y)) \le H_{k-1}^+ - (H_{k-1}^+)^{1-\varepsilon}, \tag{2.14}$$

$$\max_{b_k \le y \le z < m_{k+1}} (V(y) - V(z)) \le H_k^+ - (H_k^+)^{1-\varepsilon}. \tag{2.15}$$

Moreover,

$$\min_{x \in [\eta_k,\, m_{k+1})} V(x) \ge V(b_k) + (H_{k-1}^+)^{1-\varepsilon}, \tag{2.16}$$

$$\max_{b_k \le y \le z < \eta_k} (V(y) - V(z)) \le H_{k-1}^+ - (H_{k-1}^+)^{1-\varepsilon}. \tag{2.17}$$

We next outline the proof of (2.14) in case V is a Brownian motion. A similar argument as in the proof of Lemma 2.1 will then confirm that (2.14) holds also when V is a partial sum process. With H_{k-2}^+ measurable on the stopped σ-field at $\theta_{k-1} < \eta_{k-1}$, for $V(\cdot)$ a Brownian motion we have by the strong Markov property at η_{k-1} that conditionally on $H_{k-2}^+ = a > 0$ the process $U(s) := (V(s + \eta_{k-1}) - V(\eta_{k-1}) + a,\ 0 \le s \le \theta_k - \eta_{k-1})$ is also a Brownian motion, starting from $U(0) = a$ and killed upon first hitting 0 (at time $\theta_k - \eta_{k-1} =: S(0)$). Of course, in this case also $H_{k-1}^+ = \sup_{0 \le s \le S(0)} U(s) =: H$ and $m_k - \eta_{k-1} = \inf\{s \ge 0 : U(s) = H\} =: m_H$. Thus, denoting by P_x the probability law of a

Brownian motion $U(\cdot)$ starting at $U(0) = x$ and by $S(y) := \inf\{t \geq 0 : U(t) = y\}$ the corresponding first hitting time of y, it follows that for any $a > 0$ and $k \geq 2$,

$$P\left(\max_{m_k \leq y \leq z < \theta_k} (V(z) - V(y)) > H^+_{k-1} - (H^+_{k-1})^{1-\varepsilon} \mid H^+_{k-2} = a\right)$$
$$\leq P_a(H < \lceil a \rceil) + \sum_{h=\lceil a \rceil}^{\infty} J(a,h)$$

where for integer $h \geq 1$,

$$J(a,h) = P_a\left(h \leq H < h+1, \quad \max_{m_H \leq y \leq z < S(0)} (U(z) - U(y)) > H - H^{1-\varepsilon}\right).$$

Since $H \geq U(z)$ and $U(y) \geq 0$, the event whose probability is $J(a,h)$ requires the existence of random times $m_H < y < z < S(0)$ with $U(m_H) \geq h$, $U(y) < (h+1)^{1-\varepsilon} =: u$, $U(z) > h - (h+1)^{1-\varepsilon} =: v$ and $U(S(0)) = 0$, while $0 < U(s) < h+1$ for all $s < S(0)$. It is easy to see that $h > 2(h+1)^{1-\varepsilon}$ for any $h \geq a \geq 3^{1/\varepsilon}$, in which case by continuity of the Brownian path and the preceding reasoning,

$$J(a,h) \leq P_a(S(h) < S(0)) P_h(S(u) < S(h+1)) P_u(S(v) < S(0)) P_v(S(0) < S(h+1)) = \frac{au(h+1-v)}{h(h+1-u)v(h+1)} \leq 8ah^{-(2+2\varepsilon)}.$$

Hence, $\sum_{h \geq a} J(a,h) \leq Ca^{-2\varepsilon}$ for a finite constant $C = C(\varepsilon) \geq 1$. Further, $P_a(H < \lceil a \rceil) \leq a^{-1}$, so we conclude that

$$P\left(\max_{m_k \leq y \leq z < \theta_k} (V(z) - V(y)) > H^+_{k-1} - (H^+_{k-1})^{1-\varepsilon}\right)$$
$$\leq P(H^+_{k-2} \leq 3^{1/\varepsilon}) + 2CE((H^+_{k-2})^{-\varepsilon})$$

which is summable in k (recall that $H^+_1 \geq K_0$ and H^+_{i-1}/H^+_i, $i \geq 2$, are i.i.d. uniform (0, 1) random variables). Thus, P-almost surely for all large k,

$$\max_{m_k \leq y \leq z < \theta_k} (V(z) - V(y)) \leq H^+_{k-1} - (H^+_{k-1})^{1-\varepsilon}.$$

A similar (and easier) argument shows that, P-almost surely for all large k

$$\max_{\theta_k \leq y \leq z < b_k} (V(z) - V(y)) \leq H^+_{k-1} - (H^+_{k-1})^{1-\varepsilon},$$

yielding (2.14) when V is a Brownian motion.

The proof of (2.15) is very similar. The proofs of (2.16) and (2.17) are even easier since H^+_{k-1} is measurable on the stopped σ-field at η_k and θ_k, which is where we apply the strong Markov property when proving (2.16) and (2.17), respectively.

3. Particle in the valleys

In this section, we will consider the RWRE and give estimates on hitting times, exit times and excursions.

3.1. Hitting time

For any $x \in \mathbb{Z}_+$, define

$$T(x) := \inf\{n \geq 1 : \ X_n = x\},$$

the first hitting time of x by the particle. The inequality [4, (A.1)] states that for any $x \geq 1$,

$$E_\omega(T(x)) \leq x^2 \exp\left(\max_{0 \leq i \leq j < x} (V(j) - V(i)) \right). \tag{3.1}$$

A consequence of (3.1) is that for any $k \geq 2$ and any $\lambda \geq 1$,

$$P_\omega(T(b_k) \geq \lambda) \leq \frac{b_k^2}{\lambda} \mathrm{e}^{H_{k-1}^+}. \tag{3.2}$$

Another result we will be frequently using concerns the almost sure asymptotic behavior of $T(x)$ when $x \to \infty$. The following is a consequence of the law of the iterated logarithm for RWRE, stated in Theorems 27.8 and 27.9 of Révész [5].

Fact 3.1. *(Révész [5]). We have,*

$$\lim_{x \to \infty} \frac{\log \log T(x)}{\log x} = \frac{1}{2}, \quad \mathbb{P}\text{-a.s.}$$

Consider the k-th valley $(V(i),\ m_k \leq i < m_{k+1})$. Let a particle $(X_n,\ n \geq 0)$ start from the bottom $X_0 = b_k$ of the valley. We are interested in

$$\tau_k := \inf\{n > 0 : \ X_n \notin (m_k, m_{k+1})\},$$

the first exit time of the particle from the valley.

Lemma 3.2. *For some $c_0 < \infty$, any $k \geq 1$ and $m \geq 1$,*

$$P_\omega(\tau_k < m \mid X_0 = b_k) \leq c_0 m \mathrm{e}^{-H_k}. \tag{3.3}$$

Proof. Considering the side from which the particle exits the valley, we see that

$$P_\omega(\tau_k < m \mid X_0 = b_k) \leq P_\omega(T(m_k) < m \mid X_0 = b_k) + P_\omega(T(m_{k+1}) < m \mid X_0 = b_k),$$

hence (3.3) is just a consequence of [4, Lemma 7], the definition of H_k, and the fact that increments of V are bounded by C. □

Corollary 3.3. *For any $k \geq 1$ and $a > 0$,*

$$E_\omega\left(\mathrm{e}^{-a\tau_k} \mid X_0 = b_k\right) \leq \frac{2c_0 \mathrm{e}^{-a}}{(1 - \mathrm{e}^{-a})} \mathrm{e}^{-H_k} \leq \frac{2c_0}{a} \mathrm{e}^{-H_k}.$$

Proof. By changing the order of summation,

$$E_\omega\left(\mathrm{e}^{-a\tau_k} \mid X_0 = b_k\right) = (1-\mathrm{e}^{-a})\sum_{m=1}^{\infty}\mathrm{e}^{-am}P_\omega\left(\tau_k \le m \mid X_0 = b_k\right).$$

Replacing $P_\omega\left(\tau_k \le m \mid X_0 = b_k\right)$ by $P_\omega\left(\tau_k < m+1 \mid X_0 = b_k\right)$ and using (3.3), the corollary follows easily. □

We note for further reference that for $b < x < i$,

$$P_\omega\left(T(b) < T(i) | X_0 = x\right) = \sum_{j=x}^{i-1}\mathrm{e}^{V(j)}\left(\sum_{j=b}^{i-1}\mathrm{e}^{V(j)}\right)^{-1}. \tag{3.4}$$

This follows from direct computation, using (2.1), see also [10], formula (2.1.4).

3.2. Excursions

We collect here some elementary facts about reversible Markov chains on $\mathbb{Z}_+$ which will later be used to give estimates for excursions of the RWRE. Let $b \in \mathbb{Z}_+, b > 0$. Consider an excursion from b to b. Let $x \in \mathbb{Z}_+$, $x > 0$, $x \neq b$ and denote by $Y_{b,x}$ the number of visits to x before returning to b. The distribution of $Y_{b,x}$ is "almost geometric": we have

$$P_\omega(Y_{b,x} = m) = \begin{cases}\alpha(1-\beta)^{m-1}\beta & m = 1, 2, 3, \dots, \\ 1-\alpha, & m = 0,\end{cases}$$

where $\alpha = \alpha_{b,x} = P_\omega(T(x) < T(b) \mid X_0 = b)$, $\beta = \beta_{b,x} = P_\omega(T(b) < T(x)|X_0 = x)$. In particular,

$$E_\omega(Y_{b,x}) = \frac{\alpha}{\beta} = \frac{\mu(x)}{\mu(b)} = \frac{\mathrm{e}^{-V(x)} + \mathrm{e}^{-V(x-1)}}{\mathrm{e}^{-V(b)} + \mathrm{e}^{-V(b-1)}}, \tag{3.5}$$

where μ is the reversible measure for the Markov chain, see (2.2). Further,

$$\mathrm{Var}_\omega(Y_{b,x}) = \frac{\alpha(2-\beta-\alpha)}{\beta^2} \le \frac{2}{\beta}\frac{\mu(x)}{\mu(b)}.$$

For $x > b+1$,

$$\begin{aligned}\beta &= (1-\omega_x)P_\omega(T(b) < T(x)|X_0 = x-1)\\ &= (1-\omega_x)\left(\sum_{y=b}^{x-1}\mathrm{e}^{V(y)-V(x-1)}\right)^{-1},\end{aligned}$$

where the last formula follows from (3.4), and applies also for $x = b + 1$. Hence, for some $c_1 = c_1(\delta) > 0$, by (1.3) and (2.2),

$$\begin{aligned}\mathrm{Var}_\omega(Y_{b,x}) &\le c_1\,\mathrm{e}^{-[\,V(x)-V(b)\,]}\sum_{y=b}^{x-1}\mathrm{e}^{V(y)-V(x-1)}\\ &\le c_1\,\mathrm{e}^{-[\,V(x)-V(b)\,]}(x-b)\exp\left(\max_{b\le y\le x-1}(V(y)-V(x-1))\right). \quad (3.6)\end{aligned}$$

In the same way, one obtains, for $x < b$,

$$\mathrm{Var}_\omega(Y_{b,x}) \le c_1\,\mathrm{e}^{-[\,V(x)-V(b)\,]}(b-x)\exp\left(\max_{x\le y\le b-1}(V(y)-V(x))\right). \quad (3.7)$$

3.3. *Number of valleys seen by the particle*

Let N_n denote the number of valleys "seen" by the particle in the first n steps. More precisely,

$$N_n := \sup\left\{k : \max_{0\le i\le n} X_i \ge m_k\right\} .$$

Recall that as $k \to \infty$,

$$\frac{1}{2}\log m_k \sim \frac{1}{2}\log m_{k+1} \sim \log H_k \sim k$$

[compare (2.11) with (2.12) and use (2.4)]. In combination with Fact 3.1 this implies that $\mathbb{P}$-almost surely,

$$\log\log T(m_k) \sim \log\log T(m_{k+1}) \sim \log H_k \sim k\,, \quad k \to \infty\,.$$

Since $T(m_{N_n}) \le n < T(m_{N_n+1})$, it follows that

$$H_{N_n} = (\log n)^{1+o(1)}\,, \quad \mathbb{P}\text{-a.s.} \quad (3.8)$$

and further

$$N_n \sim \log\log n, \quad \text{for} \quad n \to \infty \quad \mathbb{P}\text{-a.s.} \quad (3.9)$$

3.4. *The particle spends most of its time in the last two valleys*

Recall that $\xi(n,x)$ denotes the local time of the RWRE in x at time n, and m_k is the beginning of the k-th valley as in (2.3). Let

$$L(n,k) := \sum_{x\in[m_k,\, m_{k+1})} \xi(n,x), \quad (3.10)$$

which is the total time the particle spends in the k-th valley during the first n steps.

The next theorem shows that the particle spends most time in the two deepest valleys, which are the two right-most valleys.

Theorem 3.4. *We have, for any $\delta < 1$,*

$$\lim_{n\to\infty} \frac{\exp\big((\log n)^\delta\big)}{n} \sum_{1\le k<N_n-1} L(n,k) = 0\ , \quad \mathbb{P}\text{-a.s.} \tag{3.11}$$

In particular,

$$\liminf_{n\to\infty} \frac{1}{n} \sup_{k\ge 1} L(n,k) \ge \frac{1}{2}\ , \quad \mathbb{P}\text{-a.s.} \tag{3.12}$$

Proof. It is clear that (3.12) follows from (3.11) by taking $\delta = 0$. Further, clearly (3.11) is a consequence of (3.8) and

$$\lim_{N\to\infty} \mathrm{e}^{(H^+_{N-2})^\delta} \max_{n\in[T(m_N),\ T(m_{N+1}))} \frac{1}{n} \sum_{1\le k<N-1} L(n,k) = 0\ , \quad \mathbb{P}\text{-a.s.} \tag{3.13}$$

holding for any $\delta < 1$.

In order to prove (3.13), we decompose the time interval $[T(m_N),\ T(m_{N+1}))$ into excursions of the particle away from b_{N-1} and m_{N-1}.

Let $\varepsilon = \varepsilon_N > 0$. Later, we will take $\varepsilon_N = \exp(-(H^+_{N-2})^\delta)$. Let

$$n^* = n^*(N) := \inf\left\{ n \ge T(m_N) :\ \sum_{1\le k<N-1} L(n,k) \ge \varepsilon\, n \right\},$$

with the notation $\inf\emptyset := \infty$. We are interested in the case $n^* < T(m_{N+1})$; thus $n^* \in [T(i), T(i+1))$ for some $i \in [m_N,\ m_{N+1})$.

We define $T^1(b_{N-1}) := T(b_{N-1})$ and inductively,

$$T^j(m_{N-1}) := \inf\Big\{ n > T^j(b_{N-1}) :\ X_n = m_{N-1} \Big\},$$
$$T^{j+1}(b_{N-1}) := \inf\Big\{ n > T^j(m_{N-1}) :\ X_n = b_{N-1} \Big\}, \quad j \ge 1.$$

For any $i \in [m_N,\ m_{N+1})$, let $M_i := \sup\{j :\ T^j(m_{N-1}) < T(i+1)\}$ (notation: $\sup\emptyset := 0$), be the total number of excursions from b_{N-1} to m_{N-1}, before reaching $i+1$.

If $n^* \in [T(i), T(i+1))$ and $M_i = 0$, we have $\sum_{1\le k<N-1} L(n^*,k) \le T(b_{N-1})$ and $n^* \ge T(i) - T(b_{N-1})$ so that

$$T(b_{N-1}) \ge \varepsilon(T(i) - T(b_{N-1})); \tag{3.14}$$

whereas if $n^* \in [T(i), T(i+1))$ and $M_i \ge 1$, then $\sum_{1\le k<N-1} L(n^*,k) \le T^1(b_{N-1}) + \sum_{j=1}^{M_i} [\,T^{j+1}(b_{N-1}) - T^j(m_{N-1})\,]$ and $n^* \ge \sum_{j=1}^{M_i} [\,T^j(m_{N-1}) - T^j(b_{N-1})\,]$ so that

$$\begin{aligned} T^1(b_{N-1}) &+ \sum_{j=1}^{M_i} [\,T^{j+1}(b_{N-1}) - T^j(m_{N-1})] \\ &\ge \varepsilon \sum_{j=1}^{M_i} \Big[T^j(m_{N-1}) - T^j(b_{N-1}) \Big]. \end{aligned} \tag{3.15}$$

We first treat the case $M_i = 0$, i.e., there is no excursion (before time $T(i+1)$) back to m_{N-1} after reaching b_{N-1}. In this case, (3.14) holds. Let

$$
\begin{aligned}
p_{i,N} &:= P_\omega\left(T(b_{N-1}) \ge \varepsilon(T(i) - T(b_{N-1}))\right) \\
&= P_\omega\left(T(b_{N-1}) \ge \frac{\varepsilon}{1+\varepsilon} T(i)\right) \\
&\le P_\omega\left(T(b_{N-1}) \ge \lambda\right) + P_\omega\left(T(i) < \frac{(1+\varepsilon)\lambda}{\varepsilon}\right),
\end{aligned}
\tag{3.16}
$$

for any $\lambda \ge 1$. Considering the first term in (3.16), we have, by (3.2),

$$
P_\omega\left(T(b_{N-1}) \ge \lambda\right) \le \frac{b_{N-1}^2}{\lambda} \exp(H_{N-2}^+).
$$

Turning to the second term in (3.16), we have

$$
\begin{aligned}
P_\omega\left(T(i) < \frac{(1+\varepsilon)\lambda}{\varepsilon}\right) &\le P_\omega\left(T(m_N) < \frac{(1+\varepsilon)\lambda}{\varepsilon} \;\Big|\; X_0 = b_{N-1}\right) \\
&\le P_\omega\left(\tau_{N-1} < \frac{(1+\varepsilon)\lambda}{\varepsilon}\right) \\
&\le \frac{c_0(1+\varepsilon)\lambda}{\varepsilon} \mathrm{e}^{-H_{N-1}},
\end{aligned}
$$

where we used (3.3) for the last inequality. Hence, plugging in the value of $\varepsilon = \mathrm{e}^{-(H_{N-2}^+)^\delta}$,

$$
\begin{aligned}
p_{i,N} &\le \frac{b_{N-1}^2}{\lambda} \exp(H_{N-2}^+) + \frac{c_0(1+\varepsilon)\lambda}{\varepsilon} \exp(-H_{N-1}) \\
&\le \frac{b_{N-1}^2}{\lambda} \exp(H_{N-2}^+) + 2c_0\lambda \exp((H_{N-2}^+)^\delta) \exp(-H_{N-1}).
\end{aligned}
$$

We choose $\lambda = \lambda_N := \exp((1/2)H_{N-1} + (1/2)H_{N-2}^+)$. Then,

$$
p_{i,N} \le (b_{N-1}^2 + 2c_0) \exp\left(-\frac{1}{2} H_{N-1} + \frac{1}{2} H_{N-2}^+ + (H_{N-2}^+)^\delta\right).
$$

Due to (2.12) and Lemma 2.1, $b_{N-1} \le (H_{N-1}^+)^3$ and $m_{N+1} \le (H_{N-1})^3$ for $N \to \infty$, so that by Lemmas 2.2 and 2.1,

$$
\sum_N \sum_{m_N \le i < m_{N+1}} P_\omega\left(n^* \in [T(i), T(i+1)),\ M_i = 0\right) < \infty, \quad P\text{-a.s.} \tag{3.17}
$$

Turning to consider $n^* \in [T(i), T(i+1))$ and $M_i \ge 1$, for $\lambda = \lambda_N > 0$ to be chosen later, and each $m \ge 1$ let

$$
A(m) := \left\{ T^1(b_{N-1}) + \sum_{j=1}^m \left[T^{j+1}(b_{N-1}) - T^j(m_{N-1}) \right] \ge m\lambda \right\}
$$

$$
B(m) := \left\{ \sum_{j=1}^m \left[T^j(m_{N-1}) - T^j(b_{N-1}) \right] < \frac{m\lambda}{\varepsilon} \right\}.
$$

Note that if $n^* \in [T(i), T(i+1))$ for some $i \geq m_N$ with $M_i \geq 1$, then (3.15) holds, and hence either $A(M_i)$ or $B(M_i)$ holds as well. Consequently, decomposing the event $A(M_i)$ according to i and the event $B(M_i)$ according to the value m of M_i, we get that

$$\begin{aligned}
&P_\omega\left(n^* \in [T(i),\ T(i+1)), \quad \text{for some } i \in [m_N,\ m_{N+1}) \quad \text{and} \quad M_i \geq 1\right) \\
&\quad \leq \sum_{i=m_N}^{m_{N+1}} P_\omega\left(A(M_i), M_i \geq 1\right) + P_\omega\left(B(M_j) \text{ for some } j \geq m_N \ \text{ and } \ M_j \geq 1\right) \\
&\quad \leq m_{N+1} \sup_{i \geq b_{N-1}} P_\omega\left(A(M_i), M_i \geq 1\right) + \sum_{m=1}^{\infty} P_\omega\left(B(m)\right) \\
&\quad =: m_{N+1} I^{(1)} + \sum_{m=1}^{\infty} I_m^{(2)}. \qquad (3.18)
\end{aligned}$$

By the strong Markov property, conditionally on ω both $T^1(b_{N-1})$ and the identically distributed random variables $T^{j+1}(b_{N-1}) - T^j(m_{N-1})$, $j \geq 1$, are independent of the value of M_i for $i \geq b_{N-1}$. Hence, by Markov's inequality

$$\begin{aligned}
I^{(1)} &\leq \sup_{m \geq 1} P_\omega\left(A(m)\right) \leq \sup_{m \geq 1} \frac{1}{m\lambda} E_\omega\left(T(b_{N-1})\right) + \frac{1}{\lambda} E_\omega\left(T(b_{N-1}) \mid X_0 = m_{N-1}\right) \\
&\leq \frac{2}{\lambda} E_\omega\left(T(b_{N-1})\right) \leq \frac{2b_{N-1}^2}{\lambda} \exp\left(H_{N-2}^+\right), \qquad (3.19)
\end{aligned}$$

where the last inequality is due to (3.1).

Further, since $T^j(m_{N-1}) - T^j(b_{N-1})$, $j \geq 1$, are i.i.d. random variables, each having the law of $T(m_{N-1})$ when starting at b_{N-1}, by Corollary 3.3, for any $a > 0$,

$$I_m^{(2)} \leq e^{am\lambda/\varepsilon}\left(E_\omega\left(e^{-aT(m_{N-1})} \mid X_0 = b_{N-1}\right)\right)^m \leq \left(e^{a\lambda/\varepsilon}\frac{2c_0 e^{-H_{N-1}}}{a}\right)^m.$$

We choose $\lambda = \lambda_N := \exp\left(H_{N-1} - 2(H_{N-2}^+)^\delta\right)$, $a = a_N := \exp\left(-H_{N-1} + (H_{N-2}^+)^\delta\right)$ and as stated before $\varepsilon = \varepsilon_N := \exp\left(-(H_{N-2}^+)^\delta\right)$. Since $a\lambda\varepsilon^{-1} = 1$ these choices result with

$$\sum_{m=1}^{\infty} I_m^{(2)} \leq \sum_{m=1}^{\infty}\left(2ec_0\exp(-(H_{N-2}^+)^\delta)\right)^m \leq c_2\exp(-(H_{N-2}^+)^\delta). \qquad (3.20)$$

In view of (3.19), these choices also lead to

$$m_{N+1} I^{(1)} \leq 2m_{N+1} b_{N-1}^2 \exp(-H_{N-1} + 2(H_{N-2}^+)^\delta + H_{N-2}^+),$$

where P-almost surely, for all large N

$$-H_{N-1} + 2(H_{N-2}^+)^\delta + H_{N-2}^+ \leq -(H_{N-2}^+)^{2\delta}$$

(see Lemma 2.2). Further, due to (2.12) and Lemma 2.1, P-almost surely, for all large N,

$$2m_{N+1}b_{N-1}^2 \le (H_{N-2}^+)^7 ,$$

yielding that

$$m_{N+1}I^{(1)} \le \exp(-(H_{N-2}^+)^\delta) .$$

Combining this with (3.20) and (3.18) yields, together with (3.17), that

$$\sum_N P_\omega \left(n^* \in [T(m_N),\ T(m_{N+1}))\right) < \infty.$$

By the Borel–Cantelli lemma, we obtain that for any $\delta < 1$,

$$\limsup_{N\to\infty} \mathrm{e}^{(H_{N-2}^+)^\delta} \max_{n\in[T(m_N),\ T(m_{N+1}))} \frac{1}{n} \sum_{1\le k<N-1} L(n,k) \le 1 , \quad \mathbb{P}\text{-a.s.}$$

Since $\delta < 1$ is arbitrary, this implies (3.13), and completes the proof of Theorem 3.4. □

4. Occupation time and local time

We have so far proved in Theorem 3.4 that ($\mathbb{P}$-almost surely for n large enough) the particle spends at least $(1/2 + o(1))n$ time in a certain valley. The goal of this section is to prove that the time spent by the particle at the bottom of this or a neighbor valley is at least a constant multiple of $n/\log\log\log n$.

There are two main points in the proof: (a) We need to investigate the ratio between the time spent in a valley (occupation time) and the time spent in the bottom of the same (or a neighbor) valley (local time). This is the main part of this section; (b) Since the valley where the particle spends at least $(1/2 + o(1))n$ time has a random number (namely, N_n or $N_n - 1$, see Section 3) and this random number depends on the environment as well as on the movement of the particle, we need a result which holds uniformly for a whole collection of valleys.

4.1. Comparison between occupation time and local time

Recall that N_n is the number of valleys seen by the particle in the first n steps. Define, for any $k \ge 1$,

$$\Lambda_k := \sum_{i=m_k}^{m_{k+1}-1} \mathrm{e}^{-[V(i)-V(b_k)]} . \tag{4.1}$$

Note that $(\Lambda_k,\ k \ge 1)$ depends only on the environment, and that

$$\inf_{k\ge 1} \Lambda_k \ge 1. \tag{4.2}$$

Here are the main estimates of this subsection, which relate occupation time with local time. In particular note that Λ_k measures the effective width of the kth valley as reflected by the ratio between the expected occupation time and the maximal expected local time among its sites (at the appropriate time $n = T(m_{k+1})$ of the particle just reaching the beginning of the next valley).

Proposition 4.1. *There exist c_3 and c_4 such that $\mathbb{P}$-almost surely for n large enough,*

$$L(n, N_n - 1) \le c_3\, \Lambda_{N_n-1}\, \xi(n, b_{N_n-1}), \tag{4.3}$$

$$L(n, N_n) \le c_4\, \Lambda_{N_n} \left[\xi(n, b_{N_n-1}) + \xi(n, b_{N_n}) \right], \tag{4.4}$$

where $L(n, k)$ is the time spent in the k-th valley as in (3.10).

Proposition 4.2. *There exists c_5 such that $\mathbb{P}$-almost surely for all large N,*

$$L(T(m_N), N - 1) \ge c_5\, \Lambda_{N-1} \max_{x\in[m_{N-1}, m_N)} \xi(T(m_N), x). \tag{4.5}$$

Remark on the proof. The basic idea of the proof of the propositions can be described as follows. For (4.5), we consider excursions of the walk away from b_{N-1} during the time interval $[T(b_{N-1}), T(m_N)]$, and let $M = M(N)$ denote the number of completed excursions (M can be 0). The random variable M, which is $\xi(T(m_N), b_{N-1})-1$, has a geometric distribution (under P_ω) and $E_\omega(M)$ is approximatively $e^{H^+_{N-1}}$. By the strong Markov property, all completed excursions make i.i.d. contributions to $\xi(T(m_N), x)$, for any x, hence also to $L(T(m_N), N-1)$. The law of large numbers says that, with ρ denoting the lifetime of an excursion,

$$\xi(T(m_N), x) \asymp M\, E_\omega(\xi(\rho, x)) \asymp \xi(T(m_N), b_{N-1}) e^{-[V(x)-V(b_{N-1})]},$$

(it was proved in Subsection 3.2 that $E_\omega(\xi(\rho, x)) \asymp e^{-[V(x)-V(b_{N-1})]}$), and similarly,

$$L(T(m_N), N-1) \asymp M \sum_{x\in[m_{N-1}, m_N)} E_\omega(\,\xi(\rho, x)\,) \asymp \xi(T(m_N), b_{N-1})\, \Lambda_{N-1}. \tag{4.6}$$

This would yield (4.5) if we take c_5 to be sufficiently small. In order to give a rigorous proof of (4.5), we need to estimate deviation probabilities for M (which is easy), and for the number of visits during a single excursion (which is done via a second moment argument).

The proof of Proposition 4.1 needs slightly more care since it involves an arbitrary time n, instead of the first hitting times $T(m_N)$ in Proposition 4.2. Both proofs go along the lines described in the preceding remark, but require certain technical adjustments. We start with a few preliminary estimates. The first is a rigorous statement of (4.6). For further needs we now provide such a statement uniformly over all $n \ge T(m_N)$, instead of just for $T(m_N)$.

Lemma 4.3. *There exist $0 < c_6 < c_3 < \infty$ such that, for any $\varepsilon > 0$, P-almost surely for all N large enough,*

$$P_\omega\left(\exists n \ge T(\eta_N),\ L(n, N) \ge c_3\, \Lambda_N \xi(n, b_N)\right) \le e^{-(H_N)^{1-\varepsilon}}, \tag{4.7}$$

$$P_\omega\left(\exists n \ge T(m_{N+1}),\ L(n, N) \le c_6\, \Lambda_N \xi(n, b_N)\right) \le e^{-(H_N)^{1-\varepsilon}}. \tag{4.8}$$

Proof of Lemma 4.3. We decompose the random walk into excursions away from $b = b_N$. That is, $T^{-1} = 0$, $T^0 := T(b)$ and

$$T^j := \inf \left\{ k > T^{j-1} : \; X_k = b \right\}, \tag{4.9}$$

are the times of consecutive visits to b, which are $\mathbb{P}$-almost surely finite on account of (1.1). Fixing $i \in [\eta_N, m_{N+1})$, consider the corresponding occupation times of the interval $[m_N, i]$, that is,

$$Z_j = Z_j(i) := \sum_{x=m_N}^{i} \xi(T^j, x) - \xi(T^{j-1}, x)\,.$$

Note that, by the strong Markov property of the walk, Z_j, $j \geq 1$, are independent non-negative random variables (under P_ω), and are also identically distributed and of finite second moment (cf. (4.22) in the sequel). Observe that

$$\overline{M} := \xi(n, b) = \inf \left\{ j : \; T^j > n \right\},$$

and $\overline{M} \geq 1$ whenever $n \geq T(b)$ (which is always the case here). Further, for $i = m_{N+1} - 1$,

$$L(n, N) \leq Z_0 + \sum_{j=1}^{\overline{M}} Z_j\,, \tag{4.10}$$

and (4.10) applies also for $i < m_{N+1} - 1$ provided $n < T(i+1)$.

Since $Z_j \geq 0$, it follows that for any $i \in [\eta_N, m_{N+1})$, $c_7 > 0$, $\ell \geq 1$ and $k_r = \ell 2^r$,

$$\begin{aligned} &P_\omega\left(\exists n \in [T(i), T(i+1)),\; L(n, N) \geq (2c_7 + 1)\overline{M}\Lambda_N\right) \\ &\leq P_\omega(\exists n \geq T(i), \overline{M} \leq \ell) + P_\omega(Z_0 \geq \ell\Lambda_N) + \sum_{r=0}^{\infty} P_\omega\left(\sum_{j=1}^{k_r} Z_j \geq c_7 k_r \Lambda_N\right) \\ &=: I_1(i) + I_2 + I_3(i)\,. \end{aligned} \tag{4.11}$$

Further, as the inequality (4.11) holds for $i = m_{N+1} - 1$ even without the condition $n < T(i+1)$, we have for $c_3 = 2c_7 + 1$ that

$$P_\omega\left(\exists n \geq T(\eta_N),\; L(n, N) \geq c_3\overline{M}\Lambda_N\right) \leq \sum_{i=\eta_N}^{m_{N+1}-1} (I_1(i) + I_2 + I_3(i))\,. \tag{4.12}$$

To estimate the term $I_1(i)$ in (4.11), let $K(b, i)$ denote the number of excursions from b to b made by the walk during the time interval $[T(b), T(i)]$, which has a geometric distribution of parameter $p = p(b, i)$, that is, $P_\omega(K = k) = (1-p)^k p$, $k = 0, 1, 2, \ldots$, where due to (3.4), for any $i > b$,

$$p(b, i) := \omega_b P_\omega(T(i) < T(b) \mid X_0 = b+1) = \omega_b \frac{e^{V(b)}}{\sum_{y=b}^{i-1} e^{V(y)}} \le e^{-W(b,i)}, \tag{4.13}$$

for $W(b, i) := \max_{b \le y < i} V(y) - V(b)$. In particular, for $i \ge \eta_N$ we have that

$$p(b_N, i) \le e^{-W(b_N, \eta_N)} \le e^{V(b_N) - V(\eta_N) + C} \le c_8 e^{-H^+_{N-1}}. \tag{4.14}$$

For any $i > b$, the event $\{n \ge T(i)\}$ implies that $\overline{M} > K(b, i)$. Hence, fixing $\varepsilon > 0$ and $\ell := \lceil p(b, i)^{-1} \exp(-(1/3)(H^+_{N-1})^{1-\varepsilon})\rceil$, we have that

$$I_1(i) \le P_\omega(K(b, i) < \ell) = 1 - (1-p)^\ell \le p\,\ell \le c_8 e^{-(1/3)(H^+_{N-1})^{1-\varepsilon}}. \tag{4.15}$$

Proceeding to deal with I_2, since the steps of the random walk within $[0, m_N - 1]$ do not matter to $Z_0(i) = Z_0(b_N)$, the latter has under P_ω the same law as that of the occupation time of $[1, b_N - m_N + 1]$ till $T(b_N - m_N + 1)$ under $P_{\widetilde{\omega}}$, where $\widetilde{\omega}_x = \omega_{x+m_N-1}$. Consequently, by (3.1) we have that P-almost surely, for all N large enough,

$$\begin{aligned} E_\omega(Z_0) &\le E_{\widetilde{\omega}}(T(b_N - m_N + 1)) \le b_N^2 \exp\left(\max_{m_N - 1 \le y \le z < b_N} (V(z) - V(y))\right) \\ &\le b_N^2\, e^{H^+_{N-1} - (H^+_{N-1})^{1-\varepsilon}}, \end{aligned}$$

with the last inequality due to (2.14). It follows that for our choice of $\ell = \ell(i, N, \varepsilon)$,

$$I_2 \le P_\omega(Z_0 \ge \ell) \le \ell^{-1} E_\omega(Z_0) \le c_8 b_N^2 e^{-(2/3)(H^+_{N-1})^{1-\varepsilon}} \tag{4.16}$$

(where the first inequality is due to (4.2) and the last one due to (4.14)).

As for the term $I_3(i)$ of (4.11), observe that in the notations of Subsection 3.2,

$$Z_1 = \sum_{x=m_N}^{i} [\,\xi(T^1, x) - \xi(T^0, x)\,] = \sum_{x=m_N}^{i} Y_{b,x}\,,$$

where, by (3.5),

$$E_\omega\Big(\xi(T^1, x) - \xi(T^0, x)\Big) = E_\omega\big(Y_{b,x}\big) = \frac{\omega_b}{\omega_x} e^{-[V(x) - V(b)]}. \tag{4.17}$$

It follows, in view of assumption (1.3), that

$$E_\omega(Z_1) \le \delta^{-1} \Lambda_N. \tag{4.18}$$

Consequently, by the independence of Z_j we get for $c_7 \ge \delta^{-1} + 1$ and $k_r = \ell 2^r$, the bound

$$I_3(i) \le \sum_{r=0}^{\infty} P_\omega\left(\sum_{j=1}^{k_r} (Z_j - E_\omega(Z_j)) \ge k_r \Lambda_N\right) \le \frac{\mathrm{Var}_\omega(Z_1)}{\Lambda_N^2} \sum_{r=0}^{\infty} \frac{1}{k_r} \le \frac{2\mathrm{Var}_\omega(Z_1)}{\ell} \tag{4.19}$$

using (4.2) in the last inequality. Observe that

$$\mathrm{Var}_\omega(Z_1) = \mathrm{Var}_\omega\left(\sum_{x=m_N}^{i} Y_{b,x}\right) \le m_{N+1} \sum_{x=m_N}^{i} \mathrm{Var}_\omega(Y_{b,x})\,.$$

Since $V(x) \ge V(b)$ for all $x \in [m_N, m_{N+1})$, we have from (3.6) that for $b = b_N$ and any $x \in (b,\ m_{N+1})$,

$$\mathrm{Var}_\omega(Y_{b,x}) \le c_1 m_{N+1} \exp\left(\max_{b\le y<x}(V(y) - V(x-1))\right). \tag{4.20}$$

Similarly, applying (3.7) instead of (3.6), we obtain that for all $x \in [m_N,\ b)$,

$$\mathrm{Var}_\omega(Y_{b,x}) \le c_1\, m_{N+1} \exp\left(\max_{x\le z<b}(V(z) - V(x))\right), \tag{4.21}$$

and of course $\mathrm{Var}_\omega(Y_{b,b}) = 0$. Summing over $x \in [m_N, i]$ we find by means of (4.20) and (4.21) that

$$\mathrm{Var}_\omega(Z_1(i)) \le c_9\, m_{N+1}^3 \mathrm{e}^{U(b,i)}\,, \tag{4.22}$$

where

$$U(b,i) := \max\left\{\max_{m_N\le y\le z<b}(V(z) - V(y)),\ \max_{b\le y\le z<i}(V(y) - V(z))\right\}.$$

Let

$$\Delta_N := \max_{i\in[\eta_N, m_{N+1})}\{U(b,i) - W(b,i)\}\,.$$

Combining (4.19) and (4.22) we see that by (4.13), for our choice of $\ell = \ell(i, N, \varepsilon)$,

$$I_3(i) \le 2p(b_N, i)\mathrm{Var}_\omega(Z_1(i))\mathrm{e}^{(1/3)(H_{N-1}^+)^{1-\varepsilon}} \le c_{10} m_{N+1}^3 \mathrm{e}^{(\Delta_N+1/3)(H_{N-1}^+)^{1-\varepsilon}}. \tag{4.23}$$

Combining (2.14) and (2.17), we deduce that P-almost surely, for all N large enough,

$$U(b,\eta_N) \le H_{N-1}^+ - (H_{N-1}^+)^{1-\varepsilon}\,.$$

Likewise, note that if $i \in (\eta_N, m_{N+1})$ then combining the preceding with (2.16) we have that

$$U(b,i) \le \max\{U(b,\eta_N), W(b,i) - (H_{N-1}^+)^{1-\varepsilon}\} \le W(b,i) - (H_{N-1}^+)^{1-\varepsilon}\,,$$

using in the last inequality the fact that if $i > \eta_N$ then

$$W(b,i) = \max_{b\le y<i} V(y) - V(b) \ge V(\eta_N) - V(b) \ge H_{N-1}^+\,.$$

Consequently, P-almost surely, for all N large enough

$$\Delta_N \le C - (H_{N-1}^+)^{1-\varepsilon}\,,$$

and thus, plugging (4.15), (4.16) and (4.23) into (4.12), we obtain that P-almost surely, for all N large

$$P_\omega\left(\exists n \geq T(\eta_N),\ L(n,N) \geq c_3\overline{M}\Lambda_N\right) \leq c_{11} m_{N+1}^4 \mathrm{e}^{-\frac{1}{3}(H_{N-1}^+)^{1-\varepsilon}} .$$

Noting that $\varepsilon > 0$ is arbitrary, in view of (2.12) and Lemma 2.1, this implies (4.7).

Moving next to the proof of (4.8), since we are not considering $n < T(m_{N+1})$ in this inequality, we set $i = m_{N+1} - 1$ for the remainder of the proof, in which case we have from (4.17) that

$$E_\omega(Z_1) = \sum_{x=m_N}^{m_{N+1}-1} \frac{\omega_{b_N}}{\omega_x} \mathrm{e}^{-[V(x)-V(b_N)]} \geq \delta \sum_{x=m_N}^{m_{N+1}-1} \mathrm{e}^{-[V(x)-V(b_N)]} \geq \delta\Lambda_N .$$

Since further $L(n,N) \geq \sum_{j=1}^{\overline{M}-1} Z_j$ in this case (regardless of n), we have similarly to (4.11), that for any $c_6 > 0$ and $k_r = \ell 2^r$, $\ell \geq 1$,

$$\begin{aligned} &P_\omega\left(\exists n \geq T(m_{N+1}),\ L(n,N) \leq c_6\overline{M}\Lambda_N\right) \\ &\quad \leq P_\omega(\exists n \geq T(m_{N+1}), \overline{M} \leq \ell) + \sum_{r=0}^{\infty} P_\omega\left(\sum_{j=1}^{k_r} Z_j \leq 4c_6 k_r \Lambda_N\right) \\ &\quad =: I_1 + I_4. \end{aligned} \tag{4.24}$$

With $E_\omega(Z_1) \geq \delta\Lambda_N$, note that if $c_6 < \delta/5$, then

$$I_4 \leq \sum_{r=0}^{\infty} P_\omega\left(\sum_{j=1}^{k_r}(Z_j - E_\omega(Z_j)) \leq -\frac{\delta}{5}k_r\Lambda_N\right) \leq \frac{c_{12}\mathrm{Var}_\omega(Z_1)}{\ell}, \tag{4.25}$$

[using in the last inequality both (4.2) and the fact that $\sum_r k_r^{-1} = 2\ell^{-1}$]. Thus, taking $\ell = \ell(i, N, \varepsilon)$ as before, in view of (4.24) and (4.25) we get (4.8) by the same argument used to complete the derivation of (4.7) (even simpler, as we neither sum over i nor consider I_2 here). □

We next show that upon the walk reaching the right end of a given valley, with high probability no point of this valley has a local time much larger than its bottom. This estimate complements (4.8) en-route to proving Proposition 4.2.

Lemma 4.4. *There exists γ finite such that for any $\varepsilon > 0$, P-almost surely for N large enough,*

$$\max_{x\in[m_N, m_{N+1})} P_\omega\left(\xi(T(m_{N+1}), x) \geq \gamma\xi(T(m_{N+1}), b_N)\right) \leq \mathrm{e}^{-(H_N)^{1-\varepsilon}}.$$

Proof of Proposition 4.2. Taking $c_5 = c_6/\gamma > 0$, the proposition follows from (4.8) and Lemma 4.4 by means of the Borel–Cantelli lemma (as $m_N \mathrm{e}^{-(H_N)^{1-\varepsilon}}$ is summable). □

Proof of Lemma 4.4. We use the path decomposition of the walk as in Lemma 4.3, with $b = b_N$, $T^{-1} = 0$, $T^0 = T(b)$ and T^j, $j \geq 1$ the times of returns of the walk to b (c.f. (4.9)). We further set $\overline{M} = \xi(n, b) \geq 1$, hereafter taking $n = T(i+1)$ for $i = m_{N+1} - 1$. Fixing $x \in [m_N, i]$, $x \neq b$, let $Y_j = \xi(T^j, x) - \xi(T^{j-1}, x)$ for $j = 0, 1, \ldots$, denote the accumulated local time at x during the j-th segment of the walk. Note that the non-negative random variables Y_j, $j \geq 1$, are i.i.d. of finite second moment, and with Y_1 having the law of $Y_{b,x}$ of Subsection 3.2, also $E_\omega(Y_1) \leq \delta^{-1}$ (c.f. (4.17)). Further, similarly to (4.10) we have that

$$\xi(n, x) \leq Y_0 + \sum_{j=1}^{\overline{M}} Y_j \,.$$

Hence, as in (4.11), for $n = T(i+1)$, $\gamma \geq 2(1 + \delta^{-1}) + 1$ and $k_r = \ell 2^r$, $\ell \geq 1$,

$$\begin{aligned} P_\omega(\xi(n, x) \geq \gamma \overline{M}) \;&\leq\; P_\omega(\exists n \geq T(i), \overline{M} \leq \ell) + P_\omega(Y_0 \geq \ell) \\ &\quad + \sum_{r=0}^{\infty} P_\omega\left(\sum_{j=1}^{k_r} (Y_j - E_\omega(Y_j)) \geq k_r \right) \\ &=: I_1 + I_2 + I_5(x)\,. \end{aligned}$$

We fix $\varepsilon > 0$ and $\ell = \ell(i, N, \varepsilon)$ as in Lemma 4.3, thus taking care of the term I_1 (c.f. (4.15)). Further, with $Y_0 \leq Z_0$ this choice also takes care of I_2 (cf. (4.16)) and just as in (4.19) we have that

$$I_5(x) \leq \frac{2\mathrm{Var}_\omega(Y_1)}{\ell} = \frac{2\mathrm{Var}_\omega(Y_{b,x})}{\ell}\,.$$

It follows from (4.20) and (4.21) that

$$\max_{x \in [m_N, i]} \mathrm{Var}_\omega(Y_{b,x}) \leq c_9 m_{N+1} \mathrm{e}^{U(b,i)}$$

[compare with the derivation of (4.22)]. For our choice of ℓ and the bound (4.13) on $p(b, i)$ it follows that P-almost surely, for any N large enough and all $x \in [m_N, i] = [m_N, m_{N+1})$,

$$I_5(x) \leq c_{13} m_{N+1} \mathrm{e}^{-2/3 (H^+_{N-1})^{1-\varepsilon}}$$

[see (4.23)]. As observed before, such estimates are all we need for the lemma [in view of (2.12) and Lemma 2.1]. □

Our next lemma is similar in spirit to Lemma 4.3. Its proof is slightly more involved since two different (consecutive) valley bottoms are relevant here. This happens for example when the occupation time of the last seen valley is to be considered, as in (4.4).

Lemma 4.5. *There exists κ finite such that for any $\varepsilon > 0$, P-almost surely for N large enough,*

$$P_\omega\left(\exists n < T(\eta_N),\; L(n, N) > \kappa\, [\, \xi(T(m_N), b_{N-1}) + \xi(n, b_N)\,]\Lambda_N\right) \leq \mathrm{e}^{-(H_N)^{1-\varepsilon}}.$$

Proof of Lemma 4.5. Clearly, it suffices to consider $n \in [T(i), T(i+1))$ for $i \in [m_N, \eta_N)$. To this end, we adopt the path decomposition and notations of Lemma 4.3 (for $b = b_N$). The random variables Z_j, $j \geq 1$ are i.i.d. and the basic inequality (4.10) applies, just taking $\overline{M} = 0$ whenever $n \in [T(i), T(i+1))$ for $i < b$. With $\overline{K} := \xi(T(m_N), b_{N-1})$, recall that $K = \overline{K} - 1$ is a geometric random variable of parameter

$$p(b_{N-1}, m_N) \leq \mathrm{e}^{-W(b_{N-1}, m_N)} \leq \mathrm{e}^{V(b_{N-1}) - V(m_N) + C} \leq c_8 \mathrm{e}^{-H^+_{N-1}} \tag{4.26}$$

[compare with (4.14)].

Recall (4.18) that $E_\omega(Z_1) \leq \delta^{-1}\Lambda_N$ and further that $\Lambda_N \geq 1$ [see (4.2)]. Hence, with $\kappa \geq 2(\delta^{-1} + 1) + 1$, adapting the derivation of (4.11) we get for $k_r = 2^r$, any $\ell \geq 1$ and $i \in [b_N, \eta_N)$ the bound

$$\begin{aligned} &P_\omega(\exists n \in [T(i), T(i+1)),\ L(n, N) > \kappa[\overline{K} + \overline{M}]\Lambda_N) \\ &\leq P_\omega(K < \ell) + P_\omega(Z_0 \geq \ell\Lambda_N) + \sum_{r=0}^{\infty} P_\omega\left(\sum_{j=1}^{k_r}(Z_j - E_\omega(Z_j)) \geq [\ell + k_r]\Lambda_N\right) \\ &=: I_1 + I_2 + I_6(i)\,. \end{aligned}$$

This applies also for $i \in [m_N, b_N)$, upon setting $I_6(i) = 0$. Fixing $\varepsilon > 0$ we take care of the term I_1 by choosing $\ell := \lceil p(b_{N-1}, m_N)^{-1} \exp(-(1/3)(H^+_{N-1})^{1-\varepsilon}) \rceil$ [see (4.15)]. By (4.26) such choice also handles the term I_2 [compare with (4.16)]. All that remains is to deal with the sum of $I_6(i)$ over $[b_N, \eta_N)$. To this end, adapting the derivation of (4.19), we get the bound

$$I_6(i) \leq \mathrm{Var}_\omega(Z_1) \sum_{r=0}^{\infty} \frac{k_r}{(\ell + k_r)^2} \leq \frac{c_{14}\mathrm{Var}_\omega(Z_1)}{\ell}\,. \tag{4.27}$$

Recall the bound (4.22) on $\mathrm{Var}_\omega(Z_1(i))$ for $i \geq b$, the monotonicity of $i \mapsto U(b, i)$ and the fact that P-almost surely $U(b, \eta_N) \leq H^+_{N-1} - (H^+_{N-1})^{1-\varepsilon}$. Together with (4.27), our choice of ℓ and the bound (4.26), this results with

$$\sum_{i=b_N}^{\eta_N - 1} I_6(i) \leq c_{15} m^4_{N+1} \mathrm{e}^{-(2/3)(H^+_{N-1})^{1-\varepsilon}}\,,$$

holding for all N large enough. As usual, by (2.12) and Lemma 2.1, this concludes the proof. □

Proof of Proposition 4.1. Our claim (4.3) amounts to having $\mathbb{P}$-almost surely for N large,

$$\max_{n \in [T(m_N),\, T(m_{N+1}))} \frac{L(n, N-1)}{\xi(n, b_{N-1})} \leq c_3\,\Lambda_{N-1}\,,$$

which in view of Lemma 2.1 follows from (4.7) by the Borel–Cantelli lemma. Similarly, since $n \mapsto \xi(n, x)$ is monotone, combining Lemma 4.5 and (4.7) we find by Lemma 2.1 that

$$\sum_N P_\omega\left(\exists n \geq T(m_N),\ L(n, N) > c_4\,[\,\xi(n, b_{N-1}) + \xi(n, b_N)\,]\Lambda_N\right) < \infty,$$

for $c_4 = \max(c_3, \kappa)$. Applying the Borel–Cantelli lemma, this obviously implies (4.4). □

4.2. The effective width of the valleys

We consider next the asymptotic growth of the effective width Λ_k [see (4.1)], of the valleys.

Proposition 4.6. *There exist constants* $0 < \gamma_- \leq \gamma_+ < \infty$ *such that*

$$\gamma_- \leq \limsup_{N\to\infty} \frac{1}{\log N} \max_{1\leq k\leq N} \Lambda_k \leq \gamma_+ , \quad P\text{-a.s.} \tag{4.28}$$

Proof. We start by proving the lower bound in (4.28). To this end, consider the events

$$E_k := \{V(b_k - i) - V(b_k) \leq c_{16}, \quad \forall\, 0 \leq i \leq c_{17} \log k\},$$

for finite positive constants c_{16} and c_{17} to be chosen later. Recall that $b_k - m_k \geq H_k^- / \log(\frac{1-\delta}{\delta})$ for the constant δ of (1.3) and P-almost surely $\log H_k^- \sim k$ for all large k (by Lemma 2.1). Consequently, the interval $[b_k - c_{17} \log k, b_k]$ lies inside the k-th valley for all k large enough, in which case the event E_k implies that $\Lambda_k \geq c_{17}\mathrm{e}^{-c_{16}} \log k$. Since E_k is adapted to the filtration $\mathcal{G}_k := \sigma\{V(i),\ 0 \leq i \leq \theta_{k+1}\}$, if

$$\sum_k P\left(E_k \,|\, \mathcal{G}_{k-1}\right) = \infty, \quad P\text{-almost surely,} \tag{4.29}$$

then by Lévy's Borel–Cantelli lemma (see, for example, [8, page 518]), we have that P-almost surely E_k occurs for infinitely many k, and therefore

$$\limsup_{k\to\infty} \frac{\Lambda_k}{\log k} \geq c_{17}\mathrm{e}^{-c_{16}}, \quad P\text{-a.s.}$$

This clearly yields the lower bound in (4.28), with $\gamma_- = c_{17}\mathrm{e}^{-c_{16}} > 0$.

Turning to prove (4.29), define, for any $\rho > 0$,

$$\eta(\rho) := \inf\left\{i > 0 :\ V_k(i) - \min_{0\leq j\leq i} V_k(j) \geq \rho\right\}, \tag{4.30}$$

$$b(\rho) := \sup\left\{i < \eta(\rho) :\ V_k(i) = \min_{0\leq j\leq \eta(\rho)} V_k(j)\right\}, \tag{4.31}$$

and the associated events

$$E(\rho, k) := \{V_k(b(\rho) - i) - V_k(b(\rho)) \leq c_{16}, \quad \forall\, 0 \leq i \leq c_{17} \log k\},$$

where $(V_k(i) := V(i + \theta_k) - V(\theta_k), i \in \mathbb{Z}_+)$ has the same law as $(V(i), i \in \mathbb{Z}_+)$. Recall that H_{k-1}^+, θ_k and $V(\theta_k)$ are $\mathcal{G}_{k-1}$-measurable while $\eta_k - \theta_k = \eta(\rho)$ and

$b_k - \theta_k = b(\rho)$ for $\rho = H^+_{k-1}$. Thus, $P(E_k|\mathcal{G}_{k-1}) = P(E(\rho, k))$ for this choice of ρ. Since P-almost surely, $\log H^+_{k-1} \sim k$ for $k \to \infty$ (by Lemma 2.1), the proof of (4.29) is reduced to showing that for some $c_{18} > 0$ and all k large enough,

$$\inf_{\rho \geq e^{k/2}} P(E(\rho, k)) \geq \frac{c_{18}}{k}. \tag{4.32}$$

To verify (4.32), recall that by assumption (1.3) the increments of the random walk V are within $[-C, C]$ for some $C = C(\delta)$ finite and positive. Further, (1.2) yields that $p_* := \min\{P(V(1) > 0),\ P(V(1) \leq -2C/d)\} > 0$ for some finite positive integer $d \geq 2$. Restricting the increments of the walk $V(i+1) - V(i)$, $i \leq j-1$ to be strictly positive if $V(i) \leq C$ and at most $-2C/d$ otherwise, followed by d increments which are at most $-2C/d$ each, results in a sample of length $j + d$ for which the event

$$F_j := \bigcup_{\ell = j+1}^{j+d} \left\{ V(0) = 0,\ V(i) \in (0, 2C],\ \ 0 \leq i \leq \ell - 1,\ V(\ell) \in (-C, 0] \right\}$$

holds. Hence, $P(F_j) \geq p_*^{j+d}$. In particular, if $c_{17} > 0$ is small enough, then $P(F_j) \geq \frac{c_{18}}{k}$, for $j = \lceil c_{17} \log k \rceil$, some $c_{18} > 0$ and all k. Setting $V(\cdot)$ for $V_k(\cdot)$, it is well known that the path $(V(i),\ 0 \leq i \leq \eta(\rho))$ can be constructed by concatenating i.i.d. excursions of the walk $V(\cdot)$, each starting at 0 and terminating at the first exit time of $(0, \rho)$. Then, $\eta(\rho)$ is the terminal time of the first excursion to exit via $[\rho, \infty)$ with $b(\rho)$ its starting time, and in concatenating the excursions en-route to the path, one adds to each excursion the (non-positive) values of all terminal points of preceding excursions. Adopting this construction, the event $E(\rho, k)$ occurs for $c_{16} = 3C < \rho$ if the last of the excursions which exit via $(-\infty, 0]$ is in F_j for $j = \lceil c_{17} \log k \rceil$, yielding the bound (4.32) in view of the independence of these excursions.

Turning to show the upper bound in (4.28), let $\eta = \eta(\rho)$ and $b = b(\rho)$ be as in (4.30)–(4.31). In the sequel we show that for some positive finite constants c_{19}, c_{20} and $r_0 \geq 1$,

$$\sup_{\rho \geq K_0} P\left(\sum_{i=0}^{\eta - 1} e^{-[V(i) - V(b)]} > c_{20} r \right) \leq e^{-c_{19} r}, \qquad \forall r \geq r_0. \tag{4.33}$$

Recall that conditional upon $\mathcal{G}_{k-1}$ the joint law of $V(i) - V(b_k)$ for $i \in [\theta_k, \eta_k]$ is the same as the unconditional joint law of $V(i) - V(b(\rho))$ for $i \in [0, \eta(\rho)]$ upon taking $\rho = H^+_{k-1}$ (which is measurable on $\mathcal{G}_{k-1}$). Since $H^+_{k-1} \geq H^+_0 \geq K_0$, it thus follows from (4.33) that $P(\sum_{i=\theta_k}^{\eta_k - 1} e^{-[V(i) - V(b_k)]} > c_{20} r) \leq e^{-c_{19} r}$, for any $k \geq 1$. So, by the Borel–Cantelli lemma, for some $\gamma_+ < \infty$ and P-almost surely for N large enough,

$$\sum_{i = \theta_N}^{\eta_N - 1} e^{-[V(i) - V(b_N)]} \leq \gamma_+ \log N.$$

Further, from the definition of θ_N and H^+_{N-1} we know that P-almost surely for N large enough,

$$\sum_{i=m_N}^{\theta_N-1} \mathrm{e}^{-[V(i)-V(b_N)]} \le \sum_{i=m_N}^{\theta_N-1} \mathrm{e}^{-H^+_{N-1}} \le b_N\, \mathrm{e}^{-H^+_{N-1}} \le \mathrm{e}^{-N}$$

(the last inequality being a consequence of (2.12) and Lemma 2.1). Also, by (2.16), for any $\varepsilon > 0$ and P-almost surely for all large N,

$$\sum_{i=\eta_N}^{m_{N+1}-1} \mathrm{e}^{-[V(i)-V(b_N)]} \le m_{N+1}\mathrm{e}^{-(H^+_{N-1})^{1-\varepsilon}} \le \mathrm{e}^{-N}.$$

Thus, we have that P-almost surely

$$\Lambda_N = \sum_{i=m_N}^{m_{N+1}-1} \mathrm{e}^{-[V(i)-V(b_N)]} \le 2\mathrm{e}^{-N} + \gamma_+ \log N,$$

for all large N, clearly yielding the upper bound in (4.28).

To complete the proof of the proposition, it thus remains only to prove (4.33). To this end, setting $\eta = \eta(\rho)$ and $b = b(\rho)$, we consider the random variables $L(j) := \#\{i < \eta :\ V(i) - V(b) \in [j, j+1)\}$, $j \in \mathbb{Z}_+$ (which depend on ρ via η and b) and the events

$$A_{j,m} := \left\{-(m+1) < V(b) \le -m,\ L(j) > c_{21}\mathrm{e}^{j/2}r\right\},$$

for $j, m \in \mathbb{Z}_+$ and $c_{21} < \infty$ to be determined in the sequel. Since $\{L(j) > c_{21}\mathrm{e}^{j/2}r\}$ is the disjoint union of $A_{j,m}$ and $\sum_{i=0}^{\eta-1} \mathrm{e}^{-[V(i)-V(b)]} \le \sum_{j=0}^{\infty} \mathrm{e}^{-j} L(j)$, it follows that

$$P\left(\sum_{i=0}^{\eta-1} \mathrm{e}^{-[V(i)-V(b)]} > c_{20}r\right) \le \sum_{j=0}^{\infty} P\left(L(j) > c_{21}\mathrm{e}^{j/2}r\right) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} P(A_{j,m}), \tag{4.34}$$

provided $c_{20} \ge c_{21} \sum_{j=0}^{\infty} \mathrm{e}^{-j/2}$.

We thus proceed to bound $P(A_{j,m})$ for all j, m and $\rho \ge K_0$. To this end, as $V(\cdot)$ is a non-degenerate random walk of zero mean and bounded increments, for large positive integer c_{22} we have that

$$q_* := \sup_{j\ge 0} P\left(\inf_{i\in[0,\,(j+c_{22})^2)} V(i) > -(j+2)\right) < 1. \tag{4.35}$$

Next, fixing $j \in \mathbb{Z}_+$, let $g = g(j) = (j + c_{22})^2 \ge 1$ and $R = R(j) = \lceil c_{21}\mathrm{e}^{j/2}r/g(j)\rceil - 1$, where c_{21} is taken sufficiently large so that $R(j) \ge 1$ for any $j \in \mathbb{Z}_+$ and $r \ge 1$. Fixing also $m \in \mathbb{Z}_+$, we consider the stopping times

$$T_0 := \inf\{i \ge 0 :\ V(i) \in (j-m-1, j-m+1)\},$$
$$T_\ell := \inf\{i \ge T_{\ell-1} + g :\ V(i) \in (j-m-1, j-m+1)\}, \quad \ell \ge 1,$$

and the associated stopped σ-fields $\mathcal{F}_\ell$. Suppose the event $A_{j,m}$ holds. Then, the random walk $(V(i),\ i \le \eta - 1)$ hits the interval $(j-m-1, j-m+1)$ more than $\lfloor c_{21} e^{j/2} r \rfloor \ge Rg$ times, and hence $T_R < \eta$. In particular, as the walk $V(\cdot)$ can not reach $[\rho, \infty)$ for $i < \eta(\rho)$ and the event $\Gamma_0 := \{T_0 < \eta\}$ must hold as well, it follows that $A_{j,m}$ is an empty set whenever $j-m-1 \ge \rho$. Further, if $A_{j,m}$ holds, then by the preceding discussion also the events

$$\Gamma_\ell := \left\{ \inf_{i \in [0,g)} V(T_{\ell-1} + i) > -(m+1) \right\}$$

hold for $\ell = 1, \dots, R$. Finally, if $A_{j,m}$ holds then $V(\eta) \ge \rho + V(b) > \rho - (m+1)$, while $V(i) > -(m+1)$ for all $i \in (T_R, \eta]$, implying that the event

$$\Gamma_* = \left\{ V(T_R + i), i \ge 0,\ \text{exits } (-(m+1), \rho - (m+1)] \text{ upwards} \right\},$$

holds as well. To summarize, we have seen that

$$A_{j,m} \subseteq \Gamma_0 \cap \bigcap_{\ell=1}^{R} \Gamma_\ell \cap \Gamma_* .$$

The following bounds apply

$$\begin{aligned} P(\Gamma_* \mid \mathcal{F}_R) &\le \sup_{x \in (j, j+2)} P\left(V(\cdot) \text{ exits } (0, \rho] \text{ upwards } \middle| V(0) = x \right) \\ &\le \frac{j+2+C}{\rho + C}, \end{aligned} \tag{4.36}$$

$$P(\Gamma_\ell \mid \mathcal{F}_{\ell-1}) \le \sup_{x \in (j-m-1, j-m+1)} P\left(\inf_{i \in [0,g)} V(i) > -(m+1) \,\middle|\, V(0) = x \right) \le q_*, \tag{4.37}$$

using (4.35) in the latter bound. Further, if $j - m + 1 \le -J\rho$ for some $J \in \mathbb{Z}_+$, then considering the first downward crossing of $-k\rho$ for $k = 1, \dots, J$, leads to

$$P(\Gamma_0) = P(T_0 < \eta) \le P\left(V(\cdot) \text{ exits } (-\rho + C, \rho) \text{ downwards} \right)^J \le \left(\frac{\rho + C}{2\rho} \right)^J .$$

Since Γ_0 is empty for $j - m - 1 \ge \rho$, this implies that for some finite c_{23} and positive c_{24},

$$P(\Gamma_0) \le c_{23} e^{-c_{24}|m-j|/\rho}, \qquad \forall \rho \ge K_0,\ j, m \in \mathbb{Z}_+ . \tag{4.38}$$

With Γ_ℓ measurable on $\mathcal{F}_\ell$, upon applying the strong Markov property at the stopping times T_ℓ, $\ell = 0, \dots, R$, we get from (4.36), (4.37) and (4.38) that

$$\begin{aligned} P\left(A_{j,m}\right) &\le P\left(\Gamma_0 \cap \bigcap_{\ell=1}^{R} \Gamma_\ell \cap \Gamma_* \right) \le c_{23} e^{-c_{24}|m-j|/\rho}\ q_*^R\ \frac{j+2+C}{\rho + C} \\ &\le c_{25} \frac{j+1}{\rho} \exp\left[-c_{26} \left(\frac{|m-j|}{\rho} + \frac{e^{j/2} r}{(j + c_{22})^2} \right) \right] . \end{aligned}$$

This implies for some finite c_{27} and all $\rho \geq K_0$,

$$\sum_{m=0}^{\infty} P(A_{j,m}) \leq c_{27}\,(j+1)\exp\left(-c_{26}\frac{e^{j/2} r}{(j+c_{22})^2}\right).$$

Plugging the latter bound into (4.34) yields (4.33), thus concluding the proof of the proposition. □

5. Proof of Theorem 1.1

We start with a preliminary result.

Lemma 5.1. *We have*

$$\lim_{n\to\infty} \frac{\xi(n, b_{N_n}) + \xi(n, b_{N_n-1})}{\max_{y<m_{N_n-1}} \xi(n, y)} = \infty, \qquad \mathbb{P}\text{-a.s.}$$

Proof. According to Proposition 4.1, we have $\mathbb{P}$-a.s. for n large enough

$$\xi(n, b_{N_n}) + \xi(n, b_{N_n-1}) \geq \max\left\{\frac{L(n, N_n)}{c_4\,\Lambda_{N_n}},\ \frac{L(n, N_n-1)}{c_3\,\Lambda_{N_n-1}}\right\}.$$

Recall that by Proposition 4.6, $\mathbb{P}$-a.s. for all n large enough

$$\max\{\Lambda_{N_n-1}, \Lambda_{N_n}\} \leq 2\gamma_+ \log N_n\,,$$

and by (3.11), for any $\delta < 1$, also $\mathbb{P}$-a.s.

$$\begin{aligned} L(n, N_n-1) + L(n, N_n) &\geq \exp\left((\log n)^{\delta}\right) \sum_{1\leq k<N_n-1} L(n,k) \\ &\geq \exp\left((\log n)^{\delta}\right) \max_{y<m_{N_n-1}} \xi(n,y)\,. \end{aligned}$$

Hence, $\mathbb{P}$-a.s. for n large enough,

$$\xi(n, b_{N_n}) + \xi(n, b_{N_n-1}) \geq c_{28}\,(\log N_n)^{-1} \exp\left((\log n)^{\delta}\right) \max_{y<m_{N_n-1}} \xi(n,y).$$

Since $N_n \sim \log\log n$ for $n \to \infty$ (see (3.9)), this proves the claim of the lemma. □

The rest of the section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1. According to a 0–1 law in [3], there exists a possibly degenerate constant $c \in [0, \infty]$ such that

$$\liminf_{n\to\infty} \frac{\xi^*(n)}{n/\log\log\log n} = c, \qquad \mathbb{P}\text{-a.s.}$$

(Though the 0–1 law was proved in [3] for transient random walk in random environment, its proof remains valid for our recurrent walk, with a reflecting barrier at the origin.)

It remains to check that $0 < c < \infty$.

We start by showing that c is positive. From Proposition 4.1 we have that $\mathbb{P}$-a.s. for n large enough,

$$L(n,\ N_n - 1) + L(n,\ N_n) \le (c_3 + c_4)[\xi(n,\ b_{N_n-1}) + \xi(n,\ b_{N_n})] \max_{k \le N_n} \Lambda_k$$

Hence, combining (3.9) with the upper bound in Proposition 4.6, we have

$$\liminf_{n\to\infty} \frac{[\,\xi(n,\ b_{N_n-1}) + \xi(n,\ b_{N_n})\,] \log\log\log n}{L(n,\ N_n - 1) + L(n,\ N_n)} \ge \frac{1}{(c_3 + c_4)\gamma_+}, \qquad \mathbb{P}\text{-a.s.}$$

Since $\xi(n,\ b_{N_n-1}) + \xi(n,\ b_{N_n}) \le 2\xi^*(n)$ and $\mathbb{P}$-almost surely $n^{-1}(L(n,\ N_n - 1) + L(n,\ N_n)) \to 1$ for $n \to \infty$ (as a consequence of Theorem 3.4), this implies that

$$\liminf_{n\to\infty} \frac{\xi^*(n) \log\log\log n}{n} \ge \frac{1}{2(c_3 + c_4)\gamma_+}, \qquad \mathbb{P}\text{-a.s.}$$

Consequently, $c \ge 1/(2(c_3 + c_4)\gamma_+) > 0$ as claimed.

Turning to show that $c < \infty$, note that if $n = T(m_N)$ then $N_n = N$ while $\xi(n, b_N) = 0$. Thus, by Lemma 5.1, $\mathbb{P}$-a.s, if $n = T(m_N)$ for N large enough, then $\xi^*(n) = \max_{x\in[m_{N-1}, m_N)} \xi(n, x)$. Consequently, by (4.5), and the trivial inequality $L(T(m_N), N - 1) \le T(m_N)$, we have that

$$\limsup_{N\to\infty} \frac{\xi^*(T(m_N))\Lambda_{N-1}}{T(m_N)} \le \frac{1}{c_5}, \qquad \mathbb{P}\text{-a.s.}$$

By the lower bound in Proposition 4.6, it follows that $\limsup_k (\log k)^{-1}\Lambda_{k-1} \ge \gamma_-$. Consequently

$$\liminf_{N\to\infty} \frac{\xi^*(T(m_N)) \log N}{T(m_N)} \le \frac{1}{c_5\,\gamma_-}, \qquad \mathbb{P}\text{-a.s.}$$

Since P-almost surely $\log\log m_N \sim \log N$ for all N large enough [see (2.13)], and $\mathbb{P}$-almost surely $\log\log T(x) \sim \frac{1}{2}\log x$ for $x \to \infty$ (see Fact 3.1), it follows that $\mathbb{P}$-almost surely $\log\log\log T(m_N) \sim \log N$ for $N \to \infty$. Therefore,

$$\liminf_{n\to\infty} \frac{\xi^*(n) \log\log\log n}{n} \le \frac{1}{c_5\,\gamma_-}, \qquad \mathbb{P}\text{-a.s.}$$

We deduce that $c \le 1/(c_5\gamma_-)$ is finite and hence conclude the proof of Theorem 1.1. □

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