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Variational approximation for Fokker–Planck equation on Riemannian manifold

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Abstract. Under the bounded geometry assumption on Riemannian manifold M, a variational approximation for Fokker–Planck equation on M is constructed by the scheme of Jordan et al. in SIAM J Math Anal 29(1):1–17, 1998. Moreover, the uniqueness and global L^p-estimate of the solution for $1 < p < \dim(M)/(\dim(M) - 1)$ are obtained for a broad class of potential.

1. Introduction

Recently there has been considerable progress in understanding a wide range of dissipative evolution equations in terms of variational problems involving the Wasserstein metric. In particular, Jordan et al. in [15] has shown that the Fokker–Planck equation, or forward Kolmogorov equation, is gradient flow for the entropy functional in 2-Wasserstein metric, and constructed a time discrete, iterative variational scheme whose solutions converge to the solution of the Fokker–Planck equation for a broad class of potential. Following their work, Carlen and Gangbo in [6] studied several constrained variational problems in the 2-Wasserstein metric for which the set of probability densities satisfying the constraint is not closed, and then in [7] devoted to the study of a variational approach to constructing solutions of the nonlinear kinetic Fokker–Planck equation, by means of "steepest descent" in the Wasserstein metric. Fokker–Planck equation plays a central role in statistical physics and in the study of fluctuations in physical and biological systems [12, 19, 20], which is closely connected with the theory of stochastic differential equations: a (normalized) solution to a given Fokker–Planck equation represents the probability density for the position (or velocity) of a particle whose motion is described by a corresponding Itô stochastic differential equation.

In [15], Jordan et al. studied the following Fokker–Planck equation on $\mathbb{R}^d$

$$\frac{\partial \rho}{\partial t} = \mathrm{div}(\rho \nabla f) + \Delta \rho, \quad \rho(0, x) = \rho_0(x), \tag{1}$$

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where ρ_0 is a smooth probability density on $\mathbb{R}^d$ and f is a positive smooth function (called potential) on $\mathbb{R}^d$ satisfying

$$|\nabla f(x)| \leqslant C(f(x)+1) \tag{2}$$

for some $C > 0$. Let us simply describe the scheme introduced in [15]. Let $\mathcal{P}$ denote the set of probability densities on $\mathbb{R}^d$ with finite second moments, i.e., a non-negative measurable function $\rho \in \mathcal{P}$ satisfy $\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x = 1$ and $\int_{\mathbb{R}^d} |x|^2\rho(x)\mathrm{d}x < +\infty$. For $\rho_1, \rho_2 \in \mathcal{P}$, the 2-Wasserstein metric is defined by

$$W_2^2(\rho_1, \rho_2) := \inf_{\pi \in \mathcal{E}(\rho_1,\rho_2)} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(\mathrm{d}x, \mathrm{d}y),$$

where $\mathcal{E}(\rho_1, \rho_2)$ consists of all probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ with marginal distribution densities ρ_1 and ρ_2. This infimum is actually attained at a unique point $\pi \in \mathcal{E}(\rho_1, \rho_2)$ (see [5, 11, 13]). The free energy functional $F(\rho)$ is defined by

$$F(\rho) := \int\limits_{\mathbb{R}^d} \rho(x)\log\rho(x)\mathrm{d}x + \int\limits_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x.$$

Assume that the initial density $\rho_0 \in \mathcal{P}$ has finite free energy. Fixing a time step $h > 0$, one inductively defines ρ_k in terms of ρ_{k-1} by choosing ρ_k to minimize the functional

$$\rho \mapsto \frac{1}{2}W_2^2(\rho_{k-1}, \rho) + hF(\rho)$$

on $\mathcal{P}$. It is shown in [15] that there is a unique minimizer $\rho_k \in \mathcal{P}$, so that each ρ_k is well defined. The time-dependent probability density $\rho^h(t,x)$ is defined by

$$\rho^h(t,x) := \sum_{k=0}^{\infty} 1_{[kh,(k+1)h)}(t)\rho_k(x).$$

It is proved in [15] that $\rho(t,\cdot) = \lim_{h\to 0} \rho^h(t,\cdot)$ exists weakly in $L^1(\mathbb{R}^d, \mathrm{d}x)$ for each t, and the resulting time dependent probability density solves the equation (1) with $\lim_{t\to 0} \rho(t,\cdot) = \rho_0$. For a wide class of potential f satisfying (2), the uniqueness was also proved.

In this paper, we consider equation (1) on an arbitrary Riemannian manifold M. In this case, $\Delta = \mathrm{div}\nabla$ is the Laplace-Beltrami operator on M. The variational approximation to equation (1) is constructed by the approach in [15]. Under the bounded geometry assumption on M, we will prove that if the potential f(positive smooth function) satisfies

$$|\nabla f(x)| \leqslant C\left(f(x) + \mathbf{d}^2(x, x_0) + 1\right)$$

for some $C > 0$, where $\mathbf{d}(\cdot,\cdot)$ is the Riemannian distance and $x_0 \in M$ is fixed, then the uniqueness for equation (1) holds. Although the method is completely parallel to that of [15], there are still some technical difficulties needed to be overcome.

One of the main difficulties is the absence of smoothness of the geodesic distance function $\mathbf{d}^2(\cdot,\cdot)$ on $M \times M$, although it is differentiable outside a set with zero volume (cf. [17]). Thus, it seems that one cannot get the core equation (40) of [15]. Instead of that, two key estimates (Lemmas 3.2 and 3.3 below) related to the distance function are proved.

This paper is organized as follows: in section 2, we give some preliminaries about the 2-Wasserstein metric and entropy function. Then, in section 3, the variational lemmas of discrete scheme are established. Lastly, in section 4, the discrete scheme is proved to converge to the solution in weak sense, and then the uniqueness and L^p-integrability of solution are proved.

2. Preliminaries

Let M be a complete and smooth Riemannian manifold (non-compact). Throughout this paper we assume that M has bounded geometry, i.e.:

(I) *Injective radius δ of M is strictly positive;*
(II) *There exists a $\kappa \geqslant 0$ such that*

$$\langle \mathrm{Ricci}_x X_x, X_x\rangle_x \geqslant -(m-1)\kappa |X_x|_x, \qquad \text{for all } x \in M \text{ and } X_x \in T_x M,$$

where m is the dimension of M.

Some notations used below are collected as follows:

$$\begin{aligned}
\langle\cdot,\cdot\rangle_x &:= \text{the Riemannian inner product at tangent space } T_x(M),\\
|\cdot|_x &:= \text{the length of a vector in } T_x(M),\\
\mathrm{d}x &:= \text{the Riemannian measure},\\
\nabla &:= \text{the gradient operator},\\
\mathrm{div} &:= \text{the dual operator of } \nabla \text{ in } L^2(M,\mathrm{d}x),\\
\Delta &:= \mathrm{div}\nabla \text{ the Laplace-Beltrami operator},\\
\mathbf{d}(\cdot,\cdot) &:= \text{the Riemannian distance},\\
B_{x_0}(r) &:= \text{the ball with center } x_0 \in M \text{ and radius } r \geqslant 0,\\
B^c_{x_0}(r) &:= \text{the complement of } B_{x_0}(r) \text{ in } M.
\end{aligned}$$

We need the following lemma about the growth of volume of balls, which may be proved directly by Bishop's comparison theorem (see [8, p.123 Theorem 3.9 and p.72 (2.48)]).

Lemma 2.1. *Let M be a complete Riemannian manifold and (II) hold, then for any $x_0 \in M$ and $r \geqslant 0$*

$$\mathrm{Vol}(B_{x_0}(r)) := \int_M 1_{B_{x_0}(n)}\mathrm{d}x \leqslant \omega_{m-1} r^m e^{\sqrt{\kappa}(m-1)r}/m,$$

where ω_{m-1} is the surface area of the unit $m-1$ sphere in $\mathbb{R}^m$. In particular, there are two positive constants c_1, c_2 depending only on κ, m such that

$$\mathrm{Vol}(B_{x_0}(r)) \leqslant c_2 e^{c_1\cdot r}, \quad r > 0. \tag{3}$$

Throughout this paper, we shall fix one point $x_0 \in M$. Define

$$V(\rho) := \int\limits_M \mathbf{d}^2\left(x, x_0\right)\rho(x)\mathrm{d}x,$$

and

$$\mathcal{K} := \left\{\rho : M \mapsto \mathbb{R}_+, \int\limits_M \rho(x)\mathrm{d}x = 1, V(\rho) < +\infty\right\}.$$

Definition 2.2. *For any $\rho_1, \rho_2 \in \mathcal{K}$, the 2-Wasserstein metric between ρ_1 and ρ_2 is defined by*

$$W_2^2\left(\rho_1, \rho_2\right) := \inf_{\pi \in \mathcal{E}(\rho_1, \rho_2)} \int\limits_{M\times M} \mathbf{d}^2(x, y)\pi\left(\mathrm{d}x, \mathrm{d}y\right), \tag{4}$$

where $\mathcal{E}\left(\rho_1, \rho_2\right)$ denotes the totality of probability measures on $M \times M$ with marginal distributions densities ρ_1 and ρ_2.

We note that the Wasserstein metric may be defined equivalently by (cf. [18])

$$W_2^2\left(\rho_1, \rho_2\right) = \inf_{\mathrm{Law}(X)=\rho_1\mathrm{d}x, \mathrm{Law}(Y)=\rho_2\mathrm{d}x} \mathbb{E}\left(\mathbf{d}^2(X, Y)\right),$$

where X, Y are random variables defined on some probability space $(\Omega, \mathcal{F}, P)$, and with values in M. The following triangular inequality holds for any $\rho_1, \rho_2, \rho_3 \in \mathcal{K}$:

$$W_2\left(\rho_1, \rho_2\right) \leqslant W_2\left(\rho_1, \rho_3\right) + W_2\left(\rho_2, \rho_3\right).$$

The convergence in the metric W_2 is equivalent to the usual weak convergence plus convergence of second moments, which may be proved by the well-known Skorohod's theorem in probability theory. Moreover, it is clear that for any $\rho_1, \rho_2 \in \mathcal{K}$, the probability measure family $\mathcal{E}\left(\rho_1, \rho_2\right)$ on $M \times M$ is tight.

The variational problem (4) is an example of Monge–Kantorovich mass transference problem with the particular cost function $c(x, y) = \mathbf{d}^2(x, y)$ (cf. [22], etc.). In that context, an infimizer $\pi_{\min} \in \mathcal{E}\left(\rho_1, \rho_2\right)$ such that

$$W_2^2\left(\rho_1, \rho_2\right) = \int\limits_{M\times M} \mathbf{d}^2(x, y)\pi_{\min}\left(\mathrm{d}x, \mathrm{d}y\right) \tag{5}$$

is referred to as an optimal transference plan. The existence of such a $\pi_{\min}$ is a consequence of the tightness of $\mathcal{E}\left(\rho_1, \rho_2\right)$ (cf. [22]). In Euclidean space, Brenier [5] has established the existence of a one-to-one optimal transference plan in the case that ρ_1 and ρ_2 have bounded support. Later, McCann [17] and Feldman and McCann [10] extended Brenier's results to Riemannian manifold for the cost functions $c(x, y) = \mathbf{d}^2(x, y)$ and $c(x, y) = \mathbf{d}(x, y)$. In the present paper, we only use the existence of infimizer in (5).

Let $S(\rho)$ be the entropy function on $\mathcal{K}$ defined by

$$S(\rho) := \int_M \rho(x) \log \rho(x)\, \mathrm{d}x.$$

Here we use the usual convention $0 \log 0 = 0$. We now prepare a lower bound estimate of the entropy function $S(\rho)$ for later use.

Lemma 2.3. *There exists a constant $C > 0$ such that for any $\rho \in \mathcal{K}$ and $\varepsilon > 0$*

$$S(\rho) \geqslant -\left(\frac{c_1^2}{\varepsilon} + C\right) - \varepsilon V(\rho), \tag{6}$$

where $c_1 > 0$ is the constant in Lemma 2.1. In particular, $S(\rho)$ is well defined and strictly convex on $\mathcal{K}$ with values in $(-\infty, +\infty)$.

Proof. Set $\sigma(x) := \mathrm{e}^{-2c_1 \mathbf{d}(x,x_0)}/Z$, where $Z := \int_M \mathrm{e}^{-2c_1 \mathbf{d}(x,x_0)} \mathrm{d}x < +\infty$ by Lemma 2.1. By the non-negativity of relative entropy and Young's inequality, we have

$$\begin{aligned}
0 &\leqslant \int_M \left(\frac{\rho}{\sigma}\right) \log\left(\frac{\rho}{\sigma}\right) \sigma \mathrm{d}x \\
&= S(\rho) + 2c_1 \int_M \mathbf{d}(x, x_0)\rho(x)\mathrm{d}x + \log Z \\
&\leqslant S(\rho) + \varepsilon V(\rho) + \frac{c_1^2}{\varepsilon} + \log Z,
\end{aligned}$$

which gives the desired result. □

The following lemma will be used to prove the existence of infimizer for the variational problem in Lemma 3.1 below, which proof is standard (cf. [3]), and the details are omitted.

Lemma 2.4. *Let $\rho_n \in \mathcal{K}$. Define the probability measures $\nu_n(\mathrm{d}x) := \rho_n(x)\mathrm{d}x$ on M. Assume that ν_n weakly converges to ν and $\sup_n(S(\rho_n) + V(\rho_n)) < +\infty$, then*

(i) $\nu(\mathrm{d}x)$ is absolutely continuous with respect to $\mathrm{d}x$, and $\rho(x) := \frac{\nu(\mathrm{d}x)}{\mathrm{d}x} \in \mathcal{K}$,
(ii) $\rho_n(x)$ weakly converges to $\rho(x)$ in $L^1(M, \mathrm{d}x)$.

3. Variational Lemmas

Let $f \in C^\infty(M)$ be positive, we define free energy functional on $\mathcal{K}$ as

$$F(\rho) := E(\rho) + S(\rho),$$

where $E(\rho) := \int_M f(x)\rho(x)\, \mathrm{d}x$.

Then we have

Lemma 3.1. *Fix $h > 0$. Let $\rho_0 \in \mathcal{K}$ satisfy $F(\rho_0) < +\infty$, then there exists a unique $\rho_{\min} \in \mathcal{K}$ minimizing the functional*

$$\mathcal{K} \ni \rho \mapsto \frac{1}{2} W_2^2 (\rho_0, \rho) + hF(\rho) =: G^{\rho_0}(\rho),$$

that is,

$$G^{\rho_0}(\rho_{\min}) = \inf_{\rho \in \mathcal{K}} G^{\rho_0}(\rho) =: G^{\rho_0}_{\min}.$$

In particular, $F(\rho_{\min}) \leqslant F(\rho_0)$.

Proof. Let X_1 and X_2 be any two random variables on M with $\mathrm{Law}(X_1) = \rho_1$ and $\mathrm{Law}(X_2) = \rho_2$, then

$$\mathbb{E}\left(\mathbf{d}^2(X_1, x_0)\right) \leqslant 2\mathbb{E}\left(\mathbf{d}^2(X_1, X_2)\right) + 2\mathbb{E}\left(\mathbf{d}^2(X_2, x_0)\right).$$

This means that

$$V(\rho_1) \leqslant 2W_2^2(\rho_1, \rho_2) + 2V(\rho_2). \tag{7}$$

From this inequality and (6), we have

$$G^{\rho_0}(\rho) \geqslant \frac{1}{4} V(\rho) - \frac{1}{2} V(\rho_0) + hS(\rho) \tag{8}$$

$$\geqslant \frac{1}{4} V(\rho) - \frac{1}{2} V(\rho_0) - hC_\varepsilon - h\varepsilon V(\rho) \tag{9}$$

which produces the boundedness of $G^{\rho_0}(\rho)$ from below.

Obviously, we have $G^{\rho_0}_{\min} \leqslant G^{\rho_0}(\rho_0) = F(\rho_0) < +\infty$. Thus for every $n \in \mathbb{N}$, we can choose $\rho_n \in \mathcal{K}$ such that

$$G^{\rho_0}(\rho_n) \leqslant G^{\rho_0}_{\min} + \frac{1}{n}. \tag{10}$$

Choosing $\varepsilon > 0$ such that $h\varepsilon < 1/8$, then from (9) we have

$$\sup_n V(\rho_n) < +\infty. \tag{11}$$

So the probability measure family $\{\rho_n(x)\mathrm{d}x\}$ is tight. On the other hand, by (8) and (10)

$$\sup_n S(\rho_n) < +\infty.$$

Thus, by Lemma 2.4, there is a subsequence ρ_{n_k} and $\rho_{\min} \in \mathcal{K}$ such that

$$\rho_{n_k} \to \rho_{\min} \text{ in } L^1(M, \mathrm{d}x) \text{ weakly}. \tag{12}$$

Now we show

$$G^{\rho_0}\left(\rho_{\min}\right) = G^{\rho_0}_{\min}. \tag{13}$$

Let $\pi^{n_k}_{\min} \in \mathcal{E}\left(\rho_{n_k}, \rho_0\right)$ be any inifimizer for the variational problem (4). By (11), we know that $\left\{\pi^{n_k}_{\min}, k \in \mathbb{N}\right\}$ is tight. Let π_0 be any accumulation of $\left\{\pi^{n_k}_{\min}, k \in \mathbb{N}\right\}$. Then, by (12) we know $\pi_0 \in \mathcal{E}\left(\rho_0, \rho_{\min}\right)$. So, for some subsequence still denoted by n_k

$$\begin{aligned} W_2\left(\rho_0, \rho_{\min}\right) &\leqslant \int\limits_{M\times M} \mathbf{d}^2(x,y)\pi_0(\mathrm{d}x,\mathrm{d}y) \\ &= \lim_{R\to\infty} \int\limits_{M\times M} \left(\mathbf{d}^2(x,y)\wedge R\right)\pi_0\left(\mathrm{d}x,\mathrm{d}y\right) \\ &= \lim_{R\to\infty}\lim_{k\to\infty} \int\limits_{M\times M} \left(\mathbf{d}^2(x,y)\wedge R\right)\pi^{n_k}_{\min}\left(\mathrm{d}x,\mathrm{d}y\right) \\ &\leqslant \liminf_{k\to\infty} W_2\left(\rho_0, \rho_{n_k}\right). \end{aligned} \tag{14}$$

Now let β_k converge to β in $L^1(M, \mathrm{d}x)$. Since $z\log z \geqslant -\mathrm{e}^{-1}$ and $z\log^+ z \geqslant 0$, by Fatou's lemma we have

$$\int\limits_{B_{x_0}(n)} \beta(x)\log\beta(x)\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{B_{x_0}(n)} \beta_k(x)\log\beta_k(x)\,\mathrm{d}x$$

$$\int\limits_{B^c_{x_0}(n)} \beta(x)\log^+\beta(x)\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{B^c_{x_0}(n)} \beta_k(x)\log^+\beta_k(x)\,\mathrm{d}x.$$

Noticing that $z \mapsto z\log z$ and $z \mapsto z\log^+ z$ are convex, by Hahn-Banach theorem, the above limits still hold if $\beta_k \to \beta$ weakly in $L^1(M, \mathrm{d}x)$. Thus

$$\int\limits_{B_{x_0}(n)} \rho_{\min}(x)\log\rho_{\min}(x)\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{B_{x_0}(n)} \rho_{n_k}(x)\log\rho_{n_k}(x)\,\mathrm{d}x \tag{15}$$

$$\int\limits_{B^c_{x_0}(n)} \rho_{\min}(x)\log^+\rho_{\min}(x)\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{B^c_{x_0}(n)} \rho_{n_k}(x)\log^+\rho_{n_k}(x)\,\mathrm{d}x. \tag{16}$$

By (11), similar to (29) of [15], we may prove that

$$\lim_{n\to\infty}\sup_k \int\limits_{B^c_{x_0}(n)} \rho_{n_k}(x)\log^-\rho_{n_k}(x)\,\mathrm{d}x = 0. \tag{17}$$

Combining (15), (16) and (17) gives

$$S\left(\rho_{\min}\right) \leqslant \liminf_{k\to\infty} S\left(\rho_{n_k}\right). \tag{18}$$

By Fatou's Lemma, we clearly have

$$E\left(\rho_{\min}\right) \leqslant \liminf_{k \to \infty} E\left(\rho_{n_k}\right). \tag{19}$$

Now equality (13) follows from (10), (14), (18) and (19).

The uniqueness follows from the strict convexity of $G^{\rho_0}(\rho)$, which follows from the linearity of $\rho \mapsto E(\rho)$, the strict convexity of $\rho \mapsto S(\rho)$ and the convexity of $\rho \mapsto W_2^2(\rho_0, \rho)$. □

The next two Lemmas play a crucial role in proving the convergence of approximation for Fokker–Planck equation on Riemannian manifold in the next section.

Lemma 3.2. *Continuing to the above lemma, for any smooth vector field X on M with compact support, we have*

$$\left| \frac{1}{2} \int\limits_{M\times M} \left[X\left(\mathbf{d}^2(x,\cdot)\right)\right](y) \cdot 1_{\{\mathbf{d}(x,y)<\delta\}} \cdot \pi_{\min}(\mathrm{d}x, \mathrm{d}y) \right.$$
$$\left. + h \int\limits_M \left[\langle \nabla f, X\rangle_x - (\mathrm{div}X)(x)\right] \rho_{\min}(x)\mathrm{d}x \right|$$
$$\leqslant \frac{\sup_{x\in M}|X|_x}{\delta} W_2^2\left(\rho_0, \rho_{\min}\right),$$

where $\pi_{\min} \in \mathcal{E}\left(\rho_0, \rho_{\min}\right)$ is any infimizer of $W_2\left(\rho_0, \rho_{\min}\right)$, δ is the injective radius of M.

Proof. Let $\phi_\tau : M \mapsto M$ be the gradient flow corresponding to X, i.e.:

$$\frac{d\phi_\tau(m)}{d\tau} = X\left(\phi_\tau(m)\right), \tau \in \mathbb{R}, \quad \phi_0(m) = m.$$

For any $\tau > 0$, let $\rho_\tau \in \mathcal{K}$ be the push forward of $\rho_{\min}$ under ϕ_τ, i.e.: for any $\xi \in C_0(M)$

$$\int\limits_M \rho_\tau(x)\xi(x)\mathrm{d}x = \int\limits_M \rho_{\min}(x)\xi\left(\phi_\tau(x)\right)\mathrm{d}x,$$

which is equivalent to

$$\left[\frac{\left(\phi_\tau^{-1}\right)_*(\mathrm{d}x)}{\mathrm{d}x}\right] \rho_\tau\left(\phi_\tau\right) = \rho_{\min}. \tag{20}$$

Then

$$\frac{1}{\tau}\left[G^{\rho_0}\left(\rho_\tau\right) - G^{\rho_0}\left(\rho_{\min}\right)\right] \geqslant 0. \tag{21}$$

Noting that

$$\frac{1}{\tau}\left[E(\rho_\tau)-E(\rho_{\min})\right]=\int\limits_M \frac{1}{\tau}\left[f(\phi_\tau(x))-f(x)\right]\rho_{\min}(x)\,\mathrm{d}x,$$

by dominated convergence theorem we have

$$\frac{d}{d\tau}E(\rho_\tau)\bigg|_{\tau=0}=\int\limits_M (Xf)(x)\rho_{\min}(x)\,\mathrm{d}x. \tag{22}$$

Next comes to $\frac{d}{d\tau}S(\rho_\tau)\big|_{\tau=0}$. We have by smooth approximation method and (20)

$$\begin{aligned}S(\rho_\tau)&=\int\limits_M \rho_\tau(x)\log(\rho_\tau(x))\mathrm{d}x\\&=\int\limits_M \rho_{\min}(x)\log(\rho_\tau(\phi_\tau(x)))\mathrm{d}x\\&=\int\limits_M \rho_{\min}(x)\log\left(\rho_{\min}(x)\left[\frac{\left(\phi_\tau^{-1}\right)_*(\mathrm{d}x)}{\mathrm{d}x}\right]^{-1}\right)\mathrm{d}x.\end{aligned}$$

Since

$$\frac{d}{d\tau}\left[\frac{\left(\phi_\tau^{-1}\right)_*(\mathrm{d}x)}{\mathrm{d}x}\right]\Bigg|_{\tau=0}=\mathrm{div}(X)(x),$$

we get

$$\begin{aligned}\frac{d}{d\tau}S(\rho_\tau)|_{\tau=0}&=-\lim_{\tau\to 0}\frac{1}{\tau}\int\limits_M \rho_{\min}(x)\log\left[\frac{\left(\phi_\tau^{-1}\right)_*(\mathrm{d}x)}{\mathrm{d}x}\right]\mathrm{d}x\\&=-\int\limits_M \rho_{\min}(x)\mathrm{div}(X)(x)\mathrm{d}x.\end{aligned} \tag{23}$$

Finally, define a probability measure π_τ on $M\times M$ by

$$\int\limits_{M\times M} g(x,y)\pi_\tau(\mathrm{d}x,\mathrm{d}y)=\int\limits_{M\times M} g\left(x,\phi_\tau(y)\right)\pi_{\min}\left(\mathrm{d}x,\mathrm{d}y\right),\quad g\in C_0(M\times M).$$

Then $\pi_\tau\in\mathcal{E}\left(\rho_0,\rho_\tau\right)$. By Definition 2.2 and (5), we have

$$\begin{aligned}&\frac{1}{2\tau}\left(W_2^2\left(\rho_0,\rho_\tau\right)-W_2^2\left(\rho_0,\rho_{\min}\right)\right)\\&\quad\leqslant\int\limits_{M\times M}\frac{1}{2\tau}\left(\mathbf{d}^2\left(x,\phi_\tau(y)\right)-\mathbf{d}^2(x,y)\right)\pi_{\min}\left(\mathrm{d}x,\mathrm{d}y\right),\end{aligned}$$

which implies that

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{2\tau}\left(W_2^2(\rho_0,\rho_\tau)-W_2^2(\rho_0,\rho_{\min})\right)\\
&\leqslant \lim_{\tau\to 0}\int\limits_{M\times M}\frac{1}{2\tau}\left(\mathbf{d}^2(x,\phi_\tau(y))-\mathbf{d}^2(x,y)\right)1_{\{\mathbf{d}(x,y)<\delta\}}\pi_{\min}(\mathrm{d}x,\mathrm{d}y)\\
&\quad+\lim_{\tau\to 0}\int\limits_{M\times M}\frac{1}{2\tau}\left(\mathbf{d}^2(x,\phi_\tau(y))-\mathbf{d}^2(x,y)\right)1_{\{\mathbf{d}(x,y)\geqslant\delta\}}\pi_{\min}(\mathrm{d}x,\mathrm{d}y)\\
&\leqslant \frac{1}{2}\int\limits_{M\times M}\left(X\left(\mathbf{d}^2(x,\cdot)\right)\right)(y)\cdot 1_{\{\mathbf{d}(x,y)<\delta\}}\cdot\pi_{\min}(\mathrm{d}x,\mathrm{d}y)\\
&\quad+\sup_{x\in M}|X|_x\int\limits_{M\times M}\mathbf{d}(x,y)\cdot 1_{\{\mathbf{d}(x,y)\geqslant\delta\}}\cdot\pi_{\min}(\mathrm{d}x,\mathrm{d}y)\\
&\leqslant \frac{1}{2}\int\limits_{M\times M}\left(X\left(\mathbf{d}^2(x,\cdot)\right)\right)(y)\cdot 1_{\{\mathbf{d}(x,y)<\delta\}}\cdot\pi_{\min}(\mathrm{d}x,\mathrm{d}y)\\
&\quad+\frac{\sup_{x\in M}|X|_x}{\delta}W_2^2(\rho_0,\rho_{\min}),
\end{aligned}\tag{24}$$

where we have used the fact $\lim_{\tau\to 0}\frac{\mathbf{d}(\phi_\tau(y),y)}{\tau}\leqslant |X|_y$ in the second step.

From (21), (22), (23) and (24), we deduce that

$$\begin{aligned}
&\frac{1}{2}\int\limits_{M\times M}\left[X\left(\mathbf{d}^2(x,\cdot)\right)\right](y)1_{\{\mathbf{d}(x,y)<\delta\}}\pi_{\min}(\mathrm{d}x,\mathrm{d}y)\\
&\qquad+h\int\limits_M\left[\langle\nabla f,X\rangle_x-(\mathrm{div}X)(x)\right]\rho_{\min}(x)\,\mathrm{d}x\\
&\qquad\geqslant-\frac{\sup_{x\in M}|X|_x}{\delta}W_2^2(\rho_0,\rho_{\min}).
\end{aligned}$$

By the symmetry, changing $X\to -X$ yields the desired inequality. □

We conclude this section by proving a simple Taylor estimate on Riemannian manifold.

Lemma 3.3. *Let ξ be a smooth function on M with compact support. Then for all $x,y\in M$ with $\mathbf{d}(x,y)$ less than the injective radius δ of M, we have*

$$\left|\xi(x)-\xi(y)-\langle\nabla\xi,\nabla\mathbf{d}^2(x,\cdot)\rangle_y/2\right|\leqslant\frac{1}{2}\sup_{x\in M}\|\mathrm{Hess}\xi(x)\|\cdot\mathbf{d}^2(x,y),$$

where Hessξ *is the usual Hessian matrix of* ξ.

Proof. For any fixed two points $x \neq y$ in M with $\mathbf{d}(x, y) < \delta$, let $\{\gamma(s), 0 \leqslant s \leqslant \mathbf{d}(x, y)\}$ be the uniquely shortest normal geodesic connecting y with x. By the usual Taylor's formula, we have

$$\begin{aligned} \xi(x) - \xi(y) &= \xi(\gamma(\mathbf{d}(x, y))) - \xi(\gamma(0)) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\xi(\gamma(s))\Big|_{s=0} \mathbf{d}(x, y) + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\xi(\gamma(s))\Big|_{s=s_0} \mathbf{d}^2(x, y) \\ &= \langle \nabla\xi, \dot{\gamma}(0)\rangle_y \cdot \mathbf{d}(x, y) + \frac{1}{2}(\mathrm{Hess}\xi)(\gamma(s_0))(\dot{\gamma}(s_0), \dot{\gamma}(s_0))\mathbf{d}^2(x, y) \end{aligned} \tag{25}$$

for some $s_0 \in [0, \mathbf{d}(x, y)]$.

We have the following claim: for any $X \in T_y(M)$

$$\langle X, \dot{\gamma}(0)\rangle_y \cdot \mathbf{d}(x, y) = \langle X, \nabla\mathbf{d}^2(x, \cdot)\rangle_y/2. \tag{26}$$

In fact, we only need to consider two cases by orthogonal projection

(i) $X = \dot{\gamma}(0)$
(ii) $\langle X, \dot{\gamma}(0)\rangle_y = 0$.

In the case of (i), (26) follows from $\langle \dot{\gamma}(0), \dot{\gamma}(0)\rangle_y = 1$ and

$$\mathbf{d}^2(x, \gamma(s)) = (\mathbf{d}(x, y) - s)^2.$$

In the case of (ii), by Gauss' Lemma (see [8, Theorem 1.8]), we know that X is tangent to the submanifold $\{z \in M : \mathbf{d}(x, z) = \mathbf{d}(x, y)\}$. On this submanifold $\mathbf{d}^2(x, \cdot)$ is constant, so

$$\langle X, \nabla\mathbf{d}^2(x, \cdot)\rangle_y = 0.$$

(26) is then proved.

Now the result follows from (25), (26) and $|\dot{\gamma}(0)|_y = 1$. □

4. Approximation for Fokker–Planck equation

Before proving our main results, we give a lemma about the L^q-estimates for the heat kernel on M and its gradient. In the following $\|\cdot\|_q$ will denote the norm in Banach space $L^q(M, \mathrm{d}x)$ for $q \geqslant 1$.

Lemma 4.1. *Assume that (I) and (II) hold, let $p_t(x, y)$ be the heat kernel on M associated with the Laplace–Beltrami operator Δ. Then for any $q \geqslant 1$, there is a constant $C := C(q, m, \delta, \kappa) > 0$ such that for any $x \in M$*

$$\begin{aligned} \|p_t(x, \cdot)\|_q &\leqslant C(t \wedge \delta)^{\frac{(1-q)m}{2q}}, \qquad t > 0, \\ \|\nabla_x p_t(x, \cdot)\|_q &\leqslant C(t \wedge \delta)^{\frac{m-(m+1)q}{2q}}, \quad t > 0, \end{aligned}$$

where m is the dimension of M, δ is the injective radius of M.

Proof. Under the conditions (I) and (II), by Li-Yau's upper bounds about the heat kernel and Croke's inequality (see [9, VIII 1.3 and V 2.15]), we have

$$p_t(x, y) \leqslant Ct^{-\frac{m}{2}}, \quad 0 < t \leqslant \delta.$$

For $t > \delta$, we have

$$p_t(x, y) = \int_M p_{t-\delta}(x, z) p_\delta(z, y) \mathrm{d}z \leqslant C\delta^{-\frac{m}{2}} \int_M p_{t-\delta}(x, z) dz = C\delta^{-\frac{m}{2}}.$$

So,

$$p_t(x, y) \leqslant C(t \wedge \delta)^{-\frac{m}{2}}, \quad 0 < t < +\infty. \tag{27}$$

Thus

$$\int_M |p_t(x, y)|^q \mathrm{d}y = \int_M |p_t(x, y)|^{q-1} p_t(x, y) \mathrm{d}y \leqslant C(t \wedge \delta)^{\frac{(1-q)m}{2}}.$$

Secondly, by Bismut's formula (see [2] or [21, p.159 Theorem 6.41]) we have

$$\nabla_x p_t(x, y) = t^{-1} p_t(x, y) \mathbb{E}\left[\int_0^t R_s dw(s) | \gamma(t) = y\right],$$

where $w(s)$ is the Brownian motion in $\mathbb{R}^m$ and $\gamma(t)$ is the Brownian motion on M,

$$\frac{\mathrm{d}R_s}{\mathrm{d}s} = -\frac{1}{2} R_s \cdot \mathrm{Ricci}_{\gamma(s)}, \quad R_0 = I.$$

Since Ricci curvature is bounded from below, we have

$$|R_s| \leqslant C.$$

So for $0 < t \leqslant \delta$, by (27) and Burkhölder's inequality we have

$$\begin{aligned}
\int_M |\nabla_x p_t(x, y)|^q \mathrm{d}y &= t^{-q} \int_M |p_t(x, y)|^q \left|\mathbb{E}\left[\int_0^t R_s dw(s) | \gamma(t) = y\right]\right|^q \mathrm{d}y \\
&\leqslant Ct^{-q-\frac{(q-1)m}{2}} \int_M \left|\mathbb{E}\left[\int_0^t R_s dw(s) | \gamma(t) = y\right]\right|^q p_t(x, y)\, \mathrm{d}y \\
&\leqslant Ct^{-q-\frac{(q-1)m}{2}} \mathbb{E}\left|\int_0^t R_s dw(s)\right|^q \\
&\leqslant Ct^{\frac{m-(m+1)q}{2}}.
\end{aligned}$$

For $t > \delta$, by Hölder's inequality and Fubini's theorem we have

$$\begin{aligned}\int_M |\nabla_x p_t(x,y)|^q \mathrm{d}y &= \int_M \left| \int_M \nabla_x p_\delta(x,z) p_{t-\delta}(z,y)\mathrm{d}z \right|_x^q \mathrm{d}y \\ &\leqslant \int_M \left(\int_M |\nabla_x p_\delta(x,z)|_x^q p_{t-\delta}(z,y)\mathrm{d}z \right) \mathrm{d}y \\ &= \int_M |\nabla_x p_\delta(x,z)|_x^q \mathrm{d}z \\ &\leqslant C\delta^{\frac{m-(m+1)q}{2}}.\end{aligned}$$

The proof is then complete. □

Let us consider the following Fokker–Planck equation

$$\frac{\partial \rho(t,x)}{\partial t} = \operatorname{div}\left(\nabla f(\cdot)\rho(t,\cdot)\right)(x) + \Delta\rho(t,x), t \geqslant 0, \quad \rho(0,x) = \rho_0(x), \tag{28}$$

where $\rho_0 \in \mathcal{K}$ and $f \in C^\infty(M)$ is positive.

To establish the existence of smooth solution, we need the following notion of weak solution.

Definition 4.2. *$\rho(t,x)$ is called a weak solution of equation (28) if and only if it holds that*

$$\begin{aligned}&\int_{\mathbb{R}_+\times M} \rho(t,x)\left(\partial_t g(t,x) - \langle \nabla f, \nabla g(t)\rangle_x + \Delta g(t,x)\right)\mathrm{d}x\mathrm{d}t \\ &\quad = \int_M \rho_0(x) g(0,x)\mathrm{d}x \end{aligned}\tag{29}$$

for all $g \in C_0^\infty(\mathbb{R}\times M)$.

The following simple lemma is needed. Since the proof is standard (cf. [16]), we omit the details.

Lemma 4.3. *Let $\rho(t,x)$ be a weak solution of equation (28). Then for any $\eta \in C_0^\infty(M)$,*

$$\begin{aligned}&\eta(y)\rho(t,y) - \int_M \eta(x)\rho_0(x)p_t(x,y)\,\mathrm{d}x \\ &\quad = \int_0^t \int_M \left(\rho(s,x)\left[\Delta\eta - \langle\nabla f, \nabla\eta\rangle_x\right]\right) p_{t-s}(x,y)\,\mathrm{d}x\mathrm{d}s \\ &\qquad + \int_0^t\int_M \rho(s,x)\langle 2\nabla\eta - \eta\nabla f, \nabla_x p_{t-s}(x,y)\rangle_x\,\mathrm{d}x\mathrm{d}s,\end{aligned}\tag{30}$$

for a.e. $(t,y) \in \mathbb{R}_+ \times M$,

where $p_t(x,y)$ is the heat kernel in Lemma 4.1.

We now define the approximation as follows: for $h > 0$, let $\rho_0^h := \rho_0$, for $\rho_{k-1}^h \in \mathcal{K}$ one defines $\rho_k^h := \rho_{\min}$ in Lemma 3.1, i.e.

$$G^{\rho_{k-1}^h}(\rho_k^h) = \inf_{\rho \in \mathcal{K}} G^{\rho_{k-1}^h}(\rho).$$

Put

$$\rho^h(t, x) := \sum_{k=0}^{\infty} 1_{[kh,(k+1)h)}(t)\rho_k^h(x).$$

We have the following existence result for equation (28).

Theorem 4.4. *Let $\rho_0 \in \mathcal{K}$ satisfy $F(\rho_0) < +\infty$. Then there is a solution $\rho(t, x) \in C^\infty(\mathbb{R}_+ \times M)$ satisfying equation (28) with the initial condition $\rho(t) \to \rho_0$ strongly in $L^1(M, \mathrm{d}x)$ as $t \downarrow 0$. Moreover,*

(i) $\rho^h(t) \to \rho(t)$ weakly in $L^1(M, \mathrm{d}x)$ for all $t > 0$,
(ii) there is a constant $C > 0$ such that for all $t \geqslant 0$

$$V(\rho(t)) \leqslant C(t^2 + 1), \quad E(\rho(t)) \leqslant C(t^2 + 1).$$

Proof. First of all, we establish some priori estimates. By the definition of ρ_k^h, we have

$$G^{\rho_{k-1}^h}\left(\rho_k^h\right) \leqslant G^{\rho_{k-1}^h}\left(\rho_{k-1}^h\right),$$

i.e.

$$\frac{1}{2}W_2^2\left(\rho_{k-1}^h, \rho_k^h\right) + hF\left(\rho_k^h\right) \leqslant hF\left(\rho_{k-1}^h\right).$$

Summing over k from 1 to N gives

$$\frac{1}{2}\sum_{k=1}^{N} W_2^2\left(\rho_{k-1}^h, \rho_k^h\right) + hF\left(\rho_N^h\right) \leqslant hF(\rho_0). \tag{31}$$

In particular,

$$F(\rho_N^h) \leqslant F(\rho_0). \tag{32}$$

Fix $T > 0$, let $h > 0$, $N \in \mathbb{N}$ satisfy $hN \leqslant T$. Then

$$\begin{aligned}
V\left(\rho_N^h\right) &\leqslant 2W_2^2\left(\rho_0, \rho_N^h\right) + 2V(\rho_0) \quad (\because (7)) \\
&\leqslant 2N\sum_{k=1}^{N} W_2^2\left(\rho_{k-1}^h, \rho_k^h\right) + 2V(\rho_0) \\
&\leqslant 2hN\left[F(\rho_0) - F\left(\rho_N^h\right)\right] + 2V(\rho_0) \quad (\because (31)) \\
&\leqslant 2T\left[F(\rho_0) + \varepsilon V\left(\rho_N^h\right) + c_1^2/\varepsilon + C\right] + 2V(\rho_0) \quad (\because (6))
\end{aligned}$$

for any $\varepsilon > 0$.

Choosing $\varepsilon = \frac{1}{4T}$, we get

$$V\left(\rho_N^h\right) \leqslant 4T\left[F(\rho_0) + 4c_1^2 T + C\right] + 4V(\rho_0) \leqslant C\left(T^2+1\right), \tag{33}$$

$$E\left(\rho_N^h\right) = F\left(\rho_N^h\right) - S\left(\rho_N^h\right) \leqslant F(\rho_0) + c_1^2 + C + V\left(\rho_N^h\right) \leqslant C(T^2+1), \tag{34}$$

where the constant C is independent of h, N and T.

Moreover, by the definition of F and (6), (32) we have

$$-C - V\left(\rho_N^h\right) \leqslant S\left(\rho_N^h\right) \leqslant F\left(\rho_N^h\right) \leqslant F\left(\rho_0\right) \tag{35}$$

and

$$\begin{aligned}\sum_{k=1}^{N} W_2^2\left(\rho_{k-1}^h, \rho_k^h\right) &\leqslant 2h\left(F(\rho_0) - F(\rho_N^h)\right) \\ &\leqslant 2h(F(\rho_0) + C + V(\rho_N^h)) \leqslant C_T h. \end{aligned} \tag{36}$$

Here the positive constant C_T is independent of h, N.

By (33) and (35), using Lemma 2.4 we conclude that there exists a measurable $\rho(t,x)$ such that, after extraction of a subsequence,

$$\rho^h \to \rho \text{ weakly in } L^1\left((0,T)\times M, \mathrm{d}t \times \mathrm{d}x\right) \text{ as } h \downarrow 0\,. \tag{37}$$

Now let us show that ρ is a weak solution of equation (28) in the sense of Definition 4.2. Let $\xi \in C_0^\infty(M)$ and $\pi_{\min}^k \in \mathcal{E}\left(\rho_{k-1}^h, \rho_k^h\right)$ be defined by (5). Then by Lemma 3.3 we have

$$\begin{aligned}
&\left|\int_M \left(\rho_k^h(x) - \rho_{k-1}^h(x)\right)\xi(x)\mathrm{d}x + \frac{1}{2}\int_{M\times M} \langle \nabla\xi, \nabla \mathbf{d}^2(x,\cdot)\rangle_y \cdot 1_{\{\mathbf{d}^2(x,y)<\delta\}} \cdot \pi_{\min}^k(\mathrm{d}x,\mathrm{d}y)\right| \\
&\quad = \left|\int_{M\times M}\left[\xi(y) - \xi(x) + \frac{1}{2}\langle\nabla\xi, \nabla\mathbf{d}^2(x,\cdot)\rangle_y \cdot 1_{\{\mathbf{d}^2(x,y)<\delta\}}\right]\pi_{\min}^k(\mathrm{d}x,\mathrm{d}y)\right| \\
&\quad \leqslant \left|\int_{M\times M}\left[\xi(y) - \xi(x) + \frac{1}{2}\langle\nabla\xi, \nabla\mathbf{d}^2(x,\cdot)\rangle_y\right]\cdot 1_{\{\mathbf{d}^2(x,y)<\delta\}} \cdot \pi_{\min}^k(\mathrm{d}x,\mathrm{d}y)\right| \\
&\qquad + \left|\int_{M\times M}\left[\xi(y) - \xi(x)\right] 1_{\{\mathbf{d}^2(x,y)\geqslant\delta\}} \cdot \pi_{\min}^k(\mathrm{d}x,\mathrm{d}y)\right| \\
&\quad \leqslant \left(\frac{1}{2}\sup_{x\in M}\|\mathrm{Hess}\xi(x)\| + \frac{2\sup_{x\in M}|\xi(x)|}{\delta^2}\right)\cdot \int_{M\times M} \mathbf{d}^2(x,y)\pi_{\min}^k\left(\mathrm{d}x,\mathrm{d}y\right) \\
&\quad = \left(\frac{1}{2}\sup_{x\in M}\|\mathrm{Hess}\xi(x)\| + \frac{2\sup_{x\in M}|\xi(x)|}{\delta^2}\right)\cdot W_2^2\left(\rho_{k-1}^h, \rho_k^h\right).
\end{aligned}$$

This together with Lemma 3.2 (therein taking $X = \nabla\xi$) yields

$$\left|\int_M \left[\left(\rho_k^h(x) - \rho_{k-1}^h(x)\right)\xi(x) - h\left(\langle\nabla f, \nabla\xi\rangle_x - \Delta\xi(x)\right)\rho_k^h(x)\right]\mathrm{d}x\right|$$

$$\leqslant \left(\frac{1}{2}\sup_{x\in M}\|\mathrm{Hess}\xi(x)\| + \frac{2\sup_{x\in M}|\xi(x)|}{\delta^2} + \frac{\sup_{x\in M}|\nabla\xi|_x}{\delta}\right)$$

$$\cdot W_2^2\left(\rho_{k-1}^h, \rho_k^h\right),$$

which together with (36) and (37) produces (29) by taking the limits. Thus we obtain the existence of weak solution.

By the famous hypoelliptic theorem of Hörmander [14, p.353, Theorem 22.2.1], a weak solution to a non-degenerate parabolic equation in an open set is necessarily smooth in that open set. Localization method gives the smoothness of $\rho(t, x)$ in $\mathbb{R}_+ \times M$.

We now prove (i). For any $T > 0$, by (36) there is a constant $C > 0$ such that for all N, N' and all $h \in [0, 1]$ with $Nh \leqslant T$ and $N'h \leqslant T$

$$W_2^2(\rho_N^h, \rho_{N'}^h) \leqslant C|Nh - N'h|.$$

Let $\xi \in C_0^\infty(M)$ and $\pi_{\min} \in \mathcal{E}\left(\rho_N^h, \rho_{N'}^h\right)$ be defined by (5), then

$$\begin{aligned}
\left|\int_M \xi(x)\left(\rho_N^h(x) - \rho_{N'}^h(x)\right)\mathrm{d}x\right| &= \left|\int_{M\times M} (\xi(x) - \xi(y))\,\pi_{\min}(\mathrm{d}x, \mathrm{d}y)\right| \\
&\leqslant \sup_{x\in M}|\nabla\xi(x)|\left|\int_{M\times M}\mathbf{d}(x, y)\pi_{\min}(\mathrm{d}x, \mathrm{d}y)\right| \\
&\leqslant \sup_{x\in M}|\nabla\xi(x)|\left|\int_{M\times M}\mathbf{d}^2(x, y)\pi_{\min}(\mathrm{d}x, \mathrm{d}y)\right|^{1/2} \\
&= \sup_{x\in M}|\nabla\xi(x)|W_2\left(\rho_N^h, \rho_{N'}^h\right) \\
&\leqslant C\sup_{x\in M}|\nabla\xi(x)|\cdot|Nh - N'h|^{1/2}. \qquad (38)
\end{aligned}$$

Hence, for any $\varepsilon > 0$ we have

$$\begin{aligned}
&\left|\int_M \xi(x)\left(\rho^h(t,x)-\rho(t,x)\right)\mathrm{d}x\right| \\
&\quad\leqslant \left|\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_M \xi(x)\left(\rho^h(t,x)-\rho^h(s,x)\right)\mathrm{d}x\mathrm{d}s\right| \\
&\qquad+\left|\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_M \xi(x)\rho^h(s,x)\mathrm{d}x\mathrm{d}s-\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_M \xi(x)\rho(s,x)\,\mathrm{d}x\mathrm{d}s\right| \\
&\qquad+\left|\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_M \xi(x)\rho(s,x)\mathrm{d}x\mathrm{d}s-\int_M \xi(x)\rho(t,x)\,\mathrm{d}x\right|.
\end{aligned}$$

By (38), the first term is less than

$$C\sup_{x\in M}|\nabla\xi(x)|\cdot|\varepsilon+h|^{1/2}.$$

By (37), the second term tends to zero as $h\downarrow 0$. By the smoothness of ρ, we get the convergence of the third term to zero as $\varepsilon\downarrow 0$. So we have

$$\int_M \xi(x)\rho^h(t,x)\mathrm{d}x\to\int_M \xi(x)\rho(t,x))\mathrm{d}x,\ \text{ for all } \xi\in C_0^\infty(M). \tag{39}$$

By (33), we have

$$\sup_h V(\rho^h(t))<\infty.$$

Consequently, (39) holds for any $\xi\in L^\infty(M)$, therefore (i) holds.

By (i) and (33), (34), a simple analysis shows that

$$\rho(t)\in\mathcal{K} \text{ for all } t>0$$

and ρ has the property (ii).

Lastly, we show that $\rho(t)\to\rho_0$ strongly in $L^1(M,\mathrm{d}x)$ as $t\downarrow 0$. By (30) and Lemma 4.1, we have for any $\eta\in C_0^\infty(M)$

$$
\begin{aligned}
&\int_M \left| \eta(y)\rho(t,y) - \int_M \eta(x)\rho_0(x)p_t(x,y)\mathrm{d}x \right| \mathrm{d}y \\
&\quad \leqslant \int_0^t \int_M \left|\rho(s,x)\left[\Delta\eta - \langle\nabla f, \nabla\eta\rangle_x\right]\right| \mathrm{d}x\mathrm{d}s \\
&\qquad + \int_0^t \int_M |\rho(s)[2\nabla\eta - \eta\nabla f]| \cdot \|\nabla_x p_{t-s}(x,\cdot)\|_{L^1}\mathrm{d}s \\
&\quad \leqslant \int_0^t \left\|\rho(s)\left[\Delta\eta - \langle\nabla f, \nabla\eta\rangle_x\right]\right\|_1 \mathrm{d}s \\
&\qquad + C\int_0^t (t-s)^{-\frac{1}{2}} \|\rho(s)|2\nabla\eta - \eta\nabla f|\|_1 \mathrm{d}s,
\end{aligned}
$$

which tends to zero as $t \downarrow 0$. On the other hand,

$$
\int_M \eta(x)\rho_0(x)p_t(x,y)\mathrm{d}x \to \eta(y)\rho_0(y) \text{ in } L^1(M, \mathrm{d}x) \text{ as } t \downarrow 0.
$$

Therefore,

$$
\eta(y)\rho(t,y) \to \eta(y)\rho_0(y) \text{ in } L^1(M, \mathrm{d}x) \text{ as } t \downarrow 0,
$$

which leads to the desired convergence by (ii). □

We have the following uniqueness result.

Theorem 4.5. *Assume that the potential f satisfies*

$$
|\nabla f(x)|_x \leqslant C_f \left(f(x) + \mathbf{d}^2(x, x_0) + 1 \right) \tag{40}
$$

for some $C_f > 0$. Then the solution constructed in Theorem 4.4 is unique. Moreover, let $m := \dim(M)$, then for any $1 < q < \frac{m}{m-1}$, there is a constant $C := C(C_f, m, q)$ such that

$$
\|\rho(t)\|_q \leqslant \begin{cases} C\left(t^{\frac{(1-q)m}{2q}} + t^{\frac{m-(m-1)q}{2q}}\right), & 0 < t \leqslant \delta, \\ C(1+t^3), & \delta < t. \end{cases}
$$

Proof. Let us first prove the uniqueness (see [15]). Let ρ_1 and ρ_2 be two solutions of equation (28). Then the difference $\rho = \rho_1 - \rho_2$ satisfies

$$
\frac{\partial\rho(t,x)}{\partial t} = \mathrm{div}\left(\rho(t,\cdot)\nabla f + \nabla\rho(t,\cdot)\right), \quad \rho(0,x) = 0. \tag{41}
$$

For $\varepsilon > 0$, set $\psi_\varepsilon(z) := \left(z^2 + \varepsilon^2\right)^{1/2}$. Then

$$\begin{aligned}\partial_t\left(\psi_\varepsilon(\rho)\right) - \operatorname{div}\left[\psi_\varepsilon(\rho)\nabla f + \nabla\left(\psi_\varepsilon(\rho)\right)\right] &= -\psi_\varepsilon''(\rho)|\nabla\rho|^2 \\ &\quad + \left(\psi_\varepsilon'(\rho)\rho - \psi_\varepsilon(\rho)\right)\Delta f \\ &\leqslant \left(\psi_\varepsilon'(\rho)\rho - \psi_\varepsilon(\rho)\right)\Delta f.\end{aligned}$$

We multiply the above inequality by a cutoff function $\eta \in C_0^\infty(M)$ and integrate over $[0, t] \times M$ to obtain

$$\begin{aligned}&\int_M \psi_\varepsilon(\rho(t,x))\eta(x)\mathrm{d}x + \int_0^t\int_M \psi_\varepsilon(\rho(s,x))\left[\langle\nabla f, \nabla\eta\rangle_x - \Delta\eta(x)\right]\mathrm{d}x\mathrm{d}s \\ &\quad\leqslant \int_0^t\int_M \left(\psi_\varepsilon'(\rho(s,x))\rho(s,x) - \psi_\varepsilon\left(\rho(s,x)\right)\right)\Delta f(x)\cdot\eta(x)\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ gives

$$\int_M |\rho(t,x)|\eta(x)\mathrm{d}x + \int_0^t\int_M |\rho(s,x)|\cdot\left[\langle\nabla f, \nabla\eta\rangle_x - \Delta\eta(x)\right]\mathrm{d}x\mathrm{d}s \leqslant 0. \quad (42)$$

Let η_n be a sequence of cutoff functions with the following properties (cf. [1])

$$\eta_n(x) = 1, \quad x \in B_{x_0}(n), \quad \eta_n(x) = 0, \quad x \in B_{x_0}^c(n+1), \quad |\nabla\eta_n| + |\Delta\eta_n| \leqslant C$$

for some $C > 0$ independent of n.

Now by the assumption and (ii) in Theorem 4.4, letting $\eta = \eta_n$ in (42) and $n \to \infty$ yields

$$\int_M |\rho(t,x)|\mathrm{d}x = 0,$$

which produces the uniqueness.

Since ρ is smooth, (30) holds for all t, y. By (40) and (ii) in Theorem 4.4, letting $\eta = \eta_n$ in (30) again and $n \to \infty$ yields

$$\rho(t,y) = \int_M \rho_0(x)p_t(x,y)\mathrm{d}x + \int_0^t\int_M \rho(s,x)\langle\nabla f, \nabla_x p_{t-s}(x,y)\rangle_x\,\mathrm{d}x\mathrm{d}s.$$

Taking the L^q-norm, and by Lemma 4.1 we obtain

$$
\begin{aligned}
\|\rho(t)\|_q &\leqslant \int\limits_M \rho_0(x)\|p_t(x,\cdot)\|_q \mathrm{d}x + \int\limits_0^t \int\limits_M \rho(s,x)|\nabla f|_x \|\nabla_x p_{t-s}(x,\cdot)\|_q \,\mathrm{d}x\mathrm{d}s \\
&\leqslant C(t\wedge\delta)^{\frac{(1-q)m}{2q}} + C\int\limits_0^t ((t-s)\wedge\delta)^{\frac{m-(m+1)q}{2q}} \\
&\quad \times \left(\int\limits_M \rho(s)(f+\mathbf{d}^2(\cdot,x_0)+1)\mathrm{d}x\right)\mathrm{d}s \\
&\leqslant C(t\wedge\delta)^{\frac{(1-q)m}{2q}} + C\int\limits_0^t ((t-s)\wedge\delta)^{\frac{m-(m+1)q}{2q}} (s^2+1)\,\mathrm{d}s \\
&\leqslant C\left[(t\wedge\delta)^{\frac{(1-q)m}{2q}} + (t^2+1)\left((t\wedge\delta)^{\frac{m-(m-1)q}{2q}} + (t-t\wedge\delta)\delta^{\frac{m-(m+1)q}{2q}}\right)\right].
\end{aligned}
$$

The proof is thus finished. □

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