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Retrieving random media

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Abstract. Benjamini asked whether the scenery reconstruction methods of Matzinger (see e.g. [21], [22], [20]) can be done in polynomial time. In this article, we give the following answer for a 2-color scenery and simple random walk with holding: We prove that a piece of the scenery of length of the order 3^n around the origin can be reconstructed – up to a reflection and a small translation – with high probability from the first $2 \cdot 3^{10\alpha n}$ observations with a constant $\alpha > 0$ independent of n. Thus, the number of observations needed is polynomial in the length of the piece of scenery which we reconstruct. The probability that the reconstruction fails tends to 0 as $n \to \infty$.

In contrast to [21], [22], and [20], the proofs in this article are all constructive. Our reconstruction algorithm is an algorithm in the sense of computer science. This is the first article which shows that the scenery reconstruction is also possible in the 2-color case *with holding*. The case with holding is much more difficult than [22] and requires completely different methods.

1. Introduction and Result

A *scenery* is a coloring of $\mathbb{Z}$ with finitely many colors. We call two sceneries ξ and ξ' *equivalent*, $\xi \approx \xi'$, if $\xi = \xi' \circ T$ where T is a translation, a reflection, or the composition of both. Let $S := (S_k)_{k \in \mathbb{N}_0}$ be a recurrent random walk on $\mathbb{Z}$. Observing the scenery along the random walk path, we obtain the color record $\chi := (\chi_k := \xi(S_k))_{k \in \mathbb{N}_0}$. The *scenery reconstruction problem* asks the following question: Given the color record χ, can we reconstruct the scenery ξ up to equivalence?

Early questions about random sceneries were raised by Benjamini and Kesten and, independently, by Keane and den Hollander. Their investigations were motivated among others by work of Kalikow [11] on the T, T^{-1} transformation. More recently, den Hollander and Steif [3] and Hoffman [7] generalized Kalikow's results. Early work on random sceneries include articles of Benjamini and Kesten [1], den Hollander [4], Howard ([8], [9], [10]), Keane and den Hollander [12], Kesten [13], and Lindenstrauss [17]. More recent contributions are due to Burdzy [2], Heicklen, Hoffman, and Rudolph [6], Levin, Pemantle and Peres [15], Levin and Peres [16].

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We refer the reader to [14] and the introductions of [25] and [24] for more details. Various contributions to the subject of scenery reconstruction have been made by Matzinger ([21],[22]), Löwe and Matzinger ([18], [19]), Löwe, Matzinger, and Merkl [20], Matzinger and Rolles [24]. In these papers, the scenery is taken random, independent of the random walk, and it is shown that for almost all realizations of the random walk path, almost all sceneries can be reconstructed up to equivalence.

The scenery reconstruction algorithms in [21], [22], [18], [19], [20], and [24] do not work in polynomial time. Benjamini asked whether some of these reconstructions can be done in polynomial time. In this article, we give the following answer to Benjamini's question: Let $\xi := (\xi_k)_{k\in\mathbb{Z}}$ with ξ_k i.i.d. uniform on $\{0, 1\}$, and let $S = (S_k)_{k\in\mathbb{N}_0}$ be a simple random walk with holding on $\mathbb{Z}$, independent of ξ. We prove that in order to reconstruct – up to a reflection and a small translation – with high probability a piece of scenery of length of the order 3^n around the origin, we need only the observations up to time $p(3^n)$ with a polynomial p, independent of n.

In order to reconstruct the whole scenery, we need infinitely many observations because the scenery is infinite. In finite time, we can never reconstruct with probability 1 a piece of scenery of length ≥ 2. As a matter of fact, the random walk stays with positive probability at the origin. Hence, we mean by reconstruction in polynomial time that there exist algorithms $\mathcal{A}_n$, $n \geq 1$, with the following properties: $\mathcal{A}_n$ obtains as input finitely many observations, namely $\chi|[0, 2\cdot 3^{10\alpha n}[$ with a constant $\alpha > 0$ and produces an output of length of the order 3^n. The probability that the reconstruction succeeds, in the sense that the output is – up to a reflection and a small translation – a piece of the scenery around the origin, tends to 1 as $n \to \infty$. The number of observations needed is polynomial in the length of the reconstructed piece of scenery. Since the scenery is assumed to be i.i.d., with probability 1 every finite piece of scenery occurs somewhere in the scenery. Thus it is crucial to reconstruct something close to the origin.

Formally, our result can be described as follows: Let $\mathcal{C} := \{0, 1\}$ denote the set of colors. For two pieces of scenery ψ and ψ' (not necessarily of the same length), we write $\psi \preceq \psi'$ if ψ is up to a possible reflection contained in ψ'. We prove:

Theorem 1.1. *There exist constants $\alpha, c_3, c_4, c_5 > 0$ and maps $\mathcal{A}_n : \mathcal{C}^{2\cdot 3^{10\alpha n}} \to \mathcal{C}^{[-3\cdot 3^n, 3\cdot 3^n]}$, $n \geq c_3$, which are measurable with respect to the canonical σ-algebras, such that for all $n \geq c_3$, the event*

$$E_n := \left\{ \xi|[-3^n, 3^n] \preceq \mathcal{A}_n(\chi|[0, 2\cdot 3^{10\alpha n}[) \preceq \xi|[-4\cdot 3^n, 4\cdot 3^n] \right\}$$

satisfies $P\left([E_n]^c\right) \leq c_4 \exp\left(-c_5 n^{0.2}\right)$.

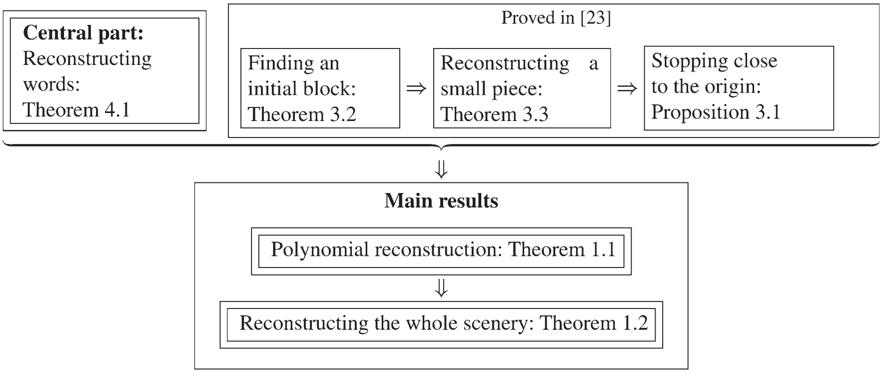
As a consequence of Theorem 1.1 the whole scenery can be reconstructed almost surely:

Theorem 1.2. *There exists a map $\mathcal{A} : \mathcal{C}^{\mathbb{N}_0} \longrightarrow \mathcal{C}^{\mathbb{Z}}$, which is measurable with respect to the canonical σ-algebras, such that $P\left(\mathcal{A}(\chi) \approx \xi\right) = 1$.*

The present article is the first article which solves the scenery reconstruction problem in the case of two colors and simple random walk with holding. We call this case the *semi-combinatorial case*. On the piece of scenery 01, the random walker can produce every pattern by jumping back and forth or staying. Thus, at many places in the scenery, the random walk can produce every possible pattern in the observations. This makes the semi-combinatorial case much more difficult than the *combinatorial case*, where with high probability words of length $c_1 n$ (with a constant $c_1 > 0$) are characteristic for certain parts of the scenery. Examples of the combinatorial case are the following articles: [22], where an i.i.d. 2-color scenery is observed along a simple random walk, [18], where a 2-dimensional scenery with many colors is observed along a simple random walk, and [20], where a scenery with sufficiently many colors is observed along a random walk on $\mathbb{Z}$ with bounded jumps. In the semi-combinatorial case, it is much more difficult than in the combinatorial case to reconstruct small pieces of the scenery. The methods used below are completely different from the techniques developed in earlier articles.

The remainder of the article is organized as follows: Section 2 collects some notation. In Section 3, we show how Theorem 1.2 follows from Theorem 1.1. Since the definition of the maps $\mathcal{A}_n$ which fulfill the claim of Theorem 1.1 is quite involved, the construction is split into several steps. In Section 3, we state the results needed for the construction of the $\mathcal{A}_n$. The crucial step consists in finding small words in the scenery; this is done in Section 4. The second important step is the construction of a partial reconstruction algorithm BigAlg^n which is treated in Section 5. In addition, we need a small piece of the scenery to get the reconstruction started and also sequences of stopping times indicating when the random walker is close to the origin. These results are proved in [23]. At the end of Section 3, we show how the results of Sections 4 and 5 together with the results from [23] imply Theorem 1.2.

The following diagram is a guide to the proofs of Theorems 1.2 and 1.1:



2. Notation

In this section, we collect some notations and conventions.

Numbers, sets, and functions: We denote by $\mathbb{N} := \{1, 2, 3, \dots\}$ the set of natural numbers and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer $\leq x$.

We write $x \wedge y$ for the minimum of $x, y \in \mathbb{R}$. For a vector $y = (y_k)_{k\in[1,m]} \in \mathbb{R}^m$ we define the l^1-norm $\|y\|_1 := \sum_{k=1}^m |y_k|$ and the l^2-norm $\|y\|_2 := \left(\sum_{k=1}^m [y_k]^2\right)^{1/2}$. The cardinality of a set D is denoted by $|D|$. We write $f|D$ for the restriction of a function f to a set D. An *integer interval* is a set of the form $I \cap \mathbb{Z}$ with an interval $I \subseteq \mathbb{R}$. In this article, intervals are always taken over the integers, e.g. $[a, b] = \{z \in \mathbb{Z} : a \le z \le b\}$.

Admissible paths: Let $I = [i_1, i_2]$ be an integer interval. We call $R \in \mathbb{Z}^I$ an *admissible piece of path* if $R_{i+1} - R_i \in \{-1, 0, 1\}$ for all $i \in [i_1, i_2 - 1]$. We call R_{i_1} the starting point, R_{i_2} the endpoint, and $|I|$ the length of R.

Measures: We define δ_x to be the Dirac measure in x. We denote the image of a measure Q under a map F by QF^{-1}.

Sceneries: We denote by $\mathcal{C} := \{0, 1\}$ the set of colors. A *scenery* is an element of $\mathcal{C}^{\mathbb{Z}}$. Let $I \subseteq \mathbb{Z}$ be an integer interval. An element of $\mathcal{C}^I$ is a *piece of scenery* or a *word*. If $\psi \in \mathcal{C}^I$, we call $|I|$ the *length* of ψ and denote it by $|\psi|$. We write $(1)_I$ for the piece of scenery in $\mathcal{C}^I$ which is identically equal to 1.

Blocks: Let $a, b \in I$ with $a < b$ and $|a - b| \ge 2$. We define $\psi \in \mathcal{C}^{[a,b]}$ to be a *block* if $\psi_a = \psi_b$ and $\psi_c \ne \psi_a$ for all $c \in]a, b[$. ψ_c is the *color* of the block. We call a the *left endpoint*, b the *right endpoint*, and $|\psi| := b - a - 1$ the *blocklength* of ψ. For instance, 01110 is a block of length 3. We set $\partial\psi := \{a, b\}$.

Let $\chi|[t_1, t_2]$ and $\xi|[a, b]$ be blocks. We say that $\chi|[t_1, t_2]$ *is generated by the random walk S on the block* $\xi|[a, b]$ if $\{S_{t_1}, S_{t_2}\} \subseteq \{a, b\}$ and $S_t \in]a, b[$ for all $t \in]t_1, t_2[$.

Equivalence of sceneries: Let $\psi \in \mathcal{C}^I$ and $\psi' \in \mathcal{C}^{I'}$ be two pieces of scenery. We say that ψ and ψ' are *equivalent* and write $\psi \approx \psi'$ iff I and I' have the same length and there exists $a \in \mathbb{Z}$ and $b \in \{-1, 1\}$ such that for all $k \in I$ we have that $a + bk \in I'$ and $\psi_k = \psi'_{a+bk}$. We call ψ and ψ' *strongly equivalent* and write $\psi \equiv \psi'$ if $I' = a + I$ for some $a \in \mathbb{Z}$ and $\psi_k = \psi'_{a+k}$ for all $k \in I$. We say ψ *occurs in* ψ' and write $\psi \sqsubseteq \psi'$ if $\psi \equiv \psi'|J$ for some $J \subseteq I'$. We write $\psi \preceq \psi'$ if $\psi \approx \psi'|J$ for some $J \subseteq I'$. If the subset J is unique, we write $\psi \preceq_1 \psi'$.

Random walks and random sceneries: Let $\Omega_2 \subseteq \mathbb{Z}^{\mathbb{N}_0}$ denote the set of admissible paths. Let $p, q > 0$ satisfy $2p + q = 1$. We denote by Q_x the distribution on Ω_2 of a random walk $(S_k)_{k\in\mathbb{N}_0}$ starting at x with i.i.d. increments distributed according to $p\delta_{-1} + q\delta_0 + p\delta_1$, i.e. S is a simple random walk with holding, and satisfies

$$p = P(S_{k+1} - S_k = 1) = P(S_{k+1} - S_k = -1),$$
$$q = P(S_{k+1} - S_k = 0)$$

for all $k \ge 0$. The scenery $\xi := (\xi_k)_{k\in\mathbb{Z}}$ is i.i.d. with $P(\xi_k = 0) = P(\xi_k = 1) = 1/2$. We assume that ξ and S are independent and realized as canonical projections on $\Omega := \mathcal{C}^{\mathbb{Z}} \times \Omega_2$ with the product σ-algebra generated by the canonical projections and probability measures $P_x := (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes\mathbb{Z}} \otimes Q_x$, $x \in \mathbb{Z}$. We abbreviate $P := P_0$. We call $\chi := (\chi_k := \xi(S_k))_{k\in\mathbb{N}_0}$ the *scenery observed along the random walk path*; sometimes we write $\xi \circ S$ instead of χ.

For a fixed scenery $\xi \in \mathcal{C}^{\mathbb{Z}}$ we set $P_{x,\xi} := \delta_\xi \otimes Q_x$, $P_\xi := P_{0,\xi}$. Thus $P_{x,\xi}$ is the canonical version of the conditional probability $P_x(\cdot|\xi)$, the distribution P conditioned on the random walk to start in x and the scenery ξ. We never work with a different version of the conditional probability $P_x(\cdot|\xi)$.

Filtration: We define $\mathcal{G} := (\mathcal{G}_n)_{n\in\mathbb{N}_0}$ with $\mathcal{G}_n := \sigma(\chi_k; k \in [0, n])$ to be the natural filtration of the observations over Ω.

Shifts: We define the shift $\theta : \mathcal{C}^{\mathbb{N}_0} \to \mathcal{C}^{\mathbb{N}_0}$, $\eta \mapsto \eta(\cdot + 1)$. We introduce the shift $\Theta : \Omega \to \Omega$, $(\xi, S) \mapsto (\xi(S_1 + \cdot), S(1 + \cdot) - S_1)$. For a set $A \subseteq \Omega$ and a random time $T \geq 0$ we set $\Theta^{-T}(A) := \{\omega : \Theta^{T(\omega)}(\omega) \in A\}$.

Constants: We denote constants by c_i, $i \geq 1$; they keep their meaning throughout the whole article. Constants c_1, c_2, c_6, c_7, and α play a special role. They are chosen as follows:

1. $c_2 > 21$,
2. $c_1 \in 4\mathbb{N}$ with $c_1 > \max\{153, 4c_2\}$,
3. $c_6 > (c_1 + 4)\ln 3$,
4. $c_7 > \max\{0, 2\ln 3 - 2c_1 \ln p + 2c_6 + 2c_1 \ln[\max_{i\in[1,5]} \|x_i^*\|_2]\}$ with x_i^* as in Definition 4.4,
5. $\alpha \in \mathbb{N}$ with $\alpha > 1 + 17c_1 + [24c_7 - 3c_1 \ln p]/\ln 3$.

3. Overview of the reconstruction

In this section, we show how Theorem 1.1 is proved using the results from Sections 4 and 5 and [23]. First we show how Theorem 1.1 implies Theorem 1.2.

Proof of Theorem 1.2. Let $\mathcal{A}_n : \mathcal{C}^{2\cdot 3^{10\alpha n}} \to \mathcal{C}^{[-3\cdot 3^n, 3\cdot 3^n]}$ be as in Theorem 1.1. We say that a sequence of pieces of sceneries $(\zeta_n \in \mathcal{C}^{I_n})_{n\geq c_3}$ converges pointwise to a scenery ζ if for all $z \in \mathbb{Z}$ there exists n_z such that $z \in I_n$ and $\zeta_n(z) = \zeta(z)$ for all $n \geq n_z$. We define

$$\mathcal{A}(\chi) := \begin{cases} \lim_{n\to\infty} \mathcal{A}_n(\chi \,|\, [0, 2\cdot 3^{10\alpha n}[) & \text{if this limit exists pointwise,} \\ (1)_{\mathbb{Z}} & \text{else.} \end{cases}$$

As a limit of measurable maps, $\mathcal{A}$ is measurable. Theorem 1.1 implies $\sum_{n=c_3}^{\infty} P([E_n]^c) \leq \sum_{n=c_3}^{\infty} c_4 \exp\left(-c_5 n^{0.2}\right) < \infty$. Hence by the Borel-Cantelli lemma, $P(\cup_{m=c_3}^{\infty} \cap_{n=m}^{\infty} E_n) = 1$. In order to prove $P(\mathcal{A}(\chi) \approx \xi) = 1$, we use the same arguments as in the proof of Theorem 3.7 of [20]. (One shows $P\left(\cup_{m=c_3}^{\infty}\cap_{n=m}^{\infty} \left\{\xi | \left[-3^n, 3^n\right] \preceq_1 \xi | \left[-4\cdot 3^{n+1}, 4\cdot 3^{n+1}\right]\right\}\right) = 1$, which implies that the reconstructed pieces of scenery $\mathcal{A}_n(\chi \,|\, [0, 2\cdot 3^{10\alpha n}[)$ fit uniquely together for all n sufficiently large and yield the scenery ξ.) □

Hence, it suffices to define maps $\mathcal{A}_n$ which fulfill the claim of Theorem 1.1. The main ingredient in the construction of $\mathcal{A}_n$ is a map BigAlg^n which obtains as input data the observations collected by the random walk up to time $2 \cdot 3^{10\alpha n}$ (as $\mathcal{A}_n$ does). In addition, BigAlg^n needs a sequence of stopping times $\tau := (\tau_k)_{k \in [1, 3^{\alpha n}]}$ and a small piece of scenery ψ. BigAlg^n produces as output a piece of scenery $w \in \mathcal{C}^{[-3 \cdot 3^n, 3 \cdot 3^n]}$ which satisfies $\xi|[-3^n, 3^n] \preceq w \preceq \xi|[-4 \cdot 3^n, 4 \cdot 3^n]$ with high probability.

The reason why we need the stopping times $(\tau_k)_{k \in [1, 3^{\alpha n}]}$ is the following: In order to be able to reconstruct the scenery in the interval $[-3^n, 3^n]$, the random walk must visit this part of the scenery many times. Otherwise, we will not have enough information for the reconstruction. Since $2 \cdot 3^{10\alpha n}$ is considerably larger than 3^n, there is a good chance, the random walk visits the interval $[-3^n, 3^n]$ often up to time $2 \cdot 3^{10\alpha n}$. However, up to time $2 \cdot 3^{10\alpha n}$, only a small fraction of the time is spent in $[-3^n, 3^n]$. The rest of the time, when the random walk is outside of $[-3^n, 3^n]$, the observations do not provide us with useful information. Hence we need to be able to determine which parts of the observations are generated by the random walk on $\xi|[-3^n, 3^n]$. Formally, the task of the stopping times $(\tau_k)_{k \in [1, 3^{\alpha n}]}$ is specified by the event $E^{n,\tau}_{\mathrm{stop}}$ defined as follows.

Definition 3.1. *For $n \in \mathbb{N}$ and a sequence $\tau = (\tau_k)_{k \geq 1}$ of $\mathcal{G}$-adapted stopping times, we define the event*

$$E^{n,\tau}_{\mathrm{stop}} := \bigcap_{k=1}^{3^{\alpha n}} \left\{ \tau_k < 3^{10\alpha n}, |S_{\tau_k}| \leq 3^n, \tau_j + 2 \cdot 3^{3n} \leq \tau_k \text{ for } j < k \right\}.$$

Besides stopping times, BigAlg^n obtains as input a piece of scenery ψ of length $\geq 2n^2 + 1$. Compared to the output of BigAlg^n, which has length of the order 3^n, ψ is very small. If $\psi \preceq \xi|[-3^n, 3^n]$, i.e. if we have with ψ some information about the underlying scenery, and if the event $E^{n,\tau}_{\mathrm{stop}}$ holds, then with high probability, BigAlg^n reconstructs a piece of scenery around the origin. More formally:

Theorem 3.1. *There exist $c_8, c_9, c_{10} > 0$ and a sequence of measurable maps*

$$\mathrm{BigAlg}^n : [0, 3^{10\alpha n}]^{[1, 3^{\alpha n}]} \times \mathcal{C}^{2 \cdot 3^{10\alpha n}} \times \bigcup_{k \geq n^2} \mathcal{C}^{[-k,k]} \to \mathcal{C}^{[-3 \cdot 3^n, 3 \cdot 3^n]}, n \in \mathbb{N},$$

such that for all $n \geq c_8$ and every sequence $\tau = (\tau_k)_{k \in [1, 3^{\alpha n}]}$ of $\mathcal{G}$-adapted stopping times

$$P\left(E^{n,\tau}_{\mathrm{stop}} \setminus E^{n,\tau}_{\mathrm{recon\ Big}}\right) \leq c_9 e^{-c_{10} n}, \text{ where}$$

$$E^{n,\tau}_{\mathrm{recon\ Big}} := \left\{ \begin{array}{l} \text{For all } \psi \in \mathcal{C}^{[-k,k]} \text{ with } k \geq n^2 \text{ and } \psi \preceq \xi|[-3^n, 3^n] \text{ we have} \\ \xi|[-3^n, 3^n] \preceq \mathrm{BigAlg}^n(\tau, \chi|\left[0, 2 \cdot 3^{10\alpha n}\right[, \psi) \preceq \xi|[-4 \cdot 3^n, 4 \cdot 3^n]. \end{array} \right\}.$$

Let us explain how BigAlg^n reconstructs a piece of the scenery. Using the stopping times τ together with the observations from its input, BigAlg^n reconstructs with high probability all words of length $c_1 n/2$ in $\xi|[-5 \cdot 3^n, 5 \cdot 3^n]$; here c_1 is a (large) constant as described in Section 2. This is the crucial step in the definition of BigAlg^n. The words cannot be extracted from χ in a simple manner. Instead we need to look at certain empirical distributions of words which then allow us to obtain information about the true distribution and finally about the words themselves. Theorem 4.1 below provides a criterion to find words in the scenery. Reconstructing the words is a hard problem under our assumptions on random walk and scenery. In fact, this part of the reconstruction is much more difficult in the present setting than in previously solved scenery reconstruction problems.

Since with high probability, each word of length $c_1 n/4$ occurs at most once in $\xi|[-5 \cdot 3^n, 5 \cdot 3^n]$, it is possible to reconstruct a piece of scenery containing $\xi|[-3^n, 3^n]$ from the collection of words of length $c_1 n/2$. The assemblage will be done as follows: We start with the small piece of scenery ψ from the input of BigAlg^n. Then we look for a word of length $c_1 n/2$ which overlaps with ψ by at least $c_1 n/4$ letters and extends ψ by at least one letter. We continue the procedure with the extended ψ.

Once we have defined BigAlg^n, we can define the map $\mathcal{A}_n$ in terms of BigAlg^n with suitable stopping times τ and a piece of scenery ψ as input. The initial piece ψ will be a piece of scenery around a long block of ξ close to the origin. Since the ideas for finding words and defining BigAlg^n are central for this paper, we decided to concentrate on these parts. The proofs concerning the stopping times and the initial piece can be found in [23].

Let $\mathsf{block}^{n+} := \xi|[b_l^{n+}, b_r^{n+}]$ designate the leftmost block of ξ of length $\geq n$ with $b_l^{n+} \geq 0$, and let $\mathsf{block}^{n-} := \xi|[b_l^{n-}, b_r^{n-}]$ denote the rightmost block of ξ of length $\geq n$ with $b_r^{n-} \leq 0$. Finally, let $\mathsf{block}^n \in \{\mathsf{block}^{n+}, \mathsf{block}^{n-}\}$ denote the block which is visited first by S.

The map $\mathcal{A}_n$ will reconstruct a piece of scenery around block^n. Thus, first we need to locate block^n. With high probability, in a large neighborhood of block^n there is no other large block in the scenery. Hence, up to a certain time horizon, long blocks in the observations χ indicate that the random walker generates the observations on block^n. The following theorem states that with high probability, there is a stopping time that stops the random walk in the set $\partial\mathsf{block}^n$.

Theorem 3.2. *([23], Theorem 3.1) For all $n \in \mathbb{N}$, there exists a $\mathcal{G}$-adapted stopping time $\nu^n(0)$, measurable with respect to $\sigma(\chi_k; k \in [0, 3^{10\alpha n}[)$, such that the probability of the event*

$$E^n_{\nu^n(0)\text{ ok}} := \{S_{\nu^n(0)} \in \partial\mathsf{block}^n\} \cap \{\nu^n(0) \leq 2 \cdot 3^{3n}\} \cap \{\partial\mathsf{block}^n \subseteq [-3^n/3, 3^n/3]\}$$

satisfies the following bound: There exist constants c_{11}, c_{12}, c_{13} such that for all $n \geq c_{11}$

$$P\left(\left[E^n_{\nu^n(0)\text{ ok}}\right]^c\right) \leq c_{12} e^{-c_{13} n^{0.3}}.$$

Next, we reconstruct a piece of scenery around block^n. We show that there is a map $\mathrm{SmallAlg}^n$ with the following properties: Given $3^{\lfloor n^{0.3}\rfloor}$ observations collected by the random walker starting in the set $\partial\mathsf{block}^n$, a piece of scenery of length of the order $3^{\lfloor n^{0.2}\rfloor}$ around block^n can be reconstructed with high probability. For our purposes, it is convenient to state this differently: For ξ in a set of probability close to 1, conditioned on the scenery ξ, $\mathrm{SmallAlg}^n$ reconstructs with high probability a piece of scenery around block^n.

Theorem 3.3. *([23], Theorem 3.2) There exist constants $c_{14}, c_{16} > 0$ and a sequence*

$$\mathrm{SmallAlg}^n : \mathcal{C}^{[0,3^{\lfloor n^{0.3}\rfloor}[} \to \mathcal{C}^{[-3\cdot 3^{\lfloor n^{0.2}\rfloor}, 3\cdot 3^{\lfloor n^{0.2}\rfloor}]}, n \geq c_{14},$$

of measurable maps such that the following holds: We set $H_i^n := \min\{k \geq 0 : S_k = b_i^n\}$ for $i \in \{l, r\}$. If we define

$$E^{n,T}_{\text{recon Small}} := \left\{\mathrm{SmallAlg}^n\left(\chi|\left[T, T+3^{\lfloor n^{0.3}\rfloor}\right[\right) \preceq \xi|[b_l^n - 3\cdot 3^{\lfloor n^{0.2}\rfloor}, b_r^n + 3\cdot 3^{\lfloor n^{0.2}\rfloor}]\right\}$$

and

$$\Xi^n := \left\{\xi \in \mathcal{C}^{\mathbb{Z}} : P_\xi\left(\left[E^{n,T}_{\text{recon Small}}\right]^c\right) \leq e^{-c_{16}n^{0.2}} \text{ for all } T \in \{H_l^n, H_r^n\}\right\},$$

then $P(\xi \notin \Xi^n) \leq e^{-c_{16}n^{0.2}}$ for all $n \geq c_{14}$.

In fact, in [23], we heavily use the ideas from the construction of BigAlg^n to define $\mathrm{SmallAlg}^n$. The piece of scenery reconstructed by $\mathrm{SmallAlg}^n$ is much smaller than the piece of scenery which $\mathcal{A}_n$ is supposed to reconstruct. The map $\mathrm{SmallAlg}^n$ is used to define stopping times $\nu^n(k)$, $k \geq 1$, which indicate when the random walk is in the interval $[-3^n, 3^n]$. Recall that $\mathcal{A}_n$ should reconstruct a piece of scenery of length of the order 3^n which is contained in $\xi|[-4\cdot 3^n, 4\cdot 3^n]$. Hence, it will be useful to have stopping times which stop the random walk in the interval $[-3^n, 3^n]$. We define

$$\psi_n := \mathrm{SmallAlg}^n(\chi|[\nu^n(0), \nu^n(0)+3^{\lfloor n^{0.3}\rfloor}[), \tag{3.1}$$

$$\mathbb{T}^n := \left\{\begin{array}{l} t \in [\nu^n(0), 3^{10\alpha n} - 3^{\lfloor n^{0.3}\rfloor}[: \exists w \in \mathcal{C}^{[-3^{\lfloor n^{0.2}\rfloor}, 3^{\lfloor n^{0.2}\rfloor}]} \text{ such that } w \preceq \psi_n \\ \text{and } w \preceq \mathrm{SmallAlg}^n(\chi|[t, t+3^{\lfloor n^{0.3}\rfloor}[) \end{array}\right\}. \tag{3.2}$$

Let $\tilde{\nu}^n(1) < \tilde{\nu}^n(2) < \cdots$ denote the points in $\mathbb{T}^n$ in increasing order. We define $\nu^n := (\nu^n(k))_{k\in[1,3^{\alpha n}]}$ by

$$\nu^n(k) := \begin{cases} \tilde{\nu}^n(2\cdot 3^{3n}k) + 3^{\lfloor n^{0.3}\rfloor} & \text{if } 2\cdot 3^{3n}k \leq |\mathbb{T}^n| \\ 3^{10\alpha n} & \text{else.} \end{cases}$$

Note that $\nu^n(k)$ depends only on $\chi|[0, 3^{10\alpha n}[$ and is a $\mathcal{G}$-adapted stopping time. In fact, in order to determine whether $t \in \mathbb{T}_n$, we need to look at $\chi|[t, t+3^{\lfloor n^{0.3}\rfloor}[$, but $\nu^n(k)$ is never defined to be t, but only $t + 3^{\lfloor n^{0.3}\rfloor}$.

The idea behind the definition of the $\nu^n(k)$'s is the following: With high probability, $\nu^n(0)$ stops the random walk in the set $\partial\mathsf{block}^n$ and ψ_n is up to a possible reflection a piece of scenery of length $6\cdot 3^{\lfloor n^{0.2}\rfloor}+1$ around block^n. The set $\mathbb{T}_n$ consists of times $t \geq \nu^n(0)$ such that SmallAlg^n applied to the observations starting at time t produces an output which agrees on a large subpiece, namely a piece of length $2\cdot 3^{\lfloor n^{0.2}\rfloor}+1$, with ψ_n. With high probability, ψ_n is typical for the scenery around block^n, and hence the random walker is in the interval $[-3^n, 3^n]$ at time t. (With high probability, block^n can be found in the piece of scenery $\xi|[-3^n/3, 3^n/3]$.) For the construction below, it will be essential that we have sufficiently many $\nu^n(k)$'s which are far enough apart from each other and all bounded by $3^{10\alpha n}$. Formally, the task of the stopping times $\nu^n(k)$ is specified by the event $E^{n,\nu^n}_{\text{stop}}$, see Definition 3.1.

Recall the definition of Ξ^n from Theorem 3.3. If the event $E^n_{\nu^n(0)\text{ ok}} \cap E^{n,\nu^n(0)}_{\text{recon Small}}$ holds and $\xi \in \Xi_n$, then with high probability the stopping times ν^n stop the random walk correctly, in the sense that the event $E^{n,\nu^n}_{\text{stop}}$ holds. This is made precise by the following proposition:

Proposition 3.1. *([23], Proposition 3.3) There exist constants c_{19}, c_{20}, c_{21} such that for all $n \geq c_{19}$*

$$P\left(\left[E^n_{\nu^n(0)\text{ ok}} \cap E^{n,\nu^n(0)}_{\text{recon Small}} \cap \{\xi \in \Xi^n\}\right] \setminus E^{n,\nu^n}_{\text{stop}}\right) \leq c_{20}e^{-c_{21}n^{0.3}}.$$

Now, we have achieved the following: Using SmallAlg^n, we can reconstruct a piece of scenery ψ_n around block^n. With high probability, $\psi_n \preceq \xi|[-3^n, 3^n]$. Furthermore, the stopping times $\nu^n(k)$ stop the random walk with high probability in the interval $[-3^n, 3^n]$. Hence, with this input data, the algorithm BigAlg^n reconstructs with high probability a piece of scenery of length of the order 3^n around the origin.

Let $n \geq c_{14}$ with c_{14} as in Theorem 3.3, and let ψ_n be as in (3.1). We define

$$\mathcal{A}_n(\chi|[0, 2\cdot 3^{10\alpha n}[) := \text{BigAlg}^n(\nu^n, \chi|[0, 2\cdot 3^{10\alpha n}[, \psi_n).$$

Proof of Theorem 1.1. We show that the maps $\mathcal{A}_n$ defined above fulfill the claim of Theorem 1.1. We have

$$\begin{aligned} P\left([E_n]^c\right) \leq{}& P\left(\left[E^{n,\nu^n}_{\text{stop}} \cap E^n_{\nu^n(0)\text{ ok}} \cap E^{n,\nu^n(0)}_{\text{recon Small}}\right] \setminus E_n\right) + P([E^n_{\nu^n(0)\text{ ok}}]^c) \\ &+P\left(\left[E^n_{\nu^n(0)\text{ ok}} \cap E^{n,\nu^n(0)}_{\text{recon Small}} \cap \{\xi \in \Xi^n\}\right] \setminus E^{n,\nu^n}_{\text{stop}}\right) \\ &+P\left([E^n_{\nu^n(0)\text{ ok}} \cap \{\xi \in \Xi^n\}] \setminus E^{n,\nu^n(0)}_{\text{recon Small}}\right) + P(\xi \notin \Xi^n). \end{aligned} \tag{3.3}$$

If $E^{n,\nu^n(0)}_{\text{recon Small}}$ holds, then $\psi_n \preceq \xi|[b^n_l - 3\cdot 3^{\lfloor n^{0.2}\rfloor}, b^n_r + 3\cdot 3^{\lfloor n^{0.2}\rfloor}]$. If in addition $E^n_{\nu^n(0)\text{ ok}}$ holds, then $\partial\mathsf{block}^n \subseteq [-3^n/3, 3^n/3]$, and consequently, $\psi_n \preceq \xi|[-3^n, 3^n]$ for all n sufficiently large. Hence, using Theorem 3.1,

$$P\left(\left[E^{n,\nu^n}_{\text{stop}} \cap E^n_{\nu^n(0)\text{ ok}} \cap E^{n,\nu^n(0)}_{\text{recon Small}}\right] \setminus E_n\right) \leq P\left(E^{n,\nu^n}_{\text{stop}} \setminus E^{n,\nu^n}_{\text{recon Big}}\right) \leq c_9 e^{-c_{10}n}$$

for all n sufficiently large. By Theorem 3.2, $P([E^n_{\nu^n(0)\text{ ok}}]^c) \leq c_{12}e^{-c_{13}n^{0.3}}$ for all $n \geq c_{11}$. Proposition 3.1 states that

$$P\left(\left[E^n_{\nu^n(0)\text{ ok}} \cap E^{n,\nu^n(0)}_{\text{recon Small}} \cap \{\xi \in \Xi^n\}\right] \setminus E^{n,\nu^n}_{\text{stop}}\right) \leq c_{20}e^{-c_{21}n^{0.3}}$$

for all $n \geq c_{19}$. Next, we estimate the second to last term in (3.3):

$$P\big([E^n_{\nu^n(0)\text{ ok}} \cap \{\xi \in \Xi^n\}] \setminus E^{n,\nu^n(0)}_{\text{recon Small}}\big) = \int_{\{\xi \in \Xi^n\}} P_\xi\big(E^n_{\nu^n(0)\text{ ok}} \setminus E^{n,\nu^n(0)}_{\text{recon Small}}\big) dP$$

$$\leq \int_{\{\xi \in \Xi^n\}} \Big[P_\xi\big(\{S_{\nu^n(0)} = b^n_l\} \setminus E^{n,\nu^n(0)}_{\text{recon Small}}\big) + P_\xi\big(\{S_{\nu^n(0)} = b^n_r\} \setminus E^{n,\nu^n(0)}_{\text{recon Small}}\big)\Big]\, dP.$$

Using the strong Markov property of the random walk and Theorem 3.3, we conclude that the last quantity is $\leq 2e^{-c_{16}n^{0.2}}$. Finally, by Theorem 3.3, $P\,(\xi \notin \Xi^n) \leq \exp[-c_{16}n^{0.2}]$ for all $n \geq c_{14}$. Combining all these estimates with (3.3), the claim follows. □

4. How we find words in the observations

In this section, we prove a sufficient condition for a word to be contained in the scenery close to the origin. First, we explain why reconstructing words is so difficult in the present setup.

Special 4-color sceneries. Assume for a moment that the scenery ξ, instead of being a 2-color scenery, is a 4-color scenery, i.e. $\xi \in \{0, 1, 2, 3\}^{\mathbb{Z}}$. Let us assume furthermore, that for two integers y, z we have $\xi_y = 2$ and $\xi_z = 3$, but $\xi_x \notin \{2, 3\}$ for all $x \in \mathbb{Z} \setminus \{y, z\}$. Then we could reconstruct the portion of the scenery ξ lying between y and z: As a matter of fact, since the random walk S is recurrent, it traverses a.s. at least once (and hence infinitely often) the shortest path from y to z. Since we are given infinitely many observations χ, the distance between y and z is the shortest time lapse that a 3 ever appears in the observations χ after a 2. When the random walk goes in the shortest possible way from y to z, it traverses the straight path from y to z. During that time, the random walk reveals in the observations the portion of ξ lying between y and z.

Simple random walk without holding. A related, but much more involved idea can still be used for 2-color sceneries. Let us next explain why the 2-color scenery reconstruction problem is much more difficult for simple random walk *with holding* than for simple random walk. So assume for the moment that S is a simple random walk, i.e. in each step S jumps one to the right or one to the left with probability 1/2. In this case, we can use instead of the extra colors 2 and 3 in the previous paragraph binary words of the form 001100 and 110011: It is easy to verify that the only possibility for the word 001100 to appear in the observations, is when 001100 occurs in the scenery (i.e. $\xi|[x, x+5] = 001100$ for some x) and the random walk traverses the straight path between x and $x + 5$. The same is true for the word 110011.

If 001100 occurred in precisely one place y of the scenery and 110011 occurred in precisely one place $z \neq y$ of the scenery, then we could reconstruct up to a reflection the piece of scenery occurring between 001100 and 110011. We would just look in the observations where the word 110011 occurs in shortest time after the word 001100. In between, we see a copy of the piece of the scenery ξ comprised between 001100 and 110011.

Of course, in an i.i.d. scenery, the word 001100 occurs a.s. infinitely often. Nevertheless, a modification of this idea was for instance used by Löwe, Matzinger, and Merkl in [20] for sufficiently many colors. They used, that with high probability certain words occur only in certain areas of the scenery, which allowed them to reconstruct the words in between.

The present problem. For a random walk with holding, the idea of patterns in the observations which tell us when we are back at the same spot like for example 001100 does not work at all. The reason is that if $\xi_z = 1$ and $\xi_{z+1} = 0$, then the random walk with holding can produce any pattern by just moving back and forth between z and $z+1$ and holding. Thus, all patterns can be produced in most places in ξ and are thus not specific for some places in the scenery. However, in the case of a random walk with holding, the same idea of getting in shortest time from a point y to a point z can be applied to the *distributions* of the observations.

A simplified version. Fix a point $x \in \mathbb{Z}$. First, assume that we have stopping times τ_k which *all* stop the random walk at the point x. The *empirical* distribution of the $3c_1n$ observations after these stopping times τ_k, i.e. the distribution

$$3^{-\alpha n} \sum_{k\in[1,3^{\alpha n}]} \delta_{\chi|[\tau_k,\tau_k+3c_1n[} \tag{4.1}$$

is an approximation of the *real* distribution

$$P_{x,\xi}(\chi|[0,3c_1n[\in \cdot) \tag{4.2}$$

of $\chi|[0, 3c_1n[$ conditioned on the scenery to be ξ and the random walk to start in x. Thus, if all the stopping times τ_k satisfy $S_{\tau_k} \in [-3^n, 3^n]$ (which is the case if the event $E^{n,\tau}_{\mathrm{stop}}$ holds), then the empirical distribution in (4.1) is an approximation of a mixture of the distributions in (4.2) where x ranges over $[-3^n, 3^n]$. In other words, it is an approximation of

$$\sum_{x\in[-3^n,3^n]} a(x)P_{x,\xi}(\chi|[0,3c_1n[\in \cdot), \tag{4.3}$$

where $a(x)$ designates the proportion of stopping times $\tau_k, k\in[1,3^{\alpha n}]$, with $S_{\tau_k}=x$.

Convention: Sometimes it will be convenient to identify a measure λ which is supported on a countable ordered set $\{s_i\}_i$ with the vector $(\lambda(\{s_i\}))_i$. In particular, we do this with $P_{x,\xi}(\chi|[0,3c_1n[\in \cdot)$.

Let $y, z \in [-3^n, 3^n]$ such that $z - y = c_1n - 1$. How can one reconstruct the word $\xi|[y, z]$ of length c_1n from the measure in (4.3)? First, we rewrite the measure in (4.3) by conditioning on the positions of the random walk at times c_1n and $2c_1n$:

$$\begin{aligned}
&\sum_{x\in[-3^n,3^n]} a(x)P_{x,\xi}(\chi|[0,3c_1n[\in \cdot)\\
&= \sum_{x,y',z'} a(x)P_{x,\xi}(\chi|[0,c_1n[\in \cdot|S_{c_1n}=y')\\
&\quad \otimes P_{y',\xi}(\chi|[0,c_1n[\in \cdot|S_{c_1n}=z')\otimes P_{z',\xi}(\chi|[0,c_1n[\in \cdot),
\end{aligned} \tag{4.4}$$

where the sum is taken over all x, y', z' such that $x \in [-3^n, 3^n]$, $|x-y'| \le c_1 n$ and $|y'-z'| \le c_1 n$. Here $\otimes$ denotes the product of two measures. If one identifies the distributions with vectors, the product measure corresponds to the tensor product of the associated vectors.

We can reconstruct $\xi|[y,z]$ from (4.3) if there is a linear component in the distribution $P_{x,\xi}(\chi|[0, 3c_1 n[\in \cdot)$ occuring only "to the left of y" and a linear component occuring only "to the right of z". By this we mean that there exist two linear functionals g^l and g^r such that

$$g^l(P_{x,\xi}(\chi|[0, c_1 n[\in \cdot|S_{c_1 n} = y')) = 0 \tag{4.5}$$

for all $x \in [-3^n, 3^n]$ and $y' > y$ and

$$g^r(P_{z',\xi}(\chi|[0, c_1 n[\in \cdot)) = 0 \tag{4.6}$$

for all $z' \le z$. When we apply the linear functional $g^l \otimes \mathrm{id} \otimes \mathrm{g^r}$ to the second sum in (4.4), all the terms disappear except for the terms with $y' = y$ and $z' = z+1$. This is the only possibility for a random walk starting at y' to be at z' at time $c_1 n$ because the interval $[y,z]$ has length $c_1 n$ and we required $|y'-z| \le c_1 n$. But when we have $y' = y$ and $z' = z+1$, then conditional on $S_0 = y'$ and $S_{c_1 n} = z'$, the random walk performs during its first $c_1 n$ steps a straight walk from the point y to z. Thus, in that case $P_{y',\xi}(\chi|[0, c_1 n[\in \cdot|S_{c_1 n} = z')$ is the atomic distribution where the atom is at the point $\xi|[y,z]$. This allows us to reconstruct $\xi|[y,z]$ provided some adequate functionals g^l and g^r exist.

The real approach. It turns out that instead of working with the distributions (4.1) and (4.2), we need to deal with slightly different distributions as will be explained now: Let $\chi^n := \chi|[0, 2\cdot 3^{10\alpha n}[$, and let $\tau = (\tau_k)_{k\in[1,3^{\alpha n}]}$ be a sequence of $\mathcal{G}$-adapted stopping times. The reader should think of stopping times such that the event $E^{n,\tau}_{\text{stop}}$ holds. Instead of taking the $3c_1 n$ observations after each stopping time, we let the random walker run freely 3^{2n} steps after each stopping time τ_k. This way, it has a chance to reach all points in $[-5\cdot 3^n, 5\cdot 3^n]$ which is important if we want to reconstruct all words from $\xi|[-5\cdot 3^n, 5\cdot 3^n]$ of length $c_1 n/2$. Then, we record the lengths of the following $c_1 n$ blocks in the observations, truncated at 5. (So if a block has length > 5, we record a 5.) Then, we record the following word from the observations of length $c_1 n/2$, extended up to the beginning of the next block. Finally, we collect the lengths of the following $c_1 n$ blocks in the observations, truncated at length 5. This way, we obtain for every stopping time τ_k a quantity $O^n(\chi|[\tau_k, \tau_k + 3^{3n}])$ which is a triple of the form (O_1^n, O_2^n, O_3^n) with $O_1^n, O_3^n \in [1,5]^{c_1 n}$ sequences of truncated block lengths and O_2^n a word over the alphabet $\{0,1\}$. Next, we look at the corresponding empirical distribution, namely

$$\hat{\mu}^{n,\tau}_{\chi^n} := 3^{-\alpha n} \sum_{k\in[1,3^{\alpha n}]} \delta_{\mathrm{O}^n(\theta^{\tau_k}\chi^n)}$$

instead of (4.1).

Let $S^n := S|[0, 2\cdot 3^{10\alpha n}[$. The real distribution $\mu^{n,\tau}_{\xi,S^n}$ of our collected information is a mixture of the distributions of $\mathrm{O}^n(\chi^n)$ under $P_{x,\xi}$, where the term

$P_{x,\xi}\left[\mathrm{O}^n(\chi^n)\right]^{-1}$, the distribution of $\mathrm{O}^n(\chi^n)$ under $P_{x,\xi}$, gets as weight the proportion of stopping times τ_k stopping the random walk at x (see Definition 4.3). Of course, $\mu^{n,\tau}_{\xi,S^n}$ cannot be obtained from χ and τ only. Thus, we need to work with the empirical distribution $\hat{\mu}^{n,\tau}_{\chi^n}$. Let $\varepsilon^{n,\tau}_{\xi,S^n}$ denote the difference between the empirical and the real distribution; so $\varepsilon^{n,\tau}_{\xi,S^n}$ is a signed measure. It will be shown in Lemma 5.8 that the probability that the stopping times τ_k stop correctly (i.e. $E^{n,\tau}_{\mathrm{stop}}$ holds), but $\varepsilon^{n,\tau}_{\xi,S^n}$ has a norm which is not exponentially small in n has an exponentially small probability. In this sense, $\hat{\mu}^{n,\tau}_{\chi^n}$ approximates $\mu^{n,\tau}_{\xi,S^n}$.

How can we reconstruct words from the scenery using the empirical distribution $\hat{\mu}^{n,\tau}_{\chi^n}$? A sufficient criterion for $w \preceq \xi[-3^{3n}, 3^{3n}]$ is given in Theorem 4.1: one needs the existence of certain linear functionals g^w_1 and g^w_3 on $(\mathbb{R}^5)^{\otimes c_1 n}$; these play the role of g^l and g^r in the above. Since the criterion is formulated in terms of the empirical distribution $\hat{\mu}^{n,\tau}_{\chi^n}$ instead of the real distribution $\mu^{n,\tau}_{\xi,S^n}$, one has the condition that $g^w_1 \otimes 1 \otimes g^w_3$ applied to the empirical distribution is small instead of being 0 as in (4.5) and (4.6).

Theorem 4.1 is the key tool for the reconstruction algorithm BigAlg^n described in Section 5. Given the observations collected by the random walk, one first considers a set Words^n of words obtained by checking conditions (5.2)–(5.5); these conditions are motivated by (4.10)–(4.13). A word belongs to Words^n if there exist functionals g^w_1 and g^w_3 with certain properties. Suitable functionals are defined in Definition 5.4. With high probability, they fulfill their task in the sense that the event $B^n_{\mathrm{functional}}$ holds (see Definition 5.9, Lemma 5.3, and the estimates in Section 5.4). With high probability, the words from Words^n can be assembled as in a puzzle game to find the scenery.

4.1. A sufficient criterion

First, we give a precise definition of O^n: Let $\eta \in \bigcup_{k\geq 3^{3n}} \mathcal{C}^{[0,k[}$; the reader should think of η as a piece of observations. We consider $\tilde{\mathrm{O}}^n_1\mathrm{O}^n_2\tilde{\mathrm{O}}^n_3$ where $\tilde{\mathrm{O}}^n_1$ consists of the first $c_1 n$ blocks of η after time 3^{2n}, O^n_2 equals the following $c_1 n/2$ observations in η extended until the next block starts, and $\tilde{\mathrm{O}}^n_3$ consists of the following $c_1 n$ blocks of η. Next, for $j = 1, 3$, we replace $\tilde{\mathrm{O}}^n_j$ by the sequence $\mathrm{O}^n_j \in \{1, 2, 3, 4, 5\}^{c_1 n}$ where the ith component equals the minimum of 5 and the length of the ith block of $\tilde{\mathrm{O}}^n_j$. Formally:

Definition 4.1. *Let $\eta \in \bigcup_{k\geq 3^{3n}} \mathcal{C}^{[0,k[}$. We abbreviate $\eta^n := \eta|\left[3^{2n}, 3^{3n}\right[$. We denote by $B_k(\eta)$ the kth block of η if η possesses at least k blocks; otherwise we set $B_k(\eta) := 101 \in \mathcal{C}^{[3^{3n},3^{3n}+3[}$. We denote by $o^n_l(\eta)$ the right end of $B_{c_1 n}(\eta^n)$. Furthermore we denote by $\tilde{o}^n_r(\eta)$ the left end of the first block of $\eta^n|\left[o^n_l(\eta)+c_1 n/2-2, 3^{3n}\right[$ and set $o^n_r(\eta) := \tilde{o}^n_r(\eta) + 1$. If $\eta^n|\left[o^n_l(\eta) + c_1 n/2 - 2, 3^{3n}\right[$ does not contain a block, then we set $o^n_r(\eta) := o^n_l(\eta)$. We define $\mathrm{O}^n := (\mathrm{O}^n_1, \mathrm{O}^n_2, \mathrm{O}^n_3)$ by*

$$\begin{aligned}
\mathrm{O}^n_1(\eta) &:= (|B_k(\eta^n)| \wedge 5)_{k\in[1,c_1 n]},\\
\mathrm{O}^n_2(\eta) &:= \eta|\left[o^n_l(\eta), o^n_r(\eta)\right],\\
\mathrm{O}^n_3(\eta) &:= (|B_k(\theta^{\tilde{o}^n_r(\eta)}(\eta))| \wedge 5)_{k\in[1,c_1 n]}.
\end{aligned}$$

The letter "O" should remind the reader of "observation". By definition, $|\mathrm{O}_2^n(\eta)| \geq c_1 n/2$ unless $o_r^n(\eta) = o_l^n(\eta)$. The following picture illustrates our definitions for $c_1 n = 6$:

$$\eta = \underbrace{1110\ldots01110010}_{\eta|[0,3^{2n}[}\,0\,\underbrace{011101000000011000\boxed{1}}_{\tilde{\mathrm{O}}_1^n(\eta)}000\underbrace{0\boxed{1}11001011110001}_{\tilde{\mathrm{O}}_3^n(\eta)}\,00111110\ldots$$

$\eta_{o_l^n}$ and $\eta_{o_r^n}$ are marked with boxes. In this example, we have $\mathrm{O}_1^n(\eta) = (3, 1, 1, 5, 2, 3)$, $\mathrm{O}_2^n(\eta) = 100001$, $\mathrm{O}_3^n(\eta) = (3, 2, 1, 1, 4, 3)$.

In the following, let $\tau = (\tau_k)_{k\in[1,3^{\alpha n}]}$ be a sequence of $\mathcal{G}$-adapted stopping times.

Definition 4.2. *For $\eta \in \mathcal{C}^{[0,2\cdot 3^{10\alpha n}[}$, we define the empirical distribution of* O^n *observed after each time τ_k, $k \in [1, 3^{\alpha n}]$:*

$$\hat{\mu}_\eta^{n,\tau} := 3^{-\alpha n} \sum_{k\in[1,3^{\alpha n}]} \delta_{\mathrm{O}^n(\theta^{\tau_k}\eta)}.$$

For $\eta \in \mathcal{C}^{\mathbb{N}_0}$, we set $\hat{\mu}_\eta^{n,\tau} := \hat{\mu}^{n,\tau}_{\eta|[0,2\cdot 3^{10\alpha n}[}$.

Definition 4.3. *For an admissible path $R \in \mathbb{Z}^{[0,2\cdot 3^{10\alpha n}[}$, let $a_R^{n,\tau}(x)$ be the proportion of $k \in \left[1, 3^{\alpha n}\right]$ with $R_{\tau_k} = x$. We define*

$$\mu_{\xi,R}^{n,\tau} := \sum_{x\in[-3^n,3^n]} a_R^{n,\tau}(x) P_{x,\xi}\left[\mathrm{O}^n(\chi)\right]^{-1},$$

$$\varepsilon_{\xi,R}^{n,\tau} := \hat{\mu}_{\xi\circ R}^{n,\tau} - \mu_{\xi,R}^{n,\tau}.$$

For an admissible path $R \in \mathbb{Z}^{\mathbb{N}_0}$, we set $\mu_{\xi,R}^{n,\tau} := \mu^{n,\tau}_{\xi,R|[0,2\cdot 3^{10\alpha n}[}$.

By definition, $\mu_{\xi,R}^{n,\tau}$ and $\hat{\mu}_\eta^{n,\tau}$ are measures on the set obs $:= [1,5]^{c_1 n} \times \mathrm{obs}_2 \times [1,5]^{c_1 n}$ with $\mathrm{obs}_2 := \{w \in \mathcal{C}^k : k \geq c_1 n/2, w_{k-1} \neq w_k, w_j = w_{k-1}$ for all $j \in [c_1 n/2 - 1, k-1]\}$. We denote by

$$\Pi_2 : \mathrm{obs} \to \mathrm{obs}_2, \quad \Pi_{1,3} : \mathrm{obs} \to [1,5]^{c_1 n} \times [1,5]^{c_1 n}$$

the canonical projections. Furthermore, we introduce the event that an observation $\mathrm{O} \in \mathrm{obs}$ has $\Pi_2(\mathrm{O})$ of length $d \geq c_1 n/2$:

$$E_{\text{block}}^{n,d} := \{\mathrm{O} \in \mathrm{obs} : [\Pi_2(\mathrm{O})]_{d-1} \neq [\Pi_2(\mathrm{O})]_d\}.$$

We order the 2^d elements of $\mathcal{C}^d$ lexicographically and denote them by v^1,v^2, $\cdots$, v^{2^d}. Let $e_{v^k} := (e_{v^k}(i))_{i\in[1,2^d]}$ be defined by $e_{v^k}(i) := \delta_k(i)$; i.e. $\{e_{v^k}; k \in [1, 2^d]\}$ is the canonical basis in $\mathbb{R}^{2^d}$. Let $\{1_{v^k}; k \in [1, 2^d]\}$ be the dual basis, i.e. $1_{v^k}(e_{v^j}) = \delta_k(j)$ for all $j, k \in [1, 2^d]$. Recall that we identify a measure λ on a countable set $\{s_i\}_i$ by $(\lambda(\{s_i\}))_i$.

Notation for linear functionals: Let $w \in \mathcal{C}^d$. For any probability measure λ on $\mathcal{C}^d$ we have $1_w(\lambda) = \lambda(w)$. In particular, if λ gives mass one to w, then $1_w(\lambda) = 1$. We denote by 1 the linear functional defined by $1(\lambda) := \sum_{i=1}^d \lambda_i$.

If g and g' are two linear functionals, we denote by $g \otimes g'$ their *tensor product*. More precisely, for row vectors $\lambda = (\lambda_i)_{i\in[1,m]} \in \mathbb{R}^m$ and $\lambda' = (\lambda'_j)_{j\in[1,m']} \in \mathbb{R}^{m'}$, we define $\lambda \otimes \lambda'$ to be the matrix

$$\lambda^t \lambda' = (\lambda_i \lambda'_j)_{i\in[1,m], j\in[1,m']}. \tag{4.7}$$

In other words, if λ and λ' represent probability distributions, then $\lambda \otimes \lambda'$ represents the corresponding product measure. If $g : \mathbb{R}^m \to \mathbb{R}$ and $g' : \mathbb{R}^{m'} \to \mathbb{R}$ are the linear functionals given by $g(\lambda) = \sum_{i=1}^m g_i \lambda_i$ and $g'(\lambda') = \sum_{j=1}^{m'} g'_j \lambda'_j$, then $g \otimes g' : \mathbb{R}^m \otimes \mathbb{R}^{m'} \to \mathbb{R}$ is the map given by

$$(g \otimes g')(\lambda \otimes \lambda') = \sum_{i,j} (g \otimes g')_{i,j} (\lambda \otimes \lambda')_{i,j} = g(\lambda) g'(\lambda') \tag{4.8}$$

where we identify g with $(g_i)_{i\in[1,m]} \in \mathbb{R}^m$ and g' with $(g'_j)_{j\in[1,m']} \in \mathbb{R}^{m'}$ and use the tensor product of vectors (4.7). Note that

$$\begin{aligned}(g\otimes g')(\rho(\lambda_1 \otimes \lambda_1{}') + (\lambda_2 \otimes \lambda_2{}')) = \rho(g \otimes g')(\lambda_1 \otimes \lambda_1{}') \\ +(g\otimes g')(\lambda_2 \otimes \lambda_2{}')\end{aligned} \tag{4.9}$$

for any scalar $\rho \in \mathbb{R}$ and $\lambda_1, \lambda_2 \in \mathbb{R}^m$, $\lambda'_1, \lambda'_2 \in \mathbb{R}^{m'}$.

The following theorem gives sufficient conditions for a word $w \in \mathcal{C}^d$ to be contained in the scenery $\xi|[-3^{3n}, 3^{3n}]$. Its proof is postponed to Section 4.3. For the definition of *positivity* for a linear functional we refer the reader to Section 4.2, in particular Definition 4.5.

Theorem 4.1. *There exists $c_{22} > 0$ such that for all $n \geq c_{22}$, $d \in [c_1 n/2, c_1 n]$, and $w \in \mathcal{C}^d$ with $w_{d-1} \neq w_d$ the following holds whenever the event $E^{n,\tau}_{\text{stop}}$ holds: Suppose there exist positive linear functionals g_1^w and g_3^w on $(\mathbb{R}^5)^{\otimes c_1 n}$ with the following properties:*

1. *Case $q \neq 0$:*

$$(g_1^w \otimes 1_w \otimes g_3^w)(\hat{\mu}^{n,\tau}_{\xi\circ S}[\cdot \cap E^{n,d}_{\text{block}}]) > 1 \tag{4.10}$$

$$(g_1^w \otimes 1 \otimes g_3^w)(\hat{\mu}^{n,\tau}_{\xi\circ S}[\cdot \cap E^{n,d-1}_{\text{block}}]) \leq 1/(5n^2) \tag{4.11}$$

$$\|g_1^w \otimes g_3^w\|_2 \cdot \|\varepsilon^{n,\tau}_{\xi,S}\|_1^{1/2} \leq 1/(2n^2). \tag{4.12}$$

2. *Case $q = 0$: (4.10), (4.12), and*

$$(g_1^w \otimes 1 \otimes g_3^w)(\hat{\mu}^{n,\tau}_{\xi\circ S}[\cdot \cap E^{n,d-2}_{\text{block}}]) \leq 1/(5n^2). \tag{4.13}$$

Then $w \preceq \xi|\left[-3^{3n}, 3^{3n}\right]$.

4.2. Positive linear functionals

For $m \in \mathbb{N}$, we denote by T_m the first hitting time of $\{-1, m\}$ by the random walk S: $T_m := \min\{k \geq 0 : S_k \in \{-1, m\}\}$. Let $\xi|[a, b]$ be a block of length m and let S^B be a random walk with $S_0^B \in \{a, b\}$ having its increments distributed as S. Then the P_ξ-distribution of the length of the first block in $\xi \circ S^B$ equals $P[T_m]^{-1}$, the distribution of T_m. For $A \subseteq \mathbb{N}_0$, we abbreviate

$$\lambda_l^m(A) := P\left(\{T_m \in A\} \cap \{S_{T_m} = -1\}\right), \ \lambda_r^m(A) := P\left(\{T_m \in A\} \cap \{S_{T_m} = m\}\right).$$

Clearly, $T_m \geq 1$ P-almost surely. So λ_l^m and λ_r^m are probability measures supported on $\mathbb{N}$. We can identify λ_l^m with the vector $\left(\lambda_l^m(\{k\})\right)_{k\in\mathbb{N}} = \left(P(T_m = k, S_{T_m} = -1)\right)_{k\in\mathbb{N}}$ and λ_r^m with the vector $\left(\lambda_r^m(\{k\})\right)_{k\in\mathbb{N}} = \left(P(T_m = k, S_{T_m} = m)\right)_{k\in\mathbb{N}}$. We compute

$$\begin{aligned}
\lambda_r^1 &= (p, pq, pq^2, pq^3, pq^4, \dots), \\
\lambda_r^2 &= (0, p^2, 2p^2q, p^4 + 3p^2q^2, 4p^4q + 4p^2q^3, \dots), \\
\lambda_r^3 &= (0, 0, p^3, 3p^3q, 2p^5 + 6p^3q^2, \dots), \\
\lambda_r^4 &= (0, 0, 0, p^4, 4p^4q, \dots), \\
\lambda_r^5 &= (0, 0, 0, 0, p^5, \dots);
\end{aligned}$$

here "…" means we are not interested in these values. We define $h : \mathbb{N}_0 \to [1, 5]$, $x \mapsto x \wedge 5$. Then, for $A \subseteq \mathbb{N}_0$,

$$\lambda_l^m h^{-1}(A) = P\left(\{T_m \wedge 5 \in A\} \cap \{S_{T_m} = -1\}\right). \tag{4.14}$$

The measures $\lambda_l^m h^{-1}$, $\lambda_r^m h^{-1}$ are supported on the set $\{1, 2, 3, 4, 5\}$. Hence we can identify them with vectors in $\mathbb{R}_+^5$.

Definition 4.4. *We define vectors in* $\mathbb{R}_+^5$*:*

$$\begin{aligned}
\vec{x}_1 &:= (p, pq, pq^2, pq^3, pq^4), \\
\vec{x}_2 &:= \lambda_r^2 h^{-1} = (0, p^2, 2p^2q, p^4 + 3p^2q^2, \lambda_r^2([5, \infty[), \\
\vec{x}_3 &:= (0, 0, p^3, 3p^3q, p^5 + 6p^3q^2), \\
\vec{x}_4 &:= \lambda_r^4 h^{-1} = (0, 0, 0, p^4, \lambda_r^4([5, \infty[), \\
\vec{x}_5 &:= (0, 0, 0, 0, 1).
\end{aligned}$$

Clearly, $\{\vec{x}_i\}_{i\in[1,5]}$ *is a basis of* $\mathbb{R}^5$. *We denote by* $\{\vec{x}_i^*\}_{i\in[1,5]}$ *the corresponding dual basis. In particular,* $\vec{x}_i^* : \mathbb{R}^5 \to \mathbb{R}$ *is a linear map with*

$$\vec{x}_i^*(\vec{x}_i) = 1 \quad \text{and} \quad \vec{x}_i^*(\vec{x}_j) = 0 \quad \text{for all } i \neq j. \tag{4.15}$$

Remark 4.1. *1. For any* $m \geq 1$ *and* $i \in \{l, r\}$ *the vector* $\lambda_i^m h^{-1}$ *can be written as a linear combination with positive coefficients of* $\vec{x}_j$, $1 \leq j \leq 5$.

2. *We have* $\vec{x}_2^*(\lambda_i^m h^{-1}) \neq 0$ *iff* $i = r$ *and* $m = 2$. *Furthermore,* $\vec{x}_4^*(\lambda_i^m h^{-1}) \neq 0$ *iff* $i = r$ *and* $m = 4$.
3. *For* $i \in \{1, 3, 5\}$, *we have* $x_i^*(\lambda_r^2 h^{-1}) = 0$ *and* $x_i^*(\lambda_r^4 h^{-1}) = 0$.

4. *The lower bound* $\vec{x}^*_{m\wedge 5}\left(\lambda_r^m h^{-1}\right) \geq (m+1)^{-1}$ *holds for all* $m \geq 1$.

Proof. First, we prove the following identities:

$$\lambda_r^1 h^{-1} = \vec{x}_1 + \lambda_r^1(]5,\infty[)\vec{x}_5, \tag{4.16}$$
$$\lambda_r^2 h^{-1} = \vec{x}_2, \tag{4.17}$$
$$\lambda_r^3 h^{-1} = \vec{x}_3 + (p^5 + \lambda_r^3(]5,\infty[))\vec{x}_5, \tag{4.18}$$
$$\lambda_r^4 h^{-1} = \vec{x}_4, \tag{4.19}$$
$$\lambda_r^m h^{-1} = (m+1)^{-1}\vec{x}_5 \qquad \text{for all } m \geq 5. \tag{4.20}$$

The identities (4.17) and (4.19) for $\lambda_r^2 h^{-1}$ and $\lambda_r^4 h^{-1}$ are true by the definitions of $\vec{x}_2$ and $\vec{x}_4$. Let $\vec{x}_i(k)$ denote the kth component of $\vec{x}_i$, $k = 1, \dots, 5$. Clearly, for $k = 1, \dots, 4$, $\lambda_r^1 h^{-1}(\{k\}) = \lambda_r^1(\{k\}) = \vec{x}_1(k)$. Furthermore,

$$\begin{aligned}
\lambda_r^1 h^{-1}(\{5\}) &= P(T_1 \wedge 5 = 5, S_{T_5} = 5) \\
&= P(T_1 = 5, S_{T_5} = 5) + P(T_1 > 5, S_{T_5} = 5) \\
&= \lambda_r^1(\{5\}) + \lambda_r^1(]5,\infty[) \\
&= \vec{x}_1(5) + \lambda_r^1(]5,\infty[).
\end{aligned}$$

Consequently, $\lambda_r^1 h^{-1} = \vec{x}_1 + \lambda_r^1(]5,\infty[)\vec{x}_5$ which is the statement in (4.16). Similarly,

$$\begin{aligned}
&\lambda_r^3 h^{-1}(\{k\}) = \lambda_r^3(\{k\}) = \vec{x}_3(k) \text{ for } k = 1, \dots, 4, \text{ and} \\
&\lambda_r^3 h^{-1}(\{5\}) = \lambda_r^3(\{5\}) + \lambda_r^3(]5,\infty[) = \vec{x}_3(5) + p^5 + \lambda_r^3(]5,\infty[).
\end{aligned}$$

Thus, (4.18) holds. For all $m \geq 5$, we have

$$\begin{aligned}
&\lambda_r^m h^{-1}(\{k\}) = 0 \text{ for } k = 1, \dots, 4, \text{ and} \\
&\lambda_r^m h^{-1}(\{5\}) = P(T_m \geq 5, S_{T_m} = m) = P(S_{T_m} = m).
\end{aligned}$$

Since for a simple symmetric random walk with holding starting at 0 the probability to reach m before -1 equals $(m+1)^{-1}$, the claim (4.20) follows.

Next, we show that the following statements hold:

$$\lambda_l^1 h^{-1} = \vec{x}_1 + \lambda_r^1(]5,\infty[)\vec{x}_5, \tag{4.21}$$
$$\lambda_l^2 h^{-1} = \vec{x}_1 + \vec{x}_3 + \lambda_l^2(]5,\infty[)\vec{x}_5, \tag{4.22}$$
$$\lambda_l^m h^{-1} = \vec{x}_1 + \vec{x}_3 + (p^5 + \lambda_l^m(]5,\infty[))\vec{x}_5 \qquad \text{for all } m \geq 3. \tag{4.23}$$

By symmetry, $\lambda_l^1 h^{-1} = \lambda_r^1 h^{-1}$; thus, (4.21) follows from (4.16). Next, we calculate using (4.14):

$$\begin{aligned}
\lambda_l^2 h^{-1} &= (p, pq, pq^2 + p^3, pq^3 + 3p^3 q, p^5 + 6p^3 q^2 + pq^4 + \lambda_l^2(]5,\infty[)) \\
&= \vec{x}_1 + \vec{x}_3 + \lambda_l^2(]5,\infty[)\vec{x}_5, \\
\lambda_l^3 h^{-1} &= (p, pq, pq^2 + p^3, pq^3 + 3p^3 q, 2p^5 + 6p^3 q^2 + pq^4 + \lambda_l^3(]5,\infty[)) \\
&= \vec{x}_1 + \vec{x}_3 + (p^5 + \lambda_l^3(]5,\infty[))\vec{x}_5.
\end{aligned}$$

Finally, let $m \geq 4$. Consider any path of our random walk, starting at 0 and ending at -1. If the path makes at most 5 steps, then it does not visit the point 3. Consequently, for $k = 1, \ldots, 4$, we have $\lambda_l^m h^{-1}(\{k\}) = P(T_m \wedge 5 = k, S_{T_m} = -1) = P(T_3 \wedge 5 = k, S_{T_3} = -1) = \lambda_l^3 h^{-1}(\{k\})$. By the same argument, $\lambda_l^m(\{5\}) = \lambda_l^3(\{5\}) = \vec{x}_1(5) + \vec{x}_3(5) + p^5$. Thus, (4.23) holds.

Part 1 of Remark 4.1 follows because all coefficients in the representations (4.16)–(4.20) and (4.21)–(4.23) are positive.

Recall (4.15). Since $\{\vec{x}_i\}_{i \in [1,5]}$ is a basis of $\mathbb{R}^5$, we have $\vec{x}_2^*(\lambda_i^m h^{-1}) \neq 0$ iff in the representation of $\lambda_i^m h^{-1}$ as a linear combination of the $\vec{x}_j$ the coefficient of $\vec{x}_2$ is non-zero. This is only the case for $\lambda_r^2 h^{-1}$ as can be seen from (4.16)–(4.20) and (4.21)–(4.23). Similarly, $\vec{x}_4^*(\lambda_i^m h^{-1}) \neq 0$ iff in the representation of $\lambda_i^m h^{-1}$ as a linear combination of the $\vec{x}_j$ the coefficient of $\vec{x}_4$ is non-zero which is only the case for $\lambda_r^4 h^{-1}$. This proves part 2.

Let $i \in \{1, 3, 5\}$, and recall (4.15). By (4.17), we have $x_i^*(\lambda_r^2 h^{-1}) = x_i^*(\vec{x}_2) = 0$, and by (4.19), we have $x_i^*(\lambda_r^4 h^{-1}) = x_i^*(\vec{x}_4) = 0$. This proves part 3 of Remark 4.1.

For $m = 1, \ldots, 4$, $\vec{x}^*_{m\wedge 5}\left(\lambda_r^m h^{-1}\right) = \vec{x}^*_m\left(\lambda_r^m h^{-1}\right) = 1 \geq (m+1)^{-1}$ because of the representations (4.16)–(4.19). On the other hand, for $m \geq 5$, $\vec{x}^*_{m\wedge 5}\left(\lambda_r^m h^{-1}\right) = \vec{x}_5^*\left(\lambda_r^m h^{-1}\right) = \vec{x}_5^*\left((m+1)^{-1}\vec{x}_5\right) = (m+1)^{-1}$ because of (4.20). This completes the proof of part 4 and thus of Remark 4.1. □

Definition 4.5. *We call a functional* $f : \left(\mathbb{R}^5\right)^{\otimes m} \to \mathbb{R}$ positive *if* $f\left(\otimes_{k=1}^m \vec{x}_{n_k}\right) \geq 0$ *for all* $n_1, n_2, \ldots, n_m \in \{1, 2, 3, 4, 5\}$.

Remark 4.2. *Let g be a positive linear functional on* $(\mathbb{R}^5)^{\otimes c_1 n}$. *If* $P_{x,\xi}[S_{o_l^n} = y] > 0$*, then* $g(P_{x,\xi}[\mathrm{O}_1^n \in \cdot | S_{o_l^n} = y]) \geq 0$. *If* $P_{x,\xi}[S_{o_r^n} = y] > 0$*, then* $g(P_{x,\xi}[\mathrm{O}_3^n \in \cdot | S_{o_r^n} = y]) \geq 0$.

Proof. Suppose $P_{x,\xi}(S_{o_l^n} = y) > 0$. By the definition of O_1^n, we can write the probability $P_{x,\xi}[\mathrm{O}_1^n \in \cdot | S_{o_l^n} = y]$ as a linear combination with positive coefficients of tensor products of the $\lambda_i^m h^{-1}$'s. Each $\lambda_i^m h^{-1}$ equals a linear combination with positive coefficients of $\vec{x}_i$, $1 \leq i \leq 5$, by Remark 4.1. The estimate $g(P_{x,\xi}[\mathrm{O}_1^n \in \cdot | S_{o_l^n} = y]) \geq 0$ follows because g is positive. The second part of the statement is proved analogously. □

4.3. Proof of Theorem 4.1

We begin with a lemma, which we need in the proof of Theorem 4.1.

Lemma 4.1. *There exists $c_{23} > 0$ such that for all $n \geq c_{23}$, for all $d \in]2, c_1 n]$, and all $x \in [0, d[$ the following hold:*

1. *If $q \neq 0$, then $P(S_d = x) \leq n^2 P(S_{d-1} = x)$.*
2. *If $q = 0$ and $P(S_{d-2} = x) > 0$, then $P(S_d = x) \leq n^2 P(S_{d-2} = x)$.*

Proof. Let $n \in \mathbb{N}$, $d \in]2, c_1 n]$, $x \in [0, d[$, and suppose $q \neq 0$. We denote by $\Pi_{d,x} \subseteq \mathbb{Z}^{[0,d]}$ the set of all admissible pieces of paths from 0 to x, and we define a map $f : \Pi_{d,x} \to \Pi_{d-1,x}$ as follows: If $\pi \in \Pi_{d,x}$ contains a holding, i.e. $\pi_y = \pi_{y-1}$ for some $y \in]0, d]$, then we define $f(\pi)$ to be the path obtained from π by removing the first holding in π. Otherwise, because of $x < d$, there exists either a step to the left followed by a step to the right or a step to the right followed by a step to the left in π. In this case, we define $f(\pi)$ to be the path obtained from π by replacing the first occurrence of such a pair of steps by a holding. For any $\pi \in \Pi_{d,x}$ we have

$$P(S|[0,d] = \pi) \leq \max\left\{q, p^2q^{-1}\right\} P(S|[0,d-1] = f(\pi)).$$

Furthermore, any $\pi' \in \Pi_{d-1,x}$ has at most $3d$ pre-images under f. Hence we obtain

$$\begin{aligned} P(S_d = x) &= \sum_{\pi\in\Pi_{d,x}} P(S|[0,d] = \pi) \leq \sum_{\pi'\in\Pi_{d-1,x}} \sum_{\pi\in f^{-1}(\pi')} P(S|[0,d] = \pi) \\ &\leq \sum_{\pi'\in\Pi_{d-1,x}} \left|f^{-1}(\pi')\right| \max\left\{q, p^2q^{-1}\right\} P(S|[0,d-1] = \pi') \\ &\leq 3d \max\left\{q, p^2q^{-1}\right\} P(S_{d-1} = x). \end{aligned}$$

Since $d \leq c_1 n$, we have $3d\max\left\{q, p^2q^{-1}\right\} \leq n^2$ for all n sufficiently large and the claim follows in the case $q \neq 0$. The case $q = 0$ is treated similarly. □

Proof of Theorem 4.1. Let $q \neq 0$. We do a proof by contradiction. Assume that for infinitely many n, there exist $d \in [c_1n/2, c_1n]$, $w \in \mathcal{C}^d$ with $w_{d-1} \neq w_d$, and positive linear functionals g_1^w and g_3^w on $(\mathbb{R}^5)^{\otimes c_1 n}$ such that (4.10)–(4.12) hold on the event $E_{\text{stop}}^{n,\tau}$, but $w \not\preceq \xi|\left[-3^{3n}, 3^{3n}\right]$.

For any linear functional $g : \mathbb{R}^m \to \mathbb{R}$ and $\lambda \in \mathbb{R}^m$, we have the estimate $|g(\lambda)| = |\sum_{i=1}^m g_i\lambda_i| \leq [\sum_{i=1}^m g_i^2]^{1/2}[\sum_{i=1}^m \lambda_i^2]^{1/2} = \|g\|_2\|\lambda\|_2$ by Hölder's inequality. Since $\varepsilon_{\xi,S}^{n,\tau}$ is the difference of two probability measures, we can identify it with a vector $\left(\varepsilon_i' - \varepsilon_i''\right)_i$, where $\varepsilon_i', \varepsilon_i'' \in [0,1]$ and i runs through a finite index set. Consequently,

$$\|\varepsilon_{\xi,S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d}]\|_2^2 \leq \|\varepsilon_{\xi,S}^{n,\tau}\|_2^2 = \sum_i \left|\varepsilon_i' - \varepsilon_i''\right|^2 \leq \sum_i \left|\varepsilon_i' - \varepsilon_i''\right| = \|\varepsilon_{\xi,S}^{n,\tau}\|_1.$$

Using this together with (4.12), we obtain

$$\begin{aligned} |(g_1^w \otimes 1_w \otimes g_3^w)(\varepsilon_{\xi,S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d}])| &\leq \|g_1^w \otimes 1_w \otimes g_3^w\|_2 \cdot \|\varepsilon_{\xi,S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d}]\|_2 \\ &\leq \|g_1^w \otimes g_3^w\|_2 \cdot \|\varepsilon_{\xi,S}^{n,\tau}\|_1^{1/2} \leq 1/(2n^2); \quad (4.24) \end{aligned}$$

here we used that $\|g_1^w \otimes 1_w \otimes g_3^w\|_2 = \|g_1^w \otimes g_3^w\|_2$. Hence, it follows from $\mu_{\xi,S}^{n,\tau} = \hat{\mu}_{\xi\circ S}^{n,\tau} - \varepsilon_{\xi,S}^{n,\tau}$ and (4.10) that

$$(g_1^w \otimes 1_w \otimes g_3^w)(\mu_{\xi,S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d}]) \geq 1 - 1/(2n^2);$$

here we used the linearity (compare (4.9)) of $g_1^w \otimes 1_w \otimes g_3^w$. Inserting the definition of $\mu_{\xi,S}^{n,\tau}$, using the linearity of $g_1^w \otimes 1_w \otimes g_3^w$ again, and using also the definition of 1_w, we obtain

$$\begin{aligned}
&1 - 1/(2n^2) \qquad (4.25)\\
&\le (g_1^w \otimes 1_w \otimes g_3^w)(\mu_{\xi,S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d}])\\
&= \sum_{x\in[-3^n,3^n]} a_S^{n,\tau}(x)(g_1^w \otimes 1_w \otimes g_3^w)(P_{x,\xi}[(\mathrm{O}_1^n(\chi), \mathrm{O}_2^n(\chi), \mathrm{O}_3^n(\chi)) \in \cdot \cap E_{\text{block}}^{n,d}])\\
&= \sum_{x\in[-3^n,3^n]} a_S^{n,\tau}(x)(g_1^w \otimes 1_w \otimes g_3^w)(P_{x,\xi}[(\mathrm{O}_1^n(\chi), \mathrm{O}_2^n(\chi) = w, \mathrm{O}_3^n(\chi)) \in \cdot]).
\end{aligned}$$

In the following, we omit dependencies on χ in the notation. Since we assumed $w \not\preceq \xi|\left[-3^{3n}, 3^{3n}\right]$, we have for any admissible path $R \in \left[-3^{3n}, 3^{3n}\right]^{[0,d[}$ with $\xi \circ R = w$ the estimate $|R_0 - R_{d-1}| < d - 1$. Consequently,

$$\begin{aligned}
&P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot]\\
= &\sum_{y,z} P_{x,\xi}[S_{o_l^n} = y, S_{o_r^n} = z, (\mathrm{O}_1^n, \mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot], \qquad (4.26)
\end{aligned}$$

where the sum is taken over all $y, z \in \left[-3^{3n}, 3^{3n}\right]$ with the property $|y-z| < d-1$ and $P_{x,\xi}[S_{o_l^n} = y, S_{o_l^n+d-1} = z] > 0$. We rewrite the addends: On the event $\{\mathrm{O}_2^n = w\}$, we have $o_r^n = o_l^n + d - 1$. Note that O_1^n depends only on the random walk up to time o_l^n, whereas $(S_{o_r^n}, \mathrm{O}_2^n, \mathrm{O}_3^n)$ depends only on $S_{o_l^n}$ and the random walk increments $S_{o_l^n+t} - S_{o_l^n}$, $t \ge 0$. Therefore, O_1^n and $(S_{o_r^n}, \mathrm{O}_2^n, \mathrm{O}_3^n)$ are independent conditioned on $S_{o_l^n} = y$. Thus, the right-hand side of equation (4.26) equals

$$\begin{aligned}
&\sum_{y,z} P_{x,\xi}[S_{o_l^n} = y, \mathrm{O}_1^n \in \cdot]\\
&\otimes P_{x,\xi}[S_{o_r^n} = z, (\mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot | S_{o_l^n} = y] \qquad (4.27)\\
= &\sum_{y,z} P_{x,\xi}[S_{o_l^n} = y, \mathrm{O}_1^n \in \cdot]\\
&\otimes \left(P_{x,\xi}[S_{o_l^n+d-1} = z | S_{o_l^n} = y] P_{x,\xi}[(\mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot | A_{y,z}^{d-1}]\right)
\end{aligned}$$

with $A_{y,z}^{d-1} := \{S_{o_l^n} = y, S_{o_l^n+d-1} = z\}$. In the last sum, the addends consist of the tensor product of the two vectors $P_{x,\xi}[S_{o_l^n} = y, \mathrm{O}_1^n \in \cdot]$ and $P_{x,\xi}[(\mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot | A_{y,z}^{d-1}]$, multiplied by the constant $P_{x,\xi}[S_{o_l^n+d-1} = z | S_{o_l^n} = y]$. Using again the Markov property of the random walk, we see that O_2^n and O_3^n are independent, conditioned on the event $A_{y,z} := \{S_{o_l^n} = y, S_{o_r^n} = z\}$. Hence,

$$\begin{aligned}
P_{x,\xi}[(\mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot | A_{y,z}^{d-1}] &= \frac{P_{x,\xi}[A_{y,z}]}{P_{x,\xi}[A_{y,z}^{d-1}]} P_{x,\xi}[(\mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot | A_{y,z}]\\
&= \frac{P_{x,\xi}[A_{y,z}]}{P_{x,\xi}[A_{y,z}^{d-1}]} P_{x,\xi}[\mathrm{O}_2^n = w \in \cdot | A_{y,z}]\\
&\quad \otimes P_{x,\xi}[\mathrm{O}_3^n \in \cdot | A_{y,z}]\\
&= P_{x,\xi}[\mathrm{O}_2^n = w \in \cdot | A_{y,z}^{d-1}] \otimes P_{x,\xi}[\mathrm{O}_3^n \in \cdot | A_{y,z}].
\end{aligned}$$

Consequently, we obtain from (4.26) and (4.27)

$$\begin{aligned}
&P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot] \\
&= \sum_{y,z} P_{x,\xi}[S_{o_l^n} = y, \mathrm{O}_1^n \in \cdot] \qquad (4.28)\\
&\otimes \big(P_{x,\xi}[S_{o_l^n+d-1} = z | S_{o_l^n} = y] P_{x,\xi}[\mathrm{O}_2^n = w \in \cdot | A_{y,z}^{d-1}]\big) \\
&\otimes P_{x,\xi}[\mathrm{O}_3^n \in \cdot | A_{y,z}].
\end{aligned}$$

In view of (4.25), the aim is to apply $g_1^w \otimes 1_w \otimes g_3^w$ to the last sum. First, we estimate 1_w applied to the middle vector in the addends of (4.28). We observe for $n \geq c_{23}$ with c_{23} as in Lemma 4.1

$$\begin{aligned}
&P_{x,\xi}[S_{o_l^n+d-1} = z | S_{o_l^n} = y] 1_w (P_{x,\xi}[\mathrm{O}_2^n = w \in \cdot | A_{y,z}^{d-1}]) \\
&\leq P_{x,\xi}[S_{o_l^n+d-1} = z | S_{o_l^n} = y] = P_{0,\xi}[S_{d-1} = z - y] \leq n^2 P_{0,\xi}[S_{d-2} = z - y] \\
&= n^2 P_{x,\xi}[S_{o_l^n+d-2} = z | S_{o_l^n} = y]; \qquad (4.29)
\end{aligned}$$

here we used the Markov property of the random walk and Lemma 4.1. Combining (4.28) with (4.29) and using Remark 4.2 yields

$$\begin{aligned}
&(g_1^w \otimes 1_w \otimes g_3^w)(P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_2^n = w, \mathrm{O}_3^n) \in \cdot]) \qquad (4.30)\\
&\leq \sum_{y,z} g_1^w [P_{x,\xi}[S_{o_l^n} = y, \mathrm{O}_1^n \in \cdot]] n^2 \\
&\times P_{x,\xi}[S_{o_l^n+d-2} = z | S_{o_l^n} = y] g_3^w [P_{x,\xi}[\mathrm{O}_3^n \in \cdot | A_{y,z}]].
\end{aligned}$$

We can enlarge the last sum by summing over *all* $y, z \in \left[-3^{3n}, 3^{3n}\right]$ with $|y - z| < d - 1$ and $P_{x,\xi}(S_{o_l^n} = y, S_{o_r^n} = z) > 0$ and not only over those with $P_{x,\xi}[S_{o_l^n} = y, S_{o_l^n+d-1} = z] > 0$. The terms added in this way are non-negative by Remark 4.2. Note that

$$P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_2^n, \mathrm{O}_3^n) \in \cdot \cap E_{\text{block}}^{n,d-1}] = \sum_{w'} P_{x,\xi}[\mathrm{O}_2^n = w', (\mathrm{O}_1^n, \mathrm{O}_3^n) \in \cdot],$$

where the sum is taken over all $w' \in \mathcal{C}^{d-1}$ with $w'_{d-1} \neq w'_{d-2}$. We use (4.28) with $d - 1$ instead of d to obtain

$$(4.30) \leq n^2 (g_1^w \otimes 1 \otimes g_3^w)[P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_2^n, \mathrm{O}_3^n) \in \cdot \cap E_{\text{block}}^{n,d-1}]]. \qquad (4.31)$$

Since the event $E_{\text{stop}}^{n,\tau}$ holds, $\sum_{x \in [-3^n, 3^n]} a_S^{n,\tau}(x) = 1$. Consequently, the estimates (4.25) and (4.31) imply

$$(1 - (2n^2)^{-1}) n^{-2} \leq (g_1^w \otimes 1 \otimes g_3^w)(\mu_{\xi \circ S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d-1}]). \qquad (4.32)$$

We can identify g_1^w with a vector $(g_1(i))_i$, g_3^w with a vector $(g_3(k))_k$, and $\varepsilon_{\xi,S}^{n,\tau}$ with a vector $(\varepsilon'_{i,j,k} - \varepsilon''_{i,j,k})_{i,j,k}$ where $\varepsilon'_{i,j,k}, \varepsilon''_{i,j,k} \geq 0$ and $\sum_{i,j,k} \varepsilon'_{i,j,k} = \sum_{i,j,k} \varepsilon''_{i,j,k} = 1$.

Consequently, using Hölder's inequality,

$$
\begin{aligned}
|(g_1^w \otimes 1 \otimes g_3^w)(\varepsilon_{\xi,S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d-1}])| &\le \sum_{i,j,k} |g_1(i)g_3(k)| \cdot |\varepsilon'_{i,j,k} - \varepsilon''_{i,j,k}| \\
&= \sum_{i,k} |g_1(i)g_3(k)| \cdot \Big[\sum_j |\varepsilon'_{i,j,k} - \varepsilon''_{i,j,k}| \Big] \\
&\le \sqrt{\sum_{i,k} |g_1(i)g_3(k)|^2} \sqrt{\sum_{i,k} \Big[\sum_j |\varepsilon'_{i,j,k} - \varepsilon''_{i,j,k}|\Big]^2} \\
&\le \|g_1^w \otimes g_3^w\|_2 \cdot \sqrt{2} \cdot \|\varepsilon_{\xi,S}^{n,\tau}\|_1^{1/2} \le \sqrt{2}/(2n^2);
\end{aligned}
$$

in the second to last inequality we used that $\sum_j |\varepsilon'_{i,j,k} - \varepsilon''_{i,j,k}| \le 2$ and for the last inequality, we used (4.12). Hence, it follows from (4.32) and $\hat{\mu}_{\xi\circ S}^{n,\tau} = \mu_{\xi\circ S}^{n,\tau} + \varepsilon_{\xi,S}^{n,\tau}$

$$
(g_1^w \otimes 1 \otimes g_3^w)(\hat{\mu}_{\xi\circ S}^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d-1}]) \ge (1 - (2n^2)^{-1})n^{-2} - 1/(\sqrt{2}n^2),
$$

which is strictly larger than $1/(5n^2)$ for n sufficiently large; thus, it contradicts assumption (4.11). Thus $w \preceq \xi | \left[-3^{3n}, 3^{3n}\right]$ and the theorem is proved in the case $q \neq 0$. If $q = 0$, one replaces $d-1$ by $d-2$ in the above argument und uses (4.13) instead of (4.11). □

5. Reconstructing a piece of scenery

Let $n \in \mathbb{N}$. The aim of this section is to define a map BigAlg^n which fulfills the claim of Theorem 3.1. Special functionals and events are needed in the proof of Theorem 3.1; their definitions are stated in Subsection 5.2. Subsection 5.3 contains the combinatorial part in the proof of Theorem 3.1, and Subsection 5.4 deals with the probabilistic estimates.

5.1. Definition of BigAlg^n

BigAlg^n takes as arguments

$$
\tau \in \left[0, 3^{10\alpha n}\right]^{[1,3^{\alpha n}]}, \ \eta \in \mathcal{C}^{2\cdot 3^{10\alpha n}}, \ \text{and } \psi \in \bigcup_{k\ge n^2} \mathcal{C}^{[-k,k]} \tag{5.1}
$$

and produces an output $\mathrm{BigAlg}^n(\tau, \eta, \psi) \in \mathcal{C}^{[-3\cdot 3^n, 3\cdot 3^n]}$. The reader should think of τ as a realization of a sequence of $3^{\alpha n}$ stopping times, η stands for $2 \cdot 3^{10\alpha n}$ observations, and ψ should be thought of as a small piece of the scenery ξ around which the reconstruction takes place. In the following, we treat τ, η, and ψ as abstract input data of BigAlg^n which need to fulfill (5.1) only.

Let τ, η, and ψ satisfy (5.1). We use the conditions of Theorem 4.1 to define a set $\mathsf{Words}^n(\tau, \eta)$ of building blocks for the scenery which we would like to reconstruct.

Definition 5.1. *Let $c_7 > 0$ be chosen as in Section 2. We define $\mathsf{Words}^n(\tau,\eta)$ to be the set of all $w \in \mathcal{C}^d$, $d \in [c_1 n/2, c_1 n]$ such that there exist positive linear functionals g_1^w and g_3^w on $(\mathbb{R}^5)^{\otimes c_1 n}$ with the following properties:*

1. *Case $q \neq 0$:*

$$\left(g_1^w \otimes 1_w \otimes g_3^w\right)(\hat{\mu}_\eta^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d}]) > 1 \tag{5.2}$$

$$(g_1^w \otimes 1 \otimes g_3^w)(\hat{\mu}_\eta^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d-1}]) \le 1/(5n^2) \tag{5.3}$$

$$\|g_1^w \otimes g_3^w\|_2 \le e^{c_7 n}. \tag{5.4}$$

2. *Case $q = 0$: (5.2), (5.4), and*

$$(g_1^w \otimes 1 \otimes g_3^w)(\hat{\mu}_\eta^{n,\tau}[\cdot \cap E_{\text{block}}^{n,d-2}]) \le 1/(5n^2). \tag{5.5}$$

The output of BigAlg^n is supposed to contain ψ in the middle and all subpieces of length $c_1 n/2$ should be contained in a possibly bigger piece of $\mathsf{Words}^n(\tau,\eta)$. Formally:

Definition 5.2. *We define $\mathsf{Output}^n(\tau,\eta,\psi) :=$*

$$\left\{\begin{array}{l} w \in \mathcal{C}^{[-3\cdot 3^n, 3\cdot 3^n]} : w|[-k,k] = \psi \; for\ k = (|\psi|-1)/2 \; and\ for\ all\ intervals\ I \\ \subset [-3\cdot 3^n, 3\cdot 3^n] \; with\ |I| = c_1 n/2 \; there\ exists\ w' \in \mathsf{Words}^n(\tau,\eta) \; such\ that\ w|I \sqsubseteq w' \end{array}\right\}.$$

We will see in the proof of Lemma 5.2 below that under appropriate conditions, there is precisely one element in $\mathsf{Output}^n(\tau,\eta,\psi)$.

Definition 5.3. *We define*

$$\mathrm{BigAlg}^n : [0, 3^{10\alpha n}]^{[1,3^{\alpha n}]} \times \mathcal{C}^{2\cdot 3^{10\alpha n}} \times \bigcup_{k \ge n^2} \mathcal{C}^{[-k,k]} \to \mathcal{C}^{[-3\cdot 3^n, 3\cdot 3^n]}$$

as follows: If $\mathsf{Output}^n(\tau,\eta,\psi) \neq \emptyset$, then we define $\mathrm{BigAlg}^n(\tau,\eta,\psi)$ to be its lexicographically smallest element. Otherwise we set $\mathrm{BigAlg}^n(\tau,\eta,\psi) := (1)_{[-3\cdot 3^n, 3\cdot 3^n]}$.

5.2. Definitions of functionals and events

Below we will need some special linear functionals. Recall the definition of $\{\vec{x}_i^*\}_{i\in[1,5]}$ from Definition 4.4.

Definition 5.4. *Let $\xi \in \mathcal{C}^{\mathbb{Z}}$.*

1. *Let $z \in \mathbb{Z}$ be such that $\xi_z \neq \xi_{z-1}$, and let $B_{i,z}^{\leftarrow}$ denote the ith block of $\xi^{\leftrightarrow}|]-\infty, z]$, where $\xi^{\leftrightarrow}$ denotes the reflected scenery, defined by $\xi_y^{\leftrightarrow} := \xi_{-y}$ for all $y \in \mathbb{Z}$. We set*

$$\tilde{g}_{z,\xi}^{n,l} := \bigotimes_{i=1}^{c_1 n} \left(|B_{i,z}^{\leftarrow}| + 1\right) \cdot \vec{x}^*_{|B_{i,z}^{\leftarrow}| \wedge 5}$$

   and call $g_{z,\xi}^{n,l} := 3^{2n} p^{-c_1 n - 2} \tilde{g}_{z,\xi}^{n,l}$ the left functional of ξ at z.

2. *Let $z \in \mathbb{Z}$ be such that $\xi_z \neq \xi_{z-1}$, and let $B_{i,z}^{\rightarrow}$ denote the ith block of $\xi|[z-1,\infty[$. We define the* right functional of ξ at z *by*

$$g_{z,\xi}^{n,r} := \bigotimes_{i=1}^{c_1 n} \left(|B_{i,z}^{\rightarrow}| + 1\right) \cdot \vec{x}^{*}_{|B_{i,z}^{\rightarrow}|\wedge 5}.$$

Clearly, $g_{z,\xi}^{n,l}$ and $g_{z,\xi}^{n,r}$ are positive linear functionals.

Definition 5.5. *Let $\xi \in \mathcal{C}^{\mathbb{Z}}$.*

1. *Let $x_1 \in \mathbb{Z}$ such that $\xi_{x_1} \neq \xi_{x_1-1}$. We call a positive linear functional g a* left limiting functional of ξ at x_1 *iff for all $x_2 > x_1$ with $\xi_{x_2-1} \neq \xi_{x_2}$ we have that for all $x \in \left[-3^n, 3^n\right]$, $P_{x,\xi}(S_{o_l^n} = x_2) > 0$ implies $g(P_{x,\xi}(\mathrm{O}_1^n \in \cdot | S_{o_l^n} = x_2)) = 0$, whilst $g(P_{x,\xi}(\mathrm{O}_1^n \in \cdot | S_{o_l^n} = x_1)) > 0$.*
2. *Let $y_1 \in \mathbb{Z}$ such that $\xi_{y_1} \neq \xi_{y_1-1}$. We call a positive linear functional g a* right limiting functional of ξ at y_1 *iff for all $y_2 < y_1$ with $\xi_{y_2} \neq \xi_{y_2-1}$ we have that for all $x \in \left[-3^n, 3^n\right]$, $P_{x,\xi}(S_{o_r^n} = y_2) > 0$ implies $g(P_{x,\xi}(\mathrm{O}_3^n \in \cdot | S_{o_r^n} = y_2)) = 0$, whilst $g(P_{x,\xi}(\mathrm{O}_3^n \in \cdot | S_{o_r^n} = y_1)) > 0$.*

In the remainder, we abbreviate

$$\chi^n := \chi|[0, 2 \cdot 3^{10\alpha n}[.$$

We define in alphabetical order events which will be needed below. The event $B^n_{\text{blocks bd}}$ holds if the lengths of any $c_1 n$ consecutive blocks are bounded in a certain sense in a region around the origin. $B^n_{\text{functional}}$ is the event that $g_{z,\xi}^{n,l}$ and $g_{z,\xi}^{n,r}$ are limiting functionals for all z not too large. $B^{n,\tau}_{\mathrm{O}_2}$ gives bounds on the length of $\mathrm{O}_2^n(\chi)$. If $B^n_{\text{scen ok}}$ holds, then for every word w of length $c_1 n/2$ there exist blocks to the left and to the right of w which are close to w. $B^n_{\text{unique fit}}$ guarantees that all words of length $c_1 n/4$ in a certain region of the scenery are distinct. Blocks of lengths 2 and 4 play a special role in the arguments below. $B^n_{\text{blocks 2,4}}$ guarantees that there are sufficiently many blocks of lengths 2 and 4 in the scenery. In Definition 5.12 we introduce a convenient notation for a sequence of blocks of lengths 2 and 4. B^n_{signals} denotes the event that certain sequences of blocks of lengths 2 and 4 can only be observed to the left or to the right of a point in the scenery. Finally, $E^{n,\tau}_{\text{Words ok}}$ is the event that all words in $\mathsf{Words}^n(\tau,\chi^n)$ are contained (up to a possible reflection) in $\xi|[-3^{3n}, 3^{3n}]$ and $\mathsf{Words}^n(\tau,\chi^n)$ contains sufficiently many words.

Definition 5.6. *Let $c_6 > 0$ be as in Section 2. Recall the definitions of $B_{i,z}^{\rightarrow}$ and $B_{i,z}^{\leftarrow}$ from Definition 5.4. We define $B^n_{\text{blocks bd}} := B^{n,\rightarrow}_{\text{bb}} \cap B^{n,\leftarrow}_{\text{bb}}$ with $B^{n,\rightarrow}_{\text{bb}} :=$*

$$\left\{\forall z \in \left[-2 \cdot 3^{3n}, 2 \cdot 3^{3n}\right] \text{ we have} \prod_{i=1}^{c_1 n} [|B_{i,z}^{\rightarrow}|+1] \le e^{c_6 n} \text{ and } \sum_{i=1}^{c_1 n} [|B_{i,z}^{\rightarrow}|+2] \le 8 c_1 n\right\},$$

and $B^{n,\leftarrow}_{\text{bb}}$ is defined by replacing "$\rightarrow$" by "$\leftarrow$" in the definition of $B^{n,\rightarrow}_{\text{bb}}$.

Definition 5.7. *Let c_2 be as in Section 2. We define*

$$B^n_{\text{blocks }2,4} := \left\{ \begin{array}{l} In\ any\ sequence\ of\ c_1 n\ consecutive\ blocks\ of\ \xi|[-7\cdot 3^n, 7\cdot 3^n] \\ there\ are\ at\ least\ c_2 n\ blocks\ of\ length\ 2\ or\ 4. \end{array} \right\}.$$

Definition 5.8. *Let c_7 be as in Section 2. We define $B^{n,\tau}_{\varepsilon} := \left\{ \|\varepsilon^{n,\tau}_{\xi,S}\|_1 \le e^{-4c_7 n} \right\}$.*

Definition 5.9. *We define $B^n_{\text{functional}} := B^{n,l}_{\text{func}} \cap B^{n,r}_{\text{func}}$ with*

$$B^{n,l}_{\text{func}} := \left\{ \begin{array}{l} For\ all\ y \in [-6\cdot 3^n, 6\cdot 3^n]\ with\ \xi_y \ne \xi_{y-1}\ the\ left\ functional\ at\ y\ is \\ a\ left\ limiting\ functional\ at\ y. \end{array} \right\},$$

$$B^{n,r}_{\text{func}} := \left\{ \begin{array}{l} For\ all\ y \in [-6\cdot 3^n, 6\cdot 3^n]\ with\ \xi_y \ne \xi_{y-1}\ the\ right\ functional\ at\ y \\ is\ a\ right\ limiting\ functional\ at\ y. \end{array} \right\}.$$

Definition 5.10. *We define the event $B^{n,\tau}_{\mathrm{O}_2} := B^{n,\tau}_{\mathrm{O}_2\text{ small}} \cap B^n_{\mathrm{O}_2\text{ large}}$ with*

$$B^{n,\tau}_{\mathrm{O}_2\text{ small}} := \left\{ \forall k \in \left[1, 3^{\alpha n}\right] : |\mathrm{O}^n_2(\theta^{\tau_k}\chi)| \le 3^n \right\},$$

$$B^n_{\mathrm{O}_2\text{ large}} := \left\{ \forall \xi \in \mathcal{C}^{\mathbb{Z}}\ and\ \forall x \in \left[-3^n, 3^n\right] : P_{x,\xi}(|\mathrm{O}^n_2(\chi)| > 3^n) \le e^{-8c_7 n} \right\}.$$

Definition 5.11. *We define $B^n_{\text{scen ok}} :=$*

$$\left\{ \begin{array}{l} For\ all\ intervals\ I \subseteq [-5\cdot 3^n, 5\cdot 3^n]\ of\ length\ c_1 n/2\ there\ exist\ y, z \in \mathbb{Z}\ such \\ that |y-z| < c_1 n,\ I \subseteq [y,z],\ \xi_y \ne \xi_{y-1},\ and\ \xi_z \ne \xi_{z-1}. \end{array} \right\}.$$

Definition 5.12. *Let $n_{2,4}$ be the number of blocks of length 2 and 4 in the piece of scenery $\xi^n := \xi|\left[-7\cdot 3^n, 7\cdot 3^n\right]$. Let $B^{2,4}_{i,y}$ be the ith block of $\xi|[y-1,\infty[$ of length 2 or 4, and let $C^{2,4}_{i,y}$ be its color. We can describe the blocks of length 2 and 4 of ξ^n by $\mathrm{col}(\xi^n) := (\mathrm{col}_i(\xi^n) := (|B^{2,4}_{i,y}|, C^{2,4}_{i,y}))_{i\in[1,n_{2,4}]}$ with $y = -7\cdot 3^n$. For $R \in [1, n_{2,4}]^I$ we have $\mathrm{col}\circ R = (\mathrm{col}_{R_i})_{i\in I}$. We set*

$$\hat{w}_{x,c_2 n,\rightarrow} := \mathrm{col}(\xi^n)|[x, x+c_2 n[, \quad \hat{w}_{x,c_2 n,\leftarrow} := \left(\mathrm{col}_{x-i}(\xi^n); i \in [0, c_2 n[\right) \tag{5.6}$$

for all x where this makes sense. For all other x, we set $\hat{w}_{x,c_2 n,\rightarrow}, \hat{w}_{x,c_2 n,\leftarrow} := ((1,1))_{i\in[0,c_2 n[}$. We denote by $\bar{\xi}^n$ the scenery obtained from ξ^n by replacing all 0's by 1's and all 1's by 0's. We define $\bar{w}_{x,c_2 n,\rightarrow}$, $\bar{w}_{x,c_2 n,\leftarrow}$ by replacing ξ^n by $\bar{\xi^n}$ in (5.6).

Definition 5.13. *We call $R \in \mathbb{Z}^{[a,b]}$ a nearest-neighbor path if $R_{i+1} - R_i \in \{-1,+1\}$ for all $i \in [a,b[$. We define $B^n_{\text{signals}} := B^{n,l}_{\text{sign}} \cap B^{n,r}_{\text{sign}}$ with*

$$B^{n,l}_{\text{sign}} := \left\{ \begin{array}{l} \forall x \in [1, n_{2,4}]\ \forall\ nearest-neighbor\ path\ R \in [1, n_{2,4}]^{[0,c_2 n[}\ with\ R_0 > x \\ we\ have\ \ \mathrm{col}(\xi^n)\circ R \notin \{\hat{w}_{x,c_2 n,\leftarrow}, \bar{w}_{x,c_2 n,\leftarrow}\} \end{array} \right\},$$

$$B^{n,r}_{\text{sign}} := \left\{ \begin{array}{l} \forall x \in [1, n_{2,4}]\ \ \forall\ nearest-neighbor\ path\ R \in [1, n_{2,4}]^{[0,c_2 n[}\ with\ R_0 < x \\ we\ have\ \ \mathrm{col}(\xi^n)\circ R \notin \{\hat{w}_{x,c_2 n,\rightarrow}, \bar{w}_{x,c_2 n,\rightarrow}\} \end{array} \right\}.$$

Definition 5.14. *For $z \in \mathbb{Z}$ and $m \in \mathbb{N}$ we define $w_{z,m,\rightarrow} := \xi|[z, z+m[$ to be the word of length m starting at z, and we denote by $w_{z,m,\leftarrow}$ the word obtained by reading $w_{z,m,\rightarrow}$ from right to left. We define*

$$B^n_{\text{unique fit}} := \left\{ \begin{array}{l} \forall z_1, z_2 \in \left[-3^{3n}, 3^{3n}\right] \text{ and } \forall i_1, i_2 \in \{\leftarrow, \rightarrow\} \text{ with } (z_1, i_1) \neq (z_2, i_2) \\ \text{we have } w_{z_1, i_1, c_1 n/4} \neq w_{z_2, i_2, c_1 n/4} \end{array} \right\}.$$

Definition 5.15. *We define* $E^{n,\tau}_{\text{Words ok}} := E^{n,\tau}_{\text{only xi}} \cap E^{n,\tau}_{\text{all words}}$ *with*

$$E^{n,\tau}_{\text{only xi}} := \left\{ \text{If } w \in \mathsf{Words}^n(\tau, \chi^n), \text{ then } w \preceq \xi | \left[-3^{3n}, 3^{3n}\right] \right\},$$
$$E^{n,\tau}_{\text{all words}} := \left\{ \begin{array}{l} \text{If } w \preceq \xi | \left[-5 \cdot 3^n, 5 \cdot 3^n\right] \text{ and } |w| = c_1 n/2, \text{ then } \exists w' \in \\ \mathsf{Words}^n(\tau, \chi^n) \text{ with } w \sqsubseteq w' \end{array} \right\}.$$

5.3. Combinatorics

Lemma 5.1. *There exists* $c_{24} > 0$ *such that for all* $n \geq c_{24}$ *the following inclusion holds:*

$$E^{n,\tau}_{\text{stop}} \cap B^n_{\text{blocks bd}} \cap B^{n,\tau}_{\varepsilon} \cap B^n_{\text{functional}} \cap B^n_{\text{scen ok}} \subseteq E^{n,\tau}_{\text{Words ok}}.$$

Proof. Let $n \in \mathbb{N}$ and suppose the events $E^{n,\tau}_{\text{stop}}$, $B^n_{\text{blocks bd}}$, $B^{n,\tau}_{\varepsilon}$, $B^n_{\text{functional}}$, and $B^n_{\text{scen ok}}$ hold.

First we show that $E^{n,\tau}_{\text{only xi}}$ holds: Let $w \in \mathsf{Words}^n(\tau, \chi^n)$. Then there exist positive linear functionals g_1^w and g_3^w such that (5.2), (5.3/5.5), and (5.4) are fulfilled. Since $B^{n,\tau}_{\varepsilon}$ holds, it follows from (5.4) that $\left\|g_1^w \otimes g_3^w\right\|_2 \cdot \|\varepsilon^{n,\tau}_{\xi,S}\|_1^{1/2} \leq e^{-c_7 n}$, which is $\leq 1/(2n^2)$ for all n sufficiently large. Consequently, the assumptions (4.10), (4.11/4.13), and (4.12) of Theorem 4.1 are satisfied, and Theorem 4.1 implies $w \preceq \xi | \left[-3^{3n}, 3^{3n}\right]$ for all n sufficiently large.

It remains to show that $E^{n,\tau}_{\text{all words}}$ holds: Let $I \subseteq \left[-5 \cdot 3^n, 5 \cdot 3^n\right]$ with $|I| = c_1 n/2$. Since $B^n_{\text{scen ok}}$ holds, there exist y, z such that $|y - z| < c_1 n$, $I \subseteq [y, z]$, $\xi_{y-1} \neq \xi_y$, and $\xi_z \neq \xi_{z-1}$. For n sufficiently large, $|y|, |z| \leq 6 \cdot 3^n$. We set $d := z - y + 1$, $w := \xi|[y, z]$, $g_1 := g_1^w := g^{n,l}_{y,\xi}$, and $g_3 := g_3^w := g^{n,r}_{z,\xi}$ and claim that w, g_1, and g_3 satisfy (5.2), (5.3/5.5), and (5.4) with $\eta = \chi^n$ which implies $w \in \mathsf{Words}^n(\tau, \chi^n)$.

Note that $\|g \otimes g'\|_2 = \|g\|_2 \|g'\|_2$ for any g, g'. Using this together with the fact that $B^n_{\text{blocks bd}}$ holds, we obtain

$$\begin{aligned} \|g_1 \otimes g_3\|_2 &\leq 3^{2n} p^{-c_1 n - 2} \prod_{i=1}^{c_1 n} \left[|B^{\rightarrow}_{i,z}| + 1\right] \cdot \left[|B^{\leftarrow}_{i,y}| + 1\right] [\max_{i \in [1,5]} \left\|\vec{x}_i^*\right\|_2]^{2c_1 n} \\ &\leq 3^{2n} p^{-2c_1 n} e^{2c_6 n} [\max_{i \in [1,5]} \left\|\vec{x}_i^*\right\|_2]^{2c_1 n} \leq e^{c_7 n} \end{aligned} \tag{5.7}$$

because $c_1 n \geq 2$ and $c_7 \geq 2 \ln 3 - 2c_1 \ln p + 2c_6 + 2c_1 \ln\left[\max_{i \in [1,5]} \left\|\vec{x}_i^*\right\|_2\right]$ (see Section 2). Hence (5.4) is satisfied.

Next, we verify (5.3) in the case $q \neq 0$. By the definition of $\mu^{n,\tau}_{\xi,S}$, we have

$$\mu^{n,\tau}_{\xi,S}[\cdot \cap E^{n,d-1}_{\text{block}}] \Pi^{-1}_{1,3} = \sum_{x \in [-3^n, 3^n]} a^{n,\tau}_S(x) P_{x,\xi}[(\mathrm{O}^n_1, \mathrm{O}^n_3) \in \cdot, o^n_r = o^n_l + d - 2].$$

With $A_{u,v} := \{S_{o_l^n} = u, S_{o_r^n} = v, o_r^n = o_l^n + d - 2\}$ the following holds

$$\begin{aligned}
&P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_3^n) \in \cdot, o_r^n = o_l^n + d - 2] \\
&= \sum_{\{u,v \in \mathbb{Z}: |u-v| \le d-2\}} P_{x,\xi}\left[A_{u,v}\right] P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_3^n) \in \cdot | A_{u,v}] \\
&= \sum_{\{u,v \in \mathbb{Z}: |u-v| \le d-2\}} P_{x,\xi}\left[A_{u,v}\right] P_{x,\xi}\left[\mathrm{O}_1^n \in \cdot | A_{u,v}\right] \otimes P_{x,\xi}[\mathrm{O}_3^n \in \cdot | A_{u,v}];
\end{aligned}$$

for the last equality we used that O_1^n and O_3^n are independent conditioned on $S_{o_l^n}$ and $S_{o_r^n}$. Let $|u - v| \le d - 2$ such that $P_{x,\xi}(A_{u,v}) > 0$. We cannot have simultaneously $u \le y$ and $v \ge z$ because $z - y = d - 1$. Hence $u > y$ or $v < z$. Recall that we chose $g_1 = g_{y,\xi}^{n,l}$ and $g_3 = g_{z,\xi}^{n,r}$. Since the event $B_{\mathrm{functional}}^n$ holds, g_1 and g_3 are left and right limiting functionals at y and z, respectively. Consequently, $g_1(P_{x,\xi}[\mathrm{O}_1^n \in \cdot | A_{u,v}]) = 0$ or $g_3(P_{x,\xi}[\mathrm{O}_3^n \in \cdot | A_{u,v}]) = 0$, and we conclude

$$(g_1 \otimes 1 \otimes g_3)(\mu_{\xi,S}^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d-1}]) = 0.$$

Hence, because of $\hat{\mu}_{\xi \circ S}^{n,\tau} = \mu_{\xi,S}^{n,\tau} + \varepsilon_{\xi,S}^{n,\tau}$ and the linearity of $g_1 \otimes g_3$, we obtain

$$\begin{aligned}
(g_1 \otimes 1 \otimes g_3)(\hat{\mu}_{\xi \circ S}^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d-1}]) &= g_1 \otimes 1 \otimes g_3(\varepsilon_{\xi,S}^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d-1}]) \\
&\le \|g_1 \otimes g_3\|_2 \cdot \|\varepsilon_{\xi,S}^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d-1}]\|_2. \qquad (5.8)
\end{aligned}$$

Since $\varepsilon_{\xi,S}^{n,\tau}$ is the difference of two probability measures and $B_\varepsilon^{n,\tau}$ holds, $\|\varepsilon_{\xi,S}^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d-1}]\|_2^2 \le \|\varepsilon_{\xi,S}^{n,\tau}\|_1 \le e^{-4c_7 n}$. Thus, using (5.8) and (5.7) yields

$$(g_1 \otimes 1 \otimes g_3)(\hat{\mu}_{\xi \circ S}^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d-1}]) \le e^{-c_7 n} \le 1/(5n^2)$$

for all n sufficiently large. Thus (5.3) holds.

Finally, we check that (5.2) holds for $q \ne 0$. Note that $\|\varepsilon_{\xi,S}^{n,\tau}\|_2 \le \|\varepsilon_{\xi,S}^{n,\tau}\|_1^{1/2} \le e^{-2c_7 n}$ because $\varepsilon_{\xi,S}^{n,\tau}$ is the difference of two probability measures and $B_\varepsilon^{n,\tau}$ holds. Since $\hat{\mu}_\chi^{n,\tau} = \mu_{\xi,S}^{n,\tau} + \varepsilon_{\xi,S}^{n,\tau}$ and $(g_1 \otimes 1_w \otimes g_3)(\varepsilon_{\xi,S}^{n,\tau}) \le \|g_1 \otimes g_3\|_2 \cdot \|\varepsilon_{\xi,S}^{n,\tau}\|_2 \le e^{-c_7 n}$ by (5.7), we obtain

$$(g_1 \otimes 1_w \otimes g_3)(\hat{\mu}_\chi^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d}]) \ge (g_1 \otimes 1_w \otimes g_3)(\mu_{\xi,S}^{n,\tau}[\cdot \cap E_{\mathrm{block}}^{n,d}]) - e^{-c_7 n}. \qquad (5.9)$$

Since $E_{\mathrm{stop}}^{n,\tau}$ holds, $\sum_{x \in [-3^n, 3^n]} a_S^{n,\tau}(x) = 1$. Hence, by the definition of $\mu_{\xi,S}^{n,\tau}$, it suffices to show that

$$(g_1 \otimes 1_w \otimes g_3)(P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_2^n, \mathrm{O}_3^n) \in \cdot, o_r^n = o_l^n + d - 1]) \ge 2 \qquad (5.10)$$

for all $x \in [-3^n, 3^n]$ because (5.10) and (5.9) imply (5.2) for all n sufficiently large.

Let y_0 and z_0 denote the right end of the $c_1 n$th block of $\xi^\leftrightarrow |]\infty, y]$ and $\xi | [z, \infty[$, respectively; recall $\xi_u^\leftrightarrow = \xi_{-u}$. I.e. y_0 is the left end of the $c_1 n$th block in ξ to the left of y. The following picture illustrates this for $c_1 n = 6$. The points y and z are marked with a box.

$$\cdots \underbrace{0}_{y_0}\, 1101000010 \underbrace{\boxed{1}1100001\boxed{0}}_{\xi|[y,z]} 1111001000110 \underbrace{1}_{z_0} \cdots$$

Let $\hat{o}_l^n$ denote the left end of the $c_1 n$th block of χ before o_l^n (here blocks are counted backwards), and let $\hat{o}_r^n$ denote the right end of the $c_1 n$th block of χ after o_r^n. Recall the definitions of $B_{i,y}^{\leftarrow}$ and $B_{i,z}^{\rightarrow}$ from Definition 5.4. We observe

$$\begin{aligned} & P_{x,\xi}\left[\left(\mathrm{O}_1^n, \mathrm{O}_2^n, \mathrm{O}_3^n\right) \in \cdot, o_r^n = o_l^n + d - 1\right] \qquad (5.11) \\ & \geq P_{x,\xi}\left[\left(\mathrm{O}_1^n, \mathrm{O}_2^n, \mathrm{O}_3^n\right) \in \cdot, o_r^n = o_l^n + d - 1, S_{\hat{o}_l^n} = y_0, S_{o_l^n} = y, S_{o_r^n} = z, S_{\hat{o}_r^n} = z_0\right] \\ & = p P_{x,\xi}\left[S_{\hat{o}_l^n} = y_0\right] \bigotimes_{i=1}^{c_1 n} \lambda_r^{|B_{i,y}^{\leftarrow}|} h^{-1} \bigotimes \left[p^d \delta_w\right] \bigotimes_{i=1}^{c_1 n} \lambda_r^{|B_{i,z}^{\rightarrow}|} h^{-1}. \end{aligned}$$

Decomposing (5.11) according to the different possible values for $S_{o_l^n}$ and $S_{o_r^n}$ and using Remark 4.2, we obtain

$$\begin{aligned} & (g_1 \otimes 1_w \otimes g_3)(P_{x,\xi}[(\mathrm{O}_1^n, \mathrm{O}_2^n, \mathrm{O}_3^n) \in \cdot, o_r^n = o_l^n + d - 1]) \\ & \geq (g_1 \otimes 1_w \otimes g_3)\Big(p P_{x,\xi}\left[S_{\hat{o}_l^n} = y_0\right] \bigotimes_{i=1}^{c_1 n} \lambda_r^{|B_{i,y}^{\leftarrow}|} h^{-1} \bigotimes [p^d \delta_w] \bigotimes_{i=1}^{c_1 n} \lambda_r^{|B_{i,z}^{\rightarrow}|} h^{-1}\Big) \\ & = 3^{2n} p^{-c_1 n + d - 1} P_{x,\xi}[S_{\hat{o}_l^n} = y_0] \prod_{i=1}^{c_1 n} \Big([|B_{i,y}^{\leftarrow}| + 1] \cdot \vec{x}^*_{|B_{i,y}^{\leftarrow}| \wedge 5}(\lambda_r^{|B_{i,y}^{\leftarrow}|} h^{-1})\Big) \\ & \quad \cdot \prod_{i=1}^{c_1 n} \Big([|B_{i,z}^{\rightarrow}| + 1] \cdot \vec{x}^*_{|B_{i,z}^{\rightarrow}| \wedge 5}(\lambda_r^{|B_{i,z}^{\rightarrow}|} h^{-1})\Big) \\ & \geq 3^{2n} p^{-1} P_{x,\xi}[S_{\hat{o}_l^n} = y_0]; \qquad (5.12) \end{aligned}$$

for the last estimate we used $d \leq c_1 n$ and the fact that $\vec{x}^*_{m \wedge 5}\left(\lambda_r^m h^{-1}\right) \geq (m+1)^{-1}$ for all $m \geq 1$ by Remark 4.1. Recall the definition of o_l^n. We have that $\hat{o}_l^n$ is the left end of the first block of $\theta^{3^{2n}}(\xi \circ S)$. If $S_{3^{2n}} = y_0$ and $S_{3^{2n}+1} = y_0 + 1$, then $S_{\hat{o}_l^n} = y_0$. (Recall that a block of ξ starts at y_0.) Using this and the local central limit theorem (see e.g. [5] Theorem (5.2), page 132) yields

$$\begin{aligned} 3^{2n} p^{-1} P_{x,\xi}[S_{\hat{o}_l^n} = y_0] & \geq 3^{2n} p^{-1} P_{x,\xi}[S_{3^{2n}} = y_0, S_{3^{2n}+1} = y_0 + 1] \\ & = 3^{2n} P_{x,\xi}[S_{3^{2n}} = y_0] \geq c_{25} 3^{2n} 3^{-n} = c_{25} 3^n \geq 2 \end{aligned}$$

for all $n \geq c_{26}$ with constants $c_{25}, c_{26} > 0$ independent of $x \in \left[-3^n, 3^n\right]$ and y_0; recall that $|y_0| \leq 7 \cdot 3^n$ for all n sufficiently large because $B^n_{\text{blocks bd}}$ holds. The estimate (5.10) follows from (5.12).

In the case $q = 0$, the above proof can be easily adapted. $\square$

Lemma 5.2. *There exists $c_{27} > 0$ such that for all $n \geq c_{27}$ the following inclusion holds:*

$$E^{n,\tau}_{\text{Words ok}} \cap B^n_{\text{unique fit}} \subseteq E^{n,\tau}_{\text{recon Big}}.$$

Proof. Let $n \in \mathbb{N}$, and suppose $E^{n,\tau}_{\text{Words ok}}$ and $B^n_{\text{unique fit}}$ hold. Let $\psi \in \cup_{k \geq n^2} \mathcal{C}^{[-k,k]}$ with $\psi \preceq \xi|[-3^n, 3^n]$. There exist $a \in [-3^n, 3^n]$ and $b \in \{-1, 1\}$ such that

$$\psi_j = \xi_{a+bj} \quad \text{and} \quad a + bj \in \left[-3^n, 3^n\right] \quad \text{for all } j \in [-k, k]. \tag{5.13}$$

We argue that $w := (\xi_{a+bj})_{j \in [-3 \cdot 3^n, 3 \cdot 3^n]} \in \mathsf{Output}^n(\tau, \chi^n, \psi)$: By (5.13), $\psi = w|[-k, k]$. Let $I \subseteq [-3 \cdot 3^n, 3 \cdot 3^n]$ be an integer interval with $|I| = c_1 n/2$. Then the image of I under the map $j \mapsto a + bj$ is again an integer interval, which is contained in $\left[-5 \cdot 3^n, 5 \cdot 3^n\right]$ for all n sufficiently large because $|a| \leq 3^n$ and $c_1 n/2 \leq 3^n$ for all n sufficiently large. Since $E^{n,\tau}_{\text{all words}}$ holds, there exists $w' \in \mathsf{Words}^n(\tau, \chi^n)$ with $w|I \sqsubseteq w'$. Hence $w \in \mathsf{Output}^n(\tau, \chi^n, \psi)$. In particular, $\mathsf{Output}^n(\tau, \chi^n, \psi) \neq \emptyset$.

It remains to show $\xi|\left[-3^n, 3^n\right] \preceq w \preceq \xi|[-4 \cdot 3^n, 4 \cdot 3^n]$ for all $w \in \mathsf{Output}^n(\tau, \chi^n, \psi)$. Let $w \in \mathsf{Output}^n(\tau, \chi^n, \psi)$. Then $w|[-k, k] = \psi$, and consequently, by (5.13),

$$w_j = \xi_{a+bj} \tag{5.14}$$

for all $j \in [-k, k]$. Suppose we prove (5.14) for all $j \in \left[-3 \cdot 3^n, 3 \cdot 3^n\right]$. Then there is precisely one element in $\mathsf{Output}^n(\tau, \chi^n, \psi)$. Since $\psi \preceq \xi|[-3^n, 3^n]$, there are more than $2 \cdot 3^n$ letters to the left and to the right of ψ in w, and consequently $\xi|[-3^n, 3^n] \preceq w$. On the other hand, in w, there are less than $3 \cdot 3^n$ letters to the left and to the right of ψ. Hence $w \preceq \xi|[-4 \cdot 3^n, 4 \cdot 3^n]$.

Thus, to finish the proof, it suffices to verify (5.14) for all $j \in \left[-3 \cdot 3^n, 3 \cdot 3^n\right]$. Suppose we know (5.14) for all $j \in [-s, s]$ for some $s \in \left[k, 3 \cdot 3^n - 1\right]$. This assumption is true for $s = k$. We set $I_l := [-s - 1, -s - 1 + c_1/2[$, $I_r :=]s + 1 - c_1 n/2, s + 1]$, $w_l := w|I_l$, and $w_r := w|I_r$. Note that w_l and w_r have both precisely $c_1 n/2 - 1$ points in common with $w|[-s, s]$; w_l and w_r extend $w|[-s, s]$ one letter to the left and to the right, respectively. The words w_l and w_r are well defined because $c_1 n/2 \leq |\psi| = 2k + 1$ for all n sufficiently large. Since $w \in \mathsf{Output}^n(\tau, \chi^n, \psi)$, there exist $w'_l, w'_r \in \mathsf{Words}^n(\tau, \chi^n)$ with $w_l \sqsubseteq w'_l$, $w_r \sqsubseteq w'_r$. Using that $E^{n,\tau}_{\text{only xi}}$ holds, we see that $w_l, w_r \preceq \xi|\left[-3^{3n}, 3^{3n}\right]$.

Suppose (5.14) does not hold for $j = -s - 1$. Let $I_{l,\xi}$ denote the image of I_l under the map $j \mapsto a + bj$. Then $\xi|I_{l,\xi} \neq w_l$; more precisely, $\xi|I_{l,\xi}$ and w_l disagree in precisely one point, namely the leftmost point. Thus we found two words of length $c_1 n/2$ in $\xi|[-3^{3n}, 3^{3n}]$ which disagree in precisely one point. Consequently, there exist $z, z' \in \left[-3^{3n}, 3^{3n}\right]$, $i, i' \in \{\leftarrow, \rightarrow\}$ with $(z, i) \neq (z', i')$ such that $\xi|I_{l,\xi} = w_{z,i,c_1 n/2}$ and $w_l = w_{z',i',c_1 n/2}$. If we restrict $w_{z,i,c_1 n/2}$ and $w_{z',i',c_1 n/2}$ to the last $c_1 n/4$ letters, we obtain two words of length $c_1 n/4$ in $\xi|\left[-3^{3n}, 3^{3n}\right]$, and these two words agree. This contradicts the fact that the event $B^n_{\text{unique fit}}$ holds. Thus (5.14) holds for $j = -s - 1$.

To see that (5.14) holds for $j = s + 1$, one applies the above argument with $\bar{w}$ defined by $\bar{w}_j := w_{-j}$ for $j \in \left[-3 \cdot 3^n, 3 \cdot 3^n\right]$ in place of w. By the induction principle, (5.14) holds for all $j \in \left[-3 \cdot 3^n, 3 \cdot 3^n\right]$. □

Lemma 5.3. *There exists c_{28} such that for all $n \geq c_{28}$ the following inclusion holds:*

$$B^n_{\text{blocks bd}} \cap B^n_{\text{blocks 2,4}} \cap B^n_{\text{signals}} \subseteq B^n_{\text{functional}}.$$

Proof. The proof will be done by contradiction. Suppose the events $B^n_{\text{blocks bd}}$, $B^n_{\text{blocks 2,4}}$, and B^n_{signals} hold, but $B^n_{\text{functional}} = B^{n,l}_{\text{func}} \cap B^{n,r}_{\text{func}}$ does not hold. Suppose $B^{n,r}_{\text{func}}$ does not hold. Then there exists $y \in [-6 \cdot 3^n, 6 \cdot 3^n]$ with $\xi_y \neq \xi_{y-1}$ such that the right functional at y is not a right limiting functional at y, i.e. there exist $y_1 < y$ with $\xi_{y_1} \neq \xi_{y_1-1}$ and $x \in [-3^n, 3^n]$ such that $g^{\xi,n}_{y,r}(P_{x,\xi}(\mathrm{O}^n_3 \in \cdot | S_{o^n_r} = y)) = 0$ or both $P_{x,\xi}(S_{o^n_r} = y_1) > 0$ and $g^{\xi,n}_{y,r}(P_{x,\xi}(\mathrm{O}^n_3 \in \cdot | S_{o^n_r} = y_1)) \neq 0$ hold.

Let R be an admissible piece of path. If $\xi \circ R$ consists of precisely k blocks, we say that R *generates k blocks on* ξ. We denote by $\xi|[b^R_{i,l}, b^R_{i,r}]$ the block of ξ on which the ith block of $\xi \circ R$ is generated. If $R_{b^R_{i,l}} = R_{b^R_{i,r}}$, we set $j^R_i := l$, otherwise we set $j^R_i := r$. We abbreviate $l^R_i := b^R_{i,r} - b^R_{i,l} - 1$. Using this notation, we have

$$P_{x,\xi}(\mathrm{O}^n_3 \in \cdot, S_{o^n_r} = y) = \sum_{(l_i, j_i)} \bigotimes_{i=1}^{c_1 n} \lambda^{l_i}_{j_i} h^{-1}, \tag{5.15}$$

where the sum is taken over all $(l_i, j_i)_{i \in [1, c_1 n]} \in (\mathbb{N} \times \{l, r\})^{[1, c_1 n]}$ such that there exists an admissible piece of path R starting at y which generates blocks with $(l^R_i, j^R_i) = (l_i, j_i)$. Since $B^n_{\text{blocks bd}}$ holds, the path which starts at y and walks $6c_1 n$ (which is $\leq 3^n$ for all n sufficiently large) steps to the right generates at least $c_1 n$ blocks on ξ, namely $B^{\rightarrow}_{i,y}$, $i \in [1, c_1 n]$. Consequently, by the definition of the right functional of ξ at y and Remark 4.1, we have $g^{\xi,n}_{y,r}(P_{x,\xi}(\mathrm{O}^n_3 \in \cdot, S_{o^n_r} = y)) > 0$. Hence, by our assumption, $P_{x,\xi}(S_{o^n_r} = y_1) > 0$ and $g^{\xi,n}_{y,r}(P_{x,\xi}(\mathrm{O}^n_3 \in \cdot | S_{o^n_r} = y_1)) \neq 0$. Writing $P_{x,\xi}(\mathrm{O}^n_3 \in \cdot, S_{o^n_r} = y_1)$ as a sum as in (5.15), we see that for at least one admissible piece of path R starting at y_1 and generating at least $c_1 n$ blocks on ξ we have $g^{\xi,n}_{y,r}(\otimes_{i=1}^{c_1 n} \lambda^{l^R_i}_{j^R_i} h^{-1}) > 0$. Inserting the definition of $g^{\xi,n}_{y,r}$, we obtain

$$0 < g^{\xi,n}_{y,r}\Big(\bigotimes_{i=1}^{c_1 n} \lambda^{l^R_i}_{j^R_i} h^{-1}\Big) = \prod_{i=1}^{c_1 n} [|B^{\rightarrow}_{i,y}| + 1] \cdot \vec{x}^*_{|B^{\rightarrow}_{i,y}| \wedge 5}\Big(\lambda^{l^R_i}_{j^R_i} h^{-1}\Big).$$

By Remark 4.1, $\vec{x}^*_2(\lambda^m_i h^{-1}) \neq 0$ iff $i = r$ and $m = 2$, and also, $\vec{x}^*_4(\lambda^m_i h^{-1}) \neq 0$ iff $i = r$ and $m = 4$. Furthermore, $x^*_i(\lambda^2_r) = 0$ and $x^*_i(\lambda^4_r) = 0$ for $i \in \{1, 3, 5\}$. Thus $|B^{\rightarrow}_{i,y}| \in \{2, 4\}$ iff $l^R_i \in \{2, 4\}$ and R crosses the block $B^{\rightarrow}_{i,y}$ from left to right. Since $|y| \leq 6 \cdot 3^n$ and $B^n_{\text{blocks bd}}$ holds, we have $B^{\rightarrow}_{i,y} \sqsubseteq \xi | [-7 \cdot 3^n, 7 \cdot 3^n]$ for all n sufficiently large and $i \in [1, c_1 n]$. Using that $B^n_{\text{blocks 2,4}}$ holds, we see that at least $c_2 n$ of the blocks $B^{\rightarrow}_{i,y}$, $i \in [1, c_1 n]$, have length 2 or 4. Hence there are $\geq c_2 n$ blocks with $l^R_i \in \{2, 4\}$.

Clearly, the color of two successive blocks in ξ, and also in the observations, must be different. Hence the colors of the blocks of length 2 or 4 among the first

$c_1 n$ blocks of $\xi \circ R$ either all agree with the colors of the blocks $B^{\rightarrow}_{i,y}$, $i \in [1, c_1 n]$, of length 2 or 4 or they have all the opposite color. But this contradicts the fact that $B^{n,r}_{\text{sign}}$ holds. A similar argument shows that the assumption that $B^{n,l}_{\text{func}}$ holds leads to a contradiction. □

5.4. Probabilistic estimates

In this section, we prove that the complements of all the basic events $B^n_{\dots}$ defined in Section 5.2 have a probability which is exponentially small in n; for some events this is only true under the assumption that $E^{n,\tau}_{\text{stop}}$ holds. We treat the events in alphabetical order.

Lemma 5.4. *There exist $c_{29}, c_{30} > 0$ such that for all $n \geq c_{29}$*

$$P\left(\left[B^n_{\text{blocks bd}}\right]^c\right) \leq 2e^{-c_{30}n}.$$

Proof. By the definition of $B^n_{\text{blocks bd}} = B^{n,\rightarrow}_{\text{bb}} \cap B^{n,\leftarrow}_{\text{bb}}$,

$$\left[B^{n,\rightarrow}_{\text{bb}}\right]^c = \bigcup_{z \in [-2\cdot 3^{3n}, 2\cdot 3^{3n}]} \left\{\prod_{i=1}^{c_1 n}[|B^{\rightarrow}_{i,z}| + 1] > e^{c_6 n}\right\} \cup \left\{\sum_{i=1}^{c_1 n}[|B^{\rightarrow}_{i,z}| + 2] > 8c_1 n\right\}.$$

For each z, the block lengths $|B^{\rightarrow}_{i,z}|$, $i \geq 1$, are i.i.d. with $P(|B^{\rightarrow}_{i,z}| = k) = 2^{-k}$, $k \geq 1$; in particular $E|B^{\rightarrow}_{i,z}| = 2$. By Chebyshev's inequality, we obtain

$$P\left(\prod_{i=1}^{c_1 n}[|B^{\rightarrow}_{i,z}| + 1] > e^{c_6 n}\right) \leq e^{-c_6 n} E\left(\prod_{i=1}^{c_1 n}[|B^{\rightarrow}_{i,z}| + 1]\right) = 3^{c_1 n} e^{-c_6 n}.$$

Furthermore, by the large deviation principle, we have

$$P\left(\sum_{i=1}^{c_1 n}[|B^{\rightarrow}_{i,z}| + 2] > 8c_1 n\right) = P\left(\sum_{i=1}^{c_1 n}|B^{\rightarrow}_{i,z}| > 6c_1 n\right) \leq e^{-c_1 n I(6)}$$

with the rate function $I(x) = (x-1)\ln(x-1) + x\ln(2/x)$. Since $I(6) > 1$, we conclude

$$P([B^{n,\rightarrow}_{\text{bb}}]^c) \leq (4 \cdot 3^{3n} + 1)\left[3^{c_1 n} e^{-c_6 n} + e^{-c_1 n}\right] \leq e^{-c_{30} n}$$

for some constant $c_{30} > 0$ for all n sufficiently large; here we used that $c_6 - (c_1 + 4)\ln 3 > 0$ by our choice of c_6 and $c_1 > 4\ln 3$. The same estimate holds for $P([B^{n,\leftarrow}_{\text{bb}}]^c)$. □

Lemma 5.5. *There exist $c_{31} > 0$ such that for all $n \in \mathbb{N}$*

$$P\left(\left[B^n_{\text{blocks 2,4}}\right]^c\right) \leq 14e^{-c_{31}n}.$$

Proof. Recall that for all z, the block lengths $|B^{\rightarrow}_{i,z}|$, $i \geq 1$, are i.i.d. with $P(|B^{\rightarrow}_{i,z}| = k) = 2^{-k}$, $k \geq 1$. Hence $P(|B^{\rightarrow}_{i,z}| \in \{2,4\}) = 2^{-2} + 2^{-4} = 5/16$. Let Y_k, $k \geq 1$, be i.i.d. Bernoulli with parameter $5/16$, and let $J(x) := (1-x)\ln\left(\frac{16(1-x)}{11}\right) + x\ln\left(\frac{16x}{5}\right)$. By the large deviation principle (see e.g. [5]), $P(\sum_{k=1}^{c_1 n} Y_k \leq c_1 n/4) \leq e^{-J(1/4)c_1 n}$. Since $c_2 < c_1/4$ and there are at most $14 \cdot 3^n$ sequences of $c_1 n$ consecutive blocks in $\xi|\left[-7\cdot 3^n, 7\cdot 3^n\right]$, we have

$$P\left(\left[B^n_{\text{blocks }2,4}\right]^c\right) \leq 14\cdot 3^n e^{-J(1/4)c_1 n} \leq 14 e^{-c_{31} n}$$

because $J(1/4)c_1 - \ln 3 > 0$. □

Recall that $3^{\alpha n} a_S^{n,\tau}(x)$ equals the number of stopping times τ_k, $k \in \left[1, 3^{\alpha n}\right]$, with $S_{\tau_k} = x$. The following lemma, which will be needed in the proof of Lemma 5.8, states that with very high probability, the stopping times stop often in x provided the event $E^{n,\tau}_{\text{stop}}$ holds.

Lemma 5.6. *There exists $c_{32} > 0$ such that for all $n \geq c_{32}$*

$$P\left(E^{n,\tau}_{\text{stop}} \cap \bigcup_{x\in[-3^n,3^n]} \left\{3^{\alpha n} a_S^{n,\tau}(x) \leq 3^{17c_1 n} e^{16 c_7 n}\right\}\right) \leq e^{-n}.$$

Proof. The proof is very similar to the proof of Lemma 6.14 in [24]. In the notation of [24], the estimate holds whenever $\alpha > 1 + \gamma - [3c_1 \ln p]/\ln 3$ with $\gamma := 17c_1 + 16c_7/\ln 3$, which is satisfied by our choice of α (see Section 2). □

The following basic large deviation estimate will be needed below.

Lemma 5.7. *Let X_i, $i \geq 1$, be i.i.d. Bernoulli with parameter δ, and let $\sigma_m := \sum_{i=1}^m X_i$. There exists a constant $c_{33} > 0$ such that for all $m \in \mathbb{N}$ and all $a > 0$*

$$P(\sigma_m \geq m(a+\delta)) \leq e^{-c_{33} m a^2}.$$

Proof. By the large deviation principle (see e.g. [5]), we have $P(\sigma_m \geq m(a+\delta)) \leq e^{-mI_\delta(a+\delta)}$ with the rate function $I_\delta(a) = a\ln\left(\frac{a}{\delta}\right) + (1-a)\ln\left(\frac{1-a}{1-\delta}\right)$. One verifies that $I_\delta(a+\delta) \geq c_{33}a^2$ for all $\delta \in]0,1[$ and $a \in]0, 1-\delta[$ with a constant $c_{33} > 0$ independent of δ and a. □

Lemma 5.8. *There exist constants $c_{34}, c_{35}, c_{36} > 0$ such that*

$$P\left(E^{n,\tau}_{\text{stop}} \setminus B^{n,\tau}_{\varepsilon}\right) \leq c_{35} e^{-c_{36} n} \quad \textit{for all } n \geq c_{34}.$$

Proof. We define for $x \in \left[-3^n, 3^n\right]$

$$\hat{\mu}^{n,\tau}_{x,\xi\circ S} := \left[3^{\alpha n} a_S^{n,\tau}(x)\right]^{-1} \sum_{k\in[1,3^{\alpha n}]} 1\{S_{\tau_k} = x\}\delta_{\mathrm{O}^n(\theta^{\tau_k}\chi)},$$

i.e. $\hat{\mu}^{n,\tau}_{x,\xi\circ S}$ is the empirical distribution of the O^n collected after times τ_k with $S_{\tau_k} = x$. Suppose the event $E^{n,\tau}_{\mathrm{stop}}$ holds. Then $|S_{\tau_k}| \leq 3^n$ for all $k \in \left[1, 3^{\alpha n}\right]$, and consequently

$$\varepsilon^{n,\tau}_{\xi,S} = \sum_{x\in[-3^n,3^n]} a^{n,\tau}_S(x) \left[\hat{\mu}^{n,\tau}_{x,\xi\circ S} - P_{x,\xi}\left[\mathrm{O}^n(\chi)\right]^{-1}\right].$$

By the triangle inequality,

$$\|\varepsilon^{n,\tau}_{\xi,S}\|_1 \leq \sum_{x\in[-3^n,3^n]} a^{n,\tau}_S(x) \left\|\hat{\mu}^{n,\tau}_{x,\xi\circ S} - P_{x,\xi}\left[\mathrm{O}^n(\chi)\right]^{-1}\right\|_1. \tag{5.16}$$

Let $\mathcal{S}$ denote the set of possible states of the random variable $\mathrm{O}^n(\chi)$ if $|\mathrm{O}^n_2(\chi)| \leq 3^n$, and let $\mathcal{S}'$ be the set of possible states of $\mathrm{O}^n(\chi)$ if $|\mathrm{O}^n_2(\chi)| > 3^n$. Recall that $\mathrm{O}^n = (\mathrm{O}^n_1, \mathrm{O}^n_2, \mathrm{O}^n_3)$ where $\mathrm{O}^n_1, \mathrm{O}^n_3 \in \{1, 2, \ldots, 5\}^{c_1 n}$ and O^n_2 is the concatenation of a word of length $< c_1 n/2$ with a block. Consequently, $|\mathcal{S}| \leq 5^{2c_1 n} 2^{c_1 n} 3^n \leq 2^{8c_1 n}$.

Recall the definition of $B^{n,\tau}_{\mathrm{O}_2}$ from Definition 5.10. Clearly,

$$P\left(E^{n,\tau}_{\mathrm{stop}} \setminus B^{n,\tau}_{\varepsilon}\right) \leq P\left(\left[E^{n,\tau}_{\mathrm{stop}} \cap B^{n,\tau}_{\mathrm{O}_2}\right] \setminus B^{n,\tau}_{\varepsilon}\right) + P\left(E^{n,\tau}_{\mathrm{stop}} \setminus B^{n,\tau}_{\mathrm{O}_2}\right). \tag{5.17}$$

We split the sum in (5.16) in two parts. Let

$$J_{\mathrm{seldom}} := \left\{x \in \left[-3^n, 3^n\right] : 3^{\alpha n} a^{n,\tau}_S(x) \leq 3^n |\mathcal{S}|^2 e^{16 c_7 n}\right\}, \quad J_{\mathrm{often}} := \left[-3^n, 3^n\right] \setminus J_{\mathrm{seldom}}.$$

By the definition of J_{seldom}, we have

$$\sum_{x\in J_{\mathrm{seldom}}} a^{n,\tau}_S(x) \left\|\hat{\mu}^{n,\tau}_{x,\xi\circ S} - P_{x,\xi}\left[\mathrm{O}^n(\chi)\right]^{-1}\right\|_1 \leq 3^{2n} 3^{(1-\alpha)n} 2^{16 c_1 n} e^{16 c_7 n} \leq e^{-8 c_7 n}, \tag{5.18}$$

where the last inequality follows from our choice of α. Next, we define the event that the contribution to $\|\varepsilon^{n,\tau}_{\xi,S}\|_1$ coming from $\mathrm{O}^n = s \in \mathcal{S}$ is small: We set for $x \in \left[-3^n, 3^n\right]$ and $s \in \mathcal{S}$

$$B^{n,\tau,s}_{x\ \mathrm{often}} := \left\{\text{If } x \in J_{\mathrm{often}},\ \text{then } \left|\hat{\mu}^{n,\tau}_{x,\xi\circ S}(\{s\}) - P_{x,\xi}\left[\mathrm{O}^n(\chi)\right]^{-1}(\{s\})\right| \leq |\mathcal{S}|^{-1} e^{-8 c_7 n}\right\}.$$

If the event $\cap_{x\in[-3^n,3^n]} \cap_{s\in\mathcal{S}} B^{n,\tau,s}_{x\ \mathrm{often}}$ holds, then

$$\sum_{x\in J_{\mathrm{often}}} a^{n,\tau}_S(x) \sum_{s\in\mathcal{S}} \left|\hat{\mu}^{n,\tau}_{x,\xi\circ S}(\{s\}) - P_{x,\xi}\left[\mathrm{O}^n(\chi)\right]^{-1}(\{s\})\right| \leq e^{-8 c_7 n}. \tag{5.19}$$

If the event $B^{n,\tau}_{\mathrm{O}_2}$ holds, then $\hat{\mu}^{n,\tau}_{x,\xi\circ S}(\{s\}) = 0$ for all $s \in \mathcal{S}'$ and consequently,

$$\begin{aligned}
&\sum_{x\in J_{\mathrm{often}}} a^{n,\tau}_S(x) \sum_{s\in\mathcal{S}'} \left|\hat{\mu}^{n,\tau}_{x,\xi\circ S}(\{s\}) - P_{x,\xi}\left[\mathrm{O}^n(\chi)\right]^{-1}(\{s\})\right| \\
&\leq \sum_{x\in J_{\mathrm{often}}} a^{n,\tau}_S(x) P_{x,\xi}(|\mathrm{O}^n_2(\chi)| > 3^n) \leq e^{-8 c_7 n}.
\end{aligned}$$

Combining the last estimate with (5.19) and (5.18), we obtain

$$\begin{aligned} E^{n,\tau}_{\text{stop}} \cap B^{n,\tau}_{\mathrm{O}_2} \cap \bigcap_{x\in[-3^n,3^n]} \bigcap_{s\in\mathcal{S}} B^{n,\tau,s}_{x\text{ often}} &\subseteq E^{n,\tau}_{\text{stop}} \cap B^{n,\tau}_{\mathrm{O}_2} \cap \left\{ \|\varepsilon^{n,\tau}_{\xi,S}\|_1 \le 3e^{-8c_7 n} \right\} \\ &\subseteq E^{n,\tau}_{\text{stop}} \cap B^{n,\tau}_{\mathrm{O}_2} \cap B^{n,\tau}_{\varepsilon} \end{aligned}$$

for all n sufficiently large. Hence, using $\Omega = \{x \in J_{\text{seldom}}\} \cup \{x \in J_{\text{often}}\}$, we obtain

$$\begin{aligned} P\left(\left[E^{n,\tau}_{\text{stop}} \cap B^{n,\tau}_{\mathrm{O}_2}\right] \setminus B^{n,\tau}_{\varepsilon}\right) &\le P\Big(E^{n,\tau}_{\text{stop}} \cap B^{n,\tau}_{\mathrm{O}_2} \cap \bigcup_{x\in[-3^n,3^n]} \bigcup_{s\in\mathcal{S}} \left[B^{n,\tau,s}_{x\text{ often}}\right]^c\Big) \quad (5.20) \\ &\le P\Big(E^{n,\tau}_{\text{stop}} \cap \bigcup_{x\in[-3^n,3^n]} \{x \in J_{\text{seldom}}\}\Big) \\ &\quad + P\Big(\bigcup_{x\in[-3^n,3^n]} \bigcup_{s\in\mathcal{S}} \left[\{x \in J_{\text{often}}\} \setminus B^{n,\tau,s}_{x\text{ often}}\right]\Big) \\ &\le P\Big[E^{n,\tau}_{\text{stop}} \cap \bigcup_{x\in[-3^n,3^n]} \{x \in J_{\text{seldom}}\}\Big] \\ &\quad + 3^{2n}|\mathcal{S}| \max_{x\in[-3^n,3^n], s\in\mathcal{S}} P\Big[\{x \in J_{\text{often}}\} \setminus B^{n,\tau,s}_{x\text{ often}}\Big]. \end{aligned}$$

It follows from $|\mathcal{S}| \le 2^{8c_1 n}$ and Lemma 5.6 that for all $n \ge c_{32}$

$$\begin{aligned} & P\Big[E^{n,\tau}_{\text{stop}} \cap \bigcup_{x\in[-3^n,3^n]} \{x \in J_{\text{seldom}}\}\Big] \\ & \le P\Big[E^{n,\tau}_{\text{stop}} \cap \bigcup_{x\in[-3^n,3^n]} \left\{3^{\alpha n} a^{n,\tau}_S(x) \le 3^{17c_1 n} e^{16c_7 n}\right\}\Big] \\ & \le e^{-n}. \end{aligned} \tag{5.21}$$

We introduce the stopping times τ^x_k when the random walker is at x: $\tau^x_1 := \min\{\tau_i : i \in \left[1, 3^{\alpha n}\right], S_{\tau_i} = x\}$, $\tau^x_{k+1} := \min\left\{\tau_i > \tau^x_k : i \in \left[1, 3^{\alpha n}\right], S_{\tau_i} = x\right\}$. The random variables $\chi|\left[\tau^x_k + 3^{2n}, \tau^x_k + 3^{3n}\right[$, $k \in [1, j]$, are i.i.d. conditioned on $E^{n,\tau}_{\text{stop}}$. Hence, by the definition of $\hat{\mu}^{n,\tau}_{x,\xi\circ S}$, $P\left(\{x \in J_{\text{often}}\} \setminus B^{n,\tau,s}_{x\text{ often}}|E^{n,\tau}_{\text{stop}}\right)$ equals a large deviation probability for sums of Bernoulli random variables and we can apply Lemma 5.7 with $m = 3^{\alpha n} a^{n,\tau}_S(x) > 3^n|\mathcal{S}|^2 e^{16c_7 n}$ and $a = |\mathcal{S}|^{-1} e^{-8c_7 n}$. Since for this choice, $ma^2 > 3^n$ we obtain

$$P\left(\{x \in J_{\text{often}}\} \setminus B^{n,\tau,s}_{x\text{ often}}\right) \le \exp(-c_{33} 3^n). \tag{5.22}$$

Combining (5.20) with (5.21), $|\mathcal{S}| \le 2^{8c_1 n}$, and (5.22), we conclude

$$P\left(\left[E^{n,\tau}_{\text{stop}} \cap B^{n,\tau}_{\mathrm{O}_2}\right] \setminus B^{n,\tau}_{\varepsilon}\right) \le 2e^{-n} \tag{5.23}$$

for all $n \ge c_{34}$ with some constant $c_{34} \ge c_{32}$. The claim of the lemma follows from (5.17), (5.23), and Lemma 5.10. □

Lemma 5.9. *There exist* $c_{37}, c_{38}, c_{39} > 0$ *such that for all* $n \geq c_{37}$

$$P\left([B^n_{\text{functional}}]^c\right) \leq c_{38} e^{-c_{39} n}.$$

Proof. By Lemma 5.3, $B^n_{\text{functional}} \subseteq \left[B^n_{\text{blocks bd}}\right]^c \cup \left[B^n_{\text{blocks 2,4}}\right]^c \cup \left[B^n_{\text{signals}}\right]^c$. The claim follows immediately from Lemmas 5.4, 5.5, and 5.12. □

Lemma 5.10. *There exist* $c_{40}, c_{41}, c_{42} > 0$ *such that for all* $n \geq c_{40}$

$$P\left(E^{n,\tau}_{\text{stop}} \setminus B^{n,\tau}_{\text{O}_2}\right) \leq c_{41} e^{-c_{42} n}.$$

Proof. Clearly,

$$P\left(E^{n,\tau}_{\text{stop}} \setminus B^{n,\tau}_{\text{O}_2}\right) \leq P\left(\left[E^{n,\tau}_{\text{stop}} \cap B^n_{\text{blocks bd}}\right] \setminus B^{n,\tau}_{\text{O}_2}\right) + P\left(\left[B^n_{\text{blocks bd}}\right]^c\right). \quad (5.24)$$

Recall that $B^{n,\tau}_{\text{O}_2} = B^{n,\tau}_{\text{O}_2\text{ small}} \cap B^n_{\text{O}_2\text{ large}}$. By definition,

$$\begin{aligned}
&P\left(\left[E^{n,\tau}_{\text{stop}} \cap B^n_{\text{blocks bd}}\right] \setminus B^{n,\tau}_{\text{O}_2\text{ small}}\right) \\
&\leq 3^{\alpha n} \max_{x\in[-3^n,3^n]} P_x(B^n_{\text{blocks bd}} \cap \{|\text{O}^n_2(\chi)| > 3^n\}) \\
&= 3^{\alpha n} \max_{x\in[-3^n,3^n]} E_x\left[1 B^n_{\text{blocks bd}} P_{x,\xi}(|\text{O}^n_2(\chi)| > 3^n)\right]. \qquad (5.25)
\end{aligned}$$

Let $x \in \left[-3^n, 3^n\right]$. Suppose the random walk starts at x and $|\text{O}^n_2(\chi)| > 3^n$. Then $\chi | \left[0, 3^{3n}\right[$ contains a block of length $\geq 3^n - c_1 n$ and this block must be generated on $\xi | \left[-2 \cdot 3^{3n}, 2 \cdot 3^{3n}\right]$. If $B^n_{\text{blocks bd}}$ holds, all blocks of $\xi | \left[-2 \cdot 3^{3n}, 2 \cdot 3^{3n}\right]$ have length $\leq 6c_1 n$. Consequently, the random walk stays time $t \geq 3^n - c_1 n$ in an interval I of length $\leq 6c_1 n$. It is known (see e.g.[23], Lemma 5.2) that

$$P(S_i \in I \text{ for all } i \in [0, t[) \leq c_{43} \exp(-c_{44} t/|I|^2)$$

with constants $c_{43}, c_{44} > 0$. Thus it follows from (5.25)

$$\begin{aligned}
P\left(\left[E^{n,\tau}_{\text{stop}} \cap B^n_{\text{blocks bd}}\right] \setminus B^{n,\tau}_{\text{O}_2\text{ small}}\right) &\leq c_{43} 3^{\alpha n} \exp\left[-\frac{c_{44}[3^n - c_1 n]}{36 c_1^2 n^2}\right] \\
&\leq e^{-n} \qquad (5.26)
\end{aligned}$$

for all n sufficiently large. Furthermore, by the above argument, $\left[E^{n,\tau}_{\text{stop}} \cap B^n_{\text{blocks bd}}\right] \setminus B^n_{\text{O}_2\text{ large}} = \emptyset$ for all n sufficiently large. Thus $P\left(\left[E^{n,\tau}_{\text{stop}} \cap B^n_{\text{blocks bd}}\right] \setminus B^{n,\tau}_{\text{O}_2}\right) \leq e^{-n}$ for all n sufficiently large. The claim follows from (5.24) and Lemma 5.4. □

Lemma 5.11. *There exist* $c_{45}, c_{46} > 0$ *such that for all* $n \geq c_{45}$

$$P\left(\left[B^n_{\text{scen ok}}\right]^c\right) \leq 12 e^{-c_{46} n}.$$

Proof. It is not hard to see that for all n sufficiently large, $B^n_{\text{scen ok}}$ contains the event $\{$All blocks of $\xi\,|\,[-6\cdot 3^n, 6\cdot 3^n]$ have length $\le c_1 n/4\}$. Consequently,

$$\begin{aligned} P\left(\left[B^n_{\text{scen ok}}\right]^c\right) &\le P\left(\exists \text{ block of } \xi\,|\,[-6\cdot 3^n, 6\cdot 3^n] \text{ of length } > c_1 n/4\right) \\ &\le 12\cdot 3^n\cdot 2^{-c_1 n/4}; \end{aligned}$$

here we used that there are $\le 12\cdot 3^n$ possible left endpoints for a block in $\xi\,|\,[-6\cdot 3^n, 6\cdot 3^n]$ and that the probability that a block starting at x has length $> c_1 n/4$ equals $2^{-c_1 n/4}$ because the scenery is i.i.d. uniformly colored. The claim follows because $c_1 > 4\ln 3/\ln 2$. □

Lemma 5.12. *There exists $c_{47} > 0$ such that for all $n \in \mathbb{N}$*

$$P\left(\left[B^n_{\text{signals}}\right]^c\right) \le 60e^{-c_{47}n}.$$

Proof. Recall the notation introduced in Definitions 5.12 and 5.13. Let $y := -7\cdot 3^n$. The sequence $(|B^{2,4}_{i,y}|, C^{2,4}_{i,y})_{i\ge 1}$ is a Markov chain under P with time-homogeneous transition probabilities. The block lengths $(|B^{2,4}_{i,y}|)_{i\ge 1}$ are i.i.d. with $P(|B^{2,4}_{i,y}| = 2) = 2^{-2}/(2^{-2}+2^{-4}) = 4/5$ and $P(|B^{2,4}_{i,y}| = 4) = 1/5$ and independent of the colors $(C^{2,4}_{i,y})_{i\ge 1}$. Note that $C^{2,4}_{i,y} \ne C^{2,4}_{i+1,y}$ iff between $B^{2,4}_{i,y}$ and $B^{2,4}_{i+1,y}$ there are $2k$ blocks of length 1, 3, or 5 for some $k \ge 0$. Recall the definition of $B^{\to}_{i,y}$ from Definition 5.4. Let $p_{2,4} := P(|B^{\to}_{i,y}| \in \{2,4\}) = 2^{-2}+2^{-4} = 5/16$ and set $q_{2,4} := 1 - p_{2,4} = 11/16$. Then

$$P(C^{2,4}_{i,y} \ne C^{2,4}_{i+1,y}) = \sum_{k=0}^{\infty} q_{2,4}^{2k} p_{2,4} = \frac{p_{2,4}}{1-q_{2,4}^2} = \frac{1}{1+q_{2,4}} = \frac{16}{27}$$

and $P(C^{2,4}_{i,y} = C^{2,4}_{i+1,y}) = 11/27$. Hence the one-step transition probabilities of the Markov chain $\text{col}_i(\xi^n)$, $i \ge 1$, are $\le \frac{4}{5}\cdot\frac{16}{27} = \frac{64}{135} < \frac{1}{2}$.

Let $x \in [1, n_{2,4}]$, let $R \in [1, n_{2,4}]^{[0,c_2 n[}$ be a nearest-neighbor path with $R_0 < x$, and let $w \in \{\hat{w}_{x,c_2n,\to}, \bar{w}_{x,c_2n,\to}\}$, $w = (w_i)_{i\in[0,c_2n[}$. We set $\mathcal{H}_k := \sigma(\text{col}_i(\xi^n); i \in [1,k])$. Clearly, $w_k \in \mathcal{H}_{x+k}$. Since R is a nearest-neighbor path, $R_k < x+k$ for all k; hence $\text{col}_{R_k} \in \mathcal{H}_{x+k-1}$ for all k. Using that w_k, $k \in [0, c_2n[$, is a Markov chain with the above specified transition probabilities, we obtain

$$\begin{aligned} P(\text{col}(\xi^n)\circ R = w) &= P(\text{col}_i(\xi^n) = w_i \ \forall i \in [0, c_2 n[) \\ &\le \prod_{i=0}^{c_2n-2} P(\text{col}_{i+1}(\xi^n) = w_{i+1} | \text{col}_i(\xi^n) = w_i) \\ &\le \left(\frac{64}{135}\right)^{c_2 n-1}. \end{aligned}$$

There are $n_{2,4} \leq 14 \cdot 3^n$ possibilities to choose x and $2^{c_2 n-1}$ possibilities to choose R. Thus, by the definition of $B^{n,r}_{\text{sign}}$,

$$
\begin{aligned}
P\left(\left[B^{n,r}_{\text{sign}}\right]^c\right) &\leq 2 \cdot 14 \cdot 3^n 2^{c_2 n-1} \cdot \left(\frac{64}{135}\right)^{c_2 n-1} \\
&\leq 30 \cdot 3^n 2^{c_2 n}\left(\frac{64}{135}\right)^{c_2 n} \\
&\leq 30 e^{-c_{47} n}
\end{aligned}
$$

for some constant $c_{47} > 0$ because $64/135 < 1/2$ and $c_2 > \ln 3/(\ln(135/128))$. The same estimate holds for $B^{n,l}_{\text{sign}}$, and the claim follows from the definition of $B^n_{\text{signals}} = B^{n,l}_{\text{sign}} \cap B^{n,r}_{\text{sign}}$. $\square$

Lemma 5.13. *There exists $c_{48} > 0$ such that for all $n \in \mathbb{N}$*

$$
P\left(\left[B^n_{\text{unique fit}}\right]^c\right) \leq 4e^{-c_{48} n}.
$$

Proof. Let $z_1, z_2 \in \left[-3^{3n}, 3^{3n}\right]$ and $i_1, i_2 \in \{\leftarrow, \rightarrow\}$ with $(z_1, i_1) \neq (z_2, i_2)$. For $k = 1, 2$, we set $o_k := +1$ if $i_k = \rightarrow$, $o_k := -1$ if $i_k = \leftarrow$, and we define $f_k(j) := z_k + o_k j$ for $j \in [0, c_1 n/4[$. As is shown in the proof of Lemma 6.8 of [24], there exists a subset $J \subseteq [0, c_1 n/4[$ of cardinality $|J| \geq c_1 n/12$ such that $f_1(J) \cap f_2(J) = \emptyset$. Consequently,

$$
\begin{aligned}
P(w_{z_1,i_1,c_1 n/4} = w_{z_2,i_2,c_1 n/4}) &\leq P(w_{z_1,i_1,c_1 n/4}|f_1(J) = w_{z_2,i_2,c_1 n/4}|f_2(J)) \\
&= 2^{-c_1 n/12}.
\end{aligned}
$$

Since there are $\leq (2 \cdot 3^{3n} + 1)^2 \leq 3^{8n}$ possibilities to choose z_1 and z_2 and ≤ 4 possibilities to choose i_1 and i_2, we conclude

$$
P\left(\left[B^n_{\text{unique fit}}\right]^c\right) \leq 4 \cdot 3^{8n} 2^{-c_1 n/12} \leq 4e^{-c_{48} n}
$$

for some constant $c_{48} > 0$ because $c_1 > 96 \ln 3/\ln 2$. $\square$

5.4.1. Proof of Theorem 3.1

Proof of Theorem 3.1 Combining Lemmas 5.2, 5.1, and 5.3 we obtain

$$
\begin{aligned}
E^{n,\tau}_{\text{stop}} \cap B^n_{\text{blocks bd}} \cap B^n_{\text{blocks 2,4}} \cap B^{n,\tau}_{\varepsilon} \cap B^n_{\text{functional}} \cap B^n_{\text{scen ok}} \cap B^n_{\text{signals}} \cap B^n_{\text{unique fit}} \\
\subseteq E^{n,\tau}_{\text{recon Big}}
\end{aligned}
$$

for all n sufficiently large. Hence

$$
\begin{aligned}
E^{n,\tau}_{\text{stop}} \setminus E^{n,\tau}_{\text{recon Big}} \subseteq \left[B^n_{\text{blocks bd}}\right]^c \cup \left[B^n_{\text{blocks 2,4}}\right]^c \cup \left[E^{n,\tau}_{\text{stop}} \setminus B^{n,\tau}_{\varepsilon}\right] \cup \left[B^n_{\text{functional}}\right]^c \\
\cup \left[B^n_{\text{scen ok}}\right]^c \cup \left[B^n_{\text{signals}}\right]^c \cup \left[B^n_{\text{unique fit}}\right]^c.
\end{aligned}
$$

The claim follows from Lemmas 5.4, 5.5, 5.8, 5.9, 5.11, 5.12, and 5.13. $\square$

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