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# Stability of the Absolutely Continuous Spectrum of Random Schrödinger Operators on Tree Graphs

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**Abstract.** The subject of this work is random Schrödinger operators on regular rooted tree graphs $\mathbb{T}$ with stochastically homogeneous disorder. The operators are of the form $H_\lambda(\omega) = T + U + \lambda V(\omega)$ acting in $\ell^2(\mathbb{T})$, with $T$ the adjacency matrix, $U$ a radially periodic potential, and $V(\omega)$ a random potential. This includes the only class of homogeneously random operators for which it was proven that the spectrum of $H_\lambda(\omega)$ exhibits an absolutely continuous (ac) component; a results established by A. Klein for weak disorder in case $U = 0$ and $V(\omega)$ given by iid random variables on $\mathbb{T}$. Our main contribution is a new method for establishing the persistence of ac spectrum under weak disorder. The method yields the continuity of the ac spectral density of $H_\lambda(\omega)$ at $\lambda = 0$. The latter is shown to converge in the $L^1$-sense over closed Borel sets in which $H_0$ has no singular spectrum. The analysis extends to random potentials whose values at different sites need not be independent, assuming only that their joint distribution is weakly correlated across different tree branches.

## Contents



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## 1. Introduction

The objective of this work is to present results on the stability of the absolutely continuous spectrum of Schrödinger operators on tree graphs, under the addition of weak but extensive disorder in the form of a random potential.

The background for this analysis is the generally known phenomenon of *Anderson localization*: the addition of extensive disorder to a linear operator results in the localization of the eigenfunctions corresponding to certain spectral regimes, where the spectral type changes to *pure-point*. The localization regime may cover the full spectral range – as is typically the case in one dimension even at arbitrarily small, but non-zero strength of the disorder [6, 14, 9, 26, 31]. A major challenge for analysts is to shed light on extended states and ac spectrum. We shall not review here the growing body of interesting works on decaying disorder, as our main focus concerns the homogeneous case. In this case the only proof of the persistence of *de-localization* – in the sense of the existence of extended states, or of absolutely continuous (ac) spectrum in a certain energy range – was obtained for the Laplacian on a regular tree perturbed by a weak random potential which is given by a collection of iid random variables [18, 20]. In this work we return to the tree setup and present a different set of tools.

### 1.1. *Statement of the main result*

We consider random Schrödinger operators on the Hilbert space $\ell^2(\mathbb{T})$ where $\mathbb{T}$ is the set of vertices of a regular rooted tree graph in which each vertex has $K \geq 2$ forward neighbors (see Subsection 1.2 for some of the basic terminology). These operators are linear and of the form

$$H_\lambda(\omega) := T + U + \lambda\, V(\omega) \tag{1.1}$$

where:

1. The operator $T$ corresponds to the adjacency matrix, i.e., the discrete version of the Laplacian without the diagonal terms:

$$\big(T\psi\big)_x := \sum_y \psi_y \qquad \text{for all } \psi \in \ell^2\big(\mathbb{T}\big), \tag{1.2}$$

   where the sum runs over all nearest neighbor vertices of $x \in \mathbb{T}$.
2. The term $U$ is a multiplication operator by a real-valued function $\{U_x\}_{x\in\mathbb{T}}$ which is radial and $\tau$-periodic in $|x|$, the distance to the root.
3. The real parameter $\lambda$ controls the strength of the random perturbation.

4. The symbol $\omega$ represents the randomness, i.e., $V(\omega)$ is a multiplication operator which is given in terms of an element $\{\omega_x\}_{x\in\mathbb{T}}$ from the probability space $\left(\mathbb{R}^{\mathbb{T}}, \mathbb{P}\right)$. Averages over that probability space will be denoted below by $\mathbb{E}[\,\cdot\,]$.

For each $\lambda$ and $\omega$ the operator $H_\lambda(\omega)$ is essentially self adjoint on the domain of functions of compact support. It is important for our discussion that the unperturbed part

$$H_0 = T + U \tag{1.3}$$

is a radially periodic Schrödinger operator on $\ell^2(\mathbb{T})$ in the sense described in 1. and 2. above. In order to prepare for the statement of our main result, let us note two facts about the spectra of such operators (see Appendix A).

**Proposition 1.1.** *Let $U$ be radial and periodic and $V(\omega)$ be* radial*, i.e., $V_x(\omega) = \omega_{|x|}$, with $\{\omega_n\}_{n\in\mathbb{N}_0}$ iid* non-constant *random variables. Then:*

1. *The ac spectrum of $H_0 = T + U$ on $\ell^2(\mathbb{T})$ consists of a finite union of closed intervals.*
2. *For every $\lambda \neq 0$ the ac spectrum of $H_\lambda(\omega) = H_0 + \lambda V(\omega)$ vanishes for almost all $\omega$.*

Our main result is that the effect on the ac spectrum is different when the perturbation is by a random potential whose values over different branches of the tree are only weakly correlated. Altogether, in this paper the random potential $V(\omega)$ is assumed to have the following properties, whose precise definitions can be found in Subsection 1.2 below.

**A1:** The probability measure $\mathbb{P}$ of the random potential is stationary under the symmetries associated with the graph endomorphisms of the rooted tree.

**A2:** The values of the potential are log-integrable: $\mathbb{E}\big[\log(1 + |V_x(\cdot)|)\big] < \infty$ for each $x \in \mathbb{T}$.

**A3:** The probability measure $\mathbb{P}$ of the random potential is weakly correlated.

These assumptions in particular ensure that the ac spectrum of $H_\lambda(\omega)$ coincides with a non-random Borel set $\Sigma_{\mathrm{ac}}(\lambda)$ for almost all $\omega$.

Following is our main result.

**Theorem 1.1.** *Let $U$ be radial and periodic and $V(\omega)$ satisfy* **A1**, **A2** *and* **A3**. *Then the random Schrödinger operator $H_\lambda(\omega) = H_0 + \lambda V(\omega)$ has the following properties.*

1. *The ac spectrum is continuous at $\lambda = 0$ in the sense that for any Borel set $I \subseteq \Sigma_{\mathrm{ac}}(0)$*
$$\lim_{\lambda\to 0} \mathcal{L}\left[I \cap \Sigma_{\mathrm{ac}}(\lambda)\right] = \mathcal{L}\left[I \cap \Sigma_{\mathrm{ac}}(0)\right] \tag{1.4}$$
*where $\mathcal{L}$ denotes the Lebesgue measure.*
2. *Over closed Borel sets $I \subseteq \Sigma_{\mathrm{ac}}(0)$ which are free of singular spectrum of $H_0$ the density of the ac component of the spectral measure associated with $\delta_0$ is $L^1$-continuous at $\lambda = 0$ in the sense that:*

$$\lim_{\lambda \to 0} \int_I \mathbb{E}\Big[\Big| \operatorname{Im} \langle \delta_0, (H_\lambda(\cdot) - E - i0)^{-1} \delta_0 \rangle \\ - \operatorname{Im} \langle \delta_0, (H_0 - E - i0)^{-1} \delta_0 \rangle - \Big|\Big] \, dE = 0. \tag{1.5}$$

*Here* $\delta_0 \in \ell^2(\mathbb{T})$ *is the indicator function supported at the root.*

*1.2. The assumptions*

A rooted tree is a connected, undirected graph with no cycles. In a slight abuse of notation, we shall use the symbol $\mathbb{T}$ for both the tree graph and the set of its vertices. The root is a particular vertex which we denote $0 \in \mathbb{T}$. For each $x \in \mathbb{T}$ we denote by $|x|$ the number of edges in the unique path connecting it to the root. In a regular tree, as those considered here, each vertex other than the root has $K+1$ neighbors, one towards the root and $K$ in what we refer to as the forward direction. The set of the forward neighbors of $x$ is denoted by $\mathcal{N}_x^+$. We say that $y \in \mathbb{T}$ is in the future of $x \in \mathbb{T}$ if the path connecting $y$ and the root runs through $x$. The subtree consisting of all the vertices in the future of $x$, with $x$ regarded as its root, is denoted by $\mathbb{T}_x$.

The symmetries referred to in **A1** are associated with endomorphisms of the rooted tree. These are mappings $s\colon \mathbb{T} \to \mathbb{T}$ preserving the adjacency relation and the orientation away from the root, i.e., neighboring vertices are mapped onto neighboring vertices, and if $x$ is in the future of $y$ then $sx$ is in the future of $sy$. To each such endomorphism corresponds a transformation $S : \mathbb{C}^{\mathbb{T}} \to \mathbb{C}^{\mathbb{T}}$ defined by $(S\omega)_x := \omega_{sx}$ for all $x \in \mathbb{T}$. A probability measure $\nu$ on $\mathbb{C}^{\mathbb{T}}$ is said to be *stationary* if for all such mappings and all bounded measurable $F : \mathbb{C}^{\mathbb{T}} \to \mathbb{C}$

$$\int_{\mathbb{C}^{\mathbb{T}}} F(S\omega)\, \nu(d\omega) = \int_{\mathbb{C}^{\mathbb{T}}} F(\omega)\, \nu(d\omega)\,. \tag{1.6}$$

The weak correlation condition required in **A3** is the subject of the following

**Definition 1.1.** *A probability measure* $\nu$ *on* $\mathbb{C}^{\mathbb{T}}$ *is said to be* weakly correlated *if there exists some* $\kappa \in (0, 1]$ *such that for any pair of vertices* $x \neq y$*, which are common forward neighbors of some vertex, and any pair of bounded measurable functions* $F, G : \mathbb{C}^{\mathbb{T}} \to [0, \infty)$*, one of which is determined by the values over the forward subtree* $\mathbb{T}_x$ *and the other determined by the values of over the forward subtree* $\mathbb{T}_y$*,*

$$\int_{\mathbb{C}^{\mathbb{T}}} F(\omega)\, G(\omega)\, \nu(d\omega) \geq \kappa \int_{\mathbb{C}^{\mathbb{T}}} F(\omega)\, \nu(d\omega) \int_{\mathbb{C}^{\mathbb{T}}} G(\omega)\, \nu(d\omega). \tag{1.7}$$

By standard approximation arguments it suffices to test (1.7) for bounded continuous $F, G$.

Clearly, the collection of probability measures $\mathbb{P}$ on $\mathbb{R}^{\mathbb{T}}$ satisfying **A1** and **A3** includes the case where $\{\omega_x\}_{x \in \mathbb{T}}$ form iid random variables. A broader collection of examples can be found within the class of Gibbs measures, informally given by

$$\nu(d\omega) = \mathcal{Z}^{-1} \exp\left(\frac{\beta}{2} \sum_{x,y\in\mathbb{T}} J_{|x-y|}(\omega_x, \omega_y)\right) \prod_{x\in\mathbb{T}} d\omega_x \,, \tag{1.8}$$

where $\mathcal{Z}$ symbolizes the normalizing factor, $|x-y|$ is the distance between $x$ and $y$, and $\{J_n\}_{n\in\mathbb{N}_0}$ is a family of real-valued, bounded, symmetric functions obeying

$$A := \sum_{n=1}^{\infty} n\, K^{n-1} \sup_{\omega,\omega'} |J_n(\omega, \omega')| < \infty \,. \tag{1.9}$$

For $\beta$ small enough, depending on the interaction [10, 13], (1.8) describes a unique Gibbs measure, and this measure satisfies **A1** and **A3** with $\kappa = e^{-3\beta A}$.

Let us also note that if $\mathbb{P}$ is stationary and weakly correlated then $\mathbb{P}$ is ergodic. As a consequence, the ac spectrum of $H_\lambda(\omega)$ coincides with a deterministic set, cf. [9, 26, 18].

### *1.3. Relation with previous results*

The topic of Anderson (de)localization on tree graphs goes back to Abou-Chacra, Anderson and Thouless [1, 2]. They noted that a "self-consistent" approach, which for general graphs can be viewed as an approximation, is exact for Cayley trees and used it to explore the location of the mobility edge for small disorder. The subject was further studied in [25, 24]. In particular, Miller and Derrida [24] argued that for energies within the spectrum of the unperturbed operator delocalized eigenfunctions should persist under weak disorder. For a summary of these findings see [18].

There have also been rigorous results on this topic. Localization, in the sense of existence of pure point spectrum, was proven at extreme energies, and at all energies for large values of the disorder parameter $\lambda$, see [3]. It was also shown in [4] that for small $\lambda$ and $U = 0$ the operator $H_\lambda(\omega)$ has only pure point spectrum for energies $|E| > K + 1$. It should be noted that this range does not include the full resolvent set of the unperturbed operator, since

$$\sigma(T) = \sigma_{\mathrm{ac}}(T) = \big[-2\sqrt{K}, 2\sqrt{K}\big]. \tag{1.10}$$

Concerning delocalization, which is the subject of this note, Klein [18, 20] established the stability of the ac spectrum in the case $U = 0$ and $\{\omega_x\}_{x\in\mathbb{T}}$ iid random variables. Using supersymmetric representations he showed that for any $0 < |E| < 2\sqrt{K}$ there exists $\lambda(E) > 0$ such that

$$\sup_{|\lambda|<\lambda(E)} \sup_{|E'|<E} \sup_{\eta>0} \mathbb{E}\Big[\big|\big\langle \delta_0, (H_\lambda(\cdot) - E' - i\eta)^{-1} \delta_0\big\rangle\big|^2\Big] < \infty. \tag{1.11}$$

In particular, (1.11) implies that for small $\lambda$ the almost sure spectrum is *purely* ac in an energy range which is contained in $\sigma_{\mathrm{ac}}(T)$ as shown in [20]. Moreover, the states in this energy range exhibit ballistic-transport behavior [19].

The results presented here address issues similar to those discussed in [20]. We do not pursue the question whether the ac spectrum is pure in the intervals under study. However, the approach we present is quite different from the technique used in the above mentioned works, and the result applies to more general situations.

## 2. A criterion for the stability of the ac spectrum

The argument which proves our main result, Theorem 1.1, yields a continuity criterion of a somewhat greater generality. To present it, we shall frame the discussion in the context of radially stationary potentials. For this purpose, we let $(\Xi, p)$ be a probability space on which there is measure preserving ergodic mapping $\mathcal{S} : \Xi \to \Xi$. Every measurable function $u : \Xi \to \mathbb{R}$ generates through

$$U_x(\theta) = u(\mathcal{S}^{|x|}\theta), \qquad \text{for all} \quad x \in \mathbb{T},\ \theta \in \Xi, \tag{2.1}$$

a potential on the tree, which is radial and stationary under radial shifts. We will subsequently refer to potentials $U(\theta)$ of the form (2.1) as *radially stationary* and assume tacitly that the function $u$ is log-integrable:

$$\int_\Xi \log(1 + |u(\theta)|)\, p(d\theta) < \infty\,. \tag{2.2}$$

To include the radially $\tau$-periodic potentials in the above setup, one may take $\Xi = \{1, \dots, \tau\}$, with $p$ the equidistribution among the $\tau$ integers, and $\mathcal{S}$ the shift $\mathcal{S}\theta := \big(\theta + 1\big) \mod \tau$.

### *2.1. The stability criterion*

Any radially stationary potential gives rise to a *radially stationary Schrödinger operator*

$$H_0(\theta) := T + U(\theta) \qquad \text{on } \ell^2(\mathbb{T}). \tag{2.3}$$

Some basic facts on the spectral properties of such operators are collected in Appendix A. As a generalization of (1.1) we consider weak perturbations of such operators by a random potential, i.e.,

$$H_\lambda(\theta, \omega) := H_0(\theta) + \lambda V(\omega) \qquad \text{on } \ell^2(\mathbb{T}). \tag{2.4}$$

For the statement of our stability criterion it is important to note that there exists a Borel set $\Sigma_{\mathrm{ac}}(0) \subseteq \mathbb{R}$ such that $\sigma_{\mathrm{ac}}\big(H_0(\theta)\big) = \Sigma_{\mathrm{ac}}(0)$ for almost all $\theta \in \Xi$, cf. Proposition A.2. Moreover, due to ergodicity there exists a Borel set $\Sigma_{\mathrm{ac}}(\lambda) \subseteq \mathbb{R}$ such that

$$\sigma_{\mathrm{ac}}\big(H_\lambda(\theta, \omega)\big) = \Sigma_{\mathrm{ac}}(\lambda) \tag{2.5}$$

for almost all $(\theta, \omega)$.

In the proof of Theorem 1.1 a significant role will be played by considerations of functions $\Gamma$ with $\operatorname{Im} \Gamma \geq 0$ satisfying the following co-cycle condition

$$\Gamma(\theta) = \frac{1}{U_0(\theta) - E - K\,\Gamma(\mathcal{S}\theta)}\,. \tag{2.6}$$

It provides an alternative formulation of the Schrödinger equation for a covariant eigenfunction. As will be discussed below, the co-cycle (2.6) has at least one solution with $\operatorname{Im} \Gamma \geq 0$ in the energy range $\Sigma_{\mathrm{ac}}(0)$. Its uniqueness in case of radially periodic $U$ is proven in Proposition A.3 of Appendix A.

**Definition 2.1.** *The Schrödinger co-cycle* (2.6) *is said to admit a unique solution in an energy range $I \subset \mathbb{R}$ if for Lebesgue-almost all $E \in I$ there exists a unique measurable function $\Gamma : \Xi \to \mathbb{C}$ satisfying* (2.6) *and* $\operatorname{Im} \Gamma(\theta) \geq 0$ *for almost every $\theta \in \Xi$.*

As is apparent from the argument, our proof of Theorem 1.1, establishes the following somewhat more general statement.

**Theorem 2.1.** *Let $U(\theta)$ be radially stationary for which the spectrum of $H_0(\theta)$ has an absolutely continuous component, and let $V(\omega)$ be a random potential satisfying* **A1**, **A2**, *and* **A3**. *Then a sufficient condition for the continuity of the ac spectrum of $H_\lambda(\theta, \omega)$ in the sense expressed in Theorem 1.1, is that the Schrödinger co-cycle* (2.6) *admits a unique solution in the energy range* $\Sigma_{\mathrm{ac}}(0)$.

It should be noted that if $U(\theta)$ is non-deterministic then $\Sigma_{\mathrm{ac}}(0) = \emptyset$ due to the relation of $\Sigma_{\mathrm{ac}}(0)$ to the ac spectrum of a one-dimensional operator discussed in Appendix A and Kotani theory [21, 30]. Examples of deterministic potentials are periodic or almost-periodic ones. For a further comment on the latter case, see Section 7.

### *2.2. Outline of the proof*

The analysis on trees is often more accessible than on other graphs since various quantities computed at the tree root satisfy recursion relations. These relate the quantity at the root to the corresponding counterparts on the subtrees to which the tree breaks upon the removal of the root site. We make use of such a relation for the diagonal elements of the forward resolvents:

$$\Gamma_x(\lambda, z, \theta, \omega) := \left\langle \delta_x \, , \left( H_\lambda^{\mathbb{T}_x}(\theta, \omega) - z \right)^{-1} \delta_x \right\rangle, \tag{2.7}$$

where $H_\lambda^{\mathbb{T}_x}(\theta, \omega)$ is the restriction of the operator $H_\lambda(\theta, \omega)$ to the Hilbert space $\ell^2(\mathbb{T}_x)$ over the forward tree graph $\mathbb{T}_x \subset \mathbb{T}$. The above is well defined for any $z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and through the Herglotz property the limit

$$\Gamma_x(\lambda, E + i0, \theta, \omega) := \lim_{\eta \downarrow 0} \Gamma_x(\lambda, E + i\eta, \theta, \omega) \tag{2.8}$$

exists for Lebesgue-almost every $E \in \mathbb{R}$, cf. Appendix B.

The forward resolvents play a diagnostic role for the problem considered here. In particular, the density of the ac component of the spectral measure of $H_\lambda(\theta, \omega)$ associated with $\delta_0 \in \ell^2(\mathbb{T})$ is given by $\pi^{-1} \operatorname{Im} \Gamma_0(\lambda, E + i0, \theta, \omega)$.

It is a significant observation that the above quantities also play another role. The products yield the off-diagonal Green function [20, Eq. (2.8)],

$$\left\langle \delta_0, \left( H_\lambda(\theta, \omega) - z \right)^{-1} \delta_x \right\rangle = \prod_{j=0}^{|x|} \Gamma_{x_j}(\lambda, z, \theta, \omega), \tag{2.9}$$

where $x_j$, with $j = 0, \ldots, |x|$, denote the vertices along the unique path joining the root $0$ $(=: x_0)$ and $x$ $(=: x_{|x|})$.

Fundamental to our discussion is the recursion relation which the forward resolvents are well know to satisfy [1, 24, 20, 12]. For each $\lambda \in \mathbb{R}$, $z \in \mathbb{C}^+$, $\omega \in \mathbb{R}^{\mathbb{T}}$ and at each vertex $x \in \mathbb{T}$, one has

$$\Gamma_x(\lambda, z, \theta, \omega) = \Big( U_x(\theta) + \lambda V_x(\omega) - z - \sum_{y \in \mathcal{N}_x^+} \Gamma_y(\lambda, z, \theta, \omega) \Big)^{-1}, \quad (2.10)$$

where $\mathcal{N}_x^+$ is the set of the forward neighbors of $x$. For $\lambda = 0$ and $E \in \mathbb{R}$ this relation boils down to (2.6) due to covariance property

$$\Gamma_x(0, z, \theta) = \Gamma_0(0, z, \mathcal{S}^{|x|}\theta)\,. \quad (2.11)$$

For almost every $E \in \Sigma_{\mathrm{ac}}(0)$ one measurable solution of (2.6) with values in $\mathbb{C}^+$ is thus provided by $\Gamma_0(0, E + i0, \theta)$, the forward resolvent corresponding to $H_0(\theta)$. The issue in the additional assumption of Theorem 2.1 is therefore the uniqueness of this solution.

The main part of our analysis is to show that the forward resolvents converge in a certain distributional sense to their unperturbed counterparts. To do so, we first prove that their distribution is sharp in the sense that

$$\Gamma_x(\lambda, E + i\eta, \theta, \omega) \;=\; \Phi_x(E, \theta) \left[ 1 + o_x(E, \theta, \omega; \lambda, \eta) \right] \quad (2.12)$$

with some $\Phi_x(E, \theta)$ which does not depend on $\omega$ and certain $o_x(E, \theta, \omega; \lambda, \eta)$ which vanish in the distributional sense for $\lambda, \eta \to 0$.

A key step in the derivation of (2.12) is the proof of the corresponding statement for just the imaginary part, $\operatorname{Im}\Gamma_x$, where we suppress the dependence on $(\lambda, E + i\eta, \theta, \omega)$. The starting point for this is the relation:

$$\log\Big( \operatorname{Im}\Gamma_x \Big) \;=\; \log\Big( K\, |\Gamma_x|^2 \Big) + \log\Big( \frac{\eta}{K} + \frac{1}{K} \sum_{y \in \mathcal{N}_x^+} \operatorname{Im}\Gamma_y \Big), \quad (2.13)$$

which follows from (2.10). The mean value of the first term on the right may be regarded as a Lyapunov exponent on the tree which vanishes at $\lambda, \eta = 0$ for most $E \in \Sigma_{\mathrm{ac}}(0)$. A relevant observation here is that while this term may exhibit a rather erratic dependence on $\lambda$ and $E$ for $\eta = 0$, its integrals over $E$ form continuous functions of $\lambda, \eta$.

The second term calls for an application of the Jensen inequality:

$$\log\Big( \frac{1}{K} \sum_{y \in \mathcal{N}_x^+} \operatorname{Im}\Gamma_y \Big) \;\geq\; \frac{1}{K} \sum_{y \in \mathcal{N}_x^+} \log \operatorname{Im}\Gamma_y\,, \quad (2.14)$$

where the inequality is strict unless $\operatorname{Im}\Gamma_y$ coincide for all $y \in \mathcal{N}_x^+$. Under the weak-correlation assumption **A3**, the above considerations lead to the conclusion that the distributions of $\operatorname{Im}\Gamma_x$ are of vanishing relative width. This is quantified below through a more thorough discussion of that notion – on which we expand in Appendix D – and a strengthened version of the Jensen inequality.

Putting the above arguments together we conclude the analog of (2.12) for $\operatorname{Im} \Gamma_x$ and use this to deduce the sharpness of the distribution of $\Gamma_x$ itself. Finally, upon substituting (2.12) into the recursion relation (2.10), we conclude that $\Phi_x(E, \theta)$ satisfies the same equation as $\Gamma_x(0, E + i0, \theta)$. Since in its dependence on $x$, $\Phi_x(E, \theta)$ is radial and covariant, i.e., $\Phi_x(E, \theta) = \Phi_0(E, \mathcal{S}^{|x|}\theta)$, our main result follows from the uniqueness of solutions of (2.6).

## 3. A Lyapunov exponent and its continuity

We shall refer to the following quantity as the Lyapunov exponent for the Schrödinger operator (1.1) on the tree

$$\gamma_\lambda(z) := -\int_\Xi \mathbb{E}\Big[\log\Big(\sqrt{K}\,\big|\Gamma_0(\lambda, z, \theta, \cdot)\big|\Big)\Big]\, p(d\theta)\,. \tag{3.1}$$

Below, we list some basic facts:

1. The Lyapunov exponent $\gamma_\lambda(z)$ is a harmonic function of $z \in \mathbb{C}^+$ as it is the negative real part of the Herglotz function
$$w_\lambda(z) := \int_\Xi \mathbb{E}\Big[\log\Big(\sqrt{K}\;\Gamma_0(\lambda, z, \theta, \cdot)\Big)\Big]\, p(d\theta)\,. \tag{3.2}$$
Assumption **A2** and Lemma B.2 ensure that $w_\lambda(z)$ and $\gamma_\lambda(z)$ are well defined. Moreover, by the symmetry assumption **A1**, $w_\lambda(z)$ remains unchanged if $\Gamma_0$ is replaced by $\Gamma_x$.
2. In view of the relation (2.9), $\gamma_\lambda(z)$ describes the typical decay rate of the Green's function along a ray, normalized so that in the absence of significant fluctuations between different branches $\gamma_\lambda(z) > 0$ assures that the Green function is square integrable. The reader is cautioned, however, that in contrast to one dimension, for random potentials on the tree the ac spectrum does not coincide with the essential closure of the set of energies on which this Lyapunov exponent vanishes. This issue is further discussed in Appendix C.

Some of the properties of the Lyapunov exponent which are of immediate relevance for our discussion are summarized in the following statement.

**Theorem 3.1.** *Let $U(\theta)$ be radially stationary, and $V(\omega)$ satisfy* **A1** *and* **A2**. *Then:*

1. *The Lyapunov exponent is positive $\gamma_\lambda(z) > 0$ for all $z \in \mathbb{C}^+$, and satisfies*
$$\lim_{\eta\to\infty} \frac{\gamma_\lambda(E + i\eta)}{\eta} = 0\,. \tag{3.3}$$
2. *For $\lambda = 0$ the Lyapunov exponent vanishes on the ac spectrum of $H_0(\theta) = T + U(\theta)$:*
$$\gamma_0(E + i0) = 0 \tag{3.4}$$
*for Lebesgue-almost every $E \in \Sigma_{\mathrm{ac}}(0)$.*

3. *For any bounded Borel set $I \subset \mathbb{R}$ the integral $\int_I \gamma_\lambda(E + i\eta)\, dE$ is continuous in $(\lambda, \eta) \in \mathbb{R} \times [0, \infty)$.*
4. *For any bounded Borel set $I \subseteq \Sigma_{\mathrm{ac}}(0)$ one has*

$$\lim_{\substack{\lambda \to 0 \\ \eta \downarrow 0}} \int_I \gamma_\lambda(E + i\eta)\, dE = 0. \tag{3.5}$$

*Proof.* 1. The positivity of $\gamma_\lambda$ on $\mathbb{C}^+$ can be seen through (2.13) and (2.14). The statement about the asymptotics derives from (B.10) in Appendix B.
2. The vanishing of $\gamma_0$ on $\Sigma_{\mathrm{ac}}(0)$ forms part of Proposition A.2 in Appendix A.
3. As a positive harmonic function which satisfies the asymptotics (3.3), $\gamma_\lambda(\cdot + i\eta)$ with $(\lambda, \eta) \in \mathbb{R} \times [0, \infty)$ can be represented as

$$\gamma_\lambda(z + i\eta) = \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|E - z|^2}\, \sigma_{(\lambda,\eta)}(dE)\,, \tag{3.6}$$

where the Borel measure $\sigma_{(\lambda,\eta)}$ is unique and satisfies $\int_{\mathbb{R}} (E^2 + 1)^{-1} \sigma_{(\lambda,\eta)}(dE) < \infty$, see [11]. Thanks to the Hergoltz property of $w_\lambda$, the harmonic conjugate of $\gamma_\lambda(\cdot + i\eta) = -\operatorname{Re} w_\lambda(\cdot + i\eta)$ has a definite sign and hence locally integrable boundary values [11, Thm. 1.1]. This implies that $\sigma_{(\lambda,\eta)}$ is purely ac [11, Thm. 3.1 & Corollary 1], and for all $(\lambda, \eta) \in \mathbb{R} \times [0, \infty)$, one has

$$\sigma_{(\lambda,\eta)}(I) = \int_I \gamma_\lambda(E + i\eta)\, dE\,. \tag{3.7}$$

The asserted continuity thus follows from the vague continuity of the measure $\sigma_{(\lambda,\eta)}$ in $(\lambda, \eta) \in \mathbb{R} \times [0, \infty)$, see [7]. By Proposition 3.1 below, a sufficient condition for the latter is the (pointwise) continuity of $\gamma_\lambda(z + i\eta)$ for all $z \in \mathbb{C}^+$. This pointwise convergence follows from the (weak) resolvent convergence

$$\lim_{\substack{\lambda' \to \lambda \\ \eta' \to \eta}} \Gamma_x(\lambda', z + i\eta', \theta, \omega) = \Gamma_x(\lambda, z + i\eta, \theta, \omega) \tag{3.8}$$

for all $\operatorname{Im} z > 0$ and all $x \in \mathbb{T}$, $\lambda \in \mathbb{R}$, $\eta \in [0, \infty)$, $\theta \in \Xi$, $\omega \in \mathbb{R}^{\mathbb{T}}$, together with the dominated convergence theorem, which is applicable thanks to (B.10) of Appendix B.
4. This is an immediate consequence of 2. and 3. □

The previous proof relied on the following convergence statement, which we recall from [15, Prop. 4.1].

**Proposition 3.1.** *Let $\sigma, \sigma_n$ be non-negative Borel measures satisfying $\int (E^2 + 1)^{-1} \sigma(dE) < \infty$ and similarly for $\sigma_n$. Assume that for all $z \in \mathbb{C}^+$*

$$\lim_{n \to \infty} \int \frac{1}{|E - z|^2}\, \sigma_n(dE) = \int \frac{1}{|E - z|^2}\, \sigma(dE)\,, \tag{3.9}$$

*then $\lim_{n \to \infty} \sigma_n = \sigma$ in the sense of vague convergence.*

For a definition of vague convergence and its implications, see [7].

## 4. Fluctuation bounds

An important tool for our analysis is the following strengthened version of the Jensen inequality for the logarithm.

**Lemma 4.1.** *Let $K \geq 2$. Then for any collection $(X_j)_{j=1}^{K}$ of positive numbers*

$$\log\left(\frac{1}{K}\sum_{j=1}^{K} X_j\right) \geq \frac{1}{K}\sum_{j=1}^{K}\log X_j + \frac{1}{2K(K-1)}\sum_{i\neq j}\left(\frac{X_i - X_j}{X_i + X_j}\right)^2 . \quad (4.1)$$

*Proof.* We expand the average inside the logarithm into an average over pairs and use Jensen's inequality to obtain

$$\log\left(\frac{1}{K}\sum_{j=1}^{K} X_j\right) \geq \frac{1}{K(K-1)}\sum_{i\neq j}\log\left(\frac{X_i + X_j}{2}\right). \quad (4.2)$$

This reduces the claimed inequality to the case $K = 2$. For a proof of the latter let $\xi_j := 2X_j/(X_1 + X_2)$, which takes values in $[0, 2]$, and let $f(\xi) := -\log\xi + (\xi - 1)$. Using the symmetry $2 - \xi_{1/2} = \xi_{2/1}$, we have

$$\log\left(\frac{X_1 + X_2}{2}\right) - \frac{1}{2}(\log X_1 + \log X_2) = \frac{1}{4}\sum_{j=1}^{2}\big(f(\xi_j) + f(2-\xi_j)\big). \quad (4.3)$$

By elementary arguments $f(\xi) + f(2-\xi) \geq (\xi - 1)^2$ for all $\xi \in (0, 2]$, which finally implies the assertion of the lemma. □

The above improvement of the Jensen inequality for $\log X$ is substantial unless the empirical distribution of $\{X_j\}$ is narrow in a sense which is quantified as follows.

**Definition 4.1.** *For $\alpha \in (0, 1/2]$ the* relative $\alpha$-width *of a probability measure $\nu$ on $(0, \infty)$ is given by*

$$\delta(\nu, \alpha) := 1 - \frac{\xi_-(\nu, \alpha)}{\xi_+(\nu, \alpha)}. \quad (4.4)$$

*where*

$$\begin{aligned} \xi_-(\nu, \alpha) &= \sup\{\xi \;:\; \nu[0, \xi) \leq \alpha\} \\ \xi_+(\nu, \alpha) &= \inf\{\xi \;:\; \nu(\xi, \infty) \leq \alpha\}. \end{aligned} \quad (4.5)$$

*For a random variable $X$ we denote*

$$\delta(X, \alpha) \;:=\; \delta(\nu_X, \alpha) \quad (4.6)$$

*where $\nu_X$ is the probability distribution of $X$, which is defined by: $\nu_X(A) = \mathbb{P}(X \in A)$.*

Thus, small $\delta(X, \alpha)$ means that the distribution of $X$ is sharp in the sense expressed in (2.12). Some useful observations about the composition laws for the relative widths of sums and products of random variables are presented in Appendix D.

We shall now apply the above tools to show that the vanishing of the Lyapunov exponent implies the sharpness of the distributions of both $\operatorname{Im} \Gamma_x$ and $|\Gamma_x|$.

**Theorem 4.1.** *Let $U(\theta)$ be radially stationary, and $V(\omega)$ satisfy* **A1**, **A2** *and* **A3**. *Then for any $x \in \mathbb{T}$, $\lambda \in \mathbb{R}$, $z \in \mathbb{C}^+$ and $\alpha \in (0, 1/2]$:*

$$\int_\Xi \delta\big(\operatorname{Im} \Gamma_x(\lambda, z, \theta, \cdot), \alpha\big)^2 \, p(d\theta) \leq \frac{8}{\kappa\, \alpha^2} \, \gamma_\lambda(z), \tag{4.7}$$

$$\left[\int_\Xi \delta\big(\big|\Gamma_x(\lambda, z, \theta, \cdot)\big|^2, \alpha\big) \, p(d\theta)\right]^2 \leq \frac{32\,(K+1)^2}{\kappa\, \alpha^2} \, \gamma_\lambda(z). \tag{4.8}$$

*Here $\kappa$ is the constant appearing in the weak-correlation condition* (1.7).

*Proof.* For a proof of (4.7), we start from (2.13) which implies the inequality

$$\mathbb{E}\Big[\log\big(\operatorname{Im} \Gamma_x\big)\Big] - 2\,\mathbb{E}\left[\log\left(\sqrt{K}\,|\Gamma_x|\right)\right] \geq \mathbb{E}\left[\log\left(\frac{1}{K} \sum_{y \in \mathcal{N}_x^+} \operatorname{Im} \Gamma_y\right)\right], \tag{4.9}$$

where again we will suppress the dependence on $(\lambda, z, \theta, \omega)$. Lemma 4.1 yields a lower bound for the right side of (4.9) consisting of a sum of two terms, $S_1$ and $S_2$. The first is

$$S_1 \ := \ \mathbb{E}\left[\frac{1}{K} \sum_{y \in \mathcal{N}_x^+} \log\big(\operatorname{Im} \Gamma_y\big)\right], \tag{4.10}$$

which, upon averaging over $\theta$ and using **A1**, will cancel the first term on the right side of (4.9). The second term is

$$S_2 \ := \ \frac{1}{2K(K-1)} \sum_{y, y' \in \mathcal{N}_x^+} \mathbb{E}\left[\left(\frac{\operatorname{Im} \Gamma_y - \operatorname{Im} \Gamma_{y'}}{\operatorname{Im} \Gamma_y + \operatorname{Im} \Gamma_{y'}}\right)^2\right]. \tag{4.11}$$

The variables $\Gamma_y$ and $\Gamma_{y'}$ appearing in the above summation $y, y' \in \mathcal{N}_x^+$ are identically distributed due to the symmetry implied by the stationarity condition **A1**. Moreover, by the weak-correlation condition **A2** the measure describing their joint distribution is bounded below by $\kappa\times$ the product measure which describes two independent copies sampled from the common distribution. Using Lemma D.2 of Appendix D, we find

$$S_2 \ \geq \ \frac{\kappa\alpha^2}{4} \left[\delta\big(\operatorname{Im} \Gamma_y, \alpha\big)\right]^2 \tag{4.12}$$

Combining the terms $S_1$ and $S_2$, and averaging (4.9) over $\theta$, one arrives at (4.7).

The second assertion (4.8) follows from squaring the following inequality,

$$\int_\Xi \delta\left(\left|\Gamma_x(\lambda, z, \theta, \cdot)\right|^2, \alpha\right) \, p(d\theta) \leq 2 \int_\Xi \delta\Big(\operatorname{Im}\Gamma_x(\lambda, z, \theta, \cdot), \frac{\alpha}{K+1}\Big) \, p(d\theta)\,, \tag{4.13}$$

applying the Jensen inequality, and inserting (4.7). For a proof of (4.13), we employ the recursion relation (2.10) and Lemma D.1, which for any $p + q = 1$ yields

$$\begin{aligned} \delta\left(|\Gamma_x|^2, \alpha\right) &\leq \delta\left(\operatorname{Im}\Gamma_x, p\alpha\right) + \delta\Big(\sum_{y \in \mathcal{N}_x^+} \operatorname{Im}\Gamma_y, q\alpha\Big) \\ &\leq \delta\left(\operatorname{Im}\Gamma_x, p\alpha\right) + \delta\left(\operatorname{Im}\Gamma_y, \frac{q\alpha}{K}\right). \end{aligned} \tag{4.14}$$

In the last inequality, we have used the fact that $\Gamma_y$ is identical in distribution to $\Gamma_{y'}$ for any $y, y' \in \mathcal{N}_x^+$. Setting $p = (K+1)^{-1}$ and averaging over $\theta$ completes the proof of (4.13). □

## 5. Distributional convergence of the forward resolvents

Our goal now is to establish that in a certain distributional sense, on which more is said below,

$$\Gamma_x(\lambda, E + i\eta, \theta, \omega) \underset{\lambda,\eta\to 0}{\xrightarrow{\mathcal{D}}} \Gamma_x(0, E + i0, \theta) \tag{5.1}$$

where the quantity on the right side is a forward resolvent of $H_0(\theta)$. The underlying reasoning for (5.1) is the observation that the fluctuation bounds of Theorem 4.1 and the information on the Lyapunov exponent in Theorem 3.1 imply that the prelimit in (5.1) exhibits very weak dependence on $\omega$ for small $\lambda$ and $\eta$. At the same time, those quantities satisfy a recursion relation which is close to (2.6). By assumption this equation has a unique solution taking values in $\mathbb{C}^+$.

There are a number of gaps in the above narrative which need to be addressed in the proof:

1. Concerning the Lyapunov exponent $\gamma_\lambda(E+i\eta)$, it is only known that the integral over $E$ tends to zero in the joint limit $\lambda, \eta \to 0$ – not that it tends to zero on some set of energies $E$.
2. The fluctuation bounds imply the narrowing of the distribution of $|\Gamma_x|$ and $\operatorname{Im}\Gamma_x$, however, the limiting value of $\Gamma_x \in \mathbb{C}^+$ could range over two distinct points.
3. The narrowing in the distribution refers only to the dependence of $\Gamma_x(\lambda, E + i\eta, \theta, \omega)$ on $\omega$ at fixed $\lambda, \eta$. That still leaves room for some rather erratic dependence on the parameters $\lambda, \eta$.
4. The uniqueness of the solution holds only for the limiting equation and within the class of perfectly radial and covariant functions of $|x|$. However, we deal with quantities which may still include additional randomness and for which the symmetry holds only in the distributional sense.

In order to bypass the first-mentioned limitation, the convergence (5.1) is derived below in the distributional sense with respect to the joint dependence on $(E, \omega)$. The latter are distributed by the product measure $\mathcal{D} := \mathcal{L}_I \otimes \mathbb{P}$ on $I \times \mathbb{R}^{\mathbb{T}}$, where $\mathcal{L}_I$ is the Lebesgue measure restricted to $I \subseteq \Sigma_{\rm ac}(0)$. In order to address the joint values of the entire collection of variables, we regard $(E, \Gamma)$ as taking values in the product space $I \times \mathbb{C}^{\mathbb{T}}$ and consider the family of finite measures $\mu^{(\theta)}_{(\lambda,\eta)}$ induced on this space by the image of $\mathcal{L}_I \otimes \mathbb{P}$ under the mapping

$$(E, \omega) \mapsto \left(E, \left\{\Gamma_x(\lambda, E + i\eta, \theta, \omega)\right\}_{x\in\mathbb{T}}\right) , \tag{5.2}$$

where $\theta \in \Xi$ and $(\lambda, \eta)$ are indexing parameters with values in $\mathbb{R} \times (0, \infty)$ or $\{(0, 0)\}$. These mappings and the corresponding measures are well-defined even along the boundary $\eta = 0$.

In the above terminology, we will establish the following

**Theorem 5.1.** *Let $U(\theta)$ be radially stationary, assume that the co-cycle* (2.6) *admits a unique solution on $\Sigma_{ac}(0)$, and let $V(\omega)$ satisfy* **A1**, **A2** *and* **A3**. *Moreover, let $I \subseteq \Sigma_{ac}(0)$ be a bounded Borel set. Then the measures on $I \times \mathbb{C}^{\mathbb{T}}$ which describe the joint distribution of $(E, \Gamma)$ induced from $\mathcal{L}_I \otimes \mathbb{P}$ by the mapping* (5.2), *satisfy for almost all $\theta \in \Xi$:*

$$\lim_{\substack{\lambda\to 0\\ \eta\downarrow 0}} \mu^{(\theta)}_{(\lambda,\eta)} = \mu^{(\theta)}_{(0,0)} \tag{5.3}$$

*in the sense of weak convergence.*

Let us note a number of elementary properties of the measures $\mu^{(\theta)}_{(\lambda,\eta)}$ and some related observations:

1. The limiting measure $\mu^{(\theta)}_{(0,0)}$ is concentrated on the graph of the function $E \mapsto \Gamma(0, E + i0, \theta) \in \mathbb{C}^{\mathbb{T}}$. Accordingly, it has the product form

$$\mu^{(\theta)}_{(0,0)}(dE\, d\Gamma) = dE\ \otimes \delta_{\Gamma(0,E+i0,\theta)}(d\Gamma). \tag{5.4}$$

2. For almost every $E \in I$ the conditional measure $\mu^{(E,\theta)}_{(\lambda,\eta)}$ equals the image of the probability measure $\mathbb{P}$ under the mapping $\omega \mapsto \Gamma(\lambda, E + i\eta, \theta, \omega) \in \mathbb{C}^{\mathbb{T}}$. Since $\mu^{(E,\theta)}_{(0,0)}$ is supported on $\Gamma(0, E + i0, \theta)$, an equivalent way of stating the conclusion (5.3) is that the forward resolvents converge in distribution, i.e., for all $x \in \mathbb{T}$ and almost all $\theta \in \Xi$

$$\mathcal{D}\text{-}\lim_{\substack{\lambda\to 0\\ \eta\downarrow 0}} \Gamma_x(\lambda, \cdot + i\eta, \theta, \cdot) = \Gamma_x(0, \cdot + i0, \theta) \tag{5.5}$$

with $\mathcal{D} := \mathcal{L}_I \otimes \mathbb{P}$, cf. Definition B.1 in Appendix B.

3. The family of measures $\mu^{(\theta)}_{(\lambda,\eta)}$ is tight [7, 8]. This is readily deduced from:

$$\inf_{t>0} \sup_{(\lambda,\eta)\in\mathbb{R}\times[0,\infty)} \mu^{(\theta)}_{(\lambda,\eta)}\big(|\Gamma_x| > t\big) = 0\,, \tag{5.6}$$

for all $x \in \mathbb{T}$ and $\theta \in \Xi$, which follows from Lemma B.1 in Appendix B.

4. The measures $\mu^{(\theta)}_{(\lambda,\eta)}$ are of constant mass, $|I| < \infty$. For a family of such measures, tightness implies that any sequence has (possibly many) weak accumulation points [8, 7]. In order to prove the claimed convergence (5.3) it suffices to show that any accumulation point of the given sequence coincides with $\mu^{(\theta)}_{(0,0)}$.

We shall follow the path indicated by the last observation. By focusing on the accumulation points $\mu^{(\theta)}$ of $\mu^{(\theta)}_{(\lambda,\eta)}$, we may take advantage of the fact that in the joint limit $\lambda, \eta \to 0$ various approximate statements which were outlined above take sharp form. In particular, the recursion relation simplifies. For the latter, we shall make use of the following general principle.

**Proposition 5.1.** *Let $(\nu_\beta)_{\beta\in J}$ be a family of finite measures on a polish space $\Upsilon$, indexed by $\beta$ which takes values in a topological space $J$. Suppose that for each $\beta \in J$*

$$\varphi(\beta, Y) = 0 \quad \nu_\beta\text{-almost surely}, \tag{5.7}$$

*with a function $\varphi : J \times \Upsilon \to \mathbb{C}$ which:*

*i) for every compact subset $K \subset \Upsilon$ is equicontinuous in $\beta$ over $J \times K$, and*
*ii) at some $\beta_0 \in J$ is continuous in $Y$, over $\Upsilon$.*

*Then, for each weak limit $\nu = \lim_{\beta\to\beta_0} \nu_\beta$:*

$$\varphi(\beta_0, Y) = 0 \quad \nu\text{-almost surely}. \tag{5.8}$$

*Proof.* Since the space $\Upsilon$ is a union of an increasing family of compact sets, it suffices to show that for any compact set $K \subset \Upsilon$

$$\int_K \psi(\beta_0, Y)\, \nu(dY) = 0 \tag{5.9}$$

with $\psi := |\varphi|/(1 + |\varphi|)$. The integral in (5.9) may be rewritten as

$$\begin{aligned}\int_K \psi(\beta_0, Y)\, \nu(dY) \le \int_K \psi(\beta, Y)\, \nu_\beta(dY) + \sup_{Y\in K} \big|\psi(\beta, Y) - \psi(\beta_0, Y)\big|\, \nu_\beta(K) \\ + \Big| \int_K \psi(\beta_0, Y) \big(\nu(dY) - \nu_\beta(dY)\big)\Big|.\end{aligned} \tag{5.10}$$

Now, under the assumption (5.7) the first term vanishes for every $\beta$, and the conditions i) and ii) imply that the second and third term vanish in the limit $\beta \to \beta_0$. □

We now have the following characterization of the possible accumulation points discussed above.

**Lemma 5.1.** *Let $\mu^{(\theta)}$ be a weak accumulation point for the family of measures $\mu^{(\theta)}_{(\lambda,\eta)}$, with parameters $(\lambda, \eta)$ in $\mathbb{R} \times (0, \infty)$ converging to $(0, 0)$, i.e.,*

$$\lim_{\substack{\lambda\to 0\\ \eta\downarrow 0}} \mu^{(\theta)}_{(\lambda,\eta)} = \mu^{(\theta)} \tag{5.11}$$

*Then:*

1. *The limiting recursion relation*

$$1 - \Big( U_x(\theta) - E - \sum_{y \in \mathcal{N}_x^+} \Gamma_y \Big) \Gamma_x = 0 \tag{5.12}$$

*holds for all* $x \in \mathbb{T}$ *and* $\mu^{(\theta)}$*-almost all* $(E, \Gamma)$.

2. *For almost all* $(E, \theta) \in I \times \Xi$ *the conditional measure* $\mu^{(E,\theta)}$

   (a) *satisfies the weak correlation condition* (1.7).

   (b) *is supported on at most two points, i.e., for all* $x \in \mathbb{T}$ *there exist* $I_x(E, \theta) \geq 0$ *and* $M_x(E, \theta) > 0$ *such that for* $\mu^{(E,\theta)}$*-almost all* $\Gamma$

$$\operatorname{Im} \Gamma_x = I_x(E, \theta) \quad \textit{and} \quad |\Gamma_x| = M_x(E, \theta) \,. \tag{5.13}$$

*Proof.* 1. The first part is a consequence of Proposition 5.1. To apply it, we let $\beta$ denote the pair $(\lambda, \eta)$, with $J$ a neighborhood of $\beta_0 := (0, 0)$ in $\mathbb{R} \times (0, \infty) \cup \{(0, 0)\}$. For $\Upsilon$ we choose $I \times \mathbb{C}^{\mathbb{T}} \times \mathbb{R}^{\mathbb{T}}$ endowed with the product topology. The measures $\nu_{(\lambda,\eta)}$ are defined as the image of $\mathcal{L}_I \otimes \mathbb{P}$ under the mapping

$$\big(E, \omega\big) \mapsto \big(E, \Gamma(\lambda, E + i\eta, \theta, \omega), V(\omega)\big) \,, \tag{5.14}$$

so that $\mu^{(\theta)}_{(\lambda,\eta)}$ coincides with the projection of $\nu_{(\lambda,\eta)}$ onto the first coordinates $(E, \Gamma)$. Finally, we set

$$\varphi\big((\lambda, \eta), E, \Gamma, V\big) := 1 - \Big( U_x(\theta) + \lambda\, V_x - E - i\eta - \sum_{y \in \mathcal{N}_x^+} \Gamma_y \Big) \Gamma_x \,, \tag{5.15}$$

so that the recursion relation (2.10) can be expressed as $\varphi\big((\lambda, \eta), E, \Gamma, V\big) = 0$.

2. We first note that (5.11) implies that there exists a set $J \subseteq I$ of full Lebesgue measure and a subsequence $\{(\lambda_k, \eta_k)\}_{k=0}^{\infty}$ of the original sequence such that

$$\lim_{k \to \infty} \mu^{(E,\theta)}_{(\lambda_k,\eta_k)} = \mu^{(E,\theta)} \tag{5.16}$$

for all $E \in J$.

   (a) The weak correlation property is inherited by weak limits, since it suffices to to verify (1.7) for bounded continuous functions $F$ and $G$. In that case the bound is implied by the continuity of the corresponding expectations under the weak convergence.

   (b) We fix $x \in \mathbb{T}$. Theorem 3.1 and Theorem 4.1 yield

$$\lim_{\substack{\lambda \to 0 \\ \eta \downarrow 0}} \int_{I \times \Xi} \delta\big(\operatorname{Im} \Gamma_x(\lambda, E + i\eta, \theta, \cdot), \alpha\big)\, dE\, p(d\theta) = 0 \tag{5.17}$$

$$\lim_{\substack{\lambda \to 0 \\ \eta \downarrow 0}} \int_{I \times \Xi} \delta\big(\big|\Gamma_x(\lambda, E + i\eta, \theta, \cdot)\big|^2, \alpha\big)\, dE\, p(d\theta) = 0 \tag{5.18}$$

for all $\alpha \in (0, 1/2]$. Lemma D.4 in Appendix D and (5.16) thus imply that for Lebesgue-almost all $E \in I$ both $\operatorname{Im} \Gamma_x$ and $|\Gamma_x|$ are $\mu^{(E,\theta)}$-almost surely constant. Note that by (5.12), $\Gamma_x \neq 0$ $\mu^{(E,\theta)}$-almost surely. $\square$

Since a line and a circle can intersect in at most two points, Eq. (5.13) ensures that the $\Gamma_x$-marginals of $\mu^{(E,\theta)}$ are supported on at most two points. In the next lemma, we will actually prove that these two points coincide. Furthermore, using the uniqueness of periodic solutions of the limiting recursion relation (5.12) we shall conclude that this point coincides with $\Gamma_x(0, E+i0, \theta)$. The chain of deductions can be presented as follows.

**Lemma 5.2.** *Assume the situation of Lemma 5.1. Then for almost all* $(E,\theta) \in I \times \Xi$ *and all* $x \in \mathbb{T}$*:*

1. *There exists* $\Phi_x(E,\theta) \in \mathbb{C}$ *with* $\operatorname{Im} \Phi_x(E,\theta) \geq 0$ *such that for* $\mu^{(E,\theta)}$*-almost surely*

$$\Gamma_x = \Phi_x(E,\theta)\,. \tag{5.19}$$

2. $\Phi_x(E,\theta) = \Phi_0(E, \mathcal{S}^{|x|}\theta)$.
3. $\Phi_x(E,\theta) = \Gamma_x(0, E+i0, \theta)$.

*Proof.* 1. By (5.13) in Lemma 5.1 there exists at most two points $\Phi_x^{\pm}(E,\theta) \in \mathbb{C}\backslash\{0\}$ with $\operatorname{Im} \Phi_x^{\pm}(E,\theta) \geq 0$ such that $\Gamma_x \in \{\Phi_x^{+}(E,\theta), \Phi_x^{-}(E,\theta)\}$ for $\mu^{(E,\theta)}$-almost all $\Gamma$. We will now prove by contradiction that these two points coincide at every $x \in \mathbb{T}$.
Assume that there exists some $y \in \mathcal{N}_x^+$ for which $\Phi_y^+(E,\theta) \neq \Phi_y^-(E,\theta)$. Since $\mu^{(E,\theta)}$ is weakly correlated, we have

$$\mu^{(E,\theta)}\Big(\bigcap_{y\in\mathcal{N}_x^+}\Big(\Gamma_y = \Phi_y^{\pm}(E,\theta)\Big)\Big) \geq \kappa \prod_{y\in\mathcal{N}_x^+} \mu^{(E,\theta)}\Big(\Gamma_y = \Phi_y^{\pm}(E,\theta)\Big) > 0, \tag{5.20}$$

and similarly $\mu^{(E,\theta)}\big(\Gamma_y = \Phi_y^+(E,\theta)$ and $\Gamma_{y'} = \Phi_y^-(E,\theta)$ for all $y \neq y' \in \mathcal{N}_x^+\big) > 0$, which implies that the image measure of $\Gamma \mapsto U_x - E - \sum_{y\in\mathcal{N}_x^+}\Gamma_y$ induced on $\mathbb{C}$ by $\mu^{(E,\theta)}$ contains at least 3 points in its support. This is however not consistent with the limiting recursion relation (5.12) since the measure induced on $\mathbb{C}$ by $\Gamma_x^{-1}$, which is equal to one described above, is supported on only two points.
2. This property follows from the invariance of $\mu_{(\lambda,\eta)}^{(E,\theta)}$ under exchange of variables on disjoint forward subtrees and its covariance under radial shifts.
3. For almost all $E \in I$ the function $\theta \to \Phi_0(E,\theta)$ is for almost all $\theta \in \Xi$ a solution of the Schrödinger co-cycle (2.6) with $\operatorname{Im} \Phi_0(E,\theta) \geq 0$. It therefore coincides with $\Gamma_0(0, E+i0, \theta)$ by the uniqueness assumption (which is verified in Appendix A for the periodic case). □

We are now ready to conclude the

*Proof (of Theorem 5.1).* In order to prove the convergence asserted in (5.3), it suffices to establish uniqueness of the accumulation point for the measures $\mu_{(\lambda,\eta)}^{(\theta)}$ with $\lambda \to 0$ and $\eta \downarrow 0$. That was done in the argument which culminated in Lemma 5.2, which shows that for almost every $\theta \in \Xi$ any such point coincides with $\mu_{(0,0)}^{(\theta)}$. □

## 6. Proof of the main result

The main results of this paper, Theorem 1.1 and Theorem 2.1, are a consequence of the convergence statements derived in the previous section. They culminate in the following

**Theorem 6.1.** *Let $U(\theta)$ be radially stationary, assume that the co-cycle* (2.6) *admits a unique solution on $\Sigma_{ac}(0)$, and let $V(\omega)$ satisfy* **A1***,* **A2** *and* **A3***. Moreover, let $I \subseteq \Sigma_{\rm ac}(0)$ be a bounded Borel set. Then for almost all $\theta \in \Xi$:*

1. *The boundary values of the forward resolvents of $H_\lambda(\theta,\omega) = T + U(\theta) + \lambda V(\omega)$ converge in distribution,*

$$\mathcal{D}\!-\!\lim_{\lambda\to 0} \Gamma_x(\lambda, \cdot + i0, \theta, \cdot) = \Gamma_x(0, \cdot + i0, \theta) \tag{6.1}$$

   *for all $x \in \mathbb{T}$, where $\mathcal{D} = \mathcal{L}_I \otimes \mathbb{P}$.*
2. *If additionally $I$ is closed and $\sigma_{\rm ac}(H_0(\theta)) \cap I = \sigma(H_0(\theta)) \cap I$, then the density of the ac component of the spectral measure associated with the forward resolvents are $L^1$-continuous at $\lambda = 0$ the sense that*

$$\lim_{\lambda\to 0} \int_I \mathbb{E}\left[\left|\,\mathrm{Im}\,\Gamma_x(\lambda, E + i0, \theta) - \mathrm{Im}\,\Gamma_x(0, E + i0, \theta)\right|\right] dE = 0 \tag{6.2}$$

   *for all $x \in \mathbb{T}$.*

*Proof.* 1. For a proof of (6.1) we let $\varepsilon > 0$ and use Fatou's lemma

$$\begin{aligned}&\mathcal{L}_I \otimes \mathbb{P}\Big(\left|\Gamma_x(\lambda, \cdot + i0, \theta, \cdot) - \Gamma_x(0, \cdot + i0, \theta)\right| > \varepsilon\Big)\\ &\quad\le \liminf_{\eta\downarrow 0}\ \mathcal{L}_I \otimes \mathbb{P}\Big(\left|\Gamma_x(\lambda, \cdot + i\eta, \theta, \cdot) - \Gamma_x(0, \cdot + i0, \theta)\right| > \varepsilon\Big) < \infty\,.\end{aligned} \tag{6.3}$$

   We now take the limit $\lambda \to 0$. Since (5.5) holds for any joint sequence of $(\lambda, \eta)$ in $\mathbb{R} \times (0, \infty)$, the right side of (6.3) converges to zero in this limit.
2. Eq. (6.2) follows from (6.1) with the help of Proposition B.2 in Appendix B. To verify its assumptions we note that the resolvent convergence (3.8) implies the weak convergence $\lim_{\lambda\to 0} \nu_\lambda^x(\theta, \omega) = \nu_0^x(\theta)$ of the spectral measures associated with $\Gamma_x$, see (B.2) and [28, Thm. VIII.24]. Moreover, since the spectrum of $H_0(\theta)$ on $I$ is purely ac, the spectral measures $\nu_0^0(\theta)$ and hence $\nu_0^x(\theta)$ for all $x \in \mathbb{T}$ are also purely ac on $I$, cf. Appendix A. □

The main result now follows from the special case $x = 0$ in the above the theorem.

*Proof (of Theorem 1.1 and Theorem 2.1).* The two statements can be proven simultaneously, as the only difference in the argument is that in the general case of Theorem 2.1 the uniqueness of solutions of (2.6) is among the assumptions, whereas for the more specific case of Theorem 1.1 this is established in Proposition A.3 of Appendix A.

1. By the non-randomness of the spectrum, it suffices to show that for any $I \subseteq \Sigma_{\mathrm{ac}}(0)$ and almost all $\theta$
$$\lim_{\lambda \to 0} \mathbb{E}\left[\mathcal{L}_I\big(\sigma_{\mathrm{ac}}(H_\lambda(\theta,\cdot))\big)\right] = \mathcal{L}_I\big(I\big)\,. \tag{6.4}$$
We start the proof of this relation by observing that
$$\begin{aligned}\mathcal{L}_I\big(I\big) &\geq \mathbb{E}\left[\mathcal{L}_I\big(\sigma_{\mathrm{ac}}(H_\lambda(\theta,\cdot))\big)\right]\\ &\geq \mathcal{L}_I \otimes \mathbb{P}\left\{0 < \operatorname{Im}\Gamma_0(\lambda, E+i0,\theta,\omega) < \infty\right\}.\end{aligned} \tag{6.5}$$
For any $\varepsilon > 0$ the set on the right side includes the collection of $(E,\omega)$ for which $\varepsilon < \operatorname{Im}\Gamma_0(0, E+i0,\theta) < \infty$ and $\big|\operatorname{Im}\Gamma_0(\lambda, E+i0,\theta,\omega) - \operatorname{Im}\Gamma_0(0, E+i0,\theta)\big| \leq \varepsilon$. Accordingly, the right side of (6.5) is bounded below by the difference of
$$\mathcal{L}_I \otimes \mathbb{P}\left\{\big|\operatorname{Im}\Gamma_0(\lambda, E+i0,\theta,\omega) - \operatorname{Im}\Gamma_0(0, E+i0,\theta)\big| \leq \varepsilon\right\} \tag{6.6}$$
and
$$\mathcal{L}_I\left\{\operatorname{Im}\Gamma_0(0, E+i0,\theta) \in [0,\varepsilon] \cup \{\infty\}\right\}. \tag{6.7}$$
As $\lambda \to 0$ the measure in (6.6) converges to $\mathcal{L}_I(I)$ for almost all $\theta$ by Theorem 6.1. Moreover, as $\varepsilon \downarrow 0$ the measure in (6.7) converges to zero for almost all $\theta$.
2. This assertion coincides with (6.2) in the special case $x = 0$.

□

## 7. Discussion

In what follows below, we include some additional comments on possible extensions of the results presented in this work.

1. *The Green function at sites other than the root.* Our discussion focused on the spectral measures associated with the vector $\delta_0 \in \ell^2(\mathbb{T})$, the vector at the root. Theorem 1.1 may, however, be extended to other sites on the tree graph, i.e., $\delta_0$ may be replaced by $\delta_x$ in (1.5). For a proof, it is useful to make the following observation. In order to study the Green function at $x \neq 0$ one may start from the collection of the forward resolvents of the "siblings" of $x$, i.e., $\{\Gamma_y\}_{|y|=|x|}$, which are then used as the input for a finite number of iterations of the suitably adjusted algebraic recursion relation (2.10). Similarly, the statement can also be concluded for the fully symmetric tree, for which the root has $K+1$, rather than $K$ forward neighbors (whose forward trees are then of the kind discussed here).
2. *Trees with non-constant branching.* The arguments presented here may easily be extended to trees where the branching numbers $\{K_x\}_{x\in\mathbb{T}}$ are non-constant but periodic functions of $|x|$. For this case, the analysis applies with only the natural adjustments.

3. *Almost periodic background operators.* The stability criterion expressed in Theorem 2.1 may be applied to radial operators with quasi-periodic potentials, for example with $H_0(\theta)$ the almost-Mathieu operator [17]. Such operators may exhibit both pure point and ac spectra in one dimension, and by Proposition A.1 also on trees. The application of our analysis to such cases requires to verify the uniqueness of covariant solutions of the Schrödinger co-cycle (2.6). This question is addressed in a subsequent publication [5].
4. *Location of the mobility edge.* We note that there is a gap between the results which address the possible location of pure point and ac spectrum for operators with weak disorder. For $H_\lambda(\omega) = T + V(\omega)$ the results of [20] as well as this work show that the location of the mobility edge for for $\lambda$ small is at $|E| \geq 2\sqrt{K}$. Conversely, it is only known [4] that the mobility edge approaches energies $|E| \leq K+1$. The limiting value of the mobility edge is still unresolved. The current guess is that it is the result of [4] which needs to be improved (we thank Y. Last for an illuminating discussion of this point).

## Appendix

### A. Schrödinger operators on tree graphs with radial potentials

In this appendix we provide some further details on the notions used in the paper and gather a few facts about the radial background operators considered in the main text.

#### *A.1. Radial Schrödinger operators*

The Schrödinger operator $H = T + U$ on $\ell^2(\mathbb{T})$ is said to be radial if the potential $U$ is multiplication by a real-valued function $\{U_x\}_{x\in\mathbb{T}}$ which has the radial symmetry of $\mathbb{T}$, i.e.,

$$U_x = U_y \qquad \text{for all } x, y \in \mathbb{T} \text{ with } |x| = |y|. \tag{A.1}$$

The spectra of radial $H$ are related to those of the corresponding half-line operators.

**Proposition A.1.** *For any radial $U$ the spectra of the operator $H = T + U$ on $\ell^2(\mathbb{T})$ and*

$$H^+ = T + K^{-1/2}\, U \qquad \text{on } \ell^2(\mathbb{N}_0) \tag{A.2}$$

*satisfy*

$$\sigma(H) \supseteq \sqrt{K}\,\sigma(H^+), \qquad \sigma_{\mathrm{ac}}(H) = \sqrt{K}\,\sigma_{\mathrm{ac}}(H^+). \tag{A.3}$$

*Proof.* The action of $H$ on $\ell^2(\mathbb{T})$ leaves invariant the subspace $\mathcal{H}_{\mathrm{rad}}$ of the radially symmetric functions. Under the partial isometry $\mathcal{U} : \mathcal{H}_{\mathrm{rad}} \to \ell^2(\mathbb{N}_0)$, defined by

$\big(\mathcal{U}\,\psi\big)_x = K^{|x|/2}\,\psi_{|x|}$, the restriction of $H$ to this subspace is unitarily equivalent to $\sqrt{K}\,H^+$. Hence $\sigma(H) \supseteq \sqrt{K}\,\sigma(H^+)$, and analogously for the ac spectrum.

Under the action of $H$, $\delta_0$ is a cyclic vector for the symmetric subspace $\mathcal{H}_{\mathrm{rad}}$. Therefore, to conclude the equality $\sigma_{\mathrm{ac}}(H) = \sqrt{K}\,\sigma_{\mathrm{ac}}(H^+)$ it suffices to show that the support of the ac component of the spectral measure associated with an arbitrary site $x \in \mathbb{T}$ coincides with that of the root. The former is concentrated on the set of energies $E \in \mathbb{R}$ for which

$$0 < \operatorname{Im}\big\langle \delta_x, (H - E - i0)^{-1}\,\delta_x \big\rangle < \infty\,, \tag{A.4}$$

and the latter on the set of energies where

$$0 < \operatorname{Im}\Gamma_0(E + i0) < \infty \tag{A.5}$$

We claim that (A.4) implies $0 < \operatorname{Im}\Gamma_y(E+i0) < \infty$ for all $y$ with $|y| = |x|+1$. The reasoning involves the radial symmetry, by which $\Gamma_x$ depends only on $|x|$, and the recursion relation. Furthermore, through the recursion relation (A.6) below this leads to (A.5). □

An alternative way to see the above unitary equivalence proceeds as follows. Since $U$ is radial, each forward resolvent $\Gamma_x$ associated with $H$ depends only on $|x|$. With a slight abuse of notation, we will let $\{\Gamma_n\}_{n\in\mathbb{N}_0}$ stand for the values of the forward resolvents along one ray in the tree. Using an analogous convention for $U$, the recursion relation (2.10) can be rewritten as

$$\Gamma_n = \frac{1}{U_n - z - K\,\Gamma_{n+1}} \qquad \text{for all } n \in \mathbb{N}_0. \tag{A.6}$$

With suitable scaling, the recursion relation (A.6) is satisfied by both the forward resolvent of $H$ and that of $H^+$. As discussed in [12], this relation has a unique solution for $\operatorname{Im} z > 0$.

### *A.2. Radially stationary Schrödinger operators*

A special case of radial Schrödinger operators are radially stationary ones with potential $U(\theta)$ defined in (2.1) of Section 2. The following proposition compiles some basic properties of such operators.

**Proposition A.2.** *For any radially stationary $U(\theta)$ the ac spectrum of the Schrödinger operator $H(\theta) = T + U(\theta)$ on $\ell^2(\mathbb{T})$ has the properties:*

1. *It is related to the ac spectrum of*

$$\widehat{H}(\theta) := T + K^{-1/2}\,U(\theta) \qquad \text{on } \ell^2(\mathbb{Z}) \tag{A.7}$$

*for almost all $\theta$,*

$$\sigma_{\mathrm{ac}}\big(H(\theta)\big) = \sqrt{K}\,\sigma_{\mathrm{ac}}\big(\widehat{H}(\theta)\big)\,. \tag{A.8}$$

2. *It can be characterized for almost all* $\theta$ *by*

$$\sigma_{\mathrm{ac}}\big(H(\theta)\big) = \overline{\{E \in \mathbb{R} \,:\, \gamma_0(E+i0) = 0\}}^{\mathrm{ess}} \tag{A.9}$$

*where* $\overline{(\cdot)}^{\mathrm{ess}}$ *denotes the Lebesgue-essential closure and*

$$\gamma_0(z) = \int_\Xi \log\left(\sqrt{K}\,\big|\Gamma_0(0,z,\theta)\big|\right)\, p(d\theta) \tag{A.10}$$

*stands for the Lyapunov exponent, cf.* (3.1).

*Proof.* 1. The first assertion is a consequence of Proposition A.1 and Kotani theory [21, 30] which in particular ensures that for almost all $\theta$, $\sigma_{\mathrm{ac}}\big(H^+(\theta)\big) = \sigma_{\mathrm{ac}}\big(\widehat{H}(\theta)\big)$ where $H^+(\theta)$ acts on $\ell^2(\mathbb{N}_0)$.
2. The second assertion follows from (A.8) and again Kotani theory [21, 30].

□

Let us note that the disappearance of the ac spectrum under arbitrarily weak random perturbations, which was claimed in Proposition 1.1, is implied by Proposition A.2 and well-known results about random Schrödinger operators in one dimension [9, 26].

### *A.3. The radial periodic case*

The radially periodic potentials $U$ fit into the framework of radially sationary ones: we choose the equidistribution $p$ on $\tau$ integers $\Xi = \{1,\dots,\tau\}$ on which $\mathcal{S}\theta = (\theta+1) \mod \tau$ acts as a periodic shift.

In this case the covariant form (2.6) of the recursion relation takes the form of a fixed point equation. More precisely, introducing the family of Möbius transformations $\mathcal{T}(E,\theta)(\Gamma) := \big(u(\theta) - E - K\Gamma\big)^{-1}$ and iterating over a period $\tau$, we obtain

$$\begin{aligned}\Gamma(\theta) &= \mathcal{T}(E,\theta)\cdots\mathcal{T}(E,\mathcal{S}^{\tau-1}\theta)\big(\Gamma(\theta)\big)\\ &=: \mathcal{S}(E,\theta)\big(\Gamma(\theta)\big) =: \frac{a(E,\theta)\,\Gamma(\theta) + b(E,\theta)}{c(E,\theta)\,\Gamma(\theta) + d(E,\theta)}\,.\end{aligned} \tag{A.11}$$

Here we used the fact that any composition of real Möbius transformations is a Möbius transformations with some coefficients $a(E,\theta)$, $b(E,\theta)$, $c(E,\theta)$, $d(E,\theta) \in \mathbb{R}$.

**Lemma A.1.** *For all, but a finite set of* $E \in \mathbb{R}$*, the fixed point equation* (A.11) *has either two non-real, complex-conjugate solutions or two real solutions.*

*Proof.* The lemma is an immediate consequences of the following observations:

i) the coefficients $a(E,\theta)$, $b(E,\theta)$, $c(E,\theta)$, $d(E,\theta)$ are polynomials in $E$ of degree at most $\tau$.

ii) if $c(E, \theta) \neq 0$ for all $\theta \in \{1, \dots, \tau\}$, then (A.11) has one or two solutions in $\mathbb{C}$, depending on the value of the discriminant

$$\begin{aligned} \varrho(E) &:= \big(\operatorname{Tr} \mathcal{S}(E, \theta)\big)^2 - 4 \det \mathcal{S}(E, \theta) \\ &= \big(a(E, \theta) - d(E, \theta)\big)^2 - 4 b(E, \theta) c(E, \theta)\,, \end{aligned} \tag{A.12}$$

which does not depend on $\theta$ for all $E$. □

The following proposition collects some facts about radially periodic Schrödinger operators. It includes a proof of the first statement in Proposition 1.1 and verifies that the co-cycle (2.6) admits a unique solution on $\Sigma_{ac}(0)$ in the periodic case.

**Proposition A.3.** *Let $U$ be radially periodic. Then the ac spectrum of the operator $H = T + U$ on $\ell^2(\mathbb{T})$ is a union of intervals. In particular,*

$$\sigma_{\mathrm{ac}}(H) = \overline{\{E \in \mathbb{R} : \varrho(E) < 0\}}, \tag{A.13}$$

*and therefore, for all but finitely many $E \in \sigma_{\mathrm{ac}}(H)$, the fixed point equation* (A.11) *has two non-real, complex-conjugate solutions.*

*Proof.* As was noted in Proposition A.1, the ac spectrum of $H$ is concentrated on those energies $E$ for which (A.4), and therefore (A.5), holds. For such energies, $\varrho(E) < 0$. If $\varrho(E) < 0$, it is clear that $E \in \sigma_{\mathrm{ac}}(H)$. This proves (A.13). As $\varrho$ is a polynomial in $E$, one has that the ac spectrum of $H$ is a union of intervals, and away from those energies for which $\varrho(E) = 0$, the fixed point equation (A.11) has two non-real, complex-conjugate solutions. □

## B. Some useful properties of resolvents

Let $\nu$ be a finite Borel measure on $\mathbb{R}$ and let $F : \mathbb{C}^+ \to \mathbb{C}^+$ stand for its Stieltjes transform given by

$$F(z) := \int_{\mathbb{R}} \frac{1}{t - z}\, \nu(dt). \tag{B.1}$$

$F$ is a Herglotz function, i.e. analytic and $\operatorname{Im} F(z) > 0$ for all $z \in \mathbb{C}^+$. In fact, every Herglotz function, which shares the property $\sup_{\eta > 0} \eta \operatorname{Im} F(i\eta) < \infty$, can be represented as a Stieltjes transform of a finite Borel measure [26, App. A]. Examples of such functions are resolvents of self-adjoint operators, in particular,

$$\Gamma_x(\lambda, z, \theta, \omega) = \int_{\mathbb{R}} \frac{1}{t - z}\, \nu_\lambda^x(dt, \theta, \omega) \tag{B.2}$$

where $\nu_\lambda^x(\theta, \omega)$ is the spectral measure associated with $H_\lambda^{\mathbb{T}_x}(\theta, \omega)$ and $\delta_x \in \ell^2(\mathbb{T}_x)$.

For the convenience of the reader, we collect some of basic properties of Herglotz functions. More information on this subject can be found in [11, 29] or [22, § 7], [9], and [26, App. A].

**Proposition B.1.** *Let $F$ be the Stieltjes transform of a finite Borel measure $\nu$ on $\mathbb{R}$.*

1. *The boundary values $F(E + i0) := \lim_{\eta \downarrow 0} F(E + i\eta)$ exist for almost every $E \in \mathbb{R}$.*
2. *The density of the ac component of $\nu$ is given by $\operatorname{Im} F(E + i0)/\pi$.*
3. *Let $a, b \in \mathbb{R}$ and $s \in (0, 1)$. Then*

$$\int_a^b |F(E + i\eta)|^s \, dE \le \frac{|b-a| + 2(1-s)^{-1}}{\cos\left(\frac{\pi}{2}s\right)} \, \nu(\mathbb{R})^s \qquad \left[ =: B_s(a,b) \right] \tag{B.3}$$

*holds uniformly in $\eta \ge 0$ and $\nu$.*

*Proof.* Assertions 1. and 2. are taken from [11, Thm. 2.2] and [22, § 7]. In both cases the proof uses the fact that $F$ can be regarded as an analytic function on the unit disk.

Assertion 3. borrows an idea from a theorem of Kolmogorov [11, Thm. 4.2] and is based on two observations concerning fractional moments:

i) If $\operatorname{Im} F \ge 0$, then $|F|^s \le \operatorname{Re}\left[e^{-is\pi/2} F^s\right] / \cos\left(\frac{\pi}{2}s\right)$.
ii) Let $\mathcal{C}_{a,b}$ be the rectangular contour joining $a \to a + i \to b + i \to b$, then

$$\int_a^b F(E + i\eta)^s \, dE = \int_{\mathcal{C}_{a,b}} F(z + i\eta)^s \, dz.$$

The claim (B.3) follows now, using the bound $|F(z + i\eta)| \le \nu(\mathbb{R})/\operatorname{Im} z$. □

This paper mainly deals with Stieltjes transforms $F(\cdot, \omega)$ of finite random Borel measures $\nu(\omega)$ which depend measurably on a parameter $\omega$ from some probability space $(\Omega, \mathbb{P})$, see for example (B.2). The following proposition establishes the equivalence of different notions of convergence of the ac spectral densities of such measures. We take the opportunity to first recall

**Definition B.1.** *For a finite-measure Borel set $I \subset \mathbb{R}$, a sequence of measurable functions $(f_n)_{n=1}^{\infty}$ on $I \times \Omega$ is said to* converge *to $f$ in $\mathcal{L}_I \otimes \mathbb{P}$*-measure *if*

$$\lim_{n \to \infty} \mathcal{L}_I \otimes \mathbb{P}\big( |f_n(E, \omega) - f(E, \omega)| > \varepsilon \big) = 0 \tag{B.4}$$

*for every $\varepsilon > 0$. We write: $\mathcal{L}_I \otimes \mathbb{P} - \lim_{n \to \infty} f_n = f$.*

It is fairly easy to see that $L^1$-convergence implies convergence in measure [7]. In the following we give a sufficient condition for the converse in our special setting.

**Proposition B.2.** *Let $(F_n(\omega))_{n=1}^{\infty}$, $F(\omega)$ be Stieltjes transforms of finite random Borel measures $(\nu_n(\omega))_{n=1}^{\infty}$, $\nu(\omega)$ on $\mathbb{R}$. Assume that there exists some closed Borel set $I \subset \mathbb{R}$ such that for almost every $\omega$*

*i)* $\lim_{n \to \infty} \nu_n(\omega) = \nu(\omega)$ *in the sense of weak convergence on $I$, and*
*ii)* $\nu(\omega)$ *is purely ac on $I$.*

*Then the convergence of the ac densities in $\mathcal{L}_I \otimes \mathbb{P}$-measure,*

$$\mathcal{L}_I \otimes \mathbb{P} - \lim_{n\to\infty} \operatorname{Im} F_n(\,\cdot + i0, \cdot) = \operatorname{Im} F(\,\cdot + i0, \cdot), \tag{B.5}$$

*implies their $L^1$-convergence*

$$\lim_{n\to\infty} \int_I \mathbb{E}\left[\left|\operatorname{Im} F_n(E+i0,\cdot) - \operatorname{Im} F(E+i0,\cdot)\right|\right] dE = 0. \tag{B.6}$$

*Proof.* Every subsequence of $(\operatorname{Im} F_n)_{n=1}^{\infty}$ has a subsequence $\left(\operatorname{Im} F_{n_k}\right)_{k=1}^{\infty}$ for which

$$\lim_{k\to\infty} \operatorname{Im} F_{n_k}(E+i0,\omega) = \operatorname{Im} F(E+i0,\omega) \tag{B.7}$$

for $\mathcal{L}_I \otimes \mathbb{P}$-almost all $(E,\omega) \in I \times \Omega$. This statement is in fact equivalent [7, Thm. 20.7] to the convergence in measure (B.5). Moreover,

$$\begin{aligned}\limsup_{n\to\infty} \int_I \mathbb{E}\left[\operatorname{Im} F_n(E+i0,\cdot)\right] dE \leq \ \pi \ \limsup_{n\to\infty} \mathbb{E}\left[\nu_n\big(I\big)\right] = \pi\, \mathbb{E}\left[\nu\big(I\big)\right] \\ = \int_I \mathbb{E}\left[\operatorname{Im} F(E+i0,\cdot)\right] dE.\end{aligned} \tag{B.8}$$

Here the first equality is a consequence of i) and ii), cf. [7, Thm. 30.12]. The last equality in (B.8) expresses the fact that by assumption ii) the measure $\nu(\omega)$ has no singular component in $I$. Eq. (B.6) now follows for subsequences fulfilling (B.7) from the improved Fatou's lemma [23]. This completes the proof since every subsequence has a subsequence for which (B.6) holds. □

In the remainder of this appendix we collect more specialized bounds on the forward resolvents (B.2). Our first estimate concerns the rareness of large values these resolvent.

**Lemma B.1.** *Let $a, b \in \mathbb{R}$ and $t > 0$. Moreover, let $x \in \mathbb{T}$ and $\lambda \in \mathbb{R}$. Then for all $\eta \geq 0$:*

$$\int_a^b \mathbb{P}\big(|\Gamma_x(\lambda, E+i\eta)| > t\big)\, dE \leq \frac{B_s(a,b)}{t^s}, \tag{B.9}$$

*where $B_s(a,b)$ was defined in* (B.3).

*Proof.* This is a consequence of the Chebychev-Markov inequality and (B.3). □

Our last estimate will guarantee the finiteness of logarithmic moments of the forward resolvents (B.2).

**Lemma B.2.** *Let $\lambda \in \mathbb{R}$, and $z \in \mathbb{C}^+$. Then for every $x \in \mathbb{T}$,*

$$\begin{aligned}\left|\log \Gamma_x(\lambda, z, \omega)\right| \leq 2 \log\big(1 + K\,(\operatorname{Im} z)^{-1}\big) + \log(1+|z|) \\ + \log(1 + |U_x| + |\lambda|\,|V_x|) + \pi.\end{aligned} \tag{B.10}$$

*Proof.* The proof is based on the fact that for any $\Gamma \in \mathbb{C}^+$ one has

$$|\log \Gamma| \leq |\log |\Gamma|| + \pi \leq \log^+ |\Gamma| + \log^+ |\Gamma|^{-1} + \pi, \tag{B.11}$$

where $\log^+ x := \max\{0, \log x\}$ denotes the positive part of the logarithm of $x > 0$. Inserting $\Gamma_x$, the first term is bounded according to $\log^+ |\Gamma_x| \leq \log^+ (\mathrm{Im}\, z)^{-1} \leq \log\left(1 + K\, (\mathrm{Im}\, z)^{-1}\right)$ for all $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}^+$. To bound the second term we employ the recursion relation (2.10) and the triangle inequality to obtain

$$\begin{aligned} \log^+ |\Gamma_x|^{-1} &= \log^+ \left| z - U_x - \lambda V_x - \sum_{y \in \mathcal{N}_x^+} \Gamma_y \right| \\ &\leq \log^+ \left( |U_x| + |\lambda|\, |V_x| + |z| + K (\mathrm{Im}\, z)^{-1} \right) \end{aligned} \tag{B.12}$$

for all $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}^+$. □

## C. Extension of a result by Kotani and Simon

For one-dimensional random Schrödinger operators, it is known [9, 26] that the ac spectrum can be characterized by the essential closure of the set of energies for which the Lyapunov exponent vanishes. This result has two parts. Ishii [16] and Pastur [27, 26] showed that the positivity of the Lyapunov exponent implies the vanishing of the ac spectrum. Their result does *not* extend to trees as is illustrated by the following two examples:

Suppose $U = 0$ and recall from (1.10) that the spectrum of $T$ on $\ell^2(\mathbb{T})$ is purely ac.

1. If $\{\omega_x\}_{x \in \mathbb{T}}$ are iid Cauchy variables with
$$\mathbb{P}\left(\omega_x \in A\right) = \frac{1}{\pi} \int_A \frac{dv}{v^2 + \sigma^2} \quad \text{with some } \sigma > 0 \text{ and all Borel } A \subset \mathbb{R},$$
then Theorem 1.1 shows that for sufficiently small $\lambda > 0$ and almost surely the spectrum of $H_\lambda(\omega) = T + \lambda V(\omega)$ has an ac component within its spectrum.
2. If $V(\omega)$ is *radial*, i.e., given by $V_x(\omega) = \omega_{|x|}$, where $\{\omega_n\}_{n \in \mathbb{N}_0}$ are iid Cauchy variables as in 1, then by Proposition 1.1, for all $\lambda > 0$ and almost every $\omega$ the spectrum of $H_\lambda(\omega) = T + \lambda V(\omega)$ has *no* ac component.

In both cases, the Lyapunov exponents are identical. In fact, a contour-integration argument shows that

$$\gamma_\lambda(E) = -\log\left(\sqrt{K}\, \left|\Gamma_0(0, E + i\lambda\sigma)\right|\right) > 0 \tag{C.1}$$

for every $E \in \sigma(T)$ and all $\lambda > 0$. Here the positivity follows from the explicit expression for $\Gamma_0(0, z)$.

The converse of the Ishii-Pastur result is due to Kotani [21], who showed that for one-dimensional, continuum random Schrödinger operators, zero Lyapunov exponent on some Borel set of positive measure implies the existence of ac spectrum there. Shortly thereafter, Simon demonstrated that an analogous result holds in the discrete setting [30]. This result partially extends to trees.

**Theorem C.1.** *Let $U(\theta)$ be radially stationary and $V(\omega)$ satisfy* **A1**–**A3**. *Let $\lambda \in \mathbb{R}$ and suppose $I \subset \mathbb{R}$ is an open interval for which $\gamma_\lambda(E + i0) = 0$ for almost every $E \in I$. Then, $\Sigma_{ac}(\lambda) \cap I \neq \emptyset$.*

*Proof.* We use a slight variation of the argument in [30]. Since $\operatorname{Re} w_\lambda(E + i0) = 0$ for almost all $E \in I$, the Schwartz-reflection principle [29] (see also [22, Lemma 7.5]) ensures that the function $w_\lambda$ defined in (3.2) has an analytic continuation through $I$ in the lower half-plane. In particular, this implies that its derivative exists through $E \in I$ and one has

$$\lim_{\eta \downarrow 0} \frac{\gamma_\lambda(E + i\eta)}{\eta} = \lim_{\eta \downarrow 0} \frac{d\gamma_\lambda(E + i\eta)}{d\eta} < \infty. \tag{C.2}$$

Using Fatou and Lemma C.1 below, one sees that

$$\begin{aligned} &\int_\Xi \mathbb{E}\left[(\operatorname{Im}\Gamma_0(\lambda, E + i0, \theta, \cdot))^{-1}\right] p(d\theta) \\ &\quad\leq \liminf_{\eta \downarrow 0} \int_\Xi \mathbb{E}\left[\left(\operatorname{Im}\Gamma_0(\lambda, E + i\eta, \theta, \cdot) + \frac{\eta}{2K}\right)^{-1}\right] p(d\theta) < \infty, \end{aligned} \tag{C.3}$$

and therefore, $\operatorname{Im}\Gamma_0(\lambda, E + i0, \theta, \omega) > 0$ for almost every $E \in I$ and $\theta, \omega$. $\square$

The proof of the previous theorem relied on the following

**Lemma C.1.** *Let $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}^+$, then*

$$\int_\Xi \mathbb{E}\left[\left(\operatorname{Im}\Gamma_0(\lambda, z, \theta, \cdot) + \frac{\operatorname{Im} z}{2K}\right)^{-1}\right] p(d\theta) \leq 2K\, \frac{\gamma_\lambda(z)}{\operatorname{Im} z}. \tag{C.4}$$

*Proof.* Again our proof is similar to Simon's argument [30]. Using (2.13) and the Jensen inequality we obtain

$$\begin{aligned} \log\left(\frac{\operatorname{Im}\Gamma_x}{K\,|\Gamma_x|^2}\right) &= \log\left(\frac{\operatorname{Im} z}{K} + \frac{1}{K}\sum_{y \in \mathcal{N}_x^+} \operatorname{Im}\Gamma_y\right) \\ &\geq \frac{1}{K}\sum_{y \in \mathcal{N}_x^+} \log\left(\frac{\operatorname{Im} z}{K} + \operatorname{Im}\Gamma_y\right). \end{aligned} \tag{C.5}$$

Taking the probabilistic expectation and using the fact that all forward neighbors are identically distributed, the right side is equal to $\mathbb{E}\log\left(\operatorname{Im} z/K + \operatorname{Im}\Gamma_y\right)$ for any $y \in \mathcal{N}_x^+$. Averaging over $\theta$, we thus obtain the estimate

$$\begin{aligned} 2\,\gamma_\lambda(z) &\geq \int_\Xi \mathbb{E}\left[\log\left(1 + \frac{\operatorname{Im} z}{K\ \operatorname{Im}\Gamma_0(\lambda, z, \theta)}\right)\right] p(d\theta) \\ &\geq \int_\Xi \mathbb{E}\left[\frac{2\operatorname{Im} z}{2K\ \operatorname{Im}\Gamma_0(\lambda, z, \theta) + \operatorname{Im} z}\right] p(d\theta)\,, \end{aligned} \tag{C.6}$$

where the last inequality follows from the fact that $\log(1+x) \geq \frac{2x}{2+x}$ for any $x \geq 0$. $\square$

## D. On relative errors

Throughout this appendix let $(\Omega, \mathbb{P})$ be some probability space and $X$ stand for a positive random variable. Our main object of interest is the relative width $\delta(X, \cdot)$ associated with the distribution of $X$, which was introduced in Definition 4.1.

The following lemma collects some useful rules of calculus associated with addition, multiplication and division of positive random variables.

**Lemma D.1.** *Let $X$ and $(X_j)_{j=1}^K$ be a collection of positive random variables. Let $\alpha$ and $(\alpha_j)_{j=1}^K$ be a collection of numbers with values in* $(0, 1/2]$. *Then:*

1. $\delta(X, \alpha_1) \leq \delta(X, \alpha_2)$ *if* $\alpha_1 \geq \alpha_2$.
2. $\delta(\eta + X, \alpha) \leq \delta(X, \alpha)$ *if* $\eta \geq 0$.
3. $\delta(1/X, \alpha) = \delta(X, \alpha)$.
4. $\delta\left(\prod_{j=1}^K X_j, \sum_{j=1}^K \alpha_j\right) \leq \sum_{j=1}^K \delta(X_j, \alpha_j)$ *if* $\sum_{j=1}^K \alpha_j \in (0, 1/2]$
5. $\delta\left(\sum_{j=1}^K X_j, \alpha K\right) \leq \delta(X, \alpha)$ *if $(X_j)_{j=1}^K$ are identically distributed and* $K\alpha \leq 1/2$.

*Proof.* Assertion 1 follows from the fact that $\alpha \mapsto \xi_{\mp}(X, \alpha)$ is monotone increasing/decreasing respectively. A proof of assertion 2 is based on the identity $\xi_{\pm}(\eta + X, \alpha) = \eta + \xi_{\pm}(X, \alpha)$ which yields

$$\delta(\eta + X, \alpha) = 1 - \frac{\eta + \xi_-(X, \alpha)}{\eta + \xi_+(X, \alpha)} \leq \delta(X, \alpha) \tag{D.1}$$

by monotonicity in $\eta \geq 0$. Assertion 3 follows from the equality $\xi_{\mp}(1/X, \alpha) = 1/\xi_{\pm}(X, \alpha)$. For a proof of assertion 4 we estimate

$$\begin{aligned} &\mathbb{P}\left(\prod_{j=1}^K X_j < \prod_{j=1}^K \xi_-(X_j, \alpha_j)\right) \\ &\quad \leq \mathbb{P}\Big(\text{There is some } j \text{ with} X_j < \xi_-(X_j, \alpha_j)\Big) \leq \sum_{j=1}^K \alpha_j \end{aligned} \tag{D.2}$$

and hence $\xi_-\left(\prod_{j=1}^K X_j, \sum_{j=1}^K \alpha_j\right) \geq \prod_{j=1}^K \xi_-(X_j, \alpha_j)$. The same lines of reasoning also yield the bound $\xi_+\left(\prod_{j=1}^K X_j, \sum_{j=1}^K \alpha_j\right) \leq \prod_{j=1}^K \xi_+(X_j, \alpha_j)$ such that

$$\begin{aligned} \delta\left(\prod_{j=1}^K X_j, \sum_{j=1}^K \alpha_j\right) &\leq 1 - \prod_{j=1}^K \frac{\xi_-(X_j, \alpha_j)}{\xi_+(X_j, \alpha_j)} \\ &\leq \sum_{j=1}^K \left(1 - \frac{\xi_-(X_j, \alpha_j)}{\xi_+(X_j, \alpha_j)}\right) = \sum_{j=1}^K \delta(X_j, \alpha_j). \end{aligned} \tag{D.3}$$

The last inequality follows by induction on $K$. For a proof of assertion 5 we estimate

$$\mathbb{P}\Big(\sum_{j=1}^{K} X_j < \sum_{j=1}^{K} \xi_-(X_j,\alpha)\Big) \le \mathbb{P}\Big(\text{There is some } j \text{ with } X_j < \xi_-(X_j,\alpha)\Big)$$
$$\le \alpha K \tag{D.4}$$

and hence $\xi_-\left(\sum_{j=1}^{K} X_j, \alpha K\right) \ge \sum_{j=1}^{K} \xi_-(X_j,\alpha) = K\xi_-(X_1,\alpha)$, because the random variables $(X_j)$ are identically distributed. Similarly, we obtain the bound $\xi_+\big(\sum_{j=1}^{K} X_j, \alpha K\big) \le K\xi_+(X_1,\alpha)$ which proves the claim. □

The next lemma employs the relative width of a single random variable as a lower bound to certain expectation values involving two identically distributed random variables under a weak correlation assumption.

**Lemma D.2.** *Let $X_1$ and $X_2$ be identically distributed positive random variables. Suppose that there exists a constant $\kappa \in (0,1]$ such that*

$$\mathbb{P}(X_1 \in A_1 \text{ and } X_2 \in A_2) \ge \kappa\, \mathbb{P}(X_1 \in A_1)\, \mathbb{P}(X_2 \in A_2) \tag{D.5}$$

*for all pairs of Borel sets $A_1, A_2 \subset (0,\infty)$. Then*

$$\mathbb{E}\left[\left(\frac{X_1 - X_2}{X_1 + X_2}\right)^2\right] \ge \frac{\kappa}{2}\big[\alpha\, \delta(X_1,\alpha)\big]^2. \tag{D.6}$$

*for all $\alpha \in (0, 1/2]$.*

*Proof.* In the event that $X_1 \le \xi_-(X_1,\alpha)$ and $X_2 \ge \xi_+(X_1,\alpha)$, one has

$$\left|\frac{X_1 - X_2}{X_1 + X_2}\right| \ge \frac{\xi_+(X_1,\alpha) - \xi_-(X_1,\alpha)}{\xi_+(X_1,\alpha) + \xi_-(X_1,\alpha)} = \frac{\delta(X_1,\alpha)}{2 - \delta(X_1,\alpha)} \ge \frac{\delta(X_1,\alpha)}{2}. \tag{D.7}$$

The same holds true if $X_2 \le \xi_-(X_1,\alpha)$ and $X_1 \ge \xi_+(X_1,\alpha)$. Therefore the left side in (D.6) is bounded from below by $\delta(X_1,\alpha)^2/4$ times

$$\begin{aligned}&\mathbb{P}\big(X_1 \le \xi_-(X_1,\alpha) \text{ and } X_2 \ge \xi_+(X_1,\alpha)\big)\\ &\quad+\mathbb{P}\big(X_2 \le \xi_-(X_1,\alpha) \text{ and } X_1 \ge \xi_+(X_1,\alpha)\big)\\ &\qquad \ge 2\kappa\, \mathbb{P}\big(X_1 \le \xi_-(X_1,\alpha)\big)\, \mathbb{P}\big(X_1 \ge \xi_+(X_1,\alpha)\big) \ge 2\kappa\alpha^2.\end{aligned} \tag{D.8}$$

□

Rather elementary considerations yield the following useful statement. We note that $\xi_\pm(\nu,\alpha)$ are defined as in (4.5) for any measure $\nu$ on $[0,\infty)$.

**Lemma D.3.** *Let $\{\nu_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $(0,\infty)$, which has a weak limit: $\lim_{n\to\infty} \nu_n = \nu$. Then, for each $\alpha \in (0, 1/2]$*

$$\limsup_{n\to\infty} \xi_-(\nu_n,\alpha) \le \xi_-(\nu,\alpha) \le \xi_+(\nu,\alpha) \le \liminf_{n\to\infty} \xi_+(\nu_n,\alpha) \tag{D.9}$$

*Furthermore, if for each $\alpha \in (0, 1/2] \cap \mathbb{Q}$*

$$\liminf_{n\to\infty} \delta(\nu_n,\alpha) = 0, \tag{D.10}$$

*then $\nu$ is supported on a single point, i.e., there is $\xi \in \mathbb{R}$ such that $\nu\{\xi\} = 1$.*

*Proof.* By its definition, $\xi_+(\nu, \alpha)$ is the smallest real number $\xi \in \mathbb{R}$ such that $\nu[\xi - \varepsilon, \infty) > \alpha$ for any $\varepsilon > 0$, and it suffices to restrict here the attention to $\varepsilon$ for which $\nu\{\xi - \varepsilon\} = 0$. For such, we may deduce that for large enough $n$ also: $\nu_n[\xi - \varepsilon, \infty) > \alpha$, and hence $\xi_+(\nu_n, \alpha) \geq \xi - \varepsilon$. From this we conclude that $\xi_+(\nu, \alpha) \leq \liminf_{n\to\infty} \xi_+(\nu_n, \alpha)$ and, by analogous reasoning, $\xi_-(\nu, \alpha) \geq \limsup_{n\to\infty} \xi_-(\nu_n, \alpha)$.

For a proof of the second assertion we distinguish two cases. If $\xi_+(\nu, \alpha) = 0$ for some $\alpha \in (0, 1/2]$, then $\nu\{0\} = 1$, because $\nu$ is supported on $[0, \infty)$. Otherwise, if $\xi_+(\nu, \alpha) > 0$ for all $\alpha \in (0, 1/2]$, then $\delta(\nu, \alpha) = 1 - \xi_-(\nu, \alpha)/\xi_+(\nu, \alpha)$ is well-defined and (D.9) implies

$$\delta(\nu, \alpha) \;\leq\; \liminf_{n\to\infty} \delta(\nu_n, \alpha)\,. \tag{D.11}$$

The claims now readily follow as, in this case, zero relative width implies zero absolute width. □

In the main text, we need the following consequence of the preceding lemma.

**Lemma D.4.** *Let $(T, \rho)$ be a finite measure space and $\{\nu_n^t\}_{n=1}^\infty$ be a measurable family of sequences of probability measures on $(0, \infty)$ which are indexed by $t \in T$. Assume that $\lim_{n\to\infty} \nu_n^t = \nu^t$ as a weak limit for almost all $t \in T$. Suppose further that*

$$\lim_{n\to\infty} \int_T \delta(\nu_n^t, \alpha)\, \rho(dt) = 0 \tag{D.12}$$

*for all $\alpha \in (0, 1/2]$. Then for almost all $t \in T$ there exists $\xi^t \in [0, \infty)$ such that $\nu^t$ is supported on $\xi^t$.*

*Proof.* From (D.12) we conclude that there exists some set $S \subseteq T$ of full $\rho$ measure and some subsequence $\{\nu_{n_k}^t\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \delta(\nu_{n_k}^t, \alpha) = 0 \tag{D.13}$$

for all $t \in S$ and all $\alpha \in (0, 1/2] \cap \mathbb{Q}$. Since the weak convergence also holds down the subsequence, i.e., $\lim_{k\to\infty} \nu_{n_k}^t = \nu^t$ at almost every $t$, the claim follows from Lemma D.3. □

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