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Construction of Gibbs measures for 1-dimensional continuum fields

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Abstract. We study 1-dimensional continuum fields of Ginzburg-Landau type under the presence of an external and a long-range pair interaction potentials. The corresponding Gibbs states are formulated as Gibbs measures relative to Brownian motion [17]. In this context we prove the existence of Gibbs measures for a wide class of potentials including a singular external potential as hard-wall ones, as well as a non-convex interaction. Our basic methods are: (i) to derive moment estimates via integration by parts; and (ii) in its finite-volume construction, to represent the hard-wall Gibbs measure on $C(\mathbb{R};\mathbb{R}^+)$ in terms of a certain rotationally invariant Gibbs measure on $C(\mathbb{R};\mathbb{R}^3)$.

1. Introduction

This paper studies $\mathbb{R}^d$-valued continuum fields $X = \{X(s), s \in \mathbb{R}\}$ over $\mathbb{R}$ with interactions prescribed by a (formal) Hamiltonian given by

$$H(X) = \frac{1}{2}\int |\nabla X(s)|^2\,ds + \int \varphi(X(s))\,ds + \frac{1}{2}\iint \psi(s-t, X(s), X(t))\,dsdt, \tag{1.1}$$

where $\nabla X = \partial X/\partial s$. This is a continuous counterpart to Ginzburg-Landau random fields over the lattice $\mathbb{Z}$ (see [23]) with an external potential φ and a long-range pair interaction potential ψ; analogously to those lattice cases, the Gibbs state associated with (1.1) would formally be given by

$$d\mu(X) = \exp\{-H(X)\}\,dX/\text{normalization}, \tag{1.2}$$

where the reference measure $dX \equiv \prod_{s\in\mathbb{R}} dX(s)$ is “Feynman’s measure”. By incorporating the first term in (1.1) into the reference measure, the measure like (1.2) can be interpreted as a local perturbation from Wiener measure; that is, we consider Brownian motion under the presence of potentials φ and ψ. The corresponding measure on the path space $C(\mathbb{R};\mathbb{R}^d)$ is formulated as a Gibbs measure relative to Brownian motion [17], and defined through the so-called DLR equation (see Definition 2.1). Similarly to the lattice field case, such a measure appears

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as equilibrium states of random time-evolutions of the fields described by Ginzburg-Landau equations (see [8, 9, 7]); [11] also studies the dynamics by using the Dirichlet form theory.

In the case $\psi = 0$, the corresponding Gibbs measure can be realized as a $P(\phi)_1$-stationary Markov process and it has been fairly understood (see, e.g., [20, 22]). On the other hand, in the non-Markovian case $\psi \neq 0$, the first mathematical treatment was done by [17]; their motivation was originated from quantum field theory, particularly from the Euclidean quantization of a certain Hamiltonian operator called Nelson's scalar field model [16]: its physical context is a quantum particle in $\mathbb{R}^d$ governed by the Schrödinger operator $H_0 = -(1/2)\Delta + \varphi$ and coupled to a free Bose field. It has been known [16] that, when the coupling is restricted to the time interval $[-T, T]$, the Euclidean quantization involves the following types of measures:

$$Z_T^{-1} \exp\left\{\iint_{|s|,|t|\le T} ds dt \int_{\mathbb{R}^d} dk \frac{\tilde{f}(k)}{f(k)} e^{-f(k)|s-t|} \cos(k\cdot(X(s)-X(t)))\right\} d\mathcal{P}^{\varphi}(X), \tag{1.3}$$

where $\mathcal{P}^{\varphi}$ denotes the law on $C(\mathbb{R};\mathbb{R}^d)$ of the $P(\phi)_1$-process associated with H_0, f and $\tilde{f}$ are some non-negative functions, and Z_T is the normalization. It is discussed in [15] that the limit measure as $T \to \infty$ (if it exists) plays an important role in the analysis of spectral structure of the operator. For details, see the references cited above.

Now we raise the question whether the family of measures as (1.3) has a limit as $T \to \infty$; that is, we discuss the existence of an infinite-volume Gibbs measure for prescribed potentials φ and ψ. There are several possible ways to show the existence: Ruelle's superstability estimates [21, 13] on lattice fields; monotonicity method developed in [17]; cluster expansions applied in [14]. Superstability estimate relies on the product structure of a reference measure and there are a number of difficulties in adapting this to the continuum case. Monotonicity uses log-concave inequalities, hence requires certain convexity on ψ, which excludes interactions as in (1.3). Cluster expansion is applicable to such an interaction when the coupling is sufficiently weak; in [14], the growth order in spatial variables was also restricted to at most quadratic. Recently, [1] has proved the existence under a mild assumption on φ; his condition is almost as weak as assuming H_0 to have an L^2-ground state, and the existence is shown in the case of bounded interactions satisfying a certain pathwise condition.

One of the main purposes of this paper is to establish the existence for a wide class of interactions. We deal with interactions of growth order $q_0 + 1$ ($q_0 \ge 0$) in spatial variables. We choose external potentials to grow polynomially of order $2p_0$ ($p_0 \ge 1$) at infinity. Roughly speaking, our result reads:

$$\text{if } p_0 \ge q_0, \text{ then the associated Gibbs measure exists.}$$

The key is to show the *localization* in its finite-volume construction (see Theorem 3.1). Our method to derive this is different from those quoted above. Its advantages are: (i) it requires no convexity on interactions; (ii) it is applicable to an arbitrary strength of interactions in the case $p_0 > q_0$. Our technique to be developed here involves (deterministic) integration by parts. So we require weak differentiability on potentials, however, many examples including Nelson's model can be treated in our framework.

Another purpose of this paper is to construct hard-wall Gibbs measures with interactions on $C(\mathbb{R}; \mathbb{R}^+)$. We deal with an external potential such as

$$\varphi(x) = \infty, \quad x < 0.$$

Under the effect of φ, the path is restricted to stay positive over the real line (the *hard wall*). Similarly to lattice fields [12, 3], a "repelling" phenomenon is also observed in the present continuum case; indeed, the paths, renormalized by the hard-wall effect, behave as 3-dimensional Bessel process, not as reflecting Brownian motion. This observation reduces the problem to the localization of certain finite-volume Gibbs measures on $C(\mathbb{R}; \mathbb{R}^3)$, to which our method applies.

This paper is organized as follows: in Section 2, we state the main result; in Section 3, we prepare a key theorem, which we prove in Section 4; in Section 5, we discuss the existence of hard-wall Gibbs measures with interactions.

Throughout this paper we use the notation (x, y) instead of $x \cdot y$ for the inner product on $\mathbb{R}^d$. We write $|x| = (x, x)^{1/2}$ and $\|x\| = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$. $\mathbb{R}^+$ denotes the set of the non-negative real numbers. We denote by $\langle \cdot \rangle$ the expectation with subscript of a reference measure. Other notation will be introduced as needed.

2. Main result

For given functions $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $\psi : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we define (φ, ψ)-*Gibbs measures* following [17]: Let $\mathcal{C} = C(\mathbb{R}; \mathbb{R}^d)$ endowed with the compact uniform topology. For a finite interval $\Lambda = [T_1, T_2] \subset \mathbb{R}$, let $\mathcal{C}_\Lambda = C(\Lambda; \mathbb{R}^d)$ and $\mathcal{C}_\Lambda^* = C(\Lambda^c; \mathbb{R}^d)$. For a given $\xi \in \mathcal{C}$, we define the local Hamiltonian $\mathcal{H}_{\Lambda,\xi} \equiv \mathcal{H}_{\Lambda,\xi}^{\varphi,\psi}$ by

$$\mathcal{H}_{\Lambda,\xi}(X) = \int_\Lambda \varphi(X(s))\, ds + \frac{1}{2} \int_{\Lambda^2} \psi(s - t, X(s), X(t))\, dsdt$$
$$+ \int_{\Lambda \times \Lambda^c} \psi(s - t, X(s), \xi(t))\, dsdt, \quad X \in \mathcal{C}_\Lambda.$$

The functions φ and ψ are called the external potential and interaction potential, respectively. We assume ψ is symmetric in the sense that, for all $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$,

$$\psi(s, x, y) = \psi(|s|, x, y), \quad \psi(s, x, y) = \psi(s, y, x). \tag{2.1}$$

Let $\pi_\Lambda : \mathcal{C} \to \mathcal{C}_\Lambda$ and $\pi_\Lambda^* : \mathcal{C} \to \mathcal{C}_\Lambda^*$ be canonical projections. For a probability measure μ on $\mathcal{C}$, define $\mu_{\Lambda,\xi}(\cdot) = \mu(\pi_\Lambda \in \cdot \,|\pi_\Lambda^*)(\xi)$. Here $\mu(\cdot|\pi_\Lambda^*)$ is the regular conditional probability with respect to the σ-field $\sigma(\pi_\Lambda^*)$.

Definition 2.1. *A probability measure μ on $\mathcal{C}$ is called a (φ, ψ)-Gibbs measure if its regular conditional probabilities satisfy the DLR equation:*

$$d\mu_{\Lambda,\xi}(X) = (Z_{\Lambda,\xi})^{-1} e^{-\mathcal{H}_{\Lambda,\xi}(X)}\, d\mathcal{W}_{\Lambda,\xi}(X) \quad \mu\text{-a.e. } \xi \in \mathcal{C}.$$

Here $\mathcal{W}_{\Lambda,\xi}$ denotes the law of Brownian bridge on $\mathcal{C}_\Lambda$ with boundary conditions $X(T_1) = \xi(T_1)$ and $X(T_2) = \xi(T_2)$, and $Z_{\Lambda,\xi} = \langle e^{-\mathcal{H}_{\Lambda,\xi}(X)} \rangle_{\mathcal{W}_{\Lambda,\xi}}$.

We proceed to the setup for the main result. We assume (A.1)–(A.3):

(A.1) Assumptions on $\boldsymbol{\varphi}$. φ is bounded from below. There exist φ_0 and φ_1 such that $\varphi = \varphi_0 + \varphi_1$, satisfying (A.1a) and (A.1b), respectively:

(A.1a) φ_0 is a continuous function such that the associated Schrödinger operator $H_0 = -(1/2)\Delta + \varphi_0$ acting on $L^2(\mathbb{R}^d; dx)$ has a strictly positive ground state f_0 of class $C^2(\mathbb{R}^d)$ satisfying the following conditions:

(i) (strict log-concavity) there exists an $\alpha > 0$ such that

$$(\zeta, \mathrm{Hess}_{u_0}(x)\zeta) \geq \alpha|\zeta|^2 \quad \text{for all } \zeta, x \in \mathbb{R}^d, \tag{2.2}$$

where $u_0 = -\log f_0$ and $\mathrm{Hess}_{u_0} = (\partial^2 u_0/\partial x_i \partial x_j)_{1\leq i,j\leq d}$ is the Hessian of u_0;

(ii) there exists a $p_0 \geq 1$ such that

$$0 < \liminf_{r\to\infty} \frac{1}{r^{2p_0}} \inf_{|x|=r} U(x), \quad \limsup_{r\to\infty} \frac{1}{r^{p_0}} \sup_{|x|=r} |\boldsymbol{V}(x)| < \infty, \tag{2.3}$$

where $U = (f_0)^{-2}\mathrm{div}(f_0\nabla f_0)$ and $\boldsymbol{V} = (f_0)^{-1}\nabla f_0$. Here div denotes the divergence.

(A.1b) $\varphi_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist $\mathsf{b} \geq 0$ and $0 \leq p_1 < p_0$ such that

$$|\nabla\varphi_1(x)| \leq \mathsf{b}\|x\|^{p_1} \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Here $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is the set of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ whose distributional derivatives $\partial f/\partial x_i$, $1 \leq i \leq d$, belong to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

(A.2) Assumptions on $\boldsymbol{\psi}$. For each fixed $s \in \mathbb{R}$ and $y \in \mathbb{R}^d$, $\psi(s, \cdot, y) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$. There exists a non-negative, integrable function ψ_0 on $\mathbb{R}$ satisfying (i) and (ii):

(i) there exists a $q_0 \geq 0$ such that, for a.e. $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$,

$$|\nabla_x\psi(s, x, y)| \leq \psi_0(s)(\|x\|^{q_0} + \|y\|^{q_0}) \quad (\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_d));$$

(ii) for a.e. $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$, $\psi(s, x, y) \geq -\psi_0(s)$.

(A.3) p_0 is strictly larger than q_0.

Remark 2.1. From (2.2), we see in particular that $|\boldsymbol{V}(x)| - \alpha|x|$ is bounded from below. We assumed $p_0 \geq 1$ in (2.3) to make our assumption consistent.

Now we state one of our main results, the existence of (φ, ψ)-Gibbs measures:

Theorem 2.1. *Assume* (A.1)–(A.3). *Then there exists a translation invariant* (φ, ψ)-*Gibbs measure* μ *satisfying*

$$\langle |X(0)|^{2p_0} \rangle_\mu < \infty.$$

Remark 2.2. In the case $p_0 = q_0$, the existence also holds if $\int_{\mathbb{R}} \psi_0(s)\, ds$ is sufficiently small. See Remark 4.3.

We give examples of potentials satisfying the above assumptions.

Example 2.1. We can construct examples of φ from f_0. Note that we may assume without loss of generality that $-(1/2)\Delta f_0 + \varphi_0 f_0 = 0$, by adding a constant to φ_0 if necessary. φ_0 is then expressed as $\varphi_0 = (1/2)\Delta f_0 / f_0$.

(i) Ornstein-Uhlenbeck processes:

$$\varphi_{\mathsf{ou}}(x) = \frac{1}{2}\sum_{i=1}^{d} a_i^2 x_i^2, \quad x \in \mathbb{R}^d,\ a_i > 0,$$

with $f_0(x) = \exp\{-(1/2)\sum_{i=1}^{d} a_i x_i^2\}$.

(ii) Double-well potentials:

$$\varphi_{\mathsf{dw}}(x) = \frac{1}{2}a^2 x^4 - \frac{1}{2}b^2 x^2, \quad x \in \mathbb{R},\ a, b > 0.$$

This is obtained by taking

$$\varphi_0(x) = \frac{1}{2}a^2(x^4 + x^2) + R(x), \quad \varphi_1(x) = -\frac{1}{2}(a^2 + b^2)x^2 - R(x)$$

with $R(x) = -a\|x\| + (a/2)\|x\|^{-1}$. In this case we take $f_0(x) = \exp\{-(a/3)\|x\|^3\}$.

Examples of ψ are:

Example 2.2. (i) Nelson's scalar field model: Typically, it is given by

$$\psi_{\mathsf{nel}}(s, x, y) = -\frac{1}{s^2 + 1 + |x - y|^2}.$$

We refer to [1]. This example corresponds to the case $q_0 = 0$.

(ii) Non-convex interactions:

$$\psi_{\mathsf{nc}}(s, x, y) = \psi_0(s)\upsilon(x - y),$$

where $\upsilon(x) = |x|^{q_0+1} + Q(|x|)$ with Q a polynomial whose degree is less than $q_0 + 1$, and ψ_0 is as in (A.2). Note that υ need not be convex.

By Theorem 2.1 we have established the existence for a large class of interactions including Example 2.2; indeed, applying the theorem to the above examples, we conclude: (i) there always exists a Gibbs measure for $(\varphi_{\mathsf{ou}}, \psi_{\mathsf{nel}})$ and for $(\varphi_{\mathsf{dw}}, \psi_{\mathsf{nel}})$; (ii) there exists a Gibbs measure for $(\varphi_{\mathsf{ou}}, \psi_{\mathsf{nc}})$ (resp. for $(\varphi_{\mathsf{dw}}, \psi_{\mathsf{nc}})$) if $q_0 < 1$ (resp. $q_0 < 2$).

3. Key theorem: Localization

To show Theorem 2.1, we prepare a key theorem, namely, the *localization* of finite-volume Gibbs measures. As was already suggested in (1.3), we will incorporate the effect of the external potential into the reference measure; this means we take a $P(\phi)_1$-process as a reference process, instead of a Brownian motion itself.

For the Schrödinger operator $H_0 = -(1/2)\Delta + \varphi_0$, recall that the associated $P(\phi)_1$-process is determined by the stochastic differential equation (SDE)

$$dX(t) = dW(t) - \nabla u_0(X(t))\, dt \tag{3.1}$$

with W a d-dimensional Brownian motion. Here $u_0 = -\log f_0$ as in (i) of (A.1a). The process X is stationary under the measure $f_0(x)^2 dx$, which we denote by $m_0(dx)$. Here we assume that f_0 is normalized so that $m_0(dx)$ is a probability measure. We denote by $\mathcal{P}^{\varphi_0}$ the law of this process on $\mathcal{C}$. For the precise definition of $P(\phi)_1$-processes and related notion, see [20, 22] and references therein; see also [2] for more detailed descriptions.

We take $\mathcal{P}^{\varphi_0}$ as a reference measure, and perturb it by φ_1 and ψ: For a finite interval $\Lambda \subset \mathbb{R}$, let $\mathcal{P}^{\varphi_0}_\Lambda$ denote the restriction of $\mathcal{P}^{\varphi_0}$ to the σ-field $\sigma(\pi_\Lambda)$. Define the finite-volume Gibbs measure $\mu^{\varphi,\psi}_\Lambda$ by

$$d\mu^{\varphi,\psi}_\Lambda(X) = (Z^{\varphi,\psi}_\Lambda)^{-1} \exp\{-\mathcal{H}^{\varphi_1,\psi}_\Lambda(X)\}\, d\mathcal{P}^{\varphi_0}_\Lambda(X), \tag{3.2}$$

where

$$\mathcal{H}^{\varphi_1,\psi}_\Lambda(X) = \int_\Lambda \varphi_1(X(s))\, ds + \frac{1}{2}\int_{\Lambda^2} \psi(s-t, X(s), X(t))\, dsdt$$

and $Z^{\varphi,\psi}_\Lambda = \langle \exp\{-\mathcal{H}^{\varphi_1,\psi}_\Lambda(X)\}\rangle_{\mathcal{P}^{\varphi_0}}$. Note that $Z^{\varphi,\psi}_\Lambda < \infty$ by assumption. $\mu^{\varphi,\psi}_\Lambda$ is a probability measure on $\mathcal{C}_\Lambda$ with free boundary condition. If we take a symmetric interval $\Lambda = [-T, T]$, each corresponding subscript Λ is replaced by T; e.g., $\mathcal{H}^{\varphi_1,\psi}_\Lambda = \mathcal{H}^{\varphi_1,\psi}_T$, $\mu^{\varphi,\psi}_\Lambda = \mu^{\varphi,\psi}_T$ and so on.

Note that by (i) of (A.2), the growth order of ψ in spatial variables x and y is less than or equal to $q_0 + 1$. Suppose we had proved that $\{\mu^{\varphi,\psi}_\Lambda\}_{\Lambda\subset\mathbb{R}}$ has the following *localization* property:

$$\sup_{\Lambda\subset\mathbb{R}} \max_{t\in\Lambda} \langle |X(t)|^{q_1}\rangle_{\mu^{\varphi,\psi}_\Lambda} < \infty \tag{$*$}$$

for some $q_1 \geq q_0 + 1$. Then, from the same argument as in [17, Sect. 4], the existence of a (φ, ψ)-Gibbs measure follows. The next theorem shows that $(*)$ is really the case:

Theorem 3.1. *Under* (A.1a)*,* (A.1b)*,* (A.2) *and* (A.3)*, the localization* $(*)$ *holds.*

Remark 3.1. As we will see in the proof, the localization $(*)$ holds with $q_1 = 2p_0$. Note that $2p_0 \geq q_0 + 1$ since we assume $p_0 \geq 1$ and $p_0 > q_0$.

4. Proof of Theorem 3.1

As was mentioned, once Theorem 3.1 is shown, then the existence result Theorem 2.1 can be obtained as a corollary, in the same way as [17]. We thus concentrate on proving Theorem 3.1. The essence of our proof is simple; we only have to use integration by parts once, which is formulated as:

$$\int_{\mathbb{R}^d} \operatorname{div}(f_0 \nabla_z f_0) e^{-\mathcal{H}_\Lambda^{\varphi_1,\psi}(X^z)}\, dz = -\int_{\mathbb{R}^d} (f_0 \nabla_z f_0, \nabla_z e^{-\mathcal{H}_\Lambda^{\varphi_1,\psi}(X^z)})\, dz,$$

where X^z denotes a path given $X(0) = z$. See (4.8) below.

We begin with the following lemma:

Lemma 4.1. *Assume* (ii) *of* (A.1a). *Then there exist constants* $\mathsf{a}_i > 0, i = 1, 2, 3,$ *such that, for all* $x \in \mathbb{R}^d$,

$$U(x) \geq \mathsf{a}_1 \|x\|^{2p_0} - \mathsf{a}_2, \quad |\boldsymbol{V}(x)| \leq \mathsf{a}_3 \|x\|^{p_0}. \tag{4.1}$$

Proof. By (2.3), there exist $c, c' > 0$ and $r > 0$ such that $U(x) \geq c|x|^{2p_0}$ and $|\boldsymbol{V}(x)| \leq c'|x|^{p_0}$ for all $|x| \geq r$. By the continuity, U is bounded from below and $|\boldsymbol{V}|$ from above on $\{|x| \leq r\}$. Combining these ends the proof. □

Remark 4.1. By definition, $\int_{\mathbb{R}^d} U(x)\, m_0(dx) = 0$, which implies that U has a negative part. So the constant a_2 above cannot be taken negative; indeed, it must satisfy $\mathsf{a}_2 \geq \mathsf{a}_1 \int_{\mathbb{R}^d} \|x\|^{2p_0}\, m_0(dx)$.

From now on we take a symmetric interval $\Lambda = [-T, T]$ for simplicity. Let

$$M_T = \max_{|t| \leq T} \langle \|X(t)\|^{2p_0} \rangle_{\mu_T^{\varphi,\psi}}.$$

Proposition 4.2. *Assume* (A.1a), (A.1b), (A.2) *and* (A.3). *Let* a_i, $i = 1, 2, 3$, *be as in Lemma 4.1. Then it holds that*

$$\mathsf{a}_1 M_T - \mathsf{a}_2 \leq \mathsf{a}_3 C_{d,\alpha} \left(\mathsf{b} M_T^{\frac{p_0+p_1}{2p_0}} + 2\bar{\psi}_0 M_T^{\frac{p_0+q_0}{2p_0}} \right) \tag{4.2}$$

for all $T > 0$. *Here* $C_{d,\alpha} = 2\sqrt{d}/\alpha$ *and* $\bar{\psi}_0 = \int_{\mathbb{R}} \psi_0(s)\, ds$.

Once this proposition is shown, then the proof of Theorem 3.1 is straightforward:

Proof of Theorem 3.1. Let us consider the following equation in $\mathsf{m} > 0$:

$$\mathsf{a}_1 \mathsf{m} - \mathsf{a}_2 = \mathsf{a}_3 C_{d,\alpha} \left(\mathsf{b} \mathsf{m}^{\frac{p_0+p_1}{2p_0}} + 2\bar{\psi}_0 \mathsf{m}^{\frac{p_0+q_0}{2p_0}} \right). \tag{4.3}$$

If $p_0 > q_0$, this equation has a unique solution m_1. By Proposition 4.2, we then obtain

$$\max_{|t| \leq T} \langle \|X(t)\|^{2p_0} \rangle_{\mu_T^{\varphi,\psi}} \leq \mathsf{m}_1 \quad \text{for all } T > 0. \tag{4.4}$$

This implies the theorem. □

For simplicity, we prove Proposition 4.2 in the case $\varphi_1 \equiv 0$; that is, we let $\mathsf{b} = 0$ in (4.2) (for the proof of the case $\varphi_1 \neq 0$, see Remark 4.2 below). We write $\mathcal{H}_\Lambda^\psi(X)$ for $\mathcal{H}_\Lambda^{\varphi_1,\psi}(X)$ accordingly. In the following we suppress potentials from the notation if there is no confusion; e.g., $Z_\Lambda^{\varphi,\psi} = Z_\Lambda$, $\mu_T^{\varphi,\psi} = \mu_T$ and so on.

For each fixed $t_0 \in [-T, T]$, let $I = \langle U(X(t_0)) \rangle_{\mu_T}$. By the estimate on U in (4.1), we have

$$\mathsf{a}_1 \langle \| X(t_0) \|^{2p_0} \rangle_{\mu_T} - \mathsf{a}_2 \leq I. \tag{4.5}$$

We also have an upper estimate on I as follows:

Proposition 4.3. *It holds that*

$$I \leq 2\mathsf{a}_3 C_{d,\alpha} \bar{\psi}_0 \langle \| X(t_0) \|^{2p_0} \rangle_{\mu_T}^{\frac{1}{2}} M_T^{\frac{q_0}{2p_0}}.$$

Proposition 4.2 is an immediate consequence of these estimates:

Proof of Proposition 4.2. By (4.5) and Proposition 4.3, we have

$$\mathsf{a}_1 \langle \| X(t_0) \|^{2p_0} \rangle_{\mu_T} - \mathsf{a}_2 \leq 2\mathsf{a}_3 C_{d,\alpha} \bar{\psi}_0 \langle \| X(t_0) \|^{2p_0} \rangle_{\mu_T}^{\frac{1}{2}} M_T^{\frac{q_0}{2p_0}}.$$

Taking the maximum over $|t_0| \leq T$ on both sides leads to (4.2). □

It now remains to prove Proposition 4.3, which we will do in a sequence of lemmas.

First we consider the disintegration of $\mathcal{P}^{\varphi_0}$ by conditioning on $X(0)$. For this purpose, let W^+ and W^-, together with a probability measure P_W, be independent d-dimensional Brownian motions starting at 0. Let $X^{z,\pm}(t) \equiv X^z(t, W^\pm)$ be the strong solutions of (3.1) starting at z. We set

$$X^z(t) = \begin{cases} X^{z,+}(t), & t \geq 0, \\ X^{z,-}(-t), & t \leq 0. \end{cases}$$

Let $\mathcal{P}^{\varphi_0}(\,\cdot\,| X(0) = z)$ denote the regular conditional probability of $\mathcal{P}^{\varphi_0}$ given $X(0) = z$. From the Markov property of $\mathcal{P}^{\varphi_0}$, we easily deduce:

Lemma 4.4. *The process* $(\{X^z(t), t \in \mathbb{R}\}, P_W)$ *has the same law as* $\mathcal{P}^{\varphi_0}(\,\cdot\,| X(0) = z)$.

Recall from Section 3 that $\mathcal{P}^{\varphi_0}(X(0) \in dz) = m_0(dz)$. For a $\mathcal{P}^{\varphi_0}$-integrable functional F on $\mathcal{C}$, we have, by Lemma 4.4 and by Fubini's theorem,

$$\begin{aligned} \langle F(X) \rangle_{\mathcal{P}^{\varphi_0}} &= \int_{\mathbb{R}^d} \langle F(X) | X(0) = z \rangle_{\mathcal{P}^{\varphi_0}} \, m_0(dz) \\ &= \left\langle \int_{\mathbb{R}^d} F(X^z) \, m_0(dz) \right\rangle_{P_W}. \end{aligned} \tag{4.6}$$

By definition, $I = Z_T^{-1} \left\langle U(X(t_0)) e^{-\mathcal{H}_T^{\psi}(X)} \right\rangle_{\mathcal{P}^{\varphi_0}}$. For convenience, we may shift t_0 to the origin by the stationarity of $\mathcal{P}^{\varphi_0}$; moreover, disintegrating $\mathcal{P}^{\varphi_0}$ as (4.6), we have

$$I = Z_\Lambda^{-1} \left\langle \int_{\mathbb{R}^d} U(z) e^{-\mathcal{H}_\Lambda^{\psi}(X^z)} \, m_0(dz) \right\rangle_{P_W}, \tag{4.7}$$

where $\Lambda = [T_1, T_2]$ with $T_1 = -(T + t_0)$ and $T_2 = T - t_0$. Recall $m_0(dz) = f_0(z)^2 \, dz$. By integration by parts formula, and by the estimate on $|\boldsymbol{V}|$ in (4.1),

$$\begin{aligned} \int_{\mathbb{R}^d} U(z) e^{-\mathcal{H}_\Lambda^{\psi}(X^z)} \, m_0(dz) &= \int_{\mathbb{R}^d} (\boldsymbol{V}(z), \nabla_z \mathcal{H}_\Lambda^{\psi}(X^z)) e^{-\mathcal{H}_\Lambda^{\psi}(X^z)} \, m_0(dz) \\ &\le \mathsf{a}_3 \int_{\mathbb{R}^d} \|z\|^{p_0} |\nabla_z \mathcal{H}_\Lambda^{\psi}(X^z)| e^{-\mathcal{H}_\Lambda^{\psi}(X^z)} \, m_0(dz). \end{aligned} \tag{4.8}$$

We shall estimate $|\nabla_z \mathcal{H}_\Lambda^{\psi}(X^z)|$ from above by using the following lemma:

Lemma 4.5. *For each $z \in \mathbb{R}^d$ and $1 \le i \le d$, let $Y_i^z(t) = (\partial X_j^z(t)/\partial z_i)_{1 \le j \le d}$. Under the condition* (i) *of* (A.1a)*, it holds that, for all z and i,*

$$\left| Y_i^z(t) \right| \le e^{-\alpha|t|}, \quad t \in \mathbb{R}.$$

Proof. By symmetry, we need only to consider the case $t \ge 0$. Then by definition, $X^z(t) \equiv X^{z,+}(t)$ satisfies

$$X^z(t) = z + W(t) - \int_0^t \nabla u_0(X^z(s)) \, ds, \quad t \ge 0. \tag{4.9}$$

Here we simply write W for W^+. Differentiating both sides of (4.9) with respect to z_i and t successively, we have

$$\frac{d}{dt} Y(t) = -\mathrm{Hess}_{u_0}(X^z(t)) Y(t),$$

where $Y = Y_i^z$. By (2.2), it then holds that

$$\frac{1}{2} \frac{d}{dt} |Y(t)|^2 = -(Y(t), \mathrm{Hess}_{u_0}(X^z(t)) Y(t)) \le -\alpha \, |Y(t)|^2,$$

which shows $|Y(t)|^2 \le |Y(0)|^2 \, e^{-2\alpha t} = e^{-2\alpha t}$. This ends the proof. □

By Lemma 4.5, we obtain the following estimate on $|\nabla_z \mathcal{H}_\Lambda^{\psi}(X^z)|$.

Lemma 4.6. *Under the assumptions* (A.1a)(i) *and* (A.2)(i)*, we have*

$$|\nabla_z \mathcal{H}_\Lambda^{\psi}(X^z)| \le \sqrt{d} \int_{\Lambda^2} \left(\|X^z(s)\|^{q_0} + \|X^z(t)\|^{q_0} \right) \Psi_\alpha(ds, dt), \tag{4.10}$$

where we set $\Psi_\alpha(ds, dt) = \psi_0(s - t) e^{-\alpha|s|} \, ds dt$.

Proof. Let Y_i^z be as in Lemma 4.5. By the symmetry (2.1) of ψ, we see

$$\frac{\partial}{\partial z_i}\mathcal{H}_\Lambda^\psi(X^z) = \int_{\Lambda^2} \left(\nabla_x \psi(s-t, X^z(s), X^z(t)), Y_i^z(s)\right) dsdt$$

for each $1 \le i \le d$. Then by (i) of (A.2) and Lemma 4.5, we have

$$\left|\frac{\partial}{\partial z_i}\mathcal{H}_\Lambda^\psi(X^z)\right| \le \int_{\Lambda^2} \psi_0(s-t)(\|X^z(s)\|^{q_0} + \|X^z(t)\|^{q_0})e^{-\alpha|s|}\, dsdt$$

for all $1 \le i \le d$. Now the assertion follows readily. □

We denote by $J(X^z)$ the RHS of (4.10). Combining this lemma and (4.8), we have

$$\int_{\mathbb{R}^d} U(z)e^{-\mathcal{H}_\Lambda^\psi(X^z)}\, m_0(dz) \le \mathsf{a}_3 \int_{\mathbb{R}^d} \|z\|^{p_0} J(X^z) e^{-\mathcal{H}_\Lambda^\psi(X^z)}\, m_0(dz).$$

Plugging this estimate into (4.7), we obtain

$$\begin{aligned} I &\le \mathsf{a}_3 \langle \|X(0)\|^{p_0} J(X)\rangle_{\mu_\Lambda} \\ &= \sqrt{d}\mathsf{a}_3 \int_{\Lambda^2} \left\langle \|X(0)\|^{p_0} \left(\|X(s)\|^{q_0} + \|X(t)\|^{q_0}\right)\right\rangle_{\mu_\Lambda} \Psi_\alpha(ds, dt). \end{aligned} \tag{4.11}$$

Since $p_0 > q_0$, the following is immediate from the (generalized) Hölder inequality:

Lemma 4.7. *Let $M_\Lambda = \max_{t\in\Lambda} \langle \|X(t)\|^{2p_0}\rangle_{\mu_\Lambda}$. We have, for all $s \in \Lambda$,*

$$\langle \|X(0)\|^{p_0} \|X(s)\|^{q_0}\rangle_{\mu_\Lambda} \le \langle \|X(0)\|^{2p_0}\rangle_{\mu_\Lambda}^{\frac{1}{2}} M_\Lambda^{\frac{q_0}{2p_0}}.$$

Now we are in a position to prove Proposition 4.3:

Proof of Proposition 4.3. By (4.11) and Lemma 4.7, we see that

$$I \le 2\sqrt{d}\mathsf{a}_3 \langle \|X(0)\|^{2p_0}\rangle_{\mu_\Lambda}^{\frac{1}{2}} M_\Lambda^{\frac{q_0}{2p_0}} \Psi_\alpha(\Lambda^2).$$

Note that $\Psi_\alpha(\Lambda^2) \le \Psi_\alpha(\mathbb{R}^2) = (2/\alpha)\bar{\psi}_0$ by definition. Now we shift the origin to t_0 and obtain the proposition. □

Remark 4.2. Let $J(X^z)$ denote the RHS of (4.10) as above. If $\varphi_1 \ne 0$, the estimate (4.10) is then replaced by

$$|\nabla_z \mathcal{H}_\Lambda^{\varphi_1,\psi}(X^z)| \le \sqrt{d}\mathsf{b} \int_\Lambda \|X^z(s)\|^{p_1} e^{-\alpha|s|}\, ds + J(X^z).$$

This follows from the assumption (A.1b) and Lemma 4.5. The rest of the proof can be proceeded similarly to the above.

Remark 4.3. Since Lemma 4.7 remains true, the inequality (4.2) also holds in the case $p_0 = q_0$. In this case the equation (4.3) has a unique solution m_2 if $\mathsf{a}_1 > 2\mathsf{a}_3 C_{d,\alpha}\bar{\psi}_0$. Then (4.4) holds with m_2, from which the existence of the associated Gibbs measures follows.

5. Construction of hard-wall Gibbs measures with interactions

In this section we discuss the existence of hard-wall Gibbs measures; that is, we consider an external potential φ^+ such that $\varphi^+(x) = \infty$ for $x < 0$.

In the following, boldfaced letters denote elements in $\mathbb{R}^3$; e.g., $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. We often write $\rho(\boldsymbol{x})$ for $|\boldsymbol{x}|$. For a function f on $\mathbb{R}^+$, $(f \circ \rho)(\boldsymbol{x})$ means $f(\rho(\boldsymbol{x}))(\equiv f(|\boldsymbol{x}|))$ as usual.

We summarize the assumptions on potentials:

(A.1′) Assumptions on $\boldsymbol{\varphi^+}$. φ^+ is bounded from below. There exist φ_0^+ and φ_1^+ such that $\varphi^+ = \varphi_0^+ + \varphi_1^+$, taking values ∞ on $\{x < 0\}$ and satisfying (A.1′a) and (A.1′b), respectively:

(A.1′a) $\varphi_0^+ \circ \rho$ satisfies (A.1a) with $d = 3$; that is, φ_0^+ is continuous on $\{x \geq 0\}$, and the Schrödinger operator $\mathsf{H}_0^\rho = -(1/2)\Delta + \varphi_0^+ \circ \rho$ acting on $L^2(\mathbb{R}^3; d\boldsymbol{x})$ has a strictly positive ground state f_0 of class $C^2(\mathbb{R}^3)$ satisfying (i) and (ii) of (A.1a).

(A.1′b) $\varphi_1^+ \circ \rho$ satisfies (A.1b) with $d = 3$.

(A.2′) Assumptions on $\boldsymbol{\psi}$. The function $\psi^\rho(s, \boldsymbol{x}, \boldsymbol{y}) := \psi(s, \rho(\boldsymbol{x}), \rho(\boldsymbol{y}))$ satisfies (A.2) with $d = 3$.

(A.3′) p_0 is strictly larger than q_0.

Remark 5.1. By definition, we see f_0 in (A.1′a) is radially symmetric; that is, there exists a g_0 on $\mathbb{R}^+$ such that $\mathsf{f}_0(\boldsymbol{x}) = g_0(|\boldsymbol{x}|)$.

Now we state the existence of hard-wall Gibbs measures:

Theorem 5.1. *Assume* (A.1′)–(A.3′). *Then there exists a* (φ^+, ψ)*-Gibbs measure.*

The existence of hard-wall Gibbs measures with interactions had been an open problem; [17] required a certain symmetry on external potentials, which excluded hard-wall ones. We have established the existence of such measures by Theorem 5.1.

To prove this theorem, we do the finite-volume construction in the same manner as Section 3: For φ_0^+, let H_0^+ be the corresponding Schrödinger operator with Dirichlet boundary condition:

$$\begin{cases} H_0^+ f = -\frac{1}{2} f''(x) + \varphi_0^+(x) f(x), & x > 0, \\ f(0) = 0. \end{cases}$$

Let g_0 be as in Remark 5.1. As we see in Lemma 5.3, a ground state f_0^+ of H_0^+ is given by $f_0^+(x) = 2\sqrt{\pi} x g_0(x)$. The constant $2\sqrt{\pi}$ is chosen so that $\int_{\mathbb{R}^+} |f_0^+|^2 \, dx = \int_{\mathbb{R}^3} |\mathsf{f}_0|^2 \, d\boldsymbol{x}$. The $P(\phi)_1$-process associated with H_0^+ is determined by the SDE

$$dX(t) = dW(t) + \frac{dt}{X(t)} - v_0'(X(t)) \, dt \tag{5.1}$$

with W a 1-dimensional Brownian motion. Here we set $v_0 = -\log g_0$. The process X is stationary under the probability measure $f_0^+(x)^2 \, dx$ (we assume f_0^+ is normalized). We denote by $\mathcal{P}^{\varphi_0^+}$ the law on $C(\mathbb{R}; \mathbb{R}^+)$ of this process. For $\mathcal{P}^{\varphi_0^+}$ and potentials φ_1^+, ψ, we define the finite-volume Gibbs measure $\mu_\Lambda^{\varphi^+, \psi}$ through

(3.2). Similarly to Theorem 2.1, the existence result Theorem 5.1 follows from the localization result of $\{\mu_\Lambda^{\varphi^+,\psi}\}_{\Lambda\subset\mathbb{R}}$:

Theorem 5.2. *Assume* (A.1′a)*,* (A.1′b)*,* (A.2′) *and* (A.3′)*. Then the localization* $(*)$ *for* $\{\mu_\Lambda^{\varphi^+,\psi}\}_{\Lambda\subset\mathbb{R}}$ *holds.*

A key to Theorem 5.2 is the following identity in law:

$$(X,\mathcal{P}^{\varphi_0^+}) \stackrel{(d)}{=} (|\boldsymbol{X}|,\mathcal{P}^{\varphi_0^+\circ\rho}), \tag{5.2}$$

where, on the RHS, $\mathcal{P}^{\varphi_0^+\circ\rho}$ denotes the law of the $P(\phi)_1$-process associated with $\mathsf{H}_0^\rho = -(1/2)\Delta + \varphi_0^+\circ\rho$, and $\boldsymbol{X}\in C(\mathbb{R};\mathbb{R}^3)$. Here is an example:

Example 5.1. When $\varphi_0^+(x) = \frac{1}{2}a^2x^2$, $x\geq 0$, the $P(\phi)_1$-process X associated with H_0^+ has the following explicit representation:

$$\{X(t), t\in\mathbb{R}\} \stackrel{(d)}{=} \left\{\frac{1}{\sqrt{2a}}e^{-at}R^{(3)}(e^{2at}), t\in\mathbb{R}\right\},$$

where $\{R^{(3)}(s), s\geq 0\}$ is a 3-dimensional Bessel process starting at 0. In view of the DLR equation, this can be seen from the fact that a Brownian bridge conditioned to be positive has the same law as a 3-dimensional Bessel bridge.

Once the identity (5.2) is verified, then Theorem 5.2 is straightforward:

Proof of Theorem 5.2. For $\varphi_0^+\circ\rho$, $\varphi_1^+\circ\rho$ and ψ^ρ, let μ_Λ^ρ be the finite-volume Gibbs measure on $C(\Lambda;\mathbb{R}^3)$ defined via (3.2):

$$d\mu_\Lambda^\rho(\boldsymbol{X}) = (Z_\Lambda^\rho)^{-1}e^{-\mathcal{H}_\Lambda^\rho(\boldsymbol{X})}\,d\mathcal{P}_\Lambda^{\varphi_0^+\circ\rho}(\boldsymbol{X}).$$

Here $\mathcal{H}_\Lambda^\rho = \mathcal{H}_\Lambda^{\varphi_1^+\circ\rho,\psi^\rho}$ and Z_Λ^ρ is the normalization. Note that, by definition, $\mathcal{H}_\Lambda^\rho(\boldsymbol{X}) = \mathcal{H}_\Lambda^{\varphi_1^+,\psi}(|\boldsymbol{X}|)$. From this and (5.2), we easily see that, for all $t\in\Lambda$,

$$\langle X(t)^{2p_0}\rangle_{\mu_\Lambda^{\varphi^+,\psi}} = \langle|\boldsymbol{X}(t)|^{2p_0}\rangle_{\mu_\Lambda^\rho}. \tag{5.3}$$

By the assumptions on potentials, we may use Theorem 3.1 to see that the localization $(*)$ for $\{\mu_\Lambda^\rho\}_{\Lambda\subset\mathbb{R}}$ holds with $q_1 = 2p_0$. Combining this with (5.3) ends the proof. □

The identity (5.2) is an immediate consequence of the following lemma:

Lemma 5.3. *(i) Let* $\{T_t^+\}_{t\geq 0}$ *and* $\{\mathsf{T}_t^\rho\}_{t\geq 0}$ *be the semi-group generated by* H_0^+ *and that generated by* H_0^ρ*, respectively. Then it holds that, for* $x>0$ *and* $\boldsymbol{x}\in\mathbb{R}^3$ *with* $|\boldsymbol{x}| = x$*,*

$$(T_t^+f)(x) = x(\mathsf{T}_t^\rho\frac{f\circ\rho}{\rho})(\boldsymbol{x}),\quad t\geq 0. \tag{5.4}$$

In particular, a ground state of H_0^+ *is given by* $xg_0(x)$*.*
(ii) Under $\mathcal{P}^{\varphi_0^+\circ\rho}$*, the process* $\{|\boldsymbol{X}(t)|, t\in\mathbb{R}\}$ *satisfies the SDE* (5.1)*. Moreover,* $|\boldsymbol{X}(0)|$ *has the same law as* $\mathcal{P}^{\varphi_0^+}(X(0)\in\cdot)$*.*

Remark 5.2. The identity (5.4) is related to the h-transform of Doob, cf. [19, Chap. VIII].

Proof. (i) Let $\mathcal{W}_x$ be the law of 1-dimensional Brownian motion starting at x. Let $\tau_0 = \inf\{t \geq 0; X(t) = 0\}$. Then T_t^+ is expressed as

$$(T_t^+ f)(x) = \langle f(X(t \wedge \tau_0)) \exp\{-\int_0^t \varphi_0^+(X(s \wedge \tau_0))\, ds\}\rangle_{\mathcal{W}_x}, \quad x > 0,$$

where $\wedge$ denotes the minimum. By the conditional equivalence between absorbing Brownian motion and 3-dimensional Bessel process (see, e.g., [10, Lemma 5.2.8]), this is rewritten as, for $\boldsymbol{x} \in \mathbb{R}^3$ with $|\boldsymbol{x}| = x$,

$$(T_t^+ f)(x) = x\langle \frac{f(|\boldsymbol{X}(t)|)}{|\boldsymbol{X}(t)|} \exp\{-\int_0^t \varphi_0^+(|\boldsymbol{X}(s)|)\, ds\}\rangle_{\mathcal{W}_{\boldsymbol{x}}^{(3)}}. \tag{5.5}$$

Here $\mathcal{W}_{\boldsymbol{x}}^{(3)}$ is the law of 3-dimensional Brownian motion starting at $\boldsymbol{x}$. Recalling $\rho(\boldsymbol{x}) = |\boldsymbol{x}|$, we see that (5.5) shows (5.4). (ii) Noting $\nabla \log \mathsf{f}_0 = -|\boldsymbol{x}|^{-1} v_0'(|\boldsymbol{x}|)\boldsymbol{x}$, we see that the process $(\boldsymbol{X}, \mathcal{P}^{\varphi_0^+ \circ \rho})$ satisfies the following SDE:

$$d\boldsymbol{X}(t) = d\boldsymbol{W}(t) - \frac{v_0'(|\boldsymbol{X}(t)|)}{|\boldsymbol{X}(t)|}\boldsymbol{X}(t)\, dt,$$

where $\boldsymbol{W}$ is a 3-dimensional Brownian motion. By Itô's formula, we have

$$d|\boldsymbol{X}(t)| = d\widetilde{W}(t) + \frac{dt}{|\boldsymbol{X}(t)|} - v_0'(|\boldsymbol{X}(t)|)\, dt, \quad \widetilde{W}(t) = \int_0^t (\frac{\boldsymbol{X}(s)}{|\boldsymbol{X}(s)|}, d\boldsymbol{W}(s)).$$

Since $\widetilde{W}$ is a Brownian motion, the former assertion is proved. For the latter, note that $\mathcal{P}^{\varphi_0^+ \circ \rho}(\boldsymbol{X}(0) \in d\boldsymbol{x}) = \mathsf{f}_0(\boldsymbol{x})^2\, d\boldsymbol{x}$, hence that $|\boldsymbol{X}(0)|$ is distributed as $f_0^+(x)^2\, dx$. This ends the proof. □

Remark 5.3. Lemma 5.3 suggests that, in the Markovian case, namely, the case $\psi = 0$, the existence of Gibbs measures for the hard-wall external potential φ^+ is reduced to that of the ground state for the Schrödinger operator defined by

$$-\frac{1}{2}\Delta + \varphi^+(|\boldsymbol{x}|) \quad \text{on } L^2(\mathbb{R}^3; d\boldsymbol{x}). \tag{5.6}$$

Now let us consider the case where φ^+ is a "single well" such as

$$\varphi^+(x) = -\frac{\beta}{(1+x^2)^\gamma}, \quad x \geq 0,$$

for $\beta, \gamma > 0$ (the *attractive* potential). The parameters β and γ control the "depth" and "width" of the well, respectively. Let $\mathsf{H}_\beta \equiv \mathsf{H}_{\beta,\gamma}$ be the Schrödinger operator defined by (5.6). Note that the essential spectrum of H_β is equal to $[0, \infty)$ (see, e.g., [18]). Let λ_β be the bottom of the spectrum of H_β. If $0 < \gamma < 1$, then $\lambda_\beta < 0$ for all $\beta > 0$ and λ_β is a simple eigenvalue; that is, the associated ground state exists. On the other hand, if $\gamma \geq 1$, then there exists a $\beta = \beta_c$ (the *threshold*) such that $\lambda_\beta \equiv 0$ for $\beta \leq \beta_c$ and $\lambda_\beta < 0$ for $\beta > \beta_c$. See [6, Chap. 8]. This phenomenon may be regarded as a counterpart to *wetting transitions* in lattice models [4, 5]. We will return to this somewhere else.

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