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Phase ordering after a deep quench: the stochastic Ising and hard core gas models on a tree

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Abstract. Consider a low temperature stochastic Ising model in the phase coexistence regime with Markov semigroup P_t. A fundamental and still largely open problem is the understanding of the long time behavior of $\delta_\eta P_t$ when the initial configuration η is sampled from a highly disordered state ν (e.g. a product Bernoulli measure or a high temperature Gibbs measure). Exploiting recent progresses in the analysis of the mixing time of Monte Carlo Markov chains for discrete spin models on a regular b-ary tree $\mathbb{T}^b$, we study the above problem for the Ising and hard core gas (independent sets) models on $\mathbb{T}^b$. If ν is a biased product Bernoulli law then, under various assumptions on the bias and on the thermodynamic parameters, we prove ν-almost sure weak convergence of $\delta_\eta P_t$ to an extremal Gibbs measure (pure phase) and show that the limit is approached at least as fast as a stretched exponential of the time t. In the context of randomized algorithms and if one considers the Glauber dynamics on a large, finite tree, our results prove fast local relaxation to equilibrium on time scales much smaller than the true mixing time, provided that the starting point of the chain is not taken as the worst one but it is rather sampled from a suitable distribution.

1. Introduction

Let $G = (V, E)$ be a countable infinite graph of bounded degree and consider, for definiteness, a continuous time stochastic Ising model (Glauber dynamics) $\{\sigma_t^\eta\}_{t \geqslant 0}$ on G with initial condition η and infinitesimal generator $\mathcal{L}$. Here η is picked from the set Ω of assignments of a ± 1 variable to each vertex $x \in V$. The main problems discussed in this paper can be formulated as follows.

Assume that the thermodynamic parameters are such that there exist multiple reversible Gibbs measures for $\mathcal{L}$. For stochastic Ising models this amounts to say that the inverse temperature β and the external field h are such that $\mu^+ \neq \mu^-$, where μ^+ and μ^- are the Gibbs measures obtained by taking infinite volume limits with pure $+$ and $-$ boundary conditions, respectively. Suppose that $\eta \in \Omega$ is distributed according to a Bernoulli product measure with parameter p, i.e. $\{\eta_x\}_{x\in V}$ is a collection of i.i.d. random variables with $\mathbb{P}(\eta_x = +1) = p$. Then:

i) Under which condition on the bias p is the Ising plus phase μ^+ the unique limit point of the law of σ_t^η as $t \to \infty$, for a.a. η ?
ii) If so, how fast does the law of σ_t^η approach μ^+ ?

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The above questions, with G some regular lattice, have their origin in the theory of "phase ordering kinetics" [6] – that is growth of order through a dynamical domain coarsening – and clearly represent basic problems in the theory of interacting particle systems. Unfortunately, a rigorous approach to these problems is still largely missing.

If the law of the starting configuration η stochastically dominates the plus phase μ^+ it is possible to use some monotonicity arguments (allowed by the ferromagnetic character of the model) to prove that μ^+ is indeed the unique limiting point of the process and that the convergence takes place faster than any inverse power of t. We refer to section 6.7 in [23] for the case $G = \mathbb{Z}^d$ and to our Lemma 2.4 below for a stronger statement in the case of regular trees. However, it is easily seen that such a stochastic domination requirement for the initial Bernoulli distribution forces the bias p to be exponentially close to 1 when $\beta \to \infty$. For $\beta < \infty$ we do not know of any result that goes beyond this simple case.

On the other hand, the extreme case $\beta = \infty$ (zero temperature Glauber dynamics) has received considerable attention in the probabilistic literature, and various kinds of graphs ($\mathbb{Z}^d$, the hexagonal lattice and the binary tree) have been considered [7, 13, 8, 14, 11]. In this case, besides the motivation from physics to study simple models of spatial domain coarsening, there is also an interesting connection with (non–linear) voter models [21]. The relevant quantities are then the probability that a given vertex flips its value finitely or infinitely many times, the probability that a given spin has not flipped before time t, the typical size of clusters of vertices with a common spin value and other related percolation questions.

Going back to our original problems, a major obstacle for progresses in the case $G = \mathbb{Z}^d$ is represented by the absence of tight bounds on the mixing time of the Glauber dynamics in finite boxes with plus boundary conditions, i.e. those boundary conditions that select the plus phase. On the contrary, when G is the regular b–ary tree, this question has been recently solved in a sharp and constructive way for various models [24]. Exploiting the results of [24] we have been able to study the above basic questions for two attractive systems on trees: the Ising model and the hard core gas (independent sets). Our results provide some answers to i) and ii) in non trivial cases. For instance we show that if the bias is sufficiently large (but independent of β) then we have the desired convergence for *all* temperatures. The paper also includes a discussion of several interesting problems that are left unsolved and that we would like to consider in future work. For simplicity we present now our main result only for the Ising model, and defer the reader to section 6 for the analogous theorem for the hard-core gas. Before stating our results we will now briefly overview the model and its basic features.

1.1. The Ising model on the b-ary tree

From now on $\mathbb{T}^b$ denotes the infinite, rooted b–ary tree, where each vertex has exactly b children ($b \geqslant 2$ is a given integer). The Ising Gibbs measure on $\mathbb{T}^b$ at inverse temperature β and external field h, formally given by

$$\mu(\sigma) \propto \exp\left[\beta\left(\sum_{xy\in E}\sigma_x\sigma_y + h\sum_x \sigma_x\right)\right],$$

where E is the set of edges of $\mathbb{T}^b$, has recently received a lot of attention as the canonical example of a statistical physics model on a "non-amenable" graph (i.e., one whose boundary is of comparable size to its volume) – see e.g. [4, 15, 10, 31, 17, 2, 3]. The phase diagram of the model in the (h, β) plane is known ([12, 22]) to be quite different from that on the cubic lattice $\mathbb{Z}^d$ (see Fig. 1).

We now recall some of its basic features. We write T_ℓ for the rooted tree obtained by removing all vertices which are at distance greater than ℓ from the root. The measures μ^+ and μ^- are obtained by imposing $+1$ and, respectively, -1 boundary data at the leaves of T_ℓ and taking the limit $\ell \to \infty$. The free measure μ^{free} is defined as the limit $\ell \to \infty$ when the boundary data at the leaves of T_ℓ are free (i.e. absent).

On the line $h = 0$ there is a first critical value $\beta_0 = \frac{1}{2}\log\left(\frac{b+1}{b-1}\right)$, marking the dividing line between uniqueness and non-uniqueness of the Gibbs measure (i.e. $\mu^+ \neq \mu^-$ as soon as $\beta \geqslant \beta_0$). Then, in sharp contrast to the model on $\mathbb{Z}^d$, there is a second critical point $\beta_1 = \frac{1}{2}\log\left(\frac{\sqrt{b}+1}{\sqrt{b}-1}\right)$ which is often referred to as the "spin-glass critical point" [9] and has different interpretations. If one considers for instance the model with $h = 0$ on the finite tree T_ℓ with i.i.d. Bernoulli random boundary data η with $p = 1/2$ at the leaves of T_ℓ, then the distribution of the magnetization at the root (as a function of η) becomes non trivial only if $\beta > \beta_1$, see [9]. In particular, as $\ell \to \infty$, for $\beta \leqslant \beta_1$ the Gibbs measure on T_ℓ with the above random boundary η, converges (weakly) a.s. to the free measure μ^{free}. Another way to look at β_1 is to say that μ^{free} is an extremal Gibbs measure iff $\beta \leqslant \beta_1$ (see [4, 15, 16, 2] and, more recently, [24]). Finally β_1 has also the interpretation of the non–reconstruction/reconstruction threshold in the context of "bit reconstruction problems" on a noisy symmetric channel [10, 27, 26].

When an external field h is added to the system, it turns out that for all $\beta > \beta_0$, there is a critical value $h_c = h_c(\beta, b) > 0$ of the field such that $\mu^+ \neq \mu^-$ iff

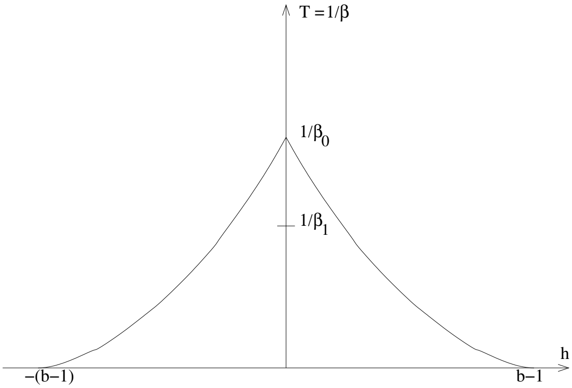

Fig. 1. The critical field $h_c(\beta, b)$. The Gibbs measure is unique above the curve

$|h| \leqslant h_c$. The Ising model on the tree at external field $h = \pm h_c$ therefore shares the following two properties with the classical Ising model on $\mathbb{Z}^d$ at zero external field: on one hand the Gibbs measure is sensitive to the choice of boundary condition; on the other hand any arbitrarily small increase of $|h|$ causes the Gibbs measure to become insensitive to the boundary condition.

1.2. The Glauber dynamics

The *Glauber dynamics* on $\mathbb{T}^b$ is the unique Markov process $\{\sigma_t^\eta\}_{t\,\geqslant\,0}$ on Ω with $\sigma_{t=0}^\eta = \eta$ and Markov generator $\mathcal{L}$ formally given by

$$(\mathcal{L}f)(\sigma) = \sum_{x\in\mathbb{T}^b} c_x(\sigma)[f(\sigma^x) - f(\sigma)]\,, \tag{1.1}$$

where σ^x denotes the configuration obtained from σ by flipping the spin at x, and $c_x(\sigma)$ denotes the flip rate at x.

Glauber dynamics on trees has received recently considerable interest [2, 24]. Results in [24] show in a rather strong form that the mixing time (see e.g. [29] for a definition) on the finite subtree T_ℓ is always $O(\ell)$ if either $\beta < \beta_1$ and h is arbitrary or if β, h are arbitrary and the boundary conditions on the leaves of T_ℓ are identically equal to $+1$ (or, by symmetry, to -1). In particular, the Glauber dynamics in the pure plus phase μ^+ always mixes fast (see [24] and section 2 below for more details).

Although all our results apply to any choice of finite–range, uniformly positive, bounded and attractive flip rates satisfying the detailed balance condition w.r.t. the Ising Gibbs measure (see [23]), for simplicity in the sequel we will work with a specific choice known as the *heat-bath* dynamics (see section 2 below for the definition). We will use the standard notation $P_t = e^{t\mathcal{L}}$ for the Markov semigroup associated to $\mathcal{L}$. The spin at x at time t with starting configuration η is denoted by $\sigma_{t,x}^\eta$ and we will often use the shortcut notation

$$\rho_{t,x}(\eta) = (P_t\sigma_x)(\eta) \tag{1.2}$$

for the expected value $\mathbb{E}(\sigma_{t,x}^\eta)$ of $\sigma_{t,x}^\eta$ given that the process starts in η.

1.3. Main results

In order to state our main results we need an extra bit of notation. We first define the set of initial configurations η such that the Glauber dynamics σ_t^η converges weakly, at a certain rate, to the plus phase μ^+.

Definition 1.1. *Given $\alpha \in (0, 1)$, Ω_α will denote the set of starting configurations $\eta \in \Omega$ such that for any $x \in \mathbb{T}^b$ there exists a time $t_0 = t_0(\eta, x) < \infty$ such that for all $t \geqslant t_0$*

$$|\,\rho_{t,x}(\eta) - \mu^+(\sigma_x)\,| \leqslant \exp(-t^\alpha)\,. \tag{1.3}$$

We will see in Corollary 2.3 below that for any $\eta \in \Omega_\alpha$ the law of the process σ_t^η converges weakly to μ^+ as $t \to \infty$.

The initial configuration η is often sampled from a Bernoulli product measure with parameter p, i.e. $\eta_x = +1$ with probability p and $\eta_x = -1$ with probability $1 - p$ independently for each $x \in \mathbb{T}^b$. We write $\mathbb{P}_p, \mathbb{E}_p$ for the corresponding probability and expectation. Finally, we need to recall the notion of partial ordering (stochastic domination) between probability measures on Ω. Given two configurations $\sigma, \eta \in \Omega$ we will write $\sigma \leqslant \eta$ iff $\sigma_x \leqslant \eta_x\ \forall x \in \mathbb{T}^b$. A function $f : \Omega \mapsto \mathbb{R}$ is called *monotone increasing (decreasing)* if $\sigma \leqslant \sigma'$ implies $f(\sigma) \leqslant f(\sigma')$ ($f(\sigma) \geqslant f(\sigma')$). Given two probability measures μ, μ' on Ω we write $\mu \leqslant \mu'$ if $\mu(f) \leqslant \mu'(f)$ for all (bounded and measurable) increasing functions f.

Our main results can now be stated as follows.

Theorem 1.2.

a) *For every $a > 0$, $b \geqslant 2$, there exists $p < 1$ such that for all $\beta \in (0, \infty)$ and $h \geqslant -h_c(\beta, b) + a$ we have $\nu(\Omega_\alpha) = 1$, for some $\alpha = \alpha(\beta, h, b) > 0$, for any initial distribution ν such that $\nu \geqslant \mathbb{P}_p$.*
b) *For every $p > \frac{1}{2}$, there exist $b_0 \in \mathbb{N}$ and $\beta_0 \in (0, \infty)$ such that for $h = 0$, $b \geqslant b_0$, $\beta \geqslant \beta_0$ we have $\nu(\Omega_\alpha) = 1$, for some $\alpha = \alpha(\beta, b, p) > 0$, for any initial distribution ν such that $\nu \geqslant \mathbb{P}_p$.*
c) *Let Ω_α^- denote the event defined in (1.3) with μ^- in place of μ^+. For every $p < 1$ there exist $b_0 \in \mathbb{N}$ and $\beta_0 \in (0, \infty)$ such that for $b \geqslant b_0$, $\beta \geqslant \beta_0$ and $h = -h_c(\beta, b)$, we have $\nu(\Omega_\alpha^-) = 1$ for some $\alpha = \alpha(\beta, b, p) > 0$, for all initial distributions ν such that $\nu \leqslant \mathbb{P}_p$.*

1.4. Remarks

Let us make some remarks on the above statements.

1. We believe that in the case $h = 0$, convergence to the plus phase should occur as soon as $p > \frac{1}{2}$. Unfortunately our bounds on p in statement *a)* are far from being sharp. However, as stated in *b)*, we can approach the critical value $\frac{1}{2}$, by taking b large. Another interesting issue is the dependence of p on h. Our technique in the proof of part a) of Theorem 1.2 breaks down in the case $h = -h_c(\beta, b)$ and the value of p in that statement approaches 1 as $a \to 0$. On the other hand, statement *c)* shows that if $h = -h_c(\beta, b)$ the critical value of p for convergence to the plus phase must approach 1 when $b \to \infty$.

2. The main arguments we use to prove Theorem 1.2 are based on two essential features of the Ising model on $\mathbb{T}^b$. The first is monotonicity which is shared by all so–called *attractive* interacting particle systems. The second is a sort of "rigidity" of pure phases for spin systems on trees. Roughly speaking this means that, as long as $h > -h_c$, if we are in the pure phase μ^+ we can add a small density of spins of the opposite ($-$) phase and this will not alter significantly the structure of μ^+. We refer to section 3 below, where we introduce *obstacles* to bound the magnetization $\rho_{t,x}(\eta)$ from below. The hard core gas model will be shown to have both these properties and our results there (see Theorem 6.1

below) will be obtained essentially by the same methods. On the other hand these techniques do not apply when the above mentioned rigidity of phases is absent, as e.g. in the Ising model on $\mathbb{Z}^d$ (see [30] for a deep investigation of the metastable behavior of this model), or when there is no attractivity, as e.g. in the q–state Potts model for $q \geqslant 3$.

3. A close check of the various probabilistic estimates needed for the proof of Theorem 1.2 and which are behind a Borel–Cantelli characterization of the set Ω_α, shows that there is also an L^2-version of Theorem 1.2, with (1.3) replaced by a bound of the form:

$$\mathbb{E}_p\big((\rho^\eta_{t,x} - \mu^+(\sigma_x))^2\big) \leqslant \exp(-t^\alpha)\,,$$

for any t large enough.

1.5. Plan of the paper

The rest of the paper is organized as follows. In section 2 we give the basic preliminaries for the proof of Theorem 1.2. In section 3 we explain our main argument. In particular, here we give the proof of Theorem 1.2 by assuming the validity of several technical claims. Section 4 and 5 deal with the proof of these claims. In section 6 we present our results for the hard core gas. Finally, some further results and open problems will be discussed in section 7.

2. Some preliminaries

Here we first collect several useful preliminaries concerning the Gibbs measure and the Glauber dynamics and then discuss some basic results on convergence to the plus phase, together with properties of the sets Ω_α introduced above.

2.1. Finite Gibbs measures on the b-ary tree

We denote by $d(x, y)$ the tree distance between two vertices $x, y \in \mathbb{T}^b$. If r is the root of the tree, we write $d(x) = d(x, r)$ for the depth of x. When A is a subset of vertices of $\mathbb{T}^b$ we set $d(x, A) = \inf_{z\in A} d(x, z)$. The boundary of A, ∂A, is defined as the set of vertices x such that $d(x, A) = 1$. $E(A)$ denotes the set of $\mathbb{T}^b$–edges (x, y) with $x, y \in A$.

The Ising spin configurations space is the set $\Omega = \{-1, +1\}^{\mathbb{T}^b}$ and its elements will be denoted by Greek letters σ, η, ξ etc. The set Ω is equipped with the standard σ–algebra $\mathcal{F}$ generated by the variables $\{\sigma_x\}_{x\in\mathbb{T}^b}$. For any finite subset $A \subseteq \mathbb{T}^b$ and any $\eta \in \Omega$, we denote by μ^η_A the Gibbs distribution over Ω conditioned on the configuration outside A being η: i.e., if $\sigma \in \Omega$ agrees with η outside A then

$$\mu^\eta_A(\sigma) \propto \exp\Big[\beta\big(\sum\nolimits_{xy\in E(A\cup\partial A)} \sigma_x\sigma_y + h\sum\nolimits_{x\in A} \sigma_x\big)\Big],$$

where β is the inverse temperature and h the external field. We define $\mu^\eta_A(\sigma) = 0$ otherwise. If the boundary configuration η is identically equal to $+1$ (-1) we will denote the corresponding conditional Gibbs distribution by μ_A^+ (μ_A^-). Whenever the set A will coincide with the finite subtree $T_\ell = \{x \in \mathbb{T}^b : d(x) \leqslant \ell\}$ we will abbreviate the symbol T_ℓ in the Gibbs measure with ℓ, i.e. μ_ℓ^η stands for $\mu_{T_\ell}^\eta$.

For a bounded measurable function $f : \Omega \to \mathbb{R}$ we denote by $\mu_A^\eta(f) = \sum_{\sigma\in\Omega} \mu_A^\eta(\sigma) f(\sigma)$ the *expectation* of f w.r.t. the distribution μ_A^η. Analogously, for any $X \in \mathcal{F}$, $\mu_A^\eta(X) := \mu_A^\eta(\mathbf{1}_X)$ where $\mathbf{1}_X$ is the characteristic function of the event X. We will write $\mathrm{Var}_{\mu_A^\eta}(f)$ or $\mathrm{Var}_A^\eta(f)$ for the *variance* $\mu_A^\eta(f^2) - \mu_A^\eta(f)^2$ and (for $f \geqslant 0$) $\mathrm{Ent}_{\mu_A^\eta}(f)$ or $\mathrm{Ent}_A^\eta(f)$ for the *entropy* $\mu_A^\eta(f \log f) - \mu_A^\eta(f) \log \mu_A^\eta(f)$ w.r.t. μ_A^η. Note that $\mathrm{Var}_A^\eta(f) = 0$ iff, conditioned on the configuration outside A being η, f does not depend on the configuration inside A. The same holds for $\mathrm{Ent}_A^\eta(f)$. We shall use the symbol μ_A for the map $\eta \to \mu_A^\eta$. Similarly, Var_A and Ent_A stand for $\eta \to \mathrm{Var}_A^\eta$ and $\eta \to \mathrm{Ent}_A^\eta$.

A probability measure μ on $\big(\Omega, \mathcal{F}\big)$ will be called a Gibbs measure for the Ising model with parameters (β, h) if

$$\mu\big(\mu_A(X)\big) = \mu(X), \qquad \text{for all } X \in \mathcal{F} \text{ and all finite sets } A \subset \mathbb{T}^b.$$

In this work a crucial role will be played by the following monotonicity property of the Gibbs measures (and of the Glauber dynamics, see below) known as *attractivity*. For any increasing bounded measurable function f:

$$(i) \qquad \text{for any } A \subset \mathbb{T}^b \text{ the map } \eta \mapsto \mu_A^\eta(f) \text{ is increasing;} \tag{2.1}$$

$$(ii) \qquad \mu_B^+(f) \leqslant \mu_A^+(f) \text{ whenever } A \subset B. \tag{2.2}$$

Recall that the "plus phase" μ^+ is obtained as the weak limit as $\ell \to \infty$ of μ_ℓ^+. Existence of this limit follows from the monotonicity properties described above. Similarly one defines the "minus phase" μ^-. It turns out that any (infinite volume) Gibbs measure μ satisfies $\mu^- \leqslant \mu \leqslant \mu^+$.

2.2. The Heat Bath dynamics on finite trees

For any finite subset $A \subseteq \mathbb{T}^b$ and any $\tau \in \Omega$ we define the Heat Bath Glauber dynamics in A with boundary condition (b.c.) τ (see e.g [23]) as the continuous time Markov chain on $\Omega_A^\tau := \{\sigma \in \{-1, 1\}^{A\cup\partial A} : \sigma = \tau \text{ on } \partial A\}$ with generator

$$(\mathcal{L}_A^\tau f)(\sigma) = \sum_{x\in A} c_x(\sigma)[f(\sigma^x) - f(\sigma)], \quad \sigma \in \Omega_A^\tau, \tag{2.3}$$

where $(\sigma^x)_y = \sigma_y$ for all $y \neq x$ and $(\sigma^x)_x = -\sigma_x$ and

$$c_x(\sigma) = \mu_{\{x\}}^\sigma(\sigma^x) = \frac{1}{1 + w_x(\sigma)}, \quad w_x(\sigma) := \exp\Big[2\beta\sigma_x\Big(h + \sum\nolimits_{y:\, d(x,y)=1} \sigma_y\Big)\Big].$$

In analogy with the infinite volume case discussed in the introduction the chain started from ξ will be denoted by $\{\sigma_t^{\xi,A,\tau}\}_{t\,\geqslant\,0}$. If $A = T_\ell$ we will simply write $\sigma_t^{\xi,\ell,\tau}$.

It is well known that there is a global pathwise coupling among the processes $\big\{(\sigma_t^{\xi;A,\tau})_{t\geq 0},\ A\subset\mathbb{T}^b,\ \xi,\tau\in\Omega\big\}$ such that, for any $A\subset B\subset\mathbb{T}^b$, any $\xi\leqslant\xi'$ and any $\tau\leqslant\tau'$:

$$\begin{aligned}
\sigma_t^{\xi;A,\tau} &\leqslant \sigma_t^{\xi';A,\tau'} && \forall t\geqslant 0\\
\sigma_t^{\xi;A,-} &\leqslant \sigma_t^{\xi;B,\tau}\leqslant \sigma_t^{\xi;A,+} && \forall t\geqslant 0
\end{aligned} \tag{2.4}$$

It is a well–known (and easily checked) fact that, for any finite $A\subset\mathbb{T}^b$ and any τ, the Glauber dynamics in A with b.c. τ is ergodic and reversible w.r.t. the Gibbs distribution μ_A^τ, i.e. for any function f

$$\lim_{t\to\infty} e^{t\mathcal{L}_A^\tau} f = \mu_A^\tau(f)\,.$$

The rate at which the above convergence takes place is often measured using two concepts from functional analysis: the *spectral gap* and the *logarithmic Sobolev constant*. We now describe these two quantities for a generic (finite or infinite volume) Gibbs measure μ.

For a local function $f:\Omega\to\mathbb{R}$ define the *Dirichlet form* of f associated to the Glauber dynamics with reversible measure μ by

$$\mathcal{D}_\mu(f)=\frac{1}{2}\sum_x \mu\big(c_x\big[f(\sigma^x)-f(\sigma)\big]^2\big)=\sum_x \mu(\mathrm{Var}_{\{x\}}(f)). \tag{2.5}$$

(The l.h.s. here is the general definition for any choice of the flip rates c_x; the last equality holds when specializing to the case of the heat-bath dynamics.) The *spectral gap* $c_{\rm gap}(\mu)$ and the *logarithmic Sobolev constant* $c_{\rm sob}(\mu)$ of μ are then defined by

$$c_{\rm gap}(\mu)=\inf_f \frac{\mathcal{D}_\mu(f)}{\mathrm{Var}_\mu(f)}\,;\qquad c_{\rm sob}(\mu)=\inf_{f\,\geqslant\,0}\frac{\mathcal{D}_\mu(\sqrt{f}\,)}{\mathrm{Ent}_\mu(f)}, \tag{2.6}$$

where the infimum in each case is over non–constant functions f.

The spectral gap $c_{\rm gap}(\mu)$ measures the rate of exponential decay as $t\to\infty$ of the variance w.r.t. μ, i.e. $c_{\rm gap}(\mu)$ is the (largest) constant such that for any f

$$\mathrm{Var}_\mu(P_t f)\leqslant e^{-2tc_{\rm gap}(\mu)}\,\mathrm{Var}_\mu(f)\,, \tag{2.7}$$

where P_t denotes the semigroup associated to the Dirichlet form $\mathcal{D}_\mu(f)$. The log–Sobolev constant $c_{\rm sob}(\mu)$ is related to the following hypercontractivity estimate (see e.g. [29]): setting $q_t:=1+e^{4c_{\rm sob}(\mu)t}$ we have, for any function f and any $t\geqslant 0$

$$\|P_t f\|_{q_t,\mu}\leqslant \|f\|_{2,\mu}\,, \tag{2.8}$$

where $\|f\|_{p,\mu}$ stands for the L^p–norm of f w.r.t. μ.

If μ is a *finite volume* Gibbs measure (i.e. $\mu=\mu_A^\tau$) then both $c_{\rm gap}(\mu)$ and $c_{\rm sob}(\mu)$ are always strictly positive (possibly depending on A,τ). The striking result of [24] is that the same is true for *any* choice of the parameters (β, h) if $\mu = \mu^+$ is the infinite volume plus phase. More precisely one has

$$\inf_\ell c_{\text{gap}}(\mu_\ell^+) > 0, \quad \inf_\ell c_{\text{sob}}(\mu_\ell^+) > 0\,. \tag{2.9}$$

Such a result does not imply however any ergodicity statement for the infinite volume Glauber dynamics. Simple monotonicity considerations show in fact that for any increasing local function f and any $t \geqslant 0$, denoting by $P_t f(-)$ and $P_t f(+)$ the processes started in the all "$-$" and all "$+$" configurations respectively:

$$P_t f(-) \leqslant \mu^-(f) \leqslant \mu^+(f) \leqslant P_t f(+)\,.$$

In particular, there is non–ergodicity whenever $\mu^- \neq \mu^+$, that is for $\beta > \beta_0$ and $|h| \leq h_c(\beta)$.

2.3. Convergence to the plus phase: preliminary results

A first important step in the proof of Theorem 1.2 is to show that convergence to μ^+ occurs when we start from all $+$ spins. Recall that $\rho_{t,x}(\eta) = \mathbb{E}(\sigma_{t,x}^\eta)$ stands for the expectation at time t under the infinite–volume dynamics started in η.

Lemma 2.1. *For all b, β, h there exist $\delta > 0$ such that the following holds. For all $x \in \mathbb{T}^b$ there exists $t_0(x) < \infty$ such that if $t \geqslant t_0(x)$ then*

$$0 \leqslant \rho_{t,x}(+) - \mu^+(\sigma_x) \leqslant \exp(-\delta\, t) \tag{2.10}$$

Proof. The left inequality is a direct consequence of monotonicity (see (2.4)) and the fact that $\mu^+(\rho_{t,x}) = \mu^+(\sigma_x)$ for all x since μ^+ is an invariant measure. We now prove the right inequality. For simplicity we only analyze the case of the root $x = r$ (the general case requires no modifications in the argument.) Fix a length scale ℓ and observe that for any η monotonicity implies $\rho_{t,r}(\eta) \leqslant \rho_{t,r}^{\ell,+}(\eta)$, with the latter denoting expectation of $\sigma_{t,r}^{\eta,\ell,+}$ (the spin at the root at time t for the dynamics in T_ℓ with $+$ b.c. at the leaves of T_ℓ and initial condition η). Let also $\phi_t^{\ell,+}$ denote the function $\eta \to \rho_{t,r}^{\ell,+}(\eta) - \mu_\ell^+(\sigma_r)$ so that

$$\rho_{t,r}(+) - \mu^+(\sigma_r) \leqslant \left[\mu_\ell^+(\sigma_r) - \mu^+(\sigma_r)\right] + \phi_t^{\ell,+}(+)\,.$$

Setting $q_{t/2} := 1+e^{2c_{\text{sob}}(\mu_\ell^+)t}$ and observing that $\phi_t^{\ell,+} = P_{t/2}\phi_{t/2}^{\ell,+}$, with P_t denoting the semigroup of the dynamics in T_ℓ with $+$ b.c., the estimate (2.8) yields

$$\|\phi_t^{\ell,+}\|_{q_{t/2},\mu_\ell^+} \leqslant \|\phi_{t/2}^{\ell,+}\|_{2,\mu_\ell^+}\,.$$

Let $\mu_\ell^+(\eta)$ denote the probability of having configuration η in T_ℓ under μ_ℓ^+. Observe that, for any η

$$\phi_t^{\ell,+}(\eta) \leqslant \left(\mu_\ell^+(\eta)\right)^{-\frac{1}{q_{t/2}}} \|\phi_t^{\ell,+}\|_{q_{t/2},\mu_\ell^+}\,.$$

It is also clear that there exists $C_1 < \infty$ such that, for all η, $\mu_\ell^+(\eta) \geqslant e^{-C_1 b^\ell}$. Moreover by (2.9) we know that $q_{t/2} \geqslant e^{c_2 t}$ for some positive constant c_2 independent of ℓ. Then, taking $\eta = +$

$$\begin{aligned} \phi_t^{\ell,+}(+) &\leqslant \left(\mu_\ell^+(+)\right)^{-\frac{1}{q_{t/2}}} \|\phi_t^{\ell,+}\|_{q_{t/2},\mu_\ell^+} \\ &\leqslant \exp\left(C_1 b^\ell e^{-c_2 t}\right) \|\phi_{t/2}^{\ell,+}\|_{2,\mu_\ell^+} . \end{aligned} \tag{2.11}$$

Set now $\ell = c_3\, t$ with $c_3 > 0$ small enough. Using (2.7) and (2.9) we therefore arrive at

$$\phi_t^{\ell,+}(+) \leqslant \exp\left(C_1 b^\ell e^{-c_2 t}\right) e^{-c_{\mathrm{gap}}(\mu_\ell^+) t} \leqslant e^{-c_4 t}$$

for a suitable constant $c_4 > 0$ and t sufficiently large. Now the claim (2.10) follows from the fact (see e.g. [24]) that in the $+$ phase the influence of plus boundary conditions decays exponentially fast at any temperature: there exists $c_5 > 0$ such that

$$|\mu^+(\sigma_r) - \mu_\ell^+(\sigma_r)| \leqslant e^{-c_5 \ell} .$$

□

The previous result allows to show that the set Ω_α is increasing, i.e. its indicator function is increasing.

Corollary 2.2. *For any $\alpha \in (0, 1)$ the event Ω_α is increasing.*

Proof. We need to show that for any pair (η', η) with $\eta' \geqslant \eta$ and $\eta \in \Omega_\alpha$, also the first component η' belongs to Ω_α. To prove the claim we observe that, for any $x \in \mathbb{T}^b$ and any $t \geq 0$, monotonicity implies

$$\rho_{t,x}(\eta) - \mu^+(\sigma_x) \leqslant \rho_{t,x}(\eta') - \mu^+(\sigma_x) \leqslant \rho_{t,x}(+) - \mu^+(\sigma_x) \tag{2.12}$$

The l.h.s. of (2.12) is $\geqslant -e^{-t^\alpha}$ for any large enough time t because $\eta \in \Omega_\alpha$ by assumption. The r.h.s. is instead bounded via Lemma 2.1 above. □

Another consequence of Lemma 2.1 is the following

Corollary 2.3. *For any $\eta \in \Omega_\alpha$ the law of the process σ_t^η converges weakly to μ^+ as $t \to \infty$.*

Proof. Observe first that by the global coupling, for any $x \in \mathbb{T}^b$ and $\eta \in \Omega$ we have

$$\mathbb{P}(\sigma_{t,x}^\eta \neq \sigma_{t,x}^+) = \mathbb{P}(\sigma_{t,x}^+ = +1) - \mathbb{P}(\sigma_{t,x}^\eta = +1) = \frac{1}{2}\left(\rho_{t,x}(+) - \rho_{t,x}(\eta)\right).$$

Let f be a function on Ω depending only on the spins in a finite set $A \subset \mathbb{T}^b$ and let $\eta \in \Omega_\alpha$. Then, using the invariance of μ^+, i.e. $\mu^+ P_t(f) = \int d\mu^+(\xi)(P_t f)(\xi) = \mu^+(f)$, for all t large enough depending on A, we have

$$
\begin{aligned}
|\mathbb{E}(f(\sigma_t^\eta)) - \mu^+(f)| &\leqslant |\mathbb{E}(f(\sigma_t^\eta) - f(\sigma_t^+))| + \left| \int d\mu^+(\xi)\mathbb{E}(f(\sigma_t^+) - f(\sigma_t^\xi))\right| \\
&\leqslant 2\,\|f\|_\infty \sum_{x\in A} \Big[\, \mathbb{P}(\sigma_{t,x}^\eta \neq \sigma_{t,x}^+) + \int d\mu^+(\xi)\mathbb{P}(\sigma_{t,x}^\xi \neq \sigma_{t,x}^+) \,\Big] \\
&= \|f\|_\infty \sum_{x\in A} \Big[\, 2\,\rho_{t,x}(+) - \rho_{t,x}(\eta) - \mu^+(\sigma_x) \,\Big] \leqslant \|f\|_\infty |A| \big[\, 2\, e^{-\delta t} + e^{-t^\alpha} \,\big].
\end{aligned}
$$

Therefore $\mathbb{E}(f(\sigma_t^\eta)) \to \mu^+(f)$ for every bounded local function and the weak convergence $\delta_\eta P_t \to \mu^+$ follows. □

Finally, the following generalization of Lemma 2.1 will also be useful. Let us define the set $\Omega_{1,\delta}$, for $\delta > 0$, as the set of $\eta \in \Omega$ such that (1.3) above holds with the stretched exponential $\exp(-t^\alpha)$ replaced by the true exponential $\exp(-\delta t)$. Lemma 2.1 then says that $+ \in \Omega_{1,\delta}$ for some $\delta > 0$.

Lemma 2.4. *For every b, β, h, there exists $\delta > 0$ such that $\nu(\Omega_{1,\delta}) = 1$ for any $\nu \geqslant \mu^+$.*

Proof. Since $\nu \geqslant \mu^+$ we have $\nu(\rho_{t,x}) \geqslant \mu^+(\sigma_x)$ for all $t \geqslant 0$. From Lemma 2.1 we then infer

$$
\begin{aligned}
&\nu(|\rho_{t,x} - \mu^+(\sigma_x)| \geqslant e^{-\delta t/4}) \leqslant e^{\delta t/2}\, \nu\big(|\rho_{t,x} - \mu^+(\sigma_x)|^2\big) \\
&\quad \leqslant e^{\delta t/2}\, \nu\big(\rho_{t,x}^2 - \mu^+(\sigma_x)^2\big) \leqslant 2\, e^{\delta t/2} \big(\rho_{t,x}(+) - \mu^+(\sigma_x)\big) \leqslant 2\, e^{-\delta t/2}\,.
\end{aligned}
$$

Therefore, the Borel-Cantelli lemma implies that there exists $\varepsilon = \varepsilon(\delta, b) > 0$ and a subset $\Omega_0 \subset \Omega$ of ν-full measure such that for all $\eta \in \Omega_0$, all integers j large enough and all $x \in T_{\varepsilon j}$ (the tree of depth $\ell = \varepsilon j$),

$$
|\rho_{j,x}(\eta) - \mu^+(\sigma_x)| \leqslant e^{-\delta j/4} \tag{2.13}
$$

To prove the lemma we will establish a bound of the type (2.13) on $|\rho_{t,r}(\eta) - \mu^+(\sigma_r)|$, i.e. at the root $x = r$, but for *all* times t large enough and not just integer ones. The case of general x is obtained by straightforward modifications. We simply write ρ_t for $\rho_{t,r}$. Then, if $\lfloor t \rfloor$ is the integer part of t:

$$
\rho_t(\eta) - \mu^+(\sigma_r) = \rho_{\lfloor t \rfloor}(\eta) - \mu^+(\sigma_r) + \int_{\lfloor t \rfloor}^t ds\, P_s g(\eta)\,, \qquad g := \mathcal{L}\sigma_r \tag{2.14}
$$

For $s \geqslant \lfloor t \rfloor$ the Markov property yields $P_s g(\eta) = P_{\lfloor t \rfloor} P_{s-\lfloor t \rfloor} g(\eta) = \mathbb{E}([P_{s-\lfloor t \rfloor} g](\sigma_{\lfloor t \rfloor}^\eta))$. On the other hand standard arguments (the so–called "finite speed of propagation" estimate) based on tail estimates for the mean one Poisson process (see e.g. [23]) show that

$$
\sup_{0 \leqslant u \leqslant 1} |P_u g(\xi) - P_u g(\xi')| \leqslant C_1 \sum_x |\xi_x - \xi'_x|\, e^{-C_2\, d(x)} \tag{2.15}
$$

for constants C_1, C_2 with the property that we can take C_2 as large as we wish provided C_1 is large accordingly. Therefore by the global coupling and the invariance of μ^+,

$$\begin{aligned} |P_s g(\eta) - \mu^+(g)| &= |\int d\mu^+(\xi)\mathbb{E}([P_{s-\lfloor t\rfloor}g](\sigma^{\eta}_{\lfloor t\rfloor}) - [P_{s-\lfloor t\rfloor}g](\sigma^{\xi}_{\lfloor t\rfloor}))| \\ &\leqslant C_1 \sum_x e^{-C_2\, d(x)} \int d\mu^+(\xi)\mathbb{E}(|\sigma^{\eta}_{\lfloor t\rfloor,x} - \sigma^{\xi}_{\lfloor t\rfloor,x}|)\,. \end{aligned}$$

To handle the last term we add and subtract $\sigma^{+}_{\lfloor t\rfloor,x}$ so that by monotonicity

$$\begin{aligned} \int d\mu^+(\xi)\mathbb{E}(|\sigma^{\eta}_{\lfloor t\rfloor,x} - \sigma^{\xi}_{\lfloor t\rfloor,x}|) &\leqslant 2\mathbb{P}(\sigma^{+}_{\lfloor t\rfloor,x} = 1) - \mathbb{P}(\sigma^{\eta}_{\lfloor t\rfloor,x} = 1) \\ &\quad - \int d\mu^+(\xi)\mathbb{P}(\sigma^{\xi}_{\lfloor t\rfloor,x} = 1) \\ &\leqslant \frac{1}{2}|\rho_{\lfloor t\rfloor,x}(\eta) - \mu^+(\sigma_x))| + (\rho_{\lfloor t\rfloor,x}(+) - \mu^+(\sigma_x)). \end{aligned}$$

Fix $j = \lfloor t\rfloor$. When $x \in T_{\varepsilon j}$ we use (2.13) for the first term above. If ε is sufficiently small the argument of Lemma 2.1 also yields

$$\rho_{\lfloor t\rfloor,x}(+) - \mu^+(\sigma_x) \leqslant e^{-\delta_1 t}\,,$$

for some $\delta_1 > 0$, uniformly in $x \in T_{\varepsilon\lfloor t\rfloor}$. In conclusion, for a suitable $\delta_2 > 0$ we have

$$|P_s g(\eta) - \mu^+(g)| \leqslant e^{-\delta_2 t} \tag{2.16}$$

for all sufficiently large t. The desired estimate now follows from (2.16) and (2.14), since $\mu^+(g) = 0$ by invariance of μ^+. □

3. Proof of Theorem 1.2

We will provide a unified proof of the three statements in Theorem 1.2. In order to be able to do so we need some preliminary observations. The first is that by the monotonicity of the events Ω_α (Corollary 2.2), statement b) in Theorem 1.2 is equivalent to

b*) *For every $p > \frac{1}{2}$ there exist b_0 and β_0 such that for $b \geqslant b_0$, $\beta \geqslant \beta_0$ and $h = 0$, we have* $\mathbb{P}_p(\Omega_\alpha) = 1$ for some $\alpha = \alpha(\beta, b) > 0$.

Similarly, performing a global spin–flip, statement c) in Theorem 1.2 can be rephrased as

c*) For every $p > 0$ there exist b_0 and β_0 such that for $b \geqslant b_0$, $\beta \geqslant \beta_0$ and $h = +h_c(\beta, b)$, we have $\mathbb{P}_p(\Omega_\alpha) = 1$ for some $\alpha = \alpha(\beta, b) > 0$.

The last observation is that we may replace statement a) in Theorem 1.2 with

a*) For every $a > 0, b \geqslant 2$, there exist $p < 1$ and $\beta_0 > 0$ such that for all $\beta \geqslant \beta_0$ and $h \geqslant -h_c(\beta, b) + a$ we have $\mathbb{P}_p(\Omega_\alpha) = 1$, for some $\alpha = \alpha(\beta, h, b) > 0$.

In other words, we are taking β large enough. To see why this is not restrictive recall that by an obvious domination argument one has $\mathbb{P}_p \geqslant \mu^+$ if

$$p \geqslant p_{\beta,h} := \frac{e^{(b+1+h)\beta}}{e^{(b+1+h)\beta} + e^{-(b+1+h)\beta}}, \quad \text{i.e.} \quad \beta \leqslant \frac{1}{2(b+1+h)} \log(\frac{p}{1-p}) .$$

Lemma 2.4 therefore implies that $\mathbb{P}_p(\Omega_\alpha) = 1$ for all $\alpha < 1$ if $p \geqslant p_{\beta,h}$. We then achieve the result of Theorem 1.2 a) from a*) above by a suitable tuning of the parameter p.

3.1. Main argument

As the convergence result of Lemma 2.1 makes clear, to prove Theorem 1.2 we need a lower bound on the quantity $\rho_{t,x}(\eta) - \mu^+(\sigma_x)$ in the three statements a*),b*) and c*) emphasized above. We shall focus only on the case $x = r$, since the case of arbitrary x is obtained with essentially no modification, see also the remark at the end of this section. Setting $\rho_t(\eta) := \rho_{t,r}(\eta)$ what we want is a bound of the form

$$\rho_t(\eta) - \mu^+(\sigma_r) \geqslant \; - \; e^{-t^\alpha} , \tag{3.1}$$

for all $t \geqslant t_0(\eta)$, $\mathbb{P}_p$–almost all η. As far as this section goes, we shall not distinguish the specific setting (a*,b* or c*), since all we do here works for the three cases without any difference. What does depend on the setting are some key estimates that will be proved in the next two sections. The latter have been emphasized as separate claims in the text (see Claims 1 to 4 below).

In order to describe the main idea behind the lower bound (3.1), we must first introduce the notion of the Ising model and the associated Glauber dynamics in a *random environment of obstacles*. Realizations of the environment are described by elements ω of Ω. We say that a vertex $x \in \mathbb{T}^b$ is an *obstacle* if $\omega_x = -1$, and that x is *free* if $\omega_x = +1$. We call $T(\omega)$ the largest connected component of the set of free vertices containing the root. Note that $T(\omega) = \emptyset$ if the root r is itself an obstacle. By construction, all vertices in $\partial T(\omega)$ are obstacles. We will be mostly concerned with the case where ω is picked according to the product Bernoulli measure $\mathbb{P}_p$, i.e. when each vertex is free with probability p, independently of all others. In this case, $\mathbb{P}_p(T(\omega)$ is infinite) is positive as soon as $p > 1/b$ and tends to 1 as $p \nearrow 1$ for fixed b, or as $b \nearrow \infty$ for fixed p, see e.g. [28].

Given a realization of obstacles ω, the Ising model among obstacles is defined as before by replacing the tree $\mathbb{T}^b$ with the random tree $T(\omega)$ and the configuration space Ω with the space

$$\mathcal{B}_\omega := \{\tau \in \Omega : \; \tau_x = -1 \, , \quad \forall \, x \notin T(\omega)\} .$$

Given a finite subset $A \subset \mathbb{T}^b$ and $\tau \in \mathcal{B}_\omega$ we denote by $\mu^\tau_{A,\omega}$ the Gibbs measure $\mu^\tau_{A \cap T(\omega)}$. We also write $\mu^\tau_{\ell,\omega}$ for the Gibbs measure $\mu^\tau_{T_\ell(\omega)}$, where we use the notation $T_\ell(\omega) := T_\ell \cap T(\omega)$. From this definition we see that obstacles act as a "minus" boundary condition. The maximal allowed configuration $\tau_+ \in \mathcal{B}_\omega$ is such that τ_+ is $+1$ at every vertex of $T(\omega)$. With slight abuse of notation, we will write

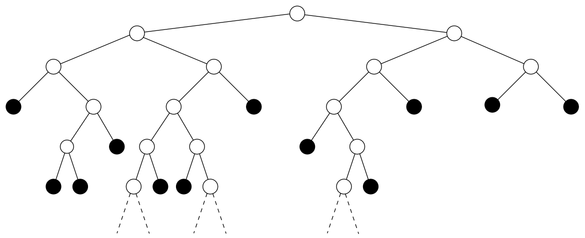

Fig. 2. Free vertices (∘) and obstacles (•) in a given realization ω on the binary tree

μ^+_ω for the Gibbs measure obtained as weak limit of $\mu^{\tau_+}_{\ell,\omega}$ as $\ell \to \infty$. Note that μ^+_ω corresponds to the equilibrium measure with "$-$" b.c. at the boundary of $T(\omega)$ and "$+$" b.c. at infinity.

Similar notations apply to the Glauber dynamics. Given a realization of obstacles ω, $A \subset \mathbb{T}^b$ and $\tau \in \mathcal{B}_\omega$, we will write $\sigma^{\xi,A,\tau}_{t,\omega}$ for the Glauber dynamics in $A \cap T(\omega)$ with boundary condition τ started from the restriction to $T(\omega)$ of the configuration $\xi \in \Omega$. If $A = T_\ell$ we simply write $\sigma^{\xi,\ell,\tau}_{t,\omega}$. When $A = T(\omega)$ the boundary condition is necessarily "$-$" and we will only write $\sigma^\xi_{t,\omega}$. In the same way we will use $\rho_{t,\omega}(\xi)$ for the expected value of $\sigma^\xi_{t,\omega}$ at the root and $\rho^{\ell,\tau}_{t,\omega}(\xi)$ for the expected value of $\sigma^{\xi,\ell,\tau}_{t,\omega}$ at the root. Monotonicity implies that, for any $\xi \leqslant \xi'$, $\sigma^\xi_{t,\omega} \le \sigma^{\xi'}_t$. In particular,

$$\rho_{t,\omega}(+) \leqslant \rho_t(\omega)\,, \qquad t \geqslant 0\,. \tag{3.2}$$

We now turn to our main argument. Fix a length scale ℓ, to be related later on to the time t, a configuration $\eta \in \Omega$ and define the associated realization $\omega = \omega(\eta, \ell)$ of obstacles by the rule:

$$\omega_x = \begin{cases} +1 & \text{if } d(x) \leqslant \ell \\ \eta_x & \text{otherwise} \end{cases} \tag{3.3}$$

Clearly $\sigma^\eta_{t,\omega} \leqslant \sigma^\eta_t$ so that

$$\rho_t(\eta) - \mu^+(\sigma_r) \geqslant [\rho_{t,\omega}(\eta) - \mu^+_\omega(\sigma_r)] - [\mu^+(\sigma_r) - \mu^+_\omega(\sigma_r)]\,. \tag{3.4}$$

If now $L = \ell^\gamma$, $\gamma > 1$, is another length scale, monotonicity shows that if we impose $+$ b.c. on the leaves of $T_L(\omega)$ we may estimate

$$\begin{aligned} &[\rho_{t,\omega}(\eta) - \mu^+_\omega(\sigma_r)] - [\mu^+(\sigma_r) - \mu^+_\omega(\sigma_r)] \\ &\quad \ge [\rho^{L,+}_{t,\omega}(\eta) - \mu^+_{L,\omega}(\sigma_r)] - [\rho^{L,+}_{t,\omega}(\eta) - \rho_{t,\omega}(\eta)] - [\mu^+(\sigma_r) - \mu^+_\omega(\sigma_r)]\,. \end{aligned} \tag{3.5}$$

Notice that in the above formula the role of the Bernoulli configuration η is twofold: it enters as the starting configuration in the first two terms but it also defines the random realization of obstacles ω, see Figure 3.

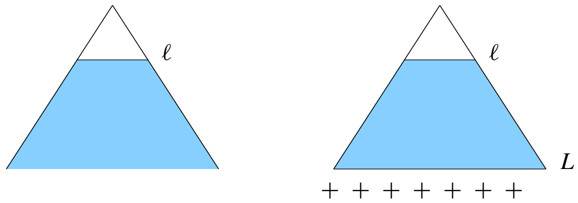

Fig. 3. Random obstacles below level ℓ: Infinite tree (left) and finite tree with $+$ boundary condition below level $L = \ell^\gamma$, $\gamma > 1$ (right)

Most of the statements that will be proved below on the r.h.s of (3.5) concern properties which hold almost surely with respect to the starting configuration η (and therefore w.r.t. ω) picked according to the Bernoulli measure $\mathbb{P}_p$. To simplify the exposition, we shall adopt the following convention: given some statements $\mathcal{E}_\ell$, $\ell \in \mathbb{N}$, we say that $\mathcal{E}_\ell$ holds $\mathbb{P}_p$–a.s. for ℓ sufficiently large whenever $\eta \in \mathcal{E}_\ell$ for all $\ell \geqslant \ell_0(\eta)$, for $\mathbb{P}_p$–a.a. $\eta \in \Omega$, or in other words, $\mathbb{P}_p(\mathcal{E}_\ell$ eventually$) = 1$. We are now in a position to explain how we will bound the three terms in the r.h.s of (3.5).

Estimate on $[\mu^+(\sigma_r) - \mu^+_\omega(\sigma_r)]$ Bounding the third term in (3.5) is a purely static problem which on the tree can be solved via a suitable recursion. In section 5 we prove

Claim 1.

$$|\mu^+(\sigma_r) - \mu^+_\omega(\sigma_r)| \leqslant e^{-2\ell}\,, \tag{3.6}$$

$\mathbb{P}_p$–a.s. for ℓ sufficiently large.

Estimate on $[\rho^{L,+}_{t,\omega}(\eta) - \mu^+_{L,\omega}(\sigma_r)]$ The first term in (3.5) is related to the speed of relaxation to equilibrium in the finite tree $T_L(\omega)$ with plus b.c. Here we need the following bound on the logarithmic Sobolev constant $c_{\rm sob}(\mu^+_{L,\omega})$.

Claim 2. *There exists $\zeta < \infty$ independent of ℓ such that*

$$c_{\rm sob}(\mu^+_{L,\omega}) \geqslant L^{-\zeta}$$

holds $\mathbb{P}_p$–a.s. for any $L \geqslant \ell$ sufficiently large.

We can repeat exactly the computation (2.11) with μ^+_ℓ replaced by $\mu^+_{L,\omega}$, $\phi^{\ell,+}_t$ by

$$\phi^{L,+}_{t,\omega}(\eta) := \rho^{L,+}_{t,\omega}(\eta) - \mu^+_{L,\omega}(\sigma_r)\,,$$

and $q_{t/2} := 1 + e^{2\,c_{\rm sob}(\mu^+_{L,\omega})\,t}$. We obtain, for any η

$$|\phi^{L,+}_{t,\omega}(\eta)| \leqslant \exp\left(C_1 b^L e^{-c_{\rm sob}(\mu^+_{L,\omega})\,t}\right) \|\phi^{L,+}_{t/2,\omega}\|_{2,\mu^+_{L,\omega}}\,. \tag{3.7}$$

for some constant $C_1 < \infty$. Assuming Claim 2 above, using $c_{\rm gap} \geqslant 2c_{\rm sob}$ (which is always true, see e.g. [29]) we estimate (3.7) with the help (2.7) and obtain, for $L = \ell^\gamma$,

$$|\rho^{L,+}_{t,\omega}(\eta) - \mu^+_{L,\omega}(\sigma_r)| = |\phi^{L,+}_{t,\omega}(\eta)| \leqslant \exp\left(C_1 b^L e^{-t/\ell^{\gamma\zeta}}\right) e^{-t/\ell^{\gamma\zeta}} \tag{3.8}$$

$\mathbb{P}_p$–a.s. for ℓ sufficiently large.

Estimate on $[\rho^{L,+}_{t,\omega}(\eta) - \rho_{t,\omega}(\eta)]$ The control of the second term in (3.5) is a true dynamical question and it involves proving that the two processes $\sigma^{\eta}_{t,\omega}$ and $\sigma^{\eta,L,+}_{t,\omega}$ remain identical at the root up to time t with large probability. This is achieved via a coupling argument together with some equilibrium estimates. The final bound will be of the form

$$\rho^{L,+}_{t,\omega}(\eta) - \rho_{t,\omega}(\eta) \leqslant t\, e^{-2\ell} \tag{3.9}$$

$\mathbb{P}_p$–a.s. for ℓ sufficiently large, provided that $\gamma > \zeta + 1$, where ζ is the constant appearing in (3.8). The argument goes as follows.

To keep the notation to a minimum, we will abbreviate the two processes $\sigma^{\eta}_{t,\omega}$ and $\sigma^{\eta,L,+}_{t,\omega}$ with ξ^1_t and ξ^2_t respectively. Set

$$\Lambda := \{x \in T(\omega) : \ d(x) = 2\ell\}, \quad \bar{\Lambda} := \{x \in T(\omega) : \ \frac{3}{2}\ell \leqslant d(x) \leqslant \frac{5}{2}\ell\}.$$

By the global coupling

$$\begin{aligned} 0 \ &\leqslant \ \rho^{L,+}_{t,\omega}(\eta) - \rho_{t,\omega}(\eta) = \mathbb{E}\Big[\xi^2_{t,r} - \xi^1_{t,r}\Big] \\ &\leqslant \ 2\,\mathbb{P}\Big[\exists s \leqslant t\,,\ \exists\, x \in \Lambda \ : \ \xi^1_{s,x} \neq \xi^2_{s,x}\Big]. \end{aligned} \tag{3.10}$$

In the last bound we have used the fact that if $\xi^1_{s,x} = \xi^2_{s,x}$ at every $x \in \Lambda$ and $s \in [0,t]$ then necessarily $\xi^1_{t,r} = \xi^2_{t,r}$ since the starting configurations are identical at time 0 for all x with $d(x) \leqslant L-1$ and any discrepancy coming from vertices y with $d(y) \geqslant L$ must cross level Λ to travel up to the root in the time interval $[0,t]$. Define A_j, $j = 0,1,\dots,\lfloor t \rfloor$, as the event

$$A_j = \{\exists x \in \bar{\Lambda} \ : \ \xi^1_{j,x} \neq \xi^2_{j,x}\}.$$

We then estimate

$$\mathbb{P}\Big[\exists s \leqslant t\,,\ \exists\, x \in \Lambda \ : \ \xi^1_{s,x} \neq \xi^2_{s,x}\Big] \leqslant \sum_{j=1}^{\lfloor t \rfloor} \mathbb{P}(A_j) \ + \sum_{j=1}^{\lfloor t \rfloor+1} \mathbb{P}\Big[A^c_{j-1} \cap B_j\Big] \tag{3.11}$$

where

$$B_j := \{\exists s \in [j-1, j]\,,\ \exists x \in \Lambda \ : \ \xi^1_{s,x} \neq \xi^2_{s,x}\}.$$

The probability of the event $A^c_{j-1} \cap B_j$ is estimated by a standard argument: the event $A^c_{j-1} \cap B_j$ implies that a discrepancy between ξ^1_{j-1} and ξ^2_{j-1} located outside $\bar{\Lambda}$, reaches in a time smaller than 1 a point $x \in \Lambda$. Since there are at most $b^{\frac{5}{2}\ell}$ possible (self–avoiding) paths from Λ to $(\bar{\Lambda})^c$ and since the rates are bounded by one, a simple tail estimate for Poisson random variables implies

$$\mathbb{P}(A^c_{j-1} \cap B_j) \leqslant c\, b^{\frac{5}{2}\ell}\, e^{-\frac{1}{2}\ell \log(\ell/c)} \leqslant e^{-3\ell} \tag{3.12}$$

for a suitable constant c and all sufficiently large ℓ. Similarly, if we look at the event A_j, we are requiring that at least one of the discrepancies at time 0 in level L travels up to level $\frac{5}{2}\ell$ in a time less than j. Therefore

$$\mathbb{P}(A_j) \leqslant c\, b^L\, e^{-(L-\frac{5}{2}\ell)\log((L-\frac{5}{2}\ell)/cj)} \leqslant e^{-L}\,, \qquad \forall j < \varepsilon L\,. \tag{3.13}$$

for some $c < \infty$ and for all $\varepsilon = \varepsilon(b, c)$ sufficiently small. We are therefore left with the estimate of $\mathbb{P}(A_j)$ for $j \geqslant \varepsilon L$. The argument for this case goes as follows.

Fix a point $x \in \bar{\Lambda}$ and recall that x is at some level between $\frac{3}{2}\ell$ and $\frac{5}{2}\ell$. Let $r_x = r(x, \ell)$ be the ancestor of x at level ℓ and let $T_{r_x}(\omega)$ be the subtree of $T(\omega)$ rooted at r_x and containing all descendants of r_x. Let also $T_{r_x,h}(\omega)$, $h \in \mathbb{N}$, be the finite subtree of $T_{r_x}(\omega)$ obtained by considering only the first h levels of $T_{r_x}(\omega)$. When $h = 2\ell$, $T_{r_x,2\ell}(\omega)$ is the tree between levels ℓ and 3ℓ, see Figure 4. Call $\nu^{+,+}_{r_x,h}$ ($\nu^{-,+}_{r_x,h}$) the Ising–Gibbs measure on $T_{r_x,h}(\omega)$ with $+$ ($-$) b.c. above the root r_x and $+$ b.c. below the leaves at level $\ell + h$ and set $\nu^{-,+}_{r_x,\infty} := \lim_{h\to\infty} \nu^{-,+}_{r_x,h}$.

Finally, we denote by ξ^3_t the Glauber dynamics evolving in $T_{r_x,2\ell}(\omega)$ with $+$ boundary conditions both above the root r_x and below the leaves of $T_{r_x,2\ell}(\omega)$ and with initial configuration η. Notice that in fact ξ^3_t starts from all pluses because, by construction, $\eta(y) = +1\ \forall y \in T(\omega),\ d(y) \geqslant \ell$.

With the above notation and using monotonicity we can now write

$$\begin{aligned}\mathbb{P}\big[\xi^1_{j,x} \neq \xi^2_{j,x}\big] &= \mathbb{P}\big[\xi^2_{j,x} = +1\big] - \mathbb{P}\big[\xi^1_{j,x} = +1\big] \\ &\leqslant \mathbb{P}\big[\xi^3_{j,x} = +1\big] - \nu^{-,+}_{r_x,\infty}(\sigma_x = +1)\,.\end{aligned} \tag{3.14}$$

The r.h.s. in (3.14) is then decomposed into the sum of two terms:

$$\mathbb{P}\big[\xi^3_{j,x} = +1\big] - \nu^{+,+}_{r_x,2\ell}(\sigma_x = +1)\,, \tag{3.15}$$

and

$$\nu^{+,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,\infty}(\sigma_x = +1)\,. \tag{3.16}$$

In order to bound the term in (3.15) we would like to argue as in (3.7) and therefore we need the following:

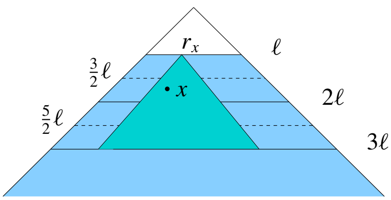

Fig. 4. The vertex $x \in \bar{\Lambda}$ and the associated tree $T_{r_x,2\ell}$

Claim 3. *There exists $\zeta < \infty$ such that*

$$\min_{x \in \bar{\Lambda}} c_{\mathrm{sob}}(\nu^{+,+}_{r_x,2\ell}) \geqslant \ell^{-\zeta}$$

holds $\mathbb{P}_p$–a.s. for ℓ sufficiently large.

The argument used in (3.7) now yields that the first term (3.15) satisfies

$$\mathbb{P}[\xi^3_{j,x} = +1] - \nu^{+,+}_{r_x,2\ell}(\sigma_x = +1) \leqslant \exp\left(c\, b^{3\ell} e^{-j/\ell^{\zeta}}\right) \exp\left(-j/\ell^{\zeta}\right). \quad (3.17)$$

Therefore, if $\gamma > \zeta + 1$, using $j \geqslant \varepsilon \ell^{\gamma}$ we may write

$$\mathbb{P}(\xi^3_{j,x} = +1) - \nu^{+,+}_{r_x,2\ell}(\sigma_x = +1) \leqslant b^{-3\ell} e^{-3\ell} \quad (3.18)$$

for ℓ large enough (independent of x). In conclusion, for any $\gamma > \zeta + 1$,

$$\sum_{x \in \bar{\Lambda}} \left\{\mathbb{P}[\xi^3_{j,x} = +1] - \nu^{+,+}_{r_x,2\ell}(\sigma_x = +1)\right\} \leqslant e^{-3\ell} \quad (3.19)$$

$\mathbb{P}_p$–a.s. for ℓ sufficiently large.

As far as the term (3.16) is concerned we will establish:

Claim 4.

$$\sum_{x \in \bar{\Lambda}} \left\{\nu^{+,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,\infty}(\sigma_x = +1)\right\} \leqslant e^{-3\ell},$$

$\mathbb{P}_p$–a.s. for ℓ sufficiently large.

Collecting (3.19) and Claim 4, we have thus shown that $\mathbb{P}(A_j) \leqslant 2e^{-3\ell}$, $j \geqslant \varepsilon L = \varepsilon \ell^{\gamma}$. Together with (3.12) and (3.13) this completes the proof of (3.9).

Conclusion. In conclusion, from (3.5), using (3.6), (3.8) and (3.9) we have the bound

$$\rho_t(\eta) - \mu^+(\sigma_r) \geqslant -\left(e^{-2\ell} + \exp\left(C_1 b^L e^{-t/\ell^{\gamma\zeta}}\right) e^{-t/\ell^{\gamma\zeta}} + t\, e^{-2\ell}\right).$$

If e.g. $\ell = t^{\alpha}$ with $\alpha > 0$ such that $\alpha\gamma(1+\zeta) < 1$ and $\gamma > \zeta + 1$, then

$$\rho_t(\eta) - \mu^+(\sigma_r) \geqslant -e^{-t^{\alpha}}, \quad (3.20)$$

$\mathbb{P}_p$–a.s. for t sufficiently large. Therefore the three steps above are sufficient to end the proof of Theorem 1.2. Note that the coefficient α depends on the various parameters (b, β, h etc.) only via the constant ζ coming from the logarithmic Sobolev inequality in Claim 2.

Remark. The above arguments show how to prove the needed lower bound on $\rho_{t,x}(\eta) - \mu^+(\sigma_x)$ when $x = r$. This choice is done merely for notational convenience. Indeed, with the environment ω defined in (3.3), we can write, as in (3.4) and (3.5)

$$\begin{aligned}&\rho_{t,x}(\eta) - \mu^+(\sigma_x)\\ &\quad \geqslant [\rho^{L,+}_{t,x,\omega}(\eta) - \mu^+_{L,\omega}(\sigma_x)] - [\rho^{L,+}_{t,x,\omega}(\eta) - \rho_{t,x,\omega}(\eta)] - [\mu^+(\sigma_x) - \mu^+_\omega(\sigma_x)]\,.\end{aligned}$$

Here $\rho_{t,x,\omega}(\eta)$ stands for the expected value of $\sigma^\eta_{t,\omega}$ at x, and $\rho^{L,+}_{t,x,\omega}(\eta)$ is the same quantity when $+$ b.c. are imposed on the leaves of $T_L(\omega)$. Notice that since ℓ and $L = \ell^\gamma$ will grow with time t our vertex x will eventually belong to the region that is free of obstacles. At this point all the estimates given for the case $x = r$ are immediately checked to hold with no change.

4. Recursive analysis among obstacles

In this section we establish a number of key estimates for the Ising Gibbs measure among obstacles. Once these results are established it will be rather easy to prove Claims 1 to 4 (see next section).

As in the previous section $\omega \in \Omega$ will denote a random realization of the obstacle–environment and μ^+_ω the associated Ising plus phase. We emphasize however that here, contrary to (3.3), ω is picked according to the product Bernoulli measure $\mathbb{P}_p$ on the *whole tree* $\mathbb{T}^b$, i.e. *each* vertex $x \in \mathbb{T}^b$ is free with probability p independently of all others.

4.1. Coupling coefficients and path weights

As in the homogeneous case treated in [24], the analysis of equilibrium properties is reduced to the study of certain coupling coefficients. For a given ω we define the ratio

$$R(\omega) = \frac{\mu^+_\omega(\sigma_r = -1)}{\mu^+_\omega(\sigma_r = +1)}\,. \tag{4.1}$$

We agree that $R(\omega) = \infty$ if $\omega_r = -1$. For every $z \in \mathbb{T}^b$ we set $R_z(\omega) := R(\theta_z\omega)$, where θ_z denotes the shift induced by the natural group action on the tree: $(\theta_z\omega)_x = \omega_{z+x}$. If $\omega_z = (\theta_z\omega)_r = +1$ and $z_1, \dots, z_b$ denote the children of $z \in T(\omega)$, one has the following easily checked recursive relation (see e.g. [1, 3]):

$$R_z(\omega) = \varepsilon^h \prod_{k=1}^{b} F_\beta(R_{z_k}(\omega))\,, \tag{4.2}$$

where, from now on, we use the following notation

$$F_\beta(a) := \frac{\varepsilon + a}{1 + \varepsilon a}\,, \qquad \varepsilon := e^{-2\beta}\,. \tag{4.3}$$

To illustrate the use of the variable R defined in (4.1), consider a vertex $z \in T(\omega)$ together with one of its ancestors y and denote by $\mu_\omega^{y,+}$ (resp. $\mu_\omega^{y,-}$) the measure μ_ω^+ conditioned to have $\sigma_y = +1$ (resp. $\sigma_y = -1$). Suppose we want to compute the total variation distance between the marginals at the vertex z, which we denote by $\|\mu_\omega^{y,+} - \mu_\omega^{y,-}\|_z$. Since the spin at z can take only two values, the latter equals $\mu_\omega^{y,+}(\sigma_z = +1) - \mu_\omega^{y,-}(\sigma_z = +1)$. If y is the parent of z, using $\mu_\omega^{y,\pm}(\sigma_z = +1) = (\varepsilon^{\pm 1} R_z(\omega) + 1)^{-1}$ we see that

$$\|\mu_\omega^{y,+} - \mu_\omega^{y,-}\|_z = K_\beta(R_z(\omega))\,, \tag{4.4}$$

where the function $K_\beta : [0, \infty) \to [0, 1]$ is defined by

$$K_\beta(a) = \frac{1}{\varepsilon\, a + 1} - \frac{1}{\varepsilon^{-1} a + 1}\,. \tag{4.5}$$

For every $\ell \in \mathbb{N}$ we define the set of descendants of y at depth ℓ:

$$D_{y,\ell}(\omega) = \{x \in T(\omega) \text{ descendant of } y \,:\, d(y, x) = \ell\} \tag{4.6}$$

To compute the total variation distance $\|\mu_\omega^{y,+} - \mu_\omega^{y,-}\|_x$ for some $x \in D_{y,\ell}(\omega)$ we may proceed as follows. Let $z_1, \dots, z_\ell = x$ be the vertices along the path from y to x. We couple the measures $\mu_\omega^{y,+}, \mu_\omega^{y,-}$ recursively in such a way that, for every $i < \ell$, given that the corresponding configurations coincide at z_i then they coincide at z_{i+1} with probability 1, while given that there is disagreement at z_i then disagreement persists at z_{i+1} with probability $\|\mu_\omega^{z_i,+} - \mu_\omega^{z_i,-}\|_{z_{i+1}} = K_\beta(R_{z_{i+1}}(\omega))$. In this way the probability of a disagreement percolating *down* the tree from y to x equals

$$\|\mu_\omega^{y,+} - \mu_\omega^{y,-}\|_x = \prod_{i=1}^{\ell} K_\beta(R_{z_i}(\omega)) \tag{4.7}$$

Moreover, if $|\sigma - \sigma'|_{y,\ell}$ denotes the Hamming distance (counting the number of disagreements) between σ and σ' restricted to the set $D_{y,\ell}(\omega)$, the above argument implies that we can find a coupling ν_ω of $\mu_\omega^{y,+}, \mu_\omega^{y,-}$ such that the expected value of $|\sigma - \sigma'|_{y,\ell}$ satisfies

$$\nu_\omega(|\sigma - \sigma'|_{y,\ell}) \leqslant \sum_{x \in D_{y,\ell}} W(\Gamma_{y,x}, \omega)\,, \tag{4.8}$$

where we introduced the path $\Gamma_{y,x}$ between y and x, consisting of the sites $z_1, \dots, z_\ell = x$ as above, and the associated weight

$$W(\Gamma_{y,x}, \omega) = \prod_{i=1}^{\ell} K_\beta(R_{z_i}(\omega))\,. \tag{4.9}$$

The rest of this section is concerned with estimates showing that, in a suitable sense, R and W are small with large probability.

4.2. Estimates on R

We write $\widetilde{\mathbb{P}}_p$ for the probability $\mathbb{P}_p$ conditioned to have $\omega_r = +1$. We want an estimate of the type

$$\widetilde{\mathbb{P}}_p\,(R > \varepsilon) \leqslant \delta\,, \tag{4.10}$$

where $\varepsilon = e^{-2\beta}$ and δ is a small parameter. We start with the setting of statement $a*$ in the proof of Theorem 1.2.

Lemma 4.1. *For any $\delta > 0$, $a > 0$, $b \geqslant 2$, there exist $p_0 < 1$ and $\beta_0 < \infty$ such that (4.10) holds for all $p \geqslant p_0$, $\beta \geqslant \beta_0$ and $h \geqslant -h_c(\beta) + a$.*

Proof. For any integer ℓ we define

$$R^\ell(\omega) = \frac{\mu^+_{\ell,\omega}(\sigma_r = -1)}{\mu^+_{\ell,\omega}(\sigma_r = +1)}\,. \tag{4.11}$$

Since $\mu^+_{\ell,\omega} \to \mu^+_\omega$, we have $R^\ell \to R$, $\ell \to \infty$, $\mathbb{P}_p$–a.s. Moreover, monotonicity implies $R^\ell(\omega) \leqslant R^{\ell+1}(\omega)$, so that the convergence is monotone. Then it is sufficient to establish (4.10) for R^ℓ in place of R, uniformly in ℓ. We will give the proof only in the case $b = 2$, since all the estimates below are easily adapted to the case of larger values of b. Recall that in general (see e.g. [12]) one has $h_c(\beta) = (b-1) + O(\beta^{-1})$, so that, replacing a with $2a$ and taking β sufficiently large,

we can assume $h \geqslant -1 + a$ without loss of generality. Let us define the probabilities

$$q_\ell^{(k)} = \widetilde{\mathbb{P}}_p\left(R^\ell > 2^{-2k}\varepsilon^{1-ka}\right), \quad k = 0, 1, 2, \ldots \tag{4.12}$$

so that $\widetilde{\mathbb{P}}_p\left(R^\ell > \varepsilon\right) = q_\ell^{(0)}$. Let now z_1, z_2 denote the two children of the root and call R_1 and R_2 the corresponding ratios, i.e.

$$R_i = \frac{\mu^+_{\ell-1,\theta_{z_i}\omega}(\sigma_r = -1)}{\mu^+_{\ell-1,\theta_{z_i}\omega}(\sigma_r = +1)}\,, \quad i = 1, 2\,. \tag{4.13}$$

Note that R_i, $i = 1, 2$ are i.i.d. random variables with the same distribution as $R^{\ell-1}$. On the event $\{\omega_r = +1\}$ the basic relation (4.2) applies and we have

$$R^\ell \leqslant \varepsilon^{a-1}\, F_\beta(R_1)\, F_\beta(R_2)\,. \tag{4.14}$$

Using the uniform bound $F_\beta \leqslant \varepsilon^{-1}$, we see that, in particular, $R^\ell \leqslant \varepsilon^{a-1}\varepsilon^{-2}$. Taking β so large that $\sqrt{\varepsilon^a} < \frac{1}{4}$ we then see that $2^{-2k}\varepsilon^{1-ka} > \varepsilon^{a-3}$ as soon as e.g. $k > k_0 := \lfloor \frac{8}{a} \rfloor$. Therefore $q_\ell^{(k)} = 0$ for every $\ell \geqslant 1$, $k > k_0$. Suppose that $R_1 \leqslant \varepsilon$. Since F_β is monotone increasing and $F_\beta(\varepsilon) \leqslant 2\varepsilon$, by (4.14) we have

$$R^\ell \leqslant 2\,\varepsilon^a F_\beta(R_2)\,.$$

Since $F_\beta(t) \leqslant \varepsilon + t$, $t \geqslant 0$, the above shows that if $R_1 \leqslant \varepsilon$ and $R^\ell > \varepsilon$ then

$$R_2 > \frac{1}{4}\,\varepsilon^{1-a}\,, \tag{4.15}$$

for large enough β. We are now able to give an estimate on $q_\ell^{(0)}$. We first remove the event $E = \{w_{z_1} = -1\} \cup \{\omega_{z_2} = -1\}$ and consider the complement $E^c = \{w_{z_1} = +1\} \cap \{\omega_{z_2} = +1\}$. The price for this is at most $2(1-p)$:

$$q_\ell^{(0)} \leqslant \widetilde{\mathbb{P}}_p\,(E) + \widetilde{\mathbb{P}}_p\left(E^c \cap \{R^\ell > \varepsilon\}\right) \leqslant 2(1-p) + \widetilde{\mathbb{P}}_p\left(E^c \cap \{R^\ell > \varepsilon\}\right).$$

On E^c we can separate the two cases: $A = \{R_1 \leqslant \varepsilon\} \cup \{R_2 \leqslant \varepsilon\}$ and $A^c = \{R_1 > \varepsilon\} \cap \{R_2 > \varepsilon\}$. In the first case the reasoning leading to (4.15) shows that we have

$$\widetilde{\mathbb{P}}_p\left(E^c \cap A \cap \{R^\ell > \varepsilon\}\right) \leqslant 2\,\widetilde{\mathbb{P}}_p\left(R_2 > \frac{1}{4}\,\varepsilon^{1-a}\right) = 2\,q_{\ell-1}^{(1)}\,.$$

In the second case we have, by independence

$$\widetilde{\mathbb{P}}_p\left(E^c \cap A^c \cap \{R^\ell > \varepsilon\}\right) \leqslant \widetilde{\mathbb{P}}_p\left(E^c \cap A^c\right) \leqslant (q_{\ell-1}^{(0)})^2\,.$$

In particular, we have obtained the bound

$$q_\ell^{(0)} \leqslant 2(1-p) + (q_{\ell-1}^{(0)})^2 + 2\,q_{\ell-1}^{(1)}\,.$$

The same reasoning as above actually shows that for any k one has

$$q_\ell^{(k)} \leqslant 2(1-p) + (q_{\ell-1}^{(0)})^2 + 2\,q_{\ell-1}^{(k+1)}\,. \tag{4.16}$$

From the monotonicity in ℓ of R^ℓ we see that $q_{\ell-1}^{(k)} \leqslant q_\ell^{(k)}$ for any k and ℓ. Therefore a simple iteration of (4.16) gives that

$$q_\ell^{(0)} \leqslant \sum_{m=0}^{j-1} 2^m \left\{2(1-p) + (q_{\ell-1}^{(0)})^2\right\} + 2^j\,q_{\ell-1}^{(j)}\,,$$

for any $j = 1, 2\ldots$. When $j = k_0 + 1$, $q_{\ell-1}^{(j)} = 0$ and we have the recursive estimate

$$q_\ell^{(0)} \leqslant 2^{k_0+1}\left\{2(1-p) + (q_{\ell-1}^{(0)})^2\right\}. \tag{4.17}$$

This implies that for every $\delta > 0$ we can choose $p_0 < 1$ and $\beta_0 < \infty$ such that $q_\ell^{(0)} \leqslant \delta$, for every $\ell \geqslant 1$, $p \geqslant p_0$ and $\beta \geqslant \beta_0$. To see this simply observe that when $\ell = 1$ the "+" boundary condition imposes $R_i = 0$ on every child z_i such that $\omega_{z_i} = +1$ and therefore $q_1^{(0)} \leqslant 2(1-p)$, which can be made arbitrarily small. Thus, climbing up the tree with the relation (4.17), we see that $\sup_{\ell \geqslant 1} q_\ell^{(0)} \leqslant \delta$ as soon as e.g. $p \geqslant 1 - 1/(2^{5+2k_0})$. □

We turn to the setting of statement $b*$ in the proof of Theorem 1.2.

Lemma 4.2. *For any $\delta > 0$, $p > \frac{1}{2}$, there exist $b_0 \in \mathbb{N}$, $\beta_0 < \infty$ and $c > 0$ such that (4.10) holds for all $\beta \geqslant \beta_0$, $b \geqslant b_0$ and $h = 0$, with $\delta = e^{-cb}$.*

Proof. Recall the definition (4.11) of $R^\ell(\omega)$. Let $z_1, \dots, z_b$ denote the children of the root r and let $m(\omega)$ stand for the number of obstacles among them: $m(\omega) = \sum_{i=1}^b \mathbf{1}_{\{\omega_{z_i} = -1\}}$. Since $p > \frac{1}{2}$, from standard large deviation estimates for the binomial distribution there exist positive numbers $a_1, a_2 > 0$ and $b_0 \in \mathbb{N}$ such that

$$\mathbb{P}_p \left(m \geqslant \left(\frac{1}{2} - a_1 \right) b \right) \leqslant e^{-a_2 b}, \tag{4.18}$$

for all $b \geqslant b_0$. Suppose now that $\omega_r = +1$ and $m < (\frac{1}{2} - a_1)b$, i.e. the root has at least $\frac{b}{2} + a_1 b$ free children. Suppose only one of these free children, say z, is such that the associated ratio R_z satisfies $R_z > \varepsilon$. Using $F_\beta \leqslant \varepsilon^{-1}$ for the ratios on the obstacles, and $F_\beta(a) \leqslant 2\varepsilon$ whenever $a \leqslant \varepsilon$, (4.2) yields

$$R^\ell \leqslant \varepsilon^{-m} (2\varepsilon)^{b-m-1} F(R_z) \leqslant 2^b \varepsilon^{2a_1 b - 1} F_\beta(R_z). \tag{4.19}$$

Since $F_\beta \leqslant \varepsilon^{-1}$ it is clear that we can take b_0, β_0 so large that in the above situation it is impossible to have $R^\ell > \varepsilon$ for all $b \geqslant b_0$ and $\beta \geqslant \beta_0$. The above discussion says, in particular, that if $m < (\frac{1}{2} - a_1)b$ and $R^\ell > \varepsilon$, then there must be at least 2 children of r with ratio greater than ε. Thus, recalling the definition of the probabilities $q_\ell^{(0)}$ (4.12), we obtain

$$q_\ell^{(0)} \leqslant e^{-a_2 b} + \sum_{n=2}^{b} \binom{b}{n} \left(q_{\ell-1}^{(0)} \right)^n . \tag{4.20}$$

Because of the $+$ boundary condition at level ℓ, the argument of (4.19) gives $q_1^{(0)} \leqslant e^{-a_2 b}$. The claim then follows by induction: Suppose $q_{\ell-1}^{(0)} \leqslant \delta$ with $\delta := e^{-a_2 b/2}$. Then (4.20) implies $q_\ell^{(0)} \leqslant e^{-a_2 b} + (1+\delta)^b - (1+\delta b) \leqslant e^{-a_2 b} + \frac{1}{2} b^2 \delta^2$, and therefore $q_\ell^{(0)} \leqslant \delta$ for b suitably large. □

Finally, for the statement $c*$ in the proof of Theorem 1.2 we need the following

Lemma 4.3. *For any $\delta > 0$, $p > 0$, there exist $b_0 \in \mathbb{N}$, $\beta_0 < \infty$ and $c > 0$ such that (4.10) holds for all $\beta \geqslant \beta_0$, $b \geqslant b_0$ and $h = +h_c(\beta)$, with $\delta = e^{-cb}$.*

Proof. As in the previous proof we denote by $m(\omega)$ the number of obstacles among the children of the root. Since $p > 0$, there exist positive numbers $a_1, a_2 > 0$ and $b_0 \in \mathbb{N}$ such that

$$\mathbb{P}(m \geqslant (1 - a_1) b) \leqslant e^{-a_2 b}, \tag{4.21}$$

for all $b \geqslant b_0$. We recall that $h_c(\beta) = (b-1) + O(\beta^{-1})$. In particular, we may assume without loss of generality that the magnetic field satisfies $h \geqslant b - 2$. If $z_1, \dots, z_b$ denote the children of the root, by (4.2) we then have

$$R^\ell \leqslant \varepsilon^{b-2} F_\beta(R_{z_1}) \cdots F_\beta(R_{z_b}).$$

Reasoning as in (4.19) we see that if $\omega_r = +1$ and $m(\omega) < (1 - a_1)b$, then we must have more than one free child with ratio greater than ε in order to produce the event $R^\ell > \varepsilon$. It follows that we may estimate the probabilities $q_\ell^{(0)}$ exactly as in (4.20). When $\ell = 1$ the + boundary condition implies $R^\ell \leqslant \varepsilon^{b-2}\varepsilon^{-m}\varepsilon^{b-m}$. Therefore on the event $m(\omega) < (1 - a_1)b$ it is impossible (for suitably large b, β) to have $R^\ell > \varepsilon$. This gives $q_1^{(0)} \leqslant e^{-a_2 b}$. As in the proof of Lemma 4.2, the desired result now follows by induction. □

4.3. Estimates on W

We turn to an estimate on the weight W introduced in (4.9). Recall the definition (4.6) of the set $D_{y,\ell}(\omega)$. Below we simply write $D_\ell = D_\ell(\omega)$ when y coincides with the root r. We also write $W(x) := W(\Gamma_{r,x}, \omega)$, for any $x \in \mathbb{T}^b$. We look for an estimate of the form: There exists $t_0 > 0$ such that

$$\mathbb{E}_p\left[\exp\left(t \sum\nolimits_{x \in D_\ell} W(x)\right)\right] \leqslant 2\,, \tag{4.22}$$

for every $t \leqslant t_0$, $\ell \geqslant 1$. The value of t_0 will depend on the parameters a, b, β in case a^*, on b, β and p in cases b^*,c^*.

We start with the setting of statement a^*.

Lemma 4.4. *For any $a > 0$, $b \geqslant 2$, there exist $\beta_0 < \infty$ and $p_0 < 1$, such that (4.22) holds for any $\beta \geqslant \beta_0$, $p \geqslant p_0$, $h \geqslant -h_c(\beta) + a$.*

Proof. The main difficulty in proving (4.22) is the non–independence of the random variables $\{K_\beta(R_z(\omega))\}_{z\in\Gamma_{y,x}}$ entering in the definition (4.9) of W. However, thanks to the tree structure of our graph, it is possible to introduce a modified weight $\widetilde{W}_\omega(\Gamma_{y,x}) := \prod_{z\in\Gamma_{y,x}} \psi_z(\omega)$ in such a way that for each ω, $W(\Gamma_{y,x}, \omega) \leqslant \widetilde{W}(\Gamma_{y,x}, \omega)$ and the random variables $\{\psi_z(\omega)\}_{z\in\Gamma_{y,x}}$ are such that a bound in terms of independent variables becomes available. The latter estimate will be based on the results in Lemma 4.1. We now describe how we construct the modified weights.

To begin with, we fix some notation: ∂B_ℓ stands for the (deterministic) set of vertices x such that $d(x) = \ell$. We also use Γ_x for the unique path from the root to x. To simplify the notation we define $W(x) = 0$ if $x \in \partial B_\ell \setminus D_\ell(\omega)$. Next, for every $x \in \partial B_\ell$ and for every vertex $z \in \Gamma_x$ we denote by Δ_z the set of all children y of z such that $y \notin \Gamma_x$. Clearly, $|\Delta_z| = b - 1$. We say that $z \in \Gamma_x$ is *regular* if $R_y \leqslant \varepsilon$ for every $y \in \Delta_z$.

Let now $u > 0$ be a small parameter to be fixed later and suppose that $z_1, z_2, \dots, z_k$ are consecutive regular sites on Γ_x, ordered in such a way that $d(z_j, r) = d(z_{j-1}, r) - 1$, see Figure 5. Let also z_0 denote the child of z_1 along Γ_x. As in the proof of Lemma 4.1 we may assume $h \geqslant -(b-1) + a$ without loss of generality. Since z_1 is regular, using $F_\beta \leqslant \varepsilon^{-1}$ for the ratio at z_0 and $F_\beta(R_y) \leqslant 2\varepsilon$ for $y \in \Delta_z$, from (4.2) we have

$$R_{z_1} \leqslant \varepsilon^{a-(b-1)} F_\beta(R_{z_0}) \prod_{y\in\Delta_{z_1}} F_\beta(R_y) \leqslant \varepsilon^{a-b}(2\varepsilon)^{b-1} = \varepsilon^{a-1}2^{b-1}\,.$$

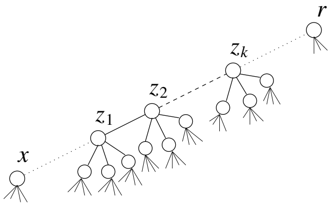

Fig. 5. k consecutive regular sites on the path Γ_x in the case $b = 4$

Similarly, for any k we have

$$R_{z_k} \leqslant \varepsilon^{a-(b-1)} F_\beta(R_{z_{k-1}}) \prod_{y \in \Delta_{z_k}} F_\beta(R_y) \leqslant 2^{b-1}\, \varepsilon^a\, F_\beta(R_{z_{k-1}})\,.$$

Using the elementary estimate $F_\beta(x) \leqslant 2(x \vee \varepsilon)$ we see that

$$R_{z_k} \leqslant 2^b\, \varepsilon^a\, (R_{z_{k-1}} \vee \varepsilon)\,, \quad k \geqslant 2\,. \tag{4.23}$$

Since $R_{z_1} \leqslant \varepsilon^{a-1} 2^{b-1}$ by the above computation, (4.23) is easily seen to show that if e.g. $k \geqslant k_0 := \lfloor \frac{4}{a} \rfloor$, then $R_{z_k} \leqslant u\varepsilon$, provided $\varepsilon \leqslant \varepsilon_0(a, b, u)$. Moreover, a simple computation gives $K_\beta(\alpha\varepsilon) \leqslant \alpha$ for every $\alpha > 0$, so that $K_\beta(R_{z_k}) \leqslant u$ in the above case.

We shall say that $z \in \Gamma_x$ is *good* if z is regular and the number of consecutive regular vertices immediately below z along Γ_x is larger or equal to $k_0 - 1$. Otherwise we say that z is *bad*.

The estimate (4.23) therefore implies that $K_\beta(R_z(\omega)) \leqslant u$ whenever z is good. Since $K_\beta \leqslant 1$ and recalling that $W(x) = 0$ if x is not connected to the root in $T(\omega)$ we may write

$$W(x) \leqslant \widetilde{W}(x) := \prod_{z \in \Gamma_x} \psi_z(\omega)\,, \qquad \psi_z(\omega) := \begin{cases} u & \omega_z = +1\,,\ z \text{ is } \textit{good} \\ 1 & \omega_z = +1\,,\ z \text{ is } \textit{bad} \\ 0 & \omega_z = -1 \end{cases} \tag{4.24}$$

We now claim that there exist $C_1 < \infty$ such that for every $\ell \geqslant 0$, for all $x \in \partial B_\ell$:

$$\mathbb{E}_p\big[\widetilde{W}(x)\big] \leqslant C_1\, (2u)^\ell\,. \tag{4.25}$$

From (4.24) we see that

$$\widetilde{W}(x) \leqslant u^{\ell - n_x(\omega)}\,, \tag{4.26}$$

where $n_x(\omega)$ stands for the number of *bad* vertices in Γ_x, $x \in \partial B_\ell$. Define, for every $z \in \Gamma_x$, $\chi_z(\omega) = 0$ if z is regular and $\chi_z(\omega) = 1$ otherwise. Note that, by construction, these are i.i.d. Bernoulli random variables. A simple deterministic bound on n_x is given by

$$n_x(\omega) \leqslant k_0 \left(1 + \sum_{z \in \Gamma_x} \chi_z(\omega) \right)\,.$$

From Lemma 4.1 we know that the probability of being irregular, for any given $z \in \Gamma_x$, is less than $\delta_1 := (b-1)\delta + (b-1)(1-p)$. Let us choose δ in Lemma 4.1 and $p < 1$ such that $\delta_1 \leqslant u^{k_0}$. We then have

$$\mathbb{E}_p\left[\widetilde{W}(x)\right] \leqslant u^{\ell-k_0}\mathbb{E}_p\left[u^{-k_0\chi_z}\right]^{\ell} \leqslant u^{\ell-k_0}(1+\delta_1 u^{-k_0})^{\ell} \leqslant u^{\ell-k_0}2^{\ell}\,. \quad (4.27)$$

The claim (4.25) then follows by taking $C_1 = u^{-k_0}$.

We are ready to prove the exponential moment estimate (4.22). For any integer k we define

$$M_k = \left(\sum_{x\in\partial B_\ell}\widetilde{W}(x)\right)^k = \sum_{x_1,\dots,x_k\in\partial B_\ell}\widetilde{W}(x_1)\cdots\widetilde{W}(x_k)\,. \quad (4.28)$$

We claim that

$$\mathbb{E}_p\left[M_k\right] \leqslant C_2^k k!\,, \qquad k = 1,2,\dots \quad (4.29)$$

for some constant $C_2 < \infty$. Note that the result (4.22) is an immediate consequence of (4.29) since the l.h.s. in (4.22) is bounded by

$$\mathbb{E}_p\left[\exp\left(tM_1\right)\right] = 1 + \sum_{k=1}^{\infty}\frac{t^k}{k!}\mathbb{E}_p M_k \leqslant \frac{1}{1-C_2 t}\,.$$

Let $x_1,\dots,x_k \in \partial B_\ell$ be given as in a generic term in the sum in (4.28). These points may be ordered by the lexicographic rule to obtain the ordered set $\tilde{x}_1 \leqslant \tilde{x}_2 \leqslant \cdots \leqslant \tilde{x}_k$. Call $\tilde{x}_0$ and $\tilde{x}_{k+1}$ the absolute leftmost and, respectively, the absolute rightmost vertex in ∂B_ℓ. Below we use $[\tilde{x}_{j-1},\tilde{x}_j)$ to denote the set of vertices $y \in \partial B_\ell$ such that y is larger or equal to $\tilde{x}_{j-1}$ but strictly less than $\tilde{x}_j$, with the agreement that, when $j = k+1$, the set $[\tilde{x}_k,\tilde{x}_{k+1})$ also includes the end point $\tilde{x}_{k+1}$, see Figure 6. With these notations we can write

$$M_{k+1}(\omega) = \sum_{j=1}^{k+1}\sum_{x_1,\dots,x_k\in\partial B_\ell}\widetilde{W}(x_1)\cdots\widetilde{W}(x_k)\sum_{y\in[\tilde{x}_{j-1},\tilde{x}_j)}\widetilde{W}(y) \quad (4.30)$$

Consider now a given $y \in [\tilde{x}_{j-1},\tilde{x}_j)$. Let T_k denote the subtree determined by the union of all paths Γ_{r,x_i}, $i = 1,\dots,k$. Let $d(y,T_k)$ denote the distance from y to T_k and write z_y for the closest ancestor of y on T_k (characterized

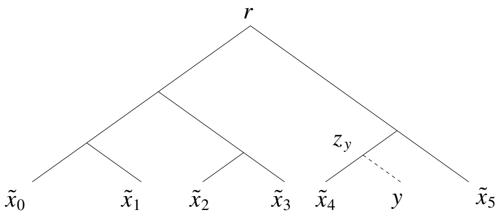

Fig. 6. Schematic picture of the tree T_k in the case $k = 4$. Here $y \in [\tilde{x}_4,\tilde{x}_5)$

by $d(z_y, y) = d(y, T_k)$). Clearly, we can estimate $\widetilde{W}(y) \leqslant \widetilde{W}(\Gamma_{z_y,y})$, where $\widetilde{W}(\Gamma_{z_y,y}) = \prod_{z\in\Gamma_{z_y,y}} \psi_z$. Now, by construction, the random variable $\widetilde{W}(\Gamma_{z_y,y})$ is independent of all the weights $\widetilde{W}(x_i)$ except for the variables ψ_z where z is either z_y or one of the $k_0 - 1$ consecutive vertices just below z_y. Let us call A_y this set of vertices. Restricting to the event that $x_1, \ldots, x_k \in D_\ell(\omega)$ we can estimate

$$\prod_{z\in A_y} (\psi_z(\omega))^{-1} \leqslant u^{-k_0}\,.$$

Therefore we have

$$\mathbb{E}_p\left[M_{k+1}\right] \leqslant u^{-k_0} \sum_{x_1,\ldots,x_k\in\partial B_\ell} \mathbb{E}_p\left[\widetilde{W}(x_1)\cdots\widetilde{W}(x_k)\right] \times \sum_{j=1}^{k+1} \sum_{y\in[\tilde{x}_{j-1},\tilde{x}_j)} \mathbb{E}_p\left[\widetilde{W}(\Gamma_{z_y,y})\right] \tag{4.31}$$

Clearly, for every integer d there at most b^d vertices y such that $d(y, T_k) = d$. Therefore, choosing $u \leqslant 1/(4b)$, from (4.25), for every pair $\tilde{x}_{j-1}, \tilde{x}_j$, we have

$$\sum_{y\in[\tilde{x}_{j-1},\tilde{x}_j)} \mathbb{E}_p\left[\widetilde{W}(\Gamma_{z_y,y})\right] \leqslant 2C_1\,. \tag{4.32}$$

From (4.31) and (4.32), setting $C_2 := 2u^{-k_0}C_1$ we obtain

$$\mathbb{E}_p\left[M_{k+1}\right] \leqslant C_2(k+1)\,\mathbb{E}_p\left[M_k\right]\,, \tag{4.33}$$

so that, for every $k \geqslant 1$ we can estimate $\mathbb{E}_p\left[M_k\right] \leqslant C_2^k k!$ as claimed in (4.29). □

We turn to the setting of statement $b*$.

Lemma 4.5. *For any $p > \frac{1}{2}$, there exist $\beta_0 < \infty$ and $b_0 \in \mathbb{N}$, such that (4.22) holds for any $\beta \geqslant \beta_0$, $b \geqslant b_0$, $h = 0$.*

Proof. The proof is essentially the same as that of Lemma 4.4, but we have to modify the definition of good and bad vertices. Given $x \in \partial B_\ell$ and $z \in \Gamma_x$ we write as before Δ_z for the set of children of z lying outside of the path Γ_x. We write also $m(\omega)$ for the number of $y \in \Delta_z$ such that $\omega_y = -1$. As in the proof of Lemma 4.2, we may use (4.18) to estimate this quantity: there exist $a_1, a_2 > 0$ such that $\mathbb{P}_p\left(m \geqslant (1-2a_1)b/2\right) \leqslant e^{-a_2 b}$, for all sufficiently large b.

Here the definition of good vertices goes as follows. We say that z is *good* if $m \leqslant (1-2a_1)b/2$ and if all the vertices $y \in \Delta_z$ such that $\omega_y = +1$ satisfy $R_y \leqslant \varepsilon$. Clearly, if z is good, from (4.2) we must have

$$R_z \leqslant \varepsilon^{-m}(2\varepsilon)^{b-1-m}\varepsilon^{-1} \leqslant 2^{b-1}\varepsilon^{2(a_1 b-1)}\,.$$

In particular, for any $u > 0$ we find b_0 and β_0 such that for all $b \geqslant b_0$ and $\beta \geqslant \beta_0$ we have $R_z \leqslant u\varepsilon$. As in the proof of Lemma 4.4 we therefore have that $K_\beta(R_z) \leqslant u$ whenever z is good. Now we can define individual weights ψ_z exactly as in (4.24) and, as before, we can estimate $W(x) \leqslant \widetilde{W}(x)$. To establish the analog of (4.25) we simply observe that ψ_z are i.i.d. random variables with the present definition of good vertices. Moreover, from Lemma 4.2 we easily infer that

$$\mathbb{P}_p\,[\,z \text{ is bad}] \leqslant e^{-c\,b} \tag{4.34}$$

for some $c > 0$. Using this we have, see (4.27)

$$\mathbb{E}_p\left[\widetilde{W}(x)\right] \leqslant (2u)^{\ell}\,, \tag{4.35}$$

as soon as $e^{-c\,b}u^{-1} \leqslant 1$. The rest of the proof goes now exactly as in Lemma 4.4. The estimate (4.31) is actually simplified by the fact that we only need to remove the vertex z_y, so that the factor u^{-k_0} is now replaced by u^{-1}. In particular, (4.33) now holds with the constant $C_2 = 2u^{-1}$. □

It remains to prove (4.22) in the setting of statement c^*.

Lemma 4.6. *For any $p > 0$, there exist $\beta_0 < \infty$ and $b_0 \in \mathbb{N}$, such that (4.22) holds for any $\beta \geqslant \beta_0$, $b \geqslant b_0$, $h = +h_c(\beta)$.*

Proof. As in the proof of Lemma 4.3 we call $m(\omega)$ the number of obstacles among the children in Δ_z for a given vertex z. We shall use the analog of estimate (4.21). Letting a_1 and a_2 be the parameters appearing there, the vertex z is now declared *good* if $m(\omega)$ satisfies $m(\omega) \leqslant (1 - a_1)b$ and all the vertices $y \in \Delta_z$ such that $\omega_y = +1$ satisfy $R_y \leqslant \varepsilon$.

With this definition of good vertices, the bounds of Lemma 4.3 now show that

$$\mathbb{P}_p\,[\,z \text{ is bad}] \leqslant e^{-c\,b}$$

for some $c > 0$ and all sufficiently large b. On the other hand, reasoning as in the proof of Lemma 4.3, an application of (4.2) gives that if z is good then

$$R_z \leqslant \varepsilon^{b-2}\varepsilon^{-m}(2\varepsilon)^{b-1-m}\varepsilon^{-1} \leqslant 2^{b-1}\varepsilon^{2a_1 b-4}\,.$$

Given $u > 0$ we then find b_0, β_0 such that $K_\beta(R_z) \leqslant u$ whenever z is good, as soon as $b \geqslant b_0$ and $\beta \geqslant \beta_0$. The rest goes exactly as in the proof of Lemma 4.5 above. □

4.4. Poincaré and Logarithmic Sobolev inequalities among obstacles

Recall the definition (2.6) of the constants $c_{\rm gap}$ and $c_{\rm sob}$. We shall focus here on the case of the measure $\mu = \mu^{+}_{L,\omega}$, i.e. the Gibbs measure with plus boundary condition below a certain level L among the obstacle environment ω. We shall write $c_{\rm gap}(L,\omega)$ and $c_{\rm sob}(L,\omega)$ for the associated constants. Here ω will be distributed according to Bernoulli(p) measure.

It is well known that, in general, $c_{\rm gap} \geqslant 2c_{\rm sob}$. On the other hand, for trees, a useful inequality established in [24] states that $c_{\rm gap} \leqslant O(\log n)c_{\rm sob}$, where n is the cardinality of the tree. In particular, Theorem 5.7 in [24] in our setting implies that for every b and every β there exists a constant $C < +\infty$ such that for every $\omega \in \Omega$ and for every L

$$c_{\text{gap}}(L,\omega) \leqslant C\,L\,c_{\text{sob}}(L,\omega) \tag{4.36}$$

Our main result here is an almost sure polynomial bound on $c_{\text{sob}}(L,\omega)$: There exists a constant $\zeta < \infty$ such that

$$c_{\text{sob}}(L,\omega) \geqslant L^{-\zeta} \tag{4.37}$$

holds $\mathbb{P}_p$–a.s. for L sufficiently large. Here the constant ζ will depend on the parameters a, b, β in case a^*, on b, β and p in cases b^*,c^*.

Theorem 4.7.

a*) *For every $a > 0$, $b \geqslant 2$, there exists $p < 1$ and β_0 such that (4.37) holds for all $\beta \geqslant \beta_0$ and $h \geqslant -h_c(\beta,b)+a$.*

b*) *For every $p > \frac{1}{2}$, there exist $b_0 \in \mathbb{N}$ and $\beta_0 < \infty$ such that (4.37) holds for $h = 0$, $b \geqslant b_0$, $\beta \geqslant \beta_0$.*

c*) *For every $p > 0$, there exist $b_0 \in \mathbb{N}$ and $\beta_0 < \infty$ such that (4.37) holds for $b \geqslant b_0$, $\beta \geqslant \beta_0$ and $h = +h_c(\beta)$.*

Proof. We will carry out the proof of the three statements simultaneously. Indeed, the key estimate we need is the exponential integrability (4.22), which holds in all cases under consideration as worked out in Lemma 4.4, Lemma 4.5 and Lemma 4.6.

Thanks to the deterministic bound (4.36) it suffices to prove the claim (4.37) with $c_{\text{sob}}(L,\omega)$ replaced by $c_{\text{gap}}(L,\omega)$. We fix a length scale ℓ_1 much smaller than L. For each vertex $x \in T_L(\omega)$, let $B_{x,\ell_1} \subset T_L(\omega)$ denote the subtree (or "block") of depth $\ell_1 - 1$ rooted at x. In this way B_{x,ℓ_1} consists of ℓ_1 levels and we understand that if x is $k < \ell_1$ levels from the bottom of $T_L(\omega)$ then B_{x,ℓ_1} has only k levels. In the end we will choose $\ell_1 = C \log L$ for some sufficiently large constant C. We define the Dirichlet form of the so-called "block–dynamics"

$$\mathcal{D}_{\ell_1,L,\omega}(f) = \sum_{x\in T_L(\omega)} \mu^+_{L,\omega}\left[\text{Var}_{B_{x,\ell_1}}(f)\right].$$

A standard argument relating the spectral gap of the heat–bath dynamics to the spectral gap of the block–dynamics (see e.g. [23]) shows that, since there are at most ℓ_1 blocks containing a given vertex x, we have

$$c_{\text{gap}}(L,\omega) \geqslant \frac{1}{\ell_1}\left\{\min_{\tau,x} c_{\text{gap}}(\mu^\tau_{B_{x,\ell_1}})\right\} \inf_f \frac{\mathcal{D}_{\ell_1,L,\omega}(f)}{\text{Var}_{\mu^+_{L,\omega}}(f)}, \tag{4.38}$$

where $c_{\text{gap}}(\mu^\tau_{B_{x,\ell_1}})$ denotes the spectral gap of the heat bath dynamics on the block B_{x,ℓ_1} with boundary condition τ (and τ is assumed to be compatible with the obstacle realization, i.e. $\tau \in \mathcal{B}_\omega$):

$$c_{\text{gap}}(\mu^\tau_{B_{x,\ell_1}}) = \inf_f \frac{\mathcal{D}_{\mu^\tau_{B_{x,\ell_1}}}(f)}{\text{Var}_{\mu^\tau_{B_{x,\ell_1}}}(f)}.$$

In general trees, according to Theorem 1.4 in [2], one has a lower bound on c_{gap} of order $n^{-\zeta}$ uniformly over the boundary condition, where n is the cardinality of the tree and $\zeta < \infty$ is a constant depending on the parameters b, β, h. In particular this implies that for all $\omega \in \Omega$ and for all sufficiently large ℓ_1

$$\frac{1}{\ell_1} \min_{\tau,x} c_{\mathrm{gap}}(\mu^{\tau}_{B_{x,\ell_1}}) \geqslant b^{-2\zeta\ell_1} \,. \tag{4.39}$$

Let $c_{\mathrm{gap}}(\ell_1, L, \omega)$ denote the spectral gap of the block–dynamics, i.e. the infimum appearing in (4.38). So far we have obtained the deterministic bound

$$c_{\mathrm{gap}}(L,\omega) \geqslant b^{-2\zeta\ell_1}\, c_{\mathrm{gap}}(\ell_1, L, \omega) \,. \tag{4.40}$$

Next we make a deterministic estimate on $c_{\mathrm{gap}}(\ell_1, L, \omega)$. To this end we use the method of [24], combined with the results we obtained in previous subsections. Given $r \in (0,1)$, we say that $\mu^+_{L,\omega}$ is (ℓ_1, r^{ℓ_1})–mixing if for every $x \in T_L(\omega)$

$$\mathrm{Var}_{\mu^+_{L,\omega}}\left(\mu^+_{L,\omega}(\sigma_x \mid \sigma_{D_{x,\ell_1}(\omega)})\right) \leqslant r^{\ell_1}\mathrm{Var}_{\mu^+_{L,\omega}}(\sigma_x) \,, \tag{4.41}$$

where $\mu^+_{L,\omega}(\sigma_x \mid \sigma_{D_{x,\ell_1}(\omega)})$ denotes the conditional expectation of σ_x given the values of σ on $D_{x,\ell_1}(\omega)$, the set of descendants of x at distance ℓ_1. A simple computation shows that (4.41) is actually equivalent to the variance mixing condition $\mathrm{VM}(\ell_1, \epsilon)$, with $\epsilon = r^{\ell_1}$, introduced in [24]. In particular, Theorem 3.2 in [24] implies that

$$c_{\mathrm{gap}}(\ell_1, L, \omega) \geqslant \frac{1}{4} \,, \tag{4.42}$$

for $\ell_1 \geqslant \ell_0$, for some finite $\ell_0 = \ell_0(r)$ as soon as $\mu^+_{L,\omega}$ is (ℓ_1, r^{ℓ_1})–mixing, with some $r \in (0,1)$. The conclusion of the theorem therefore follows from (4.40) and (4.42) if we can prove that $\mu^+_{L,\omega}$ is (ℓ_1, r^{ℓ_1})–mixing $\mathbb{P}_p$-a.s. for some $r < 1$, when $\ell_1 = C \log L$, with some $C < \infty$, for all sufficiently large L. To prove this we observe that, setting

$$g(\sigma_x) := \mu^+_{L,\omega}\left[\mu^+_{L,\omega}\left(\sigma_x \mid \sigma_{D_{x,\ell_1}(\omega)}\right) \mid \sigma_x\right] ,$$

we may write

$$\begin{aligned}
\mathrm{Var}_{\mu^+_{L,\omega}}&\left(\mu^+_{L,\omega}(\sigma_x \mid \sigma_{D_{x,\ell_1}(\omega)})\right) = \mathrm{Cov}_{\mu^+_{L,\omega}}(\sigma_x, g(\sigma_x)) \\
&= 2\,\mu^+_{L,\omega}(\sigma_x = +1)\,\mu^+_{L,\omega}(\sigma_x = -1)\;[g(+1) - g(-1)] \\
&\leqslant \frac{1}{2}\,[g(+1) - g(-1)] \,.
\end{aligned} \tag{4.43}$$

Here the notation $\mathrm{Cov}_\mu(f,g) = \mu(fg) - \mu(f)\mu(g)$ has been used for the covariance of two functions under a measure μ. Let ν denote a coupling of the measures

$\mu^+_{L,\omega}(\cdot \mid \sigma_x = +1)$ and $\mu^+_{L,\omega}(\cdot \mid \sigma_x = -1)$. We then write

$$\begin{aligned} g(+1) - g(-1) &= \sum_{\tau,\eta} \nu(\tau,\eta) \Big[\mu^+_{L,\omega}(\sigma_x \mid \tau_{D_{x,\ell_1}(\omega)}) - \mu^+_{L,\omega}(\sigma_x \mid \eta_{D_{x,\ell_1}(\omega)}) \Big] \\ &\leqslant \sum_{\tau,\eta} \nu(\tau,\eta) \sum_{y \in D_{x,\ell_1}(\omega)} 1_{\tau_y \neq \eta_y} \Big[\mu^+_{L,\omega}(\sigma_x \mid ([\tau\eta]^{y,+})_{D_{x,\ell_1}(\omega)}) \\ &\quad \Big[- \mu^+_{L,\omega}(\sigma_x \mid ([\tau\eta]^{y,-})_{D_{x,\ell_1}(\omega)}) \Big] \end{aligned}$$

with $[\tau\eta]^{y,\pm}$ denoting the interpolation between τ and η, i.e. the configuration such that, using lexicographic order on $D_{x,\ell_1}(\omega)$, $([\tau\eta]^{y,\pm})_z = \tau_z$, $z < y$, and $([\tau\eta]^{y,\pm})_z = \eta_z$, $z > y$, while $([\tau\eta]^{y,\pm})_y = \pm 1$. Recall now the definition (4.5) of the function K_β and set

$$\gamma := \sup_{x>0} K_\beta(x) = \tanh \beta\,.$$

Since $[\tau\eta]^{y,+}$ and $[\tau\eta]^{y,-}$ differ only at $y \in D_{x,\ell_1}(\omega)$, reasoning as in (4.4) and (4.7) we estimate

$$\mu^+_{L,\omega}(\sigma_x \mid ([\tau\eta]^{y,+})_{D_{x,\ell_1}(\omega)}) - \mu^+_{L,\omega}(\sigma_x \mid ([\tau\eta]^{y,-})_{D_{x,\ell_1}(\omega)}) \leqslant 2\,\gamma^{\ell_1} \quad (4.44)$$

From (4.8) we then obtain

$$g(+1) - g(-1) \leqslant 2\,\gamma^{\ell_1}\, \nu\left(|\tau - \eta|_{x,\ell_1} \right) \leqslant 2\,\gamma^{\ell_1} \sum_{y \in D_{x,\ell_1}(\omega)} W(\Gamma_{x,y}) \quad (4.45)$$

Since $\delta = \delta(\beta, h, b) := \mathrm{Var}_{\mu^+_{L,\omega}}(\sigma_x) > 0$, from (4.41), (4.43) and (4.45) we see that

$$\begin{aligned} &\mathbb{P}_p\left((\ell_1, r^{\ell_1})\text{−mixing does not hold} \right) \\ &\quad \leqslant \mathbb{P}_p\Big(\exists B_{x,\ell_1} : \sum_{y \in D_{x,\ell_1}} W(\Gamma_{x,y}) \geqslant \delta\, \gamma^{-\ell_1}\, r^{\ell_1} \Big) \\ &\quad \leqslant \sum_{x:\, d(x) \leqslant L} \mathbb{P}_p\Big(\sum_{y \in D_{x,\ell_1}} W(\Gamma_{x,y}) \geqslant \delta\, \gamma^{-\ell_1}\, r^{\ell_1} \Big)\,. \end{aligned} \quad (4.46)$$

For every x we can use the bound (4.22) on the variable $\sum_{y \in D_{x,\ell_1}} W(\Gamma_{x,y})$, so that by Markov inequality (4.46) yields

$$\mathbb{P}_p\left((\ell_1, r^{\ell_1})\text{−mixing does not hold} \right) \leqslant 2\, b^L\, \exp\left(-t_0 \delta\, r^{\ell_1} \gamma^{-\ell_1} \right)\,. \quad (4.47)$$

Setting e.g. $r = \sqrt{\gamma}$, $\ell_1 = C \log L$ with C sufficiently large we see that by the Borel Cantelli lemma we have (ℓ_1, r^{ℓ_1})–mixing $\mathbb{P}_p$-a.s. for all ℓ_1 large enough. This concludes the proof of the theorem. □

Remark. One may wonder whether the result of Theorem 4.7 captures the true behavior of the logarithmic Sobolev constant in presence of the random realization of obstacles or whether, instead, it only provides a pessimistic bound. As we show below, as soon as β is large enough (actually larger than the spin–glass critical point for the pure Ising model on $\mathbb{T}^b$ [2]), in all the three cases described in the theorem, there exists a set Ω_0 of obstacles realizations of uniformly positive probability, such that for every $\omega \in \Omega_0$ the spectral gap and a fortiori the logarithmic Sobolev constant must shrink to zero at least as fast as $L^{-\zeta'}$ for some deterministic exponent $\zeta' > 0$. A quick sketch of the proof of this fact for the setting (a^*) and e.g. $b = 2$ goes as follows.

Pick a vertex $x \in T(\omega)$ with $d(x) \leqslant \frac{L}{2}$ and denote by $T_{x,\ell}$ the finite sub-tree of $\mathbb{T}^b$ rooted at x with $\ell = \delta(M+1)\log L$ levels, $\delta \ll 1$, $M \gg 1$ but $\delta M \ll 1$. Assume that $T_{x,\ell}$ is free of obstacles, group together the sites of $\partial T_{x,\ell}$ into equal blocks according to their common ancestor in $T_{x,\ell}$ at level $\delta \log L + 1$ and order the blocks from left to right. Then impose that all vertices inside the odd blocks are obstacles while all sites z inside even blocks are not obstacles and the corresponding $R_z(\omega)$ satisfies $R_z(\omega) \le \varepsilon$ (as usual $\varepsilon = e^{-2\beta}$). Because of Lemma 4.1 the obstacles realizations that obey the above specifications have probability larger than $e^{-c|T_{x,\ell}\cup\partial T_{x,\ell}|} = e^{-cL^\alpha}$ for a suitable constant $c = c(p,\beta)$ and $\alpha = \delta(M+1)\log 2$. Since $\delta M \ll 1$ the probability of finding a vertex x with the above properties converges to one as $L \to \infty$.

Consider now the common ancestor y at level $\delta \log L$ of two odd-even neighboring blocks. It follows immediately from the recursion (4.2) and the assumptions we made on the even/odd blocks, that $|R_y - 1| \leqslant e^{-c\delta M \log L}$ for a suitable constant c. In turn, if M is large enough, that implies that the marginal of the Gibbs measure $\mu_{L,\omega}$ on the finite sub-tree rooted at x with now $\delta \log L$ levels, has a bounded (independently of L) relative density with respect to the Ising Gibbs measure on the same tree with *free boundary conditions* on its leaves. Since the latter has a spectral gap (and a fortiori a logarithmic Sobolev constant) smaller than $b^{-a(\beta)\delta\log L}$ for some positive $a(\beta)$, we conclude that for the obstacles realizations satisfying the previous conditions, $c_{\text{sob}}(\mu^+_{L,\omega}) \leqslant L^{-\zeta'}$ for some $\zeta' = \zeta'(\beta,\delta,b)$.

5. Proof of Claims (1)–(4)

The results of the previous section allow us to fill the gaps in the proof of Theorem 1.2. We refer to section 3.1 for the setting and the notation.

Claim 3. Recall the definition of the random trees $T_{z,2\ell}(\omega)$ and the associated measure $\nu^{+,+}_{z,2\ell}$, where z is a vertex at level ℓ. We have to estimate $\min_z c_{\text{gap}}(\nu^{+,+}_{z,2\ell})$. Since the b.c. above z can only affect this quantity by a constant factor (depending on β), we may replace $\nu^{+,+}_{z,2\ell}$ by the measure $\nu^+_{z,2\ell}$ with free b.c. above z. At this point, for each z we are exactly in the setting of Theorem 4.7, with $L = 2\ell$. As we have seen in the proof of that theorem (see (4.47)), there exists $\zeta < \infty$ such that

$$\mathbb{P}_p\left(c_{\text{sob}}(\nu^+_{z,2\ell}) \leqslant \ell^{-\zeta}\right) \leqslant e^{-\ell^2}, \tag{5.1}$$

for all sufficiently large values of ℓ. Since the number of z such that $d(z) = \ell$ is b^ℓ, the claim follows from the Borel–Cantelli lemma. $\square$

Claim 4. Observe that it is sufficient to prove

$$\mathbb{P}_p\left(\nu^{+,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,\infty}(\sigma_x = +1) \geqslant (3b)^{-3\ell}\right) \leqslant e^{-\ell}. \qquad (5.2)$$

Recall that $\nu^{-,+}_{r_x,2\ell}$ stands for the Gibbs measure on $T_{r_x,2\ell}(\omega)$ with $-$ b.c. above $D_\ell(\omega)$ and $+$ b.c. below $D_{2\ell}(\omega)$. Observe that, by (4.7) we have

$$\nu^{+,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,2\ell}(\sigma_x = +1) = W(\Gamma_{r_x,x}). \qquad (5.3)$$

Note that in the computation of the weigths $W(\Gamma_{r_x,x})$ appearing in (5.3) one has to take into account the $+$ b.c. at level 2ℓ. Nevertheless, we can use the same argument in the proof of (4.22), see Lemma 4.4, Lemma 4.5 and Lemma 4.6 (where the $+$ b.c. was at infinity), to show that the expectation of the expression (5.3) is estimated by $\mathbb{E}_p\widetilde{W}(\Gamma_{r_x,x}) \leqslant (2u)^{\ell/2}$, with u a small parameter, since x satisfies $d(x, r_x) \geqslant \ell/2$.

Then by Markov's inequality

$$\mathbb{P}_p\left(\nu^{+,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,2\ell}(\sigma_x = +1) \geqslant \frac{1}{2}(3b)^{-3\ell}\right) \leqslant \frac{1}{2}\, e^{-\ell} \qquad (5.4)$$

provided u is small enough. We turn to an estimate of the difference $\nu^{-,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,\infty}(\sigma_x = +1)$. Let $E_\ell(\omega)$ denote the (Ising–model) event that there exists a path Γ in $T(\omega)$ joining the sets $D_{\frac{5}{2}\ell}(\omega)$ and $D_{3\ell}(\omega)$ such that $\sigma_z = -1$ for each $z \in \Gamma$. If the sets $D_{\frac{5}{2}\ell}(\omega)$ and $D_{3\ell}(\omega)$ are not connected in $T(\omega)$ we simply set $E_\ell = \emptyset$. Observe that by monotonicity, for every ω we have

$$\nu^{-,+}_{r_x,\infty}(\sigma_x = +1 \mid E^c_\ell) \geqslant \nu^{-,+}_{r_x,2\ell}(\sigma_x = +1) \qquad (5.5)$$

The reason for the above domination is that if there is no path connecting $D_{\frac{5}{2}\ell}(\omega)$ and $D_{3\ell}(\omega)$ covered by $-$ spins, then there must exist a cut–set of $T(\omega)$, fully contained between level $\frac{5}{2}\ell$ and level 3ℓ covered by $+$ spins, and conditioned on this event $\nu^{-,+}_{r_x,\infty}$ dominates $\nu^{-,+}_{r_x,2\ell}$. We then have

$$\begin{aligned}
&\nu^{-,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,\infty}(\sigma_x = +1)\\
&\quad\leqslant \nu^{-,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,\infty}(E^c_\ell)\nu^{-,+}_{r_x,\infty}(\sigma_x = +1 \mid E^c_\ell)\\
&\quad\leqslant \nu^{-,+}_{r_x,\infty}(E_\ell)
\end{aligned} \qquad (5.6)$$

At the price of a β–dependent factor we may replace $\nu^{-,+}_{r_x,\infty}$ with the measure $\nu^{+}_{r_x,\infty}$ with free b.c. above the vertex r_x. Now we are in the familiar setting of the previous subsections. To estimate $\nu^{+}_{r_x,\infty}(E_\ell)$, suppose $\omega \in \Omega$ is such that $T(\omega)$ contains a given path $\Gamma = \{x_0, x_1, \ldots, x_h\}$ with $d(x_{j+1}) = d(x_j) + 1$, $d(x_0) = \frac{5}{2}\ell$ and $d(x_h) = 3\ell$, $h = \ell/2$. Write $\{\Gamma = -\}$ for the event $\{\sigma_{x_0} = \sigma_{x_1} = \cdots = \sigma_{x_h} = -1\}$

and let $q_j(\omega)$ denote the probability $\nu^+_{r_x,\infty}(\sigma_{x_j} = -1 \,|\, \sigma_{x_{j-1}} = -1)$. Clearly we have

$$\nu^+_{r_x,\infty}(\Gamma = -) = \nu^+_{r_x,\infty}(\sigma_{x_0} = -1) \prod_{j=1}^{h} q_j(\omega) \leqslant \prod_{j=1}^{h} q_j(\omega) \,. \tag{5.7}$$

Observe that

$$q_j(\omega) = \varepsilon^{-1} R_{x_j}(\omega)/(1 + \varepsilon^{-1} R_{x_j}(\omega)) \,. \tag{5.8}$$

As in the proof of Lemma 4.4, Lemma 4.5 and Lemma 4.6 we discriminate the vertices along Γ in good and bad vertices. We know that if z is good, then $R_z(\omega) \leqslant u\varepsilon$ and therefore, by (5.8) we have $q_j(\omega) \leqslant u$. As in (4.26) we may then estimate

$$\nu^+_{r_x,\infty}(\Gamma = -) \leqslant u^{\frac{\ell}{2} - n(\omega)} \tag{5.9}$$

where $n(\omega)$ stands for the number of bad vertices along Γ. Summing over all possible paths and estimating as in (4.27) and (4.35) we arrive at

$$\mathbb{E}_p \left[\nu^+_{r_x,\infty}(E_\ell) \right] \leqslant b^{3\ell} \, (2u)^{\frac{\ell}{2}} \leqslant \frac{1}{4} (3b)^{-3\ell} e^{-\ell} \,, \tag{5.10}$$

provided u is suitably small. From (5.6), Markov's inequality yields

$$\mathbb{P}_p \left(\nu^{-,+}_{r_x,2\ell}(\sigma_x = +1) - \nu^{-,+}_{r_x,\infty}(\sigma_x = +1) \geqslant \frac{1}{2} (3b)^{-3\ell} \right) \leqslant \frac{1}{2}\, e^{-\ell} \,. \tag{5.11}$$

This, together with (5.4), ends the proof of Claim 4. □

Let us now turn to Claim 1 and Claim 2. Here the environment ω is given by a Bernoulli(p) configuration η below level ℓ and is deterministically free of obstacles up to and including level ℓ, as prescribed by (3.3).

Claim 1. Let E_ℓ be the event that there exists a vertex x with $d(x) = \ell$, such that $\sigma_z = -1$ for every $z \in \Gamma_x$, i.e. if the root is connected to level ℓ by a path covered with $-$ spins. As in (5.5) and (5.6) we have

$$0 \leqslant \mu^+(\sigma_r) - \mu^+_\omega(\sigma_r) \leqslant \mu^+_\omega(E_\ell) \,. \tag{5.12}$$

Following (5.7) and (5.8) we estimate

$$\mu^+_\omega(E_\ell) \leqslant \sum_{x:\, d(x)=\ell} \; \prod_{z \in \Gamma_x} (\varepsilon^{-1} R_z(\omega)) \,. \tag{5.13}$$

By monotonicity we have $R_z(\omega) \leqslant R_z(\widetilde{\omega})$, where $\widetilde{\omega}$ coincides with ω (and therefore with η) below level ℓ and is given by a new (independent) Bernoulli(p) configuration η' up to and including level ℓ. We denote by $\widetilde{\mathbb{E}}_p$ the expectation over the random environment $\widetilde{\omega}$. Here we can apply the machinery developed in Lemma 4.4, Lemma 4.5 and Lemma 4.6. Namely, for a suitably small parameter $u > 0$, we can write $\varepsilon^{-1} R_z(\widetilde{\omega}) \leqslant u$ for every good vertex z. Estimating as in (5.9) and (5.11) above we have

$$\mathbb{E}_p\left[\mu_\omega^+(E_\ell)\right] \leqslant \sum_{x:\, d(x)=\ell} \widetilde{\mathbb{E}}_p\left[\prod_{z\in\Gamma_x}(\varepsilon^{-1}R_z(\widetilde{\omega}))\right] \leqslant b^\ell\,(2u)^\ell \leqslant e^{-3\ell}\,. \tag{5.14}$$

Therefore

$$\mathbb{P}_p\left(\mu_\omega^+(E_\ell) > e^{-2\ell}\right) \leqslant e^{2\ell}\,\mathbb{E}_p\left[\mu_\omega^+(E_\ell)\right] \leqslant e^{-\ell}\,. \tag{5.15}$$

Thanks to (5.12) and the Borel–Cantelli lemma, this implies the desired estimate. □

Claim 2. This is the same as the statement (4.37) appearing in Theorem 4.7, with the difference that now the environment is deterministically free of obstacles up to and including level ℓ. We can therefore repeat the argument used in the proof of Theorem 4.7 and see that what has to be established here is a version of the exponential integrability (4.22) for our new environment ω. The latter, in turn, relies on the bounds of Lemma 4.1, Lemma 4.2 and Lemma 4.3. Since the ratios R are monotonic functions of the environment, these estimates can only improve in the setting considered here and the proof of Claim 2 becomes a trivial modification of the proof of Theorem 4.7. □

6. An extension to the hard–core lattice gas (independent sets)

6.1. The hard core lattice gas

A configuration $\eta \in \Omega := \{0,1\}^{\mathbb{T}^b}$ is called an independent set if no two adjacent vertices are occupied, i.e. if $\eta_x\eta_y = 0$ for every couple $x, y \in \mathbb{T}^b$ such that $d(x,y) = 1$. We call $\bar{\Omega}$ the collection of all independent sets over the b–ary tree $\mathbb{T}^b$. In the hard–core lattice gas model $\bar{\Omega}$ is the set of allowed configurations and each such configuration $\eta \in \bar{\Omega}$ is weighted with the factor $\lambda^{|\eta|}$ where $|\eta|$ stands for the cardinality of η, i.e. the number of occupied vertices in η, and $\lambda > 0$ is the so–called activity parameter. To define the Gibbs measure we use local specifications μ_A^τ obtained by setting

$$\mu_A^\tau(\eta) \propto \lambda^{|\eta_A|}\,,$$

where A is a finite subset of $\mathbb{T}^b$, $\tau, \eta \in \bar{\Omega}$ are two allowed configurations such that $\tau_x = \eta_x$ for all $x \notin A$, and $|\eta_A| = \sum_{x\in A}\eta_x$. It is well known that the hard–core lattice gas model undergoes a phase transition at the critical activity $\lambda_c = b^b/((b-1)^{b+1})$ (see e.g. [32, 18]). For $\lambda \leqslant \lambda_c$ there is a unique phase regardless of the boundary condition on the leaves, while for $\lambda > \lambda_c$ there are (at least) two distinct phases, corresponding to the *odd* and *even* boundary conditions respectively. The even boundary condition τ^e is obtained by occupying all the vertices at even depth from the root and letting all the rest unoccupied, i.e.

$$\tau_x^e = \begin{cases} 1 & d(x) \text{ is even} \\ 0 & d(x) \text{ is odd} \end{cases}$$

The odd boundary condition τ^o is the complement $\tau^o = 1 - \tau^e$. We use the notation $\mu^e_\ell = \mu^{\tau^e}_{T_\ell}$ for the Gibbs measure on the tree of depth ℓ with even boundary condition. Similarly μ^o_ℓ denotes the Gibbs measure with odd boundary conditions. We also write $\mu^e = \lim_{\ell\to\infty} \mu^e_\ell$ and $\mu^o = \lim_{\ell\to\infty} \mu^o_\ell$. When we need to emphasize the λ–dependence we shall write μ^e_λ, μ^o_λ in place of μ^e, μ^o. Phase transition is reflected by the fact that, when $\lambda > \lambda_c$, the probability of occupation of the root differs for μ^e_λ and μ^o_λ.

6.2. *The Glauber dynamics*

The hard–core Glauber dynamics is the Markov process with Markov generator formally given by (1.1) with flip rates that are reversible w.r.t. the hard–core lattice gas Gibbs measure. As in the Ising model we restrict for simplicity to the heat–bath dynamics given by

$$c(\sigma^x) = \begin{cases} q_\lambda & \sigma_x = 1 \\ p_\lambda & \sigma^x \in \bar\Omega\,,\ \sigma_x = 0 \\ 0 & \sigma^x \notin \bar\Omega \end{cases} \qquad q_\lambda := \frac{1}{1+\lambda}\,,\ p_\lambda := \frac{\lambda}{1+\lambda}\,. \tag{6.1}$$

Here $\sigma \in \bar\Omega$ and σ^x represents the configuration σ with the occupation number at x inverted, i.e. $(\sigma^x)_y = \sigma_y$, for all $y \neq x$ and $(\sigma^x)_x = 1 - \sigma^x$. In words, only transitions within $\bar\Omega$ are allowed and the transition $\sigma \to \sigma^x$ occurs with rate $p_\lambda = \lambda/(1+\lambda)$ if x is vacant and with rate $q_\lambda = 1/(1+\lambda)$ if x is occupied. It is easily verified that detailed balance holds with this choice of rates. Moreover, for any finite subset $A \subset \mathbb{T}^b$, for any $\tau \in \bar\Omega$, the finite volume dynamics on A with boundary condition τ is ergodic and reversible w.r.t. the Gibbs measure μ^τ_A. As for the Ising Glauber dynamics we can use the spectral gap and the logarithmic Sobolev constant to estimate the rate of convergence to the stationary distribution μ^τ_A. The corresponding definitions are exactly the same as in (2.6). An important result of [24] is that the uniform bounds (2.9) hold here if we replace μ^+_ℓ with μ^e_ℓ, i.e. in the even phase one has exponential decay to equilibrium for all values of λ. Of course, the same holds for the odd phase.

6.3. *Attractivity*

It is essential for our approach that we can define a partial order on $\bar\Omega$ such that the hard–core lattice gas and its Glauber dynamics become *attractive*. Let us write $\mathbb{T}^b$ as the disjoint union of even and odd vertices, $\mathbb{T}_{\text{even}}$ and $\mathbb{T}_{\text{odd}}$, where $\mathbb{T}_{\text{even}} := \{x \in \mathbb{T}^b :\ d(x) \text{ is even}\}$ and $\mathbb{T}_{\text{odd}} := \{x \in \mathbb{T}^b :\ d(x) \text{ is odd}\}$. We define the following order on $\bar\Omega$:

$$\sigma \prec \eta \iff \begin{cases} \sigma_x \leqslant \eta_x & x \in \mathbb{T}_{\text{even}} \\ \sigma_x \geqslant \eta_x & x \in \mathbb{T}_{\text{odd}} \end{cases} \tag{6.2}$$

A function $f : \bar\Omega \to \mathbb{R}$ is called monotone increasing (decreasing) if $\sigma \prec \eta$ implies $f(\sigma) \leqslant f(\eta)$ ($f(\sigma) \geqslant f(\eta)$). We also write $\mu \leqslant \nu$, for two measures on

$\bar{\Omega}$, whenever $\mu(f) \leqslant \nu(f)$ for every monotone increasing function f. As in the Ising model it is straightforward to construct an order–preserving global path-wise coupling. Let $\sigma_t^{\xi,A,\tau}$ denote the hard–core Glauber process at time t, started in the configuration $\xi \in \bar{\Omega}$, evolved in the region A with boundary condition $\tau \in \bar{\Omega}$. We may couple the processes $\{(\sigma_t^{\xi,A,\tau})_{t\geqslant 0},\ A \subset \mathbb{T}^b,\ \xi,\tau \in \bar{\Omega}\}$ such that the following relations hold: for any $A \subset B \subset \mathbb{T}^b$, any $\xi \prec \xi'$, $\tau \prec \tau'$ and $t \geqslant 0$

$$\sigma_t^{\xi,A,\tau} \prec \sigma_t^{\xi',A,\tau'} \tag{6.3}$$

$$\sigma_t^{\xi,A,\tau^o} \prec \sigma_t^{\xi,B,\tau} \prec \sigma_t^{\xi,A,\tau^e} \tag{6.4}$$

These relations also imply the following monotonicity properties of the Gibbs measures and the associated FKG–property (see (2.1)–(2.2)):

(i) for any $A \subset \mathbb{T}^b$ the map $\eta \mapsto \mu_A^\eta(f)$ is increasing; (6.5)

(ii) $\mu_B^{\tau^e} \leqslant \mu_A^{\tau^e}$ whenever $A \subset B$. (6.6)

6.4. Results

Replacing μ^+ with μ^e we may define the sets Ω_α, $\alpha \in (0,1)$, just as in Definition 1.1. It is not difficult to check that Lemma 2.1 and therefore Corollary 2.2 hold in the present setting as well as in the Ising case. The same applies to Corollay 2.3 and Lemma 2.4.

We need to introduce the hard–core analog of the Bernoulli measures $\mathbb{P}_p$. We call $\nu_{p,\lambda}$, $p \in (0,1)$, $\lambda \geqslant 0$ the probability measure on $\bar{\Omega}$ obtained as follows: we first assign occupation numbers on $\mathbb{T}_{\text{even}}$ according to the Bernoulli(p) probability $\mathbb{P}_p$. This gives a configuration η on $\mathbb{T}_{\text{even}}$. To obtain a legal configuration (in $\bar{\Omega}$) we may now occupy only those vertices in $\mathbb{T}_{\text{odd}}$ that are at least at distance 3 from η. Call $A_\eta \subset \mathbb{T}_{\text{odd}}$ this set of available vertices. Finally put $\eta_x = 1$ with probability $p_\lambda = \lambda/(1+\lambda)$ independently for every $x \in A_\eta$.

A simple coupling argument shows that, for every $\lambda > 0$, $\nu_{p,\lambda} \geqslant \mu_\lambda^e$ as soon as $p \geqslant p_\lambda$. In particular, the argument of Lemma 2.4 shows that $\nu_{p,\lambda}(\Omega_\alpha) = 1$, for every $\alpha > 0$, for all $p \geqslant p_\lambda$. Our main result for the hard–core lattice gas is stated as follows.

Theorem 6.1.

a) *For every $b \geqslant 2$, there exists $p < 1$ such that for all $\lambda \in (0,\infty)$ we have $\nu(\Omega_\alpha) = 1$, for some $\alpha = \alpha(\lambda, b) > 0$, for any initial distribution ν such that $\nu \geqslant \nu_{p,\lambda}$.*
b) *For every $p > \frac{1}{2}$, there exist $b_0 \in \mathbb{N}$ and $\lambda_0 \in (0,\infty)$ such that for $b \geqslant b_0$, $\lambda \geqslant \lambda_0$ we have $\nu(\Omega_\alpha) = 1$, for some $\alpha = \alpha(\lambda, b, p) > 0$, for any initial distribution ν such that $\nu \geqslant \nu_{p,\lambda}$.*

6.5. Sketch of proof of Theorem 6.1

Theorem 6.1 will be proved with the same arguments used in the proof of Theorem 1.2. Below we point out the necessary (rather obvious) modifications.

The first observation is that in view of the monotonicity of Ω_α, the domination $\nu_{p,\lambda} \geqslant \mu^e_\lambda$ for $p \geqslant p_\lambda$, Lemma 2.4 allows to replace the statements in the theorem by

a*) For every $b \geqslant 2$, there exist $p < 1$ and $\lambda_0 < \infty$ such that for all $\lambda \geqslant \lambda_0$ we have $\nu_{p,\lambda}(\Omega_\alpha) = 1$, for some $\alpha = \alpha(\lambda, b) > 0$.

b*) For every $p > \frac{1}{2}$ there exist b_0 and λ_0 such that for $b \geqslant b_0$, $\lambda \geqslant \lambda_0$ we have $\nu_{p,\lambda}(\Omega_\alpha) = 1$ for some $\alpha = \alpha(\lambda, b, p) > 0$.

To repeat the argument of section 3 we need to introduce the notion of the environment of obstacles. A realization of the environment is described by $\omega \in \bar\Omega$ with the following interpretation: $x \in \mathbb{T}_{\text{even}}$ is called an *obstacle* if $\omega_x = 0$ and is said to be *free* if $\omega_x = 1$. Similarly $x \in \mathbb{T}_{\text{odd}}$ is an obstacle if $\omega_x = 1$ and is free if $\omega_x = 0$. Note that x is an obstacle in ω iff $\omega_x = \tau^o_x$. As in the Ising case ω determines the tree $T(\omega)$, i.e. the largest connected component of free vertices containing the root. We write $\mathcal{B}_\omega$ for the set of $\tau \in \bar\Omega$ such that $\tau_x = \tau^o_x$ for every $x \notin T(\omega)$. The hard–core model in a given environment ω is then obtained as before: for every $A \subset \mathbb{T}^b$ we write $\mu^\tau_{A,\omega}$ for the measure $\mu^\tau_{A\cap T(\omega)}$, where $\tau \in \mathcal{B}_\omega$. The same reasoning applies to the dynamics and we may use, as before, $\rho_{t,\omega}(\xi)$ for the expected value at the root of the occupation variable under the dynamics $\sigma^\xi_{t,\omega}$ among obstacles with starting configuration $\xi \in \bar\Omega$

We then observe, as in (3.2), that

$$\rho_{t,\omega}(\tau^e) \leqslant \rho_t(\omega)\,, \qquad t \geqslant 0\,,$$

where $\rho_t(\omega)$ denotes expectation at the root w.r.t. the dynamics in infinite volume without obstacles with starting configuration ω. We then define the environment $\omega = \omega(\eta, \ell)$ as in (3.3), where of course the $+$ configuration is replaced by τ^e. We now proceed exactly as in (3.5). Moreover, we may repeat the estimates of the three terms there without modifications. What is crucial is that the technical estimates isolated in Claims 1 to 4 can be established for the new setting. A discussion of the point is given in the next subsection.

6.6. Technical estimates

To prove the Claims 1 to 4 for the hard–core model one needs to adapt to the present setting the analysis developed in section 4. One defines the ratios R and the associated weights W in a similar way here, but the recursive relations involved in the proofs of the main estimates are model–specific and require a separate investigation. We will not provide all the details here since there is no truly new ingredient. However we give a sketch of the basic computations on the ratios R to help the interested reader in reconstructing the needed claims.

Estimates on R. Let R be defined by

$$R(\omega) = \frac{\mu^e_\omega(\sigma_r = 1)}{\mu^e_\omega(\sigma_r = 0)} \tag{6.7}$$

A simple computation gives that if $\omega_r = 1$ we have

$$R(\omega) = \lambda \prod_{i=1}^{b} \frac{1}{(1 + R_{x_i}(\omega))} \tag{6.8}$$

where $x_i, i = 1, \ldots, b$ denote the children of the root and $R_{x_i}(\omega)$ is the corresponding ratio, given as usual by the rule $R_x(\omega) = R(\theta_x \omega)$ ($\theta_x \omega$ being the environment shifted by $x \in \mathbb{T}^b$). The crucial estimate (4.10) is now replaced by

$$\widetilde{\nu}_{p,\lambda} \left(R \leqslant \sqrt{\lambda} \right) \leqslant \delta \,. \tag{6.9}$$

Here $\widetilde{\nu}_{p,\lambda}$ denotes the probability $\nu_{p,\lambda}$ conditioned to have $\omega_r = 1$ and δ is a small parameter to be fixed at a later stage. The following bound is the analogue of Lemma 4.1 in the present setting.

Lemma 6.2. *For any $\delta > 0$, $b \geqslant 2$, there exist $p_0 < 1$ and $\lambda_0 < \infty$ such that (6.9) holds for all $p \geqslant p_0$ and $\lambda \geqslant \lambda_0$.*

Proof. We shall give the proof only in the case $b = 2$. For any integer ℓ we may define the ratios $R^\ell(\omega)$ w.r.t. $\mu^e_{\ell,\omega}$ as in (4.11). We set

$$q_\ell = \widetilde{\nu}_{p,\lambda} \left(R^\ell \leqslant \sqrt{\lambda} \right) \tag{6.10}$$

Let x_1, x_2 denote the children of the root and call y_1, y_2 and y_3, y_4 the children of x_1 and x_2, respectively. Observe that the event E that $\omega_{x_i} = 0$, $i = 1, 2$ and $\omega_{y_i} = 1$, $i = 1, \ldots, 4$ has $\nu_{p,\lambda}$ probability at least $1 - 4(1 - p)$ (since it suffices to occupy all y_i's to automatically free the x_i's). Moreover for $\omega \in E$ we have

$$\begin{aligned} R^\ell(\omega) = \lambda &\left(1 + \frac{\lambda}{(1 + R_1(\omega))(1 + R_2(\omega))} \right)^{-1} \\ &\times \left(1 + \frac{\lambda}{(1 + R_3(\omega))(1 + R_4(\omega))} \right)^{-1} , \end{aligned} \tag{6.11}$$

where the ratios R_i at vertices y_i, $i = 1, \ldots, 4$ are defined by

$$R_i(\omega) = R^{\ell-2}(\theta_{y_i} \omega) \,.$$

Suppose that $R_i(\omega) > \sqrt{\lambda}$, $i = 1, 2, 3$. Then the above formula shows that for λ sufficiently large, the condition $R^\ell(\omega) \leqslant \sqrt{\lambda}$ forces $R_i(\omega) \leqslant 3$. Reasoning as in (4.16) we see that

$$q_\ell \leqslant 4(1 - p) + 6q_{\ell-2}^2 + 4\widetilde{\nu}_{p,\lambda} \left(R_1 \leqslant 3 \right) \,. \tag{6.12}$$

Using again (6.11) we see that if there is only one y_j with $R_j(\omega) \leqslant \sqrt{\lambda}$ it is impossible to have $R^\ell(\omega) \leqslant 3$. It follows that

$$\widetilde{\nu}_{p,\lambda} \left(R_1 \leqslant 3 \right) \leqslant 4(1 - p) + 6q_{\ell-4}^2 \tag{6.13}$$

Putting these estimates together and using $q_{\ell-4} \leqslant q_{\ell-2}$ we see that

$$q_\ell \leqslant 12(1 - p) + 30q_{\ell-2}^2 \,. \tag{6.14}$$

The conclusion now follows from (6.14) just as in the case of (4.17) because of the even boundary condition. $\square$

7. Open problems

We conclude by discussing an interesting open problem. Back to the Ising case with $h = 0$, let us take as initial distribution for the Glauber dynamics the symmetric product measure $\mathbb{P}_{1/2}$ that for shortness we denote by ν.

A first non trivial question is whether the law of the Glauber dynamics νP_t converges to a Gibbs measure as $t \to \infty$. In $\mathbb{Z}^d$ it is well known that this is the case (see e.g. [20]) because νP_t is translation invariant; unfortunately the Lyapunov function techniques behind the proof do not seem to apply on the tree because of the large boundary/volume ratio.

In the uniqueness region $\beta \leqslant \beta_0$ it is not difficult to check that νP_t converges weakly to the unique Gibbs measure as $t \to \infty$. More interesting is the interval $\beta \in (\beta_0, \beta_1)$, where β_1 is the spin-glass transition point discussed in section 1.1. Here the situation is more complicate due to the presence of infinitely many extremal Gibbs states.

If we recall our first characterization of β_1, it is not unreasonable to conjecture that νP_t will converge to the (extremal) free Gibbs measure μ^{free}. In fact, if we imagine that the single site Glauber dynamics is replaced by a block heat bath dynamics as in [2] then, at least for small times, each update of a block (say a large but finite subtree) replaces the Bernoulli product measure ν inside the block with a finite Gibbs measure close to μ^{free}. The case $\beta \geqslant \beta_1$ should be even more complex and one can conceive that the dynamics and coarsening of clusters of spins with opposite sign, present in the starting configuration, will play a significant role as in the $\beta = +\infty$ case [14].

Although we have no clear answers to any of the above questions, we do have some preliminary "concentration of measures" results that bring some support to the conjectured behavior in the intermediate regime (β_0, β_1).

First we show that for any local function f

$$\nu\Big(\eta;\ |P_t(f)(\eta) - \nu P_t(f)| \geqslant e^{-ct}\Big) \leqslant e^{-ce^{ct}},$$

for some $c > 0$.

In other words not too large fluctuations in the starting configuration η are completely washed out by the dynamics. In particular, for any local function f which is *odd* w.r.t. a global spin flip,

$$\lim_{t\to\infty} P_t(f)(\eta) = \mu^{\text{free}}(f) = 0 \qquad \nu\text{–a.a. } \eta\,.$$

Secondly we derive a stability result that can be roughly formulated as follows. Let $\tilde\nu$ be a perturbation of ν such that the relative entropy between ν and $\tilde\nu$ restricted to the first ℓ levels does not grow faster than $(b^\ell)^\delta$, $\delta \ll 1$. Then for any local function f

$$\lim_{t\to\infty} |\tilde\nu P_t(f) - \nu P_t(f)| = 0\,.$$

We now formalize what we just said.

Proposition 7.1. *For $h = 0$ and $\beta < \beta_1$ there exists a positive constant $c > 0$ such that, for any function f depending only on finitely many spins and any $t \geqslant 0$:*

$$\nu\Big(\eta;\ |P_t f(\eta) - \nu P_t f| \geqslant e^{-ct}\Big) \leqslant e^{-c_f e^{ct}} . \tag{7.1}$$

for a suitable constant $c_f > 0$ depending on f.

Proof. We are going to use standard Gaussian concentration bounds [19] for the measure ν of the form:

$$\nu(\eta;\ |F(\eta)| \geqslant r) \leqslant e^{-\frac{r^2}{2}} \tag{7.2}$$

for any mean zero function F with $\|F\|_{\mathrm{Lip}} \leqslant 1$, where the Lipshitz norm is defined by

$$\|F\|^2_{\mathrm{Lip}} := \sum_{x \in \mathbb{T}^b} \|F(\eta^x) - F(\eta)\|^2_\infty .$$

Therefore (7.1) follows if we can prove that for some $a = a(\beta) > 0$

$$\|P_t f\|_{\mathrm{Lip}} \leq C_f e^{-at} , \tag{7.3}$$

for a suitable constant $C_f > 0$ depending on f. Indeed, it suffices to apply (7.2) to the function $F = (P_t f - \nu P_t f)/\|P_t f\|_{\mathrm{Lip}}$, so that (7.1) follows with $c = a/2$ and $c_f = 1/2C_f^2$.

The basic tool to establish (7.3) is coupling along the lines introduced in [2]. Recall that $\tanh(\beta) < 1/\sqrt{b}$ for any $\beta < \beta_1$. Thus we can always choose $\lambda \in \big(\tanh(\beta), (b\tanh(\beta))^{-1}\big)$ in such a way that $b\lambda^2 < 1$. Given two configurations η, ξ that differ in finitely many points, define their weighted Hamming distance as

$$d_\lambda(\eta, \xi) = \sum_x \lambda^{d(x)} \mathbf{1}_{\eta_x \neq \xi_x} \tag{7.4}$$

Then a key result of [2] combined with an unpublished paper of Peres and Winkler (see section 4 of [2]) shows that under the natural coupling of the Glauber dynamics started at η and ξ

$$\mathbb{E}\big(d_\lambda(\sigma_t^\eta, \sigma_t^\xi)\big) \leqslant C e^{-ct} d_\lambda(\eta, \xi) \tag{7.5}$$

for suitable positive constants $C,\ c$. Therefore

$$\begin{aligned} \|P_t f(\eta^x) - P_t f(\eta)\|_\infty &\leqslant \sum_y \|f(\eta^y) - f(\eta)\|_\infty\, \mathbb{E}\big(\mathbf{1}_{\sigma^{\eta^x}_{t,y} \neq \sigma^\eta_{t,y}}\big) \\ &\leqslant \Big(\sum_y \lambda^{-d(y)}\, \|f(\eta^y) - f(\eta)\|_\infty\Big) e^{-ct} \lambda^{d(x)} = C_f\ e^{-ct} \lambda^{d(x)} \end{aligned} \tag{7.6}$$

Since $b\lambda^2 < 1$, the sum over x of the *square* of the r.h.s. of (7.6) converges and (7.3) follows. □

Corollary 7.2. *In the same setting as above, let $\tilde\nu$ be a probability measure on Ω and let ν_ℓ, $\tilde\nu_\ell$ be the marginals on Ω_{T_ℓ} of ν and $\tilde\nu$ respectively. Then there exists $\delta = \delta(\beta)$ such that, if $\mathrm{Ent}_\nu(\frac{d\tilde\nu_\ell}{d\nu_\ell}) \le b^{\delta\ell}$ for all ℓ, then*

$$\lim_{t\to\infty} |\tilde\nu P_t(f) - \nu P_t(f)| = 0\,. \tag{7.7}$$

Moreover the limit is attained exponentially fast.

Proof. Let $k = k(\beta)$ be so large that, with $\ell = kt$, for any large enough t

$$\| P_t(f) - e^{t\mathcal{L}_\ell^{\mathrm{free}}}(f)\|_\infty \leqslant e^{-t}\,,$$

where $\mathcal{L}_\ell^{\mathrm{free}}$ stands for the generator of the Glauber dynamics in T_ℓ with free boundary conditions. Standard results on finite speed of information propagation show that such a k exists (see e.g. [23]). Let now c_f, c be the constants appearing in Proposition 7.1 and let A_t be the set of configuration $\{\eta;\ |P_t(f)(\eta) - \nu P_t(f)| \geqslant e^{-ct}\}$. Then, by setting $h_\ell := \frac{d\tilde\nu_\ell}{d\nu_\ell}$,

$$\begin{aligned}
|\tilde\nu P_t(f) - \nu P_t(f)| &\leqslant 2e^{-t} + |\tilde\nu_\ell e^{t\mathcal{L}_\ell^{\mathrm{free}}}(f) - \nu_\ell e^{t\mathcal{L}_\ell^{\mathrm{free}}}(f)| \\
&= 2e^{-t} + |\nu\big(\big[h_\ell - 1\big]e^{t\mathcal{L}_\ell^{\mathrm{free}}}(f)\big)| \\
&\leqslant 4e^{-t} + 2e^{-ct} + e^{-c_f e^{ct}} + \|f\|_\infty \nu\big(h_\ell \mathbf{1}_{A_t}\big)
\end{aligned} \tag{7.8}$$

It remains to bound $\nu\big(h_\ell\mathbf{1}_{A_t}\big)$ and this is easily accomplished using the entropy inequality together with Proposition 7.1 and our assumption on $\mathrm{Ent}_\nu(h_\ell)$. For any $\lambda > 0$

$$\nu\big(h_\ell\mathbf{1}_{A_t}\big) \leqslant \frac{1}{\lambda}\log\Big(\nu\big(e^{\lambda\mathbf{1}_{A_t}}\big)\Big) + \frac{1}{\lambda}\mathrm{Ent}_\nu(h_\ell) \tag{7.9}$$

$$\leqslant \frac{1}{\lambda}\log\Big(1 + (e^\lambda - 1)e^{-c_f e^{ct}}\Big) + \frac{b^{\delta k t}}{\lambda} \tag{7.10}$$

If we now choose $\lambda = \frac{1}{4}c_f e^{ct}$ and $\delta < \frac{c}{k\log b}$ we see that $\nu\big(h_\ell\mathbf{1}_{A_t}\big)$ tends to zero as $t\to\infty$ exponentially fast. □

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