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Cumulants for random matrices as convolutions on the symmetric group

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Abstract. In this paper, we show that free cumulants can be naturally seen as the limiting value of "cumulants of matrices". We define these objects as functions on the symmetric group by some convolution relations involving the generalized moments. We state that some characteristic properties of the free cumulants already hold for these cumulants.

1. Introduction

D. Voiculescu introduced around 1983 a notion of freeness which plays in non commutative probability theory a role similar to independence in classical probability. Several concepts have been developed analogue to those around independence among which the free additive convolution of measures. In [19], D. Voiculescu defined a linearizing map of the free additive convolution, namely the R-series. This map can be regarded as the analogue of the logarithm of the Fourier transform for the classical theory and is of basic use in concrete calculations. The coefficients of the R-series are called *free cumulants*. R. Speicher developed a combinatorial approach for free cumulants, pointing out the connection with the lattice of non-crossing partitions, and established many of their properties (see [18], [17]; see also [13], [10], [11] and the references therein for various developments). On the other hand, D. Voiculescu ([20]) and after that several authors ([7], [12], and references therein) showed that several large independent matrices provide an asymptotic model for free random variables. Our intention is to show that free cumulants, as taken up by R. Speicher, can be naturally seen as the limiting value of scalar "cumulants of matrices", which actually already satisfy some classical properties of free cumulants. Thus, this paper attempts to draw the dotted arrows of the following diagram.

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$$\begin{tikzcd}
\text{Freeness} \arrow[rr, leftrightarrow] & & \text{Free cumulants} \\
N \times N \text{ independent random matrices} \arrow[u, "N \to +\infty"] \arrow[rr, leftrightarrow, dashed] & & \text{Matricial cumulants} \arrow[u, dashed]
\end{tikzcd}$$

Before explaining why our definition is natural, let us introduce briefly some notations. Let $\mathcal{S}_n$ be the symmetric group on $\{1, \dots, n\}$ and π be a permutation in $\mathcal{S}_n$; denoting by $\mathcal{C}(\pi)$ the set of all the disjoint cycles of π and by $\gamma_n(\pi)$ the number of cycles of π, we set for any n-tuple $\mathbf{B} = (B_1, B_2, \dots, B_n)$ of $N \times N$ complex matrices

$$r_\pi(\mathbf{B}) = r_\pi(B_1, \dots, B_n) := \prod_{C \in \mathcal{C}(\pi)} Tr\left(\prod_{j \in C} B_j\right).$$

We call *generalized moments* with order n of a set $\mathcal{X}$ of random matrices any expression $\mathbb{E}(r_\pi(X_1, \dots, X_n))$ where $X_i \in \mathcal{X}$ and $\pi \in \mathcal{S}_n$. We call *mixed generalized moments* of two sets $\mathcal{X}$ and $\mathcal{B}$ of random matrices the generalized moments of the set $\mathcal{X} \cup \mathcal{B}$; note that they can be computed from expressions $\mathbb{E}(r_\pi(B_1X_1, \dots, B_nX_n))$ with $n \in \mathbb{N}^*, \pi \in \mathcal{S}_n, B_i \in \mathcal{B} \cup \{I_N\}$ and $X_i \in \mathcal{X} \cup \{I_N\}$, denoting by I_N the $N \times N$ identity matrix.
Our intuition is based on the following results.

- If two sets of non commutative random variables $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are free in some non commutative probability space $(\mathcal{A}, \phi)$, then the distribution of $(a_1b_1, \dots, a_nb_n)$ can be described in terms of the distributions of $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. In particular, the mixed moments $\phi(a_1b_1 \cdots a_nb_n)$ can be written with the free cumulants of $(a_1, \dots, a_n)$ and the moments of $(b_1, \dots, b_n)$ or conversely (see [17] and section 2 below).
- When $\mathbf{X}$ is a n-tuple of independent Gaussian or Wishart matrices and $\mathbf{B}$ is an independent set of matrices, any mixed generalized moment $\mathbb{E}(r_\pi(B_1X_1, \dots, B_nX_n))$ can be written as a convolution on the symmetric group $\mathcal{S}_n$ of the generalized moments of $\mathbf{B}$ by one function $C_\mathbf{X}$ (see [8] and [3]). Moreover when N tends to infinity, after a suitable normalization, $C_\mathbf{X}$ converges towards the free cumulant function of respectively the semi-circular and the Marchenko-Pastur distributions.

Two questions naturally arise from the results above. For what type of matricial models $\mathbf{X}$ do we have such a convolution formula? Does the involving function $C_\mathbf{X}$ behave like a cumulant function?

Actually, dealing with two independent sets of matrices $\mathbf{X}$ and $\mathbf{B}$ such that the distribution of one tuple, $\mathbf{X}$ for example, is invariant under unitary conjugation (that is, for any unitary matrix U, $(X_1, \dots, X_n)$ and $(UX_1U^*, \dots, UX_nU^*)$ are identically distributed), we establish that any mixed generalized moment $\mathbb{E}(r_\pi(B_1X_1, \dots, B_nX_n))$ proceeds from the convolution on the symmetric group $\mathcal{S}_n$ of the generalized moments of $\mathbf{B}$ by one function $C_\mathbf{X}$. Therefore, defining our cumulant function by $C_\mathbf{X}$ is natural by analogy with the results of [17] about the multiplication of free n-tuples. We will then call by *cumulants* of $\mathbf{X}$ the collection $\{C_\mathbf{X}(\pi),\ \pi$ single cycle of

$\mathcal{S}_n, n \leq N\}$. We will show that they satisfy the expected following properties. First they do vanish as soon as the involved matrices are taken in two independent sets, one having distribution invariant under unitary conjugation; therefore they do linearize the convolution. Moreover they converge towards the free cumulants after normalization. Nevertheless, our cumulants fall outside the very general setting of [11].

The paper is organized as follows. In section 2, we recall some results about free cumulant functions as developed by R. Speicher and al. In section 3, we first state the definition of our cumulant functions and give some basic properties which are valid without any additional assumptions on the matricial models. We also give the explicite values of the cumulants for well-known matricial models. In section 4, we establish the convolution formula for mixed moments of two independent sets of random matrices, provided the distribution of one set is invariant under unitary conjugation. Section 5 deals with the linearizing property of our cumulants for models whose distribution is invariant under unitary conjugation. In section 6, we recall the asymptotic behavior of the cumulants which was already described in [3] and we mention the connection with asymptotic freeness and global fluctuations (i.e variance of traces). We end this paper by some further properties concerning the conjugation with a Gaussian matrix and the compression by a projection. These ones are the analogues of some results of A. Nica and R. Speicher in [17].

All along the paper, n is any fixed integer and we deal with sets of matrices whose generalized moments exist up to order n. We omit to mention this condition up to now on.

2. Free moments and cumulants

Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. We introduce in this section the free cumulants developed by Speicher in [18] and then by Speicher and Nica in [17]. In particular these authors characterize the freeness property and the multiplicative convolution by special relations between moments and cumulants involving non-crossing partitions. Here we translate them in terms of permutations as explained in [3].

We first present the structure of Cayley graph of $\mathcal{S}_n$: the vertex set is $\mathcal{S}_n$ and there exists an edge between two permutations σ and π if and only if $\sigma^{-1}\pi$ is a transposition. With this structure, the length of a geodesic from the permutation σ to another permutation π defines a distance between σ and π, denoted by $d_n(\sigma, \pi)$. It is known that

$$d_n(\sigma, \pi) = n - \gamma_n(\sigma^{-1}\pi) = d_n(e, \sigma^{-1}\pi). \tag{1}$$

We denote by $[e, \pi]$ the set of all the permutations lying on the geodesics from e to π. This set is characterized by the following property:

$$\sigma \in [e, \pi] \Longleftrightarrow d_n(e, \pi) = d_n(e, \sigma) + d_n(\sigma, \pi). \tag{2}$$

Moreover according to Lemma 3 in [2], for any decomposition $\pi = \prod_{i=1}^{r} \pi_i$ into disjoint cycles,

$$[e, \pi] = [e, \pi_1] \times \cdots \times [e, \pi_r]. \tag{3}$$

Let us introduce the restricted convolution on $\mathcal{S}_n$

$$f \star g(\pi) = \sum_{\sigma \in [e,\pi]} f(\sigma) g(\sigma^{-1}\pi). \tag{4}$$

The constant function $\mathbb{1}\widehat{\mathcal{S}_n}$ is $\star$-invertible with inverse function the *Möbius function* $\mu_n(e, .)$ defined by

$$\mu_n(e,\pi) = \prod \mu_{n_i}(e,\pi_i) = \prod (-1)^{n_i - 1} c_{n_i - 1}$$

when $\pi = \prod \pi_i$, where the π_i are permutations of n_i elements with disjoint supports and where $c_n = \dfrac{(2n)!}{n!(n+1)!}$ are the Catalan numbers.
Now we define *multilinear moments functionals* on $\mathcal{A}$ as the sequence $(\phi_n)_{n\in\mathbb{N}}$

$$\begin{aligned} \phi_n : \mathcal{A}^n &\to \mathbb{C} \\ (a_1, \ldots, a_n) &\mapsto \phi_n(a_1, \ldots, a_n) = \phi(a_1 \ldots a_n) \end{aligned}$$

and for $\pi = \prod_{i=1}^r \pi_i$ in $\mathcal{S}_n$ with $\pi_i = (l_{i,1}, l_{i,2}, \ldots, l_{i,n_i})$, we write

$$\phi_\pi(a_1, \ldots, a_n) := \prod_{i=1}^r \phi_{n_i}(a_{l_{i,1}}, a_{l_{i,2}}, \ldots, a_{l_{i,n_i}}).$$

The *free cumulants* $(k_n)_{n\in\mathbb{N}}$ are defined recursively on $\mathcal{A}$ by the following system of equations:

$$\phi(a_1 \ldots . a_n) = \sum_{\pi \in [e,\mathbf{1}_n]} k_\pi(a_1, \ldots, a_n) \tag{5}$$

where similarly

$$k_\pi(a_1, \ldots, a_n) := \prod_{i=1}^r k_{n_i}(a_{l_{i,1}}, a_{l_{i,2}}, \ldots, a_{l_{i,n_i}}).$$

Using (3) for a decomposition into disjoint cycles one can generalize (5) by

$$\phi_\pi(a_1, \ldots, a_n) = \sum_{\sigma \in [e,\pi]} k_\sigma(a_1, \ldots, a_n) = k(a_1, \ldots, a_n) \star \mathbb{1}\widehat{\mathcal{S}_n}(\pi). \tag{6}$$

And by Möbius inversion, (6) is equivalent to

$$k_\pi(a_1, \ldots, a_n) = \sum_{\sigma \in [e,\pi]} \phi_\sigma(a_1, \ldots, a_n) \mu_n(\sigma,\pi) = (\phi(a_1, \ldots, a_n) \star \mu(e, .))(\pi). \tag{7}$$

Nica and Speicher have described the way of getting the distribution of $(a_1 b_1, \ldots, a_n b_n)$ out of the distributions of free random variables $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. They proved in [17] the following

Proposition 2.1. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and consider random variables $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ such that $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are free. Then we have:*

$$\phi(a_1 b_1 \cdots a_n b_n) = \sum_{\sigma \in [e, \mathbf{1}_n]} k_\sigma(a_1, \ldots, a_n)\, \phi_{\sigma^{-1}\mathbf{1}_n}(b_1, \ldots, b_n).$$

From (3), we easily deduce the more general relation:

$$\phi_\pi(a_1 b_1, \ldots, a_n b_n) = \sum_{\sigma \in [e, \pi]} k_\sigma(a_1, \ldots, a_n)\, \phi_{\sigma^{-1}\pi}(b_1, \ldots, b_n). \tag{8}$$

It is a fundamental relation for our purpose since we are going to state an equivalent one for matrices.

3. Definition of matricial cumulants and basic properties

3.1. Definition

In [8] and [3], when one matricial set $\mathbf{X}$ is Gaussian or Wishart, the mixed generalized moments of $\mathbf{X}$ with an independent set $\mathbf{B}$ are expressed as a convolution between the generalized moments of $\mathbf{B}$ with some function $C_{\mathbf{X}}$ of $\mathbf{X}$. This fact led us to give a proposition for matricial cumulants in section 6.2 of [3]. Actually we will show in section 4 that such a convolution formula still exists whenever $\mathbf{X}$ (or $\mathbf{B}$) has a distribution which is invariant under unitary conjugation and that in addition it does involve the cumulant function $C_{\mathbf{X}}$ we guessed in [3]. As we will prove in a forthcoming paper that a similar decomposition of the mixed moments also occurs for orthogonally invariant matrices but requires another cumulant function, we refer $C_{\mathbf{X}}$ as the $\mathbb{U}$-*cumulant function* in the present definition.

Denote by $*$ the classical convolution operation on the space of complex functions on $\mathcal{S}_n$,

$$f * g(\pi) = \sum_{\sigma \in \mathcal{S}_n} f(\sigma) g(\sigma^{-1}\pi) = \sum_{\rho \in \mathcal{S}_n} f(\pi\rho^{-1}) g(\rho), \tag{9}$$

and by e the identity of $\mathcal{S}_n$. Recall that the $*$-unitary element is

$$\delta_e := \pi \to \begin{cases} 1 & \text{if } \pi = e \\ 0 & \text{else} \end{cases},$$

that is $f * \delta_e = \delta_e * f = f$ for all f. The inverse function of f for $*$, if there exists, is denoted by $f^{(-1)}$ and satisfies $f * f^{(-1)} = f^{(-1)} * f = \delta_e$. In particular the function $\pi \mapsto x^{\gamma_n(\pi)}$ is $*$-invertible for $n - 1 < |x|$ (see [8]). Moreover, since γ_n is central (that is, constant on the conjugacy classes), x^{γ_n} and thus $(x^{\gamma_n})^{(-1)}$ commute with any function f defined on $\mathcal{S}_n$.
We can now give the following

Definition 3.1. *For $n \leq N$, for any n-tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of random $N \times N$ complex matrices, the n-th $\mathbb{U}$-cumulant function $C_{\mathbf{X}} : \mathcal{S}_n \to \mathbb{C}, \pi \mapsto C_{\mathbf{X}}(\pi)$ is defined by the relation*

$$C_{\mathbf{X}} := \mathbb{E}(r(\mathbf{X})) * (N^{\gamma_n})^{(-1)}.$$

The $\mathbb{U}$-cumulants of X are the $C_{\mathbf{X}}(\pi)$ for single cycles π of $\mathcal{S}_n$.

For example, if $tr_N = \frac{1}{N} Tr$,

$$\begin{aligned} C_X((1)) &= \mathbb{E}(tr_N(X)) \\ C_{(X_1,X_2)}((1)(2)) &= \frac{N\mathbb{E}\{Tr(X_1)Tr(X_2)\} - \mathbb{E}\{Tr(X_1X_2)\}}{N(N^2-1)} \\ C_{(X_1,X_2)}((1\,2)) &= \frac{-\mathbb{E}\{Tr(X_1)Tr(X_2)\} + N\mathbb{E}\{Tr(X_1X_2)\}}{N(N^2-1)}. \end{aligned}$$

For simplicity since we only will consider $\mathbb{U}$-cumulant functions or $\mathbb{U}$-cumulants in this paper, we will omit the $\mathbb{U}$-feature up to now on. Moreover when $X_1 = \cdots = X_n = X$ we will also use the notation C_X for $C_{(X,\ldots,X)}$.
The following will be devoted to show that some classical properties of the free cumulants are already satisfied by our cumulants of matrices.

3.2. Basic properties and examples

We begin with elementary remarks. First of all, note that the moments of $\mathbf{X}$ with order n can be found from the cumulant function by the inverse formula:

$$\mathbb{E}(r(\mathbf{X})) = \mathbb{E}(r(I_N)) * C_{\mathbf{X}},$$

since $\mathbb{E}(r(I_N)) = N^{\gamma_n}$. This equality is to be related to the relation between moments and free cumulants of noncommutative variables (see [17]).
Now, for each π in $\mathcal{S}_n$, $(X_1, \ldots, X_n) \mapsto C_{(X_1,\ldots,X_n)}(\pi)$ is obviously n-linear. Moreover it is clear that for any unitary matrix U,

$$C_{(U^*X_1U,\ldots,U^*X_nU)}(\pi) = C_{(X_1,\ldots,X_n)}(\pi).$$

Let us also mention the action of the conjugacy in $\mathcal{S}_n$ on $C_{\mathbf{X}}$. It will be of great practical interest in the presentation of our future results. The proof of the lemma is easy and left to the reader.

Lemma 3.1. *1. For any π and σ in $\mathcal{S}_n$,*

$$C_{(X_{\sigma(1)},\ldots,X_{\sigma(n)})}(\pi) = C_{(X_1,\ldots,X_n)}(\sigma\pi\sigma^{-1}).$$

2. $C_X(\pi)$ depends only of the conjugacy class of π.

Thus the cumulants $C_X(\pi)$ of a matrix X for single cycles π of $\mathcal{S}_n$ are all equal so that we will denote by $C_n(X)$ this common value. We will call it *cumulant of order n* of the matrix X. In particular, $C_1(X) = \mathbb{E}(tr_N X)$ and $C_2(X) = \frac{N}{N^2-1}\left[\mathbb{E}\{tr_N(X^2)\} - \mathbb{E}\{(tr_N X)^2\}\right]$.

Proposition 3.1. *For any $k < n \leq N$, any π in $\mathcal{S}_n$, then*

$$C_{(X_1,\dots,X_k,I_N,\dots,I_N)}(\pi) = \begin{cases} C_{(X_1,\dots,X_k)}(\rho) & \text{if } \pi = (n)\cdots(k+1)\rho \text{ for some } \rho \in \mathcal{S}_k, \\ 0 & \text{else.} \end{cases}$$

Proof. We prove this proposition by induction on $n - k \geq 1$.
Let us prove the result for $n - k = 1$. For π in $\mathcal{S}_n$, let us define $\check{\pi}$ in $\mathcal{S}_{n-1}$ by,

- if $\pi(n) = n$, $\check{\pi} = \pi_{|\{1,\dots,n-1\}}$
- if $\pi(n) \neq n$, $\check{\pi}(i) = \pi(i)$ for $i \neq \pi^{-1}(n)$ and $\check{\pi}(\pi^{-1}(n)) = \pi(n)$.

Roughly speaking, you get $\check{\pi}$ from π by just "taking off" n. Note that

$$\begin{aligned} \mathbb{E}\left(r_\pi(X_1,\dots,X_{n-1},I_N)\right) &= \mathbb{E}\left(r_{\check{\pi}}(X_1,\dots,X_{n-1})\right) \quad \text{if } \pi(n) \neq n, \\ &= N\mathbb{E}\left(r_{\check{\pi}}(X_1,\dots,X_{n-1})\right) \quad \text{if } \pi(n) = n. \end{aligned}$$

Hence, equivalently,

$$\begin{aligned} &\sum_{\sigma\in\mathcal{S}_n} C_{(X_1,\dots,X_{n-1},I_N)}(\sigma) N^{\gamma_n(\sigma^{-1}\pi)} \\ &\quad = \sum_{\rho\in\mathcal{S}_{n-1}} C_{(X_1,\dots,X_{n-1})}(\rho) N^{\gamma_{n-1}(\rho^{-1}\check{\pi})} \qquad \text{if } \pi(n) \neq n, \\ &\quad = \sum_{\rho\in\mathcal{S}_{n-1}} C_{(X_1,\dots,X_{n-1})}(\rho) N^{\gamma_{n-1}(\rho^{-1}\check{\pi})+1} \qquad \text{if } \pi(n) = n. \end{aligned}$$

Now, let ρ in $\mathcal{S}_{n-1}$ and $\sigma = (n)\rho \in \mathcal{S}_n$. Noting that, for any π in $\mathcal{S}_n$, $\rho^{-1}\check{\pi} = \overset{\vee}{\overline{\sigma^{-1}\pi}}$, we readily get that

$$\begin{aligned} \gamma_n(\sigma^{-1}\pi) &= \gamma_{n-1}(\rho^{-1}\check{\pi}) \quad \text{if } \pi(n) \neq n, \\ &= \gamma_{n-1}(\rho^{-1}\check{\pi}) + 1 \quad \text{if } \pi(n) = n. \end{aligned}$$

Denote now by $\mathcal{A}_n$ the subset of permutations of $\mathcal{S}_n$ of the form $(n)\rho$, $\rho \in \mathcal{S}_{n-1}$. For any $\sigma = (n)\rho$ in $\mathcal{A}_n$, $\check{\sigma} = \rho$. Therefore, if π is in $\mathcal{S}_n$,

$$\sum_{\sigma\in\mathcal{S}_n} C_{(X_1,\dots,X_{n-1},I_N)}(\sigma) N^{\gamma_n(\sigma^{-1}\pi)} = \sum_{\sigma\in\mathcal{S}_n} \mathbb{1}\widehat{\mathcal{A}_n}(\sigma) C_{(X_1,\dots,X_{n-1})}(\check{\sigma}) N^{\gamma_n(\sigma^{-1}\pi)}$$

which yields (using that N^{γ_n} is $*$-invertible) that for any σ in $\mathcal{S}_n$,

$$C_{(X_1,\dots,X_{n-1},I_N)}(\sigma) = \mathbb{1}\widehat{\mathcal{A}_n}(\sigma) C_{(X_1,\dots,X_{n-1})}(\check{\sigma}) = \begin{cases} C_{(X_1,\dots,X_{n-1})}(\rho) & \text{if } \sigma = (n)\rho \\ 0 & \text{else.} \end{cases}$$

It proves the first inductive step.

Let $l \geq 2$. Let us suppose that the result is true for $n-k = l-1$. Using the induction hypothesis and then the first step, we successively get, for any σ in $\mathcal{S}_n$,

$$\begin{aligned} C_{(X_1,\dots,X_k,I_N,\dots,I_N)}(\sigma) &= \begin{cases} C_{(X_1,\dots,X_k,I_N)}(\tau) \text{ if } \sigma = (n)\cdots(k+2)\tau \\ 0 \qquad\qquad\qquad\quad \text{else} \end{cases} \\ &= \begin{cases} C_{(X_1,\dots,X_k)}(\rho) \text{ if } \sigma = (n)\cdots(k+2)(k+1)\rho \\ 0 \qquad\qquad\quad \text{else} \end{cases} \end{aligned}$$

and the proof is complete. $\square$

Proposition 3.1 together with Lemma 3.1 lead to

Corollary 3.1. *Let* $V = \{i \in \{1, \dots, n\}, X_i \neq I_N\} = \{i_1 < \cdots < i_k\}$. *Then*

$$C_{(X_1,\dots,X_n)}(\pi) = \begin{cases} C_{(X_{i_1},\dots,X_{i_k})}(\rho) & \textit{if } \pi_{|V^c} = e \textit{ and } \pi_{|V} = \rho \\ 0 & \textit{else}. \end{cases}$$

Examples (see [3], [8], [4]).

1. $X = \lambda I_N$, $\mathbb{E}(r(X)) = \lambda^n N^{\gamma_n}$, $C_X = \lambda^n \delta_e$,
   $C_1(X) = \lambda$, $\quad C_n(X) = 0 \quad$ for $n \geq 2$.
2. X is the matrix of an orthogonal projection on a p-dimensional subspace; then
$$\mathbb{E}(r(X)) = p^{\gamma_n},\ C_X = p^{\gamma_n} * (N^{\gamma_n})^{(-1)}.$$
3. X is $N(0, \sigma^2 I_{N^2})$-distributed; then with $n = 2k$,
$$C_X = \sigma^{2k} \mathbb{1}\widehat{\mathcal{P}_{2k}},\ C_2(X) = \sigma^2,\ C_n(X) = 0 \text{ for } n \neq 2$$
   where $\mathcal{P}_{2k}$ denotes the subgroup of the permutations of $\mathcal{S}_{2k}$ such that their decomposition in disjoint cycles only contains pairs.
4. X is Wishart $W(N, p, \Sigma)$-distributed; then
$$\mathbb{E}(r(X)) = p^{\gamma_n} * \mathbb{E}(r(\Sigma)),\ C_X = p^{\gamma_n} * c(\Sigma).$$
   In particular for $\Sigma = \sigma^2 I_{N^2}$,
$$\mathbb{E}(r(X)) = \sigma^{2n}\ p^{\gamma_n} * N^{\gamma_n},\ C_X = \sigma^{2n}\ p^{\gamma_n},\ C_n(X) = \sigma^{2n}\ p.$$
5. X is Wishart $W(N, p, \Sigma)$-distributed with $p > N$; then
$$\begin{aligned} \mathbb{E}(r(X^{-1})) &= (-1)^{n-\gamma_n}[(p-N)^{\gamma_n}]^{(-1)} * \mathbb{E}(r(\Sigma^{-1}))\,, \\ C_{X^{-1}} &= (-1)^{n-\gamma_n}[(p-N)^{\gamma_n}]^{(-1)} * C_{\Sigma^{-1}}. \end{aligned}$$
   In particular for $\Sigma = \sigma^2 I_{N^2}$,
$$\begin{aligned} \mathbb{E}(r(X^{-1})) &= \tfrac{(-1)^{n-\gamma_n}}{\sigma^{2n}}\ [(p-N)^{\gamma_n}]^{(-1)} * N^{\gamma_n}, \\ C_{X^{-1}} &= \tfrac{(-1)^{n-\gamma_n}}{\sigma^{2n}}[(p-N)^{\gamma_n}]^{(-1)}. \end{aligned}$$

6. X is Beta$(p, p+q)$-distributed; then

$$\mathbb{E}(r(X)) = [(p+q)^{\gamma_n}]^{(-1)} * p^{\gamma_n} * N^{\gamma_n},\ C_X = [(p+q)^{\gamma_n}]^{(-1)} * p^{\gamma_n}.$$

In subsection 4.3, we will also give the cumulants of a Haar matrix.

Note that for some of these examples, the C_n obviously linearize the convolution. Nevertheless, one can check that

$$\begin{aligned} &C_2(X_1+X_2) - C_2(X_1) - C_2(X_2) \\ &= \frac{1}{2(N^2-1)}\mathbb{E}\{Tr(X_1X_2)\} - \frac{1}{2N(N^2-1)}\mathbb{E}\{Tr(X_1)Tr(X_2)\} \end{aligned}$$

is not null for some diagonal matrices for example. Actually, Corollary 3.1 together with a relation convolution of type (15) below (when it exists!) will actually lead to the linearizing property of the C_n (see section 5). Since we are able to show the existence of such a convolution relation for models having distribution invariant under unitary conjugation, we can deduce that the C_n do linearize the convolution on such models.

4. Models with distribution invariant under unitary conjugation

The results of this section are based on the integration formula on the unitary group $\mathbb{U}(N)$. We first apply it to the computation of the mixed moments for a model having distribution invariant under unitary conjugation together with any independent set of matrices. We also do use of it to get the cumulants of a random unitary matrix following the Haar measure on $\mathbb{U}(N)$.

4.1. Integration on the unitary group

In [21], [1] and more recently [4] or [6], the authors give the following formula for integration with respect to the Haar measure on $\mathbb{U}(N)$ when $n \leq N$:

Proposition 4.1. *Let n and n' be positive integer numbers and let $\mathbf{i} = (i_1, \ldots, i_n)$, $\mathbf{i}' = (i'_1, \ldots, i'_{n'})$, $\mathbf{j} = (j_1, \ldots, j_n)$, $\mathbf{j}' = (j'_1, \ldots, j'_{n'})$ be tuples of positive integers in $\{1, \ldots N\}$. Then,*
if $n \neq n'$,

$$\int_{\mathbb{U}(N)} U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U_{i'_1 j'_1}} \ldots \overline{U_{i'_{n'} j'_{n'}}}\, dU = 0;$$

and if $n = n'$,

$$\begin{aligned} &\int_{\mathbb{U}(N)} U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U_{i'_1 j'_1}} \ldots \overline{U_{i'_n j'_n}}\, dU \\ &= \sum_{\sigma,\tau \in \mathcal{S}_n} \delta_{i_1 i'_{\sigma(1)}} \ldots \delta_{i_n i'_{\sigma(n)}} \delta_{j_1 j'_{\tau(1)}} \ldots \delta_{j_n j'_{\tau(n)}}\ Wg^U(\tau\sigma^{-1}) \end{aligned} \tag{10}$$

where Wg^U denotes the Weingarten function on $\mathcal{S}_n$.

We refer the reader to [6] for the definition and the properties of Wg^U.

4.2. Application to mixed moments

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a n-tuple of $N \times N$ complex matrices such that, for any unitary matrix U, $(UX_1U^*, \ldots, UX_nU^*)$ has the same joint distribution as $(X_1, \ldots, X_n)$. Let $\mathbf{B} = (B_1, \ldots, B_n)$ be $N \times N$ matrices which are independent with $\mathbf{X}$. We first compute $\mathbb{E}\left(r_e(B_1X_1, \ldots, B_nX_n)\right)$ using (10) and then turn to the general case.
We have for any independent $N \times N$ matrix U whose distribution is the Haar measure on the unitary group $\mathbb{U}(N)$:

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^n Tr(B_iX_i)\right) &= \mathbb{E}\left(\prod_{i=1}^n Tr(B_iUX_iU^*)\right)\\ &= \sum_{\mathbf{i},\mathbf{j},\mathbf{i}',\mathbf{j}'} \mathbb{E}\left((B_1)_{i'_1 i_1} U_{i_1 j_1} (X_1)_{j_1 j'_1} \overline{U_{i'_1 j'_1}} \cdots (B_n)_{i'_n i_n} U_{i_n j_n} (X_n)_{j_n j'_n} \overline{U_{i'_n j'_n}}\right)\end{aligned}$$

Now, one can deduce from formula (10) that

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^n Tr(B_iX_i)\right) &= \sum_{\sigma,\tau\in\mathcal{S}_n} Wg^U(\tau\sigma^{-1})\\ &\quad\times\left\{\sum_{\mathbf{i},\mathbf{i}'} \delta_{i_1 i'_{\sigma(1)}} \ldots \delta_{i_n i'_{\sigma(n)}}\ \mathbb{E}((B_1)_{i'_1 i_1} \cdots (B_n)_{i'_n i_n})\right\}\\ &\quad\times\left\{\sum_{\mathbf{j},\mathbf{j}'} \delta_{j_1 j'_{\tau(1)}} \ldots \delta_{j_n j'_{\tau(n)}}\ \mathbb{E}((X_1)_{j_1 j'_1} \cdots (X_n)_{j_n j'_n})\right\}\\ &= \sum_{\sigma,\tau\in\mathcal{S}_n} Wg^U(\tau\sigma^{-1})\mathbb{E}\left(r_\sigma(\mathbf{B})\right)\mathbb{E}\left(r_{\tau^{-1}}(\mathbf{X})\right)\\ &= \sum_{\sigma\in\mathcal{S}_n} \mathbb{E}\left(r_\sigma(\mathbf{B})\right)\left(\sum_{\tau\in\mathcal{S}_n} \mathbb{E}\left(r_\tau(\mathbf{X})\right) Wg^U(\tau^{-1}\sigma^{-1})\right)\end{aligned}$$

the last equality coming from the previous one in exchanging τ by its inverse.
Introduce for any permutation π:

$$\tilde{C}_{\mathbf{X}}(\pi) = \sum_{\tau\in\mathcal{S}_n} \mathbb{E}\left(r_\tau(\mathbf{X})\right) Wg^U(\tau^{-1}\pi) = \{\mathbb{E}\left(r(\mathbf{X})\right) * Wg^U\}(\pi)$$

We get:

$$\mathbb{E}\left(r_e(B_1X_1, \ldots, B_nX_n)\right) = \mathbb{E}\left(\prod_{i=1}^n Tr(B_iX_i)\right) = \{\mathbb{E}\left(r(\mathbf{B})\right) * \tilde{C}_{\mathbf{X}}\}(e) \quad (11)$$

Remark 1. Since Wg^U is a central function (see [6]), any function f on S_n commutes with Wg^U, so that $\tilde{C}_{\mathbf{X}} = Wg^U * \mathbb{E}\left(r(\mathbf{X})\right)$ and

$$\mathbb{E}\left(r_e(B_1X_1, \ldots, B_nX_n)\right) = \mathbb{E}\left(r(\mathbf{B})\right) * Wg^U * \mathbb{E}\left(r(\mathbf{X})\right)(e) = \tilde{C}_{\mathbf{B}} * \mathbb{E}\left(r(\mathbf{X})\right)(e)$$

Let us now consider the generalized moments with any π in $\mathcal{S}_n$. We need some preliminary results. Let us introduce the following basis $\{E_{a,b}\}_{1\le a,b\le N}$ of the $\mathbb{C}$-space of $N \times N$ complex matrices, defined by

$$(E_{a,b})_{ij} = \delta_{(a,b),(i,j)} = \delta_{a,i}\delta_{b,j}.$$

It has the property that $Tr(YE_{a,b}) = Y_{ba}$.

Lemma 4.1. *For all permutations σ, π in S_n,*

$$r_\sigma(E_{a_{\pi(1)},b_1}, \dots, E_{a_{\pi(n)},b_n}) = r_{\pi\sigma}(E_{a_1,b_1}, \dots, E_{a_n,b_n}) \tag{12}$$

The proof is let to the reader.

Now for any permutation π of S_n, let us compute $\mathbb{E}\left(r_\pi(B_1X_1, \dots, B_nX_n)\right)$. Consider first the case where the B_i are some E_{a_i,b_i}:

$$\begin{aligned}
\mathbb{E}\left(r_\pi(E_{a_1,b_1}X_1, \dots, E_{a_n,b_n}X_n)\right) &= \mathbb{E}\left(\prod_i (X_i)_{b_i,a_{\pi(i)}}\right)\\
&= \mathbb{E}\left(r_e(E_{a_{\pi(1)},b_1}X_1, \dots, E_{a_{\pi(n)},b_n}X_n)\right)\\
&\overset{(11)}{=}\sum_{\sigma\in S_n} \mathbb{E}\left(r_{\sigma^{-1}}(E_{a_{\pi(1)},b_1}, \dots, E_{a_{\pi(n)},b_n})\right)\tilde{C}_{\mathbf{X}}(\sigma)\\
&\overset{(12)}{=} \sum_{\sigma\in S_n} \mathbb{E}\left(r_{\pi\sigma^{-1}}(E_{a_1,b_1}, \dots, E_{a_n,b_n})\right)\tilde{C}_{\mathbf{X}}(\sigma)\\
&= \{\mathbb{E}\left(r(E_{a_1,b_1}, \dots, E_{a_n,b_n})\right) * \tilde{C}_{\mathbf{X}}\}(\pi)
\end{aligned}$$

By n-linearity and using Remark 1, we deduce that :

$$\mathbb{E}\left(r_\pi(B_1X_1, \dots, B_nX_n)\right) = \{\mathbb{E}(r(\mathbf{B})) * \tilde{C}_{\mathbf{X}}\}(\pi) = \{\tilde{C}_{\mathbf{B}} * \mathbb{E}(r(\mathbf{X}))\}(\pi) \tag{13}$$

Now, for $X_1 = \cdots = X_n = I_N$, which is invariant under unitary conjugations, and for any $\mathbf{B}$, we get $\mathbb{E}\left(r_\pi(\mathbf{B})\right) = \{\tilde{C}_{\mathbf{B}} * N^{\gamma_n}\}(\pi)$ so that if $n \le N$, N^{γ_n} being $*$-invertible, $\tilde{C}_{\mathbf{B}} = C_{\mathbf{B}}$. Taking now $B_1 = \cdots = B_n = I_N$, we finally get that

$$Wg^U = (N^{\gamma_n})^{(-1)} = \left(\mathbb{E}\left(r(I_N)\right)\right)^{(-1)} \tag{14}$$

Hence we have proved the following

Theorem 4.1. *Let $\mathcal{X}$ and $\mathcal{B}$ be two independent sets of $N \times N$ random complex matrices such that the distribution of $\mathcal{X}$ is invariant under unitary conjugations. Then for any $n \le N$, $\mathbf{X} = (X_1, \dots, X_n)$ a n-tuple in $\mathcal{X}$ and $\mathbf{B} = (B_1, \dots, B_n)$ in $\mathcal{B}$, we have:*

$$\mathbb{E}\left(r_\pi(B_1X_1, \dots, B_nX_n)\right) = \{\mathbb{E}(r(\mathbf{B})) * C_{\mathbf{X}}\}(\pi) = \{C_{\mathbf{B}} * \mathbb{E}(r(\mathbf{X}))\}(\pi)$$

From Theorem 4.1, we can readily get the following convolution relation which has to be related to Theorem 1.4 in [17].

Corollary 4.1. *With the hypothesis of Theorem 4.1*

$$C_{(X_1B_1,\dots,X_nB_n)} = C_{\mathbf{X}} * C_{\mathbf{B}}. \tag{15}$$

4.3. Cumulants of a n-tuple from a Haar matrix and its inverse

Proposition 4.2. *Let U be a Haar-distributed unitary random matrix.*
For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, -1\}^n$, let be $V_\varepsilon := \{i \in \{1, \ldots, n\}; \varepsilon_i = 1\}$, $V_\varepsilon^- := \{i \in \{1, \ldots, n\}; \varepsilon_i = -1\}$, $\mathcal{T}_\varepsilon := \{\xi \in \mathcal{S}_n; \xi(V_\varepsilon) = V_\varepsilon^-; \xi(V_\varepsilon^-) = V_\varepsilon\}$ (note that $\mathcal{T}_\varepsilon$ is empty for odd n). Then

$$C_{(U^{\varepsilon_1}, \ldots, U^{\varepsilon_n})}(\xi) = \mathbb{1}\mathcal{T}_\varepsilon(\xi)(N^{\gamma_{n/2}})^{(-1)}(\xi^2_{|V_\varepsilon})$$

Proof. For any π in $\mathcal{S}_n$,

$$\mathbb{E}\left(r_\pi(E_{a_1,b_1}U^{\varepsilon_1}, \ldots, E_{a_n,b_n}U^{\varepsilon_n})\right) = \mathbb{E}\left(\prod_i (U^{\varepsilon_i})_{b_i,a_{\pi(i)}}\right).$$

According to Proposition 4.1, the last quantity vanishes whenever $card V_\varepsilon^- \neq card V_\varepsilon$ (therefore in particular when n is odd). Now, for any π in $\mathcal{S}_{2p}$,

$$\begin{aligned}
&\mathbb{E}\left(r_\pi(E_{a_1,b_1}U, \ldots, E_{a_p,b_p}U, E_{a_{p+1},b_{p+1}}U^{-1}, \ldots, E_{a_{2p},b_{2p}}U^{-1})\right)\\
&= \mathbb{E}\left(\prod_{i=1}^p U_{b_i,a_{\pi(i)}} \prod_{j=1}^p \overline{U_{a_{\pi(j+p)},b_{j+p}}}\right)\\
&= \sum_{(\sigma,\tau)\in\mathcal{S}_p\times\mathcal{S}_p} \prod_{i=1}^p \delta_{b_i,a_{\pi(\sigma(i)+p)}}\delta_{a_{\pi(i)},b_{\tau(i)+p}} Wg^U(\tau\sigma^{-1})
\end{aligned}$$

where we used (10). Let us denote $\mathcal{T}_{(1,\ldots,1,-1,\ldots,-1)}$ by $\mathcal{T}$. There is a one-to-one correspondance between $\mathcal{T}$ and $\mathcal{S}_p \times \mathcal{S}_p$ defined by: $(\sigma, \tau) \in \mathcal{S}_p \times \mathcal{S}_p \mapsto \xi \in \mathcal{S}_{2p}$,

$$\forall i \in \{1, \ldots, p\} \quad \xi(i) = \tau(i) + p,$$

$$\forall i \in \{p+1, \ldots, 2p\} \quad \xi(i) = \sigma^{-1}(i-p).$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left(r_\pi(E_{a_1,b_1}U, \ldots, E_{a_p,b_p}U, E_{a_{p+1},b_{p+1}}U^{-1}, \ldots, E_{a_{2p},b_{2p}}U^{-1})\right)\\
&= \sum_{\xi\in\mathcal{S}_{2p}} \mathbb{1}\mathcal{T}(\xi)\prod_{i=1}^{2p} \delta_{b_i,a_{\pi\xi^{-1}(i)}} Wg^U(\xi^2_{|\{1,\ldots,p\}})\\
&= \sum_{\xi\in\mathcal{S}_{2p}} \mathbb{1}\mathcal{T}(\xi) r_{\pi\xi^{-1}}(E_{a_1,b_1}, \ldots, E_{a_{2p},b_{2p}}) Wg^U(\xi^2_{|\{1,\ldots,p\}}).
\end{aligned}$$

Thus, $C_{(U,\ldots,U,\ldots,U^{-1},\ldots,U^{-1})}(\xi) = \mathbb{1}\mathcal{T}(\xi)(N^{\gamma_p})^{(-1)}(\xi^2_{|\{1,\ldots,p\}})$.

Now, let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{2p})$ be in $\{1, -1\}^{2p}$ such that $card V_\varepsilon^- = card V_\varepsilon$. Then,

$$V_\varepsilon = \{i_1 < \cdots < i_p\} \text{ and } V_\varepsilon^- = \{j_1 < \cdots < j_p\}.$$

Let us define $\psi_\varepsilon \in \mathcal{S}_{2p}$ by $\forall l \in \{1, \dots, p\}$ $\psi_\varepsilon(l) = i_l$, $\psi_\varepsilon(l+n) = j_l$. By Lemma 3.1,

$$\begin{aligned} C_{(U^{\varepsilon_1},\dots,U^{\varepsilon_{2p}})}(\xi) &= C_{(U,\dots,U,\dots,U^{-1},\dots,U^{-1})}(\psi_\varepsilon^{-1}\xi\psi_\varepsilon) \\ &= \mathbb{1}\mathcal{T}(\psi_\varepsilon^{-1}\xi\psi_\varepsilon)(N^{\gamma_p})^{(-1)}(\psi_\varepsilon^{-1}\xi^2\psi_{\varepsilon|\{1,\dots,p\}}) \\ &= \mathbb{1}\mathcal{T}_\varepsilon(\xi)(N^{\gamma_p})^{(-1)}(\xi^2_{|V_\varepsilon}). \end{aligned}$$

□

5. Linearizing property of C_n on models having distribution invariant under unitary conjugation

We establish here the essential properties which lead us to adopt the name of cumulants. These properties are well-known for free cumulants (see [17] for instance).

Proposition 5.1. *Let $X_1, \dots, X_m$ be m independent matrices such that each X_i has a distribution invariant under unitary conjugation. Let $k_1, \dots, k_n$ be in $\{1, \dots, m\}^n$. For each i in $\{1, \dots, m\}$ define*

$$V_i = \{j \in \{1, \dots, n\}, k_j = i\}.$$

Let π be in $\mathcal{S}_n$. If for every i in $\{1, \dots, m\}$, $\pi(V_i) = V_i$ then

$$C_{(X_{k_1},\dots,X_{k_n})}(\pi) = \prod_{i=1}^m C_{X_i}(\pi_{|V_i}).$$

Else,

$$C_{(X_{k_1},\dots,X_{k_n})}(\pi) = 0.$$

Proof. Note that

$$C_{(X_{k_1},\dots,X_{k_n})}(\pi) = C_{(A_{k_1}B_{k_1},\dots,A_{k_n}B_{k_n})}(\pi)$$

where, if $k_i = 1$, $A_{k_i} = X_1$, $B_{k_i} = I$ and if $k_i \in \{2, \dots, m\}$, $A_{k_i} = I$, $B_{k_i} = X_{k_i}$. Hence, we have from (15):

$$\begin{aligned} C_{(X_{k_1},\dots,X_{k_n})}(\pi) &= C_{(A_{k_1},\dots,A_{k_n})} * C_{(B_{k_1},\dots,B_{k_n})}(\pi) \\ &= \sum_{\sigma \in \mathcal{S}_n} C_{(A_{k_1},\dots,A_{k_n})}(\sigma) C_{(B_{k_1},\dots,B_{k_n})}(\sigma^{-1}\pi). \end{aligned}$$

Set $V_1^c = \{i_1 < \cdots < i_k\}$. As $A_{k_i} = I$ for any i in V_1^c and $B_{k_i} = I$ for any i in V_1, according to corollary 3.1, we can conclude that, if $\pi(V_1) = V_1$, then

$$C_{(X_{k_1},\dots,X_{k_n})}(\pi) = C_{X_1}(\pi_{|V_1}) C_{(X_{k_{i_1}},\dots,X_{k_{i_k}})}(\pi_{|V_1^c})$$

and otherwise $C_{(X_{k_1},\dots,X_{k_n})}(\pi) = 0$.
Now, the result readily follows by an inductive argument on m. □

Corollary 5.1. *Let X_1 and X_2 be two independent matrices such that X_1 has a distribution invariant under unitary conjugation. For any $n \leq N$,*

$$C_n(X_1 + X_2) = C_n(X_1) + C_n(X_2).$$

6. Asymptotic behavior

For any n-tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of $N \times N$ matrices, we call *normalized generalized moments* with order n of $\mathbf{X}$ the quantities $\mathbb{E}(r_\pi^{(N)}(\mathbf{X}))$ where π is in $\mathcal{S}_n$ and

$$r_\pi^{(N)}(\mathbf{X}) := \frac{1}{N^{\gamma_n(\pi)}} r_\pi(X_1, \ldots, X_n) = \prod_{C \in \mathcal{C}(\pi)} \frac{1}{N} Tr \left(\prod_{j \in C} X_j \right).$$

We have the following equivalence.

Proposition 6.1. *When N goes to infinity, the normalized cumulants*

$$(C_{\mathbf{X}})^{(N)}(\pi) := N^{d_n(e,\pi)} C_{\mathbf{X}}(\pi)$$

tend towards the free cumulants $k_\pi(\mathbf{x})$ of noncommutative random variables $\mathbf{x} = (x_1, \ldots, x_n)$ if and only if the normalized generalized moments $\mathbb{E}(r_\pi^{(N)}(\mathbf{X}))$ converge towards the generalized moments $\phi_\pi(x_1, \ldots, x_n) = \prod_{C \in \mathcal{C}(\pi)} \phi(\prod_{i \in C} x_i)$ of $\mathbf{x}$.

In [3], a set $\mathcal{X}$ is said to satisfy *condition* $(\mathcal{C})$ if this property is valid for any tuple $\mathbf{X}$ in $\mathcal{X}$. This equivalence has been proved in lemma 6.4 therein for Hermitian matrices but still holds for any set of complex matrices. Indeed, the proof is only based on the following facts: the $*$-convolution relation $\mathbb{E}(r_\pi(\mathbf{X})) = C_{\mathbf{X}} * N^{\gamma_n(.)}(\pi)$ on the whole group $\mathcal{S}_n$ with the suitable normalization $\frac{1}{N^{\gamma_n(.)}}$ becomes the $\star$-convolution relation $\phi(\mathbf{x}) = k(\mathbf{x}) \star \mathbb{1}\mathcal{S}_n = \sum_{\sigma \in [e,\pi]} k(\mathbf{x})$ existing between moments and free cumulants of $\mathbf{x}$; conversely $C_{\mathbf{X}} = \mathbb{E}(r_.(\mathbf{X})) * (N^{\gamma_n})^{-1}$ becomes $k(\mathbf{x}) = \phi(\mathbf{x}) \star \mu_n(e, .)$, after normalization by $N^{d_n(e,.)}$ using the asymptotic result:

$$\frac{N^{2n}}{N^{\gamma_n(\pi)}} (N^{\gamma_n})^{(-1)}(\pi) = \mu_n(e, \pi) + O\left(\frac{1}{N^2}\right). \tag{16}$$

Equation (16) has been independently proved in [4].
Likewise under the hypothesis of Theorem 4.1 together with condition $(\mathcal{C})$ for $\mathcal{X}$ and $\mathcal{B}$, the mixed moments

$$\mathbb{E}(r_\pi^{(N)}(X_1 B_1, \ldots, X_n B_n)) = \frac{1}{N^{\gamma_n(\pi)}} C_{\mathbf{X}} * \mathbb{E}(r(\mathbf{B}))(\pi)$$

converge towards the $\star$-convolution

$$k(\mathbf{x}) \star \phi(\mathbf{b})(\pi) = \phi_\pi(x_1 b_1, \ldots, x_n b_n)$$

giving the mixed moments of two free sets of noncommutative variables. Thus, we get the asymptotic freeness of $\mathcal{X}$ and $\mathcal{B}$ already stated in Proposition 6.5 [3].

We now explain how one can get global fluctuations by using our matricial cumulants. The variance of two traces can easily be expressed in terms of cumulants.

Let be $n = n_1 + n_2$, $\pi_1 = (1, \ldots, n_1)$, $\pi_2 = (n_1 + 1, \ldots, n)$. Set $\mathcal{S}^1$ (respectively $\mathcal{S}^2$) the symmetric group on $\{1, \ldots, n_1\}$ (resp. on $\{n_1 + 1, \ldots, n\}$). For any n-tuple $\mathbf{A} = (A_1, \ldots, A_n)$ of $N \times N$ complex matrices, write $\mathbf{A_1} = (A_1, \ldots, A_{n_1})$, $\mathbf{A_2} = (A_{n_1+1}, \ldots, A_n)$. Then,

$$\begin{aligned}
k_2 \left(Tr(A_1 \cdots A_{n_1}), Tr(A_{n_1+1} \cdots A_n)\right) &:= \mathbb{E}(r_{\pi_1\pi_2}(\mathbf{A})) - \mathbb{E}(r_{\pi_1}(\mathbf{A_1}))\mathbb{E}(r_{\pi_2}(\mathbf{A_2})) \\
= \sum_{(\sigma_1,\sigma_2)\in\mathcal{S}^1\times\mathcal{S}^2} &\left(C_{\mathbf{A}}(\sigma_1\sigma_2) - C_{\mathbf{A_1}}(\sigma_1)C_{\mathbf{A_2}}(\sigma_2)\right) N^{\gamma_n(\sigma_1^{-1}\sigma_2^{-1}\pi_1\pi_2)} \\
+ \sum_{\sigma\in\mathcal{S}_n\setminus(\mathcal{S}^1\times\mathcal{S}^2)} &C_{\mathbf{A}}(\sigma) N^{\gamma_n(\sigma^{-1}\pi_1\pi_2)}.
\end{aligned}$$

Let $\mathbf{a}$ be some n-tuple of noncommutative random variables, $\mathbf{a_1} = (a_1, \ldots, a_{n_1})$ and $\mathbf{a_2} = (a_{n_1+1}, \ldots, a_n)$. Then if for any $(\sigma_1, \sigma_2) \in \mathcal{S}^1 \times \mathcal{S}^2$,

$$N^{d(e,\sigma_1\sigma_2)} \left\{C_{\mathbf{A}}(\sigma_1\sigma_2) - C_{\mathbf{A_1}}(\sigma_1)C_{\mathbf{A_2}}(\sigma_2)\right\} = \frac{\alpha_{\sigma_1,\sigma_2}(\mathbf{a})}{N^2} + o(\frac{1}{N^2}), \tag{17}$$

and for any σ in $\mathcal{S}_n$,

$$\lim_{N\to+\infty} (C_{\mathbf{A}})^{(N)}(\sigma) = k_\sigma(\mathbf{a}),$$

it follows that

$$\begin{aligned}
&\lim_{N\to+\infty} k_2 \left(Tr(A_1 \cdots A_{n_1}), Tr(A_{n_1+1} \cdots A_n)\right) \\
&= \sum_{\sigma_1\sigma_2\in[e,\pi_1\pi_2]} \alpha_{\sigma_1,\sigma_2}(\mathbf{a}) + \sum_{\substack{\sigma\in\mathcal{S}_n\setminus(\mathcal{S}^1\times\mathcal{S}^2) \\ d(e,\pi_1\pi_2)+2=d(e,\sigma)+d(\sigma,\pi_1\pi_2)}} k_\sigma(\mathbf{a}).
\end{aligned} \tag{18}$$

Note that (17) implies condition $(\mathcal{C}')$ in [3] and therefore the almost surely convergence of $\mathbf{A}$ towards $\mathbf{a}$. Note also that (18) is a new formulation of (43) in [15]. In particular, when $\mathbf{A}$ is a complex Wishart or Gaussian matrix, $\alpha_{\sigma_1,\sigma_2}(\mathbf{a}) = 0$ for any (σ_1, σ_2) and we recover Theorem 7.5 in [14] concerning the Wishart case. When $\mathbf{A}$ is unitary, we can similarly get Theorem 3.6 in [16].

7. Further properties

We first point out the analogues in our matricial context of the results of A. Nica and R. Speicher in [17]. In that paper the authors present some applications of their Theorem 1.4. Two of these concern conjugation with a circular element which is free from the family. Let us state their analogues in our matricial context.

The following result has to be related to Application 1.6 in [17].

Proposition 7.1. *Let G be a Gaussian random matrix with independent entries with mean zero and variance σ^2. For any integer $n \leq N$ and for any $(B_1, \ldots, B_n)$ random matrices independent with G^*G,*

$$C_{(GB_1G^*,\ldots,GB_nG^*)} = \mathbb{E}(r(\sigma^2 B_1, \ldots, \sigma^2 B_n)).$$

Proof. The matrix G^*G is a $W(N, N, \sigma^2 I)$ Wishart matrix and therefore $C_{G^*G} = \sigma^{2n} N^{\gamma_n}$. Thus we can deduce from corollary 4.1 that:

$$\begin{aligned} C_{(GB_1G^*,\dots,GB_nG^*)} &= C_{(B_1G^*G,\dots,B_nG^*G)} = C_{(G^*G)} * C_{(B_1,\dots,B_n)} \\ &= \sigma^{2n} N^{\gamma_n} * C_{(B_1,\dots,B_n)} = \sigma^{2n}\mathbb{E}(r(B_1,\dots,B_n)) \\ &= \mathbb{E}(r(\sigma^2 B_1,\dots,\sigma^2 B_n)). \end{aligned}$$

□

As to the result that this conjugation "converts orthogonality to freeness" (see Corollary 1.8 in [17]), it admits the following matricial interpretation:

Corollary 7.1. *If* $(B_1,\dots,B_n)$ *are taken among* $A_1,\dots,A_k$ *which are independent with* G^*G *and such that* $A_iA_j = 0$ *for* $i \neq j$, *then* $C_n(GB_1G^*,\dots,GB_nG^*) = 0$ *whenever there exists* i *and* j *such that* $B_i \neq B_j$.

In [17], the authors presented Application 1.11 which dealt with the compression of a family of random variables by a projection which is free with the family. Here is its matricial formulation.

Proposition 7.2. *Let* $\mathcal{M}(N)$ *denote the space of* $N \times N$ *complex matrices. Let be* $p \leq N$. *Define the contraction*

$$\Psi(N, p) : \mathcal{M}(N) \to \mathcal{M}(p),\ X \mapsto (X_{i,j})_{i,j\in\{1,\dots,p\}}.$$

Then if the distribution of $\mathbf{X} = (X_1,\dots,X_n)$ *is invariant under unitary conjugations, we have:*

$$\begin{aligned} C_{(\Psi(N,p)(X_1),\dots,\Psi(N,p)(X_n))} &= C_{\mathbf{X}}, \\ (C_n)^{(p)}_{(\Psi(N,p)(X_1),\dots,\Psi(N,p)(X_n))} &= \frac{N}{p}(C_n)^{(N)}_{(\frac{p}{N}X_1,\dots,\frac{p}{N}X_n)}. \end{aligned}$$

Proof. Let P_p be the matrix $\begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}$. The result follows from

$$\begin{aligned} \mathbb{E}(r(\Psi(N, p)(X_1),\dots,\Psi(N, p)(X_n))) &= \mathbb{E}(r(P_pX_1,\dots,P_pX_n)) \\ &= C_{\mathbf{X}} * \mathbb{E}(r(P_p,\dots,P_p)) = C_{\mathbf{X}} * p^{\gamma_n}. \end{aligned}$$

□

We now mention the straightforward action on the cumulants of an expansion of the space.

Proposition 7.3. *Let* $p \leq N$. *Define the expansion* $\Phi(p, N)$ *by*

$$\Phi(p, N) : \mathcal{M}(p) \to \mathcal{M}(N),\ X_p \mapsto Y_N := \begin{pmatrix} X_p & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$C_{Y_N} = C_{X_p} * p^{\gamma_n} * (N^{\gamma_n})^{(-1)}.$$

Finally we readily get the combined action of a random projection invariant under unitary conjugation with a contraction of the space.

Proposition 7.4. *Let p, q and N satisfy $N \leq p+q$ and let $P_p^{(p+q)}$ be the random projection of $\mathcal{M}(p+q)$ of rank p,*

$$P_p^{(p+q)} = U_{p+q} I(p,q) U_{p+q}^{-1}$$

where U_{p+q} is a $(p+q) \times (p+q)$ Haar-distributed unitary matrix and

$$I(p,q) = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}.$$

Then if the distribution of X_{p+q} is invariant under unitary conjugations and if

$$Z_N := \Psi(p+q, N)(P_p^{(p+q)} X_{p+q} P_p^{(p+q)}),$$

we have:

$$C_{Z_N} = C_{X_{p+q}} * p^{\gamma_n} * [(p+q)^{\gamma_n}]^{(-1)}.$$

Note that for $X = I_{p+q}$, we get $C_{Z_N} = p^{\gamma_n} * [(p+q)^{\gamma_n}]^{(-1)}$, which is the cumulant function of a Beta matrix Beta(p,q) (see Example 6 above). This last result agrees with [5].

The situation for orthogonally invariant matrices is much more complicated as one can see it through the computation of mixed generalized moments involving a real Wishart matrix in [9]. In a forthcoming paper, we will show that under the hypothesis of invariance under orthogonal conjugations of the distribution of one of the two concerned matricial models $\mathcal{X}$ and $\mathcal{B}$, the mixed moments can still be expressed by a convolution relation. This one relies on the integration formula on the orthogonal group stated in [6]. Note that $\mathcal{S}_n$ must be replaced there by $\mathcal{S}_{2n}$. We will be led in consequence to introduce another cumulant function $C_{\mathbf{X}}^{O}$.

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