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# Asymptotic laws for regenerative compositions: gamma subordinators and the like<sup>\*</sup>

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**Abstract.** For  $\tilde{\mathcal{R}} = 1 - \exp(-\mathcal{R})$  a random closed set obtained by exponential transformation of the closed range  $\mathcal{R}$  of a subordinator, a regenerative composition of generic positive integer  $n$  is defined by recording the sizes of clusters of  $n$  uniform random points as they are separated by the points of  $\tilde{\mathcal{R}}$ . We focus on the number of parts  $K_n$  of the composition when  $\tilde{\mathcal{R}}$  is derived from a gamma subordinator. We prove logarithmic asymptotics of the moments and central limit theorems for  $K_n$  and other functionals of the composition such as the number of singletons, doubletons, etc. This study complements our previous work on asymptotics of these functionals when the tail of the Lévy measure is regularly varying at  $0+$ .

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## 1. Introduction

For a drift-free subordinator  $(S_t, t \geq 0)$  define  $\tilde{\mathcal{R}} \subset [0, 1]$  to be the range of the *multiplicative subordinator*  $\tilde{S}_t = 1 - \exp(-S_t)$ ,  $t \geq 0$ , obtained by the exponential transform. The *gaps* in  $\tilde{\mathcal{R}}$  are the interval components of the open set  $[0, 1] \setminus \tilde{\mathcal{R}}$ . The following construction of random compositions of integers has been studied in [9–11, 13, 16]. For each  $n$ , let  $u_1, \dots, u_n$  be an independent sample from the uniform distribution on  $[0, 1]$ , also independent of  $\tilde{\mathcal{R}}$ . Define an ordered partition of the sample into nonempty blocks by assigning two sample points  $u_i$  and  $u_j$  to the same block if and only if the points fall in the same gap in  $\tilde{\mathcal{R}}$ . A composition  $C_n$  of integer  $n$  is defined by the sequence of sizes of blocks, ordered from left to right. The sequence of compositions  $(C_n)$  is a regenerative composition structure in the sense of [11, 12]. The regeneration property means that, conditionally given the first part of  $C_n$  is  $r$ , the remaining composition of  $n - r$  has the same distribution as  $C_{n-r}$ .

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The purpose of this paper is to prove central limit theorems and determine the asymptotic behaviour of moments for the number of parts  $K_n$  and some other functionals of the composition  $\mathcal{C}_n$ . We focus on the gamma subordinator with parameter  $\theta > 0$ , that is with Lévy measure  $\nu(dy) = (e^{-\theta y}/y) dy$ . More generally, we obtain similar results for Lévy measures which are sufficiently like the gamma Lévy measure in their asymptotic behaviour at both  $0+$  and  $\infty-$ . In the case of gamma subordinators our principal result (Theorem 12) specialises as follows:

**Theorem 1.** *For the gamma subordinator with parameter  $\theta > 0$  the number of parts  $K_n$  is asymptotically normal, with  $\mathbb{E} K_n \sim (\theta/2) \log^2 n$  and  $\text{var} K_n \sim (\theta/3) \log^3 n$ .*

Other variables under consideration are  $K_n(t)$ , the number of parts of a partial composition produced by the subordinator restricted to  $[0, t]$ , and the counts  $K_{n,r}$  and  $K_{n,r}(t)$  defined as the multiplicity of part  $r$  in  $\mathcal{C}_n$  and the multiplicity of part  $r$  in the partial composition, respectively.

In our previous paper [13] we studied  $K_n$  for a subordinator whose Lévy measure has regularly varying right tail  $\nu[x, \infty]$ , with some index of regular variation  $\alpha \in ]0, 1]$ . In that case we obtained limit theorems for  $K_n$  with normalisation by  $n^\alpha \ell(n)$  where  $\ell$  is a function of slow variation at  $\infty$ . According to these results,  $K_n/(n^\alpha \ell(n))$  converges with probability one and with all moments to a nondegenerate limit, which is not gaussian. Moreover,  $K_n(t)$  is of the same order of magnitude as  $K_n$ , for each  $t > 0$ . For the case of a compound Poisson process, when the Lévy measure is finite, Gnedin [9] had previously shown that  $K_n$  typically exhibits a normal limit with both expectation and variance of the order  $\log n$ , though  $K_n(t)$  remains bounded as  $n$  grows, for each  $t > 0$ . Thus the case of gamma-type subordinators may be seen as intermediate between the above two: as in the regular variation case,  $K_n(t)$  is unbounded, but, as in the case of a finite Lévy measure, its contribution to  $K_n$  is asymptotically negligible, for each  $t > 0$ . These features of the gamma case are typical for the general case of slow variation [2].

Results akin to Theorem 1 are very different from the limit theorems in [9, 13] and require other techniques. Our approach here relies on a recent version of the contraction method due to Neininger and Rüschemdorf [15]. Application of this method requires an appropriate decomposition of  $K_n$ , control of the principal asymptotics of two moments, and an estimate of the remainder term in the asymptotic expansion of the variance. We provide this background by poissonising  $\mathcal{C}_n$  and applying the Mellin transform technique to analyse integral recursions for the moments.

We also establish joint convergence to a gaussian limit for the sequence of small-part counts  $(K_{n,r}, r \geq 1)$ , as  $n \rightarrow \infty$ . This kind of convergence holds neither in the regular variation case (when the scaled  $K_{n,r}$ 's converge to multiples of the same variable) nor in the compound Poisson case (when  $(K_{n,r}, n \geq 1)$  is bounded uniformly in  $n$ , for each fixed  $r$ ). It resembles, however, Karlin's [14] central limit theorem for nonrandom frequencies in the case of regular variation with a positive index.

**2. Setup and notation**

Each *part* of  $C_n$  corresponds to a nonempty cluster of the uniform sample points within some gap. So  $K_n$  is the number of gaps occupied by at least one of  $n$  sample points, and  $K_{n,r}$  is the number of gaps that contain exactly  $r$  sample points. Keep in mind that  $K_n = \sum_r K_{n,r}$ , and  $n = \sum_r r K_{n,r}$ . Sometimes  $K_n$  is called the *length*, and  $n$  the *weight* of the composition. Similarly,  $K_n(t)$  and  $K_{n,r}(t)$  are the counts of clusters within the gaps of a smaller set  $\tilde{\mathcal{R}} \cap [0, \tilde{S}_t]$ , corresponding to the subordinator restricted to  $[0, t]$ .

For  $\nu$  the Lévy measure of  $(S_t)$  let  $\tilde{\nu}$  be the measure on  $[0, 1]$  obtained from  $\nu$  by the exponential transform  $y \mapsto 1 - e^{-y}$ . The Laplace exponent of the subordinator is given by the formulas

$$\Phi(s) = \int_0^\infty (1 - e^{-sy})\nu(dy) = \int_0^1 (1 - (1 - x)^s)\tilde{\nu}(dx), \quad \Re s \geq 0. \quad (1)$$

Considering also the binomial moments

$$\Phi(n : m) := \int_0^1 \binom{n}{m} x^m (1 - x)^{n-m} \tilde{\nu}(dx), \quad 1 \leq m \leq n, \quad (2)$$

the ratio  $\Phi(n : m)/\Phi(n)$ ,  $m = 1, \dots, n$ , gives the distribution of the first part of  $C_n$ , and the probability of a particular value of  $C_n$  is a product of factors of this type [11]. We also record the relation between power and logarithmic moments of measures  $\nu$  and  $\tilde{\nu}$ :

$$m_j := \int_0^\infty y^j \nu(dy) = \int_0^1 |\log(1 - x)|^j \tilde{\nu}(dx), \quad j = 1, 2, \dots \quad (3)$$

As in [13], we shall also consider the *poissonised* composition  $\widehat{C}_\rho$  derived from  $\tilde{\mathcal{R}}$  by the same construction as  $C_n$ , but with the set of atoms of a homogeneous Poisson point process on  $[0, 1]$  with rate  $\rho$ , instead of the uniform sample of fixed size. We denote  $\widehat{K}_\rho, \widehat{K}_{\rho,r}, \widehat{K}_\rho(t), \widehat{K}_{\rho,r}(t)$  the obvious counterparts of the fixed- $n$  variables, and introduce the Poisson transform of the Laplace exponent

$$\widehat{\Phi}(s) := \int_0^1 (1 - e^{-sx})\tilde{\nu}(dx) = e^{-s} \sum_{n=0}^\infty \frac{s^n}{n!} \Phi(n). \quad (4)$$

We shall be using throughout the abbreviations

$$L = \log \rho, \quad \log^a x = (\log x)^a.$$

Our primary concern are the compositions induced by a *gamma subordinator*, whose Lévy measure, its exponential transform and the Laplace exponent are given in terms of a parameter  $\theta > 0$  by

$$\begin{aligned} \nu(dy) &= \frac{e^{-\theta y}}{y} dy, \quad y \in ]0, \infty[, \quad \tilde{\nu}(dx) = \frac{(1 - x)^{\theta-1}}{|\log(1 - x)|} dx, \quad x \in ]0, 1], \\ \Phi(s) &= \log \left( 1 + \frac{s}{\theta} \right). \end{aligned} \quad (5)$$

The density of  $\tilde{\nu}$  has a pole and the tail  $\tilde{\nu}[x, 1]$  has a logarithmic singularity at 0. We could consider a larger family of measures which differ from (5) by a positive factor, but this would not increase generality, since multiplying the Lévy measure by such a factor does not affect the laws of  $\mathcal{R}$ ,  $C_n$ ,  $K_n$  or  $\tilde{K}_\rho$  (although it requires a linear time-change for functionals of a partial composition like  $K_n(t)$  or  $\tilde{K}_\rho(t)$ ).

More generally, we consider a drift-free subordinator with Lévy measure  $\nu$ , which has a continuous density on  $]0, \infty[$  and satisfies the following conditions (L) and (R) which we record in terms of  $\nu$  and  $\tilde{\nu}$ :

(L) either of the following four equivalent conditions holds

$$\begin{aligned} \Phi(\rho) &= L + c + O(\rho^{-\epsilon}), \text{ as } \rho \rightarrow \infty, \\ \widehat{\Phi}(\rho) &= L + c + O(\rho^{-\epsilon}), \text{ as } \rho \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} \nu[y, \infty] &= -\log y + c - \gamma + O(y^\epsilon), \text{ as } y \downarrow 0, \\ \tilde{\nu}[x, 1] &= -\log x + c - \gamma + O(x^\epsilon), \text{ as } x \downarrow 0, \end{aligned}$$

where  $c$  and  $\epsilon > 0$  are some constants and  $\gamma = -\Gamma'(1)$  is the Euler constant,

(R) either of the following two equivalent conditions holds

$$\nu[y, \infty] = O(e^{-\epsilon y}), \quad y \uparrow \infty, \quad \tilde{\nu}[x, 1] = O((1-x)^\epsilon), \quad x \uparrow 1,$$

where  $\epsilon > 0$ .

See Appendix for the equivalence of conditions in (L) (L stands for *left* and *logarithmic* at 0). Condition (R) (R for *right* or *regular* at 1) implies that all moments  $m_j$  are finite, and that  $\Phi$  is analytical for  $\Re s > -\epsilon$ .

Throughout in this paper,  $\epsilon$  or  $\delta$  are some sufficiently small positive constants whose values are context-related and may change from line to line. We denote  $c$  the constant in (L). For a generic positive constant we write  $d$ , and use  $c_j$ ,  $d_j$  for further real constants which are not important, but may appear with explicit evaluation in asymptotic expansions or other formulas. For shorthand, the right tail of  $\tilde{\nu}$  is denoted by

$$\vec{\nu}(x) = \tilde{\nu}[x, 1], \quad x \in ]0, 1]$$

and a homogeneous Poisson point process with intensity  $\rho$  on  $[0, 1]$  is denoted  $\text{PPP}(\rho)$ .

### 3. Recursions

We shall make use of two types of recursions. These apply to regenerative compositions generated by a drift-free subordinator, without any special assumptions on the Lévy measure. (They can also be readily generalised to the case with drift). The first type of recursion, suited to the Poisson framework, is based on a decomposition of  $\tilde{\mathcal{R}}$  analogous to the first-jump decomposition of a renewal process. The second type is based on splitting the range of a subordinator by its value at a stopping time.

**Integral recursions** Define a *pattern*  $E$  to be a nonempty set of positive integers. We say that a finite configuration of points within a given subinterval  $]a, b[ \in ]0, 1[$  fits in  $E$  if the cardinality of the configuration is in  $E$ . For  $E$  fixed, let

$$\pi(\rho) = \sum_{r \in E} e^{-\rho} \rho^r / r! \tag{6}$$

be the probability that the configuration of atoms of  $\text{PPP}(\rho)$  fits in the pattern  $E$ . The probability that the  $\text{PPP}(\rho)$  configuration on a subinterval  $]a, b[$  fits in  $E$  is equal to  $\pi(\rho(b - a))$ .

Consider the poissonised composition induced by an arbitrary drift-free subordinator with  $\tilde{v}\{1\} = 0$  (no killing). For  $E$  fixed, let  $N_\rho$  be the number of gaps of  $\tilde{\mathcal{R}}$  such that the  $\text{PPP}(\rho)$  configuration within the gaps fits in  $E$ . The count  $N_\rho$  is a functional of the poissonised composition  $\widehat{\mathcal{C}}_\rho$ ; in particular, defining  $E$  to be  $\{1, 2, \dots\}$  (the configuration is nonempty) or  $E = \{r\}$  (the configuration consists of  $r$  points) we obtain  $N_\rho = \widehat{K}_\rho$ , respectively  $N_\rho = \widehat{K}_{\rho,r}$ . Other possibilities may be considered, for example taking  $E = \{1, 3, 5, \dots\}$  the count  $N_\rho$  becomes the number of odd parts of the composition.

Let  $p_j(\rho) = \mathbb{P}(N_\rho = j)$  be the distribution of  $N_\rho$  for some fixed pattern  $E$ . Conditioning on the total number of Poisson atoms we are back to the fixed- $n$  composition  $\mathcal{C}_n$ , hence we can write  $p_j$  in the form

$$p_j(\rho) = e^{-\rho} \sum_{n=0}^{\infty} \frac{\rho^n}{n!} R_n(\Phi(1), \dots, \Phi(n)), \tag{7}$$

where  $R_n$ 's are some rational functions in the variables  $\Phi(1), \Phi(2), \dots$ . This follows because the law of  $\mathcal{C}_n$  is representable as a product of factors like  $\Phi(n : m) / \Phi(n)$  [11, Equations (6) and (28)]. Since each  $R_n$  is some probability we have  $0 \leq R_n \leq 1$ , hence the series converges for all  $\rho \geq 0$  and  $p_j$  may be extended to an entire function of a complex variable.

Introduce the factorial moments

$$f^{(m)}(\rho) = \mathbb{E}N_\rho(N_\rho - 1) \cdots (N_\rho - m + 1), \quad m = 0, 1, \dots$$

with the convention  $f^{(0)}(\rho) = 1$ . The following lemma is a minor variation of [13, Lemma 6.1]. In the representation of  $f^{(m)}$  akin to (7) the analogs of  $R_n$  grow not faster than  $n^m$ , thus  $f^{(m)}$  is also an entire function.

**Lemma 2.** *Let  $E$  be a pattern with the probability of occurrence  $\pi(\rho)$ . The distribution of  $N_\rho$  is uniquely determined by the integral recursion*

$$\begin{aligned} & \int_0^1 [p_j(\rho) - (1 - \pi(\rho x)) p_j(\rho(1 - x))] \tilde{v}(dx) \\ &= \int_0^1 \pi(\rho x) p_{j-1}(\rho(1 - x)) \tilde{v}(dx), \quad j = 0, 1, \dots \end{aligned} \tag{8}$$

with  $p_{-1}(\rho) = 0$  and the initial values  $p_j(0) = 1 (j = 0)$ . Similarly, the factorial moments of  $N_\rho$  satisfy the recursion

$$\int_0^1 [f^{(m)}(\rho) - f^{(m)}(\rho(1-x))] \tilde{\nu}(dx) = m \int_0^1 \pi(\rho x) f^{(m-1)}(\rho(1-x)) \tilde{\nu}(dx), \tag{9}$$

which taken together with  $f^{(0)}(\rho) = 1$  and  $f^{(m)}(0) = 1 (m = 0)$  uniquely determines them.

*Proof.* In case  $\tilde{\nu}$  a probability measure, it is the distribution of the size of the first jump of  $(\tilde{S}_t)$ . Conditionally given the jump-size  $x$ , there are  $j$  occurrences of the pattern when either the Poisson configuration within  $]0, x[$  fits in  $E$  and there are  $j - 1$  further occurrences within  $]x, 1[$ , or the configuration does not fit but there are  $j$  occurrences within  $]x, 1[$ . The recursion (2) follows from the total probability formula, the independence property of PPP and the multiplicative regeneration property of  $\tilde{\mathcal{R}}$ . By homogeneity, the recursion is also valid for arbitrary finite  $\tilde{\nu}$ .

For the general  $\tilde{\nu}$ , observe that the right side of (2) has a definite finite value, because the function  $\pi(\rho x)/x$  has no singularity at  $x = 0$  (in consequence of  $0 \notin E$ ), and because the measure  $x \tilde{\nu}(dx)$  is finite. Furthermore, using (6) and (7), we establish that the integrand in the left side expands in the sum of a similar function of  $x$  and another function proportional to  $1 - (1 - x)^n$ , hence the left side is also well defined. Expanding both sides in powers of  $\rho$  we conclude from (6) and (7) that (8) amounts to a series of polynomial identities in the variables  $(\Phi(1), \Phi(2), \dots)$ . To prove the identities it is sufficient to check that they hold for a sufficiently large range of values of the variables. But we know already that (8) is true for each finite measure  $\tilde{\nu}$  and, as  $\tilde{\nu}$  varies the range of  $(\Phi(1), \dots, \Phi(n)) \in \mathbb{R}^n$  has nonempty interior, as can be checked by just considering discrete  $\tilde{\nu}$ 's with finitely many masses. This is enough to secure polynomial identities, hence (2) can be extrapolated from the case of finite measure.

Along the same lines, expanding the integrand in the left side and integrating in  $x$  allows computing the coefficients of  $p_j$ , up to a constant term. It follows that the solution to (8) with a given initial condition is unique. The recursion for the factorial moments follows as in [13, Lemma 6.1], and the uniqueness is argued in the same way. □

Integral recursions for the distribution and the factorial moments of  $\widehat{K}_\rho$  follow by taking  $\pi(\rho) = 1 - e^{-\rho}$  which is the probability that the  $\text{PPP}(\rho)$  configuration is nonempty; we have then  $p_0(\rho) = e^{-\rho}$ . To obtain recursions for  $\widehat{K}_{r,\rho}$  we should take  $\pi(\rho) = e^{-\rho} \rho^r / r!$  (in this case no simple formula for  $p_0(\rho)$  is known).

**Splitting at an independent exponential time** Further recursions follow by splitting the range of a subordinator by its value at a stopping time. Though the fixed- $n$  version is needed to apply [15], we focus on the poissonised model, which simplifies moment computations. Transfer of results to the fixed  $n$  model then follows by elementary deppoissonisation.

For each  $t \geq 0$ , consider the sigma-algebra generated by both  $(\tilde{S}_u, u \leq t)$  and the restriction of PPP( $\rho$ ) to  $[0, \tilde{S}_t]$ . As  $t$  varies, this defines a filtration, and we can consider stopping times with respect to it. Let  $\tau$  be such a stopping time with range  $[0, \infty]$  and let  $b_\rho = \widehat{K}_\rho(\tau)$  be the number of parts of  $\mathcal{C}_\rho$  produced by the subordinator up to time  $\tau$ . The strong Markov property of  $\tilde{\mathcal{R}}$  along with the independence property of PPP imply that the number of parts satisfies the distributional equation

$$\widehat{K}_\rho \stackrel{d}{=} b_\rho + \widehat{K}'_{I_\rho}, \tag{10}$$

where  $b_\rho = \widehat{K}_\rho(\tau)$ ,  $I_\rho = \rho(1 - \tilde{S}_\tau)$  and  $(\widehat{K}'_\rho)$  is a distributional copy of  $(\widehat{K}_\rho)$ , independent of  $(b_\rho, I_\rho)$ . For example, letting  $\tau$  be the first time that a jump of the subordinator covers at least one Poisson point, the equation holds with  $b_\rho = 1$ . Identities analogous to (10) can be written also for  $\widehat{K}_{\rho,r}$  and more general pattern counts.

We shall consider in some detail the most important case when  $\tau$  is an exponential time with rate  $\lambda$ , independent of the subordinator and the PPP (in this sense  $\tau$  is a *randomised* stopping time). The stopped process has an obvious interpretation as a *killed* multiplicative subordinator, which jumps at time  $\tau$  to the terminal value 1 (thus producing the final *meander* gap in the range). In accordance with our previous notation, let  $\widehat{K}_\rho(\tau)$  be the number of parts produced by the subordinator *before*  $\tau$  (thus a possible part induced by the meander gap is not taken into account). Let  $(p_{j,\tau}(\rho), j \geq 0)$  and  $(f_\tau^{(m)}, m \geq 0)$  be the distribution and factorial moments of  $\widehat{K}_\rho(\tau)$ .

**Lemma 3.** *For  $\tau$  an exponential time with rate  $\lambda$ , independent of the process  $(\widehat{K}_\rho(t), t \geq 0)$ , the factorial moments of  $\widehat{K}_\rho(\tau)$  satisfy the recursion*

$$\begin{aligned} \lambda f_\tau^{(m)}(\rho) + \int_0^1 (f_\tau^{(m)}(\rho) - f_\tau^{(m)}(\rho(1-x))) \tilde{\nu}(dx) \\ = m \int_0^1 \pi(\rho x) f_\tau^{(m-1)}(\rho(1-x)) \tilde{\nu}(dx), \end{aligned} \tag{11}$$

which taken together with  $f_\tau^{(0)}(\rho) = 1$  and  $f_\tau^{(m)}(\rho) = 1 (m = 0)$  uniquely determines them.

*Proof.* In the case of normalised Lévy measure, the first-jump decomposition of the range of  $(\tilde{S}_t)$  yields

$$\begin{aligned} p_{j,\tau}(\rho) &= \int_0^1 \left( (1 - \pi(\rho x)) p_{j,\tau}(\rho(1-x)) + \pi(\rho x) p_{j-1,\tau}(\rho(1-x)) \right) \frac{\tilde{\nu}(dx)}{\bar{\nu}(0) + \lambda}, \\ p_{0,\tau}(\rho) &= \frac{\lambda}{\bar{\nu}(0) + \lambda} + \int_0^1 (1 - \pi(\rho x)) p_{0,\tau}(\rho(1-x)) \frac{\tilde{\nu}(dx)}{\bar{\nu}(0) + \lambda}. \end{aligned}$$

To see the extension for an arbitrary Lévy measure, substitute

$$p_{j,\tau}(\rho) = p_{j,\tau}(\rho) \frac{\lambda}{\bar{\nu}(0) + \lambda} + \int_0^1 p_{j,\tau}(\rho) \frac{\tilde{\nu}(dx)}{\bar{\nu}(0) + \lambda},$$

then rearrange terms and argue as in the proof of Lemma 2. □

If the meander gap is counted (this corresponds to  $\widehat{K}_\rho$  for killed subordinator) we should modify the recursion for moments by adding  $\lambda\pi(\rho)1(m = 1)$  to the right-hand side.

*Remark.* The distribution and moments of  $\widehat{K}_\rho(t)$  may be obtained as the inverse Laplace transform in  $\lambda$  from the analogous quantities for  $\widehat{K}_\rho(\tau)$ .

#### 4. Moments of $\widehat{K}_\rho$

**Mellin transform resolution** The recursion (9) is intrinsically related to a convolution-type integral equation

$$\int_0^1 (f(\rho) - f(\rho(1 - x)))\tilde{v}(dx) = g(\rho), \tag{12}$$

with the function  $g$  given and the function  $f$  unknown. For  $g$  analytic, there is a unique analytic solution  $f$  with given initial value  $f(0)$ . Observe that constants  $d$  constitute the null space of the integral operator, thus for each solution  $f$  the function  $f + d$  is another solution, with the initial value  $f(0) + d$ .

It is easy to write out a power series solution to (12), but such a representation itself does not help describe the large- $\rho$  behaviour. We turn therefore to asymptotic methods based on the Mellin transform. Recall that for a locally integrable real-valued function  $\phi$  on  $[0, \infty]$  the Mellin transform is defined by the integral

$$M\phi(s) = \int_0^\infty \rho^{s-1}\phi(\rho) d\rho,$$

which is assumed to converge absolutely for  $s$  in some open interval on the real axis, hence also for  $s$  in the open vertical strip based on the interval. The left convergence abscissa of  $M\phi$  is determined by the behaviour of  $\phi$  near 0, while the right convergence abscissa is determined by the behaviour of  $\phi$  at  $\infty$ .

The analysis to follow is based on the formula

$$Mf(s) = \frac{Mg(s)}{\Phi(-s)}, \tag{13}$$

which is valid in the common domain of definition of all ingredients. The formula follows by Fubini and a change of variable from

$$\begin{aligned} \int_0^\infty \rho^{s-1}d\rho \int_0^1 (f(\rho) - f(\rho(1 - x)))\tilde{v}(dx) &= Mf(s) \int_0^1 (1 - (1 - x)^{-s})\tilde{v}(dx) \\ &= Mf(s)\Phi(-s) \end{aligned}$$

according to formula (1). Another important tool is the following correspondence between asymptotic expansion of a function and singularities of its Mellin transform. See [7] for a fuller exposition of this technique.



**Lemma 4.** [7, Section 2] *Suppose the Mellin transform  $Mf$  of a function  $f$  is analytic in a strip  $a < \Re s < b$ . If  $Mf$  can be extended meromorphically through the right convergence abscissa in a larger strip  $a < \Re s < b + \epsilon$  and has finitely many poles there, and if for some  $\delta > 0$*

$$|Mf(s)| = O(|s|^{-1-\delta}) \tag{14}$$

*as  $|s| \rightarrow \infty$  in a strip  $b' < \Re s < b + \epsilon$  for some  $b' < b$ , then each pole  $z$  and each term  $d(s - z)^{-k-1}$  in the Laurent expansion of  $Mf$  at  $z$  contribute the term*

$$\frac{d(-1)^{k+1}}{k!} \rho^{-z} \log^k \rho$$

*to the asymptotic expansion of  $f$  at  $\infty$ . The remainder term of the expansion of  $f$  is then  $O(\rho^{-b-\epsilon})$ .*

*Conversely, the asymptotic expansion of  $f$  at  $\infty$  with such a remainder implies the termwise singular expansion of  $Mf$  in the strip  $a < \Re s < b + \epsilon$ . (For this direction, the decay condition (14) is not required.)*

For the expectation  $f^{(1)}(\rho) := \mathbb{E} \widehat{K}_\rho$  the formulas (9) and (13) specialise as

$$Mf^{(1)}(s) = \frac{M\widehat{\Phi}(s)}{\Phi(-s)}, \tag{15}$$

where  $\widehat{\Phi}(s)$  is defined by (4). Using Fubini and integration by parts we transform the numerator as

$$\begin{aligned} M\widehat{\Phi}(s) &= \int_0^\infty \rho^{s-1} d\rho \int_0^1 \rho e^{-\rho x} \vec{v}(x) dx = \int_0^1 \vec{v}(x) dx \int_0^\infty \rho^s e^{-\rho x} d\rho \\ &= \Gamma(s + 1) \int_0^1 x^{-s-1} \vec{v}(x) dx = -\Gamma(s) \int_0^1 x^{-s} \vec{v}(dx). \end{aligned}$$

We can now re-write formula (15) as

$$Mf^{(1)}(s) = -\Gamma(s) \frac{\Phi(-s : -s)}{\Phi(-s)} = -\Gamma(s) \frac{\int_0^1 x^{-s-1} \vec{v}(x) dx}{\int_0^1 (1-x)^{-s-1} \vec{v}(x) dx}, \tag{16}$$

where we used (1) and (2) in the form

$$\Phi(-s : -s) = -s \int_0^1 x^{-s-1} \vec{v}(x) dx, \quad \Phi(-s) = -s \int_0^1 (1-x)^{-s-1} \vec{v}(x) dx.$$

**Expectation** Assuming (L) we conclude from (16) that  $Mf^{(1)}$  is defined in the strip  $-1 < \Re s < 0$ . We focus therefore on the meromorphic continuation of  $Mf^{(1)}$  through the right convergence abscissa  $\Re s = 0$ . The same formula says that the poles of  $Mf^{(1)}$  might be caused either by poles of  $\Gamma$  or by poles of  $\Phi(-s : -s)/s$ , or by zeros of  $\Phi(-s)/s$ . We shall consider the three ingredients separately.

For  $\Re s > -1$  the gamma function has a unique pole at 0, with Laurent expansion

$$\Gamma(s) = s^{-1} - \gamma + d_1 s + O(s^2),$$

where  $d_1 = \gamma^2/2 + \pi^2/12$ . With reference to Stirling’s approximation, when  $u$  is a bounded real number and  $v$  is a large real number

$$|\Gamma(u + iv)| \sim (2\pi)^{1/2} |v|^{u-1/2} e^{-\pi|v|/2} \tag{17}$$

uniformly in  $u$ .

By assumption (R), the function

$$-\Phi(-s)/s = \int_0^1 (1-x)^{-s-1} \vec{v}(x) dx$$

is analytic for  $\Re s < \epsilon$ , and its behaviour at complex infinity is regulated by the next lemma.

**Lemma 5.** *If  $\nu$  has a continuous density on  $]0, \infty[$  and satisfies (L) and (R) then*

$$|\Phi(s)| \sim \gamma \log |s|, \quad \text{as } |s| \rightarrow \infty$$

*uniformly in any strip  $-\epsilon < \Re s < d$ , for  $\epsilon$  sufficiently small.*

*Proof.* The measure  $y \nu(dy)$  is finite, with exponential decay at infinity, thus for  $s = u + iv$  the partial derivative

$$\frac{d}{du} \Phi(u + iv) = \int_0^\infty e^{-uy} e^{-ivy} y \nu(dy)$$

is bounded for  $u > -\epsilon$ , uniformly in  $v$ . Hence  $|\Phi(u + iv) - \Phi(iv)|$  is uniformly bounded both in  $v$  and in  $u \in [-\epsilon, d]$ . Thus it is sufficient to show the asymptotics for  $u = 0$ . But for the Fourier integral

$$\Phi(iv) = iv \int_0^\infty e^{-ivy} \nu[y, \infty] dy$$

the desired asymptotics follows from the expansion (L) and smoothness of  $\nu[\cdot, \infty]$  by application of [5, Section 3.1, Theorem 1.11] (with a further reference to [1]).  $\square$

By one further elementary lemma (see Appendix), condition (L) implies that  $\Phi(-s)/s$  has no zeroes for  $\Re s \leq 0$ , which implies that there are no zeroes in an open neighbourhood of the imaginary axis, and by Lemma 5 we can actually choose a strip  $-1 < \Re s \leq \epsilon$  without zeroes. The Taylor expansion at 0 involves the logarithmic moments (3)

$$\frac{-\Phi(-s)}{s} = m_1 + m_2 \frac{s}{2!} + m_3 \frac{s^2}{3!} + O(s^3). \tag{18}$$

The integral

$$-\Phi(-s : -s)/s = \int_0^1 x^{-s-1} \bar{v}(x) \, dx$$

converges for  $\Re s < 0$  and by assumption (L) this function is meromorphic for  $\Re s < \epsilon$  and in this half-plane has the unique pole at  $s = 0$ , where the Laurent series starts with

$$\frac{-\Phi(-s : -s)}{s} = s^{-2} - (c - \gamma)s^{-1} + d_2 + o(1),$$

where  $d_2$  is some constant. The meromorphic extension is obtained from this formula, provided we substitute a suitable analytic function for the  $o(1)$  term. This follows by computing

$$\int_0^1 (|\log x| + c - \gamma + h(x))x^{-s-1} \, dx$$

with  $h(x) = O(x^\epsilon)$  and  $\Re s < 0$ , and noting that the integral of  $h$  is analytic for  $\Re s < \epsilon/2$  and it is bounded in this domain due to a maximum ridge on the real axis.

It follows that  $Mf^{(1)}$  is meromorphic in the strip  $-1 < \Re s < \epsilon$ , with a unique singularity at 0, where it has a *triple* pole. Putting the ingredients together and with a minor assistance of `Mathematica` we obtain the Laurent expansion

$$Mf^{(1)}(s) = -\frac{1}{m_1} s^{-3} + \left( \frac{c}{m_1} + \frac{m_2}{2m_1^2} \right) s^{-2} + d_3 s^{-1} + O(1)$$

where

$$d_3 = \left( \frac{-d_1 - d_2 - c\gamma + \gamma^2}{m_1} - \frac{cm_2}{2m_1^2} - \frac{m_2^2}{4m_1^3} + \frac{m_3}{6m_1^2} \right)$$

is a constant whose explicit value will not be used below. By Lemma 4, as  $\rho \rightarrow \infty$ , the asymptotic expansion of the expectation is

$$f^{(1)}(\rho) = \frac{1}{2m_1} L^2 + \left( \frac{m_2}{2m_1^2} + \frac{c}{m_1} \right) L - d_3 + O(\rho^{-\epsilon}). \tag{19}$$

The condition (14) on  $Mf^{(1)}$  required in Lemma 4 is satisfied, because this  $Mf^{(1)}$  has exponential decay as  $|s| \rightarrow \infty$  in a strip about the imaginary axis due to (17), Lemma 5 and because  $|\Phi(-s : -s)| = O(|s|^{-1})$ .

**Evaluating the right-hand side of (9).** We will use (13) once again, to compute  $f^{(2)}$ . The right-hand side of (9) is evaluated with the help of the next lemma.

**Lemma 6.** Assume (L) and (R) and suppose  $f$  is an increasing positive function of  $\rho \in [0, \infty]$ , which admits an asymptotic expansion at  $\infty$  as a polynomial in  $L$ , with a remainder  $O(\rho^{-\epsilon})$ . Then a similar expansion holds also for the function

$$h(\rho) = \int_0^1 (1 - e^{-\rho x}) f(\rho(1 - x)) \tilde{v}(dx)$$

and this expansion is obtained from that of  $f$  by replacing each  $L^k$  with

$$L^{k+1} + cL^k + \sum_{j=1}^k (-1)^j \binom{k}{j} m_j L^{k-j},$$

where  $c$  is as in (L).

*Proof.* By the assumption

$$f(\rho) = \sum_{j=0}^k d_j L^j + O(\rho^{-\epsilon}). \tag{20}$$

Start with the case  $f = L^k$ . The integral is computed by the binomial expansion of  $(L + \log(1 - x))^k$  and termwise integration. The leading term does not depend on  $x$  and, in view of (L), it contributes  $L^k(L + c) + O(\rho^{-\epsilon})$ . For the lower order terms with  $1 \leq j < k$  we have using (3)

$$\begin{aligned} & \int_0^1 (1 - e^{-\rho x}) (\log \rho)^{k-j} |\log(1 - x)|^j \tilde{v}(dx) \\ &= L^{k-j} m_j - L^{k-j} \int_0^1 e^{-\rho x} |\log(1 - x)|^j \tilde{v}(dx). \end{aligned}$$

Integrating by parts, the last integral becomes

$$\begin{aligned} & \int_0^{1-\delta} e^{-\rho x} j |\log(1 - x)|^{j-1} \tilde{v}[x, 1] \frac{dx}{(1 - x)} \\ & - \int_0^{1-\delta} \rho e^{-\rho x} |\log(1 - x)|^j \tilde{v}[x, 1] dx + O(e^{-\rho}) \end{aligned}$$

and is estimated as  $O(\rho^{-1+\epsilon})$ , by application of (L) and the formula (for  $\alpha > -1$ )

$$\int_0^{1-\delta} e^{-\rho x} x^\alpha \log^j x dx \sim \Gamma(1 + \alpha) \rho^{-\alpha-1} \log^j \rho \quad (\text{as } \rho \rightarrow \infty),$$

which is an instance of the Tauberian theorem for the Laplace transform, as found in [6].

More generally, using linearity we obtain asymptotic expansion for the *model integral*  $I(\rho)$  which corresponds to the polynomial part of  $f$ , as in (20), but with zero remainder term.

For a general  $f$  as in (20) we split the integral at  $1 - \delta$ , with  $\delta = \rho^{-1/2}$ . One checks easily using (R) that the contribution to  $I(\rho)$  of the integral over  $[1 - \delta, 1]$  is

estimated as  $O(\rho^{-\epsilon})$ . For  $x \in [0, 1 - \delta]$  we can apply expansion (20) to  $f(\rho(1 - x))$  with a remainder  $O(\rho^{-\epsilon})$ , then exploit (L) and the asymptotics of the model integral to obtain

$$\int_0^{1-\delta} (1 - e^{-\rho x})f(\rho(1 - x))\tilde{v}(dx) = I(\rho) + O(\rho^{-\epsilon}).$$

It remains to show that the integral over  $[1 - \delta, 1]$  is  $O(\rho^{-\epsilon})$  but this is easy because the assumed monotonicity yields the bound

$$\int_{1-\delta}^1 (1 - e^{-\rho x})f(\rho(1 - x))\tilde{v}(dx) < f(\rho \delta) \bar{v}(1 - \delta)$$

and this has the desired order of magnitude by (20), assumption (R) and our choice of  $\delta$ . □

Taking  $f^{(1)}$  for  $f$ , the lemma translates the expansion (19) into the formula for the right-hand side of (9) with  $m = 2$

$$g_1(\rho) = \frac{1}{m_1}L^3 + \left(\frac{3c}{m_1} + \frac{m_2}{m_1^2}\right)L^2 + d_4L + d_5 + O(\rho^{-\epsilon}) \quad (\text{for } \rho \rightarrow \infty), \tag{21}$$

where  $d_4, d_5$  are some constants, and the factor 2 in (9) is included in the definition of  $g_1$ .

**Moments of the second order** Using the direct correspondence in Lemma 4, we see that 0 is the sole singularity of  $Mg_1$ , and derive the expansion at  $s = 0$

$$Mg_1(s) = \frac{6}{m_1}s^{-4} - \left(\frac{6c}{m_1} + \frac{2m_2}{m_1^2}\right)s^{-3} + O(s^{-2}),$$

which together with (13) and (18) implies

$$Mf^{(2)}(s) = \frac{Mg_1(s)}{\Phi(-s)} = -\frac{6}{m_1^2}s^{-5} + \frac{6cm_1 + 5m_2}{m_1^3}s^{-4} + O(s^{-3}).$$

Finally, the inverse correspondence applied to  $f^{(2)}$  yields

$$f^{(2)}(\rho) = \frac{1}{4m_1^2}L^4 + \left(\frac{c}{m_1^2} + \frac{5m_2}{6m_1^3}\right)L^3 + dL^2 + O(L) \tag{22}$$

provided the bound  $|Mf^{(2)}(s)| = O(|s|^{-1-\epsilon})$  can be established. This requires some effort because for  $Mg_1$  no explicit formula is available. Postponing justification of the decay condition, and recalling

$$\mathbb{E} \widehat{K}_\rho = f^{(1)}(\rho), \quad \text{var } \widehat{K}_\rho = f^{(2)}(\rho) + f^{(1)}(\rho) - (f^{(1)}(\rho))^2$$

we have from (19) and (22)

**Theorem 7.** *If the Lévy measure of subordinator has a continuous density on  $]0, \infty[$  and satisfies the assumptions (L) and (R), then as  $\rho \rightarrow \infty$  with  $L = \log \rho$ , the asymptotic expansions of the first two central moments of  $\widehat{K}_\rho$  are*

$$\mathbb{E} \widehat{K}_\rho = \frac{1}{2 m_1} L^2 + O(L), \tag{23}$$

$$\text{var} \widehat{K}_\rho = \frac{m_2}{3 m_1^3} L^3 + O(L^2). \tag{24}$$

**Justification of the decay condition** Our plan is to decompose  $g_1 = h_1 + h_2$  so that  $h_1$  will absorb the logarithmic part of  $g_1$  at  $\infty$  and will have a manageable Mellin transform, while  $Mh_2$  will satisfy the decay condition by virtue of the following classical result.

**Lemma 8.** [19, Section 1.29] *Let  $\phi$  be analytic in a sector  $-\alpha < \text{Arg } \rho < \beta$  with  $0 < \alpha, \beta \leq \pi$ , and suppose that in this sector for some  $a < b$  and some  $\delta > 0$*

$$\phi(\rho) = O(|\rho|^{-a-\delta}) \quad \text{as } |\rho| \downarrow 0, \quad \phi(\rho) = O(|\rho|^{-b+\delta}) \quad \text{as } |\rho| \uparrow \infty.$$

*Then  $M\phi$  is analytic in the strip  $a < \Re s < b$  and satisfies*

$$\begin{aligned} M\phi(s) &= O\left(e^{-(\beta-\epsilon)\Im s}\right) \quad \text{as } \Im s \rightarrow \infty, \\ M\phi(s) &= O\left(e^{(\alpha-\epsilon)\Im s}\right) \quad \text{as } \Im s \rightarrow -\infty, \end{aligned}$$

*for each  $\epsilon > 0$  uniformly in every strip strictly inside  $a < \Re s < b$ .*

By definition

$$g_1(\rho) = \int_0^1 f^{(1)}(\rho(1-x))(1 - e^{-\rho x})\tilde{\nu}(dx),$$

which is an entire function, where

$$f^{(1)}(\rho) = e^{-\rho} \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \mathbb{E} K_n$$

and the series in  $\rho$  has all coefficients positive. The same applies to the series involved in

$$1 - e^{-\rho} = e^{-\rho} \sum_{j=1}^{\infty} \frac{\rho^j}{j!}.$$

Therefore  $|g_1(\rho)| \leq g_1(|\rho|)$  for  $\Re \rho \geq 0$  and by (21)

$$|g_1(\rho)| = O(L^3) \quad |\rho| \rightarrow \infty.$$

Writing (21) as

$$g_1(\rho) = \sum_{j=0}^3 d_j L^j + O(\rho^{-\epsilon}), \quad \rho \rightarrow \infty, \rho \in \mathbb{R}$$

and selecting

$$h_1(\rho) := d_0 e^{-1/\rho} + \log(\rho + 1) \sum_{j=1}^3 d_j \log^{j-1} \rho$$

we obtain a function which is analytic in the open halfplane  $\Re \rho > 0$  and has the same expansion in powers of  $L$  as  $g_1$  for  $\rho \rightarrow \infty$ . Hence the function  $h_2 := g_1 - h_1$  satisfies Lemma 8 with  $a = -1$ ,  $b = \epsilon$  and  $\alpha = \beta = \pi/2$ , thus

$$|Mh_2(s)| = O\left(e^{-(\pi/2-\delta)|s|}\right), \quad \text{for } -1/2 < \Re s < \epsilon, |s| \rightarrow \infty$$

for some  $\delta$ . The meromorphic continuation of  $Mh_1$  from  $-1 < \Re s < 0$  to the halfplane  $\Re s > -1$  follows from the standard Mellin transform formulas

$$\begin{aligned} M e^{-1/\cdot}(s) &= \Gamma(-s), \quad M \log(1 + \cdot)(s) = -\Gamma(s)\Gamma(-s), \quad M(\phi(\cdot) \log(\cdot))(s) \\ &= \frac{d}{ds} M\phi(s), \end{aligned}$$

which by application of (17) also imply

$$|Mh_1(s)| = O\left(e^{-(\pi/2-\delta)|s|}\right), \quad \text{for } -1/2 < \Re s < \epsilon, |s| \rightarrow \infty.$$

It follows that in a strip containing the imaginary axis  $|Mg_1(s)| = O\left(e^{-(\pi/2-\delta)|s|}\right)$ , and the same estimate holds for  $|Mf^{(2)}(s)| = |Mg_1(s)/\Phi(-s)|$  due to Lemma 5. Thus  $|Mf^{(2)}|$  decays exponentially fast and Lemma 4 is applicable. **Number of parts prior to the exponential split** Let  $\tau$  be an independent exponential time, with rate  $\lambda$ . By (11), the Mellin transform in  $\rho$  satisfies

$$M(\mathbb{E}\widehat{K}_\rho(\tau))(s) = \frac{M\widehat{\Phi}(s)}{\lambda + \Phi(-s)}.$$

The term  $\lambda$  in the denominator kills the zero of  $\Phi$  at  $s = 0$ , therefore arguing as in the analysis of (16) we see that this expression has only *double* pole at  $s = 0$ . It follows that

$$\mathbb{E}\widehat{K}_\rho(\tau) = dL + O(1)$$

for each  $\lambda > 0$ , with  $d$  depending on  $\lambda$ . Repeatedly appealing to (11) and Lemmas 5, 6 and 8 we obtain  $f^{(m)}(\rho) = O(L^m)$ , which trivially implies the following rough estimate

**Lemma 9.** *For  $\tau$  an independent exponential time, as  $\rho \rightarrow \infty$  with  $L = \log \rho$ ,*

$$\mathbb{E}(\widehat{K}_\rho(\tau))^m = O(L^m), \quad m = 1, 2, \dots$$

**Depoissonisation** In an obvious way, we can think of  $\widehat{K}_n$  and  $K_n$  as defined by means of the same uniform sample  $u_1, u_2, \dots$  for all integer  $n$ . Thus defined,  $\widehat{K}_n$  and  $K_n$  increase with  $n$ . A large deviation bound yields for  $\epsilon = n^{-1/2+\delta}$

$$\mathbb{P}(\widehat{K}_{n(1-\epsilon)} < K_n < \widehat{K}_{n(1+\epsilon)}) > 1 - e^{-n^\delta},$$

which implies  $\widehat{K}_n \sim K_n$  almost surely and with all moments. To obtain a finer result observe that the factorial moments satisfy

$$\begin{aligned} & f^{(m)}(n(1-\epsilon)) + O(e^{-n^\delta}) \\ & < \mathbb{E}K_n(K_n - 1) \cdots (K_n - m + 1) < f^{(m)}(n(1+\epsilon)) + O(e^{-n^\delta}), \end{aligned}$$

substitute  $n(1+\epsilon)$ , respectively  $n(1-\epsilon)$ , for  $\rho$  in (19) and (22) and estimate the gap between the left side and the right side by  $O(n^{-\epsilon'})$  with some fixed  $\epsilon'$ . This shows that the same asymptotic expansions are valid for the two factorial moments of  $K_n$ , thus the assertion of Theorem 7 is also valid with  $\widehat{K}_\rho$  replaced by  $K_n$ . The connection between  $\widehat{K}_n(t)$  and  $K_n(t)$  follows by the same method combined with conditioning on the value of  $S_t$ , while further averaging over  $t$  yields a fixed- $n$  counterpart of Lemma 9.

*Remark.* For  $\tau$  exponential with parameter  $\lambda$  we have

$$\mathbb{E}(\widehat{K}_\rho(\tau))^m = \lambda \int_0^\infty e^{-\lambda t} \mathbb{E}(\widehat{K}_\rho(t))^m dt,$$

and inverting the Laplace transform we see that Lemma 9 also holds with  $\tau$  replaced by arbitrary fixed  $t$ . To compare,

$$\mathbb{E}(\widehat{K}_\rho)^m \sim \frac{L^{2m}}{2^m m_1^m},$$

as can be derived for  $m > 2$  by further iterating (9) and Lemma 6. This makes precise the statement in Introduction that the contribution of each fixed finite time interval to  $\widehat{K}_\rho$  (or to  $K_n$ ) is asymptotically negligible.

### 5. Central limit theorem for $K_n$

We switch to fixed- $n$  framework, in order to apply a delicate recent result due to Neininger and Rüschemdorf, which we reproduce below with a minor adaptation and notational changes. Suppose a sequence  $(Y_n)$  of random variables satisfies

$$Y_n \stackrel{d}{=} b_n + Y'_{I_n}, \quad n \geq 1, \tag{25}$$

where  $(Y_n) \stackrel{d}{=} (Y'_n)$ , the pair  $(b_n, I_n)$  is independent of  $(Y'_n)$ , each  $I_n$  assumes values in  $\{0, \dots, n\}$ , and  $\mathbb{P}(I_n = n) < 1$  for  $n \geq 1$ . Let  $\mu_n = \mathbb{E} Y_n$ ,  $\sigma_n^2 = \text{var } Y_n$ , and  $\|\cdot\|_3$  denote the  $\mathcal{L}^3$ -norm of a random variable.



**Theorem 10.** [15, Theorem 2.1] *Assume that each  $\| Y_n \|_3 < \infty$  and that  $(Y_n)$  satisfies the recursion (25). Suppose that for some constants  $C > 0, \alpha > 0$  the following three conditions all hold:*

(i)

$$\limsup_{n \rightarrow \infty} \mathbb{E} \log \left( \frac{I_n \vee 1}{n} \right) < 0, \quad \sup_{n > 1} \left\| \log \left( \frac{I_n \vee 1}{n} \right) \right\|_3 < \infty,$$

(ii) *for some  $\lambda \in [0, 2\alpha[$  and some  $\kappa$*

$$\| b_n - \mu_n + \mu_{I_n} \|_3 = O(\log^\kappa n), \quad \sigma_n^2 = C \log^{2\alpha} n + O(\log^\lambda n),$$

(iii)

$$\alpha > \frac{1}{3} + \max \left( \kappa, \frac{\lambda}{2} \right).$$

*Then the law of*

$$\frac{Y_n - \mu_n}{\sqrt{C} \log^\alpha n}$$

*converges weakly to the standard normal distribution.*

Condition (i) says that the split  $I_n/n$  must be bounded away from 0 and 1, and the further two conditions require logarithmic growth of moments. See [15] for proof and estimates of the distance between the probability laws.

To apply the result to  $K_n$  we split the range of subordinator at an independent exponential time  $\tau$  with mean 1. Let  $b_n = K_n(\tau)$  be the number of parts of  $\mathcal{C}_n$  produced by the multiplicative subordinator in the period between 0 to  $\tau$ . Observe that by the regeneration property (25) holds for  $Y_n = K_n$ , with  $I_n$  being the number of uniform sample points larger than  $\tilde{S}_\tau$ . Keep in mind that, by virtue of depositions, the asymptotics of moments in Theorem 7 and Lemma 9 apply to  $K_n$  and  $b_n$  literally without change.

Let us check now the conditions of the theorem. Since  $1 \leq K_n \leq n$ , all absolute moments of  $K_n$  are finite. Furthermore, by the law of large numbers for coin-tossing processes the variable  $I_n/n$  converges almost surely and with all moments to  $1 - \tilde{S}_\tau$ , thus  $-\log(I_n/n)$  approaches the stopped value  $S_\tau = -\log(1 - \tilde{S}_\tau)$  of the additive subordinator, which is positive and has  $\mathbb{E}S_\tau^3 < \infty$  as a consequence of  $m_3 < \infty$ , by an easy application of the Lévy-Khintchine formula. Since logarithm is an unbounded function, condition (i) is not a straightforward consequence of the law of large numbers and needs a special justification.

**Lemma 11.** *Let  $X$  and  $I_n$  be random variables such that  $X$  assumes values in  $[0, 1]$ , and given  $X$  the distribution of  $I_n$  is Binomial( $n, 1 - X$ ). Then for each  $k = 1, 2, \dots$ , as  $n \rightarrow \infty$  we have*

$$\mathbb{E} \left| \log \left( \frac{I_n \vee 1}{n} \right) \right|^k \rightarrow \mathbb{E} |\log(1 - X)|^k, \tag{26}$$

*where the right side may be finite or infinite.*

*Proof.* Introducing the Bernstein polynomial

$$B_n(x) := \sum_{m=1}^{n-1} |\log(1 - m/n)|^k \binom{n}{m} x^m (1 - x)^{n-m},$$

the left side of (26) differs from  $\mathbb{E} [ B_n(X) ]$  by the term  $(\log n)^k \mathbb{E} X^n$ , which goes to 0 as  $n \rightarrow \infty$  provided  $\mathbb{E} |\log(1 - X)|^k < \infty$ . To estimate  $B_n(x)$  it is useful to consider

$$h_n^{(j)} := k! \sum_{1 \leq j_1 < \dots < j_k \leq n} \frac{1}{j_1 \cdots j_k}.$$

In this direction, we approximate  $|\log(1 - m/n)|^k$  by  $(h_n^{(1)} - h_{n-m}^{(1)})^k$ , then expand this by the binomial formula and approximate each term  $(h_i^{(1)})^j$  by  $h_i^{(j)}$ . The resulting expression simplifies by application of the summation formula

$$\sum_{m=1}^{n-1} h_{n-m}^{(k)} \binom{n}{m} x^m (1 - x)^{n-m} = \sum_{i=0}^k \binom{k}{i} h_n^{(k-i)} P_n^{(i)}(x),$$

(which extends [9, Equations (8) and (18)] to  $k > 2$ ), where  $P_n^{(i)}$  is the Taylor polynomial of degree  $n$  for  $|\log(1 - x)|^i$ , e.g.

$$P_n^{(1)} = \sum_{j=1}^n j^{-1} x^j, \quad P_n^{(2)} = \sum_{j=1}^n j^{-1} h_{j-1}^{(1)} x^j.$$

This readily leads to a rough bound

$$B_n(x) < 2|\log(1 - x)|^k + c_k \tag{27}$$

with a suitable constant  $c_k$ . Now, by the usual convergence properties of Bernstein polynomials we have  $B_n(x) \rightarrow |\log(1 - x)|^k$ , as  $n \rightarrow \infty$ , for each  $x \in [0, 1[$ . By dominated convergence, this taken together with (27) implies (26), provided the right side of (26) is finite. The case of infinite logarithmic moments is treated in the same way. □

Now (i) follows from the lemma applied to  $X = \tilde{S}_\tau$ . The required asymptotics of moments follow from their poissonised counterparts with  $\rho = n, L = \log n$ . Thus, from Theorem 7 we see that  $\|\mu_n - \mu_{I_n}\|_3$  is of the order of  $L$ , and by Lemma 9 also  $\|b_n\|_3 = O(L)$ , because  $\mathbb{E} b_n^3 = O(L^3)$ . By the above and Theorem 7 the parameters involved in (ii) are  $\alpha = 3/2, \lambda = 2$  and  $\kappa = 1$ , so condition (iii) is satisfied. Thus all conditions of Theorem 10 are fulfilled and we deduce the following conclusion:

**Theorem 12.** *If the Lévy measure has a continuous density on  $]0, \infty[$  and satisfies (L) and (R) then the distribution of the random variable*

$$\frac{K_n - L^2/(2m_1)}{\sqrt{m_2/(3m_1^3)} L^{3/2}}, \quad L = \log n$$

*converges weakly to the standard normal distribution, as  $n \rightarrow \infty$ .*

From  $\widehat{K}_n \sim K_n$  (almost surely) and the theorem follows that  $\widehat{K}_\rho$  is asymptotically gaussian as well.

**6. Small parts of the composition**

We have shown in [13] that in the regular variation case all  $K_{n,r}$ ,  $r = 1, 2, \dots$  are of the same order of growth as  $K_n$  and, suitably normalised, converge almost surely to multiples of the same random variable. In the compound Poisson case these variables are bounded as  $n$  grows [9], and if the increments of  $(S_r)$  are exponentially distributed (this corresponds to the Ewens partition structure [3]) the  $K_{n,r}$ 's converge to independent Poisson random variables [3]. Our next goal is proving a joint central limit theorem for the small-part counts.

**Marginal central limit theorems** The line of argument repeats that for  $K_n$ . Let  $f_r^{(m)}(\rho)$  be the  $m$ th factorial moment of  $\widehat{K}_{\rho,r}$ , e.g.  $f_1^{(1)}(\rho)$  is the mean number of singletons. Recall that the recursion (9) holds for  $f^{(m)} = f_r^{(m)}$  with  $\pi(\rho) = e^{-\rho} \rho^r / r!$ . Applying the Mellin transform to the recursion we obtain

$$Mf_r^{(1)}(s) = \frac{\Gamma(r + s)}{r!} \frac{\Phi(-s : -s)}{\Phi(-s)}, \quad -1 < \Re s < 0. \tag{28}$$

This agrees with (16) and  $K_\rho = \sum_r K_{\rho,r}$  in view of the identity

$$-\Gamma(s) = \sum_{r=1}^{\infty} \frac{\Gamma(r + s)}{\Gamma(r + 1)}, \quad -1 < \Re s < 0.$$

The right-hand side of (28) can be extended meromorphically through the imaginary axis, with a *double* pole at 0, where we have the Laurent expansion

$$Mf_r^{(1)}(s) = \frac{1}{m_1 r} s^{-2} - d_1 s^{-1} + O(1), \tag{29}$$

where

$$d_1 = \frac{2cm_1 - 2\gamma m_1 + m_2 - 2m_1 \psi^{(0)}(r)}{2m_1^2 r}$$

and  $\psi^{(0)}$  is the digamma function (the definition is recalled below in (33)). The decay at complex infinity is justified as before, thus by Lemma 4

$$f_r^{(1)}(\rho) = \frac{L}{m_1 r} + d_1 + O(\rho^{-\epsilon}). \tag{30}$$

Now we need to translate (30) into asymptotics of the right-hand side of (9) with  $m = 2$ . We note that evaluation of integrals with a factor of  $e^{-\rho x} x^r$  essentially amounts to Tauberian-type asymptotics, since for  $r > 1$  the factor  $x^r$  compensates the singularity of  $\widehat{v}$  at  $x = 0$ , and  $e^{-\rho x}$  is negligible outside each fixed vicinity of 0. In particular, we have

**Lemma 13.** *If the condition (L) holds then for  $\rho \rightarrow \infty$*

$$\int_0^1 \frac{e^{-\rho x} (\rho x)^r}{r!} \tilde{v}(dx) = \frac{1}{r} + O(\rho^{-\epsilon}).$$

*Proof.* Integrating by parts and replacing  $\vec{v}$  by the right-hand side of (L), and pushing the upper integration limit to  $\infty$  the claim is reduced to the standard integral

$$\int_0^\infty (-\log x + d) \left( \frac{e^{-\rho x} \rho^r x^{r-1}}{(r-1)!} - \frac{e^{-\rho x} \rho^{r+1} x^r}{r!} \right) dx = \frac{1}{r},$$

which does not depend on  $d$ , because

$$\int_0^\infty \left( \frac{e^{-\rho x} \rho^r x^{r-1}}{(r-1)!} - \frac{e^{-\rho x} \rho^{r+1} x^r}{r!} \right) dx = 0,$$

as is easily checked. □

Using the lemma and (30) we compute the right-hand side of (9) as

$$g(\rho) = \frac{2L}{m_1 r^2} + \frac{2d_1}{r} + O(1),$$

hence by Lemma 4

$$Mg(s) = \frac{2}{m_1 r^2} s^{-2} - \frac{2d_1}{r} s^{-1} + O(1).$$

We compute then

$$Mf_r^{(2)}(s) = \frac{Mg(s)}{\Phi(-s)} = -\frac{2}{m_1^2 r^2} s^{-3} + \left( \frac{m_2}{r^2 m_1^3} + \frac{2d_1}{r m_1} \right) s^{-2} + O(s^{-1}),$$

which yields again by Lemma 4

$$f_r^{(2)}(\rho) = \frac{1}{r^2 m_1^2} L^2 + \left( \frac{m_2}{r^2 m_1^3} + \frac{2d_1}{r m_1} \right) L + O(1)$$

(the decay condition is again checked by application of Lemma 5). This together with (30) implies

$$\text{var } K_{r,\rho} = \left( \frac{m_2}{r^2 m_1^3} + \frac{1}{r m_1} \right) L + O(1).$$

Decomposing as in (10) at rate 1 exponential time, the Mellin transform of the expectation of  $b_\rho = \widehat{K}_{\rho,r}(\tau)$  has  $\Phi(-s) + 1$  in the denominator, hence all moments of  $b_\rho$  remain bounded as  $\rho \rightarrow \infty$ . Passing to the fixed- $n$  version and applying Theorem 10 with  $\alpha = 1/2, \lambda = \kappa = 0$  we obtain

**Theorem 14.** *Under our assumptions on the Lévy measure, for large  $n$  the distribution of each  $K_{n,r}$  is approximately normal, with moments*

$$\mathbb{E}K_{n,r} = \frac{1}{m_1 r} \log n + O(1), \quad \text{var}K_{n,r} = \left( \frac{m_2}{r^2 m_1^3} + \frac{1}{r m_1} \right) \log n + O(1).$$

The same is true for  $\widehat{K}_{\rho,r}$  as  $\rho \rightarrow \infty$ .

**Joint central limit theorem** Our strategy to prove a joint central limit theorem for the small parts counts is to consider more general finite patterns and functionals of composition such as  $\sum_r a_r \widehat{K}_{\rho,r}$ . By linearity, each variable of this kind decomposes as in (10), hence the method we applied to  $\widehat{K}_{\rho,r}$  can be used to justify the normal limit. As an instance of such a functional consider

$$\widehat{K}_{\rho,[r]} := \widehat{K}_{\rho,1} + \dots + \widehat{K}_{\rho,r}$$

and define  $f_{[r]}^{(m)}(\rho)$  to be the  $m$ th factorial moment of  $\widehat{K}_{\rho,[r]}$ . The corresponding pattern is  $\{1, \dots, r\}$ , thus the moments satisfy (9) with  $\pi(\rho) = \sum_{j=1}^r e^{-\rho} \rho^j / j!$ . Obviously from (30)

$$f_{[r]}^{(1)} = \frac{h_r}{m_1} L + O(1),$$

where  $h_r := \sum_{j=1}^r 1/j$  are the harmonic numbers. Letting Mellin’s machine roll to produce  $f_{[r]}^{(1)} \rightarrow g \rightarrow Mg \rightarrow Mf_{[r]}^{(2)} \rightarrow f_{[r]}^{(2)}$  the variance is computed as

$$\text{var} K_{\rho,[r]} = \left( \frac{m_2 h_r^2}{m_1^3} + \frac{h_r}{m_1} \right) L + O(1).$$

Similar computation for the pattern  $E = \{i, j\}$  and Theorem 14 yield the covariance

$$\text{cov}(\widehat{K}_{\rho,i}, \widehat{K}_{\rho,j}) = \left( \frac{m_2}{m_1^3} \frac{1}{i j} + 1(i = j) \frac{1}{j m_1} \right) L + O(1).$$

**Theorem 15.** *Under our assumptions on  $v$ , as  $n \rightarrow \infty$ , the infinite random sequence*

$$((K_{n,r} - \mathbb{E}K_{n,r}) \log^{-1/2} n, \quad r = 1, 2, \dots, n)$$

converges in law to a multivariate gaussian sequence with the covariance matrix

$$\left( \frac{m_2}{m_1^3} \frac{1}{i j} + 1(i = j) \frac{1}{j m_1} \right)_{i,j=1}^{\infty}. \tag{31}$$

*Proof.* To justify the joint gaussian law it suffices to establish convergence to a gaussian limit for each finite linear combination

$$Q_n = \sum_r a_r K_{n,r}.$$

From the above facts about  $K_{n,r}$  and from (31) it follows that both the expectation and the variance of  $Q_n$  grow like  $\log n$ . Also, due to an obvious additivity,  $Q_n$  satisfies a distributional equation of the type (25) provided we split the range of subordinator by the value at an independent exponential time  $\tau$ . Then  $b_n$  is equal to the contribution to  $Q_n$  by the jumps of subordinator before  $\tau$ . The conditions of Theorem 10 are checked exactly as for  $K_n$  or  $K_{n,r}$ , thus by this theorem  $Q_n$  is approximately gaussian. □

### 7. The gamma case and further examples

In the case of the gamma subordinators we have

$$c = -\log \theta, \quad m_j = \frac{(j-1)!}{\theta^j},$$

and the singular expansion

$$\Phi(-s : -s) = -s^{-1} + (c - \gamma) - d_2 s + O(s^2)$$

with constant

$$d_2 = \int_0^1 \log x \frac{x(1-x)^{\theta-1} + \log(1-x)}{-x \log(1-x)} dx,$$

which does not seem to simplify. The Mellin transform of  $\mathbb{E}\widehat{K}_\rho$  is given by the formula

$$Mf^{(1)}(s) = \frac{-\Gamma(s)}{\log(1-s/\theta)} \int_0^1 \frac{x^{-s}(1-x)^{\theta-1}}{-\log(1-x)} dx,$$

which defines a function which is meromorphic for  $\Re s > -1$  with a sole singularity at  $s = 0$ . The Laurent expansion of  $Mf^{(1)}$  at  $s = 0$  involves

$$m_j = \frac{(j-1)!}{\theta^j}, \quad c = -\log \theta, \quad d_1 = \frac{\gamma^2}{2} + \frac{\pi^2}{12}.$$

Leaving only principal terms, the asymptotics of moments is

$$\mathbb{E}\widehat{K}_\rho = \frac{\theta}{2}L^2 + O(L), \quad \text{var}\widehat{K}_\rho = \frac{\theta}{3}L^3 + O(L^2)$$

as was promised in Theorem 1.

**Further examples** Another instance of a gamma-type subordinator was introduced in [10] to describe a composition resembling the ordered Ewens sampling formula. This subordinator has (multiplicative) Lévy measure

$$\tilde{\nu}(dx) = (1-x)^{\theta-1}x^{-1}dx, \quad x \in ]0, 1]$$

with parameter  $\theta > 0$ . The Laplace exponent given by the formula

$$\Phi(s) = \sum_{j=1}^{\infty} \left( \frac{1}{j + \theta - 1} - \frac{1}{j + \theta - 1 + s} \right)$$

is a function interpolating the generalised harmonic numbers

$$\Phi(n) = \sum_{j=1}^n \frac{1}{j + \theta - 1}, \tag{32}$$

which can be seen as a combinatorial analogue of the logarithmical Laplace exponent for the gamma subordinator.

The basic characteristics of subordinator are readily expressed in terms of the polygamma function

$$\psi^{(k)}(\theta) = \frac{d^{k+1} \log \Gamma(\theta)}{d\theta^{k+1}}. \tag{33}$$

Thus we have

$$\bar{v}(x) = -\log x - \psi^{(0)}(\theta) + O(x^{-1}),$$

as is shown by expanding

$$\bar{v}(x) = \int_x^1 (1 - z)^{\theta-1} z^{-1} dz = -\log x + c(\theta) - \gamma + c_1(\theta)x + \dots,$$

substituting  $\theta = 1$  to see that  $c(1) = \gamma = -\psi^{(0)}(1)$ , then differentiating in  $\theta$  and sending  $x \rightarrow 0$  to obtain

$$c'(\theta) = \int_0^1 (1 - z)^{\theta-1} z^{-1} \log(1 - z) dz = -\psi^{(1)}(\theta), \tag{34}$$

whence  $c(\theta) = -\psi^{(0)}(\theta)$ . It follows as in the Appendix or directly from (32) that the expansion at infinity is

$$\Phi(\rho) = L - \psi^{(0)}(\theta) + O(\rho^{-1}).$$

The moments are computed by further differentiating (34) as

$$m_j = \int_0^1 (1 - x)^{\theta-1} x^{-1} |\log(1 - x)|^j dx = (-1)^{j+1} \psi^{(j)}(\theta).$$

See [4] for computation of some densities related to this family of subordinators, and [18] for further examples of Lévy measures with logarithmic singularity.

**Oscillatory asymptotics for a discrete measure** The constant term  $c$  assumed in (L) cancels in the asymptotics of the moments. However, this assumption is essential by our approach. Let us assume a bounded oscillating term in place of  $c$ ,

and examine how the asymptotics could be affected. For ease of computation we consider the atomic measure

$$\tilde{\nu}(dx) = \sum_{j=1}^{\infty} \delta_{1/e^j}(dx).$$

For this measure  $\Phi(n)$  is equal to the expected maximum in a sample of  $n$  geometric random variables, which was analysed in [7, Example 12].

Denoting  $\lfloor \cdot \rfloor$ , respectively  $\{ \cdot \}$ , the integer and the fractional parts of a positive number, we have

$$\vec{\nu}(x) = \lfloor -\log x \rfloor = -\log x - \{ -\log x \}, \quad x \in ]0, 1],$$

thus there is a logarithmic singularity but the expansion (L) does not hold, because the second term oscillates between 0 and 1. Condition (R) is satisfied since  $\vec{\nu}(x) = 0$  for  $x > 1/e$  and all moments are finite,

$$m_k = \sum_{j=1}^{\infty} |\log(1 - e^{-j})|^k,$$

e.g.  $m_1 = 0.6843, m_2 = 0.2345$  (truncated at four decimals). Furthermore,

$$\widehat{\Phi}(\rho) = \sum_{j=1}^{\infty} (1 - e^{-\rho/e^j}), \quad M\widehat{\Phi}(s) = \frac{-\Gamma(s)}{1 - e^s}, \quad \Phi(s) = \sum_{j=1}^{\infty} (1 - (1 - e^{-j})^s).$$

It is seen that  $M\widehat{\Phi}$  has a *double pole* at  $s = 0$  with Laurent expansion

$$M\widehat{\Phi}(s) = \frac{1}{s^2} - \frac{\gamma + 1/2}{s} + \frac{1}{2} (1 + 6\gamma + 6\gamma^2 + \pi^2) + O(s)$$

and infinitely many *simple poles* on the imaginary axis at

$$s_k = 2\pi ik, \quad k \in \mathbb{Z} \setminus \{0\}.$$

The conclusion of Lemma 4 still holds, which can be justified by applying the inverse Mellin transform formula and integrating over increasing rectangular contours, as  $k \rightarrow \infty$ , with sides  $\Im s = 2\pi k + 1/2, \Re s = -2\pi k - 1/2, \Im s = 1/2, \Re s = -1/2$ . Thus

$$\widehat{\Phi}(\rho) = L + \gamma + 1/2 + \phi(L) + O(\rho^{-1+\epsilon}), \quad \rho \rightarrow \infty, \tag{35}$$

where the contribution of the imaginary poles amounts to the term  $\phi(L)$  given by the formula

$$\phi(u) = - \sum_{k \in \mathbb{Z} \setminus \{0\}} \Gamma(2\pi ik) e^{2\pi iku},$$



which is a periodic function with period one and a tiny amplitude. The fluctuations of the  $O(1)$  term in (35) are not asymptotically negligible, though they are very small due to the fast decay of the Fourier coefficients, e.g.

$$\Gamma(s_1) = \overline{\Gamma(s_{-1})} = (0.126 + 0.501 i) 10^{-4}.$$

(truncated at seven decimals).

It follows that  $Mf^{(1)}$  has a triple pole at 0 and simple imaginary poles  $s_k = 2\pi i k$ , hence

$$f^{(1)} = \frac{1}{2m_1} L^2 + \left( \frac{\gamma + 1/2}{m_1} + \frac{m_2}{2m_1^2} \right) L + \phi_1(L) + O(\rho^{-1+\epsilon}),$$

where  $\phi_1$  is another periodic function. Thus oscillation prevails only in the third term in the expansion of  $f^{(1)}$ . However, computing  $g_1$ , as in Lemma 6, we should include  $L^2 \widehat{\Phi}(\rho)$ , thus arriving at

$$g_1(\rho) = \frac{1}{m_1} L^3 + \phi_2(L) L^2 + \dots$$

with oscillating  $\phi_2$ . As a consequence we will have

$$f^{(2)} = \frac{1}{4m_1^2} L^4 + \phi_3(L) L^3 + \dots,$$

and this suggests that the principal  $O(L^3)$ -term in the asymptotic expansion of the variance  $\text{var} \widehat{K}_\rho$  will oscillate, though we could not establish this fact rigorously.

In a similar situation of sampling from the geometric distribution, the expectation of the number of different values in a sample also involves an oscillating second term, and the same is true for the factorial moment of order 2 [17]. However, this has no impact on the principal asymptotics of the variance: the oscillating terms cancel and the variance converges to a constant, as had been shown long ago by Karlin, see [14, Example 6, p. 385].

### 8. Appendix

**Lemma 16.** *The four conditions in (L) are equivalent.*

*Proof.* The equivalence of expansions of  $\nu$  and  $\tilde{\nu}$  follows by the change of variables  $x = 1 - e^{-y}$  and  $y = -\log(1 - x)$ . For example, assuming the expansion of  $\nu$  we obtain that of  $\tilde{\nu}$  by using

$$\tilde{\nu}[x, 1] = \nu[-\log(1 - x), \infty]$$

and substituting

$$-\log(-\log(1 - x)) = -\log(x(1 + x/2 + \dots)) = -\log x + O(x).$$

The expansions of integrals are equivalent to the expansions of measures by writing

$$\Phi(\rho)/\rho = \int_0^\infty e^{-\rho y} \nu[y, \infty] dy, \quad \widehat{\Phi}(\rho)/\rho = \int_0^1 e^{-\rho x} \widetilde{\nu}[x, 1] dx$$

and using the classical integral

$$\rho \int_0^\infty e^{-\rho y} \log(1/y) dy = L + \gamma$$

together with standard properties of the Laplace transform [6].  $\square$

**Lemma 17.** *The function*

$$\Phi(s) = \int_0^1 (1 - (1 - x)^s) \widetilde{\nu}(dx)$$

has no zeros for  $\Re s > 0$ . And if  $\Phi(s) = 0$  for purely imaginary  $s$ , then  $\widetilde{\nu}$  is atomic, with support  $\{1 - a^k, k = 1, 2, \dots\}$  for some  $0 < a < 1$ .

*Proof.* For  $\Re s > 0$  and  $x \in ]0, 1[$  we have  $|(1 - x)^s| < 1$ . Therefore  $\Re(1 - (1 - x)^s) > 0$  and the integral cannot be zero. For real  $r \neq 0$  the equality  $\Phi(ir) = 0$  is only possible when  $1 = (1 - x)^{ir}$  holds  $\widetilde{\nu}$ -almost everywhere, but such  $x$  is of the form  $x = 1 - \exp(-2\pi k/|r|)$  for some  $k = 1, 2, \dots$ .  $\square$

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