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Potential Theory of Geometric Stable Processes

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Abstract. In this paper we study the potential theory of symmetric geometric stable processes by realizing them as subordinate Brownian motions with geometric stable subordinators. More precisely, we establish the asymptotic behaviors of the Green function and the Lévy density of symmetric geometric stable processes. The asymptotics of these functions near zero exhibit features that are very different from the ones for stable processes. The Green function behaves near zero as $1/(|x|^d \log^2 |x|)$, while the Lévy density behaves like $1/|x|^d$. We also study the asymptotic behaviors of the Green function and Lévy density of subordinate Brownian motions with iterated geometric stable subordinators. As an application, we establish estimates on the capacity of small balls for these processes, as well as mean exit time estimates from small balls and a Harnack inequality for these processes.

1. Introduction

Geometric stable distributions and geometric infinitely divisible distributions were first introduced in [12]. Since their introduction they have played an important role in heavy-tail modeling of economic data, see [16] and the reference therein. Despite the wide spread applications of geometric stable processes in mathematical finance and other fields, there has not been much study of the potential theory of these processes. In this paper we take up this task. In particular, we will study the behaviors of the Green function and the Lévy density of symmetric geometric stable processes. The asymptotic behaviors of these functions near zero exhibit some

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new features that are dramatically different from the ones for stable processes. The Green function behaves near zero as $1/(|x|^d \log^2 |x|)$, while the Lévy density behaves like $1/|x|^d$.

Let $\alpha \in (0, 2]$. A Lévy process $X = (X_t, \mathbb{P}_x)$ is called a geometric strictly α -stable process if its characteristic exponent $\Psi(\xi) = -\log(\mathbb{E}_x(e^{i\xi \cdot (X_1 - X_0)}))$ is given by

$$\Psi(\xi) = \log(1 + \Phi(\xi)), \quad \xi \in \mathbb{R}^d,$$

with $\exp(-\Phi)$ being the characteristic function of some strictly α -stable distribution. In this paper we will be mainly interested in the rotationally invariant geometric strictly α -stable process in \mathbb{R}^d , that is, in the case when

$$\Psi(\xi) = \log(1 + |\xi|^\alpha), \quad \xi \in \mathbb{R}^d.$$

We will simply call these processes symmetric geometric α -stable processes. The symmetric geometric 2-stable process also goes by the name of symmetric variance gamma process and it is used by some researchers to study heavy-tailed financial models (see [15], [9] and the references therein).

Our approach to the potential theory of symmetric geometric stable processes is to realize them as subordinate Brownian motions with geometric stable subordinators, and then use Tauberian-type theorems to establish behaviors of the Green function and the Lévy density. To be more precise, for any $\alpha \in (0, 2]$, the subordinator $S = (S_t : t \geq 0)$ with the Laplace exponent

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \quad \lambda > 0$$

is called a geometric $\alpha/2$ -stable subordinator. Note that for $\alpha = 2$ the corresponding geometric stable subordinator is in fact the well-known gamma subordinator. Let $Y = (Y_t : t \geq 0)$ be a Brownian motion in \mathbb{R}^d , independent of the subordinator S . By computing the characteristic exponent, it is easy to show that the subordinate process $X = (X_t : t \geq 0)$ defined by $X_t = Y(S_t)$, $t \geq 0$, is a symmetric geometric α -stable process.

This approach has been used in [17] to study the Green function behavior of the sum of a Brownian motion and an independent α -stable process. The Laplace exponent of the corresponding subordinator is the sum of two power functions. This fact allowed for the use of Karamata's Tauberian theorem ([4], Theorem 1.7.1) and monotone density theorem ([4], Theorem 1.7.2). However, in the present case the Laplace exponent of the geometric stable subordinator is of the logarithmic type which calls for the use of more delicate de Haan's Tauberian theorem ([4], Theorem 3.7.3) and de Haan's monotone density theorem ([4], Theorem 3.6.8).

The Lévy density of the geometric $\alpha/2$ -stable subordinator is of the order $\alpha/(2x)$ for x near zero, which makes it almost integrable near zero. One consequence is that such a subordinator is very slow. This implies that the subordinate process is also slow and spends a large amount of time in a ball centered at the starting point. This fact is reflected in the behavior of the Green function near the origin which is on the brink of integrability.

The approach described above is also applicable to subordinate Brownian motions with n -iterated geometric stable subordinators, i.e., subordinators whose Laplace exponents are n -fold compositions of that of geometric stable subordinators. Iterated geometric stable subordinators and subordinate Brownian motions with iterated geometric stable subordinators give two families of very concrete Lévy processes with fat-tails and they could be very useful in applications.

The results on asymptotic behaviors of the Green functions can be used to establish estimates on the capacity of balls for the corresponding processes, as well as the exit time estimates from small balls. We present these results as a consequence of the more general results for certain symmetric Lévy processes. Finally, as an application we establish the Harnack inequality for geometric stable processes.

The content of this paper is organized as follows. In the next section we study the asymptotic behaviors of the potential density and the Lévy density at zero and infinity of geometric stable subordinators. These results are applied in Section 3 to establish the asymptotic behaviors at zero and infinity of the symmetric geometric stable processes. In Section 4, we refine the results of the previous two sections to iterated geometric stable subordinators and subordinate Brownian motions. The results of this section extend most of those of Sections 2 and 3, but the details of proofs are more cumbersome. This is why we have decided to present them separately, after the reader has become acquainted with basic ideas and techniques. In Section 5, we prove capacity estimates and the exit time estimates for Lévy processes with radially symmetric decreasing Green functions. In the last section we present the Harnack inequality for symmetric geometric stable processes.

In the paper we use following notation: If f and g are two functions, then $f \sim g$ if f/g converges to 1, and $f \asymp g$ if the quotient f/g stays bounded between two positive constants.

2. Geometric stable subordinators

In this section we assume that $\alpha \in (0, 2]$ and that $S = (S_t : t \geq 0)$ is a geometric $\alpha/2$ -stable subordinator, that is, an increasing Lévy process taking values in $[0, \infty)$ whose Laplace exponent is given by

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \quad \lambda > 0.$$

The function ϕ above can be written in the form

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu(dt), \quad \lambda > 0,$$

where μ is a σ -finite measure μ on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1) \mu(dt) < \infty.$$

The measure μ is called the Lévy measure of S . Since the function ϕ is a complete Bernstein function, the Lévy measure μ has a complete monotone density $\mu(t)$. For

definition and properties of complete Bernstein functions see, for instance, [10] or [17].

The potential measure of the subordinator S is defined by

$$U(A) = \mathbb{E} \int_0^\infty 1_{(S_t \in A)} dt, \quad (2.1)$$

and its Laplace transform is given by

$$\mathcal{L}U(\lambda) = \mathbb{E} \int_0^\infty \exp(-\lambda S_t) dt = \frac{1}{\phi(\lambda)} = \frac{1}{\log(1 + \lambda^{\alpha/2})}. \quad (2.2)$$

In the sequel we will also use U to denote the function on $[0, \infty)$ defined by $U(x) = U([0, x])$. Since $\lim_{\lambda \rightarrow \infty} \phi(\lambda) = \infty$, we must have $\mu((0, \infty)) = \infty$. Therefore by Theorem 2.3 of [17] the potential measure U has a density u which is completely monotone on $(0, \infty)$. When $\alpha = 2$, the corresponding geometric stable subordinator is the gamma subordinator, and its Lévy density is given by

$$\mu(t) = t^{-1} e^{-t}, \quad t > 0,$$

(see e.g. [18], p.45). Such an explicit formula for the Lévy density μ is not available for other values of α . The purpose of this section is to study the behaviors of the functions u and μ near zero and infinity. We will need the following versions of Tauberian and monotone density theorems. The versions when $x \rightarrow \infty$ are proved in [4], Theorem 3.7.3 and Theorem 3.6.8. We have not found in the literature the statements of the versions when $x \rightarrow 0$, but they could be proven by applying techniques from Chapter 3 of [4].

Theorem 2.1. (a) (*de Haan's Tauberian Theorem*) Let $U : (0, \infty) \rightarrow (0, \infty)$ be an increasing function. If ℓ is slowly varying at ∞ (resp. at $0+$), $c \geq 0$, the following are equivalent:

(i) As $x \rightarrow \infty$ (resp. $x \rightarrow 0+$)

$$\frac{U(\lambda x) - U(x)}{\ell(x)} \rightarrow c \log \lambda, \quad \forall \lambda > 0.$$

(ii) As $x \rightarrow \infty$ (resp. $x \rightarrow 0+$)

$$\frac{\mathcal{L}U(\frac{1}{\lambda x}) - \mathcal{L}U(\frac{1}{x})}{\ell(x)} \rightarrow c \log \lambda, \quad \forall \lambda > 0.$$

(b) (*de Haan's Monotone Density Theorem*) Let $U : (0, \infty) \rightarrow (0, \infty)$ be an increasing function with $dU(x) = u(x)dx$, where u is monotone and nonnegative, and let ℓ be slowly varying at ∞ (resp. at $0+$). Assume that $c > 0$. Then the following are equivalent:

(i) As $x \rightarrow \infty$ (resp. $x \rightarrow 0+$)

$$\frac{U(\lambda x) - U(x)}{\ell(x)} \rightarrow c \log \lambda, \quad \forall \lambda > 0.$$

(ii) As $x \rightarrow \infty$ (resp. $x \rightarrow 0+$)

$$u(x) \sim cx^{-1}\ell(x).$$

Now we are going to apply this result to establish the asymptotic behavior of the potential density at zero.

Theorem 2.2. *For any $\alpha \in (0, 2]$, we have*

$$u(x) \sim \frac{2}{\alpha x (\log x)^2}, \quad x \rightarrow 0+.$$

Proof. Recall that

$$\mathcal{L}U(\lambda) = 1/\phi(\lambda) = 1/\log(1 + \lambda^{\alpha/2}).$$

Since

$$\frac{\mathcal{L}U(\frac{1}{t\lambda}) - \mathcal{L}U(\frac{1}{\lambda})}{(\log \lambda)^{-2}} \rightarrow \frac{2}{\alpha} \log t, \quad \forall t > 0$$

as $\lambda \rightarrow 0+$, we have by (the $0+$ version of) Theorem 2.1 (a) that

$$\frac{U(tx) - U(x)}{(\log x)^{-2}} \rightarrow \frac{2}{\alpha} \log t, \quad t > 0$$

as $x \rightarrow 0+$. Now we can apply (the $0+$ version of) Theorem 2.1 (b) to get that

$$u(x) \sim \frac{2}{\alpha x (\log x)^2}$$

as $x \rightarrow 0+$. □

Remark 2.3. One can easily show that

$$u(x) \sim \frac{1}{\Gamma(\alpha/2)} x^{\alpha/2-1}, \quad x \rightarrow \infty$$

(see, e.g. [17], proof of Theorem 3.3).

Theorem 2.4. *For any $\alpha \in (0, 2]$, we have*

$$\mu(x) \sim \frac{\alpha}{2x}, \quad x \rightarrow 0+. \quad (2.3)$$

Proof. The distribution function $F_{\alpha/2}$ of the random variable S_1 is called by some authors the Mittag-Leffler distribution (see [11], for example). It follows therefore from Theorem 2.2 of [11], that

$$\mu(x) = \frac{\alpha}{2x} (1 - F_{\alpha/2}(x)), \quad x > 0.$$

Now the conclusion follows immediately. □

Since $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ is a complete Bernstein function, the function $\psi(\lambda) = \lambda/\phi(\lambda)$ is also a complete Bernstein function. Let $T = (T_t : t \geq 0)$ be a subordinator with Laplace exponent ψ and let V be the potential measure of the subordinator T . Since $\lim_{\lambda \rightarrow \infty} \psi(\lambda)/\lambda = \lim_{\lambda \rightarrow \infty} 1/\phi(\lambda) = 0$ and $\lim_{\lambda \rightarrow \infty} \psi(\lambda) = \infty$, the Lévy measure ν of T must satisfy $\nu((0, \infty)) = \infty$. Therefore by Theorem 2.3 of [17] we know that the potential measure V of T has a density v which is completely monotone on $(0, \infty)$.

Theorem 2.5. *For any $\alpha \in (0, 2)$, we have*

$$\mu(x) \sim \frac{\alpha}{2x^{\alpha/2+1}\Gamma(1-\alpha/2)}, \quad x \rightarrow \infty.$$

Proof. Since

$$\frac{1}{\psi(\lambda)} \sim \lambda^{\alpha/2-1}, \quad \lambda \rightarrow 0+,$$

we have, by Karamata's Tauberian theorem, that the potential measure V of T satisfies

$$V(x) \sim \frac{x^{1-\alpha/2}}{\Gamma(2-\alpha/2)}, \quad x \rightarrow \infty.$$

Now using Karamata's monotone density theorem we get that

$$v(x) \sim \frac{1}{x^{\alpha/2}\Gamma(1-\alpha/2)}, \quad x \rightarrow \infty.$$

It follows from Corollary 2.4.8 of [21] that

$$\mu((t, \infty)) = v(t), \quad t > 0,$$

and so we have

$$\mu((t, \infty)) \sim \frac{1}{t^{\alpha/2}\Gamma(1-\alpha/2)}, \quad t \rightarrow \infty.$$

Now applying Karamata's monotone density theorem again we get

$$\mu(t) \sim \frac{\alpha}{2t^{\alpha/2+1}\Gamma(1-\alpha/2)}, \quad t \rightarrow \infty.$$

□

It is known (see for instance [13]) that the distribution $F_{\alpha/2}$ of S_1 is absolutely continuous and the density $f_{\alpha/2}$ is decreasing on $(0, \infty)$. When $\alpha = 2$ we have

$$f_1(x) = e^{-x}, \quad x > 0.$$

In the next result we establish the asymptotic behaviors of $f_{\alpha/2}$ for $\alpha \in (0, 2)$. We will need the following fact. Let $Z = (Z_t, t \geq 0)$ be a Lévy process with characteristic exponent Φ and let τ be an exponential random variable with parameter 1 which is independent of Z . Then $X = Z(\tau)$ is a geometric infinitely divisible random variable with characteristic function $\exp(-\Psi)$, where Ψ is given by $\Psi(\xi) = \log(1 + \Phi(\xi))$. Therefore the distribution of X is equal to the 1-potential of the process Z .

Theorem 2.6. *For any $\alpha \in (0, 2)$, we have*

$$f_{\alpha/2}(x) \sim \frac{1}{\Gamma(\alpha/2)} x^{\frac{\alpha}{2}-1}, \quad x \rightarrow 0+, \quad (2.4)$$

and

$$f_{\alpha/2}(x) \sim 2\pi\Gamma\left(1 + \frac{\alpha}{2}\right) \sin\left(\frac{\alpha\pi}{4}\right) x^{-1-\frac{\alpha}{2}}, \quad x \rightarrow \infty. \quad (2.5)$$

Proof. We first prove (2.4). Since the Laplace transform of the distribution of S_1 is given by

$$\frac{1}{1 + \lambda^{\alpha/2}}, \quad \lambda > 0,$$

we can easily get from Karamata's Tauberian theorem that $F_{\alpha/2}$ is regularly varying at 0

$$F_{\alpha/2}(x) \sim \frac{1}{\Gamma(1 + \alpha/2)} x^{\frac{\alpha}{2}}, \quad x \rightarrow 0+. \quad (2.6)$$

Now we can apply Karamata's monotone density theorem to get (2.4).

Now we establish (2.5). From the paragraph preceding the theorem we know that

$$f_{\alpha/2}(x) = \int_0^\infty e^{-t} \tilde{p}_{\alpha/2}(t, x) dt = \int_0^\infty e^{-t} t^{-\frac{2}{\alpha}} \tilde{p}_{\alpha/2}\left(1, \frac{x}{t^{2/\alpha}}\right) dt$$

where $\tilde{p}_{\alpha/2}(t, x)$ is the transition density of an $\alpha/2$ -stable subordinator. It follows from [19] that

$$\tilde{p}_{\alpha/2}(1, x) \sim 2\pi\Gamma\left(1 + \frac{\alpha}{2}\right) \sin\left(\frac{\alpha\pi}{4}\right) x^{-1-\frac{\alpha}{2}}, \quad x \rightarrow \infty$$

and that for all $x > 0$

$$\tilde{p}_{\alpha/2}(1, x) \leq c(1 \wedge x^{-1-\frac{\alpha}{2}}),$$

for some positive constant $c > 0$. Now we can apply the dominated convergence theorem to arrive at (2.5). \square

3. Green functions and Lévy densities of symmetric geometric stable processes

Let $Y = (Y_t, t \geq 0)$ be a d -dimensional Brownian motion with the transition density given by

$$p_2(t, x, y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x - y|^2}{4t}\right), \quad x, y \in \mathbb{R}^d, t > 0.$$

Let $S = (S_t, t \geq 0)$ be a geometric $\alpha/2$ -stable subordinator with the Laplace exponent $\log(1 + \lambda^{\alpha/2})$, $\alpha \in (0, 2]$, and let $u(t)$ be the potential density of S . Then we know from Theorem 2.2 that

$$u(t) \sim \frac{2}{\alpha t \log^2 t}, \quad t \rightarrow 0+.$$
 (3.1)

If we assume that Y and S are independent, the symmetric geometric α -stable processes $X = (X_t, t \geq 0)$ can be obtained by $X_t = Y(S_t)$.

Throughout this section we assume that $d > \alpha$. This implies that the process X is transient (see e.g., [3], p.33). The potential operator $Gf(x) := \mathbb{E}^x \int_0^\infty f(X_t) dt$ of X has a density $G(x, y) = G(y - x)$ with

$$G(x) = \int_0^\infty p_2(t, 0, x) u(t) dt.$$

The Lévy density of X is given by

$$J(x) = \int_0^\infty p_2(t, 0, x) \mu(t) dt,$$

where $\mu(t)$ is the Lévy density of S .

In this section we will study the asymptotic behaviors of G and J . In order to establish these asymptotic behaviors we start by defining an auxiliary function. For any slowly varying function ℓ at infinity and any $\beta > 0$, let

$$f_{\ell, \beta}(y, t) := \begin{cases} \frac{\ell(1/y)}{\ell(4t/y)}, & y < \frac{t}{\beta}, \\ 0, & y \geq \frac{t}{\beta}. \end{cases}$$

The following technical lemma will be crucial in establishing the asymptotics of G and J .

Lemma 3.1. *Suppose that $w : (0, \infty) \rightarrow (0, \infty)$ is a decreasing function satisfying the following two assumptions:*

- (i) *There exist a constant $c_0 > 0$ and a continuous functions $\ell : (0, \infty) \rightarrow (0, \infty)$ slowly varying at $+\infty$ such that*

$$w(t) \sim \frac{c_0}{t \ell(1/t)}, \quad t \rightarrow 0+.$$
 (3.2)

- (ii) *If $d = 1$ or $d = 2$, then there exist a constant $c_\infty > 0$ and a constant $\gamma < d/2$ such that*

$$w(t) \sim c_\infty t^{\gamma-1}, \quad t \rightarrow +\infty.$$
 (3.3)

Let $g : (0, \infty) \rightarrow (0, \infty)$ be a function such that

$$\int_0^\infty t^{d/2-1} e^{-t} g(t) dt < \infty.$$

If there is $\beta > 0$ such that $f_{\ell, \beta}(y, t) \leq g(t)$ for all $y, t > 0$, then

$$I(x) := \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} w(t) dt \sim \frac{c_0 \Gamma(d/2)}{\pi^{d/2}} \frac{1}{|x|^d \ell(\frac{1}{|x|^2})}, \quad |x| \rightarrow 0.$$

Proof. Let us first note that the assumptions of the lemma guarantee that $I(x) < \infty$ for every $x \neq 0$. By a change of variable we get

$$\begin{aligned} \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} w(t) dt &= \frac{|x|^{-d+2}}{4\pi^{d/2}} \int_0^\infty t^{d/2-2} e^{-t} w\left(\frac{|x|^2}{4t}\right) dt \\ &= \frac{1}{4\pi^{d/2}} \left(|x|^{-d+2} \int_0^{\beta|x|^2} + |x|^{-d+2} \int_{\beta|x|^2}^\infty \right) \\ &= \frac{1}{4\pi^{d/2}} \left(|x|^{-d+2} I_1 + |x|^{-d+2} I_2 \right). \end{aligned}$$

We first consider I_1 for the case $d = 1$ or $d = 2$. It follows from the assumptions that there exists a positive constant c_1 such that $w(s) \leq c_1 s^{\gamma-1}$ for all $s \geq 1/(4\beta)$. Thus

$$\begin{aligned} I_1 &\leq \int_0^{\beta|x|^2} t^{d/2-2} e^{-t} c_1 \left(\frac{|x|^2}{4t}\right)^{\gamma-1} dt \\ &\leq c_2 |x|^{2\gamma-2} \int_0^{\beta|x|^2} t^{d/2-\gamma-1} dt \\ &= c_3 |x|^{d-2}. \end{aligned}$$

It follows that

$$\lim_{|x| \rightarrow 0} \frac{|x|^{-d+2} I_1}{\frac{1}{|x|^d \ell\left(\frac{1}{|x|^2}\right)}} = 0. \quad (3.4)$$

In the case $d \geq 3$, we proceed similarly, using the bound $w(s) \leq w(1/(4\beta))$ for $s \geq 1/(4\beta)$.

Now we consider I_2 :

$$\begin{aligned} |x|^{-d+2} I_2 &= \frac{1}{|x|^{d-2}} \int_{\beta|x|^2}^\infty t^{d/2-2} e^{-t} w\left(\frac{|x|^2}{4t}\right) dt \\ &= \frac{4}{|x|^d \ell\left(\frac{1}{|x|^2}\right)} \int_{\beta|x|^2}^\infty t^{d/2-1} e^{-t} \frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\frac{|x|^2}{4t} \ell\left(\frac{4t}{|x|^2}\right)}} \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} dt. \end{aligned}$$

Using the assumption (3.2), we can see that there is a constant $c > 0$ such that

$$\frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\frac{|x|^2}{4t} \ell\left(\frac{4t}{|x|^2}\right)}} < c,$$

for all t and x satisfying $|x|^2/(4t) \leq 1/(4\beta)$. Since ℓ is slowly varying at infinity,

$$\lim_{|x| \rightarrow 0} \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} = 1$$

for all $t > 0$. Note that

$$\frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})} = f_{\ell, \beta}(|x|^2, t).$$

It follows from the assumption that

$$t^{d/2-1} e^{-t} \frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\frac{|x|^2}{4t} \ell(\frac{4t}{|x|^2})}} \frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})} \leq c t^{d/2-1} e^{-t} g(t).$$

Therefore, by the dominated convergence theorem we have

$$\lim_{|x| \rightarrow 0} \int_{\beta|x|^2}^{\infty} t^{d/2-1} e^{-t} \frac{w\left(\frac{|x|^2}{4t}\right)}{\frac{1}{\frac{|x|^2}{4t} \ell(\frac{4t}{|x|^2})}} \frac{\ell(\frac{1}{|x|^2})}{\ell(\frac{4t}{|x|^2})} dt = \int_0^{\infty} c_0 t^{d/2-1} e^{-t} dt = c_0 \Gamma(d/2).$$

Hence,

$$\lim_{|x| \rightarrow 0} \frac{|x|^{-d+2} I_2}{\frac{4}{|x|^d \ell(\frac{1}{|x|^2})}} = c_0 \Gamma(d/2). \quad (3.5)$$

Finally, combining (3.4) and (3.5) we get

$$\lim_{|x| \rightarrow 0} \frac{I(x)}{\frac{1}{|x|^d \ell(\frac{1}{|x|^2})}} = \frac{c_0 \Gamma(d/2)}{\pi^{d/2}}.$$

□

Theorem 3.2. For any $\alpha \in (0, 2]$, we have

$$G(x) \sim \frac{\Gamma(d/2)}{2\alpha \pi^{d/2} |x|^d \log^2 \frac{1}{|x|}}, \quad |x| \rightarrow 0.$$

Proof. We apply Lemma 3.1 with $w(t) = u(t)$, the potential density of S . By (3.1), $u(t) \sim \frac{2}{\alpha t \log^2 t}$ as $t \rightarrow 0+$, so we take $c_0 = 2/\alpha$ and $\ell(t) = \log^2 t$. Moreover, by Remark 2.3, $u(t) \sim t^{\alpha/2-1}/(\Gamma(\alpha)/2)$ as $t \rightarrow +\infty$, so we can take $\gamma = \alpha/2 < d/2$. Choose $\beta = 1/2$. Let

$$f(y, t) := f_{\ell, 1/2}(y, t) = \begin{cases} \frac{\log^2 y}{\log^2 \frac{y}{4t}}, & y < 2t, \\ 0, & y \geq 2t. \end{cases}$$

Define

$$g(t) := \begin{cases} \frac{\log^2 2t}{\log^2 2}, & t < \frac{1}{4}, \\ 1, & t \geq \frac{1}{4}. \end{cases}$$

In order to show that $f(y, t) \leq g(t)$, first let $t < 1/4$. Then $y \mapsto f(y, t)$ is an increasing function for $0 < y < 2t$. Hence,

$$\sup_{0 < y < 2t} f(y, t) = f(2t, t) = \frac{\log^2 2t}{\log^2 2}.$$

Clearly, $f(y, 1/4) = 1$. For $t > 1/4$, $y \mapsto f(y, t)$ is a decreasing function for $0 < y < 1$. Hence

$$\sup_{0 < y < (2t) \wedge 1} f(y, t) = f(0, t) := \lim_{y \rightarrow 0} f(y, t) = 1.$$

Clearly,

$$\int_0^\infty t^{d/2-1} e^{-t} g(t) dt < \infty.$$

□

Remark 3.3. The asymptotic behavior of $G(x)$ as $|x| \rightarrow \infty$ was proved in [17], Theorem 3.3 to be

$$G(x) \sim \frac{1}{\pi^{d/2} 2^\alpha} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} |x|^{\alpha-d}, \quad |x| \rightarrow \infty.$$

Now we establish the asymptotic behaviors of J .

Theorem 3.4. *For every $\alpha \in (0, 2]$ we have*

$$J(x) \sim \frac{\alpha \Gamma(d/2)}{2|x|^d}, \quad |x| \rightarrow 0.$$

Proof. We again apply Lemma 3.1, this time with $w(t) = \mu(t)$, the density of the Lévy measure of S . By (2.3), $\mu(t) \sim \frac{\alpha}{2t}$ as $t \rightarrow 0+$, so we take $c_0 = \alpha/2$ and $\ell(t) = 1$. By Theorem 2.5, $\mu(t)$ is of the order $t^{-\alpha/2-1}$ as $t \rightarrow +\infty$, so we may take $\gamma = -\alpha/2$. Choose $\beta = 1/2$ and let $g = 1$. □

Theorem 3.5. *For every $\alpha \in (0, 2)$ we have*

$$J(x) \sim \frac{\alpha}{2^{\alpha+1} \pi^{d/2}} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(1 - \frac{\alpha}{2})} |x|^{-d-\alpha}, \quad |x| \rightarrow \infty.$$

Proof. Theorem 2.5 tells us that

$$\mu(t) \sim \frac{\alpha}{2\Gamma(1 - \alpha/2)} t^{-\alpha/2-1}, \quad t \rightarrow \infty.$$

Now combine this with Theorem 2.4 to get that

$$\mu(t) \leq C(t^{-1} \vee t^{-\alpha/2-1}). \quad (3.6)$$

By a simple change of variable we have

$$\begin{aligned}
 & \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) \mu(t) dt \\
 &= \frac{1}{4\pi^{d/2}} |x|^{-d+2} \int_0^\infty s^{d/2-2} e^{-s} \mu\left(\frac{|x|^2}{4s}\right) ds \\
 &= \frac{\alpha}{8\pi^{d/2} \Gamma(1-\alpha/2)} |x|^{-d-\alpha} \int_0^\infty s^{d/2-2} e^{-s} \frac{\mu\left(\frac{|x|^2}{4s}\right)}{\frac{\alpha}{2\Gamma(1-\alpha/2)} \left(\frac{|x|^2}{4s}\right)^{-\alpha/2-1}} \left(\frac{1}{4s}\right)^{-\alpha/2-1} ds \\
 &= \frac{\alpha}{2^{\alpha+1} \pi^{d/2} \Gamma(1-\alpha/2)} |x|^{-d-\alpha} \int_0^\infty s^{d/2+\alpha/2-1} e^{-s} \frac{\mu\left(\frac{|x|^2}{4s}\right)}{\frac{\alpha}{a\Gamma(1-\alpha/2)} \left(\frac{|x|^2}{4s}\right)^{-\alpha/2-1}} ds
 \end{aligned}$$

Let $|x| \geq 2$. Then by (3.6),

$$\begin{aligned}
 \frac{u\left(\frac{|x|^2}{4s}\right)}{\left(\frac{|x|^2}{4s}\right)^{-\alpha/2-1}} &\leq C \left(\left(\frac{|x|^2}{4s}\right)^{\alpha/2} \vee 1 \right) \\
 &\leq C(s^{-\alpha/2} \vee 1).
 \end{aligned}$$

It follows that the integrand in the last display above is bounded by an integrable function, so we may use the bounded convergence theorem to obtain

$$\lim_{|x| \rightarrow \infty} \frac{1}{|x|^{-d-\alpha}} \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) \mu(t) dt = \frac{\alpha}{2^{\alpha+1} \pi^{d/2}} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}, \quad (3.7)$$

which proves the result. \square

Theorem 3.6. *When $\alpha = 2$, we have*

$$J(x) \sim 2^{-d/2} \pi^{-\frac{d-1}{2}} \frac{e^{-|x|}}{|x|^{\frac{d+1}{2}}}, \quad |x| \rightarrow \infty.$$

Proof. By making a simple change of variable we get that

$$\begin{aligned}
 J(x) &= \frac{1}{2} \int_0^\infty t^{-1} e^{-t} (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2}\right) dt \\
 &= 2^{-d-1} \pi^{-d/2} |x|^{-d} \int_0^\infty s^{\frac{d}{2}-1} e^{-\frac{s}{4}-\frac{|x|^2}{s}} ds \\
 &= 2^{-d-1} \pi^{-d/2} |x|^{-d} I(|x|),
 \end{aligned}$$

where

$$I(r) = \int_0^\infty s^{\frac{d}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}} ds.$$

Using the change of variable $u = \frac{\sqrt{s}}{2} - \frac{r}{\sqrt{s}}$ we get

$$\begin{aligned} I(r) &= e^{-r} \int_0^\infty s^{\frac{d}{2}-1} e^{-(\frac{\sqrt{s}}{2} - \frac{r}{\sqrt{s}})^2} ds \\ &= e^{-r} \int_{-\infty}^\infty \frac{2(u + \sqrt{u^2 + 2r})^d}{\sqrt{u^2 + 2r}} e^{-u^2} du \\ &= 2e^{-r} r^{\frac{d-1}{2}} \int_{-\infty}^\infty \frac{u + \sqrt{u^2 + 2r}}{\sqrt{u^2 + 2r}} \left(\frac{u}{\sqrt{r}} + \sqrt{\frac{u^2}{r} + 2} \right)^{d-1} e^{-u^2} du \end{aligned}$$

Therefore by the dominated convergence theorem we get

$$I(r) \sim 2^{\frac{d}{2}+1} \sqrt{\pi} e^{-r} r^{\frac{d-1}{2}}, \quad r \rightarrow \infty.$$

Now the assertion of the theorem follows immediately. \square

Now we are going to study the asymptotic behavior of the transition density $q_\alpha(1, x, y) = q_\alpha(1, y - x)$ at infinity of the process X . From the paragraph preceding Theorem 2.6 we know that

$$q_\alpha(1, x) = \int_0^\infty e^{-t} p_\alpha(t, x) dt, \quad (3.8)$$

where $p_\alpha(t, x)$ is the transition density of a symmetric α -stable process.

Theorem 3.7. *For $\alpha \in (0, 2)$ we have*

$$q_\alpha(1, x) \sim \frac{\alpha 2^{\alpha-1} \sin \frac{\alpha\pi}{2} \Gamma(\frac{d+\alpha}{2}) \Gamma(\frac{\alpha}{2})}{\pi^{\frac{d}{2}+1} |x|^{d+\alpha}}, \quad |x| \rightarrow \infty.$$

For $\alpha = 2$ we have

$$q_2(1, x) \sim 2^{-\frac{d}{2}} \pi^{-\frac{d-1}{2}} \frac{e^{-|x|}}{|x|^{\frac{d-1}{2}}}, \quad |x| \rightarrow \infty.$$

Proof. The proof of the case $\alpha = 2$ is same as the proof of the previous theorem, so we only give the proof of the case $\alpha \in (0, 2)$. Using the scaling property we get that

$$q_\alpha(1, x) = \int_0^\infty e^{-t} t^{-\frac{d}{\alpha}} p_\alpha(1, \frac{x}{t^{1/\alpha}}) dt.$$

Now we can use Theorem 2.1 of [5] and the dominated convergence theorem to arrive at our conclusion. \square

4. Extension to iterated subordinators

In this section we extend some of the results of the previous section to iterated geometric stable subordinators and subordinate Brownian motion with iterated geometric stable subordinators.

Let $e_0 = 0$, and inductively, $e_n = e^{e_{n-1}}$, $n \geq 1$. For $n \geq 1$ define $l_n : (e_n, \infty) \rightarrow (0, \infty)$ by

$$l_n(y) = \log \log \dots \log y, \quad n \text{ times}.$$

Further, let $L_0(y) = 1$, and for $n \in \mathbb{N}$, define $L_n : (e_n, \infty) \rightarrow (0, \infty)$ by

$$L_n(y) = l_1(y)l_2(y) \dots l_n(y).$$

Note that $l'_n(y) = 1/(yL_{n-1}(y))$ for every $n \geq 1$.

Let $\alpha \in (0, 2]$. Define $\phi(y) = \phi^{(1)}(y) := \log(1 + y^{\alpha/2})$. For $n \in \mathbb{N}$ define inductively $\phi^{(n)}(y) := \phi(\phi^{(n-1)}(y))$. Let $k_n(y) := 1/\phi^{(n)}(y)$.

Lemma 4.1. *Let $t > 0$. For every $n \in \mathbb{N}$,*

$$\lim_{y \rightarrow \infty} (k_n(ty) - k_n(y))L_{n-1}(y)l_n(y)^2 = -\frac{2}{\alpha} \log t.$$

Proof. The proof for $n = 1$ is straightforward and is implicit in the proof of Theorem 2.2. We only give the proof for $n = 2$, the proof for general n is similar. Using the fact that

$$\log(1 + y) \sim y, \quad y \rightarrow 0+, \quad (4.1)$$

we can easily get that

$$\lim_{y \rightarrow \infty} \left(\log \frac{\log y}{\log(yt)} \right) \log y = - \lim_{y \rightarrow \infty} \left(\log \frac{\log y + \log t}{\log y} \right) \log y = - \log t. \quad (4.2)$$

Using (4.1) and the elementary fact

$$\log(1 + y) \sim \log y, \quad y \rightarrow \infty,$$

we get that

$$\begin{aligned} & \lim_{y \rightarrow \infty} (k_2(ty) - k_2(y))L_1(y)l_2(y)^2 \\ &= \frac{\alpha}{2} \lim_{y \rightarrow \infty} \left(\log \frac{\log(1 + y^{\alpha/2})}{\log(1 + (ty)^{\alpha/2})} \right) \frac{\log y (\log \log y)^2}{(\frac{\alpha}{2})^2 \log(\log(1 + y^{\alpha/2})) \log(\log(1 + (ty)^{\alpha/2}))} \\ &= \frac{2}{\alpha} \lim_{y \rightarrow \infty} \left(\log \frac{\log y}{\log(yt)} \right) \log y \\ &= -\frac{2}{\alpha} \log t. \end{aligned}$$

□

We will assume that $S^{(n)} = (S_t^{(n)} : t \geq 0)$ is a subordinator whose Laplace exponent is given by $\phi^{(n)}(\lambda)$. The function $\phi^{(n)}$ can be written in the form

$$\phi^{(n)}(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu^{(n)}(dt), \quad \lambda > 0$$

for some σ -finite measure $\mu^{(n)}$ on $(0, \infty)$ satisfying the

$$\int_0^\infty (t \wedge 1) \mu^{(n)}(dt) < \infty.$$

The measure $\mu^{(n)}$ is called the Lévy measure of $S^{(n)}$. Since the function $\phi^{(n)}$ is a complete Bernstein function, the Lévy measure $\mu^{(n)}$ has a complete monotone density $\mu^{(n)}(t)$ (see for instance [10]).

Note that if $S^{(n-1)}$ and S are independent subordinators with Laplace exponents $\phi^{(n-1)}$ and ϕ , respectively, then the subordinator $S^{(n-1)}(S_t)$ has the same distribution as $S_t^{(n)}$. In this way we may regard $S^{(n)}$ as an n -fold iteration of S by itself.

The potential measure of the subordinator $S^{(n)}$ is defined by

$$U^{(n)}(A) = \mathbb{E} \int_0^\infty 1_{(S_t^{(n)} \in A)} dt, \quad (4.3)$$

and its Laplace transform is given by

$$\mathcal{L}U^{(n)}(\lambda) = \mathbb{E} \int_0^\infty \exp(-\lambda S_t^{(n)}) dt = \frac{1}{\phi^{(n)}(\lambda)}. \quad (4.4)$$

In the sequel we will also use $U^{(n)}$ to denote the function on $[0, \infty)$ defined by $U^{(n)}(x) = U^{(n)}([0, x])$. Since $\lim_{\lambda \rightarrow \infty} \phi^{(n)}(\lambda) = \infty$, we must have $\mu^{(n)}((0, \infty)) = \infty$. Therefore by Theorem 2.3 of [17] the potential measure $U^{(n)}$ has a density $u^{(n)}$ which is completely monotone on $(0, \infty)$. One of the purpose of this section is to study the behaviors of the functions $u^{(n)}$ and $\mu^{(n)}$ near zero and infinity.

Theorem 4.2. *For any $\alpha \in (0, 2]$, we have*

$$u^{(n)}(x) \sim \frac{2}{\alpha x L_{n-1}(\frac{1}{x}) l_n(\frac{1}{x})^2}, \quad x \rightarrow 0+. \quad (4.5)$$

Proof. Using Lemma 4.1 we can easily see that

$$\frac{\mathcal{L}U^{(n)}(\frac{1}{t\lambda}) - \mathcal{L}U^{(n)}(\frac{1}{\lambda})}{(L_{n-1}(\frac{1}{\lambda}) l_n(\frac{1}{\lambda})^2)^{-1}} \rightarrow \frac{2}{\alpha} \log t, \quad \forall t > 0$$

as $\lambda \rightarrow 0+$. Therefore, by (the $0+$ version of) Theorem 2.1 (a) we have that

$$\frac{U^{(n)}(tx) - U^{(n)}(x)}{(L_{n-1}(\frac{1}{x}) l_n(\frac{1}{x})^2)^{-1}} \rightarrow \frac{2}{\alpha} \log t, \quad t > 0$$

as $x \rightarrow 0+$. Now we can apply (the $0+$ version of) Theorem 2.1 (b) to get that

$$u^{(n)}(x) \sim \frac{2}{\alpha x L_{n-1}(\frac{1}{x}) l_n(\frac{1}{x})^2}$$

as $x \rightarrow 0+$. □

Remark 4.3. One can easily show that

$$u^{(n)}(x) \sim \frac{1}{\Gamma((\alpha/2)^n)} x^{(\alpha/2)^n - 1}, \quad x \rightarrow \infty$$

(see, e.g. [17], proof of Theorem 3.3).

Since $\phi^{(n)}(\lambda)$ is a complete Bernstein function, the function $\psi^{(n)}(\lambda) = \lambda/\phi^{(n)}(\lambda)$ is also a complete Bernstein function. Let $T^{(n)} = (T_t^{(n)} : t \geq 0)$ be a subordinator with Laplace exponent $\psi^{(n)}$ and let $V^{(n)}$ be the potential measure of the subordinator $T^{(n)}$. Since $\lim_{\lambda \rightarrow \infty} \psi^{(n)}(\lambda)/\lambda = \lim_{\lambda \rightarrow \infty} 1/\phi^{(n)}(\lambda) = 0$ and $\lim_{\lambda \rightarrow \infty} \psi^{(n)}(\lambda) = \infty$, the Lévy measure $\nu^{(n)}$ of T must satisfy $\nu^{(n)}((0, \infty)) = \infty$. Therefore by Theorem 2.3 of [17] the potential measure $V^{(n)}$ of $T^{(n)}$ has a density $v^{(n)}$ which is completely monotone on $(0, \infty)$.

Theorem 4.4. For any $\alpha \in (0, 2)$, we have

$$\mu^{(n)}(x) \sim \frac{(\alpha/2)^n}{x^{(\alpha/2)^n + 1} \Gamma(1 - (\alpha/2)^n)}, \quad x \rightarrow \infty.$$

Proof. Since

$$\frac{1}{\psi^{(n)}(\lambda)} \sim \lambda^{(\alpha/2)^n - 1}, \quad \lambda \rightarrow 0+,$$

we have, by Karamata's Tauberian theorem, that the potential measure $V^{(n)}$ of T satisfies

$$V^{(n)}(x) \sim \frac{x^{1 - (\alpha/2)^n}}{\Gamma(2 - (\alpha/2)^n)}, \quad x \rightarrow \infty.$$

Now using Karamata's monotone density theorem we get that

$$v^{(n)}(x) \sim \frac{1}{x^{(\alpha/2)^n} \Gamma(1 - (\alpha/2)^n)} \quad x \rightarrow \infty.$$

It follows from Corollary 2.4.8 of [21] that

$$\mu((t, \infty)) = v(t), \quad t > 0,$$

and so we have

$$\mu((t, \infty)) \sim \frac{1}{t^{(\alpha/2)^n} \Gamma(1 - (\alpha/2)^n)} \quad t \rightarrow \infty.$$

Now applying Karamata's monotone density theorem again we get

$$\mu(t) \sim \frac{(\alpha/2)^n}{t^{(\alpha/2)^n + 1} \Gamma(1 - (\alpha/2)^n)} \quad t \rightarrow \infty.$$

□

Remark 4.5. Note that the previous theorem is proved for $\alpha \in (0, 2)$ only. We know that for $n = 1$, the Lévy density $\mu^{(1)}(x)$ is equal to e^{-x}/x . We expect similar behavior for $n \geq 2$ as well. Unfortunately, we were unable to find precise asymptotic behavior of the Lévy density $\mu^{(n)}(x)$ as $x \rightarrow \infty$ in case $\alpha = 2$ and $n \geq 2$. One of the difficulties is that all functions $\phi^{(n)}(\lambda)$ are of the same order λ near zero. We were unable to find in the literature a Tauberian type theorem that is applicable in this case.

Let $Y = (Y_t, t \geq 0)$ be a d -dimensional Brownian motion as in the previous section. Assume that Y and $S^{(n)}$ are independent. We define the subordinate process $X^{(n)} = (X_t^{(n)}, t \geq 0)$ by $X_t^{(n)} = Y(S_t^{(n)})$. The process $X^{(n)}$ has a transition density $q_\alpha^{(n)}(t, x, y) = q_\alpha^{(n)}(t, y - x)$ given by

$$q_\alpha^{(n)}(t, x) = \int_0^\infty p_2(t, 0, x) f_{\alpha/2}^{(n)}(t, s) ds \quad (4.6)$$

where $f_{\alpha/2}^{(n)}(t, s)$ is the density of $S_t^{(n)}$. Note that $q_\alpha^{(1)}(1, x) = q_\alpha(1, x)$, where $q_\alpha(1, x)$ was introduced in Section 3.

Throughout this section we assume that $d > 2(\alpha/2)^n$. Similarly as in the previous section, this implies that the process $X^{(n)}$ is transient. The potential operator $G^{(n)} f(x) := \mathbb{E}^x \int_0^\infty f(X_t^{(n)}) dt$ of $X^{(n)}$ has a density $G^{(n)}(x, y) = G^{(n)}(y - x) = G^{(n)}(|y - x|)$ with

$$G^{(n)}(x) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) u^{(n)}(t) dt,$$

where $u^{(n)}$ is the potential density of $S^{(n)}$. The Lévy density of $X^{(n)}$ is given by

$$J^{(n)}(x) = \int_0^\infty p_2(t, 0, x) \mu^{(n)}(t) dt,$$

where $\mu^{(n)}(t)$ is the Lévy density of S . Another expression for $J^{(n)}$ is as follows:

$$J^{(n)}(x) = \int_0^\infty p_\alpha(t, 0, x) t^{-1} e^{-t} dt \quad (4.7)$$

where p_α is the transition density of the symmetric α -stable process in \mathbb{R}^d . Note that $J^{(n)}(x)$ depends only on $|x|$. Therefore, by slightly abusing notation we will define $J^{(n)}(r) := J^{(n)}(x)$ for $r = |x| > 0$. We want to study the asymptotic behavior of $G^{(n)}$ using Lemma 3.1. In order to check the conditions of that lemma, we need some preparation.

For $n \in \mathbb{N}$, define $f_n : (0, 1/e_n) \times (0, \infty) \rightarrow [0, \infty)$ by

$$f_n(y, t) := \begin{cases} \frac{L_{n-1}(\frac{1}{y}) l_n(\frac{1}{y})^2}{L_{n-1}(\frac{4t}{y}) l_n(\frac{4t}{y})^2}, & y < \frac{2t}{e_n}, \\ 0, & y \geq \frac{2t}{e_n}. \end{cases}$$

Note that f_n is equal to the function $f_{\ell, \beta}$, defined before Lemma 3.1, with $\ell(y) = L_{n-1}(y)l_n(y)^2$ and $\beta = e_n/2$. Also, for $n \in \mathbb{N}$, let

$$g_n(t) := \begin{cases} f_n(\frac{2t}{e_n}, t), & t < 1/4, \\ 1, & t \geq 1/4. \end{cases}$$

Moreover, for $n \in \mathbb{N}$, define $h_n : (0, 1/e_n) \times (0, \infty) \rightarrow (0, \infty)$ by

$$h_n(y, t) := \frac{l_n(\frac{1}{y})}{l_n(\frac{4t}{y})}.$$

Clearly, for $0 < y < \frac{2t}{e_n} \wedge \frac{1}{e_n}$ we have that

$$f_n(y, t) = h_1(y, t) \dots h_{n-1}(y, t) h_n(y, t)^2. \quad (4.8)$$

Lemma 4.6. *For all $y \in (0, 1/e_n)$ and all $t > 0$ we have $f_n(y, t) \leq g_n(t)$. Moreover, $\int_0^\infty t^{d/2-1} e^{-t} g_n(t) dt < \infty$.*

Proof. A direct calculation of partial derivative gives

$$\frac{\partial h_n}{\partial y}(y, t) = \frac{L_n(\frac{1}{y}) - L_n(\frac{4t}{y})}{y L_{n-1}(\frac{1}{y}) L_{n-1}(\frac{4t}{y}) l_n(\frac{4t}{y})^2}.$$

The denominator is always positive. Clearly, the numerator is positive if and only if $t > 1/4$. Therefore, for $t < 1/4$, $y \mapsto h_n(y, t)$ is increasing on $(0, 2t/e_n)$, while for $t > 1/4$ it is decreasing on $(0, 2t/e_n)$.

Let $t < 1/4$. It follows from (4.8) and the fact that $y \mapsto h_n(y, t)$ is increasing on $(0, 2t/e_n)$ that $y \mapsto f_n(y, t)$ is increasing for $0 < y < 2t/e_n$. Therefore,

$$\sup_{0 < y < 2t/e_n} f_n(y, t) \leq f_n(2t/e_n, t) = g_n(t).$$

Clearly, $f_n(y, 1/4) = 1$. For $y \geq 1/4$, it follows from (4.8) and the fact that $y \mapsto h_n(y, t)$ is decreasing on $(0, 2t/e_n)$ that $y \mapsto f_n(y, t)$ is decreasing for $0 < y < 1/e_n$. Hence

$$\sup_{0 < y < \frac{2t}{e_n} \wedge \frac{1}{e_n}} f_n(y, t) = f(0, t) := \lim_{y \rightarrow 0} f_n(y, t) = 1.$$

The integrability statement of the lemma is obvious. □

Theorem 4.7. *We have*

$$G^{(n)}(x) \sim \frac{\Gamma(d/2)}{2\alpha\pi^{d/2}|x|^d L_{n-1}(1/|x|^2) l_n(1/|x|^2)^2}, \quad |x| \rightarrow 0.$$

Proof. We apply Lemma 3.1 with $v(t) = u^{(n)}(t)$, the potential density of $S^{(n)}$. By (4.5),

$$u^{(n)}(t) \sim \frac{2}{\alpha t L_{n-1}(1/t) l_n(1/t)^2}, \quad t \rightarrow 0+,$$

so we take $c_0 = 2/\alpha$ and $\ell(t) = L_{n-1}(t)l_n(t)^2$. By Remark 4.3, $u^{(n)}(t)$ is of order $t^{(\alpha/2)^n-1}$ as $t \rightarrow \infty$, so we may take $\gamma = (\alpha/2)^n < d/2$. Choose $\beta = 1/2$. The result follows from Lemma 3.1 and Lemma 4.6 \square

Remark 4.8. The asymptotic behavior of $G^{(n)}(x)$ as $|x| \rightarrow \infty$ was proved in [17], Theorem 3.3. Denote $\alpha_n = \alpha(\alpha/2)^{n-1}$. Then

$$G^{(n)}(x) \sim \frac{1}{\pi^{d/2} 2^{\alpha_n}} \frac{\Gamma(\frac{d-\alpha_n}{2})}{\Gamma(\frac{\alpha_n}{2})} |x|^{\alpha_n-d}, \quad |x| \rightarrow \infty.$$

Although we could not get the exact asymptotic behaviors of $J^{(n)}$, the following result about $J^{(n)}$ will be useful later.

Proposition 4.9. *For any $\alpha \in (0, 2)$ and $n \geq 1$, there exists a positive constant c such that*

$$J^{(n)}(r) \leq c J^{(n)}(2r), \quad \text{for all } r > 0 \quad (4.9)$$

and

$$J^{(n)}(r) \leq c J^{(n)}(r+1), \quad \text{for all } r > 1. \quad (4.10)$$

Proof. Using Theorem 4.4 and repeating the proof of (4.6) in [17], we can easily prove (4.10). We omit the details. Now we prove (4.9). Recall that $p_\alpha(t, x)$ is the transition density of a symmetric α -stable process in \mathbb{R}^d . It is well known (see Theorem 2.1 of [5]) that there exist positive constants C_1 and C_2 such that

$$C_1(1 \wedge |x|^{-(d+\alpha)}) \leq p_\alpha(1, x) \leq C_2(1 \wedge |x|^{-(d+\alpha)}), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

Using this one can easily see that there exists $C_3 > 0$ such that

$$p_\alpha(t, x) \leq c_3 p_\alpha(t, 2x), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \quad (4.11)$$

Recall that

$$J^{(1)}(x) = \frac{1}{2} \int_0^\infty p_\alpha(t, x) t^{-1} e^{-t} dt, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

Similarly as in (3.8) we have

$$q_\alpha^{(1)}(t, x) = \int_0^\infty p_\alpha(s, x) \frac{1}{\Gamma(t)} s^{t-1} e^{-s} ds, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

Combining the two displays above with (4.11) we immediately get that

$$J^{(1)}(x) \leq C_3 J^{(1)}(2x), \quad \text{for all } x \in \mathbb{R}^d, \quad (4.12)$$

$$q_\alpha^{(1)}(t, x) \leq C_3 q_\alpha^{(1)}(t, 2x), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \quad (4.13)$$

We have further that

$$J^{(2)}(x) = \frac{1}{2} \int_0^\infty p^{(1)}(t, x) \mu_{\alpha/2}(t) dt, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d$$

and

$$q_\alpha^{(2)}(t, x) = \int_0^\infty p_\alpha(s, x) f_{\alpha/2}(t, s) ds, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d,$$

where $\mu_{\alpha/2}(t)$ is the Levy density of the geometric $\alpha/2$ -subordinator. Combining the two displays above with (4.13) we immediately get that

$$J^{(2)}(x) \leq C_3 J^{(2)}(2x), \quad \text{for all } x \in \mathbb{R}^d, \quad (4.14)$$

$$q_\alpha^{(2)}(t, x) \leq C_3 q_\alpha^{(2)}(t, 2x), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \quad (4.15)$$

Now we can use induction to get that

$$J^{(n)}(x) \leq C_3 J^{(n)}(2x), \quad \text{for all } x \in \mathbb{R}^d, \quad (4.16)$$

$$q_\alpha^{(n)}(t, x) \leq C_3 q_\alpha^{(n)}(t, 2x), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \quad (4.17)$$

□

5. Capacity and exit time estimates for some symmetric Lévy processes

The purpose of this section is to establish lower and upper estimates for the capacity of balls and the exit time from balls, with respect to a class of general symmetric Lévy processes.

Suppose that $X = (X_t, \mathbb{P}_x)$ is a transient symmetric Lévy process on \mathbb{R}^d . We will assume that the potential kernel of X is absolutely continuous with a density $G(x, y) = G(|y - x|)$ with respect to the Lebesgue measure. This implies that (see Theorem 4.1.2 of [8]) the transition semigroup of X has a density with respect to the Lebesgue measure. We will assume the following conditions: $G : [0, \infty) \rightarrow (0, \infty]$ is a positive and decreasing function satisfying $G(0) = \infty$. We will have need of the following elementary lemma.

Lemma 5.1. *There exist a positive constant $C_1 = C_1(d)$ such that for every $r > 0$ and all $x \in \overline{B(0, r)}$,*

$$C_1 \int_{B(0, r)} G(|y|) dy \leq \int_{B(0, r)} G(x, y) dy \leq \int_{B(0, r)} G(|y|) dy.$$

Moreover, the supremum of $\int_{B(0, r)} G(x, y) dy$ is attained at $x = 0$, while the infimum is attained at any point on the boundary of $B(0, r)$.

Proof. The proof is elementary. We only present the proof of the left-hand side inequality for $d \geq 2$. Consider the intersection of $B(0, r)$ and $B(x, r)$. This intersection contains the intersection of $B(x, r)$ and the cone with vertex x of aperture equal to $\pi/3$ pointing towards the origin. Let $C(x)$ be that intersection. Then

$$\begin{aligned} \int_{B(0,r)} G(|y-x|) dy &\geq \int_{C(x)} G(|y-x|) dy \\ &\geq c_1 \int_{B(x,r)} G(|y-x|) dy = c_1 \int_{B(0,r)} G(|y|) dy \end{aligned}$$

where the constant c_1 depends only on the dimension d . It is easy to see that the infimum of $\int_{B(0,r)} G(x, y) dy$ is attained at any point on the boundary of $B(0, r)$. \square

Let Cap denote the (0-order) capacity with respect to X . For a measure μ denote

$$G\mu(x) := \int G(x, y) \mu(dy).$$

For any compact subset K of \mathbb{R}^d , let \mathcal{P}_K be the set of probability measures supported by K . Define

$$e(K) := \inf_{\mu \in \mathcal{P}_K} \int G\mu(x) \mu(dx).$$

Since the kernel G satisfies the maximum principle (see, for example, Theorem 5.2.2 in [6]), it follows from ([7], page 159) that for any compact subset K of \mathbb{R}^d

$$\text{Cap}(K) = \frac{1}{\inf_{\mu \in \mathcal{P}_K} \sup_{x \in \text{Supp}(\mu)} G\mu(x)} = \frac{1}{e(K)}. \quad (5.1)$$

Furthermore, the infimum is attained at the capacitary measure μ_K . The following lemma is essentially proved in [14].

Lemma 5.2. *Let K be a compact subset of \mathbb{R}^d . For any probability measure μ on K , it holds that*

$$\inf_{x \in \text{Supp}(\mu)} G\mu(x) \leq e(K) \leq \sup_{x \in \text{Supp}(\mu)} G\mu(x). \quad (5.2)$$

Proof. The right-hand side inequality follows immediately from (5.1). In order to prove the left-hand side inequality, suppose that for some probability measure μ on K it holds that $e(K) < \inf_{x \in \text{Supp}(\mu)} G\mu(x)$. Then $e(K) + \epsilon < \inf_{x \in \text{Supp}(\mu)} G\mu(x)$ for some $\epsilon > 0$. We first have

$$\int_K G\mu(x) \mu_K(dx) > \int_K (e(K) + \epsilon) \mu_K(dx) = e(K) + \epsilon.$$

On the other hand,

$$\int_K G\mu(x) \mu_K(dx) = \int_K G\mu_K(x) \mu(dx) = \int_K e(K) \mu(dx) = e(K),$$

where we have used the facts that $G\mu_K = e(K)$ quasi everywhere in K and that a measure of finite energy does not charge sets of capacity zero. This contradiction proves the lemma. \square

Proposition 5.3. *There exist positive constants $C_2 < C_3$ depending only on d , such that for all $r > 0$*

$$\frac{C_2 r^d}{\int_{B(0,r)} G(|y|) dy} \leq \text{Cap}(\overline{B(0,r)}) \leq \frac{C_3 r^d}{\int_{B(0,r)} G(|y|) dy}.$$

Proof. Let $m_r(dy)$ be the normalized Lebesgue measure on $B(0, r)$. Thus, $m_r(dy) = dy/(c_1 r^d)$, where c_1 is the volume of the unit ball. Consider $Gm_r = \sup_{x \in B(0,r)} G m_r(x)$. By Lemma 5.1, the supremum is attained at $x = 0$, and so

$$Gm_r = \frac{1}{c_1 r^d} \int_{B(0,r)} G(|y|) dy$$

Therefore from Lemma 5.2

$$\text{Cap}(\overline{B(0,r)}) \geq \frac{c_1 r^d}{\int_{B(0,r)} G(|y|) dy} \quad (5.3)$$

For the right-hand side of (5.2), it follows from Lemma 5.1 and Lemma 5.2 that

$$\text{Cap}(\overline{B(0,r)}) \leq \frac{1}{Gm_r(z)} = \frac{c_1 r^d}{\int_{B(0,r)} G(z, y) dy} \leq \frac{c_1 r^d}{C_1 \int_{B(0,r)} G(|y|) dy},$$

where in the first line, $z \in \partial B(0, r)$. \square

In the remaining part of this section we assume in addition that G is regularly varying at 0 with index $\beta < 0$. This implies that

$$\lim_{u \rightarrow 0} \frac{G(2u)}{G(u)} = 2^\beta.$$

Therefore, there exists a constant r_0 such that

$$\frac{1}{2}(2^\beta + 1) G(u) \geq G(2u), \quad 0 < u < r_0. \quad (5.4)$$

Proposition 5.4. *There exists a positive constants C_4 such that for all $r \in (0, r_0/2)$*

$$\begin{aligned} C_4 \int_{B(0,r/6)} G(|y|) dy &\leq \inf_{x \in B(0,r/6)} \mathbb{E}_x \tau_{B(0,r)} \\ &\leq \sup_{x \in B(0,r)} \mathbb{E}_x \tau_{B(0,r)} \leq \int_{B(0,r)} G(|y|) dy. \end{aligned} \quad (5.5)$$

Proof. Let $G_{B(0,r)}(x, y)$ denote the Green function of the process X killed upon exiting $B(0, r)$. Clearly, $G_{B(0,r)}(x, y) \leq G(x, y)$, for $x, y \in B(0, r)$. Therefore,

$$\begin{aligned} \mathbb{E}_x \tau_{B(0,r)} &= \int_{B(0,r)} G_{B(0,r)}(x, y) dy \\ &\leq \int_{B(0,r)} G(x, y) dy \leq \int_{B(0,r)} G(|y|) dy. \end{aligned}$$

For the left-hand side inequality, let $r \in (0, r_0/2)$, and let $x, y \in B(0, r/6)$. Then,

$$\begin{aligned} G_{B(0,r)}(x, y) &= G(x, y) - \mathbb{E}_x G(X(\tau_{B(0,r)}), y) \\ &\geq G(|y - x|) - G(2|y - x|). \end{aligned}$$

The last inequality follows because $|y - X(\tau_{B(0,r)})| \geq \frac{2}{3}r \geq 2|y - x|$. Let $c_1 = (1 - 2^\beta)/2$. By (5.4) we have that for all $u \in (0, r_0)$, $G(u) - G(2u) \geq c_1 G(u)$. Hence, $G(|y - x|) - G(2|y - x|) \geq c_1 G(|y - x|)$, which implies that $G_{B(0,r)}(x, y) \geq c_1 G(x, y)$ for all $x, y \in B(0, r/6)$. Now, for $x \in B(0, r/6)$,

$$\begin{aligned} \mathbb{E}_x \tau_{B(0,r)} &= \int_{B(0,r)} G_{B(0,r)}(x, y) dy \\ &\geq \int_{B(0,r/6)} G_{B(0,r)}(x, y) dy \\ &\geq c_1 \int_{B(0,r/6)} G(x, y) dy \\ &\geq c_1 C_1 \int_{B(0,r/6)} G(|y|) dy, \end{aligned}$$

where the last inequality follows from Lemma 5.1. \square

Example 5.5. We illustrate the last two propositions for the process $X^{(n)}$ studied in Section 4. Hence, we assume that $d > 2(\alpha/2)^n$. By a slight abuse of notation we define a function $G^{(n)} : [0, \infty) \rightarrow (0, \infty]$ by $G^{(n)}(|x|) = G^{(n)}(x)$. Note that by Theorem 4.7, G is regularly varying at zero with index $\beta = -d$. Let r_0 be the constant from (5.4) with this β . Let us first look at the asymptotic behavior of $\int_{B(0,r)} G^{(n)}(|y|) dy$ for small r . We have

$$\begin{aligned} \int_{B(0,r)} G^{(n)}(|y|) dy &= c_d \int_0^r u^{d-1} G^{(n)}(u) du \\ &\sim \frac{c_d \Gamma(d/2)}{\alpha \pi^{d/2}} \int_0^r \frac{u^{d-1}}{u^d L_{n-1}(1/u) l_n(1/u)^2} du \\ &= \frac{c_d \Gamma(d/2)}{\alpha \pi^{d/2}} \int_0^r \frac{1}{u L_{n-1}(1/u) l_n(1/u)^2} du \\ &= \frac{c_d \Gamma(d/2)}{\alpha \pi^{d/2}} \frac{1}{l_n(1/r)}, \quad r \rightarrow 0. \end{aligned}$$

It follows from Proposition 5.3 that there exist positive constants $C_5 \leq C_6$ such that for all $r \in (0, 1/e_n)$,

$$C_5 r^d l_n(1/r) \leq \text{Cap}(\overline{B(0, r)}) \leq C_6 r^d l_n(1/r).$$

Similarly, it follows from Proposition 5.4 that there exist positive constants $C_7 \leq C_8$ such that for all $r \in (0, (1/e_n) \wedge (r_0/2))$,

$$\frac{C_7}{l_n(1/r)} \leq \inf_{x \in B(0,r/6)} \mathbb{E}_x \tau_{B(0,r)} \leq \sup_{x \in B(0,r)} \mathbb{E}_x \tau_{B(0,r)} \leq \frac{C_8}{l_n(1/r)}. \quad (5.6)$$

Here we also used the fact that l_n is slowly varying.

By use of Remark 4.8 and Proposition 5.3, we can estimate capacity of large balls. It easily follows that as $r \rightarrow \infty$, $\text{Cap}(\overline{B(0, r)})$ is of the order $r^{\alpha(\alpha/2)^{n-1}}$.

6. Harnack inequality

In this section we indicate the main steps in the proof of Harnack inequality for nonnegative harmonic functions for the subordinate process. We do not provide all of the details, because they have already appeared in some other papers. The methodology was introduced in [2] and refined in [1]. We are going to use the notation and the approach from [20], combined with some results and ideas from [17] and [22].

Let $S^{(n)}$ be a subordinator whose Laplace exponent $\phi^{(n)}$ is defined in Section 4. For the case $\alpha = 2$ we assume that $n = 1$. Let Y be a d -dimensional Brownian motion independent of $S^{(n)}$, and let $X^{(n)}(t) = Y(S_t^{(n)})$. As in Section 4, we assume that $d > 2(\alpha/2)^n$. A nonnegative Borel function h on \mathbb{R}^d is said to be harmonic with respect to $X^{(n)}$ in a domain (i. e., a connected open set) $D \subseteq \mathbb{R}^d$ if it is not identically infinite in D and if for any bounded open subset $B \subset \overline{B} \subset D$,

$$h(x) = \mathbb{E}_x[h(X^{(n)}(\tau_B))], \quad \forall x \in B,$$

where $\tau_B = \inf\{t > 0 : X_t^{(n)} \notin B\}$ is the first exit time of B .

We say that the Harnack inequality holds for $X^{(n)}$ if for any domain $D \subset \mathbb{R}^d$ and any compact subset K of D , there is a constant $C > 0$ depending only on D and K such that for any nonnegative function h harmonic with respect to $X^{(n)}$ in D ,

$$\sup_{x \in K} h(x) \leq C \inf_{x \in K} h(x).$$

The following auxiliary results are needed for the proof of Harnack inequality. Let r_0 be the constant from Example 5.5.

Lemma 6.1. *There exists a positive constant C_1 such that for any $r \in (0, (1/e_n) \wedge (r_0/2))$ we have*

$$\sup_{z \in B(0, r)} \mathbb{E}_z \tau_{B(0, r)} \leq C_1 \inf_{z \in B(0, r/6)} \mathbb{E}_z \tau_{B(0, r)}.$$

Proof. This follows immediately from the estimate (5.6). \square

Note that it follows from Theorem 4.7 that there exist two positive constants C_2 and C_3 such that

$$\begin{aligned} \frac{C_2}{|x|^d L_{n-1}(1/|x|^2) l_n(1/|x|^2)^2} &\leq G^{(n)}(x) \\ &\leq \frac{C_3}{|x|^d L_{n-1}(1/|x|^2) l_n(1/|x|^2)^2}, \quad |x| < \frac{1}{e_{n+1}} \end{aligned} \quad (6.1)$$

Let us define

$$g^{(n)}(u) := \frac{1}{u^d L_{n-1}(1/u^2) l_n(1/u^2)^2}, \quad u < \frac{1}{e_{n+1}}.$$

It follows by calculus that there exists ρ_n , $0 < \rho_n < 1/e_{n+1}$, such that $u \rightarrow g^{(n)}(u)$ is decreasing on $(0, \rho_n)$. Define

$$c := \max \left\{ \frac{1}{3} \left(\frac{4C_3}{C_2} \right)^{1/d}, 1 \right\}. \quad (6.2)$$

Since $u \rightarrow L_{n-1}(1/u^2) l_n(1/u^2)^2$ is slowly varying as $u \rightarrow 0$, there exists $\tilde{\rho}_n$, $0 < \tilde{\rho}_n < 1/e_{n+1}$, such that

$$\frac{1}{2} \leq \frac{L_{n-1}(\frac{1}{u^2}) l_n(\frac{1}{u^2})^2}{L_{n-1}(\frac{1}{36c^2 u^2}) l_n(\frac{1}{36c^2 u^2})^2} \leq 2, \quad u < \tilde{\rho}_n. \quad (6.3)$$

Let

$$R_n := \min \left\{ \frac{1}{e_{n+1}}, \rho_n, \tilde{\rho}_n, \frac{r_0}{2} \right\}. \quad (6.4)$$

Then $u \rightarrow g^{(n)}(u)$ is decreasing on $(0, R_n)$, and both (6.1) and (6.3) are valid for $|x| < R_n$ and $u < R_n$ respectively.

Lemma 6.2. *Let $\beta \in (0, 1)$. There exists $C_4 > 0$ such that for any $r \in (0, (7c)^{-1} R_n)$, any closed subset A of $B(0, r)$, and any $y \in B(0, r)$*

$$\mathbb{P}_y(T_A < \tau_{B(0, 7cr)}) \geq C_4 r^\beta \frac{\text{Cap}(A)}{\text{Cap}(B(0, r))}.$$

Proof. Without loss of generality we may assume that $\text{Cap}(A) > 0$. Let $G_{B(0, 7cr)}^{(n)}$ be the Green function of the process obtained by killing $X^{(n)}$ upon exiting from $B(0, 7cr)$. If ν is the capacitary measure of A with respect to $X^{(n)}$, then we have for all $y \in B(0, r)$,

$$\begin{aligned} G_{B(0, 7cr)}^{(n)} \nu(y) &= \mathbb{E}_y[G_{B(0, 7cr)}^{(n)} \nu(X_{T_A}^{(n)}) : T_A < \tau_{B(0, 7cr)}] \\ &\leq \sup_{z \in \mathbb{R}^d} G_{B(0, 7cr)}^{(n)} \nu(z) \mathbb{P}_y(T_A < \tau_{B(0, 7cr)}) \\ &\leq \mathbb{P}_y(T_A < \tau_{B(0, 7cr)}). \end{aligned}$$

On the other hand we have for all $y \in B(0, r)$,

$$\begin{aligned} G_{B(0, 7cr)}^{(n)} \nu(y) &= \int G_{B(0, 7cr)}^{(n)}(y, z) \nu(dz) \geq \nu(A) \inf_{z \in B(0, r)} G_{B(0, 7cr)}^{(n)}(y, z) \\ &= \text{Cap}(A) \inf_{z \in B(0, r)} G_{B(0, 7cr)}^{(n)}(y, z). \end{aligned}$$

In order to estimate the infimum in the last display, note that $G_{B(0,7cr)}^{(n)}(y, z) = G^{(n)}(y, z) - \mathbb{E}_y[G^{(n)}(X_{\tau_{B(0,7cr)}^{(n)}}^{(n)}, z)]$. Since $|y - z| < 2r < R_n$, it follows by (6.1) and the monotonicity of $g^{(n)}$ that

$$G^{(n)}(y, z) \geq C_2 g^{(n)}(|z - y|) \geq C_2 g^{(n)}(2r). \quad (6.5)$$

Now we consider $G^{(n)}(X_{\tau_{B(0,7cr)}^{(n)}}^{(n)}, z)$. First note that $|X_{\tau_{B(0,7cr)}^{(n)}}^{(n)} - z| \geq 7cr - r \geq 6cr$. If $|X_{\tau_{B(0,7cr)}^{(n)}}^{(n)} - z| \leq R_n$, then by (6.1) and the monotonicity of $g^{(n)}$

$$G^{(n)}(X_{\tau_{B(0,7cr)}^{(n)}}^{(n)}, z) \leq C_3 g^{(n)}(|z - X_{\tau_{B(0,7cr)}^{(n)}}^{(n)}|) \leq C_3 g^{(n)}(6cr).$$

If, on the other hand, $|X_{\tau_{B(0,7cr)}^{(n)}}^{(n)} - z| \geq R_n$, then $G^{(n)}(X_{\tau_{B(0,7cr)}^{(n)}}^{(n)}, z) \leq G^{(n)}(w)$, where $w \in \mathbb{R}^d$ is any point such that $|w| = R_n$. Here we have used the monotonicity of $G^{(n)}$. For $|w| = R_n$ we have that $G^{(n)}(w) \leq C_3 g^{(n)}(|w|) = C_3 g^{(n)}(R_n) \leq C_3 g^{(n)}(6cr)$. Therefore

$$\mathbb{E}_y[G^{(n)}(X_{\tau_{B(0,7cr)}^{(n)}}^{(n)}, z)] \leq C_3 g^{(n)}(6cr). \quad (6.6)$$

By use of (6.5) and (6.6) we obtain

$$\begin{aligned} G^{(n)}(y, z) &\geq C_2 g^{(n)}(2r) - g^{(n)}(6cr) \\ &= g^{(n)}(2r) \left(C_2 - C_3 \frac{(2r)^d L_{n-1}(1/4r^2) l_n(1/4r^2)^2}{(6cr)^d L_{n-1}(1/36c^2r^2) l_n(1/36c^2r^2)^2} \right) \\ &= g^{(n)}(2r) \left(C_2 - C_3 \left(\frac{1}{3c} \right)^d \frac{L_{n-1}(1/4r^2) l_n(1/4r^2)^2}{L_{n-1}(1/36c^2r^2) l_n(1/36c^2r^2)^2} \right) \\ &\geq g^{(n)}(2r) \left(C_2 - C_3 \left(\frac{1}{3c} \right)^d 2 \right) \\ &\geq g^{(n)}(2r) \left(C_2 - 2C_3 \frac{C_2}{4C_3} \right) \\ &= \frac{C_2}{2} g^{(n)}(2r) = \frac{C_2}{2} \frac{1}{4r^d L_{n-1}(1/4r^2) l_n(1/4r^2)^2}, \end{aligned}$$

where in the fourth line we used (6.3). From Example 5.5 we have that $\text{Cap}(B(0, r)) \geq c_5 r^d / l_n(1/r)$. By using this in the previous display, we get

$$\begin{aligned} G^{(n)}(y, z) &\geq \frac{C_2}{8} \frac{1}{L_{n-1}(1/4r^2) l_n(1/4r^2)^2} \frac{c_5}{l_n(1/r)} \frac{1}{\text{Cap}(B(0, r))} \\ &= \frac{C_2 c_5}{8} \frac{1}{L_{n-1}(1/4r^2) l_n(1/4r^2)^2 l_n(1/r)} \frac{1}{\text{Cap}(B(0, r))} \\ &\geq C_4 r^\beta \frac{1}{\text{Cap}(B(0, r))} \end{aligned}$$

To finish the proof, note that

$$\mathbb{P}_y(T_A < \tau_{B(0,7cr)}) \geq G_{B(0,7cr)}^{(n)}(y) \geq C_4 r^\beta \frac{\text{Cap}(A)}{\text{Cap}(B(0, r))}.$$

□

Remark 6.3. It is clear from the proof that the function $r \rightarrow r^\beta$ can be replaced by a function which approaches zero more slowly.

Using Proposition 4.9 and Lemma 3.5 of [20] we immediately get the following result.

Lemma 6.4. *There exist positive constants C_5 and C_6 such that if $r \in (0, r_0/2)$, $z \in B(0, r)$ and H is a nonnegative function with support in $B(0, 2r)^c$, then*

$$\mathbb{E}_z H(X^{(n)}(\tau_{B(0,r)})) \leq C_5 (\mathbb{E}_z \tau_{B(0,r)}) \int H(y) J^{(n)}(y) dy$$

and

$$\mathbb{E}_z H(X^{(n)}(\tau_{B(0,r)})) \geq C_6 (\mathbb{E}_z \tau_{B(0,r)}) \int H(y) J^{(n)}(y) dy.$$

It follows from Lemma 6.1 and Lemma 6.4 that there exists a positive constant C_7 such that for any $r \in (0, R_n)$, any $y, z \in B(0, r/2)$ and any nonnegative function H supported in $B(0, 2r)^c$

$$\mathbb{E}_z H(X^{(n)}(\tau_{B(0,r)})) \leq C_7 \mathbb{E}_y H(X^{(n)}(\tau_{B(0,r)})). \quad (6.7)$$

Lemma 6.5. *Let $\beta \in (0, 1)$. There exists a positive constant C_8 such that for all $0 < \rho < r < 1/e_{n+1}$*

$$\frac{\text{Cap}(B(0, \rho))}{\text{Cap}(B(0, r))} \geq C_8 \left(\frac{\rho}{r}\right)^d \rho^\beta.$$

Proof. By Example 5.5

$$\frac{c_5 r^d}{l_n(1/r)} \leq \text{Cap}(B(0, r)) \leq \frac{c_6 r^d}{l_n(1/r)}$$

for every $r < 1/e_{n+1}$. Therefore,

$$\frac{\text{Cap}(B(0, \rho))}{\text{Cap}(B(0, r))} \geq \frac{c_5 \rho^d (l_n(1/\rho))}{c_6 r^d / l_n(1/r)} = \frac{c_5}{c_6} \left(\frac{\rho}{r}\right)^d \frac{l_n(1/r)}{l_n(1/\rho)}.$$

Note that $1/r > e_{n+1}$ and hence $l_n(1/r) > l_n(e_{n+1}) = 1$. Further, there exists a constant $c_7 > 0$ such that

$$\frac{1}{l_n(1/\rho)} \geq c_7 \rho^\beta \quad \text{for all } \rho \in (0, 1/e_{n+1}).$$

The lemma is proved by taking $C_8 = c_5 c_7 / c_6$. □

The following Harnack inequality is proved along the same lines as the ones in Theorem 3.1 in [22] and Theorem 4.5 in [17]. We omit the details.

Theorem 6.6. *Let R_n and c be defined by (6.4) and (6.2) respectively. Let $r \in (0, (14c)^{-1}R_n)$. There exists a constant $C_9 > 0$ such that for every $z_0 \in \mathbb{R}^d$ and every nonnegative bounded function u in \mathbb{R}^d which is harmonic with respect to $X^{(n)}$ in $B(z_0, 14cr)$ we have*

$$h(x) \leq C_9 h(y), \quad x, y \in B(z_0, r/2).$$

Following the well-known arguments, this theorem can be improved to

Theorem 6.7. *For any domain D of \mathbb{R}^d and any compact subset K of D , there exists a constant $C_{10} > 0$ such that for any function h which is nonnegative in \mathbb{R}^d and harmonic with respect to $X^{(n)}$ in D , we have*

$$h(x) \leq C_{10} h(y), \quad x, y \in K.$$

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References

1. Bass, R.F., Kassmann, M.: Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.* **357**, 837–850 (2005)
2. Bass, R.F., Levin, D.A.: Harnack inequalities for jump processes. *Potential Analysis* **17**, 375–388 (2002)
3. Bertoin, J.: *Lévy Processes*. Cambridge University Press, 1996
4. Bingham, N.H., Goldie, C.M., Teugels, J.L.: *Regular variation*. Cambridge University Press, Cambridge, 1987
5. Blumenthal, R.M., Gettoor, R.K.: Some theorems on stable processes. *Trans. Amer. Math. Soc.* **95**, 263–273 (1960)
6. Chung, K.L.: *Lectures from Markov Processes to Brownian motion*. Springer, New York, 1982
7. Fuglede, B.: On the theory of potentials in locally compact spaces. *Acta Math* **103**, 139–215 (1960)
8. Fukushima, M., Oshima, Y., Takeda, M.: *Dirichlet forms and symmetric Markov processes*. Walter De Gruyter, Berlin, 1994
9. Geman, H., Madan, D.B., Yor, M.: Time changes for Lévy processes. *Math. Finance* **11**, 79–96 (2001)
10. Jacob, N.: *Pseudo differential operators and markov processes*. Vol. **1**, Imperial College Press, London, 2001
11. Jayakumar, K., Pillai, R.N.: Characterization of Mittag-Leffler Distribution. *J. Appl. Statist. Sci.* **4**, 77–82 (1996)
12. Klebanov, L.B., Maniya, G.M., Melamed, I.A.: A problem of V. M. Zolotarev and analogues of infinitely divisible and stable distributions in a scheme for summation of a random number of random variables. *Theory Probab. Appl.* **29**, 791–794 (1984)
13. Kozubowski, T.J., Rachev, S.T.: Univariate geometric stable laws. *J. Comput. Anal. Appl.* **1**, 177–217 (1999)
14. Kunugui, K.: Étude sur la théorie du potentiel généralisé. *Osaka Math. J.* **2**, 63–103 (1950)
15. Madan, D.B., Carr, P., Chang, E.: The variance gamma process and option pricing. *European Finance Review* **2**, 79–105 (1998)

16. Rachev, S.T., Mittnik, S.: Stable paretian models in finance. Wiley, New York, 2000
17. Rao, M., Song, R., Vondraček, Z.: Green function estimates and Harnack inequality for subordinate Brownian motions. To appear in *Potential Analysis*, 2005
18. Sato, K.-I.: Lévy processes and infinitely divisible distributions. Cambridge University Press, 1999
19. Skorohod, A.V.: Asymptotic formulas for stable distribution laws. *Select. Transl. Math. Statist. and Probability Vol. 1*, 157–161 Inst. Math. Statist. and Amer. Math. Soc., Providence, R.I, 1961
20. Song, R., Vondraček, Z.: Harnack inequalities for some classes of Markov processes. *Math. Z.* **246**, 177–202 (2004)
21. Song, R., Vondraček, Z.: Potential theory of special subordinators and subordinate killed stable processes. To appear in *J. Theor. Prob.*, 2005
22. Song, R., Vondraček, Z.: Harnack inequality for some discontinuous Markov processes with a diffusion part. *Glasnik Matematički* **40**, 177–187 (2005)