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The largest eigenvalue of small rank perturbations of Hermitian random matrices

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Abstract. We compute the limiting eigenvalue statistics at the edge of the spectrum of large Hermitian random matrices perturbed by the addition of small rank deterministic matrices. We consider random Hermitian matrices with independent Gaussian entries $M_{ij}, i \leq j$ with various expectations. We prove that the largest eigenvalue of such random matrices exhibits, in the large N limit, various limiting distributions depending on both the eigenvalues of the matrix $\left(\mathbb{E}M_{ij}\right)_{i,j=1}^{N}$ and its rank. This rank is also allowed to increase with N in some restricted way.

1. Introduction and results

The aim of this paper is to investigate how a small rank perturbation of a standard $N \times N$ random matrix can affect significatively the limiting properties of the spectrum, as the size N of the matrix goes to infinity. The statistics of extreme eigenvalues is here of interest. Note that it is not clear what is meant by "a small rank perturbation of a random matrix" and we shall define it formally in the sequel. Actually, a first study of eigenvalue statistics for such perturbed random matrices has been achieved in [1]. Therein the authors consider non homogeneous Wishart random matrices $R_N = 1/NXX^*$, where X is a $p \times N$ random matrix with independent complex Gaussian entries with a *spiked* covariance matrix Σ. That is, $\Sigma - Id$ (Id is the identity matrix) is a fixed rank (independent of N) diagonal matrix, while both p and N go to infinity.
In this paper, we consider Hermitian random matrices. Let μ (resp. μ') be a probability distribution on $\mathbb{C}$ (resp. $\mathbb{R}$). A $N \times N$ random Hermitian matrix $M(\mu, \mu')$ is then a Hermitian matrix with entries being mutually independent random variables of distribution μ (resp μ') strictly above the diagonal (resp. on the diagonal). Define then $M_N(\mu, \mu') = \frac{1}{\sqrt{N}} M(\mu, \mu')$. Let also $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the ordered eigenvalues of M_N and $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ its spectral measure. A famous result of Wigner ([14]) asserts that μ_N admits a non-random limit as N goes to infinity.

Proposition 1.1. *[14] Assume that* $\int x d\mu(x) = \int x d\mu'(x) = 0$, *and that* $\int |x|^2 d\mu(x) = \sigma^2$, $\int |x|^2 d\mu'(x) < \infty$. *Then, almost surely,* $\lim_{N\to\infty} \mu_N = \tilde{\rho}_\sigma$,

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where $\tilde{\rho}_\sigma$ is the semi-circular law with parameter σ^2, defined by the density with respect to Lebesgue measure

$$\rho_\sigma(x) = \frac{2}{\pi\sigma^2}\sqrt{4\sigma^2 - x^2}1_{[-2\sigma,2\sigma]}(x). \tag{1}$$

Let $\lambda^* = 2\sigma$ be the top edge of the support of $\tilde{\rho}_\sigma$. It is then a fundamental result of [6] that, for the archetypical of Hermitian ensemble, the so-called GUE, $\lim_{N\to\infty} \lambda_1 = \lambda^*$.

Definition 1.1. *The $N \times N$ GUE with parameter σ^2 is the distribution of a $N \times N$ random matrix $M(\mu, \mu')$, if μ (resp. μ') is the centered complex (resp. real) Gaussian distribution of variance σ^2.*

The result obtained in [6] has later been precised in [13]. Consider the Airy function defined by $Ai(u) = \frac{1}{2\pi}\int_{\infty e^{i5\pi/6}}^{\infty e^{i\pi/6}} \exp\{iua + \frac{1}{3}a^3\}da$, and define the Airy kernel

$$Ai(u, v) = \int_0^\infty Ai(y+u)Ai(y+v)dy. \tag{2}$$

Definition 1.2. *The Tracy-Widom distribution is defined by the distribution function $F_2^{TW}(x) := \det(I - A_x)$, where A_x is the trace class operator acting on $L^2(x, \infty)$ with kernel $Ai(u, v)$.*

Proposition 1.2. *[13] Let λ_1 be the largest eigenvalue of $V_N = \frac{1}{\sqrt{N}}V$, where V is drawn from the GUE with parameter σ^2. Then, $\lim_{N\to\infty} P\left(\sigma^{-1}N^{2/3}\left(\lambda_1 - \lambda^*\right) \leq x\right)$ $= F_2^{TW}(x)$.*

Remark 1.1. It is shown in [12] that the above result actually holds for a wide class of random matrices $M_N(\mu, \mu')$ with centered distributions μ, μ'.

The scope of this paper is to define a suitable "small" rank perturbation of a random matrix V_N drawn from the GUE, so that the largest eigenvalue separates from "the bulk", $[-\lambda^*, \lambda^*]$, and study in this case, how it interacts with the "bulk" of eigenvalues in $[-\lambda^*, \lambda^*]$. Due to the rotational invariance of the Gaussian distribution, it is enough to consider diagonal perturbations.

1.1. The model

The model studied here is known in random matrix litterature as the deformed Wigner ensemble. The first study of such an ensemble goes back to [3] and [9].

Definition 1.3. *Given $k \in \mathbb{N}$, $r \in \mathbb{N}$ and ordered real numbers $\pi_1 > \pi_2 \geq \cdots \geq \pi_{r+1}$, a* deformed Wigner matrix *is a $N \times N$ random matrix $M_N = W_N + \frac{1}{\sqrt{N}}V$ where V is of the $N \times N$ GUE with parameter* 1 *and W_N is the diagonal matrix $W_N = diag(\pi_1, \ldots, \pi_1, \pi_2, \ldots, \pi_{r+1}, 0, \ldots, 0)$, with rank $k + r$, and where the largest eigenvalue π_1 has multiplicity k.*

Remark 1.2. We assume that $\pi_i = 0$, $\forall i \geq 2$ if $r = 0$. The $\pi_i, i = 1, \dots, r+1$ can be negative but lie in a compact set independent of N.

In this paper, we consider matrices W_N with rank $k + r$ such that

$$\lim_{N\to\infty} \frac{k+r}{N} = 0. \tag{3}$$

In particular, k and r may depend on N . Noting $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ the ordered eigenvalues of M_N and μ_N its spectral measure, condition (3) ensures that $\lim_{N\to\infty} \mu_N = \tilde{\rho}_1$, where $\tilde{\rho}_1$ is the semi-circle law defined in (1), with parameter $\sigma^2 = 1$.

1.2. Results

First, we fix the rank of W_N independently of N and identify the critical scale $\pi_1 = \pi_1^c$ for which λ_1 separates from the bulk. Results in this part are similar to those in [1]. Then, and this is the main result of the paper, we study the limiting properties of largest eigenvalues when the rank of W_N is allowed to increase with N, focusing on the case where λ_1 is separated from the bulk.

1.2.1. A fixed rank perturbation

We consider matrices W_N with fixed rank $k + r$, independent of N.

Assumption 1.1. $W_N = diag(\pi_1, \dots, \pi_1, \pi_2, \dots, \pi_r, 0, \dots, 0)$, *with* π_1 *of multiplicity* k, *such that*

- k *and* r *are given integers* independent of N,
- π_1 *is a given real number independent of* N,
- $\pi_i, i = 2, \dots, r+1$ *lie in a compact set of* $(-\infty, \pi_1)$ independent of N.

Before stating the results, we need a few definitions.

Given an integer $m \geq 1$, and a contour $\mathcal{C}$ going from $\infty e^{5i\pi/6}$ to $\infty e^{i\pi/6}$, with 0 lying above $\mathcal{C}$, we set

$$\begin{aligned} t^{(m)}(v) &= \frac{1}{2\pi} \int_{\mathcal{C}} \exp\{iua + \frac{1}{3}a^3 i\}(-ia)^{m-1} da, \\ s^{(m)}(u) &= \frac{1}{2\pi} \int_{\mathcal{C}} \exp\{iua + \frac{1}{3}a^3 i\} \frac{1}{(ia)^m} da. \end{aligned} \tag{4}$$

Given $x \in \mathbb{R}$, let also A_x be the operator acting on $L^2(x, \infty)$ with kernel $Ai(u, v)$ defined in (2), and $<\ ,\ >$ denote the standard scalar product of operators on $L^2(x, \infty)$.

Definition 1.4. *Given an integer* $k \geq 0$, F_{k+2}^{TW} *is the distribution function defined by*

$$F_{k+2}^{TW}(x) = \det(1 - A_x) \det\left(\delta_{m,n} - < \frac{1}{1-A_x} s^{(m)}, t^{(n)} >\right)_{1\leq m,n\leq k}, \quad x \in \mathbb{R}. \tag{5}$$

Remark 1.3. F_{k+2}^{TW} was proved to be distribution function in [1].

The first theorem gives a necessary condition to have $\lim_{N\to\infty} \lambda_1 = \lambda^* = 2$. Still, we prove that the limiting distribution of λ_1 depends on both the value and the multiplicity of π_1.

Theorem 1.1. *Assume Assumption 1.1 holds.*

- *If* $\pi_1 < 1$, *then,* $\lim_{N\to\infty} P\left(N^{2/3}(\lambda_1 - 2) \leq x\right) = F_2^{TW}(x)$.
- *If* $\pi_1 = 1$, *then,* $\lim_{N\to\infty} P\left(N^{2/3}(\lambda_1 - 2) \leq x\right) = F_{k+2}^{TW}(x)$.

In the next theorem, we prove that, as soon as $\pi_1 > 1$, with probability one, the largest eigenvalue λ_1 exits the support of the semi-circular law.

Definition 1.5. *Given* $k \geq 0$, *define the probability distribution*

$$F^k_{GUE,\sigma^2}(x) = \frac{1}{Z_k}\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} \prod_{1\leq i<j\leq k} |u_i - u_j|^2 \prod_{i=1}^{k} \exp\{-\frac{u_i^2}{2\sigma^2}\} du_1 \cdots du_k,$$

where Z_k *is the normalizing constant* $Z_k = \int_{\mathbb{R}^k} \prod_{1\leq i<j\leq k} |u_i - u_j|^2 \prod_{i=1}^{k} \exp\{-\frac{u_i^2}{2\sigma^2}\}$ $du_1 \cdots du_k$.

Remark 1.4. It can be shown (see e.g. [10], Chapter 5) that F^k_{GUE,σ^2} is the probability distribution of the largest eigenvalue of the $k \times k$ GUE with parameter σ^2.

Theorem 1.2. *Assume Assumption 1.1 holds with* $\pi_1 > 1$. *Then,*

$$\lim_{N\to\infty} P\left(\sigma^2(\pi_1) N^{1/2}(\lambda_1 - C(\pi_1)) \leq x\right) = F^k_{GUE,\sigma^2(\pi_1)}(x), \text{ where}$$

$$C(\pi_1) = \pi_1 + \frac{1}{\pi_1} \quad \text{and} \quad \sigma^2(\pi_1) = \frac{\pi_1^2}{\pi_1^2 - 1}. \tag{6}$$

Remark 1.5. This result should be compared with the result of [5]. Therein, the authors consider Hermitian random matrices $M_N(\mu, \mu')$, where μ, μ' are distributions with compact support such that $\int x d\mu = \int x d\mu' = m \neq 0$, $\int |x|^2 d\mu = \int |x|^2 d\mu' = \sigma^2 + m^2$. Then, for $C(\cdot)$ defined as in (6), it is proved that $\sqrt{N}\left(\lambda_1 - C(\sqrt{N}m)\right)$ has asymptotically Gaussian fluctuations $\mathcal{N}(0, \sigma^2)$. Here, we obtain that the scale at which λ_1 actually separates from the bulk (when μ, μ' are Gaussian distributions) is $m = m_N = \frac{1}{\sqrt{N}}$.

Theorem 1.2 gives the intuition that a "bulk" of k eigenvalues exits the support of the semi-circular law, provided $\pi_1 > 1$. Furthermore, these k eigenvalues seem to behave as those of a typical $k \times k$ random matrix. We now show that this still holds if k goes to infinity in some restricted way.

1.2.2. A large rank perturbation

We investigate the case where the rank of W_N is increasing with N. To our knowledge, the kind of perturbation that we now define, is new.

Let k_N, r_N be given sequences of integers such that

$$\lim_{N\to\infty} k_N = \infty, \quad \lim_{N\to\infty} \frac{k_N}{N} = 0, \text{ and } \lim_{N\to\infty} \frac{r_N}{N} = 0. \tag{7}$$

We first consider the case where $\pi_1 > 1$, so that the largest eigenvalue separates from the *bulk*.

Assumption 1.2. *$W_N =$ diag $(\pi_1, \ldots, \pi_1, \pi_2, \ldots, \pi_{r_N+1}, 0, \ldots, 0)$, with π_1 of multiplicity k_N and*

- *$(k_N)_{N\in\mathbb{N}}$ and $(r_N)_{N\in\mathbb{N}}$ satisfy (7),*

1. *$\pi_1 > 1$ is given, independent of N,*

- *$\pi_i, i = 2, \ldots, r_N + 1$ lie in a compact set of $(-\infty, \pi_1)$, independent of N.*

We first deal with local eigenvalue statistics in the "bulk" of the k_N largest eigenvalues and consider the so-called spacing function between nearest neighbor eigenvalues. Let $\rho = \rho_{\sigma^2}$ be the density of the semi-circular law (1) with parameter $\sigma^2(\pi_1)$, defined in (6). Define

$$\alpha_N = \frac{\sqrt{k_N}}{\sqrt{N}} \text{ and } \beta_N = \frac{r_N}{N}. \tag{8}$$

Let t_N be a sequence such that $\lim_{N\to\infty} t_N = \infty$, $\lim_{N\to\infty} \frac{t_N}{k_N} = 0$.

Definition 1.6. *Given $|\alpha| < 2\sigma(\pi_1)$, and for $u = C(\pi_1) + \alpha_N \frac{\alpha}{\sigma^2(\pi_1)} + \frac{\beta_N}{\pi_1} - \frac{1}{N}\sum_{i=1}^{N\beta_N} \frac{1}{\pi_1 - \pi_{i+1}}$, the "spacing function", $S_N(\alpha, s, \lambda)$, is the symmetric function which, if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$, equals*

$$S_N(\alpha, s, \lambda) = \frac{1}{2t_N}\sharp\left\{1 \leq j \leq N-1; \quad \lambda_j - \lambda_{j+1} \leq \frac{\alpha_N s}{\sigma^2 k_N \rho(\alpha)}, \right.$$
$$\left. |\lambda_j - u| \leq \frac{\alpha_N t_N}{k_N \rho(\alpha)\sigma^2}\right\}.$$

Here α is to be seen as a point in the "bulk" of (1). Then, we obtain the following result.

Theorem 1.3. *Assume Assumption 1.2 holds. Then, $\lim_{N\to\infty} \left| E S_N(\alpha, s, \lambda) - \int_0^s H''(u)du \right| = 0$, where*

$$H(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{[0,s]^m} det\Big(\frac{\sin\pi(x_i - x_j)}{\pi(x_i - x_j)}\Big)_{i,j=1}^m \prod_{i=1}^m dx_i.$$

Remark 1.6. The above theorem states that the archetypical repulsion of eigenvalues of Hermitian random matrices is exhibited amongst the k_N largest eigenvalues, in the large N limit.

Remark 1.7. The case $\pi_1 \leq 1$ has already been studied in [9] and [11] (Appendix A), showing a similar repulsion of eigenvalues (up to changes in the rescalings).

We then turn to local eigenvalue statistics at the edge.

Let α_N, β_N be given as in (8), and log be the principal branch of the logarithm. Set, for $w \in \mathbb{C} \setminus (-\infty, \pi_1]$,

$$F_u(w) := w^2/2 - uw + (1 - \alpha_N^2 - \beta_N)\log w + \alpha_N^2 \log(w - \pi_1) + \frac{1}{N}\sum_{i=1}^{N\beta_N} \log(w - \pi_{i+1}), \tag{9}$$

so that

$$F_u'(w) = w - u + \frac{1 - \alpha_N^2 - \beta_N}{w} + \frac{\alpha_N^2}{w - \pi_1} + \frac{1}{N}\sum_{i=1}^{N\beta_N} \frac{1}{w - \pi_{i+1}}, \tag{10}$$

$$F_u''(w) = 1 - \frac{1 - \alpha_N^2 - \beta_N}{w^2} - \frac{\alpha_N^2}{(w - \pi_1)^2} - \frac{1}{N}\sum_{i=1}^{N\beta_N} \frac{1}{(w - \pi_{i+1})^2}. \tag{11}$$

Note that F_u'' does not depend on u. We then define w_o as follows.

$$w_o \text{ is the largest solution of the equation } F_u''(w) = 0. \tag{12}$$

In particular, it can be shown that $w_o > \pi_1$. Finally define u_o and t_r by

$$F_{u_o}'(w_o) = 0, \quad t_r = \frac{w_o - \pi_1}{\alpha_N}, \tag{13}$$

where F_u' and w_o are respectively given by (10) and (12).

Theorem 1.4. *Assume Assumption 1.2 holds and let u_o and t_r be given by (13). Then,*

$$\lim_{N\to\infty} P\left(t_r \frac{k_N^{2/3}}{\alpha_N}(\lambda_1 - u_o) \leq x\right) = F_2^{TW}(x).$$

Remark 1.8. The above theorem states that, as long as $\alpha_N \to 0$, the suitably scaled largest eigenvalue of the deformed Wigner ensemble also behaves as the largest eigenvalue of a $k_N \times k_N$ GUE. The rescaling is such that $t_r \frac{k_N^{2/3}}{\alpha_N} = N^{2/3} \left(\frac{F_{u_o}^{(3)}(w_o)}{2}\right)^{-1/3} (1+o(1))$ and, if $r_N = N\beta_N = 0$, $u_o = C(\pi_1) + \alpha_N \frac{2}{\sigma(\pi_1)} + O(\alpha_N^2)$.

Remark 1.9. The case $\lim_{N\to\infty} \frac{k_N}{N} = \alpha \in (0, 1)$ will be the object of a subsequent paper, and is not examined here. In this context, the limiting statistics of extreme eigenvalues are determined in [2], when $W_N = \text{diag}\,(a, \dots, a, -a, \dots, -a)$, where numbers of a and $-a$ are both approximately $N/2$.

The proof of Theorem 1.4 is based on an extension of the method developed in [1] and may bring some new tools for the study of such deformed models. In particular, we can also consider the case where $\pi_1 \leq 1$.

Then , we obtain the following result.

Assumption 1.3. *$W_N =$ diag $(\pi_1, \dots, \pi_1, \pi_2, \dots, \pi_{r_N+1}, 0, \dots, 0)$, with π_1 of multiplicity k_N and*

- *$(k_N)_{N\in\mathbb{N}}$, and $(r_N)_{N\in\mathbb{N}}$, satisfy (7),*
- *$\pi_i, i = 2, \dots, r+1$ lie in a compact set of $(-\infty, \pi_1)$, independent of N,*
- *$\pi_1 \leq 1$ is given, independent of N.*

Define F_{u_o} as in (10). Let then w_o (greater than 1 here) be given as in (12) and u_o as in (13).

Theorem 1.5. *Assume Assumption 1.3 holds. Then,*

$$\lim_{N\to\infty} P\left(N^{2/3}\left(\frac{F_{u_o}^{(3)}(w_o)}{2}\right)^{-1/3}(\lambda_1 - u_o) \leq x\right) = F_2^{TW}(x).$$

Remark 1.10. If $\pi_1 < 1$ and $r_N = 0$, Theorem 1.5 proves in particular that λ_1 exhibits the archetypical behavior of the largest eigenvalue of a $N \times N$ GUE with parameter 1, as long as $k_N << N^{1/3}$. Otherwise λ_1 is slightly translated.

Remark 1.11. The proofs of Theorem 1.4 and Theorem 1.5 are very similar and the second one will only be sketched.

1.3. Sketch of the proof

Basically, the idea is to deduce the large N limit of local eigenvalue statistics of the deformed Wigner ensemble from the asymptotics of the so-called "m point correlation functions", defined as follows.

Let P_N be the joint eigenvalue distribution on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ induced by the deformed Wigner ensemble. It is known that P_N admits a density with respect to Lebesgue measure. We denote this density g. Then, given an integer $m \leq N$, the m-point correlation function, $R_N^m(\cdot)$, induced by P_N is defined by $R_N^m(x_1, \dots, x_m) = \frac{N!}{(N-m)!}\int_{\mathbb{R}^m} g(x_1, \dots, x_N) \prod_{i=m+1}^{N} dx_i$. We refer to [9], Section 4) for the use of correlation functions in the study of local eigenvalue statistics.

It happens that, for the deformed Wigner ensemble, the computation of the asymptotics of correlation functions is quite simple. This follows from beautiful results of [9], [3], [8], [7].

Proposition 1.3. *[9]The m-point correlation function of the deformed Wigner ensemble is given by* $R_N^m(x_1, \ldots, x_m) = \det\Big(K_N(x_i, x_j)\Big)_{i,j=1}^m$, *with the correlation kernel* K_N *defined by*

$$K_N(u, v) = \frac{N}{(2i\pi)^2} \int_\Gamma dz \int_\gamma dw e^{N\{\frac{w^2}{2} - vw - \frac{z^2}{2} + uz\}} \left(\frac{w}{z}\right)^{N-k-r} \times \left(\frac{w - \pi_1}{z - \pi_1}\right)^k \prod_{i=2}^{r+1} \frac{w - \pi_i}{z - \pi_i} \frac{1}{w - z}, \tag{14}$$

where Γ *encircles* 0 *and* $\pi_i, i = 1, \ldots, r+1$, *and is oriented counterclockwise, and* $\gamma = A + i\mathbb{R}$, *with A large enough to ensure that* $\Gamma \cap \gamma = \emptyset$, *is oriented from bottom to top.*

Remark 1.12. Actually, the integral representation (14) has been established in the case where W_N has pairwise distinct eigenvalues $W_{ii}, i = 1, \ldots, N$. Yet, by a straightforward use of l'Hopital's rule, one can see this formula also holds in the case where $W_{jj} = W_{kk}$, for some $j \neq k$.

Thanks to the above expression (14), the asymptotic expansion of K_N can be computed through a saddle point analysis. We then deduce the asymptotic expansion of correlation functions $R_N^m(\cdot)$ and of local eigenvalue statistics. Let us develop some of the ideas used for computing the limiting distribution of the largest eigenvalue. By an inclusion-exclusion formula, one can show that $P(\lambda_1 \leq s) = \det(I - K_N)_{L^2(s,\infty)}$, where $\det(I - K_N)_{L^2(s,\infty)}$ is the Fredholm determinant of the trace class operator acting on $L^2(s, \infty)$ with kernel K_N. First, we prove that the correlation kernel can be written as

$$K_N(x, y) = \int_0^\infty H_N(x + t) J_N(y + t) dt, \tag{15}$$

for some kernels H_N, J_N. Using a saddle point analysis, we then prove that there exist kernels H_∞, J_∞ such that

$$\lim_{N\to\infty} \int_0^\infty |H_N(x + u) - H_\infty(x + u)|^2 du = 0,$$
$$\lim_{N\to\infty} \int_0^\infty |J_N(x + u) - J_\infty(x + u)|^2 du = 0,$$

for all x in a compact interval. This ensures that $\lim_{N\to\infty} \det(I - K_N)_{L^2(s,\infty)} = \det(I - K_\infty)_{L^2(s,\infty)}$, where $K_\infty(x, y) = \int_0^\infty H_\infty(x + t) J_\infty(y + t) dt$. This eventually gives the convergence in distribution of the largest eigenvalue of the deformed Wigner ensemble.

2. Proof of Theorem 1.1

In this section, we assume that all the eigenvalues of W_N are smaller than, or equal to one. We further assume (in this section only) that $\pi_1 = 1$ and has multiplicity k. It is assumed that $k = 0$ if all the eigenvalues of W_N are strictly smaller than 1. In all cases, we assume that $\pi_i < 1 - \eta$, for $i = 2, \dots, r + 1$ where $\eta > 0$ is fixed; k and r are here given integers independent of N.

Let then some $\epsilon > 0$, that will be fixed later, be given and set

$$u = 2 + \frac{x}{N^{2/3}}, \quad v = 2 + \frac{y}{N^{2/3}}, \quad w_c = 1, \quad \tilde{w}_c = w_c + \frac{\epsilon}{N^{1/3}},$$
$$K'_N(x, y) = \frac{e^{N(u-v)\tilde{w}_c}}{N^{2/3}} K_N(u, v). \tag{16}$$

Note that the rescaled correlation kernel $K'_N(x, y)$ defines the same correlation functions as $\frac{1}{N^{2/3}} K_N(u, v)$. Define also

$$g(w) = \prod_{i=2}^{r+1} \frac{w - \pi_i}{w} \frac{1}{w^k}, \quad w \in \mathbb{C}^*, \text{ and } F(z) = z^2/2 - 2z + \log z, \quad z \in \mathbb{C} \setminus \mathbb{R}_-. \tag{17}$$

Here we use the principal branch of the logarithm and $\exp\{N \log w\}$ stands for w^N. Then, from (14), we readily obtain that $K'_N(x, y)$ can be cast to the form (15). Let Γ and γ be as in (14).

Proposition 2.1. $K'_N(x, y) = -\int_0^{\infty} H_N(x + t) J_N(y + t) dt$, *where*

$$H_N(x) = \frac{N^{1/3}}{2\pi} \int_\Gamma \frac{1}{g(z)(z - w_c)^k} \exp\{-NF(z)\} \exp\{N^{1/3}(x + t)(z - \tilde{w}_c)\} dz, \tag{18}$$

$$J_N(y) = \frac{N^{1/3}}{2\pi} \int_\gamma g(w)(w - w_c)^k \exp\{NF(w)\} \exp\{-N^{1/3}(y + t)(w - \tilde{w}_c)\} dw. \tag{19}$$

Proof. We use the fact that $\frac{1}{w - z} = N^{1/3} \int_0^{\infty} \exp\{-N^{1/3} t(w - z)\} dt$. □

We now indicate the idea of the proof, which is very similar to that in [1]. We will perform a saddle point analysis of the kernels H_N and J_N. The critical points for F satisfy $F'(w) = w + \frac{1}{w} - 2 = 0$. Such an equation admits a single critical point, $w_c = 1 = \pi_1$, and

$$F''(w_c) = 1 - \frac{1}{w_c^2} = 0, \quad F^{(3)}(w_c) = 2. \tag{20}$$

Intuitively, the leading terms of the asymptotic expansions of (18), (19) are obtained by performing the corresponding integrals on a neighborhood of width $N^{-1/3}$ of

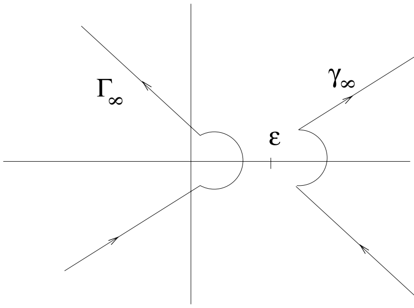

Fig. 1. Contours Γ_∞ and γ_∞

w_c. The steepest descent (resp. ascent) curve for F comes to w_c with an angle of $\pm\pi/3$ (resp $2\pi/3$) with respect to the real axis. Yet, as the integrand has a pole at w_c, one needs to deform these path so that Γ encircles w_c but does not cross γ. Essentially, we will have to show that the ascent and descent contours, deformed in such a way, still satisfy the saddle point analysis requirements. We now define the expected limiting kernels. Let Γ_∞ be the contour going from $\infty e^{-2i\pi/3}$ to $\infty e^{2i\pi/3}$, crossing the real axis on the right of the origin, oriented counterclockwise. Let γ_∞ be the contour going from $\infty e^{-i\pi/3}$ to $\infty e^{i\pi/3}$, oriented from bottom to top and crossing the real axis on the right handside of Γ_∞. A plot of these contours is given on Figure 1.

We then set

$$H_\infty(x) = \frac{\exp\{-\epsilon x\}}{2\pi} \int_{\Gamma_\infty} \exp\{xa - \frac{a^3}{3}\} \frac{1}{a^k} da, \tag{21}$$

$$J_\infty(y) = \frac{\exp\{\epsilon y\}}{2\pi} \int_{\gamma_\infty} \exp\{-yb + \frac{b^3}{3}\} b^k db. \tag{22}$$

The end of this section is devoted to the proof of the following result. □

Proposition 2.2. *Fix* $\epsilon > 0$ *and let* $Z_N = g(w_c)\exp\{NF(w_c)\}N^{-k/3}$.

For any fixed $y_o \in \mathbb{R}$, *there exists* $C > 0,\ c > 0$, *an integer* $N_o > 0$ *such that*

$$\left|Z_N H_N(x) - H_\infty(x)\right| \le \frac{C\exp\{-cx\}}{N^{1/3}},\ \textit{for any } x \ge y_o,\ N \ge N_o. \tag{23}$$

$$\left|\frac{1}{Z_N} J_N(y) - J_\infty(y)\right| \le \frac{C\exp\{-cy\}}{N^{1/3}},\ \textit{for any } y \ge y_o,\ N \ge N_o. \tag{24}$$

Remark 2.1. The fact that Proposition 2.2 implies Theorem 1.1 is proved in [1], Section 3.3. It follows in particular from the fact that $J_\infty(y) = it^{(k+1)}(y)e^{\{\epsilon y\}}$ and $H_\infty(x) = is^{(k)}(x)e^{\{-\epsilon x\}}$, where $t^{(k+1)}$, $s^{(k)}$ are defined in (4).

Remark 2.2. Before beginning the proof of Proposition 2.2, it is convenient to note that the exponential term F, given in (17), satisfies $F(z) = \overline{F(\overline{z})}$. Thus, we may only consider the parts of the contours Γ or γ lying in the upper half plane $\{z \in \mathbb{C},\ Im(z) > 0\}$. Estimates for the remaining contours are obtained by conjugation when needed. This is valid for the whole paper.

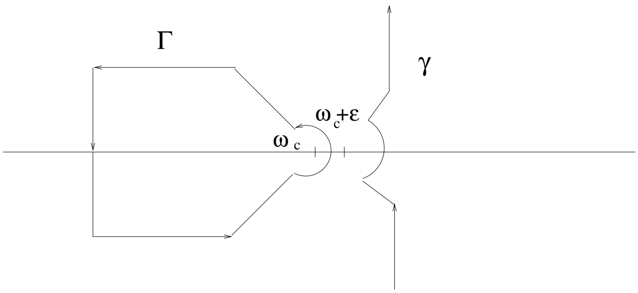

Fig. 2. Contours Γ and γ

2.1. Estimate for $Z_N H_N$.

This subsection is devoted to the proof of Formula (23). We first define an ascent curve Γ for F. We then deduce the asymptotic expansion of H_N.

2.1.1. Contour for the saddle point analysis

In this part, we give an ascent curve for F and also prove that the third order Taylor expansion of F (as heuristically explained in the preamble) can be made in some disk around w_c.
Let Γ be the contour defined in the following way.

$$\Gamma_o = w_c + \frac{\epsilon e^{i\theta}}{2N^{1/3}}, \quad \theta \in [0, 2\pi/3], \Gamma_1 = w_c + te^{i2\pi/3}, \quad \frac{\epsilon}{2N^{1/3}} \leq t \leq 2,$$
$$\Gamma_2 = \sqrt{3}i - t, \quad 0 \leq t \leq R_o, \Gamma_3 = i(\sqrt{3} - t) - R_o, \quad 0 \leq t \leq \sqrt{3}. \qquad (25)$$

Here R_o is chosen large enough so that Γ encircles all the eigenvalues $\pi_i, i = 1, \dots, r+1$, and will be fixed later. Finally define $\Gamma = \cup_{i=0}^3 \Gamma_i \cup \overline{\cup_{i=0}^3 \Gamma_i}$, oriented counterclockwise, as on Figure 2 below.

Lemma 2.1. Re(F) *increases as z along $\Gamma_1 \cup \Gamma_2$ and if $z^* = \Gamma_1 \cap \Gamma_2$,* $\min_{z \in \Gamma_2 \cup \Gamma_3} \mathrm{Re}\,(F(z)) = \mathrm{Re}\left(F(z^*)\right)$.

Proof of Lemma 2.1. For $z \in \Gamma_1$, we have that $\frac{d}{dt}\mathrm{Re}\left(F(w_c + te^{2i\pi/3})\right) = \frac{1}{2}\frac{t^2(2-t)}{1-t+t^2} \geq 0$, for $t \leq 2$. Then, along Γ_2, $\frac{d}{dt}\mathrm{Re}\left(F(\sqrt{3}i - t)\right) = t + 2 + \frac{t}{|t+\sqrt{3}+t|^2} > 2 + t$, so that Re$(F)$ achieves its minimum on Γ_2 at z^*. Finally we choose R_o such that

$$\mathrm{Re}\,(F(-R_o + it)) = \frac{R_o^2}{2} - \frac{t^2}{2} + 2R_o - \frac{1}{2}\log|R_o + it|^2$$
$$> \mathrm{Re}\left(F(z^*)\right), \quad \forall\, |t| \leq \sqrt{3}. \qquad (26)$$

This can be achieved if R_o is chosen large enough. □

We now determine some disk around w_c where the third order Taylor expansion of F holds. Let now δ be chosen so that

$$0 < \delta < 1/2 \quad \text{and} \quad \frac{\delta}{4(1-\delta)^4} \leq 1/6. \tag{27}$$

Lemma 2.2. *In the disk* $\{|z - w_c| \leq \delta\}$, $\left|F(z) - F(w_c) - \frac{F^{(3)}(w_c)}{3!}(z-w_c)^3\right| \leq \frac{F^{(3)}(w_c)}{3!}\frac{|z-w_c|^3}{2}$.

Proof of the Lemma 2.2. This follows from the Taylor expansion

$$\left|F(z) - F(w_c) - \frac{F^{(3)}(w_c)}{3!}(z-w_c)^3\right| \leq \max_{\Gamma'} \frac{|F^{(4)}(z)|}{4!}|z-w_c|^4$$
$$\leq \frac{\delta}{4!(1-\delta)^4}|z-w_c|^3 \leq \frac{|z-w_c|^3}{6}. \tag{28}$$

□

Remark 2.3. The above Lemmas imply in particular that $\forall t \leq \delta$, $\mathrm{Re}\Big(F(w_c + te^{2i\pi/3})\Big) \geq F(w_c) + t^3/6$, and $\min_{\Gamma''} \mathrm{Re}\,(F(z)) \geq F(w_c) + \delta^3/6$. Here we have used that $\delta \leq 1/2$ and (20).

The latter remark suggests that the leading term in the asymptotic expansion of H_N is given by the integral performed on the disk $\{|z - w_c| \leq \delta\}$. We thus split the contour $\Gamma = \Gamma' \cup \Gamma''$ where $\Gamma' = \Gamma \cap \{|z - w_c| \leq \delta\}$ and $\Gamma'' = \Gamma \setminus \Gamma'$. Let also Γ'_∞ be the image of Γ' under the map $z \mapsto N^{1/3}(z - w_c)$ and $\Gamma''_\infty = \Gamma_\infty \setminus \Gamma'_\infty$. We split accordingly the kernels H_N and H_∞, which we write $H_N(x) = H'_N(x) + H''_N(x)$ and $H_\infty(x) = H'_\infty(x) + H''_\infty(x)$, where

$$H'_N(x) = \frac{N^{1/3}}{2\pi}\int_{\Gamma'} \frac{e^{\{-NF(z)\}}}{g(z)(z-w_c)^k} e^{\{N^{1/3}x(z-\tilde{w}_c)\}}dz \text{ and}$$
$$H'_\infty(x) = \frac{e^{\{-\epsilon x\}}}{2\pi}\int_{\Gamma'_\infty} e^{\{xa - \frac{a^3}{3}\}}\frac{1}{a^k}da.$$

We now turn to the end of the proof of Formula (23).

2.1.2. The case x is bounded.

Formula (23) follows in this case from the following Lemma.

Lemma 2.3. *Let $y_o > 0$ be given. Then, there exists constants $C(y_o) > 0$, $N_o > 0$ such that, for any $|x| \leq y_o$, and $N \geq N_o$,*

$$|Z_N H_N(x) - H_\infty(x)| \leq \frac{C(y_o)}{N^{1/3}}.$$

Proof of Lemma 2.3. We consider the contributions of Γ' and Γ'' separately. We first prove

$$|Z_N H''(x)| \leq \exp\left\{-N\frac{\delta^3}{12}\right\}, \quad |H''_\infty(x)| \leq \exp\{-N\delta^3/6\}. \tag{29}$$

Let then $L_{\Gamma''}$ be the length of Γ'', $C(R_o) = R_o + 2$, and C_g be a constant such that

$$\frac{1}{C_g} \leq \min_{\Gamma''} |g| \leq C_g, \tag{30}$$

which is well defined since, by Assumption 1.1, $\pi_i, i = 2, \ldots, r+1$, lie in a compact interval of $(-\infty, 1)$. Then, using Remark 2.3, we have that

$$\begin{aligned}
|Z_N H''(x)| &\leq \frac{|g(w_c)|}{2\pi N^{(k-1)/3}} \int_{\Gamma''} \Big| \exp\{NF(w_c) - NF(z)\} \exp\{N^{1/3}x(z-\tilde{w}_c)\} \Big| \\
&\quad \times \frac{|dz|}{|g(z)(z-z_c)^k|} \\
&\leq \frac{|g(w_c)|}{2\pi N^{(k-1)/3}} L_{\Gamma''} \frac{C_g}{\delta^k} \exp\left\{N^{1/3} y_o C(R_o)\right\} \exp\left\{-N\frac{\delta^3}{6}\right\} \\
&\leq \exp\left\{-N\frac{\delta^3}{12}\right\},
\end{aligned} \tag{31}$$

for N large enough. This yields the first part of (29). The second inequality is straightforward from [1], formula (152), for instance.

We then turn to the contour $\Gamma' = \Gamma'_o \cup \Gamma'_1$, where $\Gamma'_o := \Gamma_o \cap \{|w - w_c| \leq \delta\} = \Gamma_o$ and $\Gamma'_1 = \Gamma' \setminus \Gamma'_o$. Here we assume that ϵ is chosen so that $\epsilon \leq \delta$ and thus $\Gamma'_o = \Gamma_o$. Here we prove that

$$|Z_N H'_N(x) - H'_\infty(x)| \leq \frac{C}{N^{1/3}}. \tag{32}$$

One has

$$\begin{aligned}
&|Z_N H'_N(x) - H'_\infty(x)| \\
&\leq \frac{N^{1/3}}{2\pi} \int_{\Gamma'} \frac{e^{N^{1/3} x \mathrm{Re}(z-\tilde{w}_c)}}{(N^{1/3}|z-w_c|)^k} \left| e^{N(-F(z)+F(w_c))} \frac{g(w_c)}{g(z)} - e^{-N(z-w_c)^3/3} \right| |dz|.
\end{aligned} \tag{33}$$

We now skip the details (given in [1], page 26).

Then for $z \in \Gamma'_o$, using (28),

$$\begin{aligned}
&\left| \exp\{N(F(w_c) - F(z))\} - \exp\{-N\frac{(w-w_c)^3}{3}\} \right| \\
&\leq \max\left(\left| e^{N(F(w_c)-F(z))} \right|, \left| e^{-N\frac{(z-w_c)^3}{3}} \right| \right) N \left| F(z) - F(w_c) - \frac{(z-w_c)^3}{3} \right| \\
&\leq NC_o |z-w_c|^4 \exp\{N\mathrm{Re}\left(\frac{(z-w_c)^3}{16}\right)\} \leq \frac{C_o \epsilon^4}{16 N^{1/3}} \exp\{\frac{\epsilon^3}{16}\},
\end{aligned}$$

where $C_o = 1/(1-\delta)^4$ is well defined since $\delta < 1/2$. Similarly $\Big|\frac{g(w_c)}{g(z)} - 1\Big| \leq \frac{C_g C_g'}{2N^{1/3}}$, where C_g is given by (30) and $C_g' = \max\{|g'(s)|, s \in \Gamma_o' \cup \Gamma_1'\}$, which is well defined since the $\pi_i, i \geq 2$ in a compact set of $(-\infty, 1)$. Thus, $\forall z \in \Gamma_o'$

$$\begin{aligned} &\Big| \exp\{N(F(w_c) - F(z))\} \frac{g(w_c)}{g(z)} - \exp\{-N \frac{(w-w_c)^3}{3}\} \Big| \\ &\leq \left(\exp\{\frac{\epsilon^3}{16}\} \frac{C_o \epsilon^4}{16} + C_g C_g'/2 \right) \frac{1}{N^{1/3}}. \end{aligned} \tag{34}$$

Using now that the length of Γ_o' is $2\pi\epsilon N^{-1/3}/3$, we obtain from (34) that there exists $C_1 > 0$ such that

$$\frac{N^{1/3}}{2\pi} \int_{\Gamma_o'} \frac{e^{N^{1/3} x \mathrm{Re}(z - \tilde{w}_c)}}{(N^{1/3}|z - w_c|)^k} \Big| e^{N(-F(z)+F(w_c))} \frac{g(w_c)}{g(z)} - e^{-N(z-w_c)^3/3} \Big| |dz| \leq \frac{C_1}{N^{1/3}}. \tag{35}$$

Similarly for $z \in \Gamma_1'$, one has that $\exp\{N^{1/3} x \mathrm{Re}(z - \tilde{w}_c)\} \leq \exp\{N^{1/3} y_o t/2 + \epsilon y_o\}$ and

$$\Big| e^{N(F(w_c)-F(z))} \frac{g(w_c)}{g(z)} - e^{-N(z-w_c)^3/3} \Big| \leq (C_o + C_g C_g')(Nt^4 + t) \exp\{-N \frac{t^3}{6}\}. \tag{36}$$

Now (see [1]), we obtain from (36) that there exists some $C_2 > 0$ such that, for N large enough,

$$\begin{aligned} &\frac{N^{1/3}}{2\pi} \int_{\Gamma_1'} \frac{e^{N^{1/3} x \mathrm{Re}(z - \tilde{w}_c)}}{(N^{1/3}|z - w_c|)^k} \Big| e^{N(-F(z)+F(w_c))} \frac{g(w_c)}{g(z)} - e^{-N(z-w_c)^3/3} \Big| |dz| \\ &\leq \frac{N^{1/3}}{\pi} (C_o^3 + C_g C_g') \int_{\frac{\epsilon}{2N^{1/3}}}^{\delta} \left(\frac{1}{N^{1/3} t} \right)^k (Nt^4 + t) \\ &\quad \times \exp\left\{ \epsilon y_o + \frac{y_o t N^{1/3}}{2} - \frac{Nt^3}{6} \right\} \leq \frac{C_2}{N^{1/3}}. \end{aligned} \tag{37}$$

Finally, combining (33), (35), and (37), we obtain (32). Using now (29), (32), we then obtain that $|Z_N H_N(x) - H_\infty(x)| \leq \frac{C(y_o)}{N^{1/3}}$, for N large enough. □

2.1.3. The case x positive

Fromula (23) follows in this case from the following Lemma.

Lemma 2.4. *Assume $x > 0$, then there exist $C > 0$, $N_o > 0$ such that for $N \geq N_o$,*

$$|Z_N H_N(x) - H_\infty(x)| \leq \frac{C \exp\{-\epsilon x/2\}}{N^{1/3}}.$$

Proof of Lemma 2.4. The thing that makes it all here is that the whole contour Γ lies on the half plane $\mathrm{Re}(z-\tilde{w}_c)<0$, where $\tilde{w}_c$ has been defined in (16). This gives that, for large positive x, the kernel $Z_N H_N$ decays exponentially, as we now explain.

For $z\in\Gamma''$, one has that $\mathrm{Re}(z-\tilde{w}_c)\leq -\frac{\epsilon}{N^{1/3}}-\frac{\delta}{2}$, yielding from (31) that

$$|Z_N H''(x)|\leq \exp\left\{-\epsilon x-\frac{\delta N^{1/3}x}{2}-N\frac{\delta^3}{12}\right\} \text{ for } N \text{ large enough.}$$

It is also easy to check that $|H''_\infty(x)|\leq \exp\{-\epsilon x-\delta\frac{N^{1/3}x}{2}-\frac{N\delta^3}{6}\}$, for N large enough.

We now consider the part of $Z_N H_N$ (resp. H'_∞) corresponding to the integral performed over Γ' (resp. Γ'_∞), along which one has that $\exp\{N^{1/3}x\mathrm{Re}(z-\tilde{w}_c)\}\leq \exp\{-\epsilon x/2\}$. Inserting the latter in (33), and performing the same computations as for the case where x lies in a compact set, we obtain that

$$|Z_N H'_N(x)-H'_\infty(x)|\leq \frac{C_2\exp\{-\epsilon x/2\}}{N^{1/3}}.$$

□

Remark 2.4. There are two crucial steps in the above proof. The first one is the definition of an ascent curve Γ, which coincides with the steepest ascent curve for F in an annulus $\{\epsilon\leq|z-w_c|\leq\delta\}$. The second step is to determine a $\delta>0$ such that Lemma 2.2 holds. This second step also ensures that we can find ϵ small enough so that Γ encircles w_c but remains on the left handside of $\tilde{w}_c$. Once these two points obtained, one only needs a good enough control of the perturbative term g along Γ, so that the end of the proof follows. This remark will be the basis for the proof of Theorem 1.4.

2.2. *Estimate for* $\frac{1}{Z_N}J_N(y)$

This subsection is devoted to the proof of Formula (24). We first define a descent curve γ for F. Then, we obtain the asymptotic expansion of J_N.

2.2.1. Contour for the saddle point analysis

We now give a descent curve for F. Define $\gamma_0=w_c+\frac{3\epsilon e^{i\theta}}{N^{\frac{1}{3}}},\ \ 0\leq\theta\leq\frac{\pi}{3}$;

$$\gamma_1=w_c+te^{i\frac{\pi}{3}},\ \ \frac{3\epsilon}{N^{\frac{1}{3}}}\leq t\leq t_o;\ \ \gamma_2=w_c+t_oe^{i\frac{\pi}{3}}+it,\ \ t\geq 0. \qquad (38)$$

Actually, we choose $t_o>\delta$, where δ is given by (27). Finally let γ be the contour $\gamma=\cup_{i=0}^2\gamma_i\cup\overline{\cup_{i=0}^2\gamma_i}$ oriented from bottom to top, as on Figure 2.

Lemma 2.5. Re(F) *is decreasing on* $\gamma_1 \cup \gamma_2$ *as* $Im(w)$ *increases. And* $\exists\, C_o > 0$ *such that, if* $w^* = w_c + t_o e^{i\pi/3}$, $\mathrm{Re}\left(F(w^* + it)\right) \leq \mathrm{Re}\left(F(w^*)\right) - \frac{C_o t^2}{2}$, $t \geq 0$.

Proof of Lemma 2.5. One has $\frac{d}{dt}\mathrm{Re}\left(F(w_c + te^{i\pi/3})\right) = \frac{-t^2(2+t)}{2(1+t+t^2)} < 0$, $\forall t > 0$. Then, along γ_2, and for $C_o = C_o = 1 - 1/|w^*|^2 > 0$, one has $\frac{d}{dt}\mathrm{Re}\left(F(w^* + it)\right) \leq -C_o(t + \frac{\sqrt{3}}{2}t_o)$. □

Let now δ be given as in (27), so that (28) still holds. We split as before the contour γ. Set $\gamma' = \gamma \cap \{|w - w_c| \leq \delta\}$ and $\gamma'' = \gamma \setminus \gamma'$. Let also γ'_∞ be the image of γ' under the map $w \mapsto N^{1/3}(w - w_c)$, and $\gamma''_\infty = \gamma_\infty \setminus \gamma'_\infty$. Define then $J''_1 = J_N - J'_N - J''_2$, and $J''_\infty = J_\infty - J'_\infty$ where

$$J'_N(y) = \frac{N^{1/3}}{2\pi}\int_{\gamma'} g(w)(w - w_c)^k \exp\{NF(w)\}\exp\{-N^{1/3}y(w - \tilde{w}_c)\}dw,$$
$$J''_2(y) = \frac{N^{1/3}}{2\pi}\int_{\gamma_2} g(w)(w - w_c)^k \exp\{NF(w)\}\exp\{-N^{1/3}y(w - \tilde{w}_c)\}dw,$$
$$J'_\infty(y) = \frac{\exp\{\epsilon y\}}{2\pi}\int_{\gamma'_\infty} \exp\{-yb + \frac{b^3}{3}\}b^k db.$$

We now prove Formula (24).

2.2.2. The case y in a compact interval

Formula (24) readily follows in this case from the following Lemma.

Lemma 2.6. *Let* $y_o > 0$ *be given. Then, there exists* $C = C(y_o) > 0$, N_o *such that for any* $|y| \leq y_o$,

$$|\frac{1}{Z_N}J_N(y) - J_\infty(y)| \leq \frac{C}{N^{1/3}}, \forall\, N \geq N_o.$$

Proof of Lemma 2.6. Let us first consider the kernel $J''_N = J''_1 + J''_2$. We now show that there exists $C > 0$ such that

$$|\frac{1}{Z_N}J''_N(y)| \leq C\exp\{-N\frac{\delta^3}{12}\} \tag{39}$$

for N large enough. The only difference from the preceding subsection is that γ'' is not of finite length. We first consider the integral performed on γ_2.

$$|\frac{1}{Z_N}J''_2(y)| \leq \frac{(N^{1/3})^{k+1}}{2\pi|g(w_c)|}\int_{\gamma_2} e^{N^{1/3}y_o\mathrm{Re}(w - \tilde{w}_c)}e^{N(\mathrm{Re}(F(w) - F(w_c)))} \times |w - w_c|^k||g(w)||dw|. \tag{40}$$

Now, using Lemma 2.5, and the fact that Re $(F(w^*)) \leq$ Re $(F(w_c)) - \delta^3/6$ (which follows from the fact that $t_o \geq \delta$ and Remark 2.3), one has

$$|\frac{1}{Z_N} J_2''(y)| \leq \frac{(N^{1/3})^{k+1}}{2\pi|g(w_c)|} e^{N^{1/3}\frac{y_o t_o}{2} - N\frac{\delta^3}{6}} \int_{i\mathbb{R}_+} e^{-N\frac{C_o t^2}{2}} \times \prod_{i=2}^{r+1} \sqrt{\frac{(1+t_o/2+A)^2 + (\sqrt{3}t_o/2+t)^2}{(1+t_o/2)^2 + (t_o\sqrt{3}/2+t)^2}} dt,$$

where A is chosen such that $|\pi_i| < A, i = 2, \ldots, r+1$. Now, under Assumption 1.1, A can be chosen independently of N. Thus, for N large enough,

$$|\frac{1}{Z_N} J_2''(y)| \leq \exp\{-N\frac{\delta^3}{12}\}. \tag{41}$$

The remaining contour $\gamma_1'' = \gamma'' \setminus \gamma_2$ has a finite length $L_{\gamma_1''}$, independent of N. Define also $\tilde{C}_g = \max_{w\in\gamma_1''}, |g(w)|$, which is uniformly bounded. Now, using that, for N large enough, Re $(F(w)) \leq F(w_c) - \delta^3/6$ and $\text{Re}(w - \tilde{w}_c) \leq t_o$, $\forall w \in \gamma_1''$, we obtain that

$$|\frac{1}{Z_N} J_1''(y)| \leq \frac{(N^{1/3})^{k+1}}{2\pi|g(w_c)|}\delta^k \tilde{C}_g L_{\gamma_1''} \exp\left\{N^{1/3} y_o t_o - N\frac{\delta^3}{6}\right\} \leq \exp\{-\frac{N\delta^3}{12}\}, \tag{42}$$

for N large enough. Combining (41) and (42) yields (39).

And inserting $b = te^{i\pi/3}$, with $t \geq \delta N^{1/3}$, in (22) yields that (see formula (188) in [1]) $|J_\infty''(y)| \leq \exp\{-\frac{N\delta^3}{6}\}$, for N large enough. Finally, mimicking the proof of (36), (34), and using the same arguments as for H_N (see Remark 2.4), it is easy to show that $\exists C_3(y_o) > 0$, such that $|\frac{1}{Z_N} J_N'(y) - J_\infty'(y)| \leq \frac{C_3(y_o)}{N^{1/3}}$. □

2.2.3. The case $y > 0$

Formula (24) follows in this case from the following Lemma.

Lemma 2.7. *There exist $C > 0$, $N_o > 0$ such that, $\forall y > 0$, and $\forall N \geq N_o$,*

$$|\frac{1}{Z_N} J_N(y) - J_\infty(y)| \leq \frac{C}{N^{1/3}} \exp\{-\frac{\epsilon y}{2}\}.$$

Proof of Lemma 2.7. We only give the mains ideas. One has

$$\begin{aligned}
&\forall w \in \gamma_2, \ \text{Re}(w - \tilde{w}_c) = \text{Re}(w^* - \tilde{w}_c) = \frac{t_o}{2} - \frac{\epsilon}{N^{1/3}};\\
&\forall w \in \gamma_1'', \ \text{Re}(w - \tilde{w}_c) \geq \frac{\delta}{2} - \frac{\epsilon}{N^{1/3}};\\
&\forall w \in \gamma_1', \ \text{Re}(w - \tilde{w}_c) = \frac{|w - w_c|}{2} - \frac{\epsilon}{N^{1/3}} \geq \frac{\epsilon}{2N^{1/3}};\\
&\forall w \in \gamma_o, \ \text{Re}(w - \tilde{w}_c) \geq \frac{\epsilon}{2N^{1/3}}.
\end{aligned} \tag{43}$$

Note that (43) explains why we choose a circle of ray 3ϵ for γ_o. Here we assume that ϵ is small enough so that $\epsilon - \delta/2 < -\epsilon/2$. This gives that the whole contour γ lies on the right of $\tilde{w}_c$. We then insert the above estimates in e.g. (40) and copy the proof of (41). The same is done for the integral performed on γ_1''. Then, one readily obtains that, for N large enough,

$$|\frac{1}{Z_N} J_N''(y)| \leq \exp\left\{\epsilon y - \frac{\delta}{2} N^{1/3} y - N\frac{\delta^3}{12}\right\}, \text{ while}$$

$$|J_\infty''(y)| \leq \exp\left\{\epsilon y - \frac{\delta N^{1/3} y}{2} - N\frac{\delta^3}{6}\right\}.$$

Finally, using (43) and mimicking the estimates of the preceding subsection, we obtain that $|\frac{1}{Z_N} J_N'(y) - J_\infty'(y)| \leq \frac{C}{N^{1/3}} \exp\{-\frac{\epsilon y}{2}\}$. □

3. Proof of Theorem 1.2

In this section, we assume that π_1 lies in a compact interval of $(1, \infty)$ and that Assumption 1.1 holds. Let now $\epsilon > 0$ be fixed and set

$$\tilde{\pi}_1 = \pi_1 + \frac{\epsilon}{\sqrt{N}}, \ u = C(\pi_1) + \frac{x}{\sigma^2\sqrt{N}}, \ v = C(\pi_1) + \frac{y}{\sigma^2\sqrt{N}}, \tag{44}$$

and let $K_N'(x, y) = \dfrac{\exp\left\{\frac{(y-x)}{\sigma^2}\tilde{\pi}_1\right\}}{\sigma^2\sqrt{N}} K_N\left(C(\pi_1) + \dfrac{x}{\sigma^2\sqrt{N}}, C(\pi_1) + \dfrac{y}{\sigma^2\sqrt{N}}\right)$ be the associated rescaled correlation kernel. Define $F_{C(\pi_1)}(w) = w^2/2 - C(\pi_1)(w - \tilde{\pi}_1) + \log w$, where we use the principal branch of the logarithm ($e^{\pm N \log w} = w^{\pm N}$). We now bring $K_N'(x, y)$ to the form (15). Let Γ and γ be contours as in Proposition 1.3.

Proposition 3.1. $K_N'(x, y) = -\int_0^\infty H_N(\frac{x+t}{\sigma^2}) J_N(\frac{y+t}{\sigma^2}) dt$, *with*

$$\begin{aligned} H_N(\frac{x}{\sigma^2}) &= \frac{\sqrt{N}}{2\pi\sigma^2} \int_\Gamma \frac{1}{(z-\pi_1)^k g(z)} \exp\left\{\sqrt{N}\frac{x}{\sigma^2}(z - \tilde{\pi}_1)\right\} \\ &\quad \times \exp\left\{-N F_{C(\pi_1)}(z)\right\} dz, \\ J_N(\frac{y}{\sigma^2}) &= \frac{\sqrt{N}}{2\pi\sigma^2} \int_\gamma (w-\pi_1)^k g(w) \exp\left\{-\sqrt{N}\frac{y}{\sigma^2}(w - \tilde{\pi}_1)\right\} \\ &\quad \times \exp\left\{N F_{C(\pi_1)}(w)\right\} dw. \end{aligned} \tag{45}$$

We briefly indicate the idea of the proof of Theorem 1.2. Here, the critical points to be considered satisfy $F'_{C(\pi_1)}(w) = w + 1/w - C(\pi_1) = 0$. They are given by π_1 and $1/\pi_1$ and are non degenerate. One can check that

$$F''(\pi_1) = 1 - \frac{1}{\pi_1^2} = \frac{1}{\sigma^2(\pi_1)} > 0. \tag{46}$$

We will show that, as Γ has to encircle the pole π_1, the contribution of the sole pole π_1 will give the leading term in the asymptotic expansion. In the subsequent, we note $\sigma^2 = \sigma^2(\pi_1)$ and define now the expected limiting kernels.

Let $\gamma_\infty = 2\epsilon + i\mathbb{R}$ (resp. $\Gamma_\infty = \epsilon e^{i\theta}, 0 \leq \theta \leq 2\pi$,) oriented from bottom to top (resp. counterclockwise). Set then

$$\begin{aligned} H_\infty(\frac{x}{\sigma^2}) &= \frac{1}{2\pi\sigma^2} \exp\{-\epsilon\frac{x}{\sigma^2}\} \int_{\Gamma_\infty} \frac{1}{a^k} \exp\left\{-\frac{a^2}{2\sigma^2} + \frac{x}{\sigma^2}a\right\} da, \\ J_\infty(\frac{y}{\sigma^2}) &= \frac{1}{2\pi\sigma^2} \exp\{\epsilon\frac{y}{\sigma^2}\} \int_{\gamma_\infty} s^k \exp\left\{\frac{s^2}{2\sigma^2} - \frac{y}{\sigma^2}s\right\} ds. \end{aligned} \tag{47}$$

The aim of the rest of this section is to prove the following result.

Proposition 3.2. *Assume $\epsilon > 0$ is fixed, and set $Z_N = g(\pi_1)N^{-k/2}\exp\{NF_{C(\pi_1)}(\pi_1)\}$. For any fixed $y_o \in \mathbb{R}$, there exists constants $C > 0$, $c > 0$, and an integer $N_o > 0$ such that*

$$\left|\frac{1}{Z_N} J_N(\frac{y}{\sigma^2}) - J_\infty(\frac{y}{\sigma^2})\right| \leq \frac{C\exp\{-c\frac{y}{\sigma^2}\}}{\sqrt{N}}, \text{ for any } y \geq y_o,\ N \geq N_o. \tag{48}$$

$$\left|Z_N H_N(\frac{x}{\sigma^2}) - H_\infty(\frac{x}{\sigma^2})\right| \leq \frac{C\exp\{-c\frac{x}{\sigma^2}\}}{\sqrt{N}}, \text{ for any } x \geq y_o,\ N \geq N_o. \tag{49}$$

Remark 3.1. The fact that Proposition 3.2 implies Theorem 1.2 follows from the equality

$$-\exp\{\epsilon\frac{(x-y)}{\sigma^2}\} \int_0^\infty H_\infty(\frac{x+u}{\sigma^2}) J_\infty(\frac{y+u}{\sigma^2}) du = K(x, y), \tag{50}$$

where $K(x, y)$ is the correlation kernel of the $k \times k$ GUE with parameter σ^2.

Formula (50) follows from (14) and a simple change of variables. Another proof of (50) is given in [1], Section 4.

The proof of Proposition 3.2 will be obtained in the following subsections.

3.1. Estimate for $\frac{1}{Z_N} J_N(\frac{y}{\sigma^2})$

This subsection is devoted to the proof of formula (48). The details of the proof will be skipped since the scheme is exactly the same as in the preceding Section. The key points are the following Lemmas. In the first one, we give the descent curve for $F_{C(\pi_1)}$. In the second one, we determine a disk where the second order Taylor expansion of $F_{C(\pi_1)}$ holds.

Let γ_1 be the contour $\gamma_1 = \pi_1 + \frac{2\epsilon}{\sqrt{N}} + it, t \in \mathbb{R}_+$, $\gamma = \gamma_1 \cup \overline{\gamma_1}$ and set $\pi_1' = \pi_1 + \frac{2\epsilon}{\sqrt{N}}$.

Lemma 3.1. *There exists $C_o > 0$ such that* $\mathrm{Re}\left(F_{C(\pi_1)}(\pi_1' + it)\right) \leq F_{C(\pi_1)}(\pi_1') - C_o t^2/2, \forall t \in \mathbb{R}$.

Proof of Lemma 3.1. $\frac{d}{dt}\mathrm{Re}\left(F_{C(\pi_1}(\pi_1' + it)\right) = -t\left(1 - \frac{1}{|\pi_1' + it|^2}\right) \leq -C_o t$ since $\pi_1' > \pi_1$ lies in a compact interval of $(1, \infty)$. □

Let now δ be such that

$$\frac{\delta}{(\pi_1 - \delta)^3} \leq \frac{1}{4\sigma^2(\pi_1)} \text{ and } \delta \leq \pi_1/2. \tag{51}$$

Lemma 3.2. *In the disk* $\{|w - \pi_1| \leq \delta\}$, *one has*

$$\left|F_{C(\pi_1)}(w) - F_{C(\pi_1)}(\pi_1) - \frac{F''_{C(\pi_1)}(\pi_1)}{2}(w - \pi_1)^2\right| \leq \frac{F''_{C(\pi_1)}(\pi_1)}{4}|w - \pi_1|^2. \tag{52}$$

Proof of Lemma 3.2. It is proved as Lemma 2.2. □

As before, we now split the contour into two parts. Let $\gamma' = \gamma \cap \{|w - \pi_1| \leq \delta\}$ and $\gamma'' = \gamma \setminus \gamma'$. Let also γ'_∞ be the image of γ' under the map $w \mapsto \sqrt{N}(w - \pi_1)$ and $\gamma''_\infty = \gamma_\infty \setminus \gamma'_\infty$. Set now $J_N(\frac{y}{\sigma^2}) = J'_N(\frac{y}{\sigma^2}) + J''_N(\frac{y}{\sigma^2})$,

$J_\infty(\frac{y}{\sigma^2}) = J'_\infty(\frac{y}{\sigma^2}) + J''_\infty(\frac{y}{\sigma^2})$, where

$$J'_N(\frac{y}{\sigma^2}) = \frac{\sqrt{N}}{2\pi\sigma^2}\int_{\gamma'}(w - \pi_1)^k g(w)\exp\left\{-\sqrt{N}\frac{y}{\sigma^2}(w - \tilde{\pi}_1)\right\}\exp\left\{NF_{C(\pi_1)}(w)\right\}dw$$

and $J'_\infty(\frac{y}{\sigma^2}) = \frac{1}{2\pi\sigma^2}\exp\{\epsilon\frac{y}{\sigma^2}\}\int_{\gamma'_\infty} s^k \exp\left\{\frac{s^2}{2\sigma^2} - \frac{y}{\sigma^2}s\right\}ds.$

We only give the main steps of the proof. Let $y_o > 0$ be given and assume first that y lies in the interval $[-y_o, y_o]$. Then we show that there exists $C > 0$, such that for N large enough,

$$\left|\frac{1}{Z_N}J''_N(y)\right| \leq C\exp\{-N\frac{\delta^2}{24}\}, \quad \left|J''_\infty(y)\right| \leq C\exp\{-N\frac{\delta^2}{24}\},$$
$$|\frac{1}{Z_N}J'_N(\frac{y}{\sigma^2}) - J'_\infty(\frac{y}{\sigma^2})| \leq \frac{C}{\sqrt{N}}. \tag{53}$$

Here we have to take care of the fact that γ does not exactly go through the critical point π_1. Consider first γ'' and let $w^* = \gamma \cap \{|w - \pi_1| = \delta\}$. From Lemma 3.1,

$$\mathrm{Re}\big(F(w^* + it) - F(w^*)\big) \le -C_o \frac{t^2}{2},\ \forall t > 0. \tag{54}$$

Furthermore, as N goes to infinity, $w^* \to \pi_1 + i\delta$, so that, for N large enough, by Lemma 2.2,

$$\mathrm{Re}\left(F_{C(\pi_1)}(w^*)\right) - \mathrm{Re}\left(F_{C(\pi_1)}(\pi_1)\right) \le -\frac{\delta^2}{12}. \tag{55}$$

Combining (54), (55), and Remark 2.4, we obtain the first inequality in (53), for N large enough. The second inequality is straightforward. Conversely, Lemma 3.2 and Remark 2.4 give the last inequality in (53), since the perturbative term $|g(w)(w - w_c)^k|$ is uniformly bounded along γ'. This yields (48) in this case. Finally, we use the fact that $\mathrm{Re}(w - \tilde{\pi}_1) > C$, $\forall w \in \gamma$, for some constant $C > 0$, and the same arguments as in the preceding Section, to obtain (48) in the case $y > 0$.

3.2. Estimate for $Z_N H_N(\frac{x}{\sigma^2})$

This subsection is devoted to the proof of formula (49). We examine $Z_N H_N(\frac{x}{\sigma^2})$ as a residue integral and show that the sole residue at $z = \pi_1$ gives the leading term in the asymptotic expansion. We thus split the contour accordingly. Let Γ'' be a contour that encloses 0 and π_i, $i = 2, \dots, r+1$ but not π_1, oriented counterclockwise. Then we readily obtain the following Proposition.

Proposition 3.3. *$Z_N H_N(\frac{x}{\sigma^2}) = H_1(\frac{x}{\sigma^2}) + H_2(\frac{x}{\sigma^2})$ where*

$$\begin{aligned} H_2(\frac{x}{\sigma^2}) &= \frac{g(\pi_1)e^{-\epsilon\frac{x}{\sigma^2}}}{2\pi\sigma^2(\sqrt{N})^{k-1}} \int_{\Gamma''} \frac{\exp\{\sqrt{N}\frac{x}{\sigma^2}(z-\pi_1)\}}{(z-\pi_1)^k g(z)} \\ &\quad \times \exp\left\{-N(F_{C(\pi_1)}(z) - F_{C(\pi_1)}(\pi_1))\right\}dz, \\ H_1(\frac{x}{\sigma^2}) &= \frac{e^{-\epsilon\frac{x}{\sigma^2}}}{\sigma^2} \int_{\Gamma_\infty} \frac{\exp\{\frac{x}{\sigma^2}a\}}{a^k} \frac{g(\pi_1)}{g(\pi_1 + \frac{a}{\sqrt{N}})} \\ &\quad \times \exp\left\{-N\left(F_{C(\pi_1)}(\pi_1 + \frac{a}{\sqrt{N}}) - F_{C(\pi_1)}(\pi_1)\right)\right\}da. \end{aligned} \tag{56}$$

The proof of Formula (49) is now divided into two facts, in which we examine separately the two kernels H_1 and H_2. First we show that $H_1(\frac{x}{\sigma^2})$ behaves as $H_\infty(\frac{x}{\sigma^2})$.

Fact 3.1. *Given any fixed $y_o \in \mathbb{R}$, there exists constants $C > 0$, $c > 0$, $N_o > 0$ such that*

$$|H_1(\frac{x}{\sigma^2}) - H_\infty(\frac{x}{\sigma^2})| \le \frac{C\exp\{-c\frac{x}{\sigma^2}\}}{\sqrt{N}},\ \textit{for any } x \ge y_o,\ N \ge N_o.$$

Proof of Fact 3.1. We only explain the main changes from [1], since the proof follows the same steps. For any l , the derivatives $F^{(l)}_{C(\pi_1)}(\pi_1)$, $g^{(l)}(\pi_1)$ are all bounded, and $|g(\pi_1)| > 0$ thanks to Assumption 1.1. Thus, by a straightforward Taylor expansion, we have that

$$\int_{\Gamma_\infty} \frac{1}{a^k} \frac{g(\pi_1)}{g(\pi_1 + \frac{a}{\sqrt{N}})} \exp\left\{-N\left(F_{C(\pi_1)}(\pi_1 + \frac{a}{\sqrt{N}}) - F_{C(\pi_1)}(\pi_1)\right)\right\} \times \exp\left\{\frac{x}{\sigma^2}a\right\}da$$
$$= \int_{\Gamma_\infty} \frac{1}{a^k} \exp\left\{\frac{x}{\sigma^2}a - \frac{1}{2\sigma^2}a^2\right\}\left(1 + \sum_{j=1}^{k-1} \frac{q_j(a)}{(\sqrt{N})^j}\right) da, \tag{57}$$

for some polynomials q_j, $j = 1, \ldots, k-1$ independent of N. Now (57) and (56) give Fact 3.1. □

We now turn to the asymptotics of the kernel H_2.

Fact 3.2. *For any fixed $y_o \in \mathbb{R}$, there exists $C > 0$, $c > 0$, $N_o > 0$ such that*

$$\left|H_2(\frac{x}{\sigma^2})\right| \leq c \exp\{-\epsilon x - CN\}, \text{ for any } N \geq N_o,\ x \geq y_o.$$

Proof of Fact 3.2. The proof is obtained by a saddle point analysis of the kernel H_2. We define the suitable contour Γ'', that depends on some constants η, R, θ_o, x_o^* that will be fixed later. Set $\pi^* = \max(1, \pi_2)$ and define

$$\Gamma_1'' = \frac{\pi_1 + \pi^*}{2} + iy, \quad y \leq \eta,$$
$$\Gamma_2'' = \frac{\pi_1 + \pi^*}{2} + i\eta - x, \quad 0 \leq x \leq \frac{\pi_1 + \pi^*}{2} - x_o^*,$$
$$\Gamma_3'' = \frac{C(\pi_1)}{2}e^{i\theta}, \quad \theta_o \leq \theta \leq \frac{\pi}{2}, \Gamma_4'' = i\frac{C(\pi_1)}{2} - x, \quad 0 \leq x \leq R$$
$$\Gamma_5'' = i(\frac{C(\pi_1)}{2} - t) - R, \quad 0 \leq t \leq \frac{C(\pi_1)}{2}.$$

Set $\Gamma'' = \cup_{i=1}^5 \Gamma_i'' \cup \overline{\cup_{i=1}^5 \Gamma_i''}$. A plot of the contours $\Gamma = \Gamma_\infty \cup \Gamma''$ and γ is given on Figure 3 below.

Remark 3.2. Here, we make some preliminary restrictions on η and R, that will be fixed in the following Lemma. We assume that η is small enough so that the curve $x + i\eta$, $1 \leq x \leq \frac{\pi_1+\pi^*}{2}$ crosses the circle of ray $\frac{C(\pi_1)}{2}$. As $\frac{C(\pi_1)}{2} > 1$, we will then choose some $\eta \leq \sqrt{(\frac{C(\pi_1)}{2})^2 - 1}$. Given such a η, we call $x^* = x^*(\eta) = \frac{C(\pi_1)}{2}e^{i\theta_o} = x_o^* + i\eta$ this intersection. Moreover, R is chosen large enough to enclose all the π_i, $i = 2, \ldots, r+1$.

The crucial step in the proof of Fact 3.2 is the following Lemma.

Lemma 3.3. *There exists $0 < \eta \leq \sqrt{(\frac{C(\pi_1)}{2})^2 - 1}$, $R > 0$ for which*

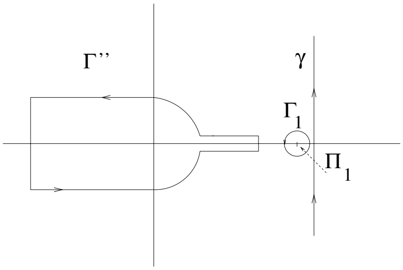

Fig. 3. Contours Γ and γ

- *there exists* $C = C(\eta) > 0$ *such that for any* $z \in \Gamma_1'' \cup \Gamma_2''$, $\mathrm{Re}\big(F_{C(\pi_1)}(z) - F_{C(\pi_1)}(\pi_1)\big) \geq C > 0$.
- $\mathrm{Re}\left(F_{C(\pi_1)}\right)$ *achieves its minimum on* $\Gamma_3'' \cup \Gamma_4'' \cup \Gamma_5''$ *at* $x^* = x^*(\eta)$ *defined in Remark 3.2.*

Proof of Lemma 3.3. Consider first $\Gamma_1'' \cup \Gamma_2''$. The function $x \mapsto \mathrm{Re}\big(F_{C(\pi_1)}(x) - F_{C(\pi_1)}(\pi_1)\big)$ is decreasing on the interval $[\frac{1}{\pi_1}, \pi_1]$. Thus, for any $x \in [1, \pi^*]$, which is a compact interval of $(\frac{1}{\pi_1}, \pi_1)$,

$$\mathrm{Re}\left(F_{C(\pi_1)}(x) - F_{C(\pi_1)}(\pi_1)\right) \geq \mathrm{Re}\left(F_{C(\pi_1)}(\pi^*) - F_{C(\pi_1)}(\pi_1)\right) \geq C_o > 0.$$

As $F'_{C(\pi_1)}$ is uniformly bounded in a compact set away from 0, we can now choose η small enough so that $\mathrm{Re}\left(F_{C(\pi_1)}(z)\right) \geq F_{C(\pi_1)}(\pi_1) + \frac{C_o}{2}$, $\forall z = x + iy$, with $x \in [1, \pi^*]$, $|y| \leq \eta$.

Now, along Γ_3'', $\frac{d}{d\theta}\mathrm{Re}\left(F_{C(\pi_1)}(\frac{C(\pi_1)}{2}e^{i\theta})\right) = \sin\theta C(\pi_1)^2/2(1-\cos\theta) > 0$, since $\theta \geq \theta_o > 0$. Along Γ_4'', for $z = iC(\pi_1)/2 - x$, $x > 0$, $\frac{d}{dx}\mathrm{Re}\left(F_{C(\pi_1)}(\frac{iC(\pi_1)}{2} - x)\right) = C(\pi_1) + x + \frac{x}{|\frac{iC(\pi_1)}{2} - x|^2} > 0$. Along Γ_5'', and for $z = -R + it$, $t \leq \frac{C(\pi_1)}{2}$ $\mathrm{Re}\left(F_{C(\pi_1)}(z)\right) = \frac{R^2}{2} + C(\pi_1)R - \frac{t^2}{2} - \frac{1}{2}\log(|R + it|^2)$. We can then choose R large enough so that along Γ_5'', $\mathrm{Re}\left(F_{C(\pi_1)}(z)\right) \geq \mathrm{Re}\left(F_{C(\pi_1)}(x^*)\right)$. □

Now, we fix η and R so that Lemma 3.3 holds. Then, one has

$$\mathrm{Re}\left(F_{C(\pi_1)}(z) - F_{C(\pi_1)}(\pi_1)\right) \geq C > 0 \text{ and } \mathrm{Re}(z - \tilde{\pi}_1) > \epsilon, \quad \forall z \in \Gamma''.$$

Using now the fact that Γ'' is a fixed (independent of N) length contour along which $|1/g|$ is uniformly bounded, it is then straightforward to obtain Fact 3.2 from Lemma 3.3. □

Combining Fact 3.2 with Fact 3.1 gives formula (49), which finally proves Proposition 3.2.

4. Proof of Theorem 1.3

In the whole Section, we assume that π_1 lies in a compact interval of $(1, \infty)$. We further make the simplifying assumption

$$W_N = \text{ diag } (\pi_1, \dots, \pi_1, 0, \dots, 0),$$

with π_1 of multiplicity k_N such that $\frac{k_N}{N} \to 0$, $k_N \to \infty$ as N goes to infinity. The changes to be made in the case where W_N admits eigenvalues between 0 and π_1 (including the case where the number of these eigenvalues is increasing with N) will be indicated at the end of this section.

Let Γ be a contour encircling the poles π_1 and 0, oriented counter clockwise and $\gamma = A + it, t \in \mathbb{R}$, such that $\Gamma \cap \gamma = \emptyset$. Then the correlation kernel is now given by

$$K_N(u, v) = \frac{N}{(2i\pi)^2} \int_\Gamma dz \int_\gamma dw e^{-N(z^2/2-uz)+N(w^2/2-wv)} \times \left(\frac{w}{z}\right)^{N-k_N} \left(\frac{\pi_1 - w}{\pi_1 - z}\right)^{k_N} \frac{1}{w-z}.$$

Let us briefly indicate the idea of the proof of Theorem 1.3. Let $C(\pi_1)$ and $\sigma^2(\pi_1)$ be defined by (6) and α_N be defined by (8). The idea is to make a second order Taylor expansion around π_1. If $w = \pi_1 + \alpha_N s$, and $u = C(\pi_1) + \frac{\alpha}{\sigma^2}\alpha_N$, for some $|\alpha| < 2\sigma(\pi_1)$, then the exact exponential term, F_u, defined by

$$F_u(w) := w^2/2 - wu + (1-\alpha_N^2)\log w + \alpha_N^2 \log(w - \pi_1) \tag{58}$$

satisfies $$F_u(\pi_1 + \alpha_N s) = Ct(\pi_1) + k_N\left(\frac{s^2}{2\sigma^2} - \frac{\alpha s}{\sigma^2} + \log s + \alpha_N G(s)\right),$$
for some constant term $Ct(\pi_1)$ depending on π_1 and a function G that should not grow much. The function $H(s) = \frac{s^2}{2\sigma^2} - \frac{\alpha s}{\sigma^2} + \log s$ is then the exponential term of the correlation kernel (14) of the GUE with parameter $\sigma^2 = \sigma^2(\pi_1)$. Thus, suitably rescaled, the k_N largest eigenvalues of the deformed Wigner ensemble should exhibit the same fluctuations as the eigenvalues of a $k_N \times k_N$ GUE with parameter σ^2. That is what we now show.

Let then ρ be the density of the semi-circular law with parameter $\sigma^2 = \sigma^2(\pi_1)$ defined in (1). Let x_o, y_o be fixed and set

$$\begin{aligned} u &= C(\pi_1) + \frac{\alpha_N x}{\sigma^2},\ x = \alpha + \frac{x_o}{k_N\rho(\alpha)}; \\ v &= C(\pi_1) + \frac{\alpha_N y}{\sigma^2},\ y = \alpha + \frac{y_o}{k_N\rho(\alpha)}. \end{aligned} \tag{59}$$

For u, v satisfying (59), we here consider the rescaled correlation kernel

$$K'_N(x, y) = \frac{\alpha_N}{k_N\sigma^2\rho(\alpha)} \exp\{-N\frac{(x-y)}{\sigma^2}\alpha_N\pi_1\} K_N(u, v). \tag{60}$$

The aim of the rest of this section is to obtain the following result.

Proposition 4.1. *Assume* $\alpha = 2\sigma\cos\theta_c$ *in (59), (60), with* $0 < |\theta_c| < \pi$, *and let* $t^{\pm}_{c,\alpha} = \sigma\cos\theta_c$. *Then,*

$$\lim_{N\to\infty} K'_N(x,y)\exp\left\{(y_o - x_o)\mathrm{Re}\left(\frac{t^{+}_{c,\alpha}}{\sigma^2}\right)\right\} = \frac{\sin\pi(x_o - y_o)}{\pi(x_o - y_o)}. \tag{61}$$

Remark 4.1. Theorem 1.3 is then an easy consequence of Proposition 4.1 (see e.g. [4], Section 6).

Before beginning the proof of Proposition 4.1, it is convenient to make here the following simplifying assumptions. We assume that $N \geq N_o$, where N_o is such that

$$\forall N \geq N_o, \forall |t| \leq 2\sigma + 1, |\pi_1 + \alpha_N t| \geq \frac{\pi_1}{2}, |\pi_1 + \alpha_N t - 1| \geq \frac{\pi_1 - 1}{2}. \tag{62}$$

4.1. Rewriting the kernel

In this subsection, we first split the kernel into two subkernels, since the idea is to prove that the asymptotics of $K'_N(x,y)$ is lead by the integral performed on a neighborhood of π_1. Then we obtain an integral representation of these subkernels suitable for the saddle point analysis.

Let then Γ_1 (resp. Γ_2) be the circle of ray σ (resp. 1) centered at π_1 (resp. the origin). Both contours are oriented counterclockwise. Let F_u be given by (58), and define the kernels

$$\begin{aligned} K_{N,1}(u,v) &= \alpha_N \exp\{-N\alpha_N\pi_1\frac{(x-y)}{\sigma^2}\}\int_{\Gamma_1} dz \int_{\gamma} dw \\ &\quad \times \exp\{-NF_u(z) + NF_v(w)\}\frac{1}{w-z}, \\ K_{N,2}(u,v) &= \alpha_N \exp\{-N\alpha_N\pi_1\frac{(x-y)}{\sigma^2}\}\int_{\Gamma_2} dz \int_{\gamma} dw \\ &\quad \times \exp\{-NF_u(z) + NF_v(w)\}\frac{1}{w-z}. \end{aligned} \tag{63}$$

Proposition 4.2. *Let* $K'_N(x,y)$ *be given by (60). Then,* $K'_N(x,y) = K'_{N,1}(x,y) + K'_{N,2}(x,y)$, *where*

$$\begin{aligned} K'_{N,1}(x,y) &= \frac{1}{k_N\sigma^2\rho(\alpha)(2i\pi)^2}K_{N,1}(u,v) \textit{ and} \\ K'_{N,2}(x,y) &= \frac{1}{k_N\sigma^2\rho(\alpha)(2i\pi)^2}K_{N,2}(u,v). \end{aligned} \tag{64}$$

As $x \simeq y \simeq \alpha$ in (59), it is not hard to see that the two integrands $F_u(w)$, $F_v(w)$ have the same critical points lying around π_1. While this should not cause any trouble for the saddle point analysis of $K'_{N,2}$, this prevents that of $K'_{N,1}$, because of the singularity $\frac{1}{w-z}$. Thus, we have to remove the singularity of the kernel $K'_{N,1}$. This is the object of the following Proposition.

Set, for $s \in \mathbb{C}$ such that $\mathrm{Re}(\pi_1 + \alpha_N sx) > 0$, $\forall x \in [0,1]$,

$$G(s) = \alpha_N^2 s^3 \int_0^1 \frac{(1-x)^2}{(\pi_1 + \alpha_N sx)^3} dx - s \int_0^1 \frac{1}{\pi_1 + s\alpha_N x} dx. \tag{65}$$

Proposition 4.3. *Assume $N \geq N_o$, with N_o defined in (62), then with the rescalings (59),*

$$\begin{aligned} K'_{N,1}(x,y) = \frac{k_N}{(2i\pi)^2(y_o - x_o)} &\int_{\Gamma'_1}\int_{\gamma'} \\ &\times \exp\left\{k_N\left(\frac{t^2-2yt}{2\sigma^2} + \alpha_N G(t) - \frac{s^2-2sx}{2\sigma^2} - \alpha_N G(s)\right)\right\} \\ &\times \left(\frac{t}{s}\right)^{k_N}\left(1 - \exp\left\{\frac{s(y_o - x_o)}{\sigma^2\rho(\alpha)}\right\}\right) \\ &\times \frac{1}{s}\left(\frac{s+t-y}{\sigma^2} + \alpha_N\left(\frac{tG'(t) - sG'(s)}{t-s}\right)\right) ds dt, \end{aligned} \tag{66}$$

where Γ'_1 is a circle of ray σ around the origin and $\gamma' = A + i\mathbb{R}$, with $A \geq -2\sigma - 1$.

Remark 4.2. Γ'_1 can now cross γ'.

Proof of Proposition 4.3. Assume that $\gamma'' = A + i\mathbb{R}$, $A > 0$ large enough not to cross a circle of ray σ around π_1. We first show the formula

$$\begin{aligned} K_{N,1}(u,v) = \frac{k_N}{(2i\pi)^2} &\int_{\Gamma'_1} ds \int_{\gamma''} dt \left(\frac{t}{s}\right)^{k_N} \frac{1}{t-s} \\ &\times \exp\left\{k_N\left(\frac{t^2-2yt}{2\sigma^2} + \alpha_N G(t) - \frac{s^2-2sx}{2\sigma^2} - \alpha_N G(s)\right)\right\}. \end{aligned} \tag{67}$$

Define $\tilde{F}_u(z) := z^2/2 - uz + \log z$. Here we choose the principal branch of the logarithm. We now make the change of variables $z = \pi_1 + \alpha_N s$. Then one has that $\tilde{F}_u(\pi_1 + \alpha_N s) = \tilde{F}_{C(\pi_1)}(\pi_1 + \alpha_N s) - (u - C(\pi_1))(\pi_1 + \alpha_N s)$. Performing now a Taylor expansion for the real and imaginary part gives

$$\begin{aligned} \tilde{F}_{C(\pi_1)}(\pi_1 + \alpha_N s) = \tilde{F}_{C(\pi_1)}(\pi_1) &+ \frac{\tilde{F}''_{C(\pi_1)}(\pi_1)}{2}\alpha_N^2 s^2 \\ &+ \alpha_N^3 s^3 \int_0^1 \frac{(1-x)^2}{(\alpha_N sx + \pi_1)^3} dx. \end{aligned} \tag{68}$$

Finally, as $\pi_1 + \alpha_N s$ does not reach $\mathbb{R}_-$, because of (62), we can write $\alpha_N^{k_N} \frac{s^{k_N}}{(\pi_1 + \alpha_N s)^{k_N}} = \alpha_N^{k_N} s^{k_N} \exp\left\{-k_N\left(\log(\pi_1) + \alpha_N s \int_0^1 \frac{1}{\pi_1 + \alpha_N sx} dx\right)\right\}$. Thus we obtain (67) for contours Γ'_1 and γ'' chosen as above (as neither $\pi_1 + \alpha_N t$ nor $\pi_1 + \alpha_N s$ reaches $\mathbb{R}_-$) .

Finally, we use the same method as in [9] to remove the singularity. In (67), we make the change of variables $s \mapsto \beta s$, $t \mapsto \beta t$ for β real close to one. Thanks to Cauchy's theorem, we can deform back these contours to γ and Γ. Taking the derivative at $\beta = 1$ gives

$$\begin{aligned} K_{N,1}(u,v) = &-\frac{k_N^2}{(2i\pi)^2}\int_{\Gamma_1'} ds \int_{\gamma''} dt \\ &\times \exp\left\{k_N\left(-\frac{s^2-2sx}{2\sigma^2} - \alpha_N G(s) + \frac{t^2-2ty}{2\sigma^2} + \alpha_N G(t)\right)\right\} \\ &\times\left(\frac{t^2-s^2}{\sigma^2} + \frac{xs-yt}{\sigma^2} + \alpha_N t G'(t) - \alpha_N s G'(s)\right)\left(\frac{t}{s}\right)^{k_N}\frac{1}{t-s}. \end{aligned} \tag{69}$$

Now this gives, using (67) and for the rescalings (59),

$$\begin{aligned} \frac{d((\frac{x}{\sigma^2}-\frac{y}{\sigma^2})K'_{N,1}(x,y))}{d(\frac{x}{\sigma^2})} =& -\frac{k_N^2}{(2i\pi)^2}\int_{\Gamma_1'}\int_{\gamma''} \\ &\times \exp\left\{k_N\left(\frac{t^2-2ty}{2\sigma^2} + \alpha_N G(t) - \frac{s^2-2sx}{2\sigma^2} - \alpha_N G(s)\right)\right\} \\ &\left(\frac{t}{s}\right)^{k_N}\left(\frac{s+t-y}{\sigma^2} + \alpha_N\frac{tG'(t)-sG'(s)}{t-s}\right) ds dt. \end{aligned} \tag{70}$$

Solving (70) with an integration by parts, we obtain finally Proposition 4.3 (we can then move γ'' to γ'). □

4.2. A study of critical points

In this part, under Assumption 1.2, we show that the exact critical points of the integrands, in $K'_{N,1}$ and $K'_{N,2}$, lie on a curve that is almost the circle of ray $\sigma(\pi_1)$ around π_1, provided α_N tends to 0. Furthermore, we prove that the relevant critical points for the saddle point analysis are well approximated by those of H_{α/σ^2} if

$$H_u(t) := \frac{t^2}{2\sigma^2} - ut + \log t. \tag{71}$$

Consider the exact exponential term to be analyzed, $F_u(w) := w^2/2 - uw + (1 - \alpha_N^2)\log(w) + \alpha_N^2 \log(w - \pi_1)$. The equation $F'_u(w) = w - u + (1-\alpha_N^2)/w + \alpha_N^2/(w-\pi_1) = 0$ admits three solutions. One is real, in the interval $(0, \pi_1)$, and two others $w_N^{\pm}$ that are conjugate. We now study these critical points $w_N^{\pm}$. It is an easy fact that any critical point w for F_u with non zero imaginary part satisfies the equation

$$\frac{1-\frac{1}{|w|^2}}{1-\frac{1}{|w-\pi_1|^2}} = \frac{-\alpha_N^2}{1-\alpha_N^2}. \tag{72}$$

Then the solution of (72) define one or two (depending on α_N^2) curves encircling 0 and π_1.

Consider now a sequence α_N such that $\lim_{N\to\infty} \alpha_N = 0$. We now show that critical points for F_u almost lie on the curve $C_1 = \{\pi_1 + \alpha_N \sigma e^{i\theta}, 0 \leq \theta \leq 2\pi\}$, where $\sigma = \sigma(\pi_1)$ in the following.

Lemma 4.1. *Let u be given by (59) with $\alpha = 2\sigma\cos(\theta_c)$, $0 < |\theta_c| < \pi$. Then, the critical points $w_N^{\pm}$ are non real and $\exists\, C(\pi_1) > 0$ such that $w_N^{\pm} = \pi_1 + \alpha_N t_N^{\pm}$ with $|t_N^{\pm} - \sigma e^{i\pm\theta_c}| \leq C(\pi_1)\alpha_N + \frac{|x_o|}{k_N\rho(\alpha)}$.*

Proof of Lemma 4.1. If $u = C(\pi_1) + \alpha_N\alpha/\sigma^2$, then

$$\begin{aligned} F_u'(\pi_1 + \alpha_N t) &= \alpha_N\left(H'_{\frac{\alpha}{\sigma^2}}(t) + \alpha_N G'(t)\right) \\ &= \alpha_N\left(H'_{\frac{\alpha}{\sigma^2}}(t) + \alpha_N \frac{t^2 - \pi_1^2}{\pi_1^2(\pi_1 + \alpha_N t)}\right), \end{aligned} \tag{73}$$

with $H'_{\frac{\alpha}{\sigma^2}}(t) = \frac{t}{\sigma^2} - \frac{\alpha}{\sigma^2} + \frac{1}{t}$. Set now $T_o = max\{2\sigma(\pi_1), 4\pi_1^2\}$. As π_1 lies in a compact set of $(1,\infty)$, it is not hard to see that, for $|t| < T_o$, and N large enough so that $\alpha_N T_o < \pi_1/2$, there exists $C(\pi_1) > 0$ such that $\left|\left(\frac{t^2 - \pi_1^2}{\pi_1^2(\pi_1 + \alpha_N t)}\right)^{(l)}\right| \leq C(\pi_1),\quad 0 \leq l \leq 4$. Thus, if now u is now given as in (59),

$$\begin{aligned} &\left|\frac{F_u'(\pi_1 + \alpha_N t)}{\alpha_N} - H'_{\frac{\alpha}{\sigma^2}}(t)\right| \leq \alpha_N C(\pi_1) + \frac{|x_o|}{k_N\rho(\alpha)}, \\ &|F_u''(\pi_1 + \alpha_N t) - H''_{\frac{\alpha}{\sigma^2}}(t)| \leq \alpha_N C(\pi_1). \end{aligned} \tag{74}$$

Now, if $\alpha = 2\sigma\cos\theta_c$, with $0 < |\theta_c| < \pi$, $H_{\frac{\alpha}{\sigma^2}}$ admits two critical points that are conjugate, and given by $t_{c,\alpha}^{\pm} = \sigma e^{i\pm\theta_c}$. Thus using (74), we obtain Lemma 4.1. □

4.3. Estimate for $K'_{N,1}$

This subsection is devoted to the proof of the following Proposition. Let $K'_{N,1}$ be the kernel defined in Proposition 4.2.

Proposition 4.4. *Assume $\alpha = 2\sigma\cos\theta_c$, with $0 < |\theta_c| < \pi$, and let $t_{c,\alpha}^{\pm} = \sigma e^{i\pm\theta_c}$.*

$$\lim_{N\to\infty} K'_{N,1}(x,y)\exp\left\{(y_o - x_o)\mathrm{Re}\left(\frac{t_{c,\alpha}^{+}}{\sigma^2}\right)\right\} = \frac{\sin\pi(x_o - y_o)}{\pi(x_o - y_o)}.$$

Proof of Proposition 4.4. The proof is organized as follows. As the correlation kernel given in Proposition 4.3 is not of the form (15), we analyze the double integral "simultaneously". First we define ascent and descent contours for H_{α/σ^2} and show that the perturbative terms, due to G defined in (65), do not grow too much. We then slightly deform these contours to go through the effective critical points of F_v, so that we can then perform the saddle point analysis.

Remark 4.3. From now on, as $t^{\pm}_{c,\alpha}$, as well as $t^{\pm}_N$, are conjugate, we may drop the $\pm$ sign (when possible) in the following, if results proved for t^+_N hold for t^-_N up to conjugation.

Set $\gamma' = t^+_{c,\alpha} + it, t \in \mathbb{R}$, oriented from bottom to top. Let also $0 < \epsilon << Im(t^+_{c,\alpha})$ be given.

Lemma 4.2. *One has* $\max\left\{\left|e^{k_N H_{\frac{\alpha}{\sigma^2}}(t^+_{c,\alpha}+it)}\right|, -Im(t^+_{c,\alpha}) \le t \le -Im(t^+_{c,\alpha})+\epsilon\right\} = \left|e^{k_N H_{\frac{\alpha}{\sigma^2}}(\mathrm{Re}(t^+_{c,\alpha})+i\epsilon)}\right|$ *and there exists* $c_o > 0$ *such that* $\left|e^{k_N H_{\frac{\alpha}{\sigma^2}}(t^+_{c,\alpha}+it)}\right| \le \left|e^{k_N H_{\frac{\alpha}{\sigma^2}}(t^+_{c,\alpha})}\right| e^{-c_o k_N t^2}$, $\forall t \in [-Im(t^+_{c,\alpha}) + \epsilon, \infty]$.

Proof. This follows from the fact that $\frac{d}{dt}\log\left|e^{H_{\frac{\alpha}{\sigma^2}}(\sigma\cos\theta_c+it)}\right| = -t\left(\frac{1}{\sigma^2} - \frac{1}{\sigma^2\cos^2\theta_c + t^2}\right)$ ($t > 0$ if $\theta_c = \frac{\pi}{2}$). And $t \mapsto |e^{H(\sigma\cos\theta_c+it)}|, 0 \le t \le \epsilon$, is a decreasing function if ϵ is small enough. □

We now show that $\mathrm{Re}(F_u)$ decreases faster than (resp. almost as) $H_{\frac{\alpha}{\sigma^2}}$ on γ', if $t > 0$ is large enough and $\mathrm{Re}\left(t^{\pm}_{c,\alpha}\right) \ge 0$ (resp $\mathrm{Re}\left(t^{\pm}_{c,\alpha}\right) < 0$). Let ϵ be as in Lemma 4.2, $\eta > 0$ (small) be given.

Lemma 4.3. *There exist* $T_o > 0$, N_1 *depending on* π_1 *only,* $C_o > 0$, $C_{T_o} > 0$ *such that, for* $N \ge N_1$,

$$\begin{aligned}
&\left|e^{\{NF_u(\pi_1+\alpha_N(t^+_{c,\alpha}+it))\}}\right| \\
&\quad \le e^{\{N\mathrm{Re}\left(Fu(\pi_1+\alpha_N t^+_{c,\alpha})\right)-k_N C_o\epsilon^2/8\}},\ t \in [-Im(t^+_{c,\alpha}), -Im(t^+_{c,\alpha})+\epsilon], \qquad (75)\\
&\left|e^{\{NF_u(\pi_1+\alpha_N(t^+_{c,\alpha}+it))\}}\right| \\
&\quad \le e^{\{N\mathrm{Re}\left(Fu(\pi_1+\alpha_N t^+_{c,\alpha})\right)-k_N C_o t^2/4\}},\ t \in [-Im(t^+_{c,\alpha})+\epsilon, -\eta] \cup [\eta, T_o], \qquad (76)\\
&\left|e^{N\{F_u(\pi_1+\alpha_N(t^+_{c,\alpha}+it))\}}\right| \\
&\quad \le \left|e^{N\{Fu(\pi_1+\alpha_N t^+_{c,\alpha})-k_N C_o T_o^2/4-k_N C_{T_o}(t^2-T_o^2)/4\}}\right|,\ T_o \le t. \qquad (77)
\end{aligned}$$

Proof of Lemma 4.3. We first examine the case where $\mathrm{Re}(t^{\pm}_{c,\alpha}) = \sigma cos\theta_c > 0$. Using that for $t > T_o = \max(4\pi_1^2, 2\sigma(\pi_1))$, $Im(G'(t)) > 0$, we obtain that for $t > T_o$,

$$\frac{d}{dt}\mathrm{Re}\left(\frac{F_u(\pi_1+\alpha_N(t^+_{c,\alpha}+it))}{\alpha_N^2}\right) < -Im\left(H'_\alpha(t^+_{c,\alpha}+it)\right) \le -C_{T_o}t, \qquad (78)$$

where $C_{T_o} = 1/\sigma^2 - 1/|t^+_{c,\alpha} + iT_o|^2 \ge 1/\sigma^2 - 1/|\sigma e^{i\theta_c} + iT_o|^2 > 0$. Now, G, G' are uniformly bounded on a compact set K (independant of N) containing

$\gamma' \cap \{|Im(w)| \leq T_o + 2\sigma\}$. Thus, using Lemma 4.2, we know that $\exists C_o > 0$ such that, for N large enough,

$$\frac{d}{dt}\text{Re}\left(H_{\frac{\alpha}{\sigma^2}}(t^+_{c,\alpha} + it) + \alpha_N G(t^+_{c,\alpha} + it)\right)$$
$$\leq -C_o t/2, \forall t \in [-Im(t^+_{c,\alpha}) + \epsilon, -\eta] \cup [\eta, T_o].$$

This gives (76). The fact that G is bounded on K also gives that (75) holds for N large enough. Combining (76) with (78) gives then (77). This proves Lemma 4.3 in this case.

If $\text{Re}(t^{\pm}_{c,\alpha}) = \sigma \cos\theta_c \leq 0$, (75) and (76) are proved as above. One can then check that $\exists\, C(\pi_1) > 0$ such that $Im\left(G'(\text{Re}(t^+_{c,\alpha}) + iT)\right) \geq -C(\pi_1)T$, provided $\text{Re}(\pi_1 + \alpha_N t^+_{c,\alpha}) \geq \pi_1/2$. This holds for N large enough and we can then find $N_1 > 0$, such that $C_{T_o} - C(\pi_1)\alpha_N > \dfrac{C_{T_o}}{2}$, $\forall N \geq N_1$. Thus for $N \geq N_1$, and $t \geq T_o$, one has that $\dfrac{d}{dt}\text{Re}\left(\dfrac{F_u(\pi_1 + \alpha_N(t^+_{c,\alpha} + it))}{\alpha_N^2}\right) < -\dfrac{C_{T_o}t}{2}$. This finishes the proof of Lemma 4.3. □

We now turn to the second contour. Define then $\Gamma'_1 = \sigma e^{i\theta}, \theta \in [0, 2\pi]$, oriented counterclockwise. Note that Γ'_1 describes the curve of critical points for H_x when x describes $[-2\sigma, 2\sigma]$.

Lemma 4.4. *Assume that* $\alpha = 2\sigma\cos\theta_c$. *Then, there exists* $c_o > 0$ *such that, for any* $\theta \in [0, 2\pi]$,
$|e^{-k_N H_{\alpha/\sigma^2}(\sigma e^{i\theta})}| \leq |e^{-k_N H_{\alpha/\sigma^2}(\sigma e^{i\theta_c})}| e^{-k_N c_o(\theta-\theta_c)^2}$.

Proof. If $|\theta| < \pi$, $\dfrac{d}{d\theta}\text{Re}\left(H_{\alpha/\sigma^2}(\sigma e^{i\theta})\right) = 2\sin\theta(\cos\theta_c - \cos\theta)$. The computation for $\theta = \pi$ is here left. □

As the contour Γ'_1 lies in a fixed compact set away from the singularities of G, we know that the contribution of G will not perturb the saddle point analysis on Γ'_1.

Before performing the asymptotic expansion of $K'_{N,1}(x, y)$, one should take care of the remaining terms, which should not explode due to the perturbation G. Set

$$h(s,t) = \frac{\exp\{\frac{s(x_o - y_o)}{\sigma^2\rho(\alpha)}\} - 1}{y_o - x_o}, \quad K_y(s) = H_{\frac{y}{\sigma^2}}(s) + \alpha_N G(s), \tag{79}$$

$$g(s,t) = \frac{1}{s}\left(\frac{s+t-y}{\sigma^2} + \alpha_N \frac{tG'(t) - sG'(s)}{t-s}\right) = \frac{tK'_y(t) - sK'_y(s)}{s(t-s)}. \tag{80}$$

Then $K'_{N,1}(x, y) = \dfrac{k_N}{(2i\pi)^2}\displaystyle\int_{\Gamma'}\int_{\gamma} h(s,t)g(s,t)e^{\{-k_N K_y(t) + k_N K_y(s)\}} ds dt.$

We have to check that the function $g(s, t)$ will not perturb the saddle point analysis. As the contour Γ'_1 is compact and for $|Im(w)| \leq T_o$, the functions $G(t)$, $G'(t)$

are bounded by some constant depending on π_1 only. Thus $g(s,t)$ is bounded on $\Gamma_1' \cup (\gamma \cap \{|Im(w)| \leq T_o\})$. Note also that along γ', $\frac{1}{|\pi_1 + \alpha_N t|} \leq \frac{2}{\pi_1}$ so that $|G'(t)| \leq \alpha_N t^2$. Thus, there exists some constant $C > 0$ such that, for $t > T_o$, using Lemma 4.3,

$$|g(s,t)| \left| \frac{\exp\{NF_u(\pi_1 + \alpha_N(t_{c,\alpha} + it))\}}{\exp\{NF_u(\pi_1 + \alpha_N t_{c,\alpha})\}} \right| \leq Ct^3 \exp\left\{-Ck_N \frac{t^2}{4}\right\} \leq \exp\left\{-Ck_N \frac{t^2}{8}\right\},$$

for N large enough. This is the needed estimate to perform the saddle point analysis.

Now, and this is the core of the argument, we slightly deform the contours γ and Γ_1' to contours γ_N and Γ_N that go through the effective critical points $t_N^{\pm}$ of K_y. By Lemma 4.1, these contours lie within a C^1 distance of γ (resp. Γ_1') smaller than $C\alpha_N$ for some constant $C > 0$. Furthermore, γ_N and Γ_N coincide with γ and Γ_1' outside the disks $|t - t_{c,\alpha}^{\pm}| < \eta$ (η small). Then , by Proposition 4.3, Lemmas 4.2, 4.3, 4.4 and (77), we obtain, by a standard saddle point argument that

$$\lim_{N\to\infty} K_{N,1}'(x,y) = \sum_{b,d=\pm 1} \frac{sgn(b)}{(2i\pi)^2} \frac{2\pi \exp\{k_N(K_y(t_N^b) - K_y(t_N^d))\}}{i\sqrt{K_y''(t_N^b)K_y''(t_N^d)}} g(t_N^b, t_N^d) h(t_N^b, t_N^d). \quad (81)$$

Now, the critical points are conjugate, thus $K_y(t_N^+) = \overline{K_y(t_N^-)}$. Using (80), one can check that

$$g(t_N^+, t_N^-) = g(t_N^-, t_N^+) = 0, \quad g(t_N^{\pm}, t_N^{\pm}) = K_y''(t_N^{\pm}),$$

so that only the contribution of equal critical points have to taken into account in (81). By Lemma 4.1, one has $Im(t_N^{\pm}) = \pm\pi\sigma^2\rho(\alpha) + o(1)$, so that for h given by (79),
$\frac{h(t_N^-, t_N^-) - h(t_N^+, t_N^+)}{2i} \exp\{(y_o - x_o)\mathrm{Re}(t_N^+/\sigma^2)\} = \frac{\sin\pi(x_o - y_o)}{\pi(x_o - y_o)}$. This yields Proposition 4.4. □

4.4. *Estimate for* $K_{N,2}'(x,y)$

This subsection is devoted to the proof of the following Proposition. Let $K_{N,2}'$ be the kernel defined in Proposition 4.2.

Proposition 4.5. *There exists* $C_o > 0,\ N_o > 0$ *such that*

$$|K_{N,2}'(x,y)| \leq \exp\{-C_o N/a2\},\ \forall N \geq N_o.$$

Proof of Proposition 4.5. We first show that the function $\frac{1}{w - z}$ is bounded as $z \in \Gamma_2$ and $w = \pi_1 + \alpha_N t, t \in \gamma'$. By (62), we can assume that the image of γ'

under the map $t \mapsto \pi_1 + \alpha_N t$ lies in the half plane $\mathrm{Re}(w) > (\pi_1 + 1)/2$. Thus, for $z \in \Gamma_2$ and $w = \pi_1 + \alpha_N t, t \in \gamma'$, $\frac{1}{|w - z|} \leq \frac{2}{\pi_1 - 1}$. Now, for N large enough, $\min_{\Gamma_2} \mathrm{Re}(F_u(z)) = \mathrm{Re}(F_u(1))$ and 1 lies in a compact set of $(1/\pi_1, \pi_1)$. Then, we have that (as $t_N^+ = \overline{t_N^-}$, we can consider $t_N{}^+$ only, and drop the $+$ sign from now on)

$$\begin{aligned}
&\exp\{-NF_u(1) + NF_v(\pi_1 + \alpha_N t_N)\} \\
&\quad = \Bigg[\frac{\exp\left\{N\left((\pi_1 + \alpha_N t_N)^2/2 - C(\pi_1)(\pi_1 + \alpha_N t_N)\right)\right\}}{\exp\{N(1/2 - C(\pi_1))\}} (\pi_1 + \alpha_N t_N)^N \\
&\qquad \times \exp\{N(C(\pi_1) - v)(\pi_1 + \alpha_N t_N) - N(C(\pi_1) - u)\} (\pi_1 + \alpha_N t_N)^{-k_N} \\
&\qquad \times \left(\frac{\alpha_N t_N}{1 - \pi_1}\right)^{k_N} \Bigg] \qquad (82)
\end{aligned}$$

Now, it is easy to see that the term in the [ ] in $|(82)| \leq e^{C\alpha_N N}$, for some constant $C > 0$. Finally, using that $\left(\frac{\pi_1 + \alpha_N t_N}{\pi_1}\right)^N = \exp\left\{N\alpha_N \int_0^{t_N/\pi_1} \frac{du}{1 + \alpha_N u}\right\} \leq \exp\{N\alpha_N C'\}$, for some constant $C' > 0$ and N large enough, we obtain that there exists a constant C and N_o such that for $N \geq N_o$,

$$\begin{aligned}
&\left|\exp\{-NF_u(1) + NF_v(\pi_1 + \alpha_N t_N)\}\right| \\
&\quad \leq \exp\{N\left(\pi_1^2/2 - C(\pi_1)\pi_1\right) - N\left(1/2 - C(\pi_1)\right) + C\alpha_N N\}\pi_1^N.
\end{aligned}$$

Now, $\exists\ C_o > 0$ such that $\exp\{N\left(\pi_1^2/2 - C(\pi_1)\pi_1\right)\}\pi_1^N \exp\{-N(1/2 - C(\pi_1))\} \leq \exp\{-C_o N\}$. This follows from the fact that the function $f : x \mapsto x^2/2 - C(\pi_1)x + \log x, x \geq 1$ is strictly decreasing in the interval $(1/\pi_1, \pi_1)$, as π_1 lies in a compact interval of $(1, \infty)$. Therefore for N large enough $K'_{N,2}(x, y) \leq \exp\{-C_o N/2\}$. □

Finally, combining Propositions 4.2, 4.4 and Proposition 4.5 yields Theorem 1.3.

4.5. Extensions

We now explain the changes to be made to prove Theorem 1.3 in the case where W_N has eigenvalues $\pi_i, i = 2, \ldots, r_N + 1$ distinct of 0, under Assumption 1.2. The exponential term to be analyzed is given by

$$\tilde{F}_u(w) = F_u(w) - \beta_N \log(w) + \frac{1}{N} \sum_{i=1}^{N\beta_N} \log(w - \pi_{i+1}), \qquad (83)$$

where F_u is given as in (58). Let also u be given as in Definition 1.6. Then, there exist constants, depending on π_1 only, such that, for all t in a given compact set of $\mathbb{C}^*$,

$$N\tilde{F}_u(\pi_1 + \alpha_N t) = NCt(\pi_1) + \beta_N Ct'(\pi_1) + k_N H_{\frac{\alpha}{\sigma^2}}(t) + k_N O(\alpha_N + \beta_N). \qquad (84)$$

Define $G_1(t) := \frac{1}{k_N}\big(N\tilde{F}_u(\pi_1+\alpha_N t)-NCt(\pi_1)-\beta_N Ct'(\pi_1)-k_N H_{\frac{\alpha}{\sigma^2}}(t)\big)$, which plays the role of the function $\alpha_N G$ defined in (65). Let also $\tilde{t}_N^{\pm}$ be the critical points for $t \mapsto \tilde{F}_u(\pi_1 + \alpha_N t)$, and ρ be the density of the semi-circular law with parameter $\sigma^2(\pi_1)$ as before. As G_1 and its three first derivatives have no singularity in a given compact neighborhood K_o of 0, we readily have that

$$Im(\tilde{t}_N^{\pm}) = \pi\rho(\alpha) + O(\alpha_N + \beta_N). \tag{85}$$

Furthermore, defining $u_i = C(\pi_1) + \frac{\alpha}{\sigma^2}\alpha_N$, and given any compact set K of $\mathbb{C} \setminus \{0, \pi_2, \dots, \pi_{r+1}\}$, it is easy to check that there exists a constant $C(K)$ such that

$$|\tilde{F}_u^{(l)}(w) - F_{u_i}^{(l)}(w)| \le C(K)\beta_N,\ \forall w \in K,\ l = 0, \dots, 3. \tag{86}$$

Now formulas (84), (85) and (86) readily give that the asymptotics of $K'_{N,1}$ is unchanged. One simply replaces the function $\alpha_N G(\cdot)$ with the function $G_1(\cdot)$ in the proof of Proposition 4.4. For the proof of Proposition 4.5, we choose Γ_2' to be the circle of ray $\pi^* = \max\{\pi_2+(\pi_1-\pi_2)/2, 1\}$ completed by some contour encircling the $\pi_i < 0$. The latter contour lies in a fixed compact set K of $\mathbb{C}\setminus\{0, \pi_2, \dots, \pi_{r+1}\}$, by Assumption 1.2. Then $\mathrm{Re}(\tilde{F}_u(w)) > \mathrm{Re}(\tilde{F}_u(\pi^*)) - C(K)\beta_N$, $\forall w \in \Gamma_2'$. The fact that $\mathrm{Re}(\tilde{F}_u(\pi^*)) > \mathrm{Re}(\tilde{F}_u(\pi_1+\alpha_N\tilde{t}_N^{\pm}))$ now follows from the same arguments as in the proof of Proposition 4.5. This finishes the proof of Theorem 1.3 in this case.

5. Proofs of Theorem 1.4 and Theorem 1.5

In this Section we first prove Theorem 1.4 under the following simplifying assumptions. We assume that $\pi_1 > 1$ is given independently of N and that $W_N = \mathrm{diag}\,(\pi_1, \dots, \pi_1, 0, \dots, 0)$, with π_1 of multiplicity k_N, for some sequence k_N satisfying (7). Changes to be made in the case where W_N has eigenvalues distinct of 0 and π_1, or to prove Theorem 1.5, will be indicated in subsection 5.4 below. With the above assumption, F_u, defined by (9), becomes

$$F_u(w) = \frac{w^2}{2} - uw + (1-\alpha_N^2)\log(w) + \alpha_N^2 \log(w - \pi_1). \tag{87}$$

The basic idea for the study of the correlation kernel at the edge is to perform a third order Taylor expansion of F_u close to the degenerate critical point w_o defined by $F_u'(w_o) = F_u''(w_o)$. This point is close to $\pi_1+\alpha_N\sigma$, which is the degenerate critical point of $H_{2/\sigma}$. The ascent or descent curves for $F_u(\pi_1+\alpha_N t)$ should then be those for $H_{2/\sigma}$ slightly modified in a neighborhood of width $k_N^{-1/3}$ of $\pi_1 + \alpha_N\sigma$, to go through the exact degenerate critical point. This simple analysis can be achieved as long as $k_N << N^{3/7}$. This is the regime where the bulk of $N - k_N$ eigenvalues does not interfere with the k_N largest eigenvalues. For the other regimes, one will have to define new contours, that are descent or ascent paths for F_u, and show that the Taylor expansion can still be made in a neighborhood of w_o.

We will however see that the asymptotic expansion is still lead in some way by $H_{2/\sigma}$.

We set as in (13), $w_o = \pi_1 + \alpha_N t_r$ and consider the rescalings

$$u = u_o + x\left(\frac{\nu_N}{2}\right)^{1/3}\frac{\alpha_N}{k_N^{2/3}}, \quad v = u_o + y\left(\frac{\nu_N}{2}\right)^{1/3}\frac{\alpha_N}{k_N^{2/3}}, \tag{88}$$

where

$$\nu_N = \alpha_N F_{u_o}^{(3)}(\pi_1 + \alpha_N t_r) = \frac{2}{t_r^3} + \alpha_N \frac{1-\alpha_N^2}{(\pi_1 + \alpha_N t_r)^3}. \tag{89}$$

Let $\epsilon > 0$ be given. From now on, we consider the rescaled correlation kernel

$$K'_N(x, y) = \frac{\alpha_N}{k_N^{2/3}}\left(\frac{\nu_N}{2}\right)^{1/3} K_N(u, v)\exp\left\{-N(u-v)\left(\pi_1 + \alpha_N(t_r + \frac{\epsilon}{k_N^{2/3}})\right)\right\}. \tag{90}$$

The end of this section is now devoted to the proof of Theorem 1.4. This proof is here indirect, since we will first split the correlation kernel into two subkernels. These subkernels are then analyzed separately, using the same scheme as in Section 2.

Before beginning the proof of Theorem 1.4, it is convenient to make the following assumption on N. Let then $t_c = \sigma$ be the degenerate critical point for $H_{2/\sigma}$ and define sequences μ_N, μ'_N by

$$u_o = C(\pi_1) + \alpha_N \frac{2}{\sigma}(1 + \mu_N), \ t_r = t_c(1 + \mu'_N).$$

Then it is easy to check that there exists some constant C, depending on π_1 only, such that $|\mu_N|, |\mu'_N| \leq C\alpha_N$. Let also R_o and ν_N be defined as in (26) and (89). From now on, we assume that $N \geq N_o$, where N_o is such that

$$\forall N \geq N_o, \quad \forall |t| \leq 2\sigma R_o + 1, \ |\pi_1 + \alpha_N t| \geq \frac{\pi_1}{2}, \text{ and } |\pi_1 + \alpha_N t - 1| \geq \frac{\pi_1 - 1}{2}$$
$$t_r \in [\frac{\sigma}{2}, \frac{3\sigma}{2}], \ |\mu'_N| \leq \frac{1}{2}, \ |\mu_N| \leq \frac{1}{2}, \quad \frac{3}{t_c^3} \geq \nu_N \geq \frac{1}{t_c^3}. \tag{91}$$

5.1. Rewriting the kernel

In this subsection, we split the kernel $K'_N(x, y)$, defined in (90), into two subkernels, to get rid of the integrals performed away from a small neighborhood of π_1. Then we bring these subkernels to the form (15). Set $\tilde{t}_r = t_r + \frac{\epsilon}{k_N^{2/3}}$, and let

$\exp\{-NF_u(w)\}$ stand for $\exp\{-Nw^2/2+wu\}\dfrac{w^{k_N-N}}{(w-\pi_1)^{k_N}}$. Define the kernels

$$J_N(y) = k_N^{1/3}(\frac{\nu_N}{2})^{1/3}\int_{\gamma'} \times \exp\left\{NF_{u_o}(\pi_1+\alpha_N t)\right\}\exp\left\{-k_N^{1/3}y(\frac{\nu_N}{2})^{1/3}(t-\tilde{t}_r)\right\}dt, \quad (92)$$

$$H_N(x) = k_N^{1/3}(\frac{\nu_N}{2})^{1/3}\int_{\Gamma_1'} \times \exp\left\{-NF_{u_o}(\pi_1+\alpha_N s)\right\}\exp\left\{k_N^{1/3}x(\frac{\nu_N}{2})^{1/3}(s-\tilde{t}_r)\right\}ds, \quad (93)$$

$$H_N''(x) = k_N^{1/3}(\frac{\nu_N}{2})^{1/3}\int_{\Gamma''} \times \exp\left\{-NF_{u_o}(\pi_1+\alpha_N s)\right\}\exp\left\{k_N^{1/3}x(\frac{\nu_N}{2})^{1/3}(s-\tilde{t}_r)\right\}ds, \quad (94)$$

where Γ_1' is a contour encircling 0 not crossing $\gamma' := a+i\mathbb{R}$, $a>0$ and Γ'' is such that its image under the map $t \mapsto \pi_1+\alpha_N t$, is the circle of ray one centered at the origin. Both Γ_1' and Γ'' are oriented counterclockwise and γ' is oriented from bottom to top.

Proposition 5.1. *$K_N'(x,y) = K_N^1(x,y)+K_N^2(x,y)$, with*

$$K_N^1(x,y) = -\int_0^\infty H_N(x+u)J_N(y+u)du \text{ and}$$
$$K_N^2(x,y) = -\int_0^\infty H_N''(x+u)J_N(y+u)du.$$

Proof of Proposition 5.1. We first split the contour Γ into the contours $\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_1 is encircling π_1 and crosses the real axis at $\pi_1\pm\sigma\alpha_N$. Γ_2 is a contour encircling 0. Then, let $\gamma = A+i\mathbb{R}$ with $A>0$ large enough so that $\gamma\cap\Gamma_1=\emptyset$. We call K_N^1 the part of the integral formula defining (90) integrated on Γ_1, and γ. Then we obtain

$$\begin{aligned} K_N^1(x,y) &= \frac{N\alpha_N}{(2i\pi)^2 k_N^{2/3}}(\frac{\nu_N}{2})^{1/3}\int_{\Gamma_1}dz\int_\gamma dw \\ &\quad\times\frac{w^{N-k_N}(w-\pi_1)^{k_N}}{z^{N-k_N}(z-\pi_1)^{k_N}}\frac{1}{w-z}\frac{\exp\left\{Nw^2/2-Nu_o w-N(v-u_o)(w-\tilde{\pi}_1)\right\}}{\exp\left\{Nz^2/2-Nu_o z-N(u-u_o)(z-\tilde{\pi}_1)\right\}} \\ &= \frac{k_N^{2/3}}{(2i\pi)^2}(\frac{\nu_N}{2})^{2/3}\int_{\Gamma_1'}ds\int_{\gamma'}dt\int_0^\infty du \\ &\quad\times\exp\left\{NF_{u_o}(\pi_1+\alpha_N t)-NF_{u_o}(\pi_1+\alpha_N s)\right\} \\ &\quad\times\exp\left\{-k_N^{1/3}(y+u)(\frac{\nu_N}{2})^{1/3}(t-\tilde{t}_r)+k_N^{1/3}(x+u)(\frac{\nu_N}{2})^{1/3}(s-\tilde{t}_r)\right\}. \end{aligned} \quad (95)$$

The last equality follows from a change of variables. □

We now set

$$Z_N = \exp\{N F_{u_o}(\pi_1 + \alpha_N t_r)\}. \tag{96}$$

The end of this section is aimed at obtaining the asymptotics of the rescaled kernels $Z_N H''_N$, $Z_N H_N$, and $1/Z_N J_N$. It is then straightforward to deduce the asymptotics for the correlation kernel (90).

5.2. Estimate for $Z_N H''_N$

The aim of this subsection is to prove the following Proposition. Let H''_N be the kernel defined in (94), Z_N as in (96).

Proposition 5.2. *For any fixed $y_o \in \mathbb{R}$, $\exists\, C > 0, c > 0, C' > 0$, an integer $N_o > 0$ such that*

$$|Z_N H''_N(x)| \leq \frac{C \exp\{-cx\}}{k_N^{1/3}} \exp\{-C'N\}, \textit{ for any } x \geq y_o,\ N \geq N_o. \tag{97}$$

Proof of Proposition 5.2. Let Γ'' be such that its image under the map $\pi_1 + \alpha_N t$ is the circle of ray one, oriented counterclockwise. Then, it is easy to see that $\min_{\Gamma''} \mathrm{Re} F_{u_o}(\cdot) = F_{u_o}(1)$. Now, one can check that $F'_{u_o}(x) = -\dfrac{(x-\alpha_o)(x-(\pi_1+\alpha_N t_r))^2}{x(\pi_1 - x)}$, where $\alpha_o < 1$ is the second critical point, of mutliplicity one, of F_{u_o}. Thus for N large enough, as π_1 lies in a compact interval of $(1,\infty)$, one has that $\mathrm{Re} F'_{u_o}(x) < 0\ \forall\, x \in (1, \pi_1)$. Let then $0 < \eta_1 < \eta_2 < (\pi_1 - 1)/2$ be given and set $I = [1+\eta_1, 1+\eta_2]$. Then, there exist N_o and $\eta > 0$, depending on π_1 only, such that $|F'_u(x)| > 2\eta$, $\forall x \in I$ and $\eta_2 < \pi_1 - \alpha_N t_r$, $\forall N \geq N_o$. From this, we deduce that there exists $\eta' > 0$ such that $\Big|\exp\{-N F_{u_o}(1)\}\Big| \leq \Big|\exp\{-N(F_{u_o}(\pi_1 - \alpha_N t_r) + 2\eta')\}\Big|$. Now there exists $C > 0$ such that $|F_{u_o}(\pi_1 + \alpha_N t_r) - F_{u_o}(\pi_1 - \alpha_N t_r)| \leq C\alpha_N$, so that, for N large enough,

$$\Big|\exp\{-N F_{u_o}(1)\}\Big| \leq \Big|\exp\{-N F_{u_o}(\pi_1 + \alpha_N t_r) - N\eta'\}\Big|.$$

Let then $y_o > 0$ be given and assume first that $x \in [-y_o, y_o]$. Using (94) and the fact that the contour Γ'' is of length $\frac{2\pi}{\pi_1 \alpha_N}$, we can see that for N large enough, $|Z_N H''_N(x)| \leq \dfrac{k_N^{1/3}}{\alpha_N} \exp\left\{\dfrac{k_N^{1/3}}{\alpha_N}(\pi_1 + 2\sigma) y_o - N\eta'\right\}$, which goes to zero as N goes to infinity (since $k_N^{1/3}/\alpha_N << \sqrt{N}$). Thus, for N large enough, $|Z_N H''_N(x)| \leq \exp\{-N\dfrac{\eta'}{4}\}$. This yields Proposition 5.2 in this case. The case where x is positive is handled as in the preceding sections. Indeed, $\mathrm{Re}(s - \tilde{t}_r) \leq -\frac{(\pi_1 - 1)}{4\alpha_N} - \epsilon$ along Γ'', for N large enough. Thus, we readily obtain from the above proof that, for $x > 0$ and N large enough, $|Z_N H''_N(x)| \leq \exp\{-N\dfrac{\eta'}{4} - \epsilon x\}$. □

5.3. Estimate for $Z_N H_N$, $1/Z_N J_N$

The aim of this subsection is to obtain the following estimates for the kernels H_N and J_N defined in (93) and (92).

Proposition 5.3. *Assume $\epsilon > 0$ is fixed and let ν_N be given by (89), Z_N by (96). For any fixed $y_o \in \mathbb{R}$, $\exists\, C > 0, c > 0, N_o > 0$ such that for any $y \geq y_o$, $x \geq y_o$ and $N \geq N_o$,*

$$\left|\frac{J_N(y)}{Z_N} - ie^{\epsilon y(\frac{\nu_N}{2})^{1/3}} Ai(y)\right| \leq \frac{C \exp\{-cy\}}{k_N^{1/3}} \text{ and}$$
$$\left|Z_N H_N(x) - ie^{-\epsilon x(\frac{\nu_N}{2})^{1/3}} Ai(x)\right| \leq \frac{C \exp\{-cx\}}{k_N^{1/3}}.$$

The proof of Proposition 5.3 is divided into three parts. First, we establish three basic lemmas that enable us to get rid of some negligible parts of the contours and to perform the third order Taylor expansion. In the second part, we give the contours needed to perform the saddle point analysis and obtain, in the last part, the asymptotic expansion of the kernels H_N and J_N.

5.3.1. Preliminary lemmas

In this part we prove that there exists a disk, $D = D(t_r, \delta')$, such that the exponential term is driven by $H_{2/\sigma}$ outside D, and by its third order Taylor expansion inside D. First, we fix the left frontier of Γ_1 and show that on this frontier, the exponential term behaves as $\exp\{H_{2/\sigma}\}$ despite the artificial singularity we have introduced (due to the log). This is the object of the following Lemma.

Lemma 5.1. *Let R_o be defined in (26) and assume $t = \sigma(-R_o + ix)$, $|x| \leq \sqrt{3}$. Then, $\exists C_o(\pi_1) > 0$ depending on π_1 only such that*

$$\begin{aligned}&|\exp\{NF_{u_o}(\pi_1 + \alpha_N t_r) - NF_{u_o}(\pi_1 + \alpha_N t)\}|\\ &\quad \leq |\exp\{k_N(H_{2/\sigma}(t_r) - H_{2/\sigma}(t))\}| \exp\{C_o(\pi_1)\alpha_N k_N\},\end{aligned}$$

where $\exp\{-k_N H_{2/\sigma}(-R_o)\}$ *stands for* $\exp\{-k_N \frac{R_o^2+4\sigma R_o}{2\sigma^2}\}(-R_o)^{-k_N}$.

Proof of Lemma 5.1. We set $t = -R + ix$ where $R = \sigma R_o$ and $x \in [0, \sigma\sqrt{3}]$. The case where $x \in [-\sigma\sqrt{3}, 0]$ is obtained by using that $F_u(w) = \overline{F}_u(\bar{w})$. As $N \geq N_o$, where N_o has been defined in (91), $\pi_1 + \alpha_N t$ does not lie on the negative real axis, thus by a straightforward Taylor expansion

$$\begin{aligned}
&\exp\left\{NF_{u_o}(\pi_1+\alpha_N t_r) - NF_{u_o}(\pi_1+\alpha_N(-R+ix))\right\}\\
&\quad = \exp\left\{k_N\left(\frac{t_r^2-(-R+ix)^2}{2}\right)\right\}\left(\frac{t_r}{-R+ix}\right)^{k_N}\\
&\qquad \times \exp\left\{-\frac{2k_N}{\sigma}(1+\mu_N)(t_r-(-R+ix))\right\}\\
&\qquad \times \exp\left\{N\alpha_N\left(\int_{(-R+ix)/\pi_1}^{t_r/\pi_1}\frac{du}{1+\alpha_N u} - (t_r+R-ix)/\pi_1\right)\right\}\\
&\qquad \times \left(\frac{1+\alpha_N t_r/\pi_1}{1+\alpha_N(-R+ix)/\pi_1}\right)^{-k_N} \qquad (98)
\end{aligned}$$

where we have used that $u_o - C(\pi_1) = \alpha_N \dfrac{2}{\sigma}(1+\mu_N)$. Now

$$\begin{aligned}
&\int_{(-R+ix)/\pi_1}^{t_r/\pi_1}\frac{du}{1+\alpha_N u} - \frac{t_r+R-ix}{\pi_1}\\
&\quad = -\frac{\alpha_N}{2}\frac{1}{\pi_1^2}(t_r^2-(-R+ix)^2) - \alpha_N^2\int_{(-R+ix)/\pi_1}^{t_r/\pi_1}\frac{u^2}{1+\alpha_N u}. \qquad (99)
\end{aligned}$$

Inserting (99) in (98) yields

$$\begin{aligned}
&\exp\left\{NF_{u_o}(\pi_1+\alpha_N t_r) - NF_{u_o}(\pi_1+\alpha_N(-R+ix))\right\}\\
&\quad = \exp\left\{k_N H_{2/\sigma}(t_r) - k_N H_{2/\sigma}(-R+ix)\right\}\times\left(\frac{1+\alpha_N(-R+ix)/\pi_1}{1+\alpha_N t_r/\pi_1}\right)^{k_N} \qquad (100)\\
&\qquad \times \exp\left\{-\frac{2}{\sigma}\mu_N k_N(t_r+R-ix) - \alpha_N k_N\int_{(-R+ix)/\pi_1}^{t_r/\pi_1}\frac{u^2}{1+\alpha_N u}\right\} \qquad (101)
\end{aligned}$$

Now, as $N \geq N_o$, (101) is $O((\alpha_N+\mu_N)k_N)$, and this O is uniform, since π_1 lies in a compact interval of $(1,\infty)$. Indeed, we can choose a segment S for the u-path from $-R+ix$ to t_r, of length smaller than $R^2+3\sigma^2+t_r^2 \leq \sigma^2(R_o^2+3)+t_r^2$, which is uniformly bounded. Thus, as $t_r \in [\frac{\sigma}{2}, \frac{3\sigma}{2}]$, there exists $C_1(\pi_1, R_o) > 0$ such that $\displaystyle\int_S \frac{|u|^2}{|1+\alpha_N u/\pi_1|}|du| \leq C_1(\pi_1, R_o)$. The remaining bracket in (100) is obviously bounded. This finishes the proof of Lemma 5.1. □

In the following lemma, we prove that, in a suitably chosen compact set of $\mathbb{C}$, $NF_{u_o}(\pi_1+\alpha_N t)$ behaves, up to constants or lower order terms, as $k_N H_{2/\sigma}(t)$. Let $\delta' > 0$ be given and define

$$t_r^*(\Gamma_1') = t_r(1+\delta')e^{2i\pi/3},\ t_c^*(\Gamma_1') = t_c(1+\delta')e^{2i\pi/3}, \qquad (102)$$

$$t_r^*(\gamma') = t_r(1+\delta')e^{i\pi/3},\ t_c^*(\gamma') = t_c(1+\delta')e^{i\pi/3}. \qquad (103)$$

Define also $D(\Gamma_1')$ (resp. $D(\gamma')$) to be the segment joining $t_r^*(\Gamma_1')$ to $t_c^*(\Gamma_1')$ ($t_r^*(\gamma')$ to $t_c^*(\gamma')$). Let finally R_o be chosen as in Lemma 5.1 and $\eta > 0$ be given.

Lemma 5.2. *There exists constants* $Ct(\pi_1)$ *depending on* π_1 *only, and* $C > 0$ *(depending on* η *and* π_1*) such that*

$$|NF_{u_o}(\pi_1 + \alpha_N t) - NCt(\pi_1) - k_N H_{2/\sigma}(t)| \leq C\alpha_N k_N, \quad \forall \eta < |t| \leq 2\sigma R_o,$$
$$|NF_{u_o}(\pi_1 + \alpha_N t_r) - NF_{u_o}(\pi_1 + \alpha_N t_c)| \leq C\alpha_N k_N,$$
$$|H_{2/\sigma}(t_r^*(\Gamma_1')) - H_{2/\sigma}(t)| \leq C\alpha_N, \ \forall t \in D_{\Gamma_1'}, \text{ and}$$
$$|H_{2/\sigma}(t_r^*(\gamma')) - H_{2/\sigma}(t)| \leq C\alpha_N, \forall t \in D_{\gamma'}.$$

Proof of Lemma 5.2. One has $\dfrac{d}{dt} N\text{Re}\left(F_{u_o}(\pi_1 + \alpha_N t)\right) = k_N\Big(\text{Re}(H'_{2/\sigma}(t) + \dfrac{2}{\sigma}\mu_N + \alpha_N G'(t))\Big)$. The first estimate follows from the fact that G and $H_{2/\sigma}$ are uniformly bounded in the annulus considered. Combining the first estimate and the inequality $|H_{2/\sigma}(t_r) - H_{2/\sigma}(t_c)| \leq \mu_N^3$ (which follows from the facts that $H'_{2/\sigma}(t) = \frac{(t-t_c)^2}{t\sigma^2}$ and t_r, t_c are greater than $\sigma/2$), yields the second estimate. The last ones follow from the fact that both $|t_c^*(\Gamma_1') - t_r^*(\Gamma_1')| \leq C'\alpha_N$ and $|t_c^*(\gamma') - t_r^*(\gamma')| \leq C'\alpha_N$ for some constant C', and that $|H'_{2/\sigma}|$ is bounded on the two segments considered. □

In the third lemma, we then determine a disk where the third order Taylor expansion for the exact exponential term $F_u(.) = F_{u,N}(.)$, depending on N, can still be made. Let δ be given by (27).

Lemma 5.3. *There exist* $0 < \delta' < \delta/2 < 1$, N_1 *independent of* δ', *a constant* $C_o = C_o(\pi_1) > 0$, *such that, for any* $N \geq N_1$, *for any* $t \in D(t_r, \delta') := \{|t - t_r| \leq t_r\delta'\}$

$$|F_{u_o}^{(4)}(\pi_1 + \alpha_N t)\alpha_N^2| \leq C_o,$$
$$\left|F_{u_o}(\pi_1 + \alpha_N t) - F_{u_o}(\pi_1 + \alpha_N t_r) - \frac{\alpha_N^3(t - t_r)^3}{3!}F_{u_o}^{(3)}(\pi_1 + \alpha_N t_r)\right|$$
$$\leq \frac{|\alpha_N(t_r - t)|^3}{24}|F_{u_o}^{(3)}(\pi_1 + \alpha_N t_r)|$$

Remark 5.1. The above Lemma implies in particular, for N large enough (to ensure that $\nu_N = F_{u_o}^{(3)}(\pi_1 + \alpha_N t_r)\alpha_N \geq 1/t_r^3$), that $\text{Re}\big(NF_{u_o}(\pi_1 + \alpha_N t_r^*(\gamma')) - NF_{u_o}(\pi_1 + \alpha_N t_r)\big) \leq -k_N \delta'^3/8$, and $\text{Re}\big(NF_{u_o}(\pi_1 + \alpha_N t_r^*(\Gamma_1')) - NF_{u_o}(\pi_1 + \alpha_N t_r)\big) \geq k_N \delta'^3/8$.

Proof of Lemma 5.3. We prove the second inequality of Lemma 5.3 (the first one will be established within this proof). This inequality will be established if we find $\delta' > 0$ such that

$$\frac{1}{4!}\max_{D(t_r,\delta')}\left|F_{u_o}^{(4)}(t)\right||\alpha_N(t_r - t)|^4 \leq \frac{F_{u_o}{}^{(3)}(\pi_1 + \alpha_N t_r)}{24}|\alpha_N(t - t_r)|^3.$$

Assume $\delta' < 1/2$, then, as $t_r \in [\frac{\sigma}{2}, \frac{3\sigma}{2}]$, $D(t_r, \delta') \subset D(t_c, \frac{\delta'+1}{2})$. Define then $v_o = \dfrac{2}{\sigma}(1 + \mu_N)$, so that $u_o = C(\pi_1) + \alpha_N v_o$, and let H_{v_o} be given by (71). Then, $F'_{u_o}(\pi_1 + \alpha_N t) = \alpha_N(H'_{v_o}(t) + \alpha_N G'(t))$, where $G'(t) = \dfrac{t^2 - \pi_1^2}{\pi_1^2(\pi_1 + \alpha_N t)}$.

Now, as $N \geq N_o$, for $t \in D(t_r, \delta') \subset D(t_c, \frac{\delta'+1}{2})$, as $\pi_1 + \alpha_N t \geq \pi_1/2$, there exists constants $C_3(\pi_1) > 0$, $C_4(\pi_1) > 0$, depending on π_1 only, such that

$$\max_{t \in D(t_c, \frac{\delta'+1}{2})} |G^{(4)}(t)| \leq C_4(\pi_1), \ \ |G^{(3)}(t_r)| \leq C_3(\pi_1).$$

Note that this gives the first inequality in Lemma 5.3 with $C_o = C_4(\pi_1)$. Furthermore, one has $\max_{t \in D(t_r, \delta')} |H_{v_o}^{(4)}(t)| = \dfrac{6}{t_r^4(1-\delta')^4}$. Thus to prove Lemma 5.3, it is enough to determine δ' such that

$$\forall t \in D(t_r, \delta'), \ \ \frac{1}{4!}|t-t_r|^4 \left(\frac{6}{t_r^4(1-\delta')^4} + \alpha_N C_4(\pi_1) \right) + \frac{\alpha_N}{24} C_3(\pi_1)|t-t_r|^3 \\ \leq \frac{|t-t_r|^3}{24} H_{v_o}^{(3)}(t_r). \tag{104}$$

Let now $0 < \delta' < 1$ be such that $\dfrac{\delta'}{(1-\delta')^4} < \dfrac{1}{32}$. As $H_{v_o}^{(3)}(t_r) = 2/t_r^3$, we then have that

$$\frac{6}{24 t_r^4 (1-\delta')^4}|t-t_r|^4 < \frac{6}{32}|t-t_r|^3 \frac{H_{v_o}^{(3)}(t_r)}{24}, \quad \forall t \in D(t_r, \delta'). \tag{105}$$

And there exists N_2, depending on π_1 only, such that, as $t_r \in [\frac{\sigma}{2}, \frac{3\sigma}{2}]$, and $\delta' < 1$,

$$\frac{3!}{4!}\delta' t_r C_4(\pi_1)\alpha_N + \frac{\alpha_N}{24} C_3(\pi_1) \leq \frac{\alpha_N}{24}\left(18\sigma C_4(\pi_1) + C_3(\pi_1)\right) \\ \leq \frac{2}{96\sigma^3} \leq \frac{2}{96 t_r^3} = \frac{1}{4}\frac{H_{v_o}^{(3)}(t_r)}{24}. \tag{106}$$

Formulas (105) and (106) now imply (104). This finishes the proof of Lemma 5.3. □

5.3.2. Contours

We now define the contours Γ_1' and γ', suitable for the saddle point analysis of H_N and J_N. Let δ be given by (27) and $\delta' \leq \delta/2$ be chosen so that Lemma 5.3 holds. From now on, we assume that N is large enough to ensure that $D(t_r, \delta') \subset D(t_c, \delta)$. Let then Γ_σ and γ_σ be the image of the contours defined in Figure 2 under the map $t \mapsto \sigma t$. Then Γ_σ (resp. γ_σ) is an ascent (resp. descent) curve for $H_{2/\sigma}$, as $H_{2/\sigma}(\sigma t) = F(t) + \log \sigma$ where F has been defined in (17).
We now define the contour Γ_1', which coincides with Γ_σ outside $D(t_c, \delta)$. Let then

$$\Gamma'_{1,i} = \Gamma_\sigma \cap D(t_c, \delta)^c; \ \ \Gamma'_{1,0} = t_r + \frac{\epsilon}{2k_N^{1/3}} e^{i\theta}, 0 \leq \theta \leq 2\pi/3;$$

$$\Gamma'_{1,1} = t_r + t e^{2i\pi/3}, \ \frac{\epsilon}{2k_N^{1/3}} \leq t \leq \delta' t_r.$$

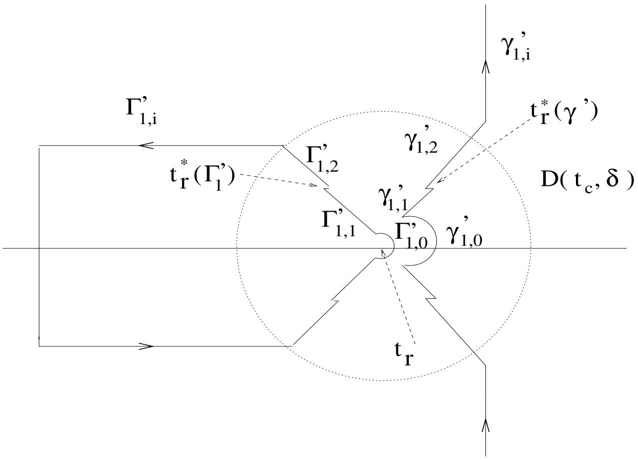

Fig. 4. Contours Γ_1' and γ'

Let then $t_r^*(\Gamma_1')$ and $t_c^*(\Gamma_1')$ be given as in (102) and note that they are the respective endpoints of Γ_{11}' and Γ_σ. We then join $t_r^*(\Gamma_1')$ to $t_c^*(\Gamma_1')$ by a segment (of length smaller than $C\alpha_N$), and finally join $t_c^*(\Gamma_1')$ to $t_c(1+\delta)e^{2i\pi/3}$ along Γ_σ. We call $\Gamma_{1,2}'$ this last contour. Finally we set $\Gamma_1' = \Gamma_{1,i}' \cup \Gamma_{1,0}' \cup \Gamma_{1,1}' \cup \Gamma_{1,2}' \cup \overline{\Gamma_{1,0}' \cup \Gamma_{1,1}' \cup \Gamma_{1,2}'}$, and this contour is oriented counterclockwise. Similarly, γ' is the contour γ_σ modified in the disk $D(t_c, \delta)$, in the following way. Let then $\gamma_{1,i}' = \gamma_\sigma \cap D(t_c, \delta)^c$; $\gamma_{1,0}' = t_r + \frac{3\epsilon}{2k_N^{1/3}} e^{i\theta}, 0 \leq \theta \leq \pi/3$;

$$\gamma_{1,1}' = t_r + te^{i\pi/3}, \ \frac{\epsilon}{2k_N^{1/3}} \leq t \leq \delta' t_r;$$

Let also $t_r^*(\gamma')$ and $t_c^*(\gamma')$ be given by (103). We then join $t_r^*(\gamma')$ to $t_c^*(\gamma')$ by a segment (of length smaller than $C\alpha_N$), and finally join $t_c^*(\gamma')$ to $t_c(1+\delta)e^{i\pi/3}$ along γ_σ. We call $\gamma_{1,2}'$ this last contour and define $\gamma' = \gamma_{1,i}' \cup \gamma_{1,0}' \cup \gamma_{1,1}' \cup \gamma_{1,2}' \cup \overline{\gamma_{1,0}' \cup \gamma_{1,1}' \cup \gamma_{1,2}'}$, oriented from bottom to top. A plot of the contours Γ_1' and γ' is given on Figure 4.

Remark 5.2. There exists $\eta > 0$ such that $\gamma' \cap D(0, \eta) = \emptyset$ and $\Gamma_1' \cap D(0, \eta) = \emptyset$.

The contours defined above coincide with the steepest ascent and descent curves for F_{u_o} in a small disk $D(t_r, \delta')$, where the third order Taylor expansion is known to hold. Thus we now introduce the expected limiting kernels. Let $\Gamma_{\infty,N}$ (resp. $\gamma_{\infty,N}$) be a contour such that it coincides with the image of Γ_1' (resp. γ') under the map $t \mapsto k_N^{1/3}(t - t_r)$, in the disk $D(t_r, \delta')$, and then follows the curve $te^{\pm i2\pi/3}$, $|t| \geq \delta'$, (resp. $te^{\pm i\pi/3}$, $|t| \geq \delta'$). Set then

$$H_{\infty,N}(x) := (\frac{\nu_N}{2})^{1/3} \exp\{-\epsilon x(\frac{\nu_N}{2})^{1/3}\} \int_{\Gamma_{\infty,N}} \exp\{x(\frac{\nu_N}{2})^{1/3}a - \frac{a^3}{3!}\nu_N\}da, \tag{107}$$

$$J_{\infty,N}(y) := (\frac{\nu_N}{2})^{1/3} \exp\{\epsilon y(\frac{\nu_N}{2})^{1/3}\} \int_{\gamma_{\infty,N}} \exp\{-y(\frac{\nu_N}{2})^{1/3}b - \frac{b^3}{3!}\nu_N\}db. \tag{108}$$

Then, $H_{\infty,N}(x)=i\exp\{-\epsilon x(\frac{\nu_N}{2})^{1/3}\}Ai(x)$ and $J_{\infty,N}(y)=i\exp\{\epsilon y(\frac{\nu_N}{2})^{1/3}\}Ai(y)$. We now split the contours. Set $H'_N(x) = H_N(x) - H''_{N,2}(x)$, $J'_N(y) = J_N(y) - J''_{N,2}(y)$, where

$$\begin{aligned} H''_{N,2}(x) &= k_N^{1/3}(\frac{\nu_N}{2})^{1/3} \int_{\Gamma'_1 \cap D(t_r,\delta')^c} \exp\left\{-NF_{u_o}(\pi_1+\alpha_N s)\right\} \\ &\quad \times \exp\left\{k_N^{1/3}x(\frac{\nu_N}{2})^{1/3}(s-\tilde{t}_r)\right\}ds, \\ J''_{N,2}(x) &= k_N^{1/3}(\frac{\nu_N}{2})^{1/3} \int_{\gamma' \cap D(t_r,\delta')^c} \exp\left\{NF_{u_o}(\pi_1+\alpha_N t)\right\} \\ &\quad \times \exp\left\{-k_N^{1/3}y(\frac{\nu_N}{2})^{1/3}(t-\tilde{t}_r)\right\}dt. \end{aligned}$$

Similarly, $H''_{\infty,N}(x)$ (resp. $J''_{\infty,N}(y)$) is the part of (107) (resp (108) corresponding to the integral performed on the curve $te^{\pm i2\pi/3}$, $|t| \geq \delta'$ (resp. $te^{\pm i\pi/3}$, $|t| \geq \delta'$).

5.3.3. Saddle point estimates

We now prove Proposition 5.3 in the case x and y lie in a fixed compact interval; the case where they are positive follows from arguments similar to those of the preceding sections.

We first show that the contribution of the contour outside $D(t_r, \delta')$ is negligible, because the exponential term behaves as $k_N H_{2/\sigma}(t)$ outside this disk.

Fact 5.1. *Let $y_o > 0$ be fixed and assume that $x, y \in [-y_o, y_o]$. There exists $N_1 > 0$ such that,*

$$|Z_N H''_{N,2}(x)| \leq \exp\{-\frac{k_N}{16}\delta'^3\},\ |H''_{\infty,N}(x)| \leq \exp\{-k_N\frac{\delta'^3}{12}\}, \forall N \geq N_1, \tag{109}$$

$$|\frac{1}{Z_N}J''_{N,2}(y)| \leq \exp\{-\frac{k_N}{16}\delta'^3\},\ |J''_{\infty,N}(y)| \leq \exp\{-k_N\frac{\delta'^3}{12}\}, \forall N \geq N_1. \tag{110}$$

Proof of Fact 5.1. We first prove (109) and consider $\Gamma'_1 \cap D(t_r,\delta')^c$. Lemma 5.3 and Remark 5.1 first ensure that $\mathrm{Re}\left(NF_{u_o}(\pi_1+\alpha_N t_r^*(\Gamma'_1))\right) - NF_{u_o}(\pi_1+\alpha_N t_r) \geq k_N\frac{\delta'^3}{8}$. Let $\eta > 0$ be chosen as in Remark 5.2. Then, from Lemma 5.2, we obtain that $\forall t \in \Gamma'_{12} \cap \Gamma_\sigma^c$,

$$\begin{aligned}
&N\mathrm{Re}\left(F_{u_o}(\pi_1+\alpha_N t)-F_{u_o}(\pi_1+\alpha_N t_r)\right)\\
&\quad > N\mathrm{Re}\left(F_{u_o}(\pi_1+\alpha_N t)-F_{u_o}(\pi_1+\alpha_N t_r^*(\Gamma_1'))\right)+k_N\frac{\delta'^3}{8}\\
&\quad \geq k_N\frac{\delta'^3}{8}-C\alpha_N k_N\geq k_N\frac{\delta'^3}{16},
\end{aligned}\tag{111}$$

for N large enough. Similarly for $t\in\Gamma_{12}'\cap\Gamma_\sigma$, using Lemma 5.1, Lemma 5.2, and the fact that Γ_σ is an ascent curve for $H_{2/\sigma}$, we obtain that $\mathrm{Re}\big(NF_{u_o}(\pi_1+\alpha_N t)-NF_{u_o}(\pi_1+\alpha_N t_r)\big)\geq k_N\frac{\delta'^3}{16}$ for N large enough. Combining the latter inequality and (111), we obtain that $|Z_N H_{N,2}''(x)|\leq\exp\left\{-k_N\frac{\delta'^3}{8}+k_N^{1/3}y_o+C\alpha_N k_N\right\}$, for some constant C uniformly bounded. Thus, for N large enough, we obtain the first part of (109). The second part is straightforward using that $\nu_N\in[1/\sigma,3/\sigma]$.

We now turn to the proof of (110). Remark 5.1 also ensures that

$$\mathrm{Re}\left(NF_{u_o}(\pi_1+\alpha_N t_r^*(\gamma'))-NF_{u_o}(\pi_1+\alpha_N t_r)\right)\leq -k_N\frac{\delta'^3}{8}.$$

Let then t_o be chosen as in (38) and large enough so that $(1+t_o/2)^2<3t_o^2/4$. Using again Lemma 5.2, we have that $\forall t\in\gamma'\cap\{|Im(t)|<t_o\sqrt{3}\sigma/2\}$

$$\mathrm{Re}\left(NF_{u_o}(\pi_1+\alpha_N t)-NF_{u_o}(\pi_1+\alpha_N t_r)\right)\leq -k_N\frac{\delta'^3}{8}+C\alpha_N k_N\leq -k_N\frac{\delta'^3}{16},\tag{112}$$

for N large enough. And for $t\in\gamma'$, with $t=t(s)=t_c+t_o e^{i\pi/3}\sigma+is$, $s\geq 0$, it is easy to check that, as $(1+t_o/2)^2<3t_o^2/4$, there exists $C>0$ depending on π_1 only, such that

$$\mathrm{Re}\frac{d}{ds}NF_{u_o}(\pi_1+\alpha_N t(s))<-k_N Im\left(H_{2/\sigma}(t(s))\right)\leq -k_N C\,Im(t(s)).\tag{113}$$

Now, (112) and (113) give that $\left|\frac{1}{Z_N}J_{N,2}''(y)\right|\leq\exp\left\{-k_N\frac{\delta'^3}{8}+k_N^{1/3}y_o+C\alpha_N k_N\right\}$, which proves the first part of (110). The second part of (110) is easy to check. □

We now show that the contribution from the contours in the disk $D(t_r,\delta')$ gives the leading term of the asymptotic expansion for both kernels $Z_N H_N$ and $1/Z_N J_N$.

Fact 5.2. *Let $y_o>0$ be fixed and assume $x,y\in[-y_o,y_o]$. Then, $\exists\, C=C(y_o)>0$, N_o, such that $\forall N\geq N_o$, one has*

$$|Z_N H_N'(x)-H_{\infty,N}'(x)|\leq\frac{C}{k_N^{1/3}},\quad \frac{1}{Z_N}|J_N'(y)-J_{\infty,N}'(y)|\leq\frac{C}{k_N^{1/3}}.\tag{114}$$

Proof of Fact 5.2. We will only prove the first inequality of (114), since the second follows from similar arguments. Then,

$$|Z_N H_N'(x) - H_{\infty,N}'(x)| \leq \frac{k_N^{1/3}}{2\pi}\left(\frac{\nu_N}{2}\right)^{1/3} \int_{\Gamma_{1,0}'\cup\Gamma_{1,1}'} e^{k_N^{1/3} y_o (\frac{\nu_N}{2})^{1/3}\mathrm{Re}(t-\tilde{t}_r)} \times |e^{-NF_{u_o}(\pi_1+\alpha_N t)+NF_{u_o}(\pi_1+\alpha_N t_r)} - e^{-k_N\nu_N(t-t_r)^3/3!}|\,|dt|. \quad (115)$$

We first consider the $\Gamma_{1,0}'$ integral in (115). Thus $t = t_r + \dfrac{\epsilon}{2k_N^{1/3}}e^{i\theta}$, and using Lemma 5.3 to mimick the proof of (34), we obtain that

$$\Big| \exp\{NF_{u_o}(\pi_1+\alpha_N t) - NF_{u_o}(\pi_1+\alpha_N t_r)\} - \exp\{-k_N\frac{\nu_N(t-t_r)^3}{3!}\}\Big| \leq \tilde{C}_o \exp\{\nu_N\epsilon^3\}\frac{1}{k_N^{1/3}}.$$

Now, we use the fact that $\nu_N \leq \frac{3}{\sigma^3(\pi_1)}$, by (91), to obtain that there exists $C > 0$ so that in (115)

$$\frac{k_N^{1/3}}{2\pi}\left(\frac{\nu_N}{2}\right)^{1/3}\int_{\Gamma_{1,0}'} e^{k_N^{1/3} y_o (\frac{\nu_N}{2})^{1/3}\mathrm{Re}(t-\tilde{t}_r)}\Big|e^{-NF_{u_o}(\pi_1+\alpha_N t)+NF_{u_o}(\pi_1+\alpha_N t_r)} - e^{-k_N\nu_N(t-t_r)^3/3!}\Big||dt| \leq \frac{C}{k_N^{1/3}}.$$

And for $t = t_r + pe^{i2\pi/3} \in \Gamma_{1,1}'$, there exists C_o, depending on π_1 only, such that, by Lemma 5.3,

$$|\exp\{NF_{u_o}(\pi_1+\alpha_N t_r) - NF_{u_o}(\pi_1+\alpha_N t)\} - \exp\{-k_N(t-t_r)^3\frac{\nu_N}{3!}\}| \leq \exp\{-k_N p^3\frac{\nu_N}{4!}\}C_o(k_N p^4 + p).$$

Now, following the same scheme as in Section 2, we obtain that in (115)

$$\frac{k_N^{1/3}}{2\pi}\left(\frac{\nu_N}{2}\right)^{1/3}\int_{\Gamma_{1,1}'} e^{k_N^{1/3} y_o (\frac{\nu_N}{2})^{1/3}\mathrm{Re}(t-\tilde{t}_r)}\Big|e^{-NF_{u_o}(\pi_1+\alpha_N t)+NF_{u_o}(\pi_1+\alpha_N t_r)} - e^{-k_N\nu_N(t-t_r)^3/3!}\Big||dt|$$
$$\leq C_o k_N^{1/3}\int_{\frac{\epsilon}{2k_N^{1/3}}}^{\delta' t_r} dp(k_N p^4 + p) \times \exp\left\{\epsilon y_o\left(\frac{\nu_N}{2}\right)^{1/3} + \left(\frac{\nu_N}{2}\right)^{1/3} k_N^{1/3} y_o \frac{p}{2} - k_N\frac{\nu_N p^3}{4!}\right\} \leq \frac{C}{k_N^{1/3}}.$$

Here, we have used that both ν_N and t_r are uniformly bounded. This finally gives from (115) that $|Z_N H_N'(x) - H_{\infty,N}'(x)| \leq \dfrac{C}{k_N^{1/3}}$. This proves (114). □

Combining formulas (114),(109) and (110) yield then Proposition 5.3 in the case x or y lie in a fixed compact interval. The case where $x > 0$ (resp. $y > 0$), is analyzed in a similar way than in the preceding sections, using the fact that the whole contour Γ_1 (resp. γ') lies on the left (resp. right) handside of $\tilde{t}_r$. The detail is left. This finishes the proof of Theorem 1.4.

5.4. Extensions

In this part, we explain how the proof has to be modified to consider more general diagonal perturbations W_N. It is easy to see that the core of the proof of Theorem 1.4 are the three Lemmas obtained in Subsection 5.3.1.

We now indicate the main changes to prove Theorem 1.4 under Assumption 1.2, when some eigenvalues of W_N differ from 0 or π_1. Let $\tilde{F}_u$ be given by (83) and $\tilde{w}_o$, $\tilde{u}_o$ be defined as in (12) and (13). Let also F_u be as in (87) and set set $\tilde{G} = \tilde{F}_u - F_u$. Then, under assumption 1.2, there exist sequences μ'_N, η_N, μ''_N, a constant $C > 0$, such that

$$\tilde{w}_o = \pi_1 + \alpha_N \sigma(\pi_1)(1 + \mu'_N + \eta_N), \qquad \text{where } |\mu'_N| \le C\alpha_N \text{ and } |\eta_N| \le C\beta_N,$$
$$\tilde{u}_o = C(\pi_1) + \tilde{G}'(\pi_1) + \alpha_N \frac{2}{\sigma(\pi_1)} + C\alpha_N \mu''_N, \qquad \text{with } |\tilde{G}'(\pi_1)| \le C\beta_N$$
$$\text{and } \lim_{N\to\infty} \mu''_N = 0.$$

This implies that $\tilde{t}_r = \dfrac{w_o - \pi_1}{\alpha_N}$ still lies in an arbitrarily small neighborhood of $t_c = \sigma(\pi_1)$ and also gives that $0 < \lim_{N\to\infty} \alpha_N \tilde{F}^{(3)}_{\tilde{u}_o}(\tilde{w}_o) < \infty$. And, given a compact set K of $\mathbb{C}^*$, there exist positive constants C_o, C_1, C, depending on π_1 and K, and a sequence μ_N with $\lim_{N\to\infty} \mu_N = 0$, such that,

$$N\tilde{F}_{\tilde{u}_o}(\pi_1 + \alpha_N t) = NC_o + r_N C_1 + k_N H_{2/\sigma}(t) + O(\mu_N k_N), \quad \forall t \in K, \tag{116}$$
$$|\tilde{F}^{(l)}_{\tilde{u}_o}(\pi_1 + \alpha_N t) - F^{(l)}_{\tilde{u}_o}(\pi_1 + \alpha_N t)| \le C\beta_N, \quad \text{for } l = 3, 4, \quad \forall t \in K. \tag{117}$$

In this case, Lemma 5.2 (resp. Lemma 5.3) follows from (116) (resp. (117)). We also choose Γ'' as in the proof of Fact 3.2. The end of the proof is a simple rewriting of the arguments used in the preceding subsections. This gives Theorem 1.4 in this case.

We now indicate the idea of the proof of Theorem 1.5, when $0 < \pi_1 \le 1$. For ease of explanatory, we here assume that $r_N = 0$. Assume first that $\pi_1 < 1$. Then, the exponential to be considered is given by $F_{u_o}(w) = w^2/2 - u_o w + (1-\alpha_N^2)\log w + \alpha_N^2 \log(w - \pi_1)$. Let then w_o and u_o be defined as in (12) and (13). Then there exists some sequences C_N, C'_N, ν_N such that

$$w_o = 1 + \alpha_N^2 C_N, \quad u_o = 2 + \alpha_N^2 C'_N, \quad F^{(3)}_{u_o}(w_o) = \nu_N, \text{ with}$$

$$\lim_{N\to\infty} C_N = C_o := \frac{1}{2}\left(\frac{1}{(1-\pi_1)^2} - 1\right), \quad \lim_{N\to\infty} C'_N = \frac{1}{1-\pi_1} - 1,$$
$$\lim_{N\to\infty} \nu_N = 2.$$

The function that now leads the exponential term is

$$F(w) = \frac{w^2}{2} - 2w + \log w, \tag{118}$$

and given any compact set K of $\mathbb{C} \setminus \{0, \pi_1\}$, we have that

$$\left| F_{u_o}^{(l)}(w) - F^{(l)}(w) \right| \leq C(K)\alpha_N^2, \quad \forall l = 0, \dots, 4, \tag{119}$$

where $C(K)$ depends on the compact set K only. Formula (119) ensures that Lemma 5.3 can be established, as $F^{(3)}(2) = 2 > 0$. It also readily gives Lemma 5.2. We choose the contours Γ and γ as in Section 2, slightly modified in a small disk around w_o. Then replacing, in the whole Section 5, the function $H_{2/\sigma}$ with F defined above, it is not hard to deduce Theorem 1.5.

If $\pi_1 = 1$ then there exists sequences μ_N, μ'_N such that

$$w_o = 1 + 2^{-1/3}\alpha_N^{2/3}(1 + \mu'_N), \quad u_o = 2 + 3\alpha_N^{4/3}2^{-2/3}(1 + \mu_N),$$
$$\text{with } \lim_{N\to\infty} \mu_N^{(')} = 0,$$
$$F_{u_o}^{(3)}(w_o) = \nu_N, \text{ with } \lim_{N\to\infty} \nu_N = 6. \tag{120}$$

Let then K be a given compact set of $\mathbb{C}^*$. By as straightforward Taylor expansion, one has that $F_{u_o}(1 + x\alpha_N^{2/3}) = Ct(N) + \alpha_N^2 H(x) - \alpha_N^2 \log(1 + \alpha_N^{2/3}x) + O(\alpha_N^{2/3}|x|)^4$, $\forall\, x \in K$, where $Ct(N)$ depends on N only and

$$H(x) = x^3/3 - 3x2^{-2/3} + \log x. \tag{121}$$

The function H admits the degenerate critical point $x_c = 2^{-1/3}$, and $H^{(3)}(x_c) = 6$, $H^{(4)}(x_c) = 2^{1/3} \times 12$. Set then $G(x) = \left(F_{u_o}(1 + x\alpha_N^{2/3}) - Ct(N) + \alpha_N^2 H(x)\right)/\alpha_N^2$. Then there exists $C > 0$ such that $|G^{(l)}(x)| \leq C\alpha_N^{2/3}$, $\forall x \in K, \forall l = 0, \dots, 4$. This ensures that Lemma 5.2 can be established in a suitably chosen neighborhood of width $\alpha_N^{2/3}$ of w_o. Lemma 5.3 also holds in some disk centered at $1 + \alpha_N^{2/3}x_c$ of ray $\delta'\alpha_N^{2/3}$, for some $\delta' > 0$. Now, the steepest descent and ascent curves for H can be computed. Indeed, one can check that $\frac{d}{dt}\mathrm{Re}\left(H(x_c + te^{2i\pi/3})\right) = \frac{t^2(t^2 - 2x_c t + 3x_c^2)}{t^2 - x_c t + t^2} > 0$, $\forall t \neq 0$. Then, the contours for the saddle point analysis are chosen as follows. Here, for short, we do not make the change of variables $w \to 1 + \alpha_N^{2/3}x$ to define the contours as in Subsection 5.3. Let $t_o > \delta'$ be given and define

$$\Gamma_{1,+} = \{1 + \alpha_N^{2/3}(x_c + te^{2i\pi/3}), \quad 0 \leq t \leq 2x_c\}$$
$$\cup\{1 + \alpha_N^{2/3}\sqrt{3}x_c e^{i\theta}, \quad \pi/2 \leq \theta \leq \pi\},$$

$$\Gamma'' = 1/2e^{i\theta}, \quad 0 \le \theta \le 2\pi,$$
$$\gamma_+ = \{1 + \alpha_N^{2/3}(x_c + te^{i\pi/3}), \quad 0 \le t \le 2t_o\}$$
$$\cup \{1 + \alpha_N^{2/3}(x_c + 2t_o e^{i\pi/3}) + it, \quad t \ge 0\},$$

and set $\Gamma_1 = \Gamma_{1,+} \cup \overline{\Gamma_{1,+}}$, $\gamma = \gamma_+ \cup \overline{\gamma_+}$. We then slightly modify the contours Γ_1 and γ in a small neighborhood of width $\alpha_N^{2/3}$ of w_o, as in Subsection 5.3. Then, considering the rescalings $u = u_o + \alpha_N^{4/3} k_N^{-2/3} y = u_o + N^{-2/3} y$, it is enough to replace $H_{2/\sigma}$ with H defined in (121) and α_N with $\alpha_N^{2/3}$ in the whole Section 5. The fact that the contribution of Γ'' is negligible is also clear. This is because, far from $w = 1$, the exponential term $F_u(\cdot)$ behaves as F defined in (118). The proof of Theorem 1.5 is then straightforward.

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