Probab. Theory Relat. Fields 135, 457–469 (2006)
Digital Object Identifier (DOI) 10.1007/s00440-005-0463-2

Jérôme Depauw

Degree two Brownian Sheet in Dimension three

Received: 13 April 2005 / Revised version: 23 May 2005 /
Published online: 17 August 2005 –

Abstract. In this work, we construct a degree two Brownian Sheet in dimension three which is obtained from the ordinary Brownian Sheet in $\mathbb{R}^2$ in the same way that Paul Lévy has obtained his two-parameter Brownian motion from the ordinary Brownian motion.

1. Introduction

There are two different generalizations of the notion of Brownian motion to the two dimensional parameters space. The first one is Paul Lévy's two-parameter Brownian motion (See [6]). It is a centered Gaussian field $(B_x)_{x\in\mathbb{R}^2}$ defined by the facts that the variance of B_x is equal to the Euclidean length $\|x\|$, and that the $\mathbb{R}^2$ action S on the space of the trajectories $B = (B_x)_{x\in\mathbb{R}^2}$ defined by $(S_u B)_x = B_{x+u} - B_u$ preserves the law of the Gaussian field $(B_x)_{x\in\mathbb{R}^2}$. It has the following property: for any 1-dimensional linear subspace $D \subset \mathbb{R}^2$ and any $u \in \mathbb{R}^2$, the Gaussian process $(S_u B_x)_{x\in D}$ has the law of the standard 1-parameter Brownian motion (the question of the continuity being not considered). The second generalization is the (degree 2) Brownian sheet (See [9]), which is usually defined as a centered Gaussian field $(B_x)_{x\in\mathbb{R}^2_+}$ with covariance $\mathrm{cov}(B_x, B_{x'})$ equal to $\min(x_1, x'_1) \cdot \min(x_2, x'_2)$, where $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$. Let us note that $\mathrm{cov}(B_x, B_{x'})$ is the area of the intersection of the rectangles $[0, x_1] \times [0, x_2]$ and $[0, x'_1] \times [0, x'_2]$. The Brownian sheet may also be defined as the Gaussian field indexed by the set of triangular surfaces, with covariance equal to the area of the intersection of the triangles. In this paper, we will generalize the notion of degree 2 Brownian sheet on triangles, to dimension 3. Following Paul Lévy's way, it is a centered Gaussian field which has the property that any of its restriction to a 2-dimensional subspace of $\mathbb{R}^3$ has the law of the degree 2 Brownian sheet on $\mathbb{R}^2$ described above. This Brownian sheet is used as a typical example of cocycle of degree 2 in the theory developed in [3]. The main part of the work is to determine an extension, to dimension 3, of the function of covariance of the Brownian sheet in $\mathbb{R}^2$, and to check that it is of positive type.

2. Degree two Brownian sheet in $\mathbb{R}^2$

As we said in Introduction, we will consider the Brownian sheet on triangles in the plan (instead of rectangles with sides parallel to the axes).

J. Depauw: Laboratoire de Mathématiques et Physique Théorique, Faculté des Sciences, Université F. Rabelais de Tours, Parc de Grandmont, 37000 Tours, France.
e-mail: `jerome.depauw@univ-tours.fr`

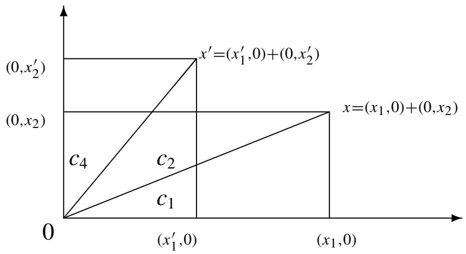

Fig. 1. The two descriptions of degree 2 Brownian sheet in $\mathbb{R}^2$

Definition 1. *The degree* 2 *Brownian sheet in* $\mathbb{R}^2$ *is a centered Gaussian field* $(B_{x,y})_{x,y\in\mathbb{R}^2\times\mathbb{R}^2}$, *such that the covariance* $\operatorname{cov}(B_{x,y}, B_{x',y'})$ *is the signed area of the intersection of the oriented triangle of vertices* 0, x, $x+y$ *with the oriented triangle of vertices* 0, x', $x'+y'$, *the sign being positive if the two triangles have the same orientation, negative otherwise.*

In order to be sure that the preceding definition makes sense, it is necessary to check that the function which defines the covariance is of positive type. It will be a particular case of Theorem 1 of the next section.

The connection between our definition of the Brownian sheet and the standard one can be deduced from the following remark: for $x=(x_1,x_2)\in\mathbb{R}_+^2$, the rectangle $[0,x_1]\times[0,x_2]$ can be decomposed in two triangles, the positively oriented triangle of vertices 0, $(0,x_1)$ and x, and the negatively oriented triangle of vertices 0, $(x_2,0)$ and x. This leads to the following formula. Let $(B_{x,y})_{x,y\in\mathbb{R}^2\times\mathbb{R}^2}$ be a degree 2 Brownian sheet as above defined. Let $(B_x)_{x\in\mathbb{R}_+^2}$ be the Gaussian field defined by

$$\text{For}\quad x=(x_1,x_2),\qquad B_x=B_{(x_1,0),(0,x_2)}-B_{(0,x_2),(x_1,0)}$$

Then $(B_x)_x$ is a standard degree 2 Brownian sheet as defined in Introduction. Indeed, for $x=(x_1,x_2)$, let us denote temporarily $B_1=B_{(x_1,0),(0,x_2)}$, $B_2=-B_{(0,x_2),(x_1,0)}$, and similarly B_i', $i=1,2$, for $x'=(x_1',x_2')$. We have to calculate $\operatorname{cov}(B_x,B_{x'})=\operatorname{cov}(B_1+B_2,\ B_1'+B_2')$. Expanding this covariance leads to

$$\operatorname{cov}(B_1,\ B_1')+\operatorname{cov}(B_2,\ B_1')+\operatorname{cov}(B_1,\ B_2')+\operatorname{cov}(B_2,\ B_2').$$

Let us denote by c_i, $i=1,\dots,4$ each of the four above covariances, and suppose for example that $x_1>x_1'$ and $x_2<x_2'$. By Definition 1, these covariances are respectively equal to the areas drawn on the figure 1 (and $c_3=0$). Hence we have $\sum_{i=1}^4 c_i=x_1'x_2$, as excepted. The proofs of the other cases are similar. This checks that $(B_x)_x$ is a standard Brownian sheet.

We need the analog of the shift S_u defined in Introduction. It is given by the following proposition:

Proposition 1. *The* $\mathbb{R}^2$ *action* T *defined on the space of the trajectories* $B=(B_{x,y})_{x,y\in\mathbb{R}^2}$ *by*

$$(T_uB)_{x,y}=B_{u,x}+B_{u+x,y}-B_{u,x+y}\tag{1}$$

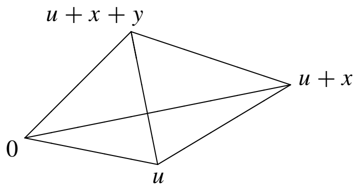

Fig. 2. Shift of triangles

preserves the law of the Brownian sheet $(B_{x,y})_{x,y\in\mathbb{R}^2}$. *More precisely, the covariance* $\mathrm{cov}((T_u B)_{x,y}, (T_{u'} B)_{x',y'})$ *is the signed area of the intersection of the oriented triangle of vertices* u, $u+x$, $u+x+y$ *with the oriented triangle of vertices* u', $u'+x'$, $u'+x'+y'$.

Proof. The fact that T is an action of the group $\mathbb{R}^2$ can be checked by a simple calculus. The expanding of $(T_v(T_u(B)))_{x,y}$ leads to $3\times 3 = 9$ terms; 6 of them can be simplified, and the 3 other ones give $(T_{u+v}B)_{x,y}$. Before getting into the details of the proof of the stationarity of T, let us give its geometrical interpretation. Let us denote $\mathcal{T}(u, x, y)$ the plane surface delimited by the triangle of first vertex u and edges x and y, that is to say of vertices u, $u+x$ and $u+x+y$. Let $\chi_{u,x,y}$ be its signed characteristic function: $\chi_{u,x,y}(z) = \pm 1$ if $z \in \mathcal{T}(u, x, y)$ and $= 0$ otherwise, the sign being the sign of the determinant of (x, y). This leads to the formula

$$z-\text{a.e.} \qquad \chi_{u,x,y}(z) = \chi_{0,u,x}(z) + \chi_{0,u+x,y}(z) - \chi_{0,u,x+y}(z). \tag{2}$$

as we can check it on Figure 2. Hence the transformation T_u defined in the above proposition is the analog of the shift of the triangles $\mathcal{T}$, for the trajectories B.

The stationarity of T will be proved if we check that

$$\mathrm{cov}(B_{x,y}, B_{x',y'}) = \mathrm{cov}((T_u B)_{x,y}, (T_u B)_{x',y'}). \tag{3}$$

But we have, by definition of the law of $(B_{x,y})_{x,y}$,

$$\mathrm{cov}(B_{x,y}, B_{x',y'}) = \int \chi_{0,x,y} \cdot \chi_{0,x',y'} \, d\lambda, \tag{4}$$

where λ is Lebesgue measure on $\mathbb{R}^2$. This formula can be generalized as following:

$$\mathrm{cov}((T_u B)_{x,y}, (T_{u'} B)_{x',y'}) = \int \chi_{u,x,y} \cdot \chi_{u',x',y'} \, d\lambda. \tag{5}$$

Indeed, using (2), the right member leads to $3\times 2 = 6$ terms. Applying (4) to each of them, we obtain the expanding of

$$\mathrm{cov}(B_{u,x} + B_{u+x,y} - B_{u,x+y},\ B_{u',x'} + B_{u'+x',y'} - B_{u',x'+y'}).$$

Then, using (1), assertion (5) follows.

Now (3) is a readily consequence of the invariance of λ under the shift. □

We now establish an other expression of the covariance $\text{cov}(B_{x,y}, B_{x',y'})$ which can easily be generalized to dimension 3.

Lemma 1. *Let $(B_{x,y})_{x,y}$ be the degree 2 Brownian sheet in $\mathbb{R}^2$. We have*

$$\text{cov}(B_{x,y}, B_{x',y'}) = \frac{-1}{2\pi} \int_{\partial\mathcal{T}(0,x,y)} \int_{\partial\mathcal{T}(0,x',y')} \ln(\|M - N\|) \, \langle d\vec{\ell}(M), d\vec{\ell}'(N)\rangle$$

where $d\vec{\ell}$ and $d\vec{\ell}'$ are respectively the infinitesimal tangential fields of the boundary of triangles $\mathcal{T}(0, x, y)$ and $\mathcal{T}(0, x', y')$, and $\langle d\vec{\ell}, d\vec{\ell}'\rangle$ is the tensor-product of theses two-dimensional valued one-forms (see the explicit expression at line (6) below).

Proof. Note that the function $t \mapsto \ln|t|$ is integrable on the neighborhood of 0 in $\mathbb{R}$, hence the above integral is well defined. Let $u \mapsto (M_1(u), M_2(u)), u \in [0, 1]$ (resp. $v \mapsto (N_1(v), N_2(v)), v \in [0, 1]$) be a parameterization of the boundary $\partial\mathcal{T}(0, x, y)$ (resp. $\partial\mathcal{T}(0, x', y')$). We have to prove that the signed area of the intersection of the triangles $\mathcal{T}(0, x, y)$ and $\mathcal{T}(0, x', y')$ can be expressed as

$$\frac{-1}{2\pi} \int_{u=0}^{1} \int_{v=0}^{1} \frac{1}{2} \ln\Big(\sum_{i=1}^{2}\big(M_i(u) - N_i(v)\big)^2\Big)\Big(\sum_{i=1}^{2} \frac{dM_i}{du}\frac{dN_i}{dv}\Big) \, du dv. \tag{6}$$

Integrating first in dv, we have to calculate

$$J_i(u) = \int_{v=0}^{1} \frac{1}{2} \ln\Big(\big(M_1(u) - N_1(v)\big)^2 + \big(M_2(u) - N_2(v)\big)^2\Big) \frac{dN_i}{dv} \, dv,$$

for $i = 1, 2$. Assume that $\mathcal{T}(0, x', y')$ is positively oriented. By the Green-Riemann formula

$$\int_{\partial\mathcal{T}(0,x',y')} \Big(P \, dN_1 + Q \, dN_2\Big) = \iint_{\mathcal{T}(0,x',y')} \Big(\frac{\partial Q}{\partial N_1} - \frac{\partial P}{\partial N_2}\Big) \, dN_1 dN_2,$$

the integrals $J_i(u)$ become

$$J_1(u) = \iint_{\mathcal{T}(0,x',y')} \frac{M_2(u) - N_2}{\big(M_1(u) - N_1\big)^2 + \big(M_2(u) - N_2\big)^2} \, dN_1 dN_2,$$

$$J_2(u) = -\iint_{\mathcal{T}(0,x',y')} \frac{M_1(u) - N_1}{\big(M_1(u) - N_1\big)^2 + \big(M_2(u) - N_2\big)^2} \, dN_1 dN_2.$$

Hence, the integral $I = \frac{-1}{2\pi} \int_0^1 \sum_{i=1}^{2} J_i(u) \frac{dM_i}{du} \, du$ above can be written

$$\frac{1}{2\pi} \iint_{\mathcal{T}(0,x',y')} \int_{u=0}^{1} \frac{-(M_2(u) - N_2)\frac{dM_1}{du} + (M_1(u) - N_1)\frac{dM_2}{du}}{\big(M_1(u) - N_1\big)^2 + \big(M_2(u) - N_2\big)^2} \, du \, dN_1 dN_2.$$

Note that the function of the variable u inside the integral is

$$\frac{d}{du} \arctan \frac{M_2(u) - N_2}{M_1(u) - N_1}.$$

Then the integral in du is equal to $\pm 2\pi$ if the point (N_1, N_2) is inside the curve $\partial\mathcal{T}(0, x, y)$, and 0 otherwise. This means that

$$I = \iint_{\mathcal{T}(0,x',y')} \chi_{0,x,x+y}(N_1, N_2)\, dN_1 dN_2$$

which is the excepted equality, since the triangle $\mathcal{T}(0, x', y')$ was supposed positively oriented. □

3. Generalization of the degree two Brownian sheet to $\mathbb{R}^3$

The notions of plane triangular surface $\mathcal{T}(u, x, y)$, of centered Gaussian field $(B_{x,y})_{x,y}$, and the definition of the action $(T_u)_u$ by formula (1) can be generalized to $\mathbb{R}^3$ without modifications. But the geometrical interpretation of $(T_u)_u$ is changed. Indeed, formula (2) is replaced by

$$\partial\mathcal{T}(u, x, y) = \partial\mathcal{T}(0, u, x) + \partial\mathcal{T}(0, u + x, y) - \partial\mathcal{T}(0, u, x + y). \tag{7}$$

This formula has rigorous sense in the abelian free group generated by the family of the oriented segments $([v, v + z])_{v,z\in\mathbb{R}^3\times\mathbb{R}^3}$. A little picture shows that it is true: in Figure 2, suppose that the points 0, u, $u + x$ and $u + x + y$ are not necessary in the same plan, and look at the oriented boundaries of the triangles. Hence T_u is analogous, for trajectories of B, to the shift of the boundaries of the triangles, not of the surfaces themselves.

The generalization of the degree two Brownian sheet to dimension 3 can be defined as follows:

Definition 2. *A degree* 2 *Brownian sheet in dimension* 3 *is a centered Gaussian field* $(B_{x,y})_{x,y\in\mathbb{R}^3\times\mathbb{R}^3}$ *with the following properties:*

1. *the law of the Gaussian field* $B = (B_{x,y})_{x,y\in\mathbb{R}^3\times\mathbb{R}^3}$ *is preserved by the action* $(T_u)_{u\in\mathbb{R}^3}$*;*
2. *for any* 2 *dimensional linear subspace* $P \subset \mathbb{R}^3$ *and any* $u \in \mathbb{R}^3$, *the Gaussian field* $(T_u B_{x,y})_{x,y\in P\times P}$ *has the law of the degree* 2 *Brownian sheet in dimension* 2 *described above.*

For any points u, x, y, u', x', y' in $\mathbb{R}^3$, let us denote $c(u, x, y, u', x', y')$ the following integral:

$$c(u, x, y, u', x', y') = \frac{-1}{2\pi} \int_{\partial\mathcal{T}(u,x,y)} \int_{\partial\mathcal{T}(u',x',y')} \ln(\|M - N\|)\, \langle d\vec{\ell}(M), d\vec{\ell}'(N)\rangle$$

We will prove the following assertion:

Theorem 1. *The moment condition*

$$\operatorname{cov}(B_{x,y}, B_{x',y'}) = c(0, x, y, 0, x', y') \tag{8}$$

defines a centered Gaussian field $(B_{x,y})_{x,y\in\mathbb{R}^3\times\mathbb{R}^3}$ *which is a degree* 2 *Brownian sheet in dimension* 3.

The proof of this theorem is presented in three steps:

- For the 2 dimensional restriction property, we have to check that, if x, y, x', y' belong to the same linear plan of $\mathbb{R}^3$, then $c(0, x, y, 0, x', y')$ is the signed area of the intersection of the oriented triangles $\mathcal{T}(0, x, y)$ and $\mathcal{T}(0, x', y')$.
- For the stationarity, we have to check that

$$\mathrm{cov}(B_{x,y}, B_{x',y'}) = \mathrm{cov}((T_u B)_{x,y}, (T_u B)_{x',y'}). \tag{9}$$

- For the compatibility condition of Kolmogorov theorem (see [1], or [2] for the Gaussian version), we have to prove that for any finite family of real coefficients $(a_i)_{i\le n}$, and any finite families of points $(x_i)_{i\le n}$ and $(y_i)_{i\le n}$, the double sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j c(0, x_i, y_i, 0, x_j, y_j) \tag{10}$$

is non negative (note that c is obviously symmetric).

The first step is a direct consequence of Lemma 1.

The second directly follows from Formula (7), and from the fact that the integral which defines $c(u, x, y, u', x', y')$ is taken on the boundaries of the triangles: we use (7) to calculate $c(u, x, y, u', x', y')$, and (1) to calculate $\mathrm{cov}((T_u B)_{x,y}, (T_{u'} B)_{x',y'})$. Applying (8) to each terms, we obtain

$$c(u, x, y, u', x', y') = \mathrm{cov}((T_u B)_{x,y}, (T_{u'} B)_{x',y'}).$$

But, in the case $u = u'$, it is easy to see that

$$c(u, x, y, u, x', y') = c(0, x, y, 0, x', y')$$

which gives (9). □

The third step is the most delicate one. We begin with a smooth version of it.

Lemma 2. *For any vector fields $\vec{f}, \vec{g}$ of class $\mathcal{C}^\infty$ with compact support, let $b(\vec{f}, \vec{g})$ be the integral*

$$b(\vec{f}, \vec{g}) = \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} -\ln(\|M - N\|) \cdot \langle \mathrm{curl}\vec{f}(M), \mathrm{curl}\vec{g}(N)\rangle \, dM dN, \tag{11}$$

where $\mathrm{curl}\vec{f}$ *is the vector field whose coordinates are defined by* $\mathrm{curl}_k\vec{f}(v) = \frac{\partial f_j}{\partial v_i} - \frac{\partial f_i}{\partial v_j}$ *for any cyclic permutation* (i, j, k) *of* $(1, 2, 3)$. *Then, for any vector field $\vec{f}$ of class $\mathcal{C}^\infty$ with compact support, $b(\vec{f}, \vec{f})$ is non negative.*

The function $v \mapsto -\ln\|v\|$ and its partial derivatives of order one and two are locally integrable in $\mathbb{R}^3$. Hence, expanding the above scalar product to

$$\sum_{i=1}^{3}\Big(\sum_{\substack{j=1\\ j\ne i}}^{3} \frac{\partial f_i}{\partial v_j}(M)\frac{\partial g_i}{\partial v_j}(N)\Big) - \sum_{\substack{i,j=1\\ i\ne j}}^{3} \frac{\partial f_j}{\partial v_i}(M)\frac{\partial g_i}{\partial v_j}(N),$$

and using two integrations by parts, we get

$$b(\vec{f}, \vec{g}) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} f_i(M) K_{i,j}(M-N) g_j(N)\, dM dN \tag{12}$$

where $K_{i,j}(v)$ is defined for $v = (v_i)_{1 \le i \le 3}$ by

$$K_{i,j}(v) = \begin{cases} -\dfrac{\partial^2}{\partial v_i \partial v_j}(\ln \|v\|) & \text{if } i \neq j; \\ \displaystyle\sum_{\substack{k=1 \\ k \neq i}}^{3} \dfrac{\partial^2}{\partial v_k \partial v_k}(\ln \|v\|) & \text{if } i = j. \end{cases} \tag{13}$$

This leads to $K_{i,j}(v) = 2\dfrac{v_i v_j}{\|v\|^4}$. The proof of Lemma 2 will be obtained by Fourier analysis. Also we first calculate the Fourier transform of the function $K_{i,j}$. For $k \ge 0$ integer, and $\alpha \in]0, n[$, where $n = 3$ for us, we denote by $\gamma_{k,\alpha}$ the following constant:

$$\gamma_{k,\alpha} = i^k \pi^{n/2-\alpha} \frac{\Gamma(k/2 + \alpha/2)}{\Gamma(k/2 + n/2 - \alpha/2)}.$$

We also use the following standard notation: $\delta_{i,j} = 1$ if $i = j$ and $= 0$ if not. For any function ϕ on $\mathbb{R}^3$, let us denote $\hat{\phi}$ its Fourier transform, defined by:

$$\hat{\phi}(\eta) = \int_{\mathbb{R}^3} \phi(v) e^{2i\pi\langle \eta, v\rangle}\, dv.$$

Lemma 3. *We have* $\hat{K}_{i,j}(\eta) = \dfrac{\pi}{\|\eta\|}(\delta_{i,j} - \dfrac{\eta_i \eta_j}{\|\eta\|^2})$*, where* $\eta = (\eta_i)_{1 \le i \le 3}$*, in the sense that for any function* ϕ *of class* $\mathcal{C}^\infty$ *with compact support,*

$$\int_{\mathbb{R}^3} K_{i,j}(M)\phi(M)\, dM = \int_{\mathbb{R}^3} \frac{\pi}{\|\eta\|}(\delta_{i,j} - \frac{\eta_i \eta_j}{\|\eta\|^2})\hat{\phi}(\eta)\, d\eta. \tag{14}$$

For a function ψ *of class* $\mathcal{C}^\infty$ *with compact support, let us define the function* $\mathcal{K}_{i,j}(\psi)$ *by*

$$\mathcal{K}_{i,j}(\psi)(M) = \int_{\mathbb{R}^3} K_{i,j}(M-N)\psi(N)\, dN.$$

Then its Fourier transform is $\mathcal{K}_{i,j}(\psi)\hat{}(\eta) = \frac{\pi}{\|\eta\|}(\delta_{i,j} - \frac{\eta_i \eta_j}{\|\eta\|^2})\hat{\psi}(\eta)$*, in the sense that for any function* ϕ *of class* $\mathcal{C}^\infty$ *with compact support, we have*

$$\int_{\mathbb{R}^3} \mathcal{K}_{i,j}(\psi)(M)\phi(M)\, dM = \int_{\mathbb{R}^3} \frac{\pi}{\|\eta\|}(\delta_{i,j} - \frac{\eta_i \eta_j}{\|\eta\|^2})\hat{\psi}(\eta)\overline{\hat{\phi}(\eta)}\, d\eta. \tag{15}$$

Proof of Lemma 3. The equality (14) of this lemma is a direct corollary of the lemma page 73 of [10], which can be cited as follows:

Lemma(Stein). Let k be a non negative integer. Let P_k be a homogeneous polynomial of degree k, defined on $\mathbb{R}^n$. Let us suppose that $\Delta P_k = 0$, where Δ denotes the standard Laplacian operator. Let $\alpha \in]0, n[$. Then for any function ϕ of class $\mathcal{C}^\infty$ with compact support, we have

$$\int_{\mathbb{R}^n} \frac{P_k(x)}{\|x\|^{n+k-\alpha}} \phi(x)\, dx = \gamma_{k,\alpha} \int_{\mathbb{R}^n} \frac{P_k(\eta)}{\|\eta\|^{k+\alpha}} \hat{\phi}(\eta)\, d\eta.$$

In the case $i \neq j$, Stein's lemma can be applied directly, with $n = 3, k = 2, \alpha = 1$ and the homogeneous polynomial $P(v) = 2v_i v_j$ of degree 2, which is harmonic. Since $\gamma_{2,1} = -\pi/2$, its gives

$$\hat{K}_{i,j}(\eta) = -\pi \frac{\eta_i \eta_j}{\|\eta\|^3}.$$

In the case $i = j$ (= 1 for example) the polynomial $P(v) = 2v_1^2$ is not harmonic, but $K_{1,1}(v)$ can be decomposed as follows:

$$K_{1,1}(v) = \frac{2}{3} \frac{2v_1^2 - v_2^2 - v_3^2}{\|v\|^4} + \frac{2}{3} \frac{1}{\|v\|^2}.$$

Stein'o lemma can be applied to the first member with $k = 2, \alpha = 1$ and $P(v) = 2v_1^2 - v_2^2 - v_3^2$, and to the second member with $k = 0, \alpha = 1$ and $P(v) = 1$. Since $\gamma_{0,1} = \pi$, this leads to

$$\hat{K}_{1,1}(\eta) = \pi \frac{\eta_2^2 + \eta_3^2}{\|\eta\|^3}.$$

These are the desired equalities. This proves (14). Note that (14) can be rewritten as

$$\int_{\mathbb{R}^3} K_{i,j}(N)\psi(M - N)\, dN = \int_{\mathbb{R}^3} \frac{\pi}{\|\eta\|} (\delta_{i,j} - \frac{\eta_i \eta_j}{\|\eta\|^2}) \hat{\psi}(\eta) e^{-2i\pi\langle \eta, M\rangle}\, d\eta;$$

multiplying by $\phi(M)$ and integrating in dM, we obtain (15). The proof Lemma 3 is complete. □

For a vector field $\vec{f} = (f_i)_{1 \le i \le j}$, let us denote $\hat{f} = (\hat{f}_i)_{1 \le i \le 3}$ the vector field of its Fourier transforms. It follows from (12) that

$$b(\vec{f}, \vec{f}) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} f_i(M) \mathcal{K}_{i,j}(f_j)(M)\, dM;$$

hence (15) leads to $b(\vec{f}, \vec{f}) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \frac{\pi}{\|\eta\|} (\delta_{i,j} - \frac{\eta_i \eta_j}{\|\eta\|^2}) \overline{\hat{f}_i(\eta)} \hat{f}_j(\eta)\, d\eta$. This can be rewritten as

$$b(\vec{f}, \vec{f}) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{\pi}{\|\eta\|} \Big(\langle \hat{f}(\eta), \hat{f}(\eta) \rangle^2 - |\langle \frac{\eta}{\|\eta\|}, \hat{f}(\eta) \rangle|^2 \Big) d\eta$$

where $\langle , \rangle$ denotes the standard Hermitian product on $\mathbb{C}^3$. This is clearly positive, by the Cauchy-Schwarz inequality in $\mathbb{C}^3$, hence Lemma 2 is proved. □

The fact that the function $c(0, x, y, 0, z, t)$ is of positive type can be deduced from Lemma 2 by an approximation argument. We will use the following lemma of approximation.

Lemma 4. *Let x, y, z and t be points in $\mathbb{R}^3$. For any $\zeta > 0$, there exists two vector fields of class $\mathcal{C}^\infty$ with compact support $\vec{f}$ and $\vec{g}$ such that $|c(0, x, y, 0, z, t) - b(\vec{f}, \vec{g})| < \zeta$.*

In this lemma, the vector fields $\vec{f}$ and $\vec{g}$ will respectively be close to the fields of normal vectors of the triangles $\mathcal{T}(0, x, y)$ and $\mathcal{T}(0, z, t)$, and curl$\vec{f}$ and curl$\vec{g}$ will respectively be close to the fields of tangential vectors of the boundaries $\partial\mathcal{T}(0, x, y)$ and $\partial\mathcal{T}(0, z, t)$. Let us construct $\vec{f}$ for example. Let ψ be a infinitely derivable non negative function of integral 1, whose support is compact and contains 0. For any $\epsilon > 0$, let ψ_ϵ be the function defined by $\psi_\epsilon(v) = \frac{1}{\epsilon^3}\psi(\frac{v}{\epsilon})$. Up to a rotation, we can suppose that the points $x = (x_i)_i$ and $y = (y_i)_i$ are such that $x_3 = 0$ and $y_3 = 0$. For a point $M = (M_i)_i$ of $\mathbb{R}^3$, let $\vec{f}(M) = (f_i(M))_i$ be defined by $f_1 = f_2 = 0$ and

$$f_3(M) = \pm \int_{\mathcal{T}(0,x,y)} \psi_\epsilon(M_1 - M_1', M_2 - M_2', M_3)\, dM_1' dM_2',$$

the sign being the sign of $x_1 y_2 - x_2 x_1$. Let us suppose it is positive. This mean that the triangle $\mathcal{T}(0, x, y)$, which is contained in the plan $M_3' = 0$, has the orientation given by the axes $((0, M_1'), (0, M_2'))$. By derivation under the integral, we have $\mathrm{curl}\vec{f} = (\mathrm{curl}_i \vec{f})_i$ with

$$\begin{aligned}
\mathrm{curl}_1 \vec{f}(M) &= \int_{\mathcal{T}(0,x,y)} \frac{\partial \psi_\epsilon}{\partial v_2}(M_1 - M_1', M_2 - M_2', M_3)\, dM_1' dM_2';\\
\mathrm{curl}_2 \vec{f}(M) &= -\int_{\mathcal{T}(0,x,y)} \frac{\partial \psi_\epsilon}{\partial v_1}(M_1 - M_1', M_2 - M_2', M_3)\, dM_1' dM_2';\\
\mathrm{curl}_3 \vec{f}(M) &= 0. \qquad (16)
\end{aligned}$$

By the Green-Riemann formula applied separately on these integrals, it leads to

$$\mathrm{curl}_i \vec{f}(M) = \int_u \psi_\epsilon(M_1 - M_1'(u), M_2 - M_2'(u), M_3) \frac{d}{du} M_i'(u)\, du \qquad (17)$$

for $i = 1, 2$ and $\mathrm{curl}_3 \vec{f}(M) = 0$, where $M'(u)$ is a parameterization of the boundary $\partial\mathcal{T}(0, x, y)$. If the triangle $\mathcal{T}(0, x, y)$ is not in the plan $M_3' = 0$, these formulas may be generalized as follows

$$\begin{aligned}
\vec{f}(M) &= \left(\int_{0<r<s<1} \psi_\epsilon(M - rx - sy)\, dr ds \right) \cdot x \times y;\\
\mathrm{curl}_i \vec{f}(M) &= \int_u \psi_\epsilon(M - M'(u)) \frac{d}{du} M_i'(u)\, du, \qquad i = 1, \dots, 3, \qquad (18)
\end{aligned}$$

where $x \times y$ is the cross-product on $\mathbb{R}^3$. In the same way, we associate to the second triangle $\mathcal{T}(0, z, t)$ the function $\vec{g}$ defined by

$$\begin{aligned} \vec{g}(N) &= \left(\int_{0<r<s<1} \psi_\epsilon(N - rz - st)\, drds\right) \cdot z \times t; \\ \mathrm{curl}_i\, \vec{g}(N) &= \int_v \psi_\epsilon(N - N'(v)) \frac{d}{dv} N'_i(v)\, dv, \end{aligned} \tag{19}$$

where $v \mapsto N'(v)$ is a parameterization of $\partial \mathcal{T}(0, z, t)$. We consider the integral

$$b(\vec{f}, \vec{g}) = \frac{-1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln \|M - N\| \Big(\sum_{i=1}^{3} \mathrm{curl}_i\, \vec{f}(M) \mathrm{curl}_i\, \vec{g}(N)\Big)\, dMdN.$$

Let us replace $\mathrm{curl}_i\, \vec{f}(M)$ and $\mathrm{curl}_i\, \vec{g}(N)$ by the expressions obtained from (18) and (19). By the Fubini theorem, it can be written as an integral in $dudv$ of an integral in $dMdN$:

$$b(\vec{f}, \vec{g}) = \frac{-1}{2\pi} \int_u \int_v I(u, v) \sum_{i=1}^{3} \frac{dM'_i}{du} \frac{dN'_i}{dv}\, dudv. \tag{20}$$

with

$$I(u, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln(\|M - N\|) \cdot \psi_\epsilon(M - M'(u)) \cdot \psi_\epsilon(N - N'(v))\, dMdN.$$

By change of variables $(M, N) \mapsto (P, Q)$ defined by $\begin{cases} M = P + M'(u) \\ N = P - Q + N'(v) \end{cases}$, this leads to

$$I(u, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln \|Q + M'(u) - N'(v)\| \psi_\epsilon(P) \psi_\epsilon(P - Q)\, dPdQ.$$

Let us replace it in (20) and use again Fubini theorem, we obtain

$$b(\vec{f}, \vec{g}) = \int_{\mathbb{R}^3} \phi_\epsilon(Q) H(Q)\, dQ \tag{21}$$

with

$$\begin{aligned} H(Q) &= \frac{-1}{2\pi} \int_u \int_v \ln \|Q + M'(u) - N'(v)\| \sum_{i=1}^{3} \frac{dM'_i}{du} \frac{dN'_i}{dv}\, dudv, \\ \phi_\epsilon(Q) &= \int_{\mathbb{R}^3} \psi_\epsilon(P) \psi_\epsilon(P - Q)\, dP. \end{aligned} \tag{22}$$

But H is a continuous function of Q. Hence, when ϵ goes to zero, the integral (21) tends to $H(0)$, which is $c(0, x, y, 0, z, t)$. This proves Lemma 4. □

We will now deduce that (10) is non negative. Let us consider a finite family of real coefficients $(a_i)_{i \leq n}$, and a finite family of points $(x_i)_{i \leq n}$ and $(y_i)_{i \leq n}$. For any $\epsilon > 0$, let $\vec{f_i}$ be the vector field associated to the triangle $\mathcal{T}(0, x_i, y_i)$ as in the proof of Lemma 4. For a fixed $\zeta > 0$, we choose ϵ small enough such that for any $i, j \leq n$,

$$|b(\vec{f_i}, \vec{f_j}) - c(0, x_i, y_i, 0, x_j, y_j)| < \zeta.$$

Let $\vec{f} = \sum_{i=1}^n a_i \vec{f_i}$. The above inequality implies

$$|b(\vec{f}, \vec{f}) - \sum_i \sum_j a_i a_j c(0, x_i, y_i, 0, x_j, y_j)| < (\sum_i |a_i|)^2 \zeta.$$

By the non negativity of $b(\vec{f}, \vec{f})$ stated by Lemma 2, it follows

$$\sum_i \sum_j a_i a_j c(0, x_i, y_i, 0, x_j, y_j) \geq -(\sum_i |a_i|)^2 \zeta.$$

Since it is true for any ζ, the double sum is non negative. This proves the third and last step of Theorem 1. □

4. Remarks and Comments

A natural question after the construction presented is this paper concerns the uniqueness of the determination of the degree 2 Brownian sheet in $\mathbb{R}^3$. The covariance of Lévy's two-parameter Brownian motion readily follows from its definition: $\mathrm{cov}(B_x, B_y) = \frac{1}{2}(\|x\| + \|y\| - \|x - y\|)$. The analogue calculus doesn't work for the degree two Brownian sheet. Indeed, from Definition 2 it follows that the variance $\|B_{x,y}\|_2^2$ is equal to the area $\frac{1}{2}\|x \wedge y\|$ of the triangle $\mathcal{T}(0, x, y)$. Since the law of the field $(B_{x,y})_{x,y}$ is preserved by T_u, we have also

$$\begin{aligned}\frac{1}{2}\|x \wedge y\| &= \|B_{u,x} + B_{u+x,y} - B_{u,x+y}\|_2^2 \\ &= \mathrm{cov}(B_{u,x}, B_{u+x,y}) - \mathrm{cov}(B_{u,x}, B_{u,x+y}) - \mathrm{cov}(B_{u+x,y}, B_{u,x+y}) \\ &\quad + \frac{1}{2}\|u \wedge x\| + \frac{1}{2}\|(u + x) \wedge y\| + \frac{1}{2}\|u \wedge (x + y)\|. \end{aligned} \tag{23}$$

But this formula doesn't allow us to determinate $\mathrm{cov}(B_{x,y}, B_{z,t})$. It is an open question to know whether the definition of the degree 2 Brownian sheet in dimension 3 implies the moment condition of Theorem 1 or not.

There are other representations of the degree 2 Brownian sheet in dimension 3. For example, geometrically, it can be viewed as being a centered Gaussian field indexed by the set $\mathcal{S}$ of the compact two-dimensional oriented $\mathcal{C}^1$-sub-manifolds with boundary. For $\Sigma, \Sigma' \in \mathcal{S}$, the covariance $\mathrm{cov}(B_\Sigma, B_{\Sigma'})$ is

$$\mathrm{cov}(B_\Sigma, B_{\Sigma'}) = \frac{-1}{2\pi} \int_{\partial\Sigma} \int_{\partial\Sigma'} \ln(\|M - N\|) \, \langle d\vec{\ell}(M), d\vec{\ell}'(N) \rangle. \tag{24}$$

Note that B_Σ depends only on the boundary $\partial\Sigma$. Indeed, the variance of $B_\Sigma - B_{\Sigma'}$ is equal to

$$\mathrm{cov}(B_\Sigma, B_\Sigma) + \mathrm{cov}(B_{\Sigma'}, B_{\Sigma'}) - 2\mathrm{cov}(B_\Sigma, B_{\Sigma'}).$$

If $\partial\Sigma = \partial\Sigma'$, this variance is zero and we have $B_\Sigma = B_{\Sigma'}$. In the same way, if the manifold Σ has no boundary (a sphere for instance), then $B_\Sigma = 0$. Hence, by lemma 1, if $\Sigma, \Sigma' \in \mathcal{S}$ are such that there boundaries are contained in the same plan, then the covariance $\mathrm{cov}(B_\Sigma, B_{\Sigma'})$ is the signed area of the intersection of $\bar{\Sigma}$ and $\bar{\Sigma}'$, where $\bar{\Sigma}$ and $\bar{\Sigma}'$ are the plane surfaces such that $\partial\Sigma = \partial\bar{\Sigma}$ and $\partial\Sigma' = \partial\bar{\Sigma}'$.

Let us come back to the Gaussian field $(B_{x,y})_{x,y}$ on triangles studied in this paper. It has connexion with homology theory. Indeed, let us denote Ω the space of trajectories $B = (B_{x,y})_{x,y}$, and define the function G on $\Omega \times \mathbb{R}^3 \times \mathbb{R}^3$ by $G(B, x, y) = B_{x,y}$. We have, by definition of T

$$G(T_u B, x, y) = G(B, u, x) + G(B, u + x, y) - G(u, x + y).$$

This is the equation which defines the algebraic notion of cocycle of degree 2 for the action T (see [4], [7] or [3]). This fact is analogous to the fact that, for the 2-parameter Lévy's Brownian motion $(B_x)_x$ defined in Introduction, the function F defined by $F(B, x) = B_x - B_0$ verifies the equation

$$F(B, u + x) = F(B, u) + F(S_u B, x).$$

This means exactly that F is an algebraic cocycle of degree 1 for the action S. Let us note that the more "natural" degree in dimension 3 is the degree 3: the Brownian cocycle of degree 3 in dimension 3 is obtained from the ordinary Brownian sheet on tetrahedra of $\mathbb{R}^3$. Indeed, this field is the centered Gaussian field $(B_{x,y,z})_{x,y,z\in\mathbb{R}^3}$ such that the covariance $\mathrm{cov}(B_{x,y,z}, B_{x',y',z'})$ is the signed volume of the intersection of the tetrahedron of vertices $0, x, x + y, x + y + z$ with the tetrahedron of vertices $0, x', x' + y', x' + y' + z'$. Since the $\mathbb{R}^3$-action $(U_u)_u$ defined by

$$(U_u B)_{x,y,z} = B_{u+x,y,z} - B_{u,x+y,z} + B_{u,x,y+z} - B_{u,x,y}$$

is analogous, on trajectories B, to the shift of tetrahedra, it is stationary. Now the function H defined by $H(B, x, y, z) = B_{x,y,z}$ is, by definition of U, a cocycle of degree 3 for U, that is to say satisfies

$$\begin{aligned} H(U_u B, x, y, z) &= H(B, u + x, y, z) - H(B, u, x + y, z) \\ &\quad + H(B, u, x, y + z) - H(B, u, x, y). \end{aligned}$$

A natural extension of this work is the case of higher dimension, and higher degree. Following the same method, with some harder calculus on the matrix $K = (K_{i,j})_{i,j}$, we have checked the case of the degree 2 in dimension n. Concerning the higher degree, and using for example the geometrical point of view, we propose the following function of covariance: a degree k Brownian sheet in dimension n ($k \le n$) may be a centered Gaussian field indexed by the set $\mathcal{S}$ of compact k-dimensional oriented $\mathcal{C}^1$-sub-manifolds of $\mathbb{R}^n$ with boundary such that for $\Sigma, \Sigma' \in \mathcal{S}$,

$$\mathrm{cov}(B_\Sigma, B_{\Sigma'}) = \int_{\partial\Sigma} \int_{\partial\Sigma'} G_k(M - N)\, \langle d\vec{\sigma}(M), d\vec{\sigma}'(N)\rangle \qquad (25)$$

where G_k is the extension to $\mathbb{R}^n$ of the radial decreasing Green function of $\mathbb{R}^k$:

$$G_k(v) = \begin{cases} -\|v\|/2 & \text{if } k = 1; \\ -(\ln \|v\|)/(2\pi) & \text{if } k = 2; \\ (\frac{\Gamma(k/2-1)}{4\pi^{k/2}})/(\|v\|^{k-2}) & \text{if } k \geq 3. \end{cases}$$

The detailed check of the fact that the equality (25) is appropriate is in progress.

References

1. Billingsley, Y.: Probability and Measure. Wiley, 1979
2. Dacunha-Castelle, D., Duflo, M.: Probabilités et Statistiques. Masson, 1983
3. Depauw, J.: Cocycles alternés de degré 2 d'actions mesurables stationnaires de $\mathbb{R}$, $\mathbb{R}^2$ et $\mathbb{R}^3$. preprint
4. Feldman, J., Moore, C.C.: Ergodic equivalence relations, cohomology, and von Neumann algebra. I. Trans. Amer. Math. Soc. **234**, 289–324 (1977)
5. Godement, R.:Théorie des faiceaux. Publications de l'Institut de Mathématiques de l'Université de Strasbourg XIII. Hermann, Paris, 1964
6. Lévy, P.: Processus stochastiques et mouvement brownien. Gauthier-Villars, Paris, 1965
7. Mackey, G.W.: Ergodic Theory And Virtual Groups. Math. Ann. **166**, 187–207 (1966)
8. Mac Lane, S.: Homology. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer-Verlag, 1967
9. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes, and Martingales. Cambridge University Press, 2000
10. Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, 1970