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Distributional limit theorems in infinite ergodic theory

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Abstract. We present a unified approach to the Darling-Kac theorem and the arcsine laws for occupation times and waiting times for ergodic transformations preserving an infinite measure. Our method is based on control of the transfer operator up to the first entrance to a suitable reference set rather than on the full asymptotics of the operator. We illustrate our abstract results by showing that they easily apply to a significant class of infinite measure preserving interval maps. We also show that some of the tools introduced here are useful in the setup of pointwise dual ergodic transformations.

1. Introduction

The study of ergodic and probabilistic properties of dynamical systems with an infinite invariant measure has recently led to a number of interesting results which generalize classical theorems for null-recurrent Markov chains to the weakly dependent processes generated by certain types of infinite measure preserving transformations. In the present paper we shall focus on three distributional limit theorems, the Darling-Kac theorem for ergodic sums of integrable functions, the arcsine law for occupation times of sets of infinite measure, and the (Dynkin-Lamperti) arcsine law for waiting times, and present a natural unified approach to them. The following example illustrates the limit theorems we are going to consider by specializing them to the case of Boole's transformation on $\mathbb{R}$, where we obtain results analogous to well known classical facts about the coin tossing random walk (cf. Chapter III of [Fe1]).

Example 1.1 (Distributional limit theorems for Boole's transformation). The map $T : \mathbb{R} \to \mathbb{R}$ given by $Tx := x - \frac{1}{x}$ preserves Lebesgue measure λ and is conservative ergodic, cf. [AW] or [Sch]. For measurable functions $f : \mathbb{R} \to \mathbb{R}$ let $\mathbf{S}_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, $n \in \mathbb{N}$. Fix any Borel probability measure $\nu \ll \lambda$.

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The *Darling-Kac theorem* shows that for the occupation times of any Borel subset $E \subseteq \mathbb{R}$ of finite positive measure, as $n \to \infty$,

$$\nu\left(\left\{\frac{\pi}{\sqrt{2n}}\mathbf{S}_n(1_E) \leq \lambda(E)\,t\right\}\right) \longrightarrow \frac{2}{\pi}\int_0^t e^{-\frac{y^2}{\pi}}\,dy, \quad t \geq 0.$$

(Here 1_E may be replaced by any integrable function f with $\lambda(f) = \int_{\mathbb{R}} f\,d\lambda > 0$.) The *arcsine law for occupation times* implies that the proportion of time spent on a half-line converges to the *classical arcsine distribution*,

$$\nu\left(\left\{\frac{1}{n}\mathbf{S}_n(1_A) \leq t\right\}\right) \longrightarrow \frac{2}{\pi}\arcsin\sqrt{t}, \quad t \in [0, 1],$$

where A is any Borel set with $\lambda(A \,\triangle\, (0, \infty)) < \infty$. The *arcsine law for waiting times* finally provides us with a similar result for $\mathbf{Z}_n(E)(x)$, the time of the last visit of the orbit $(T^k x)_{k\geq 0}$ to the set E up to step n (and 0 if there was no visit at all), showing that

$$\nu\left(\left\{\frac{1}{n}\mathbf{Z}_n(E) \leq t\right\}\right) \longrightarrow \frac{2}{\pi}\arcsin\sqrt{t}, \quad t \in [0, 1],$$

for every bounded $E \subseteq \mathbb{R}$ with $\lambda(E) > 0$.

For the specific transformation T of the example, these statements follow from earlier work in [A1], [T6], and [T4] respectively. The purpose of the present paper is to develop an approach to these limit theorems in a general abstract setup, based on, and improving, ideas from [T6]. Our assumptions are of a different type than those used in [A1], [T4], and constitute a generalization of the abstract condition which can be extracted from [T6]. They allow simple direct verification for an important class of examples. Moreover, the proofs themselves have a very clear and natural common structure. In the final section we point out that some of the ideas employed here are also of interest in the setup of [A1] and [T4] (pointwise dual ergodic transformations).

2. Preliminaries

In order to formulate our results, we need to fix some notation and recall a number of important concepts. Throughout the paper, *all measures are understood to be σ-finite*. The key to an understanding of the stochastic properties of a (typically non-invertible) *nonsingular transformation* T on a measure space $(X, \mathcal{A}, m)$, i.e. of a measurable map $T : X \to X$ for which $m \circ T^{-1} \ll m$, often lies in the study of the long-term behaviour of its *transfer operator* $\widehat{T} : L_1(m) \to L_1(m)$ describing the evolution of measures under the action of T on the level of densities: $\widehat{T}u := d(\nu \circ T^{-1})/dm$, where ν has density u w.r.t. m. Equivalently, $\int_X u \cdot (v \circ T)\,dm = \int_X \widehat{T}u \cdot v\,dm$ for all $u \in L_1(m)$ and $v \in L_\infty(m)$, i.e. $v \longmapsto v \circ T$ is the dual of $\widehat{T}$. The operator $\widehat{T}$ naturally extends to $\{u : X \to [0, \infty)\ \mathcal{A}\text{-measurable}\}$. It is a linear Markov operator, $\int_X \widehat{T}u\,dm = \int_X u\,dm$ for $u \geq 0$. The system is conservative and ergodic iff $\sum_{k\geq 0} \widehat{T}^k u = \infty$ a.e. for all $u \in L_1^+(m) := \{u \in L_1(m) : u \geq 0$

and $m(u) > 0\}$. Invariance of m under T means that $\widehat{T}1 = 1$, and we will denote the measure by μ in this case. When dealing with L_1-functions, uniform convergence will always be understood mod m. Similarly, we will simply write inf for the essential infimum etc.

If, for some measurable function $H \geq 0$ supported on $Y \in \mathcal{A}$, there is some $K \in \mathbb{N}_0$ such that $\inf_Y \sum_{k=0}^{K} \widehat{T}^k H > 0$, then H will be called *uniformly sweeping (in K steps)* for Y.

If ν is a probability measure on $(X, \mathcal{A})$, $(R_n)_{n\geq 1}$ is a sequence of measurable real-valued functions on X, and R is a random variable taking values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, then distributional convergence of $(R_n)_{n\geq 1}$ to R w.r.t. ν will be denoted by $R_n \overset{\nu}{\Longrightarrow} R$. *Strong distributional convergence* $R_n \overset{\mathcal{L}(m)}{\Longrightarrow} R$ on $(X, \mathcal{A}, m)$ means that $R_n \overset{\nu}{\Longrightarrow} R$ for all probability measures $\nu \ll m$.

A function $a : (L, \infty) \to (0, \infty)$ is *regularly varying of index* $\rho \in \mathbb{R}$ at infinity, written $a \in \mathcal{R}_\rho$, if a is measurable and $a(ct)/a(t) \to c^\rho$ as $t \to \infty$ for any $c > 0$, and we shall interpret sequences (a_n) as functions on $\mathbb{R}_+$ via $t \longmapsto a_{[t]}$. *Slow variation* means regular variation of index 0. $\mathcal{R}_\rho(0)$ is the family of functions $r : (0, \varepsilon) \to \mathbb{R}_+$ regularly varying of index ρ at zero (same condition as above, but for $t \searrow 0$). We refer to Chapter 1 of [BGT] for a collection of basic results.

Let T be a conservative ergodic measure preserving transformation (*c.e.m.p.t.*) on $(X, \mathcal{A}, \mu)$. For any $Y \in \mathcal{A}$, $\mu(Y) > 0$, the *first entrance (resp. return) time* of Y is[1] $\varphi : X \to \mathbb{N} \cup \{\infty\}$, given by $\varphi(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$, and we let $T_Y x := T^{\varphi(x)} x$, $x \in X$. The restricted measure $\mu\,|_{Y\cap\mathcal{A}}$ is invariant under the *first return map*, T_Y restricted to Y. On the level of densities this means that

$$1_Y = \sum_{k\geq 1} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \quad \text{a.e.} \tag{2.1}$$

If $\mu(Y) < \infty$, it is natural to regard φ as a random variable on the probability space $(X, \mathcal{A}, \mu_Y)$, where $\mu_Y(E) := \mu(Y)^{-1}\mu(Y \cap E)$, and $\mu(X) = \infty$ is equivalent to $\int \varphi \, d\mu_Y = \infty$ by Kac' formula (integrate (2.3) below).

If $\mu(X) = \infty$, a good understanding of T frequently depends on its behaviour relative to a suitable *reference set* Y of finite measure, defined through some distinctive property. Specifically, the asymptotic behaviour of the *return distribution of* Y, i.e. that of the *(first) return probabilities* $f_k(Y) := \mu_Y(\{\varphi = k\})$, $k \in \mathbb{N}$, is a crucial feature determining the stochastic properties of the system. For distributional limit theorems to hold, regular variation of $f_k(Y)$ or, more generally, of the *tail probabilities* $q_n(Y) := \sum_{k>n} f_k(Y) = \mu_Y(\{\varphi > n\})$, $n \in \mathbb{N}_0$, or the *wandering rate* of Y, given by $w_N(Y) := \mu(Y) \sum_{n=0}^{N-1} q_n(Y) = \sum_{n=0}^{N-1} \mu(Y \cap \{\varphi > n\}) = \int_Y (\varphi \wedge N) \, d\mu$, $N \geq 1$, is decisive.

To formulate the key assumption characterizing our reference sets $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, we define

$$Y_0 := Y \quad \text{and} \quad Y_n := Y^c \cap \{\varphi = n\},\, n \geq 1.$$

[1] We suppress the dependence of φ on the usually fixed set Y in our notation.

The standard proof of T_Y-invariance of $\mu\mid_{Y\cap\mathcal{A}}$ shows that $\mu(Y_n) = \mu(Y)\,q_n(Y)$ for $n \geq 0$. We will need a pointwise version of this. Notice that for any $A \in \mathcal{A}$ we have $1_A = \widehat{T}1_{T^{-1}A}$ a.e., and hence

$$1_{Y_n} = \widehat{T}1_{Y\cap\{\varphi=n+1\}} + \widehat{T}1_{Y_{n+1}} \quad \text{a.e. for } n \in \mathbb{N}_0. \tag{2.2}$$

Since $Y_n \subseteq T^{-n}Y$ and $\mu(Y) < \infty$ we have $\mu(Y_n) \searrow 0$, and repeated application of (2.2) yields

$$1_{Y_n} = \sum_{k>n} \widehat{T}^{k-n} 1_{Y\cap\{\varphi=k\}} \quad \text{a.e. for } n \in \mathbb{N}_0, \tag{2.3}$$

generalizing (2.1). Observing that $\bigcup_{n=0}^{N-1} T^{-n}Y = \bigcup_{n=0}^{N-1} Y_n$ (pairwise disjoint), we see

$$w_N(Y) = \mu\left(\bigcup_{n=0}^{N-1} T^{-n}Y\right) = \int_Y \left(\sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n}\right) d\mu \quad \text{for } N \geq 1. \tag{2.4}$$

The condition we are going to impose on the reference set Y is that

$$\frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \quad \text{converges uniformly on } Y \text{ as } N \to \infty. \tag{2.5}$$

The limit function $H : Y \to [0, \infty)$, the *asymptotic entrance density of* Y, automatically is a bounded probability density w.r.t. μ. (It is the uniform limit of a sequence of bounded functions.) In addition, we will assume that H is uniformly sweeping for Y.

The examples discussed in Section 8 actually have the property that

$$\frac{1}{f_k(Y)} \cdot \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \qquad \text{converges uniformly on } Y \text{ as } k \to \infty. \tag{2.6}$$

By (2.3), this implies uniform convergence of $\frac{1}{q_n(Y)} \cdot \widehat{T}^n 1_{Y_n}$, $n \geq 1$, which in turn entails (2.5).

3. Main results

We are now ready to state the abstract distributional limit theorems which are the main results of the present paper.

Perhaps the most basic question about some c.e.m.p.t. T on $(X, \mathcal{A}, \mu)$ is that for the asymptotic behaviour of ergodic sums $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 1$, of measurable functions f. If μ is finite (and w.l.o.g. normalized), Birkhoff's ergodic theorem provides us with a strong law of large numbers asserting that $n^{-1}\mathbf{S}_n(f) \longrightarrow \mu(f)$ a.e. for any $f \in L_1(\mu)$. The picture is fundamentally different if T preserves an infinite measure μ: Not only will we have $n^{-1}\mathbf{S}_n(f) \longrightarrow 0$ a.e. for any $f \in L_1(\mu)$, but it is in fact impossible to find any sequence (a_n) of normalizing constants for which $a_n^{-1}\mathbf{S}_n(f)$ has nontrivial a.e. limits for $f \in L_1^+(\mu)$, cf. Section

2.4 of [A0]. However, the Darling-Kac theorem shows that there may still be (a_n) such that $a_n^{-1}\mathbf{S}_n(f)$ converges in distribution.

We let $\mathcal{M}_\alpha$, $\alpha \in [0,1]$, denote a non-negative real random variable distributed according to the *(normalized) Mittag-Leffler distribution of order* α, which can be characterized by its moments

$$\mathbb{E}\left[\mathcal{M}_\alpha^r\right] = r!\,\frac{(\Gamma(1+\alpha))^r}{\Gamma(1+r\alpha)}\,, \quad r \in \mathbb{N}_0.$$

Our Darling-Kac theorem for infinite m.p.t.s reads as follows:

Theorem 3.1 (Darling-Kac theorem). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, and assume there is some $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, such that*

$$\frac{1}{w_N(Y)}\sum_{n=0}^{N-1}\widehat{T}^n 1_{Y_n} \to H \qquad \text{uniformly on } Y \text{ as } N \to \infty, \text{ with } H : Y \to [0,\infty) \text{ uniformly sweeping,} \tag{3.1}$$

and that

$$(w_N(Y)) \in \mathcal{R}_{1-\alpha}\ \text{for some}\ \alpha \in [0,1]. \tag{3.2}$$

Then

$$\frac{1}{a_n}\mathbf{S}_n(f) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mu(f)\cdot\mathcal{M}_\alpha \quad \text{for all } f \in L_1(\mu) \text{ s.t. } \mu(f) \neq 0, \tag{3.3}$$

where

$$a_n := \frac{1}{\mu(Y)}\int_Y \mathbf{S}_n(1_Y)\,d\mu_Y \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\cdot\frac{n}{w_n(Y)} \quad \text{as } n \to \infty.$$

Remark 3.1 (Weak law and dual statement). Notice that $\mathcal{M}_1 = 1$, so that for $\alpha = 1$ the result provides us with a generalized weak law of large numbers. For $\alpha \in (0,1)$ the conclusion (3.3) of the theorem is equivalent to strong distributional convergence

$$\frac{1}{b_j}\sum_{i=0}^{j-1}\varphi_E \circ T_E^i \overset{\mathcal{L}(\mu_E)}{\Longrightarrow} (\mu(E)\cdot\Gamma(1+\alpha))^{-\frac{1}{\alpha}}\cdot\mathcal{G}_\alpha$$

of the j-th return time of an arbitrary $E \in \mathcal{A}$, $0 < \mu(E) < \infty$, $\varphi_E(x) := \min\{i \geq 1 : T^i x \in E\}$, where b is asymptotically inverse to a, and $\mathcal{G}_\alpha$ is a random variable distributed according to the *one-sided stable law of index* α, characterized by

$$\mathbb{E}\left[e^{-s\mathcal{G}_\alpha}\right] = e^{-s^\alpha}, \quad s > 0.$$

Ergodic sums of non-integrable functions will exhibit a different behaviour. We shall content ourselves with occupation times $\mathbf{S}_n(1_A)$ of sets with $\mu(A)=\infty$. The situation $\mu(A^c)<\infty$ being trivial, we are going to compare pairs A_1, A_2 of disjoint sets of infinite measure. The additional structure enabling us to derive a strong result again involves the dynamics relative to a reference set Y: As in [ATZ] we say that two disjoint sets $A_1, A_2 \subseteq X$ are *dynamically separated by* $Y \subseteq X$ *(under the action of* T*)* if $x \in A_1$ and $T^n x \in A_2$ (resp. $x \in A_2$ and $T^n x \in A_1$) imply the existence of some $k = k(x) \in \{0, \ldots, n\}$ for which $T^k x \in Y$ (i.e. T-orbits can't pass from one set to the other without visiting Y). In the present paper, Y will always be disjoint from $A_1 \cup A_2$, with all these sets measurable and $\mu(Y) < \infty$. The latter condition prevents, for example, trivial periodicities between components of infinite measure (like $A_1 = 2\mathbb{Z}$ and $A_2 = 2\mathbb{Z}+1$ in the case of the simple random walk on the integer lattice). Defining $w_N(Y, A_i) := \sum_{n=0}^{N-1} \mu(Y \cap T^{-1}A_i \cap \{\varphi > n\})$, $N \geq 1$, we will see (cf. (6.6) below) that if $X = A_1 \cup Y \cup A_2$ (disjoint), then

$$w_N(Y, A_i) = \mu(Y \cap T^{-1}A_i) + \sum_{n=1}^{N-1} \mu(Y_n \cap A_i). \tag{3.4}$$

For $\alpha, \beta \in (0,1)$ we let $\mathcal{L}_{\alpha,\beta}$ denote a random variable with (values in $[0,1]$ and) distribution given by

$$\begin{aligned}\Pr(\{\mathcal{L}_{\alpha,\beta} \leq t\}) &= \frac{b \sin \pi\alpha}{\pi} \int_0^t \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{b^2 x^{2\alpha} + 2b x^\alpha (1-x)^\alpha \cos \pi\alpha + (1-x)^{2\alpha}}\, dx \\ &= \frac{1}{\pi\alpha} \operatorname{arccot}\left(\frac{((1-t)/t)^\alpha}{b \sin \pi\alpha} + \cot \pi\alpha \right), \quad t \in (0,1],\end{aligned}$$

where $b := (1-\beta)/\beta$. Continuously extending this family, we let $\mathcal{L}_{\alpha,1} := 1$ and $\mathcal{L}_{\alpha,0} := 0$, $\alpha \in [0,1]$, and $\mathcal{L}_{1,\beta} := \beta$, $\Pr(\mathcal{L}_{0,\beta} = 1) = \beta = 1 - \Pr(\mathcal{L}_{0,\beta} = 0)$. These variables satisfy $\mathbb{E}[\mathcal{L}_{\alpha,\beta}] = \beta$ and $\operatorname{Var}[\mathcal{L}_{\alpha,\beta}] = (1-\alpha)\beta(1-\beta)$, cf. [L1] and Section 3 of [T6], where the relation to one-sided stable variables $\mathcal{G}_\alpha$ is discussed, too.

Theorem 3.2 (Arcsine law for occupation times). *Let* T *be a c.e.m.p.t. on the* σ*-finite measure space* $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, *and* Y *be as in Theorem 3.1, satisfying (3.1) and (3.2). Assume further that* $X = A_1 \cup Y \cup A_2$ *(measurable and pairwise disjoint), where* $\mu(A_1) > 0$ *and* Y *dynamically separates* A_1 *and* A_2*, and that*

$$\frac{1}{w_N(Y, A_1)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{A_1 \cap Y_n} \to H_1 \quad \text{uniformly on } Y \text{ as } N \to \infty, \text{ with } H_1 : Y \to [0,\infty) \text{ uniformly sweeping,} \tag{3.5}$$

and

$$\frac{w_N(Y, A_1)}{w_N(Y)} \longrightarrow \beta \in [0,1] \quad \text{as } N \to \infty. \tag{3.6}$$

Then

$$\frac{1}{n}\mathbf{S}_n(1_A) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{L}_{\alpha,\beta} \tag{3.7}$$

for all $A \in \mathcal{A}$ *satisfying* $\mu(A \bigtriangleup A_1) < \infty$.

Remark 3.2. In the $\alpha = 1$, $\beta \in (0,1)$ case this gives a non-trivial weak law of large numbers for the occupation times of the infinite measure set A. The question of the pointwise (a.e.) behaviour in such situations has been discussed in [ATZ].

The following observations are very useful in applying the theorems, cf. Section 8 below. The first enables us to deduce our conditions if we know about smaller components partitioning Y^c.

Remark 3.3. Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, $X = Y \cup \bigcup_{j\in J} B_j$ (measurable and pairwise disjoint), where $0 < \mu(Y) < \infty$, J is finite, and $\mu(B_j) > 0$ for all $j \in J$. Suppose that Y dynamically separates B_i and B_j whenever $i \neq j$. If, for all $j \in J$,

$$\frac{1}{w_N(Y, B_j)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{B_j \cap Y_n} \to D_j \quad \begin{array}{l}\text{uniformly on } Y \text{ as } N \to \infty, \text{ with} \\ D_j : Y \to [0,\infty) \text{ uniformly sweeping,}\end{array} \tag{3.8}$$

and

$$\frac{w_N(Y, B_j)}{w_N(Y)} \longrightarrow \beta_j \in [0,1] \quad \text{as } N \to \infty, \tag{3.9}$$

then T satisfies (3.1) with $H = \sum_{j\in J} \beta_j D_j$. Moreover, for any partition $J = J_1 \cup J_2$, the sets $A_i := \bigcup_{j\in J_i} B_j$ are dynamically separated by Y, and if $\sum_{j\in J_1} \beta_j > 0$, then A_1 satisfies (3.5) and (3.6) with $\beta = \sum_{j\in J_1} \beta_j$ and $H_1 = \beta^{-1} \sum_{j\in J_1} \beta_j D_j$.

The second provides us with an important way to find or recognize good components A_i in systems known to have property (3.1).

Remark 3.4. Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, and Y be as in Theorem 3.1, satisfying (3.1). Assume further that $X = A_1 \cup Y \cup A_2$ (measurable and pairwise disjoint), and that there are disjoint sets $E_1, E_2 \in \mathcal{A} \cap Y$ with $TA_j \setminus A_j \subseteq E_j$, $j \in \{1,2\}$. Then Y separates A_1 and A_2, and if $1_{E_1} \cdot H$ is uniformly sweeping for Y, then (3.5) and (3.6) are satisfied with $\beta = \int_{E_1} H \, d\mu > 0$ and $H_1 = \beta^{-1} 1_{E_1} H$. Moreover, $A_j = \bigcup_{n\geq 1} Y_n \cap T^{-n} E_j$ (mod μ), $j \in \{1,2\}$, which indicates how to construct dynamically separated pairs starting from subsets of Y. (To see this, verify that $A_j \cap Y_n = Y_n \cap T^{-n} E_j$ and hence $\widehat{T}^n 1_{A_j \cap Y_n} = 1_{E_j} \widehat{T}^n 1_{Y_n}$ for $n \geq 1$.)

In many situations (see Example 1.1 and Section 8) there are natural candidates A_i which can be shown to fulfill the conditions of Theorem 3.2. Still we will show, using the preceding remark, that in the situation of our Darling-Kac theorem there are always sets satisfying the arcsine law:

Proposition 3.1 (Existence of sets satisfying the arcsine law). *Let T be a c.e.m.p.t. on the nonatomic σ-finite measure space $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, and Y be as in Theorem 3.1, satisfying (3.1) and (3.2). Then, for any $\beta \in (0, 1)$, there are pairs (A_1, A_2) satisfying the assumptions of Theorem 3.2.*

The second arcsine limit theorem we discuss involves the times at which orbits visit a good set. For $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, we define the $\mathbb{N}_0$-valued variables $\mathbf{Z}_n(Y)$, $n \in \mathbb{N}_0$, on X by $\mathbf{Z}_n(Y)(x) := \max(\{0\} \cup \{1 \leq k \leq n : T^k x \in Y\})$. In the language of renewal theory, $n - \mathbf{Z}_n(Y)$ is the *spent waiting time* if the process is inspected at time n. If μ is a probability measure, the ergodic theorem immediately shows[2] that

$$\frac{1}{n}\mathbf{Z}_n(Y) \longrightarrow 1 \text{ a.e.}$$

The Dynkin-Lamperti arcsine theorem describes the asymptotic behaviour of these renewal-theoretic random variables in infinite measure preserving situations: For $\alpha \in (0, 1)$ we let $\mathcal{Z}_\alpha$ denote a random variable (with values in $[0, 1]$) distributed according to the $B(\alpha, 1-\alpha)$*-distribution* (sometimes called the *generalized arcsine distribution*), i.e.

$$\Pr(\{\mathcal{Z}_\alpha \leq t\}) = \frac{\sin \pi\alpha}{\pi} \int_0^t \frac{dx}{x^{1-\alpha}(1-x)^\alpha}, \quad t \in [0, 1].$$

Continuously extending this family to $\alpha \in [0, 1]$ we let $\mathcal{Z}_0 := 0$ and $\mathcal{Z}_1 := 1$. We are going to prove the following version of the Dynkin-Lamperti theorem for the reference set Y. (For more specific maps, like the one in Example 1.1, it is easy to extend the result to a large family of sets, see Proposition 7.1 and Remark 8.1 below.)

Theorem 3.3 (Arcsine law for waiting times). *Let $(X, \mathcal{A}, \mu)$, T, and Y be as in Theorem 3.1, satisfying (3.1) and (3.2). Then*

$$\frac{1}{n}\mathbf{Z}_n(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha. \tag{3.10}$$

Remark 3.5 (Alternative formulations). Statement (3.10) is equivalent to assertions about other renewal theoretic variables (cf. [Dy], [T4]): Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, and for $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, define $\mathbf{Y}_n(Y)(x) := \min\{k > n : T^k x \in Y\} = \varphi(T^n x) + n$, $x \in X$, $n \in \mathbb{N}_0$, so that $\mathbf{Y}_n(Y) - n$ is the *residual waiting time*. Due to $\{\mathbf{Z}_n(Y) \leq k\} = \{\mathbf{Y}_k(Y) > n\}$, (3.10) holds iff

$$\frac{1}{n}\mathbf{Y}_n(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha^{-1}, \tag{3.11}$$

or, equivalently, $(\varphi \circ T^n)/n \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha^{-1} - 1$. Moreover, letting $\mathbf{V}_n(Y) := \mathbf{Y}_n(Y) - \mathbf{Z}_n(Y)$ denote the *total waiting time*, (3.10) and (3.11) imply

$$\frac{1}{n}\mathbf{V}_n(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{V}_\alpha, \tag{3.12}$$

[2] Since $\mathbf{Z}_n(Y) = \sum_{i=0}^{S_n - 1} \varphi \circ T_Y^i$ with $S_n := \sum_{j=1}^{n} 1_Y \circ T^j$.

where $\mathcal{V}_0 := \infty$, $\mathcal{V}_1 := 0$, and $\mathcal{V}_\alpha$, $\alpha \in (0,1)$, has distribution given by

$$\Pr(\{\mathcal{V}_\alpha \le t\}) = \frac{\sin \pi\alpha}{\pi} \int_0^t \frac{1-(\max(1-x,0))^\alpha}{x^{1+\alpha}}\,dx, \quad t \ge 0.$$

(In the situation of [T4] the converse implication holds as well.)

For a c.e.m.p.t. T on $(X, \mathcal{A}, \mu)$ the asymptotics of the wandering rate $(w_N(Y))$ in general depends on the set Y, and there are no sets with maximal rate, provided μ is non-atomic (cf. Proposition 3.8.2 of [A0]). Still, there may be sets $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, with *minimal wandering rate*, meaning that $\underline{\lim}_{N\to\infty} w_N(Z)/w_N(Y) \ge 1$ for all $Z \in \mathcal{A}$, $0 < \mu(Z) < \infty$. If such sets Y exist, $w_N(T) := w_N(Y)$, $N \ge 1$, defines the *wandering rate of* T (up to asymptotic equivalence), whose asymptotic proportionality class is an isomorphism invariant (cf. [T2]). The following result shows that $(w_N(Y))$ may be replaced by $(w_N(T))$ in the assumptions of Theorems 3.1, 3.2, and 3.3 (compare Theorem 3 in [T2] and Theorem 4.1 in [ADU]).

Proposition 3.2 (Minimal wandering rates). *Let T be a c.e.m.p.t. on the σ-finite measure space $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$. If $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, satisfies (3.1), then Y has minimal wandering rate.*

Proof. Take any $Z \in \mathcal{A}$, $0 < \mu(Z) < \infty$, and let $Y^N := \bigcup_{n=0}^{N-1} T^{-n}Y = \bigcup_{n=0}^{N-1} Y_n$, $Z^N := \bigcup_{n=0}^{N-1} T^{-n}Z$, $N \ge 1$. Then,

$$w_N(Y) = \mu(Y^N) \le \mu(Z^N) + \mu(Y^N \setminus Z^N) = w_N(Z) + \mu(Y^N \setminus Z^N).$$

Taking into account that $Z^N \supseteq T^{-n}Z$, $0 \le n < N$, we get

$$\begin{aligned}\mu(Y^N \setminus Z^N) &= \sum_{n=0}^{N-1} \mu(Y_n \setminus Z^N) \\ &\le \sum_{n=0}^{N-1} \mu(Y_n \cap T^{-n}(Y \setminus Z)) = \int_{Y\setminus Z} \left(\sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right) d\mu.\end{aligned}$$

Therefore, for all $N \ge 1$,

$$\frac{w_N(Z)}{w_N(Y)} \ge 1 - \int_{Y\setminus Z} g_N\,d\mu \ge 1 - \sup_{l\ge 1} \int_{Y\setminus Z} g_l\,d\mu,$$

where $g_N := w_N(Y)^{-1} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n}$, $N \ge 1$. Applying this estimate to Z^L for fixed $L \ge 1$ and using $w_N(Z^L) \le w_N(Z) + L\,\mu(Z)$, we obtain

$$\frac{w_N(Z)}{w_N(Y)} \ge 1 - \sup_{l\ge 1} \int_{Y\setminus Z^L} g_l\,d\mu - \frac{L\,\mu(Z)}{w_N(Y)} \quad \text{for } N \ge 1,$$

and thus

$$\underline{\lim}_{N\to\infty} \frac{w_N(Z)}{w_N(Y)} \ge 1 - \sup_{l\ge 1} \int_{Y\setminus Z^L} g_l\,d\mu.$$

By (3.1), however,

$$\lim_{L\to\infty}\sup_{l\geq 1}\int_{Y\setminus Z^L} g_l\,d\mu = 0,$$

and our result follows. □

Remark 3.6. This argument shows in fact that uniform integrability of the sequence $(w_N(Y)^{-1}\sum_{n=1}^{N-1}\widehat{T}^n 1_{Y_n})_{N\geq 1}$ is sufficient for Y to have minimal wandering rate.

We finally emphasize the difference to earlier work on Darling-Kac and Dynkin-Lamperti-type results for m.p.t.s: The original proof (cf. [A0], [A1]) of the dynamical Darling-Kac theorem applies to c.e.m.p.t.s T on $(X, \mathcal{A}, \mu)$, which are *pointwise dual ergodic (p.d.e.)*, meaning that there exists a sequence $(a_n) = (a_n(T))$ in $\mathbb{R}_+$ (the *return sequence* of T) such that

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\widehat{T}^k u \longrightarrow \mu(u) \quad \text{a.e. on } X \text{ for each } u \in L_1(\mu). \tag{3.13}$$

The same is true for the Dynkin-Lamperti theorem (cf. [T4]) which in addition requires the sets under consideration to be *uniform sets*, i.e. the convergence in (3.13) has to be uniform on Y for some $u \in L_1^+(\mu)$. These are *conditions about the full asymptotics of the transfer operator*. Checking them for specific systems like interval maps with indifferent fixed points is a nontrivial matter, cf. [A2], [T3], and [Z2]. (We shall revisit the proof of the Darling-Kac theorem for p.d.e. transformations in Section 9.)

In [T6] a different approach, based on property (2.6), has been used to derive a version of the arcsine law for occupation times for certain infinite measure preserving interval maps (including Boole's transformation). Here, we will develop this method more systematically, showing that the weaker condition (2.5) is a suitable starting point for all three limit theorems. On the other hand, we demonstrate that for a large class of infinite measure preserving interval maps these *conditions which only concern the dynamics up to the first entrance to Y* can be verified with little effort.

4. Outline of the approach and analytic tools

We give a brief sketch of our method and provide a few auxiliary results which will be used in the sequel. To begin with, we recall an important fact concerning strong distributional convergence: Given distributional convergence w.r.t. some probability measure $\nu \ll m$, strong distributional convergence is automatic if the random variables are *asymptotically invariant in measure*[3] under an ergodic nonsingular transformation T on $(X, \mathcal{A}, m)$.

[3] On a σ-finite measure space $(X, \mathcal{A}, m)$ convergence in measure w.r.t. m, $V_n \stackrel{m}{\longrightarrow} V$, means convergence in measure, $V_n \stackrel{\nu}{\longrightarrow} V$, for all probability measures $\nu \ll m$.

Proposition 4.1 (Strong distributional convergence). *Let T be a nonsingular ergodic transformation on the σ-finite measure space $(X, \mathcal{A}, m)$. Assume that $R_n : X \to \mathbb{R}$, $n \geq 1$, are measurable functions satisfying*

$$R_n \circ T - R_n \xrightarrow{m} 0 \quad or \quad \frac{R_n \circ T}{R_n} \xrightarrow{m} 1. \tag{4.1}$$

If $R_n \overset{\nu}{\Longrightarrow} R$ for some probability measure $\nu \ll m$ and some random variable R taking values in $\overline{\mathbb{R}}$, then $R_n \overset{\mathcal{L}(m)}{\Longrightarrow} R$.

(See [Ea] for the probability preserving case, and [A1] or Section 3.6 of [A0] for the case of nonsingular T and ergodic sums $R_n = a_n^{-1}\mathbf{S}_n(f)$. As pointed out in [T4], the argument given in the latter reference actually applies to the more general situation considered here.) This remarkable observation shows that many distributional limit theorems for dynamical systems, which are usually formulated in terms of the invariant measure, extend at once to arbitrary absolutely continuous initial distributions ν. Moreover, we shall see that it also is a strong tool for establishing distributional limit theorems in the first place.

The random variables occurring in our results all satisfy the asymptotic invariance condition $R_n \circ T - R_n \xrightarrow{\mu} 0$. Therefore it is enough to prove distributional convergence w.r.t. one particular initial distribution $\nu \ll \mu$, which we will choose to be concentrated on the reference set Y. Since in each case the distribution of the limiting variable R is determined by its moments, $R_n \overset{\nu}{\Longrightarrow} R$ follows as soon as the moments of the R_n converge to the moments of R, i.e. $\int_X R_n^r \, d\nu \to \mathbb{E}[R^r]$ as $n \to \infty$ for all $r \geq 1$. All variables R_n we are going to consider here are non-negative.

To establish convergence of moments, we are essentially going to use the following scheme: We dissect trajectories of points in the reference set Y at their first return to Y, thus obtaining a recursion formula which, on each $Y \cap \{\varphi = k\}$, expresses R_n in terms of $R_{n-k} \circ T^k$, and automatically gives corresponding formulae for the R_n^r. These *dissection identities* being convolution-like, we pass to Laplace transforms, turning them into product form. The implicit recursive relations for the Laplace transforms of the moments involve the $\widehat{T}^k 1_{Y \cap \{\varphi = k\}}$ and $\widehat{T}^n 1_{Y_n}$. Our condition (2.5) together with regular variation now enables us to derive explicit asymptotic recursions for the transforms. Technically, this step is taken care of by Lemmas 4.2 and 4.3 below.

We will, however, encounter a problem with the asymptotic recursions thus obtained: They involve a change of measure, and express the moments of the R_n w.r.t. one probability measure in terms of its lower-order moments w.r.t. a different measure. This will be resolved by means of an important consequence of Proposition 4.1, the equivalent moments principle, Lemma 4.4 below. Employing this we end up with a proper asymptotic recursion formula for the transforms of the moments w.r.t. one particular measure. Completing the proofs then is a matter of asymptotic analysis.

We supply a number of important analytic tools. Throughout we use the convention that for $a_n, b_n \geq 0$ and $\vartheta \in [0, \infty)$,

$$a_n \sim \vartheta \cdot b_n \quad \text{as } n \to \infty \quad \text{means} \quad b_n > 0 \text{ for } n \geq n_0 \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = \vartheta,$$

even if $\vartheta = 0$ (and analogously for functions and $f(s) \sim \vartheta \cdot g(s)$ as $s \searrow 0$). We shall heavily depend on Karamata's Tauberian theorem for discrete Laplace transforms and the Monotone Density theorem for regularly varying functions, cf. Corollary 1.7.3 of [BGT]. We will need the following version:

Proposition 4.2 (Karamata's Tauberian Theorem, KTT). *Let (b_n) be a sequence in $[0, \infty)$ such that for all $s > 0$, $B(s) := \sum_{n\geq 0} b_n \, e^{-ns} < \infty$. Let $\ell \in \mathcal{R}_0$ and $\rho, \vartheta \in [0, \infty)$. Then*

$$B(s) \sim \vartheta \left(\frac{1}{s}\right)^{\rho} \ell\left(\frac{1}{s}\right) \quad \textit{as } s \searrow 0, \tag{4.2}$$

iff

$$\sum_{k=0}^{n-1} b_k \sim \frac{\vartheta}{\Gamma(\rho+1)} \, n^{\rho} \ell(n) \quad \textit{as } n \to \infty. \tag{4.3}$$

If (b_n) is eventually monotone and $\rho > 0$, then both are equivalent to

$$b_n \sim \frac{\vartheta\rho}{\Gamma(\rho+1)} \, n^{\rho-1} \ell(n) \quad \textit{as } n \to \infty. \tag{4.4}$$

Remark 4.1. In Corollary 1.7.3 of [BGT], the last equivalence is stated under the additional assumption $\vartheta > 0$. This is, however, an unnecessary restriction. The way we have written the constant in (4.4), the implication (4.3)⇒(4.4) remains true even for $\rho = 0$ (but to conclude that $(b_n) \in \mathcal{R}_{\rho-1}$ one clearly needs $\vartheta\rho > 0$). The implication (4.4)⇒(4.3) requires $\rho > 0$, but does not depend on the monotonicity condition.

We will also exploit the Monotone Density theorem in the form of the following differentiation rules. To formulate them, define

$$c_{\rho,r} := \rho(\rho+1)\dots(\rho+r-1) = (-1)^r r! \binom{-\rho}{r}$$

for $\rho \in \mathbb{R}$ and $r \in \mathbb{N}_0$, and let $c_{\rho,-1} := 0$. Notice that

$$c_{\rho,r} - r \, c_{\rho,r-1} = c_{\rho-1,r} \quad \text{for all } r \in \mathbb{N}_0. \tag{4.5}$$

Lemma 4.1 (Differentiation lemma). ***a)*** *Let $f : (0, \eta) \to (0, \infty)$ be continuously differentiable, $g \in \mathcal{R}_0(0)$, and let $\rho \in \mathbb{R}$, $\vartheta \in [0, \infty)$. If f' is monotone, then*

$$f(s) \sim \vartheta \cdot s^{\rho} g(s) \quad \textit{as } s \searrow 0$$

implies

$$f'(s) \sim \vartheta\rho \cdot s^{\rho-1} g(s) \quad \text{as } s \searrow 0.$$

b) *Consequently, if* $b_n \geq 0$, $n \geq 0$, *are such that* $B(s) := \sum_{n\geq 0} b_n e^{-ns} < \infty$ *for* $s > 0$, *and if*

$$B(s) \sim \vartheta \cdot G(s) \quad \text{as } s \searrow 0$$

with $G \in \mathcal{R}_{-\rho}(0)$, *and* $\rho, \vartheta \in [0, \infty)$, *then, for* $r \in \mathbb{N}_0$,

$$(-1)^r B^{(r)}(s) = \sum_{n\geq 0} n^r b_n \, e^{-ns} \sim \vartheta \cdot c_{\rho,r} \left(\frac{1}{s}\right)^r G(s) \quad \text{as } s \searrow 0. \tag{4.6}$$

(Unless explicitely stated otherwise, We agree that $0^0 := 1$ in coefficients of power series.) Next, we provide the two lemmas mentioned above.

Lemma 4.2 (Integrating transforms I). *Let* T *be a nonsingular transformation on the* σ*-finite measure space* $(X, \mathcal{A}, m)$, $Y \in \mathcal{A}$ *with* $0 < m(Y) < \infty$, *and* H *a nonnegative measurable function, supported on and uniformly sweeping in* $K \in \mathbb{N}_0$ *steps for* Y. *Suppose that* $R_n : X \to [0, \infty)$, $n \geq 0$, *are measurable satisfying*

$$0 < \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} < \infty \quad \text{for all } s > 0,$$

and that for all $k \in \{0, \ldots, K\}$,

$$\int_Y R_n \circ T^k \cdot H \, dm = O\left(\int_Y R_{n+k} \cdot H \, dm \right) \quad \text{as } n \to \infty.$$

Let $v_n : Y \to [0, \infty)$, $n \geq 0$, *be bounded measurable functions such that for all* $s > 0$ *we have* $0 < \sum_{n\geq 0} (\int_Y v_n \, dm) \, e^{-ns} < \infty$. *If*

$$\frac{\sum_{k=0}^n v_k}{\sum_{k=0}^n \int_Y v_n \, dm} \longrightarrow H \qquad \text{uniformly on } Y \text{ as } n \to \infty, \tag{4.7}$$

then

$$\int_Y \left(\sum_{n\geq 0} v_n \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} R_n \, e^{-ns} \right) dm$$
$$\sim \sum_{n\geq 0} \left(\int_Y v_n \, dm \right) e^{-ns} \cdot \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \quad \text{as } s \searrow 0.$$

Condition (4.7) obviously holds if $\int_Y v_n \, dm$ is eventually positive and

$$\sum_{n\geq 0} \int_Y v_n \, dm = \infty \quad \text{and} \quad \frac{v_n}{\int_Y v_n \, dm} \longrightarrow H \qquad \text{uniformly on } Y. \tag{4.8}$$

Proof. We have to show that

$$\int_Y H_s \cdot \sum_{n\geq 0} R_n \, e^{-ns} \, dm \sim \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \quad \text{as } s \searrow 0,$$

where

$$H_s := \frac{\sum_{n\geq 0} v_n \, e^{-ns}}{\sum_{n\geq 0} (\int_Y v_n \, dm) \, e^{-ns}} = \frac{\sum_{n\geq 0} \left(\sum_{k=0}^n v_k \right) e^{-ns}}{\sum_{n\geq 0} \left(\sum_{k=0}^n \int_Y v_k \, dm \right) e^{-ns}}.$$

Recalling that the functions v_n (and hence also H) are bounded, it is straightforward to verify that

$$H_s \longrightarrow H \qquad \text{uniformly on } Y \text{ as } s \searrow 0.$$

Therefore, given $\varepsilon > 0$ there is some $s_\varepsilon > 0$ such that for $s \in (0, s_\varepsilon)$,

$$\left| \int_Y H_s \cdot \sum_{n\geq 0} R_n \, e^{-ns} \, dm - \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \right| \leq \varepsilon \sum_{n\geq 0} \left(\int_Y R_n \, dm \right) e^{-ns},$$

and the proof will be complete if we show that

$$\sum_{n\geq 0} \left(\int_Y R_n \, dm \right) e^{-ns} = O\left(\sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \right) \quad \text{as } s \searrow 0.$$

Since H is uniformly sweeping in K steps for Y, we have $C \sum_{k=0}^K \widehat{T}^k H \geq 1$ a.e. on Y for some $C > 0$. Therefore, for all $n \geq 0$,

$$\int_Y R_n \, dm \leq C \sum_{k=0}^{K} \int_Y (R_n \circ T^k) \cdot H \, dm \leq \widetilde{C} \sum_{k=0}^{K} \int_Y R_{n+k} \cdot H \, dm,$$

which implies

$$\sum_{n\geq 0} \left(\int_Y R_n \, dm \right) e^{-ns} \leq \left(\widetilde{C} \sum_{k=0}^{K} e^{ks} \right) \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \quad \text{for } s > 0.$$

□

Besides this elementary observation, we will also make use of a more sophisticated version which covers derivatives and also provides us with a monotone density result. This result also turns out to be useful in other situations, see Section 9. We state it as a separate lemma since it is worth pointing out that the easy Lemma 4.2 suffices if we content ourselves with the stronger assumption (2.6) instead of (2.5) in the arcsine theorems.

Lemma 4.3 (Integrating transforms II). *Let $(X, \mathcal{A}, m)$, T, Y, H, and (R_n) be as in Lemma 4.2, and let $v_n : Y \to [0, \infty)$, $n \geq 0$, be bounded measurable functions with $\int_Y \sum_{n\geq 0} v_n\, dm > 0$, and $b_n \geq 0$, $n \geq 0$, be constants such that $B(s) := \sum_{n\geq 0} b_n\, e^{-ns} \in \mathcal{R}_{-\rho}(0)$ for some $\rho \in [0, \infty)$.*

a) *Assume that*

$$\frac{\sum_{k=0}^{n} v_k}{\sum_{k=0}^{n} \int_Y v_k\, dm} \longrightarrow H \quad \begin{array}{l}\text{uniformly on } Y\\ \text{as } n \to \infty,\end{array} \tag{4.9}$$

and that for some $\vartheta \in [0, \infty)$,

$$\sum_{k=0}^{n} \int_Y v_k\, dm \sim \vartheta \cdot \sum_{k=0}^{n} b_k \quad \text{as } n \to \infty. \tag{4.10}$$

Then, for all $r \in \mathbb{N}_0$,

$$\int_Y \left(\sum_{n\geq 0} n^r v_n\, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} R_n\, e^{-ns} \right) dm \tag{4.11}$$
$$\sim \vartheta \cdot (-1)^r r! \binom{-\rho}{r} \left(\frac{1}{s}\right)^r B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm \right) e^{-ns} \quad \text{as } s \searrow 0.$$

b) *If, moreover, $v_n \searrow 0$ a.e. on Y as $n \to \infty$, so that $v_n = \sum_{k>n} u_k$ with $u_n \geq 0$, $n \geq 1$, measurable, then, for all $r \geq 1$,*

$$\int_Y \left(\sum_{n\geq 1} n^r u_n\, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} R_n\, e^{-ns} \right) dm \tag{4.12}$$
$$\sim \vartheta \cdot (-1)^{r-1} r! \binom{1-\rho}{r} \left(\frac{1}{s}\right)^{r-1} B(s)$$
$$\cdot \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm \right) e^{-ns} \quad \text{as } s \searrow 0.$$

Proof. **a)** Suppose first that $r = 0$. By Lemma 4.2 and (4.10) we find

$$\int_Y \left(\sum_{n\geq 0} v_n\, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} R_n\, e^{-ns} \right) dm$$
$$\sim \left(\sum_{n\geq 0} \left(\int_Y v_n\, dm \right) e^{-ns} \right) \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm \right) e^{-ns}$$
$$\sim \vartheta \cdot B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm \right) e^{-ns} \quad \text{as } s \searrow 0.$$

If $r \geq 1$ we let $V_n := \sum_{k=0}^{n} v_k$, $B_n := \sum_{k=0}^{n} b_k$, $n \geq 0$. On Y we have, for $s > 0$,

$$\sum_{n\geq 1} n^r v_n \, e^{-ns} = (1-e^{-s}) \sum_{n\geq 0} (n+1)^r V_n \, e^{-ns} - \sum_{n\geq 0} \left((n+1)^r - n^r\right) V_n \, e^{-ns}.$$

Since

$$\frac{V_n}{\int_Y V_n \, dm} \longrightarrow H \qquad \text{uniformly on } Y \text{ as } n \to \infty \tag{4.13}$$

(and $(n+1)^r \sim n^r$), Lemma 4.2 implies

$$\int_Y \left(\sum_{n\geq 0} (n+1)^r V_n \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} R_n \, e^{-ns} \right) dm$$
$$\sim \left(\sum_{n\geq 0} n^r \left(\int_Y V_n \, dm \right) e^{-ns} \right) \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \quad \text{as } s \searrow 0.$$

Now

$$\sum_{n\geq 0} B_n e^{-ns} = \frac{1}{1-e^{-s}} B(s) \sim \frac{1}{s} B(s) \in \mathcal{R}_{-(\rho+1)}(0)$$

as $s \searrow 0$, so that by (4.10) and part b) of Lemma 4.1,

$$\sum_{n\geq 0} n^r \left(\int_Y V_n \, dm \right) e^{-ns} \sim \vartheta \cdot \sum_{n\geq 0} n^r B_n \, e^{-ns} \sim \vartheta \cdot c_{\rho+1,r} \left(\frac{1}{s} \right)^{r+1} B(s)$$

as $s \searrow 0$. Consequently,

$$(1-e^{-s}) \int_Y \left(\sum_{n\geq 0} (n+1)^r V_n \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} R_n \, e^{-ns} \right) dm$$
$$\sim \vartheta \cdot c_{\rho+1,r} \left(\frac{1}{s} \right)^{r} B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \quad \text{as } s \searrow 0.$$

Due to (4.13) and $((n+1)^r - n^r) \sim r \, n^{r-1}$ as $n \to \infty$, we can conclude analogously that

$$\int_Y \left(\sum_{n\geq 0} \left((n+1)^r - n^r\right) V_n \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} R_n \, e^{-ns} \right) dm$$
$$\sim \vartheta \cdot r \, c_{\rho+1,r-1} \left(\frac{1}{s} \right)^{r} B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H \, dm \right) e^{-ns} \quad \text{as } s \searrow 0.$$

Therefore

$$\frac{\int_Y \left(\sum_{n\geq 1} n^r v_n\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} R_n\, e^{-ns}\right)\, dm}{\left(\frac{1}{s}\right)^r B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm\right) e^{-ns}} \longrightarrow \vartheta \cdot (c_{\rho+1,r} - r\, c_{\rho+1,r-1})$$
$$= \vartheta \cdot c_{\rho,r}$$

as $s \searrow 0$, which completes the proof of (4.11).

b) We need to sharpen (4.11) to get (4.12) for $r \geq 1$. To do so, we use the identity

$$\sum_{n\geq 1} n^r u_n\, e^{-ns} = e^{-s} \sum_{j=0}^{r-1} \binom{r}{j} \sum_{n\geq 0} n^j v_n\, e^{-ns} - (1-e^{-s}) \sum_{n\geq 0} n^r v_n\, e^{-ns} \quad \text{on } Y,$$

which is straightforward from $u_n = v_{n-1} - v_n$, $n \geq 1$. According to (4.11) we have, as $s \searrow 0$,

$$e^{-s} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 0} n^j v_n\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} R_n\, e^{-ns}\right) dm$$
$$\sim \vartheta \cdot \sum_{j=0}^{r-1} \binom{r}{j} c_{\rho,j} \left(\frac{1}{s}\right)^j B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm\right) e^{-ns}$$
$$\sim \vartheta \cdot r\, c_{\rho,r-1} \left(\frac{1}{s}\right)^{r-1} B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm\right) e^{-ns}$$

and

$$(1 - e^{-s}) \int_Y \left(\sum_{n\geq 0} n^r v_n\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} R_n\, e^{-ns}\right) dm$$
$$\sim \vartheta \cdot s\, c_{\rho,r} \left(\frac{1}{s}\right)^r B(s) \sum_{n\geq 0} \left(\int_Y R_n \cdot H\, dm\right) e^{-ns}.$$

Combinings these observations with $r\, c_{\rho,r-1} - c_{\rho,r} = -c_{\rho-1,r} = (-1)^{r-1} r! \binom{1-\rho}{r}$, our claim (4.12) follows.

□

To conclude this preparatory section we prove the crucial equivalent moments principle (cf. Lemma 4 of [T6]). As it is of some independent interest we give a version which is somewhat more general than what we actually need below.

Lemma 4.4 (Equivalent moments principle). *Let T be a nonsingular ergodic transformation on the σ-finite measure space $(X, \mathcal{A}, m)$, and let $R_n : X \to [0, \infty)$, $n \geq 1$, be measurable, satisfying (4.1). Suppose that $\nu, \nu^* \ll m$ are probability measures on $(X, \mathcal{A})$ such that for all $r \in \mathbb{N}_0$ the sequences $(\int_X R_n^r\, d\nu)_{n\geq 1}$ and $(\int_X R_n^r\, d\nu^*)_{n\geq 1}$ are bounded, and assume that $\underline{\lim}_{n\to\infty} \int_X R_n\, d\nu > 0$. Then*

$$\lim_{n\to\infty} \frac{\int_X R_n^r\, d\nu^*}{\int_X R_n^r\, d\nu} = 1 \qquad \text{for all } r \in \mathbb{N}_0.$$

Proof. Take some $p \in \mathbb{N}$ and let (n_k) be a subsequence of indices such that

$$\rho := \lim_{k\to\infty} \frac{\int_X R_{n_k}^p \, d\nu^*}{\int_X R_{n_k}^p \, d\nu} \in [0,\infty]$$

exists. We show that necessarily $\rho = 1$. By Helly's compactness theorem and Proposition 4.1 there is some subsequence (m_l) of (n_k) and some random variable R taking values in $[0,\infty]$ such that $R_{m_l} \overset{\mathcal{L}(m)}{\Longrightarrow} R$. Since $\sup_{n\geq 1} \int_X R_n^r \, d\nu < \infty$ for each $r \in \mathbb{N}_0$, we conclude that $\mathbb{E}[R^r] < \infty$ and $\lim_{l\to\infty} \int_X R_{m_l}^r \, d\nu = \mathbb{E}[R^r]$ for all $r \geq 0$. The same is true for ν^*. As $\underline{\lim}_{n\to\infty} \int_X R_n \, d\nu > 0$, we know that $\mathbb{E}[R] \in (0,\infty)$, and hence $\mathbb{E}[R^r] \in (0,\infty)$ for all $r \in \mathbb{N}_0$ (and in particular for $r = p$). Hence $\rho = 1$. □

5. The Darling-Kac theorem

Suppose that the assumptions of Theorem 3.1 are satisfied. As a consequence of Proposition 4.1 for $R_n := a_n^{-1}\mathbf{S}_n(f)$ and Hopf's ratio ergodic theorem (see [KK] or [Z5] for short proofs), the conclusion of our theorem follows as soon as there is any $f \in L_1^+(\mu)$ and any $\nu \ll \mu$ for which $a_n^{-1}\mathbf{S}_n(f) \overset{\nu}{\Longrightarrow} \mu(f)\,\mathcal{M}_\alpha$, and we will choose $f = 1_Y$ and $\nu := \mu_Y$, thus considering the occupation times[4] $S_n := \sum_{j=1}^n 1_Y \circ T^j$, $n \geq 0$. As the Mittag-Leffler laws are determined by their moments, the theorem can be proved by showing that

$$\int_Y \left(\frac{S_n}{a_n}\right)^r d\mu_Y \longrightarrow \mu(Y)^r\, \mathbb{E}[\mathcal{M}_\alpha^r] = \mu(Y)^r\, r!\, \frac{(\Gamma(1+\alpha))^r}{\Gamma(1+r\alpha)}\,, \qquad r \in \mathbb{N}_0. \quad (5.1)$$

We proceed along the lines sketched above. The dissection identity is

$$S_n = \begin{cases} 1 + S_{n-k} \circ T^k & \text{on } Y \cap \{\varphi = k\},\ 1 \leq k \leq n, \\ 0 & \text{on } Y \cap \{\varphi > n\}, \end{cases} \qquad \text{for } n \geq 0, \qquad (5.2)$$

which leads to

Lemma 5.1 (Splitting moments at the first return). *Let T be a c.e.m.p.t. of $(X,\mathcal{A},\mu)$, consider $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, and define $S_n := \sum_{j=1}^n 1_Y \circ T^j$, $n \geq 0$. For all $r \geq 1$ and $s > 0$ we then have*

$$\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} S_n^r\, e^{-ns}\right) d\mu$$
$$= \frac{1}{1-e^{-s}} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} S_n^j\, e^{-ns}\right) d\mu.$$

[4] Working with S_n rather than $\mathbf{S}_n(1_Y)$ leads to slightly nicer formulae.

Proof. According to the dissection identity (5.2) and the fact that $\widehat{T}^k 1_{Y\cap\{\varphi=k\}} = 0$ a.e. on Y^c, we obtain for $n \geq 0$ and $r \geq 1$,

$$\begin{aligned}
\int_Y S_n^r \, d\mu &= \sum_{k=1}^{n} \int_{Y\cap\{\varphi=k\}} (1+S_{n-k})^r \circ T^k \, d\mu \\
&= \sum_{k=1}^{n} \int_Y \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \cdot (1+S_{n-k})^r \; d\mu \\
&= \sum_{k=1}^{n} \int_Y \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \cdot \left(\sum_{j=0}^{r} \binom{r}{j} S_{n-k}^j \right) d\mu \\
&= \sum_{j=0}^{r} \binom{r}{j} \int_Y \sum_{k=1}^{n} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \cdot S_{n-k}^j \, d\mu .
\end{aligned}$$

Passing to generating functions, and separating the $j = r$ term from the rest of the sum, we conclude that for $s > 0$,

$$\begin{aligned}
&\sum_{n\geq 0} \left(\int_Y S_n^r \, d\mu \right) e^{-ns} \\
&\quad = \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} S_n^j \, e^{-ns} \right) d\mu \\
&\qquad + \int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} S_n^r \, e^{-ns} \right) d\mu .
\end{aligned}$$

Observe that the first expression on the right-hand side agrees with the one in the identity we wish to prove. Recalling (2.1) and (2.3), we see that

$$\begin{aligned}
1_Y - \sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} &= \sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} - \sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \\
&= (1-e^{-s}) \sum_{n\geq 0} \left(\sum_{k>n} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \right) e^{-ns} \\
&= (1-e^{-s}) \sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns} \quad \text{a.e.} \qquad (5.3)
\end{aligned}$$

and our assertion follows. □

Condition (3.1) now enables us to convert this *implicit recursion formula* into a simpler *explicit asymptotic recursion formula*. The price we pay is a change of measure.

Lemma 5.2 (Asymptotic recursion). *If, in the situation of the previous lemma,*

$$\frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \to H \qquad \textit{uniformly on } Y \textit{ as } N \to \infty, \textit{ with } H : Y \to [0,\infty) \textit{ uniformly sweeping,}$$

then, for any $r \geq 1$,

$$\sum_{n\geq 0}\left(\int_Y S_n^r \cdot H\, d\mu\right) e^{-ns} \sim \frac{r}{s\, Q_Y(s)} \sum_{n\geq 0}\left(\int_Y S_n^{r-1}\, d\mu_Y\right) e^{-ns},$$

as $s \searrow 0$, where $Q_Y(s) := \sum_{n\geq 0} q_n(Y)\, e^{-ns}$, $s > 0$.

Proof. As a consequence of Lemma 4.2 applied to $R_n := S_n^r$ and $v_n := \widehat{T}^n 1_{Y_n}$, we find for the left-hand side of Lemma 5.1 that

$$\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} S_n^r\, e^{-ns}\right) d\mu \tag{5.4}$$
$$\sim \mu(Y) Q_Y(s) \sum_{n\geq 0}\left(\int_Y S_n^r \cdot H\, d\mu\right) e^{-ns}$$

as $s \searrow 0$. To deal with the right-hand side, we use (5.3) and the identity

$$(1 - e^{-s}) Q_Y(s) = 1 - F_Y(s)$$

for $F_Y(s) := \sum_{k\geq 1} f_k(Y)\, e^{-ks}$, $s > 0$, to see that

$$1_Y - \sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}}\, e^{-ns} = (1 - F_Y(s))\, \frac{\sum_{n\geq 0} \widehat{T}^n 1_{Y_n}\, e^{-ns}}{Q_Y(s)} \quad \text{a.e.} \tag{5.5}$$

As in the proof of Lemma 4.2 we have

$$\frac{\sum_{n\geq 0} \widehat{T}^n 1_{Y_n}\, e^{-ns}}{Q_Y(s)} \longrightarrow \mu(Y) \cdot H \qquad \text{uniformly on } Y \text{ as } s \searrow 0,$$

and since $1 - F_Y(s) \to 0$ as $s \searrow 0$, we conclude from (5.5) that

$$\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}}\, e^{-ns} \longrightarrow 1 \qquad \text{uniformly on } Y \text{ as } s \searrow 0.$$

Hence, for $0 \leq j < r$, we obtain as $s \searrow 0$,

$$\int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} S_n^j\, e^{-ns}\right) d\mu \sim \int_Y \left(\sum_{n\geq 0} S_n^j\, e^{-ns}\right) d\mu. \tag{5.6}$$

We claim that on the right-hand side of Lemma 5.1 the term with $j = r - 1$ dominates the others, thus determining the asymptotics. To see this, notice that for $0 \leq j < r - 1$ we have $\int_Y S_n^j\, d\mu = o(\int_Y S_n^{r-1}\, d\mu)$ as $n \to \infty$ since $S_n \to \infty$ a.e. on X. Combining this with (5.4) and (5.6) our assertion follows. □

The proof of the theorem makes use of the second of the following simple observations (see also 2.10.2 and 2.10.3 of [BGT]).

Lemma 5.3. *Let $(b_n)_{n\geq 0}$ be a non-negative sequence and let $B(s) := \sum_{n\geq 0} b_n\, e^{-ns}$, $s > 0$. Then*

$$\sum_{n=0}^{N-1} b_n = O\left(B\left(\frac{1}{N}\right)\right) \quad \text{as } N \to \infty.$$

If, moreover, (b_n) is increasing, then

$$b_n = O\left(\frac{1}{n} B\left(\frac{1}{n}\right)\right) \quad \text{as } n \to \infty.$$

Proof. For $n \geq 1$,

$$\sum_{n=0}^{N-1} b_n \leq e \sum_{n=0}^{N-1} b_n\, e^{-\frac{n}{N}} \leq e\, B\left(\frac{1}{N}\right).$$

If the sequence is increasing, then

$$n\, b_n \leq \sum_{k=n}^{2n} b_k \leq e^2 \sum_{k=n}^{2n} b_k\, e^{-k/n} \leq e^2 B\left(\frac{1}{n}\right).$$

□

Proof of Theorem 3.1. We are going to convert the formula of Lemma 5.2 into an actual recursion formula by showing that for all $r \geq 1$,

$$\int_Y S_n^r \cdot H\, d\mu \sim \int_Y S_n^r\, d\mu_Y \qquad \text{as } n \to \infty. \tag{5.7}$$

In view of Lemma 5.2 and the fact that trivially $\sum_{n\geq 0} \left(\int_Y S_n^0\, d\mu_Y\right) e^{-ns} \sim s^{-1}$, this immediately implies that for all $r \geq 0$,

$$\sum_{n\geq 0} \left(\int_Y S_n^r\, d\mu_Y\right) e^{-ns} \sim \frac{r!}{s} \left(\frac{1}{s\, Q_Y(s)}\right)^r \qquad \text{as } s \searrow 0. \tag{5.8}$$

To establish (5.7), we apply the equivalent moments principle, Lemma 4.4. We first claim that for all $r \in \mathbb{N}_0$,

$$\int_Y S_n^r \cdot H\, d\mu \asymp \int_Y S_n^r\, d\mu_Y \quad \text{as } n \to \infty, \tag{5.9}$$

i.e. that the ratio is asymptotically bounded away from zero and infinity. Choose $K \in \mathbb{N}_0$ such that $\sum_{k=0}^{K} \widehat{T}^k H$ is bounded away from zero (mod μ) on Y. Since this function is also bounded above, we obviously have

$$\int_Y S_n^r \cdot \left(\sum_{k=0}^{K} \widehat{T}^k H\right) d\mu \asymp \int_Y S_n^r\, d\mu_Y \quad \text{as } n \to \infty \text{ for any } r \in \mathbb{N}_0.$$

On the other hand,

$$\begin{aligned}\int_Y S_n^r \cdot H\, d\mu &\le \int_Y S_n^r \cdot \left(\sum_{k=0}^{K} \widehat{T}^k H\right) d\mu \le \sum_{k=0}^{K} \int_Y S_n^r \circ T^k \cdot H\, d\mu \\ &\le \sum_{k=0}^{K} \int_Y (S_n + k)^r \cdot H\, d\mu \le C_r \int_Y S_n^r \cdot H\, d\mu + K_r\end{aligned}$$

for constants C_r and K_r, $r \in \mathbb{N}_0$, and (5.9) follows.

Using (5.9) and Lemma 5.2 we see by induction that

$$\sum_{n\ge 0} \left(\int_Y S_n^r\, d\mu_Y\right) e^{-ns} = O\left(\frac{1}{s}\left(\frac{1}{s\,Q_Y(s)}\right)^r\right) \quad \text{as } s \searrow 0$$

for each $r \in \mathbb{N}_0$. As a consequence of the second statement of Lemma 5.3, therefore

$$\int_Y S_n^r\, d\mu_Y = O\left(\left(\frac{n}{Q_Y(1/n)}\right)^r\right) \quad \text{as } n \to \infty \text{ for any } r \in \mathbb{N}_0. \tag{5.10}$$

Since $(w_n(Y))$ is regularly varying of index $1-\alpha$, $\alpha \in [0,1]$, we have, for $s > 0$,

$$Q_Y(s) = \left(\frac{1}{s}\right)^{1-\alpha} \ell\left(\frac{1}{s}\right), \quad \text{and} \quad w_n(Y) \sim \frac{\mu(Y) n^{1-\alpha}\ell(n)}{\Gamma(2-\alpha)} \quad \text{as } n \to \infty \tag{5.11}$$

with ℓ slowly varying at infinity. Thus

$$\int_Y S_n^r\, d\mu_Y = O\left(\left(\frac{n^\alpha}{\ell(n)}\right)^r\right) \quad \text{as } n \to \infty \text{ for any } r \in \mathbb{N}_0. \tag{5.12}$$

The $r = 1$ case of Lemma 5.2 together with monotonicity of $(\int_Y S_n \cdot H\, d\mu)_{n\ge 1}$ yields, via KTT,

$$\int_Y S_n \cdot H\, d\mu \sim \frac{1}{\Gamma(1+\alpha)} \frac{n^\alpha}{\ell(n)} \quad \text{as } n \to \infty. \tag{5.13}$$

In view of (5.9), (5.12), and (5.13) the sequence given by $R_n := n^{-\alpha}\ell(n) S_n$, $n \ge 1$, satisfies the conditions of Lemma 4.4 with respect to the probability measures in question and we conclude that (5.7), and hence also (5.8), holds.

Using (5.11) we thus have

$$\sum_{n\ge 0} \left(\int_Y S_n^r\, d\mu_Y\right) e^{-ns} \sim r! \left(\frac{1}{s}\right)^{1+\alpha r} \left(\frac{1}{\ell(\frac{1}{s})}\right)^r \quad \text{as } s \searrow 0$$

for all $r \ge 0$. By KTT and monotonicity of the sequences $(\int_Y S_n^r\, d\mu_Y)_{n\ge 1}$, therefore (5.1) holds, where, due to (5.11), (a_n) is any sequence satisfying

$$a_n \sim \frac{n^\alpha}{\mu(Y)\Gamma(1+\alpha)\,\ell(n)} \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{n}{w_n(Y)} \quad \text{as } n \to \infty.$$

Taking $r = 1$ in (5.1) we see that (a_n) can be chosen as stated in the theorem. □

6. The arcsine law for occupation times

Suppose that the assumptions of Theorem 3.2 are satisfied. If $\mu(A_1) < \infty$, then clearly $\beta = 0$ and the conlusion follows from the ergodic theorem. We therefore assume that $\mu(A_1) = \infty$. Again appealing to the ergodic theorem we see that it is enough to consider $A := A_1$. Due to Proposition 4.1 we need only prove distributional convergence w.r.t. the particular probability measure $\nu \ll \mu$ with density H. By boundedness of the variables under consideration, it suffices to prove convergence of the moments, i.e.

$$\int_Y \left(\frac{S_n}{n}\right)^r \cdot H\,d\mu \longrightarrow \mathbb{E}[\mathcal{L}_{\alpha,\beta}^r] \quad \text{as } n \to \infty \text{ for all } r \geq 1. \tag{6.1}$$

The dissection identity for $S_n := \sum_{j=1}^n 1_A \circ T^j$, $n \geq 0$, reads as follows

$$S_n = \begin{cases} k-1+S_{n-k} \circ T^k & \text{on } Y \cap T^{-1}A \cap \{\varphi = k\},\ 1 \leq k \leq n, \\ S_{n-k} \circ T^k & \text{on } Y \cap T^{-1}A^c \cap \{\varphi = k\},\ 1 \leq k \leq n, \\ n & \text{on } Y \cap T^{-1}A \cap \{\varphi > n\}, \\ 0 & \text{on } Y \cap T^{-1}A^c \cap \{\varphi > n\}, \end{cases} \quad \text{for } n \geq 0, \tag{6.2}$$

which results in

Lemma 6.1 (Splitting moments at the first return). *Let T be a c.e.m.p.t. of $(X, \mathcal{A}, \mu)$, and assume that $X = A \cup Y \cup B$ (measurable and pairwise disjoint) such that $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, dynamically separates A and B. Let $S_n := \sum_{j=1}^n 1_A \circ T^j$, $n \geq 1$, then, for $r \geq 1$ and $s > 0$,*

$$(1 - e^{-s}) \int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n}\, e^{-ns}\right) \left(\sum_{n\geq 0} S_n^r\, e^{-ns}\right) d\mu$$

$$= e^{-s} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j}\, \widehat{T}^{n+1} 1_{Y \cap T^{-1}A \cap \{\varphi = n+1\}}\, e^{-ns}\right) \left(\sum_{n\geq 0} S_n^j\, e^{-ns}\right) d\mu$$

$$+ \sum_{n\geq 1} n^r \mu(Y \cap T^{-1}A \cap \{\varphi > n\})\, e^{-ns}.$$

Proof. Analogous to Lemma 5.1, compare Lemma 1 of [T6]: For $n \geq 0$ and $r \geq 1$,

$$\int_Y S_n^r\, d\mu = \sum_{k=1}^n \int_{Y \cap T^{-1}A \cap \{\varphi = k\}} (k - 1 + S_{n-k})^r \circ T^k\, d\mu$$

$$+ \sum_{k=1}^n \int_{Y \cap T^{-1}A^c \cap \{\varphi = k\}} S_{n-k}^r \circ T^k\, d\mu + n^r \mu(Y \cap T^{-1}A \cap \{\varphi > n\})$$

$$= \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \sum_{k=1}^n (k-1)^{r-j} \cdot \widehat{T}^k 1_{Y \cap T^{-1}A \cap \{\varphi = k\}}\, S_{n-k}^j\, d\mu$$

$$+ \int_Y \sum_{k=1}^n \widehat{T}^k 1_{Y \cap \{\varphi = k\}} \cdot S_{n-k}^r\, d\mu + n^r \mu(Y \cap T^{-1}A \cap \{\varphi > n\}).$$

Therefore, for $s > 0$,

$$\sum_{n\geq 0}\left(\int_Y S_n^r\,d\mu\right)e^{-ns}$$
$$= e^{-s}\sum_{j=0}^{r-1}\binom{r}{j}\int_Y\left(\sum_{n\geq 1} n^{r-j}\,\widehat{T}^{n+1}1_{Y\cap T^{-1}A\cap\{\varphi=n+1\}}\,e^{-ns}\right)\left(\sum_{n\geq 0} S_n^j\,e^{-ns}\right)d\mu$$
$$+\int_Y\left(\sum_{n\geq 1}\widehat{T}^n 1_{Y\cap\{\varphi=n\}}\,e^{-ns}\right)\left(\sum_{n\geq 0} S_n^r\,e^{-ns}\right)\,d\mu$$
$$+\sum_{n\geq 1} n^r\,\mu(Y\cap T^{-1}A\cap\{\varphi>n\})\,e^{-ns}.$$

Using (5.3), the assertion follows. □

Lemma 6.2 (Asymptotic recursion). *Under the assumptions of Theorem 3.2, we have for* $r \geq 1$*, as* $s \searrow 0$*,*

$$\frac{1}{r!}\sum_{n\geq 0}\left(\int_Y S_n^r\cdot H\,d\mu\right)e^{-ns}$$
$$\sim (-1)^r\beta\left[\sum_{j=0}^{r-1}(-1)^{j+1}\binom{\alpha}{r-j}\left(\frac{1}{s}\right)^{r-j}\cdot\frac{1}{j!}\sum_{n\geq 0}\left(\int_Y S_n^j\cdot H_1\,d\mu\right)e^{-ns}\right.$$
$$\left.+\binom{\alpha-1}{r}\left(\frac{1}{s}\right)^{r+1}\right].$$

Proof. As, in particular, all assumptions of Theorem 3.1 are fulfilled, we find for the left-hand side of Lemma 6.1, exactly as in the proof of Lemma 5.2, that

$$(1-e^{-s})\int_Y\left(\sum_{n\geq 0}\widehat{T}^n 1_{Y_n}\,e^{-ns}\right)\cdot\left(\sum_{n\geq 0} S_n^r\,e^{-ns}\right)d\mu \tag{6.3}$$
$$\sim \mu(Y)\,s\,Q_Y(s)\sum_{n\geq 0}\left(\int_Y S_n^r\cdot H\,d\mu\right)e^{-ns},\quad\text{as } s\searrow 0.$$

Turning to the right-hand side of Lemma 6.1, we first consider the rightmost sum. Letting $Q_{Y,A}(s) := \sum_{n\geq 0}\mu_Y(Y\cap T^{-1}A\cap\{\varphi>n\})\,e^{-ns}$, $s>0$, we have

$$Q_{Y,A}(s)\sim\beta\cdot Q_Y(s)\quad\text{as } s\searrow 0,$$

since $w_n(Y,A)\sim\beta\cdot w_n(Y)$ as $n\to\infty$. Due to $(w_n(Y))\in\mathcal{R}_{1-\alpha}$, we have

$$Q_Y(s)=\left(\frac{1}{s}\right)^{1-\alpha}\ell\left(\frac{1}{s}\right),\quad s>0,$$

with ℓ slowly varying at infinity, and thus

$$Q_{Y,A}(s) \sim \beta \cdot \left(\frac{1}{s}\right)^{1-\alpha} \ell\left(\frac{1}{s}\right), \quad \text{as } s \searrow 0.$$

Therefore, according to Lemma 4.1, as $s \searrow 0$,

$$\sum_{n\geq 1} n^r \mu(Y \cap T^{-1}A \cap \{\varphi > n\})\, e^{-ns} \sim (-1)^r \beta\, r! \binom{\alpha - 1}{r} \left(\frac{1}{s}\right)^r \mu(Y)\, Q_Y(s). \tag{6.4}$$

For the other summands on the right-hand side of Lemma 6.1, we fix some $j \in \{0, \dots, r-1\}$. We claim that we can apply part b) of Lemma 4.3 with $R_n := S_n^j$, $u_n := \widehat{T}^{n+1} 1_{Y\cap T^{-1}A\cap\{\varphi=n+1\}}$, $\vartheta := \beta$, and $\rho := 1 - \alpha$, thereby obtaining

$$\begin{aligned}
&\int_Y \left(\sum_{n\geq 1} n^{r-j}\, \widehat{T}^{n+1} 1_{Y\cap T^{-1}A\cap\{\varphi=n+1\}}\, e^{-ns}\right) \left(\sum_{n\geq 0} S_n^j\, e^{-ns}\right) d\mu \\
&\sim (-1)^{r-j-1} \beta\, (r-j)! \binom{\alpha}{r-j} \left(\frac{1}{s}\right)^{r-j-1} \mu(Y)\, Q_Y(s) \\
&\quad \cdot \sum_{n\geq 0} \left(\int_Y S_n^j \cdot H_1\, d\mu\right) e^{-ns}
\end{aligned}$$

as $s \searrow 0$. Combining this with (6.3) and (6.4), our assertion then follows.

It remains to check that the assumptions of Lemma 4.3 are satisfied. We claim that for $n \geq 1$,

$$v_{n-1} = \sum_{k>n} \widehat{T}^k 1_{Y\cap T^{-1}A\cap\{\varphi=k\}} = \widehat{T}^n 1_{A\cap Y_n} \quad \text{a.e. for } n \geq 1. \tag{6.5}$$

To verify this, notice that for $1 \leq l \leq k-1$, $k \geq 2$, we have, due to dynamical separation,

$$Y \cap T^{-1}A \cap \{\varphi = k\} = Y \cap T^{-l}A \cap \{\varphi = k\},$$

and hence $\widehat{T}^l 1_{Y\cap T^{-1}A\cap\{\varphi=k\}} = 1_A \widehat{T}^l 1_{Y\cap\{\varphi=k\}}$ a.e.. Consequently, by (2.3),

$$1_{A\cap Y_n} = \sum_{k>n} 1_A \widehat{T}^{k-n} 1_{Y\cap\{\varphi=k\}} = \sum_{k>n} \widehat{T}^{k-n} 1_{Y\cap T^{-1}A\cap\{\varphi=k\}} \quad \text{a.e. for } n \geq 1, \tag{6.6}$$

as required (hence (3.4)). It is then clear from our assumption (3.5) that

$$\frac{\sum_{k=0}^n v_k}{\sum_{k=0}^n \int_Y v_k\, d\mu} = \frac{\sum_{k=1}^{n+1} \widehat{T}^k 1_{A\cap Y_k}}{\sum_{k=1}^{n+1} \mu(A \cap Y_k)} \longrightarrow H_1 \quad \begin{array}{l}\text{uniformly on } Y\\ \text{as } n \to \infty.\end{array}$$

Moreover, $\sum_{k=0}^n \int_Y v_k\, d\mu \sim \beta \cdot w_n(Y)$ as $n \to \infty$ with $(w_n(Y)) \in \mathcal{R}_{1-\alpha}$, and $B(s) = \mu(Y) Q_Y(s)$. The remaining assumptions of Lemma 4.3 clearly being fulfilled, we are done. □

Proof of Theorem 3.2. We first recall that according to Proposition 1 of [T6] (and by elementary considerations for the boundary cases), if $\alpha, \beta \in [0, 1]$,

$$\mathbb{E}[\mathcal{L}^r_{\alpha,\beta}] = (-1)^r \beta \left[\sum_{j=0}^{r-1} (-1)^{j+1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{L}^j_{\alpha,\beta}] + \binom{\alpha-1}{r} \right], \quad r \geq 1. \tag{6.7}$$

(Conversely, for $\alpha, \beta \in (0, 1)$, the density of $\mathcal{L}_{\alpha,\beta}$ can be reconstructed from this by inverting its Stieltjes transform which can be calculated explicitely from (6.7), compare [L1].) Taking $r = 1$ in the conclusion of the previous lemma, we see that

$$\sum_{n\geq 0} \left(\int_Y S_n \cdot H \, d\mu \right) e^{-ns} \sim \beta \cdot \left(\frac{1}{s} \right)^2 \quad \text{as } s \searrow 0. \tag{6.8}$$

Due to monotonicity of $(\int_Y S_n \cdot H \, d\mu)_{n\geq 1}$ we can conclude (cf. KTT) that

$$\int_Y S_n \cdot H \, d\mu \sim \beta \cdot n \quad \text{as } n \to \infty. \tag{6.9}$$

For $\beta = 0$ this means $n^{-1} \int_Y S_n \cdot H \, d\mu \to 0$, and hence $n^{-1} S_n \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{L}_{\alpha,\beta} = 0$, as required.

Assume now that $\beta > 0$. To obtain a proper recursion formula from Lemma 6.2, we apply Lemma 4.4 to the sequence given by $R_n := n^{-1} S_n$, $n \geq 1$. As (R_n) is uniformly bounded and by (6.9) satisfies $\underline{\lim}_{n\to\infty} \int_Y R_n \cdot H \, d\mu > 0$, we obtain

$$\int_Y S^r_n \cdot H \, d\mu \sim \int_Y S^r_n \cdot H_1 \, d\mu \quad \text{as } n \to \infty \text{ for } r \geq 0. \tag{6.10}$$

Hence the recursion obtained in Lemma 6.2 becomes

$$\begin{aligned} \frac{1}{r!} \sum_{n\geq 0} \left(\int_Y S^r_n \cdot H \, d\mu \right) e^{-ns} &\sim (-1)^r \beta \left[\binom{\alpha-1}{r} \left(\frac{1}{s} \right)^{r+1} \right. \\ &\left. + \sum_{j=0}^{r-1} (-1)^{j+1} \binom{\alpha}{r-j} \left(\frac{1}{s} \right)^{r-j} \cdot \frac{1}{j!} \sum_{n\geq 0} \left(\int_Y S^j_n \cdot H \, d\mu \right) e^{-ns} \right] \end{aligned}$$

for $r \geq 1$ as $s \searrow 0$. (This is also true in the cases $\alpha \in \{0, 1\}$ since not all of the $\binom{\alpha}{r}, \ldots, \binom{\alpha}{1}, \binom{\alpha-1}{r}$ vanish simultaneously.) Starting from the trivial $r = 0$ case, $\sum_{n\geq 0} (\int_Y S^0_n \cdot H \, d\mu) \, e^{-ns} \sim s^{-1}$, induction on r together with (6.7) then shows that

$$\sum_{n\geq 0} \left(\int_Y S^r_n \cdot H \, d\mu \right) e^{-ns} \sim r! \, \mathbb{E}[\mathcal{L}^r_{\alpha,\beta}] \left(\frac{1}{s} \right)^{r+1} \quad \text{as } s \searrow 0. \tag{6.11}$$

(Since we assumed $\beta > 0$, all the $\mathbb{E}[\mathcal{L}^r_{\alpha,\beta}]$ are positive.) KTT and monotonicity of $(\int_Y S^r_n \cdot H \, d\mu)_{n\geq 1}$ now show that (6.11) implies (6.1) as required. □

We conclude this section showing that there are many situations in which Theorem 3.2 applies.

Proof of Proposition 3.1. Suppose that the bounded function $H : Y \to [0, \infty)$ is uniformly sweeping in K steps. Due to Remark 3.4, it is enough to show that for any $\beta \in (0, 1)$ there is some set $E_1 \subseteq Y$ with $\int_{E_1} H\, d\mu = \beta$ for which $1_{E_1} \cdot H$ is uniformly sweeping, and since the latter property is preserved if we enlarge the set, we need only check that $\int_{E_1} H\, d\mu$ can be made arbitrarily small.

Fix $\varepsilon > 0$ and take any $C \in Y \cap \mathcal{A}$ with $0 < \int_C H\, d\mu < \varepsilon/2$. As T is conservative ergodic, we have $\sum_{l\geq 0} \widehat{T}^l(1_C \cdot H) = \infty$ a.e., implying that there are $L \in \mathbb{N}_0$ and $Z \in Y \cap \mathcal{A}$ satisfying $\inf_{Y\setminus Z} \sum_{l=0}^{L} \widehat{T}^l(1_C \cdot H) > 0$ and $\mu(Z) < \varepsilon/(2(K+1)\sup H)$. By assumption, $\sum_{k=0}^{K} \widehat{T}^k(1_{T^{-k}Z} \cdot H) = 1_Z \sum_{k=0}^{K} \widehat{T}^k H$ has positive infimum on Z, and hence the same is true for $\sum_{k=0}^{K} \widehat{T}^k(1_F \cdot H)$, where $F := Y \cap \bigcup_{k=0}^{K} T^{-k}Z$. Since $\mu(F) \leq (K+1)\mu(Z) < \varepsilon/(2\sup H)$, we see that $\int_F H\, d\mu < \varepsilon/2$, and $E_1 := C \cup F$ is a suitable choice. □

7. The arcsine law for waiting times

Suppose that the assumptions of Theorem 3.3 are satisfied. Due to our Proposition 4.1 and Lemma 1 in [T4], it is enough to prove that $n^{-1}\mathbf{Z}_n(Y) \overset{\nu}{\Longrightarrow} \mathcal{Z}_\alpha$ for one probability measure $\nu \ll \mu$. We shall use the measure ν given by the asymptotic entrance density H of Y, and henceforth abbreviate $Z_n := \mathbf{Z}_n(Y)$. Since for any $\alpha \in [0, 1]$, $\mathcal{Z}_\alpha$ is a bounded random variable, its distribution is determined by its moments $\mathbb{E}[\mathcal{Z}_\alpha^r] = (-1)^r \binom{-\alpha}{r}$, $r \in \mathbb{N}_0$, and it suffices to prove

$$\int_Y \left(\frac{Z_n}{n}\right)^r \cdot H\, d\mu \longrightarrow \mathbb{E}[\mathcal{Z}_\alpha^r] \quad \text{as } n \to \infty. \tag{7.1}$$

The dissection identity for Z_n is

$$Z_n = \begin{cases} k + Z_{n-k} \circ T^k & \text{on } Y \cap \{\varphi = k\},\ 1 \leq k \leq n, \\ 0 & \text{on } Y \cap \{\varphi > n\}, \end{cases} \quad \text{for } n \geq 0, \tag{7.2}$$

leading to

Lemma 7.1 (Splitting moments at the first return). *Let T be a c.e.m.p.t. of $(X, \mathcal{A}, \mu)$, consider $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, and define $Z_n := \mathbf{Z}_n(Y)$. For all $r \geq 1$ and $s > 0$ we then have*

$$\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} Z_n^r\, e^{-ns}\right) d\mu$$
$$= \frac{1}{1-e^{-s}} \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j}\, \widehat{T}^n 1_{Y\cap\{\varphi=n\}}\, e^{-ns}\right) \cdot \left(\sum_{n\geq 0} Z_n^j\, e^{-ns}\right) d\mu.$$

Proof. Due to the dissection identity (7.2) and the fact that $\widehat{T}^k 1_{Y\cap\{\varphi=k\}} = 0$ a.e. on Y^c, we get for $n \geq 1$ and $r \geq 1$, by the same calculation as in the proof of Lemma 5.1 ,

$$\int_Y Z_n^r \, d\mu = \sum_{k=1}^{n} \int_{Y\cap\{\varphi=k\}} (k + Z_{n-k})^r \circ T^k \, d\mu$$
$$= \sum_{j=0}^{r} \binom{r}{j} \int_Y \sum_{k=1}^{n} k^{r-j} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \cdot Z_{n-k}^j \, d\mu.$$

Consequently, for $s > 0$,

$$\sum_{n\geq 0} \left(\int_Y Z_n^r \, d\mu \right) e^{-ns}$$
$$= \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n\geq 1} n^{r-j} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} Z_n^j \, e^{-ns} \right) d\mu$$
$$+ \int_Y \left(\sum_{n\geq 1} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} Z_n^r \, e^{-ns} \right) d\mu.$$

Recalling identity (5.3), the assertion follows easily. □

We now exploit our condition (3.1) together with regular variation of the wandering rate to turn this implicit recursion formula into an explicit asymptotic recursion formula (again involving a change of measure).

Lemma 7.2 (Asymptotic recursion). *If, in the situation of the previous lemma,*

$$\frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \to H \qquad \text{uniformly on } Y \text{ as } N \to \infty, \text{ with } H : Y \to [0,\infty) \text{ uniformly sweeping,}$$

and $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$, $\alpha \in [0,1]$, *then, for any* $r \geq 1$, *as* $s \searrow 0$,

$$\frac{1}{r!} \sum_{n\geq 0} \left(\int_Y Z_n^r \cdot H \, d\mu \right) e^{-ns}$$
$$\sim \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{\alpha}{r-j} \left(\frac{1}{s} \right)^{r-j} \cdot \frac{1}{j!} \sum_{n\geq 0} \left(\int_Y Z_n^j \cdot H \, d\mu \right) e^{-ns}.$$

Proof. Observe first that

$$Z_n \circ T^k \leq Z_{n+k} \quad \text{for all } n, k \in \mathbb{N}_0. \tag{7.3}$$

Lemma 4.2, with $R_n := Z_n^r$ and $v_n := \widehat{T}^n 1_{Y_n}$ as in the proof of Lemma 5.2, yields

$$\int_Y \left(\sum_{n\geq 0} \widehat{T}^n 1_{Y_n} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} Z_n^r \, e^{-ns} \right) \, d\mu \tag{7.4}$$
$$\sim \mu(Y) Q_Y(s) \sum_{n\geq 0} \left(\int_Y Z_n^r \cdot H \, d\mu \right) e^{-ns}$$

as $s \searrow 0$. Turning to the right-hand side of the preceding lemma, we fix $j \in \{0, \dots, r-1\}$ and apply part b) of Lemma 4.3 with $u_n := \widehat{T}^n 1_{Y\cap\{\varphi=n\}}$, $n \geq 1$ (so that $v_n = \widehat{T}^n 1_{Y_n}$), $\rho := 1-\alpha$, and $\vartheta := 1$, to see that

$$\int_Y \left(\sum_{n\geq 1} n^{r-j} \widehat{T}^n 1_{Y\cap\{\varphi=n\}} \, e^{-ns} \right) \cdot \left(\sum_{n\geq 0} Z_n^j \, e^{-ns} \right) \, d\mu$$
$$\sim (-1)^{r-j-1} (r-j)! \binom{\alpha}{r-j} \left(\frac{1}{s} \right)^{r-j-1} \mu(Y) \, Q_Y(s) \cdot \sum_{n\geq 0} \left(\int_Y Z_n^j \cdot H \, d\mu \right) e^{-ns}$$

as $s \searrow 0$. Combining these observations with Lemma 7.1, our assertion follows. □

Here the measure changed on both sides, and we can immediately continue to exploit the recursion formula.

Proof of Theorem 3.3. Using the identity $\sum_{j=0}^r \binom{\alpha}{r-j}\binom{-\alpha}{j} = 0$, $r \in \mathbb{N}$, we see that the moments of $\mathcal{Z}_\alpha$ satisfy the recursion formula

$$\mathbb{E}[\mathcal{Z}_\alpha^r] = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{\alpha}{r-j} \mathbb{E}[\mathcal{Z}_\alpha^j] \quad \text{for } r \in \mathbb{N}. \tag{7.5}$$

An induction based on Lemma 7.2 therefore shows that for any $r \in \mathbb{N}_0$,

$$\sum_{n\geq 0} \left(\int_Y Z_n^r \cdot H \, d\mu \right) e^{-ns} \sim r! \left(\frac{1}{s} \right)^{r+1} \mathbb{E}[\mathcal{Z}_\alpha^r] \qquad \text{as } s \searrow 0. \tag{7.6}$$

By KTT and monotonicity of the sequence $(\int_Y Z_n^r \cdot H \, d\mu)_{n\geq 1}$, this asymptotic equation implies (7.1), as required. □

To deal with subsets of our reference set Y, we will use the following observation. To formulate it, we note that T_Y is a nonsingular map from $(X, \mathcal{A}, \mu)$ to $(Y, Y \cap \mathcal{A}, \mu \mid_{Y\cap\mathcal{A}})$, with transfer operator $\widehat{T}_Y : L_1(\mu) \to L_1(\mu \mid_{Y\cap\mathcal{A}})$.

Proposition 7.1 (Dynkin-Lamperti law for subsets). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, and assume that $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$ satisfies*

$$\frac{1}{n} \mathbf{Z}_n(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha$$

for some $\alpha \in [0,1]$. *If there is some probability density u such that the sequence*

$$\left(\widehat{T}_Y\left(\widehat{T}^n u\right)\right)_{n\in\mathbb{N}} \quad \textit{is uniformly integrable,} \tag{7.7}$$

then every $E \in Y \cap \mathcal{A}$ *with* $\mu(E) > 0$ *satisfies*

$$\frac{1}{n}\mathbf{Z}_n(E) \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha.$$

Proof. Fix any $E \in Y \cap \mathcal{A}$ with $\mu(E) > 0$, and let τ denote its first entrance (return) time. Recalling Remark 3.5, our assertion is equivalent to $(\tau \circ T^n)/n \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha^{-1} - 1$. Since it is easy to check that $(\tau \circ T^{n+1} - \tau \circ T^n)/n \stackrel{\mu}{\to} 0$ (notice $\tau \circ T - \tau = (1_Y\, \tau) \circ T - 1$), this is the same as

$$\frac{\tau \circ T^n}{n} \stackrel{\nu}{\Longrightarrow} \mathcal{Z}_\alpha^{-1} - 1,$$

where ν is the measure with density u, cf. Proposition 4.1. Observe that $\tau = \varphi + (1_{Y\setminus E}\, \tau) \circ T_Y$. Since, by assumption, $(\varphi \circ T^n)/n \stackrel{\nu}{\Longrightarrow} \mathcal{Z}_\alpha^{-1} - 1$, it suffices to check that

$$\frac{\tau \circ T^n - \varphi \circ T^n}{n} = \frac{(1_{Y\setminus E}\, \tau) \circ T_Y \circ T^n}{n} \stackrel{\nu}{\longrightarrow} 0 \quad \text{as } n \to \infty,$$

or, equivalently, that for any $c > 0$,

$$\int_{Y \cap \{1_{Y\setminus E}\, \tau > c\,n\}} \widehat{T}_Y\left(\widehat{T}^n u\right) d\mu \longrightarrow 0 \quad \text{as } n \to \infty.$$

Due to (7.7) this is an immediate consequence of $\mu(Y \cap \{1_{Y\setminus E}\, \tau > c\,n\}) \to 0$. □

8. Application to interval maps with indifferent fixed points

An important family of infinite measure preserving dynamical systems is given by piecewise $\mathcal{C}^2$ interval maps with indifferent (neutral) fixed points. We are going to show that the approach developed above applies to them in a very natural way. The following class of transformations has been studied in [Z1], [Z2] generalizing earlier work from [A0], [A2], [ADU], and [T1]-[T3]. Notations and terminology below are the same as in [Z1], [Z2], except that (as above) $\widehat{T}$ denotes the transfer operator w.r.t. the invariant measure μ. Throughout this section λ will denote one-dimensional Lebesgue measure, and $\mathcal{B}$ will be the Borel-σ-field of the space under consideration. If $E \subseteq \mathbb{R}$ is a finite union of intervals, we let $\mathrm{BV}(E)$ denote the space of real-valued functions of bounded variation on E.

A *piecewise monotonic system* is a triple (X, T, ξ), where X is the union of some finite family of disjoint bounded open intervals, ξ is a collection of nonempty pairwise disjoint open subintervals with $\lambda(X \setminus \bigcup \xi) = 0$, and $T : X \to X$ is a map such that $T\mid_Z$ is continuous and strictly monotonic for each $Z \in \xi$. We let ξ_n denote the family of *cylinders of rank n*, that is, the nonempty sets of the form $\bigcap_{i=0}^{n-1} T^{-i} Z_i$ with $Z_i \in \xi$. If $W \subseteq Z \in \xi_n$, we let $f_W := (T^n\mid_W)^{-1}$ be the inverse of the branch $T^n \mid_W$. Our maps will be $\mathcal{C}^2$ on each $Z \in \xi$ and satisfy *Adler's condition*

$$T''/(T')^2 \text{ is bounded on } \bigcup \xi, \tag{8.1}$$

as well as the *finite image condition*

$$T\xi = \{TZ : Z \in \xi\} \text{ is finite.} \tag{8.2}$$

There is a finite set $\zeta \subseteq \xi$ of cylinders Z having an *indifferent fixed point* x_Z as an endpoint (i.e. $\lim_{x \to x_Z, x \in Z} Tx = x_Z$ and $\lim_{x \to x_Z, x \in Z} T'x = 1$), and each x_Z is a *one-sided regular source*, i.e.

$$\text{for } x \in Z,\ Z \in \zeta, \text{ we have } (x - x_Z)T''(x) \geq 0. \tag{8.3}$$

The second endpoint of $Z \in \zeta$ will be denoted by y_Z. Our maps are *uniformly expanding on sets bounded away from* $\{x_Z : Z \in \zeta\}$, in the sense that letting $X_\varepsilon := X \setminus \bigcup_{Z \in \zeta} ((x_Z - \varepsilon, x_Z + \varepsilon) \cap Z)$ we have

$$\left|T'\right| \geq \rho(\varepsilon) > 1 \quad \text{on } X_\varepsilon \quad \text{ for each } \varepsilon > 0. \tag{8.4}$$

Following [Z1], [Z2], we call (X, T, ξ) an *AFN-system* if it satisfies (8.1)-(8.4).

Henceforth we assume that T is conservative ergodic and $\zeta \neq \varnothing$ (a *basic* AFN-system in the sense of [Z2]). (See Theorem 1 in [Z1] for ergodic decompositions.) The system then has an invariant measure $\mu \ll \lambda$ with $\mu(X) = \infty$ whose density $d\mu/d\lambda$ has a version $h(x) = h_0(x)G(x)$, where

$$G(x) := \begin{cases} \frac{x - x_Z}{x - f_Z(x)} & \text{for } x \in Z \in \zeta \\ 1 & \text{for } x \in X \setminus \bigcup \zeta, \end{cases}$$

and $0 < \inf_X h_0 \leq \sup_X h_0 < \infty$, and h_0 has bounded variation on each X_ε, $\varepsilon > 0$. For $Z \in \zeta$ we let $B_Z := f_Z(Z)$, $Z \in \zeta$, and $Z(1) := Z \setminus B_Z$. We are going to show that

$$Y = Y(T) := X \setminus \bigcup_{Z \in \zeta} B_Z = \bigcup_{W \in \xi \setminus \zeta} W \cup \bigcup_{Z \in \zeta} Z(1) \quad (\text{mod } \lambda), \tag{8.5}$$

is a suitable reference set for T. It is clear that Y dynamically separates the (infinite measure) components $B_Z = Y^c \cap Z$, $Z \in \zeta$, of its complement, so that we are in the situation of Remark 3.3 with $X = Y \cup \bigcup_{Z \in \zeta} B_Z$. Our aim is to check the sufficient conditions (3.8) and (3.9) given there.

The first one is taken care of by the following stronger result. For $B \in \mathcal{B} \cap Y^c$ we define $f_k(Y, B) := \mu_Y(Y \cap T^{-1}B \cap \{\varphi = k\})$, $k \geq 1$.

Theorem 8.1 (Return properties of AFN maps). *Let (X, T, ξ) be a basic AFN-system, and Y as in (8.5). Then for each $Z \in \zeta$ there is some probability density $D_Z \in \mathrm{BV}(Y)$, positive on $Z(1)$, such that*

$$\frac{1}{f_k(Y, B_Z)} \cdot \widehat{T}^k \left(1_{Y \cap T^{-1} B_Z \cap \{\varphi = k\}}\right) \longrightarrow \mu(Y)\ D_Z \quad \begin{array}{l}\textit{uniformly on } Y \\ \textit{as } k \to \infty,\end{array} \tag{8.6}$$

and any $D \in \mathrm{BV}(Y)$ with $D \geq 0$ and $\int_Y D\, d\lambda > 0$ is uniformly sweeping for Y.

The key to this theorem is a lemma about the asymptotic behaviour of high iterates of (the inverse branch of) T near an indifferent fixed point, cf. Lemma 2 of [T6], or Theorem 17 of [Z3].

Lemma 8.1 (Asymptotic shape of high iterates at a regular source). *Let* $f : [0, y] \to \mathbb{R}$ *be* $\mathcal{C}^1$*, satisfying* $0 < f(x) < x$*,* $f'(x) > 0$ *on* $(0, y]$*,* $f'(0) = 1$*, and let* f *be concave on* $[0, \eta]$ *for some* $\eta > 0$*. Then there exists a positive continuous function* g *on* $(0, y]$*, non-increasing on* $(0, \eta]$*, such that*

(i) $(f^n)' \sim (f^n(y) - f^{n+1}(y)) \cdot g$ *as* $n \to \infty$ *uniformly on each* $(\varepsilon, y]$, $\varepsilon > 0$,

(ii) $\frac{f'(x)}{x - f(x)} \leq g(x) \leq \frac{1}{x - f(x)}$ *on* $(0, \eta]$*, and*

(iii) $\int_{f(x)}^{x} g(t)\, dt = 1$ *for all* $x \in (0, y]$.

Proof of Theorem 8.1. Instead of directly using $\widehat{T}$ it will be convenient to deal with the dual operator $\mathbf{P}$ of T w.r.t. Lebesgue measure λ (the *Perron Frobenius operator*). The two are related via $\widehat{T}^n u = \mathbf{P}^n(hu)/h$, $n \in \mathbb{N}_0$, and $\mathbf{P}^n$ has an explicit representation $\mathbf{P}^n u = \sum_{Z \in \xi_n} (u \circ f_Z) \cdot \mid f_Z' \mid$. We shall henceforth use the version given by the expression on the right-hand side. Fix any $Z \in \zeta$.

a) By the finite image condition (8.2), there are $L \in \mathbb{N}$ (w.l.o.g. $L \geq 2$) and $\emptyset \neq \eta \subseteq \xi$ such that if $l \geq L$, then $T(W \cap Y) \supseteq Z \cap \{\varphi \geq l\}$ for $W \in \eta$, while $T(W \cap Y) \cap (Z \cap \{\varphi \geq l\}) = \emptyset$ for $W \in \xi \setminus \eta$. Clearly, $Z \cap \{\varphi \geq l\} = B_Z \cap \{\varphi \geq l\}$ if $l \geq 2$. For $k > L$ therefore

$$\begin{aligned} \mathbf{P}\left(1_{Y \cap T^{-1} B_Z \cap \{\varphi \geq k\}} \cdot h\right) &= 1_{Z \cap \{\varphi \geq k-1\}} \sum_{W \in \eta} \mathbf{P}\left(1_W\, h\right) \\ &= 1_{Z \cap \{\varphi \geq k-1\}} \sum_{W \in \eta} (h \circ f_W) \cdot \mid f_W' \mid \end{aligned}$$

(for all $W \in \eta$ we have $1_{Z \cap \{\varphi \geq k-1\}} \mathbf{P}(1_{W \cap Y}\, h) = 1_{Z \cap \{\varphi \geq k-1\}} \mathbf{P}(1_W\, h)$). Observe that the restriction to $Z \cap \{\varphi \geq k-1\}$ of each $h \circ f_W$, $W \in \eta$, is of bounded variation with positive infimum. Adler's condition (8.1) implies that the same is true for the restriction of the sum $V := \sum_{W \in \eta} (h \circ f_W) \cdot \mid f_W' \mid$ on the right-hand side. (As $\sup \mid f_W' \mid \leq e^{-\lambda(X) a} \inf \mid f_W' \mid$ with $a := \sup \mid T''/(T')^2 \mid$, and $\int_{TW} \mid f_W' \mid d\lambda = \lambda(W)$.) Now, on Y,

$$\begin{aligned} \widehat{T}^k\left(1_{Y \cap T^{-1} B_Z \cap \{\varphi = k\}}\right) &= h^{-1} \cdot \mathbf{P}^k\left(1_{Y \cap T^{-1} B_Z \cap \{\varphi \geq k\}} \cdot h\right) \\ &= h^{-1} \cdot \mathbf{P}^{k-1}\left(\mathbf{P}\left(1_{Y \cap T^{-1} B_Z \cap \{\varphi \geq k\}} \cdot h\right)\right) \\ &= h^{-1} \cdot \mathbf{P}^{k-1}\left(1_{Z \cap \{\varphi \geq k-1\}} \cdot V\right) \\ &= 1_{Z(1)}\, h^{-1} \cdot \left(V \circ f_Z^{k-1}\right) \left(f_Z^{k-1}\right)'. \end{aligned}$$

Notice that the limit $V(x_Z) := \lim_{x \to x_Z, x \in Z} V(x) \in (0, \infty)$ exists since $V \in \mathrm{BV}(Z \cap \{\varphi \geq k-1\})$, and recall that h is bounded on Y. By Lemma 8.1, there is some positive continuous function g_Z on $Z(1)$ such that $\left(f_Z^{k-1}\right)' \sim$

$\left| f_Z^k(y_Z) - f_Z^{k+1}(y_Z) \right| \cdot g_Z$ uniformly on $Z(1)$ as $k \to \infty$. Consequently, we also have

$$\widehat{T}^k \left(1_{Y \cap T^{-1} B_Z \cap \{\varphi = k\}} \right) \sim 1_{Z(1)}\, h^{-1} \cdot V(x_Z) \left| f_Z^k(y_Z) - f_Z^{k+1}(y_Z) \right| \cdot g_Z$$

uniformly on Y as $k \to \infty$, and letting $D_Z := 1_{Z(1)}\,(g_Z/h)/\int_{Z(1)}(g_Z/h)\,d\mu \in \mathrm{BV}(Y)$ completes the proof of (8.6).

b) We check that D is uniformly sweeping for Y by showing that there is some $K \in \mathbb{N}_0$ such that $\inf_Y \sum_{k=0}^{K} \mathbf{P}^k D > 0$, which suffices since $0 < \inf_Y h \le \sup_Y h < \infty$. Due to our assumptions on D, there is some nondegenerate interval $I \subseteq Y$ such that $\inf_I D > 0$ (by bounded variation, D is lower semicontinuous mod λ). As T has bounded derivative on each cylinder and satisfies (8.2), we have $\inf_{T^k(I)} \mathbf{P}^k D > 0$ for all $k \in \mathbb{N}_0$. Our claim therefore follows once we prove that

$$\text{for any interval } I \subseteq Y \text{ there is some } K = K(I) \in \mathbb{N}_0 \text{ s.t. } \bigcup_{k=0}^{K} T^k I \supseteq Y. \quad (8.7)$$

Standard arguments (compare e.g. Lemma 10 of [Z1]) show that the induced map T_Y on Y is uniformly expanding and satisfies (8.1) and (8.2), implying that for any interval $I \subseteq Y$ there is some $L \in \mathbb{N}$ s.t. $\bigcup_{l=0}^{L-1} T_Y^l I \supseteq Y$. However, as T satisfies (8.2), we see that given any interval $I \subseteq Y$, we have $T_Y I \subseteq \bigcup_{m=1}^{M} T^m I$ for some $M = M(I) \in \mathbb{N}$, and that $T^j I$ and $T_Y I$ are finite unions of intervals. Together, these observations yield (8.7). □

Given a basic AFN system (X, T, ξ) we take Y as in (8.5). To ensure regular variation of wandering rates and condition (3.9), we assume that for each $Z \in \zeta$ there are $a_Z \neq 0$ and $p_Z \in [1, \infty)$ such that

$$Tx = x + a_Z\,|x - x_Z|^{1+p_Z} + o\left(|x - x_Z|^{1+p_Z}\right) \qquad \text{as } x \to x_Z \text{ in } Z, \quad (8.8)$$

and let $p := \max\{p_Z : Z \in \zeta\}$. Then (as in [T2] or Theorem 3 of [Z2]), as $n \to \infty$,

$$w_N(Y, B_Z) \sim \frac{h_0(Z)}{|a_Z|^{1/p_Z}} \cdot \begin{cases} \log N & \text{if } p_Z = 1, \\ p_Z^{-1/p_Z} \frac{p_Z}{p_Z - 1} \cdot N^{1-1/p_Z} & \text{if } p_Z > 1, \end{cases}$$

where $h_0(Z) := \lim_{x \to x_Z, x \in Z} h_0(x) = |a_Z| \lim_{x \to x_Z, x \in Z} |x - x_Z|^{p_Z}\, h(x) \in (0, \infty)$ exists, cf. [Z2], p. 1534. Of course, $w_N(Y) \sim \sum_{Z \in \zeta} w_N(Y, B_Z)$. (For the asymptotics of $f_k(Y, B_Z)$ see e.g. Remark 1 in [Z4].) In particular, condition (3.9) is satisfied, and we can apply our abstract Theorem 3.1 to obtain a Darling-Kac theorem for AFN-systems (compare Theorem 5 of [Z2]).

Corollary 8.1 (Darling-Kac theorem for AFN maps). *Let (X, T, ξ) be a basic AFN-system satisfying (8.8), and $\alpha := 1/p$. Then*

$$\frac{1}{a_n} \mathbf{S}_n(f) \stackrel{\mathcal{L}(\mu)}{\Longrightarrow} \mu(f) \cdot \mathcal{M}_\alpha \quad \text{for all } f \in L_1(\mu) \text{ s.t. } \mu(f) \neq 0,$$

where $a_n \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{n}{w_n(Y)}$, $n \geq 1$.

Again appealing to Remark 3.3, we can also apply Theorem 3.2 to extend the arcsine law of [T6] to a considerably larger family of AFN-systems. Given $\varnothing \subsetneqq \eta \subseteq \zeta$ we let $A_\eta := \bigcup_{Z\in\eta} B_Z$.

Corollary 8.2 (Arcsine law for neighbourhoods of neutral fixed points). *Let (X, T, ξ) be a basic AFN-system satisfying (8.8) and let $\alpha := 1/p$. Suppose that $\varnothing \subsetneqq \eta \subseteq \zeta$. Then*

$$\frac{w_N(Y, A_\eta)}{w_N(Y)} \longrightarrow \beta := \frac{\sum_{Z\in\eta, p_Z=p} h_0(Z)\,|a_Z|^{-\alpha}}{\sum_{Z\in\zeta, p_Z=p} h_0(Z)\,|a_Z|^{-\alpha}} \in [0, 1] \quad \text{as } N \to \infty,$$

and

$$\frac{1}{n}\mathbf{S}_n(1_A) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{L}_{\alpha,\beta}$$

for all $A \in \mathcal{B}$ with $\mu(A \triangle A_\eta) < \infty$. Here $\beta \notin \{0, 1\}$ iff $\max\{p_Z : Z \in \eta\} = \max\{p_Z : Z \in \zeta \setminus \eta\}$ *(and hence $= p$).*

While unions of neighbourhoods of different x_Z, $Z \in \zeta$, are the most obvious candidates for components of infinite measure in the regime of the arcsine law for occupation times, Remark 3.4 provides us with a very general method for finding further examples. In fact, Proposition 3.1 promises sets satisfying the arcsine law even for maps with a single indifferent fixed point, and our general construction amounts to splitting neighbourhoods in this case. We illustrate this in the simplest setup:

Example 8.1 (Arcsine law for split neighbourhoods). For fixed $p \geq 1$ let

$$Tx := \begin{cases} x + 2^p x^{1+p} & \text{for } x \in (0, 1/2) \\ 2x - 1 & \text{for } x \in (1/2, 1), \end{cases}$$

which defines a basic AFN map satisfying (8.8) for its single indifferent fixed point at $x = 0$. For $\gamma \in (0, 1)$ we let $z := 1 - \gamma/2 \in (1/2, 1)$, denote the inverse of $T\mid_{(0,1/2)}$ by f, and consider the set

$$A := \bigcup_{n\geq 0} f^n(z, 1).$$

Employing Remark 3.4, we see that

$$\frac{1}{n}\mathbf{S}_n(1_A) \overset{\mathcal{L}(\lambda)}{\Longrightarrow} \mathcal{L}_{\alpha,\beta}$$

where $\alpha := 1/p$ and $\beta = \beta(\gamma)$ with β an increasing homeomorphism of $(0, 1)$ onto itself. To obtain examples with $\alpha = 0$ and arbitrary $\beta \in (0, 1)$, play the same game using the map

$$Tx := \begin{cases} x + 2x^2 e^{2-1/x} & \text{for } x \in (0, 1/2) \\ 2x - 1 & \text{for } x \in (1/2, 1). \end{cases}$$

We finally turn to the arcsine theorem for waiting times of AFN-maps. Our abstract Theorem 3.3 immediately implies

Corollary 8.3 (Dynkin-Lamperti law for AFN reference sets). *Let* (X, T, ξ) *be a basic AFN-system satisfying (8.8) and let* $\alpha := 1/p$. *Then*

$$\frac{1}{n}\mathbf{Z}_n(Y) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mathcal{Z}_\alpha.$$

Remark 8.1 (Extension to a larger class of sets). In Corollary 8.3, the set Y can be replaced by any $E \in \mathcal{E}(T) := \{E \in \mathcal{B} : E \subseteq X_\varepsilon \text{ for some } \varepsilon > 0\}$ with $\mu(E) > 0$, cf. [T4] and Theorem 11 of [Z2]. This sharper statement can also be recovered in our setup. For example, in the Markov case it is not hard to see that $(\widehat{T}_Y(\widehat{T}^n 1_Y))$ is uniformly bounded, hence satisfying condition (7.7) of Proposition 7.1 (and we may assume w.l.o.g. that $E \subseteq Y$).

9. Distributional limit theorems for pointwise dual ergodic transformations

Earlier work on distributional limit theorems for infinite measure preserving transformations in [A0], [A1], and [T4] assumed T to be pointwise dual ergodic (p.d.e.), cf. (3.13). Some of the tools developed above can be used to simplify the arguments there. We demonstrate this for the p.d.e. version of the Darling-Kac theorem.

Theorem 9.1 (Darling-Kac theorem for p.d.e. systems; [A0],[A1]). *Let* T *be a c.e.m.p.t. on the* σ*-finite measure space* $(X, \mathcal{A}, \mu)$. *If* T *is pointwise dual ergodic with return sequence* $(a_n) \in \mathcal{R}_\alpha$, $\alpha \in [0, 1]$, *then*

$$\frac{1}{a_n}\mathbf{S}_n(f) \overset{\mathcal{L}(\mu)}{\Longrightarrow} \mu(f) \cdot \mathcal{M}_\alpha \quad \textit{for all } f \in L_1(\mu) \textit{ s.t. } \mu(f) \neq 0. \tag{9.1}$$

We first review our proof of Theorem 3.1 in the light of the account of Theorem 9.1 given in [A0]: Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$ and call $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, a *moment set* (for T), if there exists some $U : (0, \eta) \to (0, \infty)$ such that

$$\sum_{n\geq 0}\left(\int_Y S_n^r \, d\mu_Y\right) e^{-ns} \sim \frac{r!}{s} U(s)^r \quad \text{as } s \searrow 0 \text{ for all } r \in \mathbb{N}_0. \tag{9.2}$$

Choosing $r = 1$ we see that necessarily $U(s) \sim U_Y(s) := \sum_{n\geq 0} u_n(Y)\, e^{-ns}$, where $u_n(Y) := \mu_Y(T^{-n}Y)$, $n \geq 0$. Hence we may w.l.o.g. replace U by U_Y in the definition of a moment set (as in Section 3.6 of [A0]). By KTT one sees that (9.2) implies (5.1) if $U_Y \in \mathcal{R}_\alpha(0)$ (cf. [DK] or Theorem 3.6.4 of [A0]):

$$\begin{gathered}\text{If } T \text{ has a moment set } Y \text{ with } U_Y \in \mathcal{R}_\alpha(0), \text{ then} \\ \text{(9.1) holds with } a_n := \mu(Y)^{-1} \textstyle\sum_{j=0}^{n-1} u_j(Y),\, n \geq 1.\end{gathered} \tag{9.3}$$

If T is p.d.e., then there are sets $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, satisfying

$$\left\| \frac{1}{a_n} \sum_{j=0}^{n-1} \widehat{T}^j 1_Y \right\|_{L_\infty(Y)} \leq M < \infty \quad \text{for } n \geq 0,$$

and the main step of the proof of Theorem 9.1 in [A0] is to show (cf. Theorem 3.7.2 there) that any such Y is a moment set. Applying Theorem 9.1 to examples like those of Section 8 requires in addition to show that $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$ implies $U_Y \in \mathcal{R}_\alpha(0)$ (cf. Proposition 3.8.7 of [A0]).

Our argument includes both steps and can be summarized as follows: If $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, satisfies (3.1) and (3.2), then Y is a moment set with $U(s) = 1/(s\,Q_Y(s))$.

Finally, we show that the equivalent moments principle, Lemma 4.4, offers a way to overcome the main difficulty in the proof of Theorem 9.1.

Proof of Theorem 9.1 (revised). An Egorov-type argument shows that there is some $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, with

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\widehat{T}^k u \longrightarrow 1 \quad \text{uniformly on } Y \text{ as } n \to \infty \tag{9.4}$$

for a bounded probability density u satisfying $\inf_Y u > 0$.

Let $S_n := \sum_{k=1}^{n} 1_Y \circ T^k$, $n \geq 0$, and fix some $r \in \mathbb{N}$. Using

$$\binom{S_n}{r} = \sum_{k=1}^{n}\left(1_Y\binom{S_{n-k}}{r-1}\right)\circ T^k, \quad \text{for } n \in \mathbb{N}_0,$$

(cf. [A0], [A1]), and Lemma 4.2 ($v_n = \widehat{T}^n u$, $H = \mu(Y)^{-1}1_Y$, $K = 0$), we obtain

$$\begin{aligned}\sum_{n\geq 0}\left(\int_X \binom{S_n}{r}\cdot u\,d\mu\right)e^{-ns} &= \int_Y\left(\sum_{n\geq 1}\widehat{T}^n u\,e^{-ns}\right)\left(\sum_{n\geq 0}\binom{S_n}{r-1}e^{-ns}\right)d\mu \\ &\sim U(s)\cdot\sum_{n\geq 0}\left(\int_Y\binom{S_n}{r-1}d\mu_Y\right)e^{-ns}\end{aligned} \tag{9.5}$$

as $s \searrow 0$, where $U(s) := \sum_{n\geq 0}\left(\int_{T^{-n}Y} u\,d\mu\right)e^{-ns}$, $s > 0$. Since $S_n \to \infty$ a.e., it is easy to check that $\int_X S_n^r \cdot u\,d\mu \sim r!\int_X \binom{S_n}{r}\cdot u\,d\mu$ as $n \to \infty$ (and analogously for $d\mu_Y$), showing that (9.5) is equivalent to

$$\sum_{n\geq 0}\left(\int_X S_n^r\cdot u\,d\mu\right)e^{-ns} \sim rU(s)\cdot\sum_{n\geq 0}\left(\int_Y S_n^{r-1}\,d\mu_Y\right)e^{-ns} \text{ as } s \searrow 0. \tag{9.6}$$

The point now is to replace $u\,d\mu$ by $d\mu_Y$ on the left-hand side via Lemma 4.4, which can be achieved in much the same way as in the proof of Theorem 3.1, noting that there is some $C > 0$ such that $\int_X S_n^r\,d\mu_Y \leq C\cdot\int_X S_n^r\cdot u\,d\mu$ for all $r, n \in \mathbb{N}_0$. Thus (9.6) becomes a recursion formula showing that Y is a moment set, and (9.3) finishes the proof. □

Remark 9.1 (The arcsine law for waiting times of p.d.e. systems; [T4]). The thesis [Eb] contains a short proof of the Dynkin-Lamperti arcsine law for Markov chains. Part a) of Lemma 4.3 enables us to use the same argument for p.d.e. transformations.

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