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Classification and decomposition of Quantum Markov Semigroups

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Abstract. We show that a QMS on a σ-finite von Neumann algebra $\mathcal{A}$ can be decomposed as the sum of several "sub"-semigroups corresponding to transient and recurrent projections. We discuss two applications to physical models.

1. Introduction

The analysis of transient and recurrent states is a key step in the study of Markov processes, where the concepts of transience and recurrence are closely connected with Potential Theory for semigroups on L^∞ spaces (see e.g. [2], [14]). These are the typical commutative von Neumann algebras. In the theory of open quantum systems, however, models of irreversible evolutions are given by means of positive and identity-preserving semigroups on an arbitrary von Neumann algebra. A mathematical theory parallel to the classical theory of Markov processes and semigroups, however, is still missing. It seems therefore reasonable to provide the non-commutative generalizations of classical notions like transience, recurrence and decomposition of semigroups into transient and recurrent parts. This paper is aimed at clarifying these notions for Quantum Dynamical Semigroups (QDS) and providing mathematical tools for the study of evolution equations (master equations) for open quantum systems.

A QDS $\mathcal{T}$ on a von Neumann algebra $\mathcal{A}$ is a weak* continuous semigroup of normal completely positive maps $\{\mathcal{T}_t\}_{t\geq 0}$ on $\mathcal{A}$; if $\mathcal{T}$ is identity-preserving, then it is *Markov* (i.e. it is a QMS). In the work [11] transience and recurrence are defined as the natural extension of the corresponding classical concepts and irreducible semigroups are shown to be either transient or recurrent. Our intention here is to find the decomposition of a QMS into "sub"-semigroups corresponding to classes of transient and recurrent states. To this end we start by defining the *fast recurrent projection* p_R determined by supports of normal invariant states. We show that states with support contained in $p_R\mathcal{A}p_R$ do not leave p_R under the action of $\mathcal{T}$ (see Thm. 1) and establish the ergodic properties of the reduced QMS $\mathcal{T}^{p_R}$ on the subalgebra $p_R\mathcal{A}p_R$ (see Thm. 3). Moreover, under appropriate hypotesis, we can write $\mathcal{T}^{p_R}$ as a direct sum of irreducible "sub-QMS" each one with a unique faithful normal invariant state (Thm. and Prop. 5).

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Then we define the *transient projection* p_T by means of range projections of potentials (see [11]). It turns out that a QMS can be either transient or recurrent according to $p_T = \mathbf{1}$ or $p_T = 0$ respectively. Further, to distinguish between fast and slow recurrence, we introduce the projection p_{R_0} such that $p_T^{\perp} = p_R + p_{R_0}$, calling a semigroup fast recurrent if $p_R = \mathbf{1}$ and slow recurrent if $p_{R_0} = \mathbf{1}$. We show then that, when $\mathcal{A}$ is σ-finite, the von Neumann subalgebra $p_T \mathcal{A} p_T$ is invariant for $\mathcal{T}$ (see Cor. 2); moreover, the restriction of $\mathcal{T}$ to $p_T \mathcal{A} p_T$ is a transient semigroup. On the other hand, we can construct a recurrent QMS on $p_T^{\perp} \mathcal{A} p_T^{\perp}$ (see Prop. 10) which contains the fast recurrent "sub"-QMS $\mathcal{T}^{p_R}$.

As an application we determine p_T, p_R and p_{R_0} in two physical models: a one-mode radiation and an atom with two-degenerate levels.

2. Notations and preliminaries

In this paper $\mathcal{A}$ is a von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$, endowed with a trace $tr(\cdot)$; we denote by $\mathbf{1}$ the unit of $\mathcal{A}$. A state ω on $\mathcal{A}$ is called *normal state* if it is σ-weakly continuous or, equivalently, if $\omega(\sup_\alpha a_\alpha) = \sup_\alpha \omega(a_\alpha)$ for any increasing net $(a_\alpha)_\alpha$ of positive elements in $\mathcal{A}$ with an upper bound; we denote by $\mathcal{A}_*$ the *predual* of $\mathcal{A}$, that is the space of all σ-weakly continuous linear functional on $\mathcal{A}$. We recall also that ω is a normal state if and only if there exists a density matrix ρ, that is, a positive trace-class operator of $\mathcal{H}$ with a unit trace, such that $\omega(a) = tr(\rho a)$ for all $a \in \mathcal{A}$. For all normal state ω on $\mathcal{A}$, the *support projection* $s(\omega)$ is the smallest projection in $\mathcal{A}$ such that $\omega(s(\omega)a) = \omega(as(\omega)) = \omega(a)$ for any $a \in \mathcal{A}$ (c.f. [7], Prop. 3); if $s(\omega) = \mathbf{1}$, we say that ω is *faithful*. A family $\mathcal{G}$ of normal states on $\mathcal{A}$ is called *faithful* if $a \in \mathcal{A}$, a positive and $\omega(a) = 0$ for all $\omega \in \mathcal{G}$ implies $a = 0$.

Given a QDS $\mathcal{T}$ on $\mathcal{A}$, its *infinitesimal generator* is the operator $\mathcal{L}$ whose domain $D(\mathcal{L})$ is the vector space of elements a in $\mathcal{A}$ for which there exists an element b in $\mathcal{A}$ such that $b = \lim_{t \to 0} t^{-1}(\mathcal{T}_t(a) - a)$ in the weak* topology, and $\mathcal{L}(a) = b$; the *predual semigroup* of $\mathcal{T}$ is the semigroup $\mathcal{T}_*$ of operators in $\mathcal{A}_*$ defined by $(\mathcal{T}_{*t}(\omega))(a) = \omega(\mathcal{T}_t(a))$ for every $a \in \mathcal{A}$ and $\omega \in \mathcal{A}_*$. Since any map $\mathcal{T}_{*t}$ is clearly weak continuous on $\mathcal{A}_*$, by a well-known fact (see, for instance [4] Cor. 3.1.8), $\mathcal{T}_*$ is a strongly continuous semigroup in the Banach space $\mathcal{A}_*$; moreover, if $\mathcal{T}$ is Markov, $\mathcal{T}$ and $\mathcal{T}_*$ are semigroups of contractions (see [9], Prop. 2.10.3).

We say that a normal state ω on $\mathcal{A}$ is *invariant* if $\mathcal{T}_{*t}(\omega) = \omega$ for all $t \geq 0$ and we denote by $\mathcal{F}(\mathcal{T}_*)_1$ the set of normal invariant states on $\mathcal{A}$.

3. Subharmonic projections and the fast recurrent projection

A positive operator a is *subharmonic* (resp. *superharmonic*, resp. *harmonic*) if $\mathcal{T}_t(a) \geq a$ (resp. $\mathcal{T}_t(a) \leq a$, resp. $\mathcal{T}_t(a) = a$) for all $t \geq 0$; we denote by $\mathcal{F}(\mathcal{T})$ the set of harmonic elements of $\mathcal{T}$. We call a QDS $\mathcal{T}$ *irreducible* if $\mathcal{T}$ has no non-trivial subharmonic projections.

We introduce now some results that we shall often use in this paper.

Lemma 1. *Given a positive element $a \in \mathcal{A}$ and a projection p, $p^{\perp} a p^{\perp} = 0$ implies $a = pap$.*

Proof. Let $u \in p(\mathcal{H})^\perp$ and $v \in p(\mathcal{H})$; since a is positive, we have $\langle \lambda u + v, a(\lambda u + v)\rangle = 2\Re\langle \lambda u, av\rangle + \langle v, av\rangle \geq 0$ for all $\lambda \in \mathbb{C}$. Therefore $\langle u, av\rangle = 0$ for all $u \in p(\mathcal{H})^\perp$, $v \in p(\mathcal{H})$, that is $p^\perp a p = 0$. It follows that $pap^\perp = 0$ as well, and so $a = pap$. □

Proposition 1. *A normal state ω on $\mathcal{A}$ is faithful on $s(\omega)\mathcal{A}\,s(\omega)$.*

Proof. Let $p = s(\omega)$ and suppose $\omega(a) = 0$, where $a \in p\mathcal{A}p$, $a \geq 0$; if q_n denotes the spectral projection of a associated with the interval $]1/n, \|a\|]$, $(n \geq 1)$, then $\omega(q_n) \leq n\omega(a) = 0$ implies $q_n \leq p^\perp$ for all $n \geq 1$. Since $q_n \leq na \leq n\|a\|p$, this means that $q_n = 0$ for all $n \geq 1$; hence $q := \sup_n q_n = 0$. But q is the projection onto the closure of the range of a, so $a = 0$. □

Theorem 1. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$. If $\omega \in \mathcal{A}_*$ is an invariant state, then its support projection is subharmonic.*

Proof. Let $p := s(\omega)$ and fix $t \geq 0$. From the invariance of ω it follows $\omega(p - p\mathcal{T}_t(p)p) = \omega(p - \mathcal{T}_t(p)) = 0$ and then $p\mathcal{T}_t(p)p = p$, because $p\mathcal{T}_t(p)p \leq p$ and ω is faithful on $p\mathcal{A}p$ (see Prop. 1). Therefore, the projection $p^\perp$ satisfies $p\mathcal{T}_t(p^\perp)p = 0$, so $\mathcal{T}_t(p^\perp) = p^\perp\mathcal{T}_t(p^\perp)p^\perp$ by Lemma 1. This implies $\mathcal{T}_t(p^\perp) \leq p^\perp$ and consequently $\mathcal{T}_t(p) \geq p$. □

Proposition 2. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$ and let p_1, p_2 be subharmonic projections in $\mathcal{A}$ with $p_1 \geq p_2$. If p_1 is the support projection of a normal invariant state ω_1, then we have $(p_1 - p_2)\mathcal{T}_t(p_2)(p_1 - p_2) = 0$ for all $t \geq 0$. In particular, if $p_2 \in D(\mathcal{L})$, $(p_1 - p_2)\mathcal{L}(p_2)(p_1 - p_2) = 0$.*

Proof. Since any $(\mathcal{T}_t(p_i))_{t\geq 0}$ $(i = 1, 2)$ is an increasing positive net with upper bound $\mathbf{1}$, there exists $x_i \in \mathcal{A}$ such that $x_i = \text{w}^*\text{-}\lim_t \mathcal{T}_t(p_i)$, $x_i \geq p_i$, $i = 1, 2$ and $x_1 \geq x_2 \geq 0$. Therefore, from the invariance of ω_1 it follows $\omega_1(x_1 - x_2) = \lim_t \omega_1(\mathcal{T}_t(p_1 - p_2)) = \omega_1(p_1 - p_2)$, i.e. $\omega_1(x_1 - p_1) = \omega_1(x_2 - p_2)$. But $\omega_1(x_1) = \lim_t \omega_1(\mathcal{T}_t(p_1)) = \omega_1(p_1)$, so $\omega_1(x_2 - p_2) = 0$. Since ω_1 is faithful on $p_1\mathcal{A}p_1$ (Prop. 1), this means that $p_1(x_2 - p_2)p_1 = p_1(x_2 - p_2) = (x_2 - p_2)p_1 = 0$ by Lemma 1, and then

$$0 = (p_1 - p_2)(x_2 - p_2)(p_1 - p_2) \geq (p_1 - p_2)(\mathcal{T}_t(p_2) - p_2)(p_1 - p_2) \geq 0$$

which implies $(p_1 - p_2)\mathcal{T}_t(p_2)(p_1 - p_2) = (p_1 - p_2)p_2(p_1 - p_2) = 0$ for all $t \geq 0$. Deriving at $t = 0$ we get also the last statement. □

Prop. 2 provides us with a good rule to test whether, given two comparable subharmonic projections, their upper bound is the support of an invariant state. It will be very useful to find the normal invariant states in section 7.

Notation. For any $\omega \in \mathcal{A}_*$ and p projection of $\mathcal{A}$, we denote by $p\omega p$ the element of $\mathcal{A}_*$ defined as $p\omega p(a) = \omega(pap)$ for all $a \in \mathcal{A}$, and by $p\mathcal{A}_*p$ the set of $p\omega p$ as ω varies in $\mathcal{A}_*$. Then we can identify the normal states on $p\mathcal{A}p$ with the normal states on $\mathcal{A}$ whose support is smaller than p.

Theorem 2. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$. A projection p in $\mathcal{A}$ is subharmonic if and only if*

$$p\mathcal{T}_t(a)p = p\mathcal{T}_t(pap)p \quad \forall\, a \in \mathcal{A},\ t \geq 0. \tag{1}$$

Proof. If p is subharmonic, then for any $\omega \in \mathcal{A}_*$ we have $(\mathcal{T}_{*t}(p\omega p))(p^{\perp}) = \omega(p\mathcal{T}_t(p^{\perp})p) = 0$, so $s(\mathcal{T}_{*t}(p\omega p)) \leq p$ for all $t \geq 0$; therefore, given $a \in \mathcal{A}$, if q_t is the support projection of $\mathcal{T}_{*t}(p\omega p)$, the equalities

$$\begin{aligned}\omega(p\mathcal{T}_t(pap)p) &= (\mathcal{T}_{*t}(p\omega p))(pap) = (\mathcal{T}_{*t}(p\omega p))(q_t papq_t)\\ &= (\mathcal{T}_{*t}(p\omega p))(q_t aq_t) = (\mathcal{T}_{*t}(p\omega p))(a) = \omega(p\mathcal{T}_t(a)p)\end{aligned}$$

hold for all $t \geq 0$. Since $\omega \in \mathcal{A}_*$ is arbitrary, this means that $p\mathcal{T}_t(a)p = p\mathcal{T}_t(pap)p$ for all $t \geq 0$. Conversely, if (1) holds, taking $a = \mathbf{1}$ we get $p = p\mathcal{T}_t(p)p$, that is $p\mathcal{T}_t(p^{\perp})p = 0$ for all $t \geq 0$; we can then conclude that p is subharmonic by Lemma 1. □

If $\mathcal{T}$ is a QMS on $\mathcal{A}$ and p is a subharmonic projection in $\mathcal{A}$, it follows by Thm. 2 that we can construct a QMS $\mathcal{T}^p$ on $p\mathcal{A}p$ by defining

$$\mathcal{T}_t^p(a) := p\mathcal{T}_t(a)p$$

for all $a \in p\mathcal{A}p,\ t \geq 0$. $\mathcal{T}^p$ is called *reduced semigroup* associated with the subharmonic projection p. It is easy to check that its predual semigroup is the restriction of $\mathcal{T}$ to the subspace $p\mathcal{A}_*p$.

We want to construct a subharmonic projection p_R, called the fast recurrent projection, determined by supports of normal invariant states on $\mathcal{A}$.

Definition 1. *Given a family $(p_i)_{i\in I}$ of projections in $\mathcal{A}$, we denote by $\sup_{i\in I} p_i$ the projection (in $\mathcal{A}$) onto the closure of the linear space of $\mathcal{H}$ generated by the ranges of p_i's.*

Proposition 3. *Let $(p_i)_{i\in I}$ be a family of subharmonic projections for a QMS $\mathcal{T}$ on $\mathcal{A}$. The projection $p = \sup_{i\in I} p_i$ is also subharmonic for $\mathcal{T}$.*

Proof. Fix $t \geq 0$; then $\mathcal{T}_t(p^{\perp}) \leq \mathcal{T}_t(p_i^{\perp}) \leq p_i^{\perp}$ for all $i \in I$. Hence, we have $p_i(\mathcal{H}) = \ker p_i^{\perp} \subseteq \ker \mathcal{T}_t(p^{\perp})$ for all $i \in I$, so $p(\mathcal{H}) \subseteq \ker \mathcal{T}_t(p^{\perp})$; it follows that $\mathcal{T}_t(p^{\perp})p = 0$ and then $p\mathcal{T}_t(p^{\perp})p = 0$. Therefore, by Lemma 1 we get $\mathcal{T}_t(p^{\perp}) \leq p^{\perp}$. □

Definition 2. *We call* fast recurrent projection *associated with a QMS $\mathcal{T}$ the projection $p_R = \sup_{i\in I} p_i$, where the p_i's are the support projections of all invariant states of $\mathcal{T}$.*

Since the support projections of normal invariant states are subharmonic (Thm. 1), Prop. 3 implies that p_R is also subharmonic, so we can consider the reduced semigroup $\mathcal{T}^{p_R}$ associated with p_R.

Proposition 4. *The family $\mathcal{F}(\mathcal{T}_*)_1$ is faithful on $p_R\mathcal{A}p_R$.*

Proof. Let $a \in p_R \mathcal{A} p_R$ be a positive element such that $\omega(a) = 0$ for all $\omega \in \mathcal{F}(\mathcal{T}_*)$; then $s(\omega)as(\omega) = as(\omega) = s(\omega)a = 0$ for all ω in $\mathcal{F}(\mathcal{T}_*)$ by Prop. 1 and Lemma 1. Therefore, since

$$a\left(\sum_{j\in F} u_j\right) = \sum_{j\in F} a\left(s(\omega_j)u_j\right) = 0$$

if F is a finite subset of I and $u_j \in s(\omega_j)(\mathcal{H})$, with $\omega_j \in \mathcal{F}(\mathcal{T}_*)$, $j \in F$, we find $a(u) = 0$ for $u \in p_R(\mathcal{H})$. Clearly $a(u) = ap_R(u) = 0$ if $u \in \ker p_R$, so that $a = 0$. □

Since any $\mathcal{T}$-invariant state is clearly also $\mathcal{T}^{p_R}$-invariant, Prop. 4 assures the existence of a faithful family of normal invariant states for $\mathcal{T}^{p_R}$; so, the application of the mean ergodic Thm. of [13] to $\mathcal{T}^{p_R}$ leads to the following.

Theorem 3. *For all $a \in \mathcal{A}$ the limit*

$$\mathcal{E}(a) := \text{w}^*\text{-}\lim_t \frac{1}{t}\int_0^t p_R \mathcal{T}_s(a) p_R \, ds$$

exists and it defines a $p_R\mathcal{T}p_R$-invariant normal conditional expectation onto the von Neumann subalgebra $\mathcal{F}(\mathcal{T}^{p_R})$ of $p_R\mathcal{A}p_R$ such that $\mathcal{E}\circ\mathcal{T}_t = \mathcal{E}$ for all $t \geq 0$. A normal state ω on $\mathcal{A}$ is $\mathcal{T}$-invariant if and only if $\omega\circ\mathcal{E} = \omega$.

We recall that a von Neumann algebra $\mathcal{A}$ on $\mathcal{H}$ is *σ-finite* if there exists a countable subset S of $\mathcal{H}$ which is separating for $\mathcal{A}$ (i.e. for any $a \in \mathcal{A}$, $au = 0$ for all $u \in S$ implies $a = 0$) (see [4], Prop. 2.5.6). If $\mathcal{A}$ is σ-finite and p is a projection in $\mathcal{A}$, then $p\mathcal{A}p$ is also σ-finite on $p(\mathcal{H})$ because S separating for $\mathcal{A}$ implies $\{pe : e \in S\}$ separating for $p\mathcal{A}p$.

Theorem 4. *If $\mathcal{A}$ is σ-finite, then there exists a normal invariant state with support p_R.*

Proof. Let $\{e_n\}_{n\geq 1}$ be a countable subset of $p_R(\mathcal{H})$ which is separating for $p_R\mathcal{A}p_R$; by definition of p_R, for any $n, m \geq 1$ there exist a finite set $F_{n,m} \subseteq \mathbb{N}$, $\omega_i \in \mathcal{F}(\mathcal{T}_*)_1$ ($i \in F_{n,m}$) and $x_{n,m} \in \text{span}\{s(\omega_i) : i \in F_{n,m}\}$ such that $\|e_n - x_{n,m}\| < m^{-1}$. Therefore,

$$\omega(a) := \sum_{n,m\geq 1} \frac{1}{2^{n+m}|F_{n,m}|} \sum_{i\in F_{n,m}} \omega_i(a)$$

defines a normal invariant state on $\mathcal{A}$ by Beppo Levi Theorem. We prove that ω is faithful on $p_R\mathcal{A}p_R$ (i.e. $s(\omega) = p_R$). Let $a \in \mathcal{A}_+$ such that $\omega(a) = 0$; then $\omega_i(a) = 0$ for all $i \in F_{n,m}$, $n, m \geq 1$. This implies $s(\omega_i)a = as(\omega_i) = 0$ for all $i \in F_{n,m}$, $n, m \geq 1$ by Lemma 1, so that $p_{n,m}a = ap_{n,m} = 0$ for all $n, m \geq 1$, where $p_{n,m}$ is the orthogonal projection onto the closure of $\text{span}\{s(\omega_i) : i \in F_{n,m}\}$. Fix $n \geq 1$, since $ax_{n,m} = ap_{n,m}x_{n,m} = 0$, we have

$$\|ae_n\| \leq \|ae_n - ax_{n,m}\| + \|ax_{n,m}\| \leq \|a\| \cdot \|e_n - x_{n,m}\| \leq m^{-1}\|a\|$$

for all $m \geq 1$, so that $ap_Re_n = 0$. This means $ap_R = 0 = p_Rap_R$. □

Corollary 1. *If $\mathcal{A}$ is σ-finite, then $p_R = \mathbf{1}$ if and only if there exists a faithful normal invariant state on $\mathcal{A}$.*

We want now to see when we can decompose p_R as sum of orthogonal $\mathcal{T}^{p_R}$-invariant projections (p_n) such that any restriction of $\mathcal{T}^{p_R}$ to the subalgebra $p_n\mathcal{A}p_n$ is irreducible. In this case, since $p_n\mathcal{A}p_n$ is $\mathcal{T}^{p_R}$-invariant, we have

$$\mathcal{T}_t^{p_R}(x) = p_n\mathcal{T}_t^{p_R}(x)p_n = p_n\mathcal{T}_t(x)p_n \quad \forall\ x \in p_n\mathcal{A}p_n,$$

so that the restriction of $\mathcal{T}^{p_R}$ to $p_n\mathcal{A}p_n$ is the reduced semigroup $\mathcal{T}^{p_n}$ for all $n \geq 0$. Moreover, given $\omega \in \mathcal{F}(\mathcal{T}_*)_1$ with $\omega(p_n) \neq 0$ (which exists by virtue of Prop. 4), we get $p_n\omega p_n(\mathcal{T}_t^{p_n}(x)) = \omega(\mathcal{T}_t(x)) = \omega(x)$ for all $x \in p_n\mathcal{A}p_n$. Hence, $\omega_n := \omega(p_n)^{-1}p_n\omega p_n$ is a normal $\mathcal{T}^{p_n}$-invariant state; also, from the irreducibility of $\mathcal{T}^{p_n}$ it follows that ω_n is faithful on $p_n\mathcal{A}p_n$, so that it is unique by Thm. 1 of [12]. As a consequence, $\mathcal{T}^{p_R}$ is the direct sum of the irreducible "sub-QMSS" $\mathcal{T}^{p_n}$ each one with a unique faithful normal invariant state.

To see when this decomposition is possible we need to introduce the extremal states of $\mathcal{F}(\mathcal{T}_*)_1$.

Lemma 2. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$; if ω is a normal state on $\mathcal{A}$ and p is a subharmonic projection such that $p \geq s(\omega)$, then:*

1. *ω is $\mathcal{T}$-invariant if and only if ω is $\mathcal{T}^p$-invariant;*
2. *ω is extremal in $\mathcal{F}(\mathcal{T}_*)_1$ if and only if ω is extremal in $\mathcal{F}(\mathcal{T}_*^p)_1$.*

Proof. 1. If ω is $\mathcal{T}^p$-invariant, then

$$\omega(\mathcal{T}_t(a)) = \omega(p\mathcal{T}_t(a)p) = \omega(\mathcal{T}_t^p(pap)) = \omega(pap) = \omega(a)$$

for all $a \in \mathcal{A}$, $t \geq 0$, for $s(\omega) \leq p$ and p is subharmonic. The converse is clear since $\mathcal{T}_*^p$ is the restriction of $\mathcal{T}_*$ to $p\mathcal{A}_*p$.

2. Let ω be extremal in $\mathcal{F}(\mathcal{T}_*)_1$ and suppose $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ in $p\mathcal{A}p$ with $0 < \lambda < 1$ and $\omega_1, \omega_2 \in \mathcal{F}(\mathcal{T}_*^p)_1$; since $s(\omega_i), s(\omega) \leq p$, $i = 1, 2$, this equality holds on all $\mathcal{A}$. Therefore, we get $\omega = \omega_1 = \omega_2$ by virtue of point 1. On the other hand, if ω is extremal in $\mathcal{F}(\mathcal{T}_*^p)_1$ and $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ with $0 < \lambda < 1$ and $\omega_i \in \mathcal{F}(\mathcal{T}_*)_1$, then $s(\omega_i) \leq s(\omega) \leq p$, that is any ω_i belongs to $\mathcal{F}(\mathcal{T}_*^p)_1$; hence, we have $\omega(a) = \omega(pap) = \omega_i(pap) = \omega_i(a)$ for all $a \in \mathcal{A}$ and $i = 1, 2$, i.e. $\omega = \omega_1 = \omega_2$. □

Theorem 5. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$. The following facts are equivalent:*

1. *there exists a set of pairwise orthogonal projections $(p_n)_{n\in N}$, card $N \leq \aleph_0$, such that $p_R = \sum_{n\in N} p_n$, $\mathcal{T}_t^{p_R}(p_n) = p_n$ and the restriction of $\mathcal{T}^{p_R}$ to the subalgebra $p_n\mathcal{A}p_n$ is irreducible for all $n \in N$;*
2. *there exists a sequence $(\omega_n)_{n\in N}$ of extremal points of $\mathcal{F}(\mathcal{T}_*)_1$ such that $s(\omega_m)s(\omega_n) = 0$ for $n \neq m$ and $\sum_{n\in N} s(\omega_n) = p_R$.*

Proof. 1) $\Rightarrow$ 2) Fix $n \in N$. By the above remarks, there exists a unique faithful normal $\mathcal{T}^{p_n}$-invariant state on $p_n\mathcal{A}p_n$, so that it is extremal in $\mathcal{F}(\mathcal{T}_*^{p_n})_1$ by Thm. 1 of [12]; we can then conclude the proof by virtue of Lemma 2.

2) $\Rightarrow$ 1) Set $p_n := s(\omega_n)$ for all $n \in N$; then $(p_n)_{n\in N}$ is a sequence of pairwise orthogonal $\mathcal{T}^{p_R}$-harmonic projections (since any p_n is subharmonic by Thm. 1 and ω_n is a faithful invariant state on $p_n\mathcal{A}p_n$) such that $p_R = \sum_{n\in N} p_n$.
Fix $n \in N$, since ω_n is extremal in $\mathcal{F}(\mathcal{T}_*)_1$ it is also extremal in $\mathcal{F}(\mathcal{T}_*^{p_n})_1$ by Lemma 2; therefore, $s(\omega_n) = p_n$ implies $\mathcal{T}^{p_n} = \mathcal{T}^{p_R}_{|_{p_n\mathcal{A}p_n}}$ irreducible by Thm. 1 of [12]. □

Lemma 3. *If $\mathcal{T}$ is a QMS on $\mathcal{A}$ and ω, σ are extremal states of $\mathcal{F}(\mathcal{T}_*)_1$ such that $[s(\omega), s(\sigma)] = 0$, then $s(\omega)s(\sigma) = 0$.*

Proof. Set $p = s(\omega)$ and $q = s(\sigma)$. The condition $[p, q] = 0$ implies $p \wedge q = pq$; since $p \wedge q$ is $\mathcal{T}^{p_R}$-invariant and ω, σ are extremal, by Thm. 1 of [12] we have either $p \wedge q = 0$ or else $p \wedge q = p$, and correspondingly $pq = 0$ or $p = q$. But $p = q$ means $\sigma = \omega$ by Lemma 1 of [12], for ω is the unique faithful $\mathcal{T}^p$-invariant state on $p\mathcal{A}p$. □

Proposition 5. *Suppose $\mathcal{A}$ σ-finite and let $\mathcal{T}$ be a QMS on $\mathcal{A}$. The equivalent conditions of Thm. 5 are satisfied in one of the following cases:*

- *$\mathcal{F}(\mathcal{T}_*)$ is finite-dimensional,*
- *$\mathcal{A}$ is commutative and the family of extremal states of $\mathcal{F}(\mathcal{T}_*)_1$ is faithful on $p_R\mathcal{A}p_R$.*

Proof. Let $\{\omega_n\}_{n\in N}$; card $N \leq \aleph_0$, be a maximal family of extremal states of $\mathcal{F}(\mathcal{T}_*)_1$ with orthogonal supports. Set $q := \sum_{n\in N} s(\omega_n) \leq p_R$ and $q' := p_R - q$. If $q \neq p_R$, we want to construct an extremal state σ of $\mathcal{F}(\mathcal{T}_*)_1$ such that $s(\sigma)s(\omega_n) = 0$ for all $n \in N$; this contradicts the maximality of $\{\omega_n\}_{n\in N}$.

Let $\rho \in \mathcal{F}(\mathcal{T}_*)_1$ be such that $\rho(q') \neq 0$ (which exists by virtue of Prop. 4). Since q' is $\mathcal{T}^{p_R}$-harmonic and $s(\rho) \leq p_R$, we have

$$q'\rho q'(\mathcal{T}_t(a))=\rho(q'\mathcal{T}_t^{p_R}(q'aq')q') = \rho(\mathcal{T}_t^{p_R}(q'aq')) = \rho(\mathcal{T}_t(q'aq')) = q'\rho q'(a)$$

for all $a \in \mathcal{A}$, $t \geq 0$, that is $\omega := \rho(q')^{-1}q'\rho q'$ is a normal invariant state. Therefore, if $\mathcal{F}(\mathcal{T}_*)$ is finite-dimensional, and since ω is a convex combination of extremal points of $\mathcal{F}(\mathcal{T}_*)_1$ by Thm. 2.3.15 of [4], we have $q' \geq s(\omega) \geq s(\sigma)$ for some σ extremal in $\mathcal{F}(\mathcal{T}_*)_1$, which means $s(\sigma)s(\omega_n) = 0$ for all $n \in N$.
On the other hand, if $\mathcal{A}$ commutative and the family of extremal states of $\mathcal{F}(\mathcal{T}_*)_1$ is faithful on $p_R\mathcal{A}p_R$, we can choose ρ extremal such that $\rho(q') \neq 0$; therefore, since $s(\omega_n) \leq q$, Lemma 3 implies $s(\rho)s(\omega_n) = 0$ for all $n \in N$. □

4. Potential and the transient projection

In this section we study the part of a QMS $\mathcal{T}$ without invariant states as well as the projections in which the system spends a small amount of time; therefore, we need to introduce a *potential* associated to $\mathcal{T}$, which really represents the time of sojourn of a pure state in a projection.

Our reference on quadratic forms is the book of Kato [15].

Definition 3. *Given a positive operator $x \in \mathcal{A}$ we define the* form-potential of x *as a quadratic form* $\mathfrak{U}(x)$ *by*

$$\mathfrak{U}(x)[u] = \int_0^\infty \langle u, \mathcal{T}_s\,(x)\,u\rangle ds, \quad For\ all\ u \in D(\mathfrak{U}(x)),$$

where the domain $D(\mathfrak{U}(x))$ is the set of all $u \in \mathcal{H}$ s.t. $\int_0^\infty \langle u, \mathcal{T}_s\,(x)\,u\rangle ds < \infty$.

This is clearly a symmetric and positive form and by Thm. 3.13a, and Lemma 3.14a of [15] it is also closed. Therefore, when it is densely defined, it is represented by a self-adjoint operator (see Thm. 2.1, Thm. 2.6 and Thm. 2.23 of [15]). This motivates the following definition.

Definition 4. *For all positive $x \in \mathcal{A}$ such that $D(\mathfrak{U}(x))$ is dense, we call* potential of x *the self-adjoint operator $\mathcal{U}(x)$ which represents $\mathfrak{U}(x)$. We set also $\mathcal{A}_{int} := \{\, x \in \mathcal{A}_+ : \mathcal{U}(x)$ is bounded $\}$ and we call $\mathcal{T}$*-integrable *(or* integrable*) its elements.*

Since $D(\mathcal{U}(x)^{1/2}) = D(\mathfrak{U}(x))$ by [15] Thm. 2.23, taken $x \in \mathcal{A}_{int}$, we have $D(\mathfrak{U}(x)) = \mathcal{H}$ and then $\langle u, \mathcal{U}(x)u\rangle = \int_0^\infty \langle u, \mathcal{T}_s(x)u\rangle ds$ for all $u \in \mathcal{H}$.

We recall that a closed operator A is affiliated with a von Neumann algebra $\mathcal{A}$ if $a'D(A) \subseteq D(A)$ and $a'A \subseteq Aa'$ for all $a' \in \mathcal{A}'$.

Proposition 6. *Let $\mathcal{T}$ be a QMS and let $x \in \mathcal{A}$ be positive. Then the orthogonal projection onto the closure of $D(\mathfrak{U}(x))$ and the projection onto $\mathcal{K}(x) = \{u \in D(\mathfrak{U}(x)) :\ \mathfrak{U}(x)[u] = 0\}$ are subharmonic.*

In particular, if $\mathcal{T}$ is irreducible, then $D(\mathfrak{U}(x))$ is either dense or $\{0\}$.

We refer to [11], Prop. 2 and 4 for the proof.

The following Thm. extends some results in [11] to the general case, when $D(\mathfrak{U}(x))$ is not dense in $\mathcal{H}$.

Theorem 6. *Let x be a positive element of $\mathcal{A}$ with $D := D(\mathfrak{U}(x)) \neq \{0\}$; then there exists a positive self-adjoint operator X on the Hilbert space $\overline{D}$ with $D(X) \subseteq D$, $D(X^{1/2}) = D(\mathfrak{U}(x))$ and*

$$\langle u, Xu\rangle = \int_0^\infty \langle u, \mathcal{T}_t(x)u\rangle dt \quad \forall\, u \in D(X), \tag{2}$$

$$\|X^{1/2}u\|^2 = \int_0^\infty \langle u, \mathcal{T}_t(x)u\rangle dt \quad \forall\, u \in D(\mathfrak{U}(x)). \tag{3}$$

Moreover, if p is the orthogonal projection onto $\overline{D}$, then:

- (i) *X is affiliated with $p\mathcal{A}p$;*
- (ii) *if $\tilde{X}u := X(p+X)^{-1}pu$ for all $u \in \mathcal{H}$, then $\tilde{X}$ is superharmonic and $\mathcal{T}_t^p(\tilde{X})$ converges strongly to 0 as $t \to \infty$;*
- (iii) *if $X_t = \int_0^t \mathcal{T}_r(x)\,dr$ for all $t \geq 0$ and $\hat{x} := s - \lim_{t\to\infty} X_t(\mathbf{1} + X_y)^{-1}$, then $\hat{x}$ is superharmonic and $p\hat{x}p = \tilde{X}$;*
- (iv) *if $u \in D$, then $p\hat{x}pu = 0$ implies $\mathfrak{U}(x)[u] = 0$.*

Proof. The form $\mathfrak{Q}[u] := \int_0^\infty \langle u, \mathcal{T}_s(x)u\rangle ds$ for all $u \in D$ is a symmetric, positive and closed form on D. Therefore it is represented by a self-adjoint operator $X : D(X) \subseteq \overline{D} \to \overline{D}$ which satisfies (2) and (3).

(i) Fix $y \in (p\mathcal{A}p)'$ and define $\tilde{X}_t = pX_t p$ for all $t \geq 0$; clearly, both $\tilde{X}_t$ and ${\tilde{X}_t}^{1/2}$ belong to $p\mathcal{A}p$. Given any $u \in D$, we have

$$\int_0^t \langle yu, p\mathcal{T}_s(x)pyu\rangle ds = \langle yu, \tilde{X}_t yu\rangle = \langle y{\tilde{X}_t}^{1/2}u, y{\tilde{X}_t}^{1/2}u\rangle \leq \|y\|^2\langle u, \tilde{X}_t u\rangle.$$

Taking the supremum on $t \geq 0$, it follows that, if $u \in D(\mathfrak{U}(x)) = D(X^{1/2})$, then $pyu = yu \in D$.
Now, if $v, u \in D(X) \subseteq D$, then $y^*v,\ yu \in D$ and

$$\int_0^t \langle y^*v, p\mathcal{T}_s(x)pu\rangle ds = \int_0^t \langle p\mathcal{T}_s(x)py^*v, u\rangle ds = \int_0^t \langle p\mathcal{T}_s(x)pv, yu\rangle ds,$$

so that, letting $t \to \infty$, we get $\langle y^*v, Xu\rangle = \langle Xv, yu\rangle$. Hence, $\langle v, y\,Xu\rangle = \langle Xv, yu\rangle$ which implies $yu \in D(X)$ and $Xyu = y\,Xu$ by Thm. of representation (see [15]); therefore $y\,X \subseteq Xy$, i.e. X is affiliated with $p\mathcal{A}p$.

(ii) We first notice that $\tilde{X} = f(X)$ with $f(z) = \frac{z}{1+z}$ for all $z \geq 0$, so it belongs to $p\mathcal{A}p$ because f is bounded on $[0, \infty)$.
Since p is subharmonic (see Prop. 6.i), Thm. 2 implies

$$p\mathcal{T}_t\left(\tilde{X}_s\right)p = \int_0^s p\mathcal{T}_{t+r}(x)p\,dr = \int_t^{t+s} p\mathcal{T}_r(x)p\,dr = \tilde{X}_{t+s} - \tilde{X}_t \quad (4)$$

for any $s, t \geq 0$, so

$$p\mathcal{T}_t\left(\tilde{X}_s\right)p \leq \tilde{X}_{t+s}. \quad (5)$$

Since $p\mathcal{T}_t(\cdot)p$ is 2-positive and $p\mathcal{T}_t(p)p = p = \mathbf{1}_{p\mathcal{A}p}$, by Lemma 1.4.2 of [5] we get

$$\left(p + p\mathcal{T}_t\left(\tilde{X}_s\right)p\right)^{-1} \leq p\mathcal{T}_t\left((p + \tilde{X}_s)^{-1}\right)p.$$

From (5) we have then

$$\left(p + \tilde{X}_{t+s}\right)^{-1} \leq p\mathcal{T}_t\left((p + \tilde{X}_s)^{-1}\right)p.$$

Therefore

$$p\mathcal{T}_t\left(\tilde{X}_s(p + \tilde{X}_s)^{-1}\right)p = p - p\mathcal{T}_t\left((p + \tilde{X}_s)^{-1}\right)p \leq p - \left(p + \tilde{X}_{t+s}\right)^{-1}$$
$$= \tilde{X}_{t+s}\left(p + \tilde{X}_{t+s}\right)^{-1}.$$

Since $\tilde{X}_{t+s}u$ converges to Xu as $s \to \infty$ for all $u \in D(X)$ and $D(X)$ is dense in $p(\mathcal{H})$, Thms. 4.2, 4.5 of [17] imply that

$$\lim_s (p + \tilde{X}_{t+s})^{-1}u = (p + X)^{-1}u \tag{6}$$

for all $u \in \mathcal{H}$, $t \geq 0$; hence, letting $s \to \infty$, the inequality $p, \mathcal{T}_t(\tilde{X})p \leq \tilde{X}$ holds, because the map $\mathcal{T}_t(\cdot)$ is normal. Finally, since (4) implies

$$p\mathcal{T}_t\left(\tilde{X}_s(p + \tilde{X}_s)^{-1}\right)p \leq p\mathcal{T}_t\left(\tilde{X}_s\right)p = \tilde{X}_{t+s} - \tilde{X}_t,$$

for all $u \in D$ we get

$$\langle u, p\mathcal{T}_t\left(\tilde{X}_s(p + \tilde{X}_s)^{-1}\right)pu\rangle \leq \int_t^{t+s} \langle u, p\mathcal{T}_r(x)\, pu\rangle dr;$$

letting $s \to \infty$ again,

$$\langle u, p\mathcal{T}_t(\tilde{X})pu\rangle \leq \int_t^{\infty} \langle u, p\mathcal{T}_r(x)\, pu\rangle dr,$$

thus, $\langle u, p\mathcal{T}_t(\tilde{X})pu\rangle$ vanishes for all $u \in D$, as t goes to infinity. Since D is dense in $p(\mathcal{H})$ and the operators $p\mathcal{T}_t(\tilde{X})p$ are uniformly bounded in norm by $\|\tilde{X}\|$, the last statement follows.

(*iii*) Since the map $s \mapsto X_s(\mathbf{1} + X_s)^{-1} = \mathbf{1} - (\mathbf{1} + X_s)^{-1}$ is increasing and $X_s(\mathbf{1}+X_s)^{-1} \leq \mathbf{1}$ for all $s \geq 0$, there exists the strong limit $\hat{x}$ of $X_s(\mathbf{1}+X_s)^{-1}$; clearly, $\hat{x}$ belongs to $\mathcal{A}$ and it is positive. Moreover, arguing as in the above point, we can show that $\hat{x}$ is superharmonic.
Since $\tilde{X}_s$ is positive, $p + \tilde{X}_s$ is invertible on $p(\mathcal{H})$ and we have

$$p(\mathbf{1} + X_s)^{-1}u = p(p + \tilde{X}_s)^{-1}u \quad \forall\, u \in p(\mathcal{H}). \tag{7}$$

Indeed, if E_s is the spectral measure of X_s, $\tilde{E}_s(B)u := pE_s(B)u \in p(\mathcal{H})$ defines the spectral measure of $\tilde{X}_s$ for any set $B \in \mathcal{B}(\mathbb{R}_+)$ and $u \in p(\mathcal{H})$; hence we get

$$\begin{aligned}\langle u, p(\mathbf{1} + X_s)^{-1}u\rangle = \langle u, (\mathbf{1} + X_s)^{-1}u\rangle &= \int (1 + \lambda)^{-1} dE_{s_{u,u}}(\lambda) \\ &= \int (1 + \lambda)^{-1} d\tilde{E}_{s_{u,u}}(\lambda) = \langle u, (p + \tilde{X}_s)^{-1}u\rangle\end{aligned}$$

for all $u \in p(\mathcal{H})$, which is (7). Therefore, (6) implies that

$$\begin{aligned} p\hat{x}pu &= \lim_s pX_s(\mathbf{1} + X_s)^{-1}u = \lim_s(pu - p(\mathbf{1} + X_s)^{-1}u) \\ &= \lim_s(pu - p(p + \tilde{X}_s)^{-1}u) = pu - p(p + X)^{-1}u = pX(p+X)^{-1}u \end{aligned}$$

for all $u \in p(\mathcal{H})$, that is $p\hat{x}p = \tilde{X}$.

(*iv*) Let $u \in D$; then $p\hat{x}pu = X(p+X)^{-1}u$ by the above point. Hence, if E is the spectral measure of X, $p\hat{x}pu = 0$ means

$$0 = \langle u, X(p+X)^{-1}u\rangle = \int_0^\infty \lambda(1+\lambda)^{-1} dE_{u,u}(\lambda),$$

that is $E_{u,u} = \delta_0$; it follows that $0 = \|X^{1/2}u\|^2 = \mathfrak{U}(x)[u]$. □

Remark 1. When $D(\mathfrak{U}(x))$ is dense in $\mathcal{H}$, then $X = \mathcal{U}(x)$.

Theorem 7 (Riesz Decomposition). *Let x be a positive element in $\mathcal{A}$; if x is superharmonic and $\mathcal{T}_t(x)$ is weakly* convergent to 0 as $t \to \infty$, then for any $\lambda > 0$ there exists $y_\lambda \in \mathcal{A}_{int}$ such that $\mathcal{R}_\lambda(x) = \mathcal{U}(y_\lambda)$, where $\mathcal{R}_\lambda$ is the resolvent of $\mathcal{T}$.*

Proof. Let $\lambda > 0$; since x is superharmonic, we have $\mathcal{R}_\lambda(x) \le \lambda^{-1}x$ and $\mathcal{T}_t(\mathcal{R}_\lambda(x)) \le \mathcal{R}_\lambda(x)$ for all $t \ge 0$. It follows that w*- $\lim_t \mathcal{T}_t(\mathcal{R}_\lambda(x)) \le \lambda^{-1}$w*- $\lim_t \mathcal{T}_t(x) = 0$. Therefore, since for all $t \ge 0$

$$\int_0^t \mathcal{T}_s(-\mathcal{L}(\mathcal{R}_\lambda(x)))ds = \mathcal{R}_\lambda(x) - \mathcal{T}_t(\mathcal{R}_\lambda(x)),$$

getting $t \to \infty$, we obtain $\mathcal{U}(-\mathcal{L}(\mathcal{R}_\lambda(x))) = \mathcal{R}_\lambda(x)$, with $-\mathcal{L}(\mathcal{R}_\lambda(x)) \ge 0$ because $\mathcal{R}_\lambda(x)$ is superharmonic. We can then put $y_\lambda = -\mathcal{L}(\mathcal{R}_\lambda(x)) \in \mathcal{A}_{int}$. □

We introduce now the transient projection.

For each operator x on $\mathcal{H}$, we call *range projection* of x and denote it by $[x]$ the orthogonal projection onto the closure of $x(\mathcal{H})$; it is well-known that $x \in \mathcal{A}$ implies $[x] \in \mathcal{A}$.

Definition 5. *We call* transient projection *associated with the QMS $\mathcal{T}$ the projection p_T in $\mathcal{A}$ defined by $p_T := \sup\{\, p : p \in \mathcal{P}\,\}$ where*

$$\mathcal{P} := \{\, p \text{ projection in } \mathcal{A} : \exists\, x \in \mathcal{A}_{int} \text{ s.t. } p = [\mathcal{U}(x)]\,\}.$$

The transient projection is orthogonal to the supports of normal invariant states, that is

Proposition 7. *We have $p_T \le p_R^\perp$.*

Proof. Given $p = [\mathcal{U}(x)]$ with $x \in \mathcal{A}_{int}$ and ω a normal invariant state, we have

$$\omega(\mathcal{U}(x)) = \int_0^\infty \omega(\mathcal{T}_s(x))ds = \int_0^\infty \omega(x)ds,$$

which implies $\omega(\mathcal{U}(x)) = 0$. Since ω is faithful on the subalgebra $s(\omega)\mathcal{A}\,s(\omega)$, this means that $s(\omega)\mathcal{U}(x) = 0$ by Lemma 1, i.e. $\overline{\mathcal{U}(x)(\mathcal{H})} \subseteq \ker s(\omega)$; then $p(\mathcal{H}) \subseteq \ker p_R$, so $p \le p_R^\perp$ for all $p \in \mathcal{P}$, which implies $p_T \le p_R^\perp$. □

By Prop. 6 any projection p in $\mathcal{P}$ is superharmonic, but it is not clear whether the supremum of a family of superharmonic projections is still superharmonic. However, we will prove that p_T is superharmonic when $\mathcal{A}$ is σ-finite.

Lemma 4. *If $e \in p_T(\mathcal{H})$, then there exists $x \in \mathcal{A}_{int}$ such that $e \in \mathrm{Ran}(\mathcal{U}(x))$.*

Proof. Since $p_T(\mathcal{H})$ is the closure of the union of $p(\mathcal{H})$ as $p \in \mathcal{P}$, for any $n \geq 1$ there exists $u_n \in p_n(\mathcal{H})$, $p_n \in \mathcal{P}$, such that $\|e - u_n\| < n^{-1}$; suppose $p_n = [\mathcal{U}(x_n)]$ with $x_n \in \mathcal{A}_{int}$ for all $n \geq 1$ and put

$$x := \sum_{n\geq 1} 2^{-n}(\|x_n\| + \|\mathcal{U}(x_n)\| + 1)^{-1} x_n.$$

Then $x \in \mathcal{A}_{int}$ and $\ker \mathcal{U}(x) = \bigcap_{n\geq 1} \ker \mathcal{U}(x_n)$; so, if we define $p = \sup_n p_n$, we have $p = [\mathcal{U}(x)]$. Moreover, since $u_n \in p_n(\mathcal{H})$ and $p_n \leq p$ for all $n \geq 1$, we get

$$\|e - pe\| \leq \|e - u_n\| + \|pu_n - pe\| < n^{-1} + \|u_n - e\| < 2n^{-1}$$

for all $n \geq 1$, which implies $pe = e$, that is $e \in p(\mathcal{H})$. □

Theorem 8. *If $\mathcal{A}$ is σ-finite, there exists an increasing sequence $(p_n)_{n\geq 0}$ in $\mathcal{P}$ such that $p_T = \sup_{n\geq 0} p_n$. Moreover $p_T \in \mathcal{P}$.*

Proof. Let $\{e_n\}_{n\geq 0}$ be a countable subset of $\mathcal{H}$ which is separating for $\mathcal{A}$ and suppose $p_T e_n \in \overline{\mathcal{U}(x_n)(\mathcal{H})}$ for some $x_n \in \mathcal{A}_{int}$, $n \geq 0$ (see Lemma 4). Define $y_n := \sum_{k=0}^{n} x_k$ and put $p_n := [\mathcal{U}(y_n)]$ for all $n \geq 0$; then any y_n belongs to $\mathcal{A}_{int}$ and $(p_n)_{n\geq 0}$ is an increasing sequence in $\mathcal{P}$. Moreover, since

$$\ker \mathcal{U}(y_n) = \cap_{k=0}^{n} \ker \mathcal{U}(x_n)$$

for all $n \geq 0$, we have $p_T e_n \in \overline{\mathcal{U}(x_n)(\mathcal{H})} \subseteq \overline{\mathcal{U}(y_n)(\mathcal{H})} = p_n(\mathcal{H})$, which implies $(p_T - \sup_{m\geq 0} p_m) p_T e_n = 0$ for all $n \geq 0$, so $p_T = \sup_{n\geq 0} p_n$ because $\{p_T e_n\}_{n\geq 0}$ is separating for $p_T \mathcal{A} p_T$ and $p_T - \sup_{n\geq 0} p_n \in p_T \mathcal{A} p_T$.

Finally, if

$$y := \sum_{n\geq 0} 2^{-n}(\|y_n\| + \|\mathcal{U}(y_n)\| + 1)^{-1} y_n,$$

it is clear that $y \in \mathcal{A}_{int}$ and $\ker \mathcal{U}(y) = \cap_{n\geq 0} \ker \mathcal{U}(y_n) = \ker p_T$, so that $[\mathcal{U}(y)] = p_T$, i.e. $p_T \in \mathcal{P}$. □

Corollary 2. *If $\mathcal{A}$ is σ-finite, then the transient projection p_T is superharmonic. In particular, the subalgebra $p_T \mathcal{A} p_T$ is $\mathcal{T}_t$-invariant for all $t \geq 0$.*

Proof. By Thm. 8 we have $p_T = \mathrm{w}^*\text{-}\lim_n p_n$, $p_n \in \mathcal{P}$ for all $n \geq 0$; since any p_n satisfies $\mathcal{T}_t(p_n) \leq p_n \leq p_T$ for all $t \geq 0$, letting $n \to \infty$ the inequality $\mathcal{T}_t(p_T) \leq p_T$ holds for all $t \geq 0$.

Finally, if x is a positive element of $\mathcal{A}$, $x = p_T x p_T$, we have $x \leq \|x\| p_T$, so $0 \leq p_T^{\perp} \mathcal{T}_t(x) p_T^{\perp} = 0$ for all $t \geq 0$, because p_T is superharmonic; it follows then by Lemma 1 that $\mathcal{T}_t(x) = p_T \mathcal{T}_t(x) p_T$ for all $t \geq 0$, i.e. any $\mathcal{T}_t(x)$ belongs to $p_T \mathcal{A} p_T$. □

We can then consider the restriction of $\mathcal{T}$ to the subalgebra $p_T \mathcal{A} p_T$. Notice that, if $(p_n)_{n\geq 0}$ is a sequence of projections as in Thm. 8, then the map $t \mapsto \langle u, \mathcal{T}_t(p_n) u\rangle$ is integrable on $[0, \infty)$ for all $u \in \mathcal{H}$; this implies that $\mathcal{T}_t(p_n)$ is strongly convergent to 0 as $t \to \infty$. Therefore, we have the following

Corollary 3. *If $\mathcal{A}$ is σ-finite, then the restriction of $\mathcal{T}$ to $p_T \mathcal{A} p_T$ has no normal invariant states.*

Proof. If ω is a normal invariant state on $p_T \mathcal{A} p_T$, then $\lim_n \omega(p_n) = \omega(p_T) = 1$ for every $t \geq 0$, so there exists $m > 0$ such that $\omega(\mathcal{T}_t(p_m)) = \omega(p_m) > 1/2$ for all $t \geq 0$. Since $\mathcal{T}_t(p_m)$ is uniformly bounded in t and converges strongly to 0 as $t \to \infty$, we get the contradiction $0 > 1/2$. □

Proposition 8. *If $\mathcal{A}$ is σ-finite, then*

$$p_T = \{p \text{ projection of } \mathcal{A} : p \in \mathcal{A}_{int}\}.$$

Proof. By Thm. 8 we have $p_T = [\mathcal{U}(x)]$ with $x \in \mathcal{A}_{int}$.
Fix $\lambda > 0$ and put $y = \mathcal{R}_\lambda(x)$. It is easy to see that $y \in \mathcal{A}_{int}$ and $[\mathcal{U}(x)] = [y]$; therefore, if $p_n := E^y(]\frac{1}{n}, \|x\|])$, we get that $p_n \leq ny$, so that it belongs to $\mathcal{A}_{int}$ for all $n \geq 1$ and $\sup p_n = [y] = p_T$. □

5. Decomposition of QMS

In this section we define in first the slow recurrent projection p_{R_0} and introduce the transient, fast and slow recurrent semigroups in terms of p_T, p_R, p_{R_0}; we will show that it is possible to decompose a QMS as the sum of a transient and a recurrent part.

Definition 6. *The projection $p_{R_0} = p_R^\perp - p_T$ is called* slow recurrent projection *associated with the QMS $\mathcal{T}$.*

Definition 7. *We call a QMS $\mathcal{T}$* transient *if $p_T = \mathbf{1}$,* recurrent *if $p_T = 0$,* fast recurrent *if $p_R = \mathbf{1}$ and* slow recurrent *if $p_{R_0} = \mathbf{1}$.*

Notice that we can also define p_T, p_R and p_{R_0} for a QDS $\mathcal{T}$ on $\mathcal{A}$ such that $\mathcal{T}_t(\mathbf{1}) \leq \mathbf{1}$ for all $t \geq 0$; since it is easy to check that this projections satisfy the same properties, we can introduce the concepts of transience and recurrence for such semigroup too.

Prop. 8 implies that, when the von Neumdun algebra $\mathcal{A}$ is σ-finite, the definition of transient QMS is equivalent with the one given in [11]; instead, it is not yet clear if the same holds for the recurrent QMS. In order to prove this, starting with a positive element x such that $\mathfrak{U}(x)[u] > 0$ for some $u \in D(\mathfrak{U}(x))$ we have to construct a non-zero integrable element. This is done in the following

Lemma 5. *Let x be a positive non-zero element in $\mathcal{A}$ such that there exists $u_0 \in D(\mathfrak{U}(x))$ with $\mathfrak{U}(x)[u_0] > 0$ and let $X_t = \int_0^t \mathcal{T}_r(x)\,dr$ for all $r \geq 0$. If $y = s - \lim_{t\to\infty} X_t(\mathbf{1} + X_t)^{-1}$, then $z = -\mathcal{L}(\mathcal{R}_\lambda(y))$ is non-zero and integrable for all $\lambda > 0$.*

Proof. Since y is superharmonic by Thm. 6.iii, $\mathcal{R}_\lambda(y)$ is also superharmonic ($\lambda > 0$); therefore, $z := -\mathcal{L}(\mathcal{R}_\lambda(y))$ is positive and

$$\int_0^t \mathcal{T}_s(z)ds = \int_0^t \mathcal{T}_s(-\mathcal{L}(\mathcal{R}_\lambda(y)))ds = \mathcal{R}_\lambda(y) - \mathcal{T}_t(\mathcal{R}_\lambda(y)) \leq \mathcal{R}_\lambda(y)$$

for all $t \geq 0$, that is z is integrable (letting $t \to \infty$). To conclude the proof it is enough to prove that $z \neq 0$.

If $z = 0$, we have $\mathcal{T}_t(\mathcal{R}_\lambda(y)) = \mathcal{R}_\lambda(y)$ for all $t \geq 0$; since the map $t \mapsto e^{-\lambda s}\mathcal{T}_s(y - \mathcal{T}_t(y))$ is positive and continuous, this means that y is harmonic. Therefore, if $D := D(\mathfrak{U}(x))$ and p is the orthogonal projection onto the closure of D, $p\mathcal{T}_t(pyp)p = p\mathcal{T}_t(y)p = pyp$ holds for all $t \geq 0$, p being subharmonic by Prop. 6. But $p\mathcal{T}_t(pyp)p$ converges strongly to 0 as $t \to \infty$ by Thm. 6.*ii* and *iii*, so that $pyp = 0$; by virtue of *iv* of the same Thm., this implies $\mathfrak{U}(x)[u] = 0$ for all $u \in D$, which is a contradiction since $\mathfrak{U}(x)[u_0] > 0$ for $u_0 \in D(\mathfrak{U}(x)) = D$. □

Proposition 9. *A QMS $\mathcal{T}$ is recurrent if and only if for each positive $x \in \mathcal{A}$ and $u \in \mathcal{H}$ either $u \notin D(\mathfrak{U}(x))$ or $u \in D(\mathfrak{U}(x))$ and $\mathfrak{U}(x)[u] = 0$.*

Proof. $\Leftarrow$) It is trivial since we have $\mathcal{A}_{int} = \{0\}$, so $\mathcal{P} = \{0\}$ and $p_T = 0$.
$\Rightarrow$) If there exist a positive element x in $\mathcal{A}$, $x \neq 0$, and $u \in D(\mathfrak{U}(x))$ such that $\mathfrak{U}(x)[u] > 0$, by Lemma 5 we can find $z \in \mathcal{A}_{int}$, $z \neq 0$. Therefore $0 \leq [z] \leq [\mathcal{U}(z)] \leq p_T = 0$ which is a contradiction. □

As a consequence, definitions 7.2 and 3 of [11] are equivalent.

In general, a QMS $\mathcal{T}$ is not type 1, 2, 3, 4, but, if $\mathcal{A}$ is σ-finite, we can write it as sum of a transient QDS (which is the restriction of $\mathcal{T}$ to the subalgebra $p_T\mathcal{A}p_T$) and a recurrent QMS (which is the reduced semigroup $\mathcal{T}^{p_T^\perp}$ associated to the subharmonic projection $p_T^\perp$). To see it, we show the following

Proposition 10. *For all positive $x \in p_T^\perp\mathcal{A}p_T^\perp$ and $u \in \mathcal{H}$ we have either $p_T^\perp u \notin D(\mathfrak{U}(x))$ or $p_T^\perp u \in D(\mathfrak{U}(x))$ and $\mathfrak{U}(x)[p_T^\perp u] = 0$.*

Proof. Suppose that there exist a positive x in $p_T^\perp\mathcal{A}p_T^\perp$ and $u \in \mathcal{H}$ such that $p_T^\perp u \in D(\mathfrak{U}(x))$ and $\mathfrak{U}(x)[p_T^\perp u] > 0$; we can then also assume that u is a non zero element of $p_T^\perp(\mathcal{H})$. If $D = D(\mathfrak{U}(x))$ and p is the orthogonal projection onto $\overline{D}$, by Lemma 5 we can find $z \in \mathcal{A}_{int} \setminus \{0\}$, $z = -\mathcal{L}(\mathcal{R}_\lambda(y))$, where the superharmonic element y is the strong limit of $X_s(\mathbf{1} + X_s)^{-1}$, $X_s = \int_0^s \mathcal{T}_h(x)dh$.

Since $-\mathcal{L}(\mathcal{R}_\lambda(y)) = (\lambda - \mathcal{L} - \lambda)(\lambda - \mathcal{L})^{-1}y = y - \lambda\mathcal{R}_\lambda(y)$ belongs to $\mathcal{A}_{int}$, then $[y - \lambda\mathcal{R}_\lambda(y)] \leq [\mathcal{U}(y - \lambda\mathcal{R}_\lambda(y))] \leq p_T$, that is $(y - \lambda\mathcal{R}_\lambda(y))p_T^\perp = 0$; but

$$0 = \langle v, (y - \lambda\mathcal{R}_\lambda(y))v\rangle = \int_0^\infty \lambda e^{-\lambda s}\langle v, (y - \mathcal{T}_s(y))v\rangle ds \quad \forall\, v \in p_T^\perp(\mathcal{H})$$

means $(y - \mathcal{T}_s(y))v = 0$ for all $v \in p_T^\perp(\mathcal{H})$ and $s \geq 0$, i.e. $(y - \mathcal{T}_s(y))p_T^\perp = 0$. In particular

$$pypu = pyp_T^\perp pu = p\mathcal{T}_t(y)p_T^\perp pu = p\mathcal{T}_t(y)pu = p\mathcal{T}_t(pyp)pu$$

holds for all $t \geq 0$, where $p_T^\perp pu = u = pu$ (since $u = p_T^\perp u \in D \subseteq p(\mathcal{H})$) and $\mathcal{T}_t(p) \leq p$ have been used. Letting $t \to \infty$, by virtue of Thm. 6.*ii* and *iii* we get $pypu = 0$, and this implies the contradiction $\mathfrak{U}(x)[u] = 0$ by *iv* of the same Thm.
□

Theorem 9. *If $\mathcal{A}$ is σ-finite and $\mathcal{T}$ is a QMS on $\mathcal{A}$, then the restriction of $\mathcal{T}$ to $p_T\mathcal{A}p_T$ is a transient QDS on $p_T\mathcal{A}p_T$ while $\mathcal{T}^{p_T^\perp}$ is a recurrent QMS on $p_T^\perp\mathcal{A}p_T^\perp$. Moreover $\mathcal{T}^{p_T^\perp}$ contains the fast recurrent "sub"-QMS $\mathcal{T}^{p_R}$ on $p_R\mathcal{A}p_R$.*

Proof. $\mathcal{T}_{|p_T\mathcal{A}p_T}$ is transient by Thm. 8 and Prop. 8; since the form-potential of any positive $x \in p_T^\perp\mathcal{A}p_T^\perp$ is

$$\int_0^\infty \langle u, \mathcal{T}_s^{p_T^\perp}(x)u\rangle ds = \int_0^\infty \langle p_T^\perp u, \mathcal{T}_s(x)p_T^\perp u\rangle ds = \mathfrak{U}(x)[p_T^\perp u]$$

for all $u \in \mathcal{H}$ such that this integral is convergent, we can also conclude that $\mathcal{T}^{p_T^\perp}$ is recurrent by Prop. 9 and 10. Finally, since any normal $\mathcal{T}$-invariant state belongs to $p_R\mathcal{A}_*p_R$ (because its support is $\leq p_R$) and it is also $\mathcal{T}^{p_R}$-invariant, we get $\sup\{s(\omega) : \omega \in \mathcal{F}(\mathcal{T}_*^{p_R})_1\} = p_R$, so that the last statement follows. □

Instead, it is not yet clear if we can decompose $\mathcal{T}^{p_T^\perp}$ as sum of a fast and a slow recurrent semigroup.

We now study better the evolution of a pure state φ_u defined by a density matrix $\rho = |u\rangle\langle u|$ with $u \in p(\mathcal{H})$, $\|u\| = 1$, and $p \in \{p_T, p_R, p_{R_0}\}$.

Notice that, since the map $t \mapsto \langle u, \mathcal{T}_t(p)u\rangle$ is positive and continuous on $[0, +\infty)$, we have $\mathfrak{U}(p)[u] > 0$ when u belongs to the range of p.

The following statement is immediate

Proposition 11. *Suppose $\mathcal{A}$ σ-finite. If $\mathcal{T}$ is a QMS on $\mathcal{A}$, then:*

1. *$\mathfrak{U}(p_T)[u] = 0$ for $u \in p_T^\perp(\mathcal{H})$;*
2. *if $u \in p_T^\perp(\mathcal{H})$, then $u \notin D(\mathfrak{U}(p_T^\perp))$;*
3. *if $u \in p_R(\mathcal{H})$, then $u \notin D(\mathfrak{U}(p_R))$;*
4. *for $u \in p_R^\perp(\mathcal{H})$ either $u \notin D(\mathfrak{U}(p_R))$ or $u \in D(\mathfrak{U}(p_R))$ and $\mathfrak{U}(p_R)[u] = 0$;*
5. *for $u \in p_T(\mathcal{H})$ either $u \notin D(\mathfrak{U}(p_T^\perp))$ or $u \in D(\mathfrak{U}(p_T^\perp))$ and $\mathfrak{U}(p_T^\perp)[u] = 0$;*
6. *$\mathfrak{U}(p_{R_0})[u] = 0$ for $u \in p_R(\mathcal{H})$;*
7. *if $u \in p_{R_0}(\mathcal{H})$, then $u \notin D(\mathfrak{U}(p_{R_0}))$.*

Proof. 1, 2, 3 are trivial because p_T is superharmonic while p_R and $p_T^\perp$ are subharmonic.

4. Let $u \in p_R^\perp(\mathcal{H}) \cap D(\mathfrak{U}(p_R))$ and show that $\mathfrak{U}(p_R)[u] = 0$. If $\mathfrak{U}(p_R)[u] > 0$ and we let $\omega_u = tr(|u\rangle\langle u|\cdot)$, there exists $t_u > 0$ s.t. $(\mathcal{T}_{*t_u}(\omega_u))(p_R) = \langle u, \mathcal{T}_{t_u}(p_R)u\rangle > 0$. The subharmonicity of p_R implies
$$\mathfrak{U}(p_R)[u] \geq \int_0^\infty (\mathcal{T}_{*t_u}(\omega_u))\mathcal{T}_s(p_R)ds \geq \int_0^\infty \mathcal{T}_{*t_u}(\omega_u)(p_R)ds = +\infty,$$
so that $u \notin D(\mathfrak{U}(p_R))$, which is a contradiction.
5. It is enough to argue as in 4, for $p_T^\perp$ is also subharmonic.
6. It is clear because $p_{R_0} \leq p_R^\perp$ and $p_R^\perp$ is superharmonic.
7. It follows by Prop. 10. □

Remark 2. The hypotesis "$\mathcal{A}$ σ-finite" was used only in the proof of 1.

Writing, for any projection p in $\mathcal{A}$, $\mathfrak{U}(p)[u] = +\infty$ when $u \notin D(\mathfrak{U}(p))$, if $\mathcal{A}$ is σ-finite we can then summarize the situation in this way:

	$\tau(p_T)[u]$	$\tau(p_T^{\perp})[u]$
$u \in p_T(\mathcal{H})$	$l > 0, +\infty$	$0, +\infty$
$u \in p_T^{\perp}(\mathcal{H})$	0	$+\infty$

where $\tau(p)[u] := \mathfrak{U}(p)[u]$ and the norm one vector u belongs either to $p_T(\mathcal{H})$ or $p_T^{\perp}(\mathcal{H})$. Since for all projections $p \in \mathcal{A}$ $\mathfrak{U}(p)[u] = \int_0^{\infty}\langle u, \mathcal{T}_s(p)u\rangle ds$ represents the time of sojourn of the state $tr(|u\rangle\langle u|\cdot)$ ($\|u\| = 1$) in p (see [11]) and any normal state ω is defined by a density matrix $\sum_k \lambda_k|e_k\rangle\langle e_k|$ with $e_k \in s(\omega)(\mathcal{H})$, the table should be read as follows:

- starting from a transient (support in $p_T\mathcal{A}p_T$) state, the semigroup $\mathcal{T}_*$ spends a finite or an infinite amount of time in p_T but, if it leaves p_T to come into $p_T^{\perp}$, (i.e. its support is in $p_T^{\perp}\mathcal{A}p_T^{\perp}$), it stays there forever;
- starting from a recurrent state, the semigroup $\mathcal{T}_*$ cannot leave $p_T^{\perp}$.

Moreover, by Prop. 11.iv and vii, we have that:

- starting from a slow recurrent (support in $p_{R_0}\mathcal{A}p_{R_0}$) state, the system spends an infinite amount of time in p_{R_0}, it cannot enter in p_T, but it can spend a null or an infinite amount of time in p_R;
- starting from a fast recurrent state, the semigroup $\mathcal{T}_*$ cannot leave p_R.

It is not clear if, starting from a transient state, the system can spend a finite amount of time in p_{R_0}.

6. The finite-dimensional case

In this section we suppose that $\mathcal{A}$ acts on a finite-dimensional Hilbert space $\mathcal{H}$ and analyze the properties of the recurrent and transient projections.

As for the Markov chains with a finite state space, we are going to show that $p_R \neq 0$ and $p_{R_0} = 0$. Moreover, p_T is integrable.

Notice that, if $\dim\mathcal{H} < +\infty$, then $\mathcal{T}$ has an invariant state by the Markov-Kakutani Theorem. Therefore, we have the following

Lemma 6. *If* $\dim\mathcal{H} < +\infty$ *and* $\mathcal{T}$ *is a QMS on* $\mathcal{A}$, *then* $p_R \neq 0$.

Lemma 7. *If* $\dim\mathcal{H} < +\infty$ *and* $\mathcal{T}$ *is a QMS on* $\mathcal{A}$, *then* $p_R^{\perp} \in \mathcal{A}_{int}$. *In particular, the net* $\{\mathcal{T}_t(p_R^{\perp})\}_t$ *is convergent to* 0 *as* t *goes to* ∞.

Proof. Since $\{\mathcal{T}_t(p_R^{\perp})\}_{t\geq 0}$ is a positive decreasing net in $p_R^{\perp}\mathcal{A}p_R^{\perp}$ ($p_R^{\perp}$ is superharmonic), it is convergent toward a positive element $x_0 \in p_R^{\perp}\mathcal{A}p_R^{\perp}$ which is clearly harmonic. Let

$$\mathcal{S}_n := \frac{1}{n}\sum_{k=1}^{n}\mathcal{T}_{*k} \quad (n \geq 1),$$

passing to subsequences if necessary, we can suppose that $\{\mathcal{S}_n(\omega)\}_n$ is convergent for all $\omega \in \mathcal{A}_* = \mathcal{A}^*$; if ω is a normal state on $\mathcal{A}$ and $\mathcal{S}(\omega) = \lim_n \mathcal{S}_n(\omega)$, we have $\mathcal{S}(\omega) \in \mathcal{F}(\mathcal{T}_*)_+$, so $s(\mathcal{S}(\omega)) \leq p_R$. Hence

$$\omega(x_0) = \lim_n \frac{1}{n} \sum_{k=1}^{n} \omega(\mathcal{T}_1^k(x_0)) = \mathcal{S}(\omega)(x_0) = \mathcal{S}(\omega)(s(\mathcal{S}(\omega))x_0) = 0,$$

which implies $x_0 = 0$. Since $\mathcal{H}$ is finite-dimensional, this means that $\mathcal{T}_t(p_R^\perp)$ is norm-convergent to 0, and then $\|\mathcal{T}_{t_0}(p_R^\perp)\| < 1$ for some $t_0 > 0$; therefore, we get $\|\mathcal{T}_t(p_R^\perp)\| \leq \|\mathcal{T}_{t_0}(p_R^\perp)\| < 1$ for all $t \geq t_0$, so that

$$\int_0^\infty \|\mathcal{T}_t(p_R^\perp)\| dt \leq t_0 + \int_{t_0}^\infty \|\mathcal{T}_t(p_R^\perp)\| dt < \infty,$$

i.e. $p_R^\perp \in \mathcal{A}_{int}$. □

Theorem 10. *If* $\dim \mathcal{H} < +\infty$ *and* $\mathcal{T}$ *is a QMS on* $\mathcal{A}$*, then* $p_{R_0} = 0$.

Proof. By Lemma 7 it follows that w*- $\lim_{t\to\infty} \mathcal{T}_t(p_R^\perp) = 0$, so $\mathcal{R}_\lambda(p_R^\perp) = \mathcal{U}(y_\lambda)$ for some $y_\lambda \in \mathcal{A}_{int}$ by Thm. 7.

Therefore we have $p_R^\perp \leq [\mathcal{R}_\lambda(p_R^\perp)] \leq p_T \leq p_R^\perp$, i.e $p_{R_0} = 0$. □

Corollary 4. *If* $\dim \mathcal{H} < +\infty$ *and* $\mathcal{T}$ *is a QMS on* $\mathcal{A}$*, then* $p_T \in \mathcal{A}_{int}$.

The following Prop. is useful to see when a superharmonic projection is integrable.

Proposition 12. *Let* $p \in \mathcal{A}$ *be a superharmonic projection such that there exist* $s, \epsilon > 0$ *with* $p - \mathcal{T}_s(p) \geq \epsilon p$*; then* $\|\mathcal{T}_{s|_{p\mathcal{A}p}}\| < 1$ *and*

$$\int_0^\infty \|\mathcal{T}_{t|_{p\mathcal{A}p}}\| dt < \infty. \tag{8}$$

In particular $p \in \mathcal{A}_{int}$.

Proof. We set $S_t := \mathcal{T}_{t|_{p\mathcal{A}p}}$ for all $t \geq 0$; then $\{S_t\}_{t\geq 0}$ is a QDS on $p\mathcal{A}p$, because p is superharmonic. Moreover

$$\|S_s\| = \|S_s(p)\| \leq 1 - \epsilon < 1.$$

Finally, since we can write any $t \geq 0$ as $t = k_t s + r$ with $k_t = [t/s]$ and $0 \leq r < s$, we have

$$\int_0^\infty \|S_t\| dt \leq \int_0^\infty \|S_s\|^{k_t} dt = \int_0^\infty \|S_s\|^{\frac{t-r}{s}} dt = \int_0^\infty e^{\frac{t-r}{s} \ln \|S_s\|} dt;$$

but $\ln \|S_s\| < 0$ because $\|S_s\| < 1$, and $t - r \geq 0$ (indeed if $t < s$, $r = t$, otherwise $r < s \leq t$), so (8) follows.

The last statement is a trivial consequence. □

7. Examples

1) We follow [1]. Let $\mathcal{H} := \ell^2(\mathbb{N})$ and $\mathcal{A} = \mathcal{B}(\mathcal{H})$. If $\{e_j\}_{j\geq 0}$ is an orthonormal basis of $\mathcal{H}$, we define

$$\mathcal{L}(x) = -\frac{1}{2}\sum_{\{h,k:h<k\}} \gamma_{k,h}^- \left(\{|e_k\rangle\langle e_k|, x\} - 2|e_k\rangle\langle e_h|\, x\, |e_h\rangle\langle e_k|\right) + i[H, x]$$

for all $x \in \mathcal{A}$, where $\{\cdot,\cdot\}$ denotes the anticommutant, $H = \sum_{j\geq 0}\delta_j|e_j\rangle\langle e_j|$ and the positive constants δ_j, $\gamma_{k,h}^-$ $(h, k, j \geq 0,\ h < k)$ satisfy

$$\sup_{j\geq 0}\delta_j < \infty, \quad \sup_{k\geq 0}\sum_{h<k}\gamma_{k,h}^- < \infty. \tag{9}$$

Since (9) ensure its boundedness, $\mathcal{L}$ is the generator of a uniformly continuous semigroup $\mathcal{T}$; moreover, the operators

$$G = -\frac{1}{2}\sum_{k\geq 0}\sum_{h<k}\gamma_{k,h}^-|e_k\rangle\langle e_k| - iH, \quad L_{h,k} = \sqrt{\gamma_{k,h}^-}|e_h\rangle\langle e_k|$$

if $h < k$ and $L_{h,k} = 0$ if $h \geq k$, are bounded by (9), so that $\mathcal{L}$ can be represented in the Lindblad form and so $\mathcal{T}$ is a QDS on $\mathcal{A}$. Finally, it is Markov because $\mathcal{L}(\mathbf{1}) = 0$.
If we suppose for simplicity that $\gamma_{k,k+1}^- > 0$ for all $k \geq 0$, then $p_R = |e_0\rangle\langle e_0|$. Namely, by $\mathcal{L}_*(|e_0\rangle\langle e_0|) = 0$ follows immediately that $|e_0\rangle\langle e_0| \leq p_R$; to prove the conversely, we find the subharmonic projections of $\mathcal{T}$. If p is a non trivial subharmonic projection, then it fulfills $p^\perp L_{h,k}p = 0$ for all $h < k$ by Thm. III.1 of [10], so, in particular, we have either $pe_k = 0$ or $p^\perp e_{k-1} = 0$, for $\gamma_{k,k-1} > 0$. It is easy to check that this means $p = p_d$ for some $d \geq 0$, where $p_d := \sum_{j=0}^d |e_j\rangle\langle e_j|$; on the other hand, since any p_d satisfy also $p_d^\perp G p_d = 0$, the set of subhamonic projections is $\{0, p_d : d \geq 0\}$ by Thm. III.1 and Lemma III.1 of [10].
If p_d is the support of a normal invariant state for some $d \geq 1$, by Prop. 2 we get $\sum_{s=0}^{d-1}\gamma_{d,s}^-|e_d\rangle\langle e_d| = (p_d - p_{d-1})\mathcal{L}(p_{d-1})(p_d - p_{d-1}) = 0$, but this is impossible since $\gamma_{d,d-1}^- > 0$. Therefore, the only projection which can be support of an invariant normal state is $p_0 = |e_0\rangle\langle e_0|$, i.e. $p_R = |e_0\rangle\langle e_0|$. We want now to prove that $p_{R_0} = 0$.
Let $\mathcal{T}^d$ be the reduced semigroup associated with p_d, $d \geq 0$, and let p_R^d, p_T^d be the fast recurrent and the transient projection of $\mathcal{T}^d$ respectively; since $\mathcal{T}^d$ is a QMS on $p_d\mathcal{A}p_d$ which acts on the finite dimensional Hilbert space $p_d(\mathcal{H}) = \ell^\infty(\{1, \ldots, d\})$, by Lemma 7 and Thm. 10 it follows $p_T^d = (p_R^d)^\perp$ and $\mathcal{T}_t^d(p_R^d) \nearrow p_d$ for all $d \geq 0$. But $p_R^d = p_R = |e_0\rangle\langle e_0|$ for all $d \geq 0$ because any $\mathcal{T}^d$-invariant state is clearly also $\mathcal{T}$-invariant, so $\mathcal{T}_t^d(p_R) \nearrow p_d$; therefore, if $x := \text{w}^*\text{-}\lim_t \mathcal{T}_t(p_R)$, $p_d x p_d = \text{w}^*\text{-}\lim_t p_d\mathcal{T}_t(p_R)p_d = \text{w}^*\text{-}\lim_t \mathcal{T}_t^d(p_R) = p_d$ holds. Letting $d \to \infty$ it is easy to show that this means $x = \mathbf{1}$, i.e. $\mathcal{T}_t(p_R) \nearrow \mathbf{1}$. We can then conclude that $p_T = p_R^\perp$ as in the proof of Thm. 10.

The second example is a model for an atom with two-degenerate levels; it is given in [3] and its greater complexity lies in the fact that is not easy to find the invariant states.

2) Let $\mathcal{H} = \mathbb{C}^{2F_-+1} \oplus \mathbb{C}^{2F_++1}$, $F_+ = F_- + 1$, $2F_- \in \mathbb{N}$. We denote by $\{e_j^{\pm}\}_{j=-F_\pm,\ldots,F_\pm}$ the orthonormal basis of $\mathcal{H}$, where $e_l^- = (e_{l+F_-+1}, 0)$ for $l = -F_-, \ldots, F_-$, $e_k^+ = (0, e_{k+F_++1})$ for $k = -F_+, \ldots, F_+$ and $\{e_i\}_{i=1,\ldots,4F_-+4}$ is the canonical basis of $\mathbb{C}^{4F_-+4}$. Let us denote by $P_\pm$ the projections onto $\mathbb{C}^{2F_\pm+1}$, $P_\pm = \sum_{j=-F_\pm}^{F_\pm} |e_j^\pm\rangle\langle e_j^\pm|$.
If $\mathcal{K}$ is a separable Hilbert space with orthonormal basis $\{z_k\}_{k\geq 1}$, we define an operator on $\mathcal{A} = \mathcal{B}(\mathcal{H})$ by

$$\begin{aligned}\mathcal{L}_B(x) = &\frac{B}{8}\left([(P_+ - P_-)x, P_+ - P_-] + [P_+ - P_-, x(P_+ - P_-)] + i[H, x]\right)\\ &+\frac{1}{2}\sum_{k\geq 1}\left(2D(z_k)^* x D(z_k) - xD(z_k)^* D(z_k) - D(z_k)^* D(z_k)x\right),\end{aligned}$$

where

$$Q_m = \sum_{l=-F_-}^{F_-} c_{l,m}|e_l^-\rangle\langle e_{l+m}^+| \quad (m = -1, 0, 1),$$

$$D(z_k) = \sum_{m=-1}^{1} \alpha_{k,m} Q_m + \Omega e^{i\delta}(\beta_{k,+}P_+ + \beta_{k,-}P_-),$$

$$H = \frac{\gamma}{2}(P_+ - P_-) + \frac{i}{4}\Omega\left[e^{i\delta}(1 + e^{2i\delta_-})Q_1^* - e^{-i\delta}(1 + e^{-2i\delta_-})Q_1\right],$$

$B, \Omega, \delta \geq 0$, $\delta_\pm \in [0, 2\pi)$ and the complex constants $c_{l,m}, \alpha_{k,m}, \beta_{k,\pm}$ satisfy

1. $\sum_{k\geq 1} \beta_{k,-}\overline{\alpha_{k,m}} = 0$ for $m = -1, 0$, $\operatorname{rank}(\{\alpha_{k,m}\}_{m=-1,0,1,\ k\geq 1}) = 3$;
2. $\sum_{m=-1}^{1} Q_m^* Q_m = P_+$ and $c_{l,m} \neq 0$ for $l = -F_-, \ldots, F_-$, $m = -1, 0, 1$.

It is easy to check that 2 implies $|c_{F_-,1}| = |c_{-F_-,-1}| = 1$ and $|c_{l,m}| < 1$ for $(l, m) \notin \{(F_-, 1), (-F_-, -1)\}$.

Since $\mathcal{L}$ is represented in the form of Lindblad taking $L_0 = \sqrt{B}2^{-1}(P_+ - P_-)$, $L_k = D(z_k)$ for $k \geq 1$ and $G = -2^{-1}\sum_{k\geq 0} L_k^* L_k - iH$, it is the generator of an uniformly continuous QDS $\mathcal{T}$ on $\mathcal{A}$; $\mathcal{T}$ is Markov because $\mathcal{L}(\mathbf{1}) = 0$.

We find the subharmonic projections to determine p_R; if p is a such projection, by Thm. III.1 and Lemma III.1 of [10] it follows that $p(\mathcal{H})$ is invariant for any L_k and G, i.e. p fulfills:

a) $pL_k p = L_k p$ for all $k \geq 0$;
b) $pGp = Gp$.

For $k = 0$ in *a*), we get $pP_\pm p = P_\pm p$, because $P_+ = \mathbf{1} - P_-$; hence, for $k \geq 1$, $p\sum_{m=-1}^{1} \alpha_{k,m} Q_m p = \sum_{m=-1}^{1} \alpha_{k,m} Q_m p$ holds, which means $pQ_m p = Q_m p$ for all $m = -1, 0, 1$ by 1. Therefore, *b*) implies that $p(\mathcal{H})$ is invariant for

$$\frac{1}{2}\sum_{m,n=-1}^{1} \epsilon Q_m^* Q_n - \frac{1}{2}\Omega e^{i\delta}\zeta Q_1^*,$$

$\epsilon := \sum_{k\geq 1} \overline{\alpha_{k,m}}\alpha_{k,n}$, $\zeta := \sum_{k\geq 1} \beta_{k,-}\overline{\alpha_{k,1}} - \frac{1}{2}(1 + e^{2i\delta_-})$, since 1 holds. Finally, using that $pP_- = (P_-p)^* = (pP_-p)^* = pP_-p = P_-p$ and $Q_nP_- = 0$ for all $n = -1, 0, 1$, we obtain $pQ_1^*p = Q_1^*p$.

As a consequence, we claim that $p = p_j$ for some $j \in \{-F_-, \dots, F_-\}$, where $p_j := \sum_{i=j}^{F_-}(|e_{i+1}^+\rangle\langle e_{i+1}^+| + |e_i^-\rangle\langle e_i^-|)$.

Indeed, if $\{f_1, \dots, f_s\}$ is the orthonormal basis of $p(\mathcal{H})$,

$$f_i = \sum_{l=-F_-}^{F_-} \lambda_{i,l}e_l^- + \sum_{k=-F_+}^{F_+} \mu_{i,k}e_k^+,$$

put $j := \min\{l = -F_-, \dots, F_- : \exists\, i \text{ s.t. } \lambda_{i,l} \neq 0\}$, we have clearly $P_-p(\mathcal{H}) \subseteq \text{span}\{e_j^-, \dots, e_{F_-}^-\}$; moreover, if there exists $k_0 < j + 1$ such that $\mu_{i,k_0} \neq 0$ for some $i \in \{1, \dots, s\}$, by $\sum_{k=-F_+}^{F_+} \mu_{i,k}c_{k-1}e_{k-1}^- = Q_1P_+f_i \in p(\mathcal{H})$ we get a contradiction since the coefficient of $e_{k_0-1}^-$ is $\mu_{i,k_0}c_{k_0-1} \neq 0$ and $k_0 - 1 < j$. Therefore every f_i belongs to $\text{span}\{e_{j+1}^+, \dots, e_{F_+}^+, e_j^-, \dots, e_{F_-}^-\}$, that is $p \leq p_j$. On the other hand, since $pQ_1 = (Q_1^*p)^* = (pQ_1^*p)^* = pQ_1p = Q_1p$, $Q_1^*p = pQ_1^*$ holds too, we have $pQ_1^*Q_1 = Q_1^*Q_1p$, so that p commutes with any spectral projections of the self-adjoint operator $Q_1^*Q_1$; because $|c_{F_-,1}|^2 = 1$ is a simple eigenvalue of $Q_1^*Q_1$ by an above remark, this means that p commutes in particular with $|e_{F_+}^+\rangle\langle e_{F_+}^+|$, i.e. $pe_{F_+}^+ = \nu e_{F_+}^+$ with $\nu \in \{0, 1\}$. It follows that $\nu c_{F_-,1}e_{F_-}^- = Q_1pe_{F_+}^+ = pQ_1e_{F_+}^+ = c_{F_-,1}pe_{F_-}^-$, that is $pe_{F_-}^- = \nu e_{F_-}^-$; moreover, since $Q_0pe_{F_-}^+ = pQ_0e_{F_-}^+ = c_{F_-,0}\nu e_{F_-}^-$ holds, if we let $pe_{F_-}^+ = \sum_{l=j}^{F_-}(a_le_l^- + b_{l+1}e_{l+1}^+)$, we obtain

$$\sum_{l=j}^{F_--1} b_{l+1}c_{l+1,0}e_{l+1}^- = \nu c_{F_-,0}e_{F_-}^-,$$

so $b_{l+1} = 0$ for all $l \in \{j, \dots, F_- - 2\}$, $b_{F_-} = \nu$ and $pe_{F_-}^+ = \sum_{l=j}^{F_-} a_le_l^- + \nu e_{F_-}^+$. Finally, by

$$\sum_{l=j}^{F_-} a_l\overline{c_{l,1}}e_{l+1}^+ = Q_1^*pe_{F_-}^+ = pQ_1^*e_{F_-}^+ = 0$$

we infer $a_l = 0$ for all $l \in \{j, \dots, F_-\}$, and consequently $pe_{F_-}^+ = \nu e_{F_-}^+$. Therefore, by iteration, we have $pe_{l+1}^+ = \nu e_{l+1}^+$ for all $l \in \{j, \dots, F_-\}$, which implies $pe_l^- = \nu e_l^-$ for all $l \in \{j, \dots, F_-\}$ by application of Q_1. Since $p \neq 0$, this shows that e_{l+1}^+ and e_l^- belong to $p(\mathcal{H})$ for all $l \in \{j, \dots, F_-\}$, that is $p = p_j$.

Hence, since $(p_j - p_{F_-})\mathcal{L}(p_{F_-})(p_j - p_{F_-}) \neq 0$ for all $j \in \{-F_-, \dots, F_- - 1\}$, Prop. 2 entails that p_j cannot be the support of an invariant normal state for $j \neq F_-$, so $p_R = 0$ or $p_R = p_{F_-}$. We can then conclude that $p_R = p_{F_-}$ by Lemma 6.

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