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Central limit theorems for random polytopes

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Abstract. Let K be a smooth convex set. The convex hull of independent random points in K is a random polytope. Central limit theorems for the volume and the number of i dimensional faces of random polytopes are proved as the number of random points tends to infinity. One essential step is to determine the precise asymptotic order of the occurring variances.

1. Introduction and main results

Let K be a smooth convex set of volume one, and let $X(n)$ be a Poisson point process in $\mathbb{R}^d$ of intensity n. The intersection of K with $X(n)$ consists of uniformly distributed random points $X_1, \ldots, X_N$. (A more precise definition is given in Section 2.) Define the random polytope Π_n as the convex hull $[X_1, \ldots, X_N] = [K \cap X(n)]$ of these random points and denote by $V(\Pi_n)$ the volume of Π_n. What can be said about the distribution function of $V(\Pi_n)$? In particular, does the distribution function of $V(\Pi_n)$ satisfy a central limit theorem as n tends to infinity?

This problem turns out to be surprisingly difficult even in simple cases. Although the intersection of a Poisson point process with sets of volume one is a well investigated subject nothing is known about the precise distribution function in question. Hence it is impossible to deduce central limit theorems from properties of the distribution function of $V(\Pi_n)$. Nevertheless, in the planar case and if K is a ball Hsing [16] succeeded in proving a central limit theorem for the volume of random polygons, and Groeneboom [13] obtained a central limit theorem for the number of vertices of the random polygon Π_n in this case. It seems that the methods cannot be applied to solve the problem in higher dimensions.

Here we generalize these results and prove a central limit theorem in all dimensions for arbitrary smooth convex sets K. Let $\mathcal{K}^2_+$ be the set of compact convex sets $K \in \mathbb{R}^d$, $d \geq 2$, with nonempty interior, boundary of differentiability class $\mathcal{C}^2$, and positive Gaussian curvature. By $\mathbb{P}(A)$ we denote the probability of A, by $\mathbb{E}$ the expectation and by Var the variance of a random variable. Φ denotes the distribution function of the normal distribution.

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Theorem 1. *Let $K \in \mathcal{K}_+^2$ and let Π_n be the convex hull of the intersection of K with a Poisson point process of intensity n. Then there is a constant $c_1(K)$ such that*

$$\left| \mathbb{P}\left(\frac{V(\Pi_n) - \mathbb{E}V(\Pi_n)}{\sqrt{\operatorname{Var} V(\Pi_n)}} \leq x \right) - \Phi(x) \right| \leq c_1(K)\, n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+\frac{2}{d+1}} n.$$

The asymptotic behavior of the occuring expectation is well understood. It follows easily from well known results on random polytopes (see Lemma 1) that

$$\mathbb{E}V(\Pi_n) = V(K) - \gamma_d \Omega(K) n^{-\frac{2}{d+1}}(1 + o(1)), \tag{1}$$

as $n \to \infty$, where the constant γ_d only depends on the dimension and is known explicitly, and $\Omega(K)$ denotes the affine surface area of K. But it turned out to be difficult to deduce more information about the asymptotic behavior of the variance of functionals of random polytopes. From Theorem 3 we deduce in Lemma 1 that there are positive constants $c_2(K)$, $c_3(K)$ depending only on K such that

$$c_2(K) n^{-1-\frac{2}{d+1}} \leq \operatorname{Var} V(\Pi_n) \leq c_3(K) n^{-1-\frac{2}{d+1}}. \tag{2}$$

Thus at least the asymptotic order of the variance occuring in the denominator in Theorem 1 is known.

A second functional of Π_n which is of fundamental interest is the number of i-dimensional faces $f_i(\Pi_n)$, $i = 0, \ldots, d-1$. The analogous question here is to determine the asymptotic distribution function of $f_i(\Pi_n)$ as n tends to infinity. It should be mentioned that there are close connections between the number of vertices $f_0(\Pi_n)$ and the volume $V(\Pi_n)$ of the random polytope. Efron's identity [11] states for a Poisson point process that

$$\mathbb{E}f_0(\Pi_n) = n(V(K) - \mathbb{E}V(\Pi_n)),$$

and thus the results concerning $\mathbb{E}V(\Pi_n)$ can be used to determine the expected number of vertices $f_0(\Pi_n)$. Corresponding to (1),

$$\mathbb{E}f_0(\Pi_n) = \gamma_d \Omega(K) n^{1-\frac{2}{d+1}}(1 + o(1))$$

as $n \to \infty$. Lemma 2 gives an analogous formula for all $i \in \{0, \ldots, d-1\}$,

$$\mathbb{E}f_i(\Pi_n) = \gamma_{d,i} \Omega(K) n^{1-\frac{2}{d+1}}(1 + o(1)) \tag{3}$$

as $n \to \infty$, and also the asymptotic order of the variance is stated there: there are positive constants $c_4(K)$, $c_5(K)$ depending only on K such that

$$c_4(K) n^{1-\frac{2}{d+1}} \leq \operatorname{Var} f_i(\Pi_n) \leq c_5(K) n^{1-\frac{2}{d+1}}. \tag{4}$$

The second main theorem of this paper gives a central limit theorem for $f_i(\Pi_n)$, $i \in \{0, \ldots, d-1\}$. It generalizes the central limit theorem proved by Greoneboom [13] for $d = 2$ and K a ball.

Theorem 2. *Let $K \in \mathcal{K}^2_+$ and define Π_n as the convex hull of the intersection of K with a Poisson point process of intensity n. Then there is a constant $c_6(K)$ such that for $i \in \{0, \dots, d-1\}$*

$$\left| \mathbb{P}\left(\frac{f_i(\Pi_n) - \mathbb{E} f_i(\Pi_n)}{\sqrt{\operatorname{Var} f_i(\Pi_n)}} \le x \right) - \Phi(x) \right| \le c_6(K)\, n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+3i+\frac{2}{d+1}} n.$$

Our paper is motivated by a classical problem going back to Sylvester in 1861: Given a convex set K and $n \in \mathbb{N}$, choose precisely n random points $X_1, \dots, X_n$ independently and according to the uniform distribution in K. Let P_n be their convex hull $[X_1, \dots, X_n]$. What is the distribution function of $V(P_n)$ and $f_i(P_n)$?

Numerour papers are dedicated solely to the question to determine the expectation of these quantities and the asymptotic behaviour of these expectations as n tends to infinity. Out of a large number of contributions we only mention the work of Rényi and Sulanke [23], Wieacker [31], Bárány [4], and Schütt [29] who proved that

$$\mathbb{E} V(P_n) = V(K) - \gamma_d \Omega(K) n^{-\frac{2}{d+1}} (1 + o(1)),$$

as $n \to \infty$, where γ_d is the constant occuring in (1). Efron's identity [11] implies the analogous result for the numbers of vertices of P_n,

$$\mathbb{E} f_0(P_n) = \gamma_d \Omega(K) n^{1-\frac{2}{d+1}} (1 + o(1))$$

as $n \to \infty$. In [22] this was generalized to all $i \in \{0, \dots, d-1\}$:

$$\mathbb{E} f_i(P_n) = \gamma_{d,i} \Omega(K) n^{1-\frac{2}{d+1}} (1 + o(1))$$

as $n \to \infty$ for $K \in \mathcal{K}^2_+$.

In contrast to the large number of results dealing with the first moment of $V(P_n)$ and $f_i(P_n)$ there are only few results concerning the variance of $V(P_n)$ and $f_i(P_n)$ or higher moments. It was proved only recently [21], [22] that for $K \in \mathcal{K}^2_+$

$$\operatorname{Var} V(P_n) \le c_7(K) n^{-1-\frac{2}{d+1}}$$

with a positive constant $c_7(K)$, and

$$\operatorname{Var} f_i(P_n) \le c_9(K) n^{1-\frac{2}{d+1}}$$

for $i \in \{0, \dots, d-1\}$ with a positive constant $c_9(K)$.

Buchta ([9], Corollary 2 and 3) used a generalization of Efron's identity to show that if K fulfills stronger differentiability assumptions then

$$\operatorname{Var} V(P_n) \ge c_8(K) n^{-\frac{5}{3}}$$

for $d = 2$ with a constant $c_8(K) > 0$, and

$$\operatorname{Var} f_0(P_n) \ge c_{10}(K) n^{1-\frac{2}{d+1}}$$

for $d \ge 4$ with a constant $c_{10}(K) > 0$.

One of the essential ingredients in proving central limit theorems for random polytopes is to generalize these results to all dimensions. This determines the precise order of the variance of $V(P_n)$ and $f_i(P_n)$ in arbitrary dimensions and for all $i \in \{0, \dots, d-1\}$.

Theorem 3. *Let $K \in \mathcal{K}^2_+$ and choose n random points in K independently and according to the uniform distribution. Then there are positive constants $c_7(K)$, $c_8(K)$ depending only on K such that*

$$c_8(K) n^{-1-\frac{2}{d+1}} \le \operatorname{Var} V(P_n) \le c_7(K) n^{-1-\frac{2}{d+1}}.$$

The analogous result for the number of faces of P_n reads as follows:

Theorem 4. *Let $K \in \mathcal{K}^2_+$ and choose n random points in K independently and according to the uniform distribution. Then there are positive constants $c_9(K)$, $c_{10}(K)$ depending only on K such that for $i \in \{0, \dots, d-1\}$*

$$c_{10}(K) n^{1-\frac{2}{d+1}} \le \operatorname{Var} f_i(P_n) \le c_9(K) n^{1-\frac{2}{d+1}}.$$

In Lemma 1 and Lemma 2 we prove that these results concerning the expectation and the variance of P_n imply formulae (1) – (4) for Π_n stated above.

The problem to determine the precise distribution function at least in one of the cases mentioned above is nontrivial. Starting with, the only cases where a distribution function is known explicitly concerns $V(P_3)$ in the planar case, and if K is a triangle (Alagar [1]), a circular disc or a parallelogram (Henze [15]); for arbitrary n, or in higher dimensions it is hopeless to expect explicit formulae.

The first who succeeded in proving a limit theorem was Schneider [25]. He obtained in the planar case for points on the boundary of K a strong law of large numbers for $V(P_n)$ if K is smooth. In [21] this was generalized to arbitrary dimensions: $(V(K) - V(P_n))n^{2/(d+1)}$ tends with probability one to an explicitly known constant depending on K as n tends to infinity. Finally, as mentioned above, Hsing [16] proved a central limit theorem for $V(P_n)$ but only in the planar case and if K is a ball.

Here we strengthen these results. By Lemma 3 the distribution function of $V(\Pi_n)$ approximates the distribution function of $V(P_n)$ and thus Theorem 1 implies a central limit theorem for $V(P_n)$.

Theorem 5. *Let $K \in \mathcal{K}^2_+$ and choose n random points in K independently and according to the uniform distribution. Then there are numbers c_n bounded between two positive constants depending on K, and a constant $c_{11}(K)$ such that*

$$\left| \mathbb{P}\left(\frac{V(P_n) - \mathbb{E}V(P_n)}{\sqrt{c_n\, n^{-1-\frac{2}{d+1}}}} \le x \right) - \Phi(x) \right| \le c_{11}(K)\, n^{-\frac{1}{2(d+1)}} \ln^{2+\frac{2}{d+1}} n.$$

Note that the denominator in Theorem 5 has the same order of magnitude as the standard deviation of $V(P_n)$. In fact, the denominator equals the standard deviation of $V(\Pi_n)$ which is of the same asymptotic order as the standard deviation of $V(P_n)$. It is an open problem whether $\operatorname{Var} V(P_n)$ equals asymptotically $\operatorname{Var} V(\Pi_n)$. Since we are not able to prove this equality, we can not prove that the denominator equals the standard deviation of $V(P_n)$.

The second theorem of this paper implies in the same way a central limit theorem for $f_i(P_n)$, $i \in \{0, \dots, d-1\}$, in arbitrary dimensions:

Theorem 6. *Let $K \in \mathcal{K}_+^2$ and choose n random points in K independently and according to the uniform distribution. Then there are numbers d_n bounded between two positive constants depending on K, and a constant $c_{12}(K)$ such that for $i \in \{0, \dots, d-1\}$*

$$\left| \mathbb{P}\left(\frac{f_i(P_n) - \mathbb{E} f_i(P_n)}{\sqrt{d_n n^{1-\frac{2}{d+1}}}} \le x \right) - \Phi(x) \right| \le c_{12}(K)\, n^{-\frac{1}{2(d+1)}} \ln^{2+3i+\frac{2}{d+1}} n.$$

Again the denominator equals the standard deviation of the number of i-dimensional faces of a random polytope Π_n.

As a general reference for Poisson point processes we mention the books of Schneider and Weil [27] and Stoyan, Kendall, and Mecke [28]. For more information on random polytopes we refer to a recent survey article by Schneider [26], and for a comparison of random polytopes and best approximating polytopes to a survey article by Gruber [14]. In particular, if the underlying convex set K itself is a polytope, we mention the work of Bárány and Buchta [7] dealing with the expectation of $f_i(P_n)$, and Groeneboom [13] and Cabo and Groeneboom [10] who obtained in the planar case central limit theorems for $f_0(\Pi_n)$ (but the stated asymptotic value for the variance in [10] appears to be incorrect, see Hüsler [18], page 111, and for a corrected version Buchta [9] and Finch and Hueter [12]). To the best of our knowledge the only central limit theorem holding in arbitrary dimensions is due to Hueter [17] for $f_0(P_n)$ where the random points are chosen with respect to the d-dimensional normal distribution. For (central) limit theorems dealing with important and highly interesting investigations of random convex hulls closely related to our problem we mention the work of Bárány [5] [6], and Bárány, Rote, Steiger, and Zhang [8].

2. Approximating P_n by Π_n

In Section 4 we prove central limit theorems concerning functions of Π_n, the distribution functions of $(V(\Pi_n) - \mathbb{E}V(\Pi_n))/\sqrt{\operatorname{Var} V(\Pi_n)}$ and $(f_i(\Pi_n) - \mathbb{E}f_i(\Pi_n))/\sqrt{\operatorname{Var} f_i(\Pi_n)}$ tend to the distribution function Φ of the normal distribution. To deduce from these results the corresponding results for P_n we have to show that expectation, variance, and distribution function of these functions of Π_n and P_n are sufficiently close if n tends to infinity.

We start by recalling some elementary facts about a Poisson point process. Let A be a measurable subset of $\mathbb{R}^d$. Then the intersection of the Poisson point process $X(\lambda)$ of intensity λ with A consists of random points $\{X_1, \dots, X_N\} = A \cap X(\lambda)$ where the number of random points N is Poisson distributed with intensity $\lambda V_d(A)$ and, conditioning on N, the points $X_1, \dots, X_N$ are independently uniformly distributed in A. If A and B are two disjoint subsets of $\mathbb{R}^d$ then the two point sets $\{X_1, \dots, X_N\} = A \cap X(\lambda)$ and $\{Y_1, \dots, Y_M\} = B \cap X(\lambda)$ are independent: N

and M are independently Poisson distributed and the points X_i and Y_j are chosen independently.

As we mentioned in the introduction, respectively going to prove in Section 3 we have

$$V(K) - \mathbb{E}V(P_n) = \gamma_d \Omega(K) n^{-\frac{2}{d+1}} (1 + o(1)), \quad \text{and} \quad \operatorname{Var} V(P_n) \approx n^{-1-\frac{2}{d+1}}.$$

Here $f(n) \approx g(n)$ means that there are two constants $a, b > 0$ such that $a\, g(n) \leq f(n) \leq b\, g(n)$. It is the aim of this section to show that in all these results P_n can be replaced by Π_n:

Lemma 1. *Let Π_n be the convex hull of the intersection of K with the Poisson point process $X(n)$ of intensity n. Then*

$$V(K) - \mathbb{E}V(\Pi_n) = \gamma_d \Omega(K) n^{-\frac{2}{d+1}} (1 + o(1)) \tag{5}$$

as $n \to \infty$, and

$$\operatorname{Var} V(\Pi_n) \approx n^{-1-\frac{2}{d+1}}. \tag{6}$$

We prove (5) by showing that $V(K) - \mathbb{E}V(\Pi_n) = (V(K) - \mathbb{E}V(P_n))(1 + o(1))$. It is also generally believed (but apparently unproved) that $\operatorname{Var} V(P_n) = \operatorname{Var} V(\Pi_n)(1 + o(1))$ and $\operatorname{Var} f_i(P_n) = \operatorname{Var} f_i(\Pi_n)(1 + o(1))$. (Such an equality was stated without prove in the paper by Cabo and Groeneboom [10].) If there would exist a stronger statement than Theorem 3 saying that $\operatorname{Var} V(P_n) n^{1+2/(d+1)}$ converges to a constant as $n \to \infty$, the method developed in this section would suffice to prove this equality. Yet we are not able to obtain this strengthening of Theorem 3.

Proof. The result for the expectation of $V(\Pi_n)$ follows easily from the corresponding result for P_n since

$$V(K) - \mathbb{E}V(\Pi_n) = \sum_{|k-n| \leq n^{\frac{7}{8}}} (V(K) - \mathbb{E}V(P_k))\, e^{-n} \frac{n^k}{k!} + \sum_{|k-n| \geq n^{\frac{7}{8}}} (V(K) - \mathbb{E}V(P_k))\, e^{-n} \frac{n^k}{k!}$$

and by Chebyshev's inequality $\mathbb{P}(|N - n| \leq n^{7/8}) \geq 1 - n^{-3/4}$. Because $V(K) - \mathbb{E}V(P_k) \leq 1$ the second sum is of order $n^{-3/4}$. Observe that for each k occurring in the first sum we have $k^{-2/(d+1)} = n^{-2/(d+1)}(1 + o(1))$. Combining this proves (5).

In the next step we show that the result of Theorem 3 concerning bounds on the variance $\operatorname{Var} V(P_n)$ implies analogous bounds for $\operatorname{Var} V(\Pi_n)$. In the following N always denotes the number of points of the intersection of K with the Poisson point process $X(n)$ which is Poisson distributed with parameter n. We need some preparations: By Stirling‘s formula $k! \geq k^k e^{-k}$, and as $(ne/x)^x$ is monoton for $x < n$ we have

$$\mathbb{P}\left(N \le \frac{n}{2}\right) = e^{-n} \sum_{k=0}^{\frac{n}{2}} \frac{n^k}{k!} \le e^{-n} \sum_{k=0}^{\frac{n}{2}} \left(\frac{en}{k}\right)^k \le \frac{n+2}{2} \left(\frac{e}{2}\right)^{-\frac{n}{2}}.$$

Analogously

$$\mathbb{P}(N \ge 3n) = e^{-n} \sum_{k=3n}^{\infty} \frac{n^k}{k!} \le e^{-n} \sum_{k=3n}^{\infty} \left(\frac{en}{k}\right)^k \le e^{-n} \sum_{k=0}^{\infty} \left(\frac{e}{3}\right)^k = e^{-n} \frac{3}{3-e}. \tag{7}$$

We decompose $\operatorname{Var} V(\Pi_n)$,

$$\operatorname{Var} V(\Pi_n) = \mathbb{E}\operatorname{Var}(V(\Pi_n)|N) + \operatorname{Var}\mathbb{E}(V(\Pi_n)|N)$$

and prove in a first step that the second term is of negligible order.

$$\begin{aligned}
\operatorname{Var}\mathbb{E}(V(\Pi_n)|N) &= \mathbb{E}\,(\mathbb{E}(V(\Pi_n)|N))^2 - (\mathbb{E}\,\mathbb{E}(V(\Pi_n)|N))^2 \\
&= \sum_{j=\frac{n}{2}}^{\infty} \sum_{k=\frac{n}{2}}^{\infty} \Big((\mathbb{E}V(P_k))^2 - \mathbb{E}V(P_k)\mathbb{E}V(P_j)\Big)\; e^{-2n}\frac{n^{k+j}}{k!j!} \\
&\quad + O\left(n\left(\frac{e}{2}\right)^{-\frac{n}{2}}\right) \\
&= \sum_{j=\frac{n}{2}}^{\infty} \sum_{k=j}^{\infty} \big(\mathbb{E}V(P_k) - \mathbb{E}V(P_j)\big)^2\; e^{-2n}\frac{n^{k+j}}{k!j!} \\
&\quad + O\left(n\left(\frac{e}{2}\right)^{-\frac{n}{2}}\right).
\end{aligned}$$

It is essential to know that $\mathbb{E}V(P_k) - \mathbb{E}V(P_j)$ is small if k, j is large. Precisely this was proved in Section 12 in [21]: There is a constant c_d depending on the dimension such that

$$\mathbb{E}V(P_{k+1}) - \mathbb{E}V(P_k) = c_d \Omega(K) k^{-1-\frac{2}{d+1}}(1 + o(1))$$

as $k \to \infty$ and thus $\mathbb{E}V(P_{k+1}) - \mathbb{E}V(P_k) \le c_1 k^{-1-2/(d+1)}$. This implies a smoothness condition for $\mathbb{E}V(P_k)$: for $k > j$

$$\mathbb{E}V(P_k) - \mathbb{E}V(P_j) = \sum_{i=j}^{k-1} (\mathbb{E}V(P_{i+1}) - \mathbb{E}V(P_i)) \le c_1(k-j)j^{-1-\frac{2}{d+1}}.$$

(Throughout this paper constants c_k may depend only on the dimension d, on the dimension i of the faces we are interested in, and on the convex set K, and are independent of anything else.) This gives us the following:

$$
\begin{aligned}
\mathrm{Var}\mathbb{E}(V(\Pi_n)|N) &\le c_1^2 \sum_{j=\frac{n}{2}}^{\infty} \sum_{k=j}^{\infty} (k-j)^2 j^{2(-1-\frac{2}{d+1})}\, e^{-2n} \frac{n^{k+j}}{k!j!} \\
&\quad + O\left(n \left(\frac{e}{2}\right)^{-\frac{n}{2}}\right) \\
&\le c_1^2 n^{2(-1-\frac{2}{d+1})} \mathrm{Var} N + O\left(n \left(\frac{e}{2}\right)^{-\frac{n}{2}}\right) \\
&= O(n^{-1-\frac{4}{d+1}}).
\end{aligned}
$$

As for the first term of $\mathrm{Var}\, V(\Pi_n)$ we know that $\mathrm{Var}\, V(P_n)$ can be estimated by $n^{-1-2/(d+1)}$. Thus by (7)

$$
\begin{aligned}
\mathbb{E}\mathrm{Var}(V(P_N)|N) &\approx \mathbb{E} N^{-1-\frac{2}{d+1}} \\
&= \mathbb{E}\left(N^{-1-\frac{2}{d+1}} I\left(\frac{n}{2} < N \le 3n\right)\right) \\
&\quad + O\left(\mathbb{P}\left(N \le \frac{n}{2},\, 3n < N\right)\right) \\
&= \mathbb{E}\left(N^{-1-\frac{2}{d+1}} I\left(\frac{n}{2} < N \le 3n\right)\right) \\
&\quad + O\left(\frac{n+1}{2}\left(\frac{e}{2}\right)^{-\frac{n}{2}} + e^{-n}\right) \\
&\approx n^{-1-\frac{2}{d+1}}
\end{aligned}
$$

which proves (6). □

In exactly the same way we prove

Lemma 2. *Let Π_n be the convex hull of the intersection of K with the Poisson point process $X(n)$ of intensity n. Then for $i \in \{0, \dots, d-1\}$*

$$
\mathbb{E} f_i(\Pi_n) = \gamma_{d,i}\Omega(K) n^{1-\frac{2}{d+1}} (1 + o(1)) \tag{8}
$$

as $n \to \infty$, and

$$
\mathrm{Var}\, f_i(\Pi_n) \approx n^{1-\frac{2}{d+1}} \tag{9}
$$

for $i = 0, \dots, d-1$.

Proof. It follows from $\mathbb{E} f_i(P_k) = \gamma_{d,i}\Omega(K) k^{1-2/(d+1)} \le c_2 k$ that

$$
\begin{aligned}
\mathbb{E} f_i(\Pi_n) &= \sum_{|k-n| \le n^{\frac{7}{8}}} \mathbb{E} f_i(P_k)\, e^{-n} \frac{n^k}{k!} \;+\; O\left(\sum_{|k-n| \ge n^{\frac{7}{8}}} k\, e^{-n} \frac{n^k}{k!}\right) \\
&= \gamma_{d,i}\Omega(K) n^{1-\frac{2}{d+1}} (1 + o(1))\, \mathbb{P}(|N-n| \le n^{\frac{7}{8}}) + O(n^{\frac{1}{4}})
\end{aligned}
$$

which proves (8).

As for $\operatorname{Var} f_i(\Pi_n)$ we start as before with

$$\operatorname{Var} f_i(\Pi_n) = \mathbb{E}\operatorname{Var}(f_i(\Pi_n)|N) + \operatorname{Var}\mathbb{E}(f_i(\Pi_n)|N)$$

and use that

$$\mathbb{E}\left(NI\left(N \le \frac{n}{2}\right)\right) \le \sum_{k=0}^{\frac{n}{2}} k\, e^{-n} \left(\frac{en}{k}\right)^k \le \frac{n^2}{4} \left(\frac{e}{2}\right)^{-\frac{n}{2}}$$

and

$$\mathbb{E}(NI(N \ge 3n)) \le \sum_{k=0}^{\infty} k\, e^{-n} \left(\frac{e}{3}\right)^k = e^{-n} \frac{3e}{(3-e)^2}.$$

Again the second term is of negligible order.

$$\begin{aligned}
\operatorname{Var}\mathbb{E}(f_i(\Pi_n)|N) &= \sum_{j=\frac{n}{2}}^{\infty} \sum_{k=j}^{\infty} \left(\mathbb{E}f_i(P_k) - \mathbb{E}f_i(P_j)\right)^2 \ e^{-2n} \frac{n^{k+j}}{k!j!} \\
&\quad + O\left(\frac{n^2}{4}\left(\frac{e}{2}\right)^{-\frac{n}{2}}\right) \\
&\le c_3 \sum_{j=\frac{n}{2}}^{\infty} \sum_{k=j}^{\infty} (k-j)^2 j^{2(-\frac{2}{d+1})} \ e^{-2n} \frac{n^{k+j}}{k!j!} \\
&\quad + O\left(\frac{n^2}{4}\left(\frac{e}{2}\right)^{-\frac{n}{2}}\right) \\
&= O(n^{1-\frac{4}{d+1}})
\end{aligned}$$

because it follows from (27) in [22] and Theorem 6 in [21] that

$$\mathbb{E}f_i(P_{k+1}) - \mathbb{E}f_i(P_k) \le \binom{d+1}{i+1}\mathbb{E}F_k(X) = \binom{d+1}{i+1} c_d \Omega(K) k^{-\frac{2}{d+1}} (1+o(1))$$

as $k \to \infty$ which shows for $k > j$

$$\mathbb{E}f_i(P_k) - \mathbb{E}f_i(P_j) \le c_3 (k-j) j^{-\frac{2}{d+1}}.$$

Estimating $\operatorname{Var}(f_i(\Pi_n)|N)$ by $N^{1-2/(d+1)}$ which follows from Theorem 4, and in the range $N \le n/2,\ N \ge 3n$ by N we get

$$\mathbb{E}\operatorname{Var}(f_i(P_N)|N) \approx \mathbb{E}N^{1-\frac{2}{d+1}} \approx n^{1-\frac{2}{d+1}}$$

which is (9). □

It remains to prove that also the distribution functions of $V(\Pi_n)$, respectively $f_i(\Pi_n)$, are sufficiently close to the distribution functions of $V(P_n)$, respectively $f_i(P_n)$. The following fact will be essential:

Let A, B be arbitrary events. Then

$$|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq 1 - \mathbb{P}(A). \tag{10}$$

This follows from

$$\mathbb{P}(B) - (1 - \mathbb{P}(A)) \leq \mathbb{P}(A)\mathbb{P}(B \mid A) \leq \mathbb{P}(B).$$

The estimate (10) will be used in the case where B is a suitable indicator function, and thus $\mathbb{P}(B)$ is the distribution function of $V(\Pi_n)$, $V(P_n)$, $f_i(\Pi_n)$, or $f_i(P_n)$, and where A is the event that the boundary of P_n and Π_n are close to the boundary of K.

To show that the boundary of P_n and Π_n is close to the boundary of K with high probability, denote by $K(\varepsilon)$ the inner parallel set of ∂K,

$$K(\varepsilon) = \{x \in K \mid \delta^H(x, \partial K) \leq \varepsilon\}$$

where δ^H denotes the Hausdorff distance. Our aim is to show that, with high probability, ∂P_n and $\partial \Pi_n$ are contained in $K(\varepsilon_n)$ with

$$\varepsilon_n = \left(\frac{2d \ln n}{d_3 n}\right)^{\frac{2}{d+1}}$$

where d_3 is the constant appearing in Lemma 5.

Let A_n be the event that $\partial P_n \subset K(\varepsilon_n)$. Then the probability of the complement of A_n is the probability that at least one facet of P_n has distance at least ε_n from the boundary of K, i.e., the hyperplane which is the affine hull of this facet cuts of from K a cap of height ε_n which contains no other random point. By Lemma 5 the volume of this cap is bounded by $d_3\varepsilon_n^{(d+1)/2} = 2dn^{-1}\ln n$ and thus we have

$$1 - \mathbb{P}(A_n) \leq \binom{n}{d}(1 - 2dn^{-1}\ln n)^{n-d}.$$

Analogously let A_π be the event that $\partial \Pi_n \subset K(\varepsilon_n)$. Then

$$1 - \mathbb{P}(A_\pi) = \sum_{k=0}^{\infty}(1 - \mathbb{P}(A_k))\,\frac{n^k}{k!}e^{-n} \leq d\,\frac{n^{d-1}}{(d-1)!}\,e^{-n} + \frac{1}{d!}n^{-d}.$$

Thus with high probability $K(\varepsilon_n)$ contains the vertices of the random polytopes P_n and Π_n: by (10)

$$|\mathbb{P}(V(P_n) \leq x) - \mathbb{P}(V(P_n) \leq x \,|A_n)| \leq c_4\, n^{-d} \tag{11}$$

and

$$|\mathbb{P}(V(\Pi_n) \leq x|A_\pi) - \mathbb{P}(V(\Pi_n) \leq x)| \leq c_5\, n^{-d}. \tag{12}$$

Given A_n and A_π the volume of the random polytopes only depends on the set of random points in $K(\varepsilon_n)$. Set $p = V(K(\varepsilon_n))$ which is bounded by $S(K)\varepsilon_n$, where $S(K)$ is the surface area of K. Using the Poisson approximation of the binomial distribution we have

$$\begin{aligned}
&|\mathbb{P}(V(P_n) \le x | A_n) - \mathbb{P}(V(\Pi_n) \le x \mid A_\pi)| \\
&\quad = \sum_{k=0}^{\infty} \mathbb{P}\left(V(\Pi_n) \le x \mid A_\pi,\ \sharp\{K(\varepsilon_n) \cap X(n)\} = k\right) \\
&\quad\quad \times \left| \frac{(np)^k}{k!} e^{-np} - \binom{n}{k} p^k (1-p)^{n-k} \right| \qquad (13) \\
&\quad \le \sum_{k=0}^{\infty} \left| \frac{(np)^k}{k!} e^{-np} - \binom{n}{k} p^k (1-p)^{n-k} \right| \\
&\quad \le 2p \le c_6\, n^{-\frac{2}{d+1}} \ln^{\frac{2}{d+1}} n. \qquad (14)
\end{aligned}$$

The last estimate giving a bound $2p$ is due to Vervaat [30]. Combining (11), (12), and (14) we obtain an estimate for $\mathbb{P}(V(P_n) \le x) - \mathbb{P}(V(\Pi_n) \le x)$. Precisely the same proof holds for the number of i-dimensional faces instead of volume.

Lemma 3. *Let P_n be the convex hull of n random points chosen independently according to the uniform distribution in K, and let Π_n be the convex hull of the intersection of K with a Poisson point process of intensity n. Then there are constants c_7, c_8 depending on K such that*

$$|\mathbb{P}(V(P_n) \le x) - \mathbb{P}(V(\Pi_n) \le x)| \le c_7\, n^{-\frac{2}{d+1}} \ln^{\frac{2}{d+1}} n$$

and

$$|\mathbb{P}(f_i(P_n) \le x) - \mathbb{P}(f_i(\Pi_n) \le x)| \le c_8\, n^{-\frac{2}{d+1}} \ln^{\frac{2}{d+1}} n$$

for $i = 0, \dots, d-1$.

From this it is clear that Theorem 5 follows from Theorem 1, and Theorem 6 follows from Theorem 2.

3. A lower bound for the variance

In this section we show that

$$\operatorname{Var} V(P_n) \approx n^{-1-\frac{2}{d+1}}$$

and

$$\operatorname{Var} f_i(P_n) \approx n^{1-\frac{2}{d+1}}$$

for all $i = 0, \dots, d-1$. It is already known [21] that there are constants $c_7(K)$ and $c_9(K)$ such that $\operatorname{Var} V(P_n) \le c_7(K)\, n^{-1-\frac{2}{d+1}}$ and $\operatorname{Var} f_i(P_n) \le c_9(K)\, n^{1-\frac{2}{d+1}}$ so it remains to prove the reverse inequalities:

$$\operatorname{Var} V(P_n) \ge c_8(K)\, n^{-1-\frac{2}{d+1}}$$

and

$$\operatorname{Var} f_i(P_n) \geq c_{10}(K)\, n^{1-\frac{2}{d+1}}.$$

Proof of Theorem 3. Clearly it is sufficient to prove these inequalities for $n \geq n_0$ with some $n_0 = n_0(K)$.

By $H(u,t)$ we denote the hyperplane $\{x \in \mathbb{R}^d,\ \langle x,u\rangle = t\}$ and by $H^+(u,t)$ the corresponding halfspace $\{x \in \mathbb{R}^d,\ \langle x,u\rangle \geq t\}$. Let K be a smooth convex set and $H(u, h_K(u))$ be a supporting hyperplane, i.e., the intersection of K with $H^+(u, h_K(u))$ is a point y on the boundary ∂K of K, and $h_K(u)$ is the support function of K at u. We denote the intersection of K with $H^+(u, h_K(u) - h)$ by $C^K(y,h)$ and call $C^K(y,h)$ a *cap* of K of height h.

Let E be the standard paraboloid in $\mathbb{R}^d$,

$$E = \left\{ x \in \mathbb{R}^d \mid \langle x, e_d\rangle \geq \sum_{j=1}^{d-1} \langle x, e_j\rangle^2 \right\}$$

and set

$$\tfrac{1}{2}E = \left\{ x \in \mathbb{R}^d \mid \langle x, e_d\rangle \geq 2\sum_{j=1}^{d-1} \langle x, e_j\rangle^2 \right\} \subset E \subset$$

$$\subset 2E = \left\{ x \in \mathbb{R}^d \mid \langle x, e_d\rangle \geq \frac{1}{2}\sum_{j=1}^{d-1} \langle x, e_j\rangle^2 \right\}.$$

Choose a simplex S_0 in the cap $C^{\frac{1}{2}E}(0,1)$ in the following way: the base is a regular simplex with vertices on $\partial(\frac{1}{2}E) \cap H(e_d, (32d^2)^{-1})$ and the apex is the origin. It is elementary to see that the cone $\{\lambda x \in \mathbb{R}^d \mid x \in S_0,\ \lambda \in \mathbb{R}_+\}$ generated by S_0 contains $2E \cap H(e_d, 1)$, since the inball of the $(d-1)$-dimensional base of S_0 is a ball of radius $(8d^2)^{-1}$.

By continuity there is a cap $C^{\frac{1}{2}E}(0,\delta_0)$ and closed sets $C^1, \dots, C^d \subset C^{\frac{1}{2}E}(0,1)$ (e.g. suitable caps whose centers are the vertices of the base of S_0) with $V(C^{\frac{1}{2}E}(0,\delta_0)) = V(C^1) = \cdots = V(C^d) = c_0$ sufficiently small such that for all $Y \in C^{\frac{1}{2}E}(0,\delta_0)$, $x_i \in C^i$, $i = 1, \dots, d$, we have that the simplex $[Y, x_1, \dots, x_d]$ is close to S_0, in particular that

$$\{\lambda x \mid x \in [Y, x_1, \dots, x_d],\ \lambda \in \mathbb{R}_+\} \supset 2E \cap H(e_d, 1).$$

Choose the point Y at random according to the uniform distribution in $C^{\frac{1}{2}E}(0,\delta_0)$. Then there is a constant $c_9 > 0$ such that

$$\operatorname{Var}_Y V([Y, x_1, \dots, x_d]) \geq c_9$$

for all possible locations of $x_i \in C^i$. Here the variance is taken with respect to the random variable Y. Now let Q be a paraboloid of the form

$$Q = \left\{ x \in \mathbb{R}^d \mid \langle x, e_d\rangle \geq \frac{1}{2}\sum_{j=1}^{d-1} k_j \langle x, e_j\rangle^2 \right\}, \tag{15}$$

put $\kappa = \prod k_j$, and apply an affinity which maps $C^E(0,1)$ onto $C^Q(0,h)$ and leaves the coordinate axis invariant. Then there is a cap $C^{\frac{1}{2}Q}(0, h\,\delta_0)$ and sets D^i – the images of $C^{\frac{1}{2}E}(0,\delta_0)$ and C^i – with

$$V(C^{\frac{1}{2}Q}(0, h\,\delta_0)) = V(D^i) = 2^{\frac{d-1}{2}}\,\kappa^{-\frac{1}{2}} h^{\frac{d+1}{2}} c_0 \tag{16}$$

such that for all $Y \in C^{\frac{1}{2}Q}(0, h\,\delta_0)$ and $x_i \in D^i$

$$\{\lambda x \mid x \in [Y, x_1, \ldots, x_d],\ \lambda \in \mathbb{R}_+\} \supset 2Q \cap H(e_d, h)\,. \tag{17}$$

Choosing the point Y at random according to the uniform distribution in $C^{\frac{1}{2}Q}(0, h\,\delta_0)$ we have

$$\mathrm{Var}_Y V([Y, x_1, \ldots, x_d]) \geq 2^{d-1}\kappa^{-1} h^{d+1} c_9 \tag{18}$$

for all possible locations of $x_i \in D^i$ by the homogeneity of the variance.

Since $K \in \mathcal{K}^2_+$ at each boundary point $y \in \partial K$ there is a paraboloid $Q(y)$ osculating ∂K at y which is the image of a paraboloid Q given by (15) under a linear map. Thus the principal curvatures k_i of $\partial Q(y)$ at y are given by the principal curvatures $k_i(y)$ of ∂K at y, and $\kappa = \kappa(y)$ is the Gaussian curvature at the boundary point y. We denote by $\frac{1}{2}Q(y)$, respectively $2Q(y)$, the paraboloid touching ∂K at y having twice, respectively half, the principal curvatures of ∂K at y. Since $K \in \mathcal{K}^2_+$ there is a constant $h_0 = h_0(K)$ such that for all $h \leq h_0$ and for all $y \in \partial K$

$$C^{\frac{1}{2}Q(y)}(y, h) \subset C^K(y, h) \subset C^{2Q(y)}(y, h)$$

(see Section 5). By (16)–(18) we have the following: there is a cap $C^{\frac{1}{2}Q(y)}(y, h\,\delta_0)$ and sets $D^i(y)$ with

$$V(C^{\frac{1}{2}Q(y)}(y, h\,\delta_0)) = V(D^i(y)) = 2^{\frac{d-1}{2}}\,\kappa(y)^{-\frac{1}{2}} h^{\frac{d+1}{2}} c_0 \tag{19}$$

such that $h \leq h_0$ and for all $Y \in C^{\frac{1}{2}Q(y)}(y, h\,\delta_0)$ and $x_i \in D^i(y)$

$$\begin{aligned}\{\lambda x \mid x \in [Y, x_1, \ldots, x_d],\ \lambda \in \mathbb{R}_+\} &\supset 2Q(y) \cap H(u, h_K(u) - h)\\ &\supset K \cap H(u, h_K(u) - h)\end{aligned} \tag{20}$$

where u is the outer normal vector of K at y. Choosing the point Y at random according to the uniform distribution in $C^{\frac{1}{2}Q(y)}(y, h\,\delta_0)$ we have

$$\mathrm{Var}_Y V([Y, x_1, \ldots, x_d]) \geq 2^{d-1}\kappa(y)^{-1} h^{d+1} c_9 \tag{21}$$

for all possible locations of $x_i \in D^i(y)$.

Choose n random points $X_1, \ldots, X_n$ in K according to the uniform distribution. We now use the results concerning the cap coverings proved in the appendix. Set

$$m = \left\lfloor n^{1-\frac{2}{d+1}} \right\rfloor$$

and choose according to Lemma 6 a set of points $y_1, \dots, y_m \in \partial K$, and at these points corresponding disjoint caps C_j, $j = 1, \dots, m$, of height $h_m \approx m^{-2/(d-1)}$. Assume that n is sufficiently large such that $h_m \le h_0$. As stated above for each point y_j we have a cap $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$ and sets $D^i(y_j)$ with the properties (19)–(21). For $j \in \{1, \dots, m\}$ let A_j be the event that exactly one random point is contained in $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$ and in each set $D^i(y_j)$ and no other point in C_j. Then

$$\begin{aligned} I\!\!P(A_j) &= \binom{n}{d+1} I\!\!P(X_i \in D^i(y_j),\ i \le d) I\!\!P(X_{d+1} \in C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)) \\ &\quad \times I\!\!P(X_l \notin C_j,\ l \ge d+2) \\ &= \binom{n}{d+1} \left(V(C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)) \prod_{i=1}^{d} V(D^i(y_j)) \right) \left(1 - V(C_j)\right)^{n-d-1}. \end{aligned}$$

By (19), Lemma 4, and Lemma 6 the volumes of the caps $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$, C_j, and of the sets $D^i(y_j)$ are bounded by functions depending on h_m and the Gaussian curvature $\kappa(y_j)$. Since the Gaussian curvature is bounded between two positive constants, there are constants $c_{10}, c_{11} > 0$ such that

$$I\!\!P(A_j) \ge c_{10}\, n^{d+1} n^{-d-1} (1 - d_7 n^{-1})^{n-d-1} \ge c_{11} > 0$$

and

$$I\!\!E\left(\sum_{j=1}^{m} I(A_j) \right) = \sum_{j=1}^{m} I\!\!P(A_j) \ge c_{11} m. \tag{22}$$

Denote by $\mathcal{F}$ the position of all random points of $\{X_1, \dots, X_n\}$ except those which are contained in caps $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$ with $I(A_j) = 1$. Then

$$\begin{aligned} \operatorname{Var} V(P_n) &= I\!\!E \operatorname{Var}(V(P_n) \,|\, \mathcal{F}) + \operatorname{Var} I\!\!E(V(P_n) | \, \mathcal{F}) \\ &\ge I\!\!E \operatorname{Var}(V(P_n) \,|\, \mathcal{F}). \end{aligned}$$

Assume that $I(A_j) = I(A_k) = 1$ for some $j, k \in \{1, \dots, m\}$ and further without loss of generality that X_j, respectively X_k, is the unique point in $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$, respectively $C^{\frac{1}{2}Q(y_k)}(y_k, h_m\,\delta_0)$. By construction the points X_j and X_k are vertices of P_n, and by (20) there is no edge between X_j and X_k. Hence the change of volume of P_n if X_j is moved, is independent of the change of volume if X_k is moved. This independence structure of P_n implies

$$\operatorname{Var}(V(P_n) \,|\, \mathcal{F}) = \sum_{I(A_j)=1} \operatorname{Var}_{X_j} V(P_n)$$

where the variance is taken with respect to the random variable $X_j \in C^{\frac{1}{2}Q(y_j)} \times (y_j, h_m\,\delta_0)$, and we sum over all $j = 1, \ldots, m$ with $I(A_j) = 1$. Combining this with (21), with the definition of h_m, that $\kappa(y_j)$ is bounded, and with (22) implies

$$\begin{aligned}\operatorname{Var} V(P_n) &\geq \mathbb{E}\left(\sum_{I(A_j)=1} 2^{d-1}\kappa(y)^{-1} h_m^{d+1}\, c_9\right)\\ &\geq c_{12} n^{-2}\mathbb{E}\left(\sum_{j\in \mathrm{J}} I(A_j)\right)\\ &\geq c_{11} c_{12}\, n^{-1-\frac{2}{d+1}}\end{aligned}$$

which is Theorem 3. □

Proof of Theorem 4. We use exactly the same method with one exception: instead of choosing one random point Y in $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$ we choose two random points Y, Z. (The convex hull of $x_1, \ldots, x_d$ with only one random point Y in $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$ is a simplex and thus $f_i([Y, x_1, \ldots, x_d])$ would be a constant. But $[Y, Z, x_1, \ldots, x_d]$ can either be a simplex or can have both points Y, Z as vertices and thus $f_i([Y, Z, x_1, \ldots, x_d])$ can at least attain two values with positive probability.) Then it is immediate that (20) remains unchanged

$$\begin{aligned}\{\lambda x \mid x \in [Y, Z, x_1, \ldots, x_d],\ \lambda \in \mathbb{R}_+\} &\supset 2Q(y) \cap H(u, h_K(u) - h)\\ &\supset K \cap H(u, h_K(u) - h)\end{aligned}$$

whereas (21) is replaced by

$$\operatorname{Var}_{Y,Z} f_i([Y, Z, x_1, \ldots, x_d]) \geq c_{13} > 0$$

for all $i = 0, \ldots, d-1$ since f_i is invariant under affine transformations. For $j \in \{1, \ldots, m\}$ let A_j be the event that exactly two random points are contained in $C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0)$ and one in each set $D^i(y)$. Then

$$\begin{aligned}\mathbb{P}(A_j) &= \binom{n}{d+2}\mathbb{P}(X_i \in D^i(y_j),\ i \leq d)\\ &\quad \times \mathbb{P}(X_{d+1}, X_{d+2} \in C^{\frac{1}{2}Q(y_j)}(y_j, h_m\,\delta_0))\mathbb{P}(X_l \notin C_j,\ l \geq d+3)\\ &\geq c_{14} > 0\end{aligned}$$

and

$$\mathbb{E}\left(\sum_{j=1}^{m} I(A_j)\right) = \sum_{j=1}^{m} \mathbb{P}(A_j) \geq c_{14} m.$$

Thus

$$\begin{aligned}\operatorname{Var} f_i(P_n) &\geq \mathbb{E}\operatorname{Var}(f_i(P_n)\,|\,\mathcal{F})\\ &= \mathbb{E}\left(\sum_{I(A_j)=1}\operatorname{Var}_{X_j,X_{j'}} f_i(P_n)\right)\\ &\geq c_{13}\mathbb{E}\left(\sum_{j\in \mathrm{J}} I(A_j)\right)\\ &\geq c_{13}c_{14}\, n^{1-\frac{2}{d+1}}\end{aligned}$$

which is Theorem 4. □

4. Central limit theorems for random points in *K*

The main tool for proving the central limit theorem for $V(\Pi_n)$ and $f_i(\Pi_n)$ is a central limit theorem for dependency graphs due to Rinott [24]. Dependency graphs are defined as follows: Let V_j, $j \in \mathcal{V}$, be a collection of random variables. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be a dependency graph for V_j if for any pair of disjoint sets $A_1, A_2 \subset \mathcal{V}$ such that no edge in $\mathcal{E}$ has one endpoint in A_1 and the other in A_2, the sets of random variables $\{V_i, i \in A_1\}$ and $\{V_i, i \in A_2\}$ are independent.

Theorem 7. **(Rinott)** *Let V_j, $j \in \mathcal{V}$, be random variables having a dependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Set $V = \sum_{j\in\mathcal{V}} V_j$, denote the maximal degree of $\mathcal{G}$ by D and suppose that $|V_j - \mathbb{E}V_j| \leq B$ a.s. Then*

$$\left|\mathbb{P}\left(\frac{V-\mathbb{E}V}{\sqrt{\operatorname{Var} V}} \leq x\right) - \Phi(x)\right| \leq \frac{1}{\sqrt{2\pi}}\frac{DB}{\sigma(V)} + 16\,\frac{|\mathcal{V}|^{\frac{1}{2}} D^{\frac{3}{2}} B^2}{\sigma^2(V)} + 10\,\frac{|\mathcal{V}|\, D^2 B^3}{\sigma^3(V)}.$$

where $\sigma^2(V) = \operatorname{Var} V$.

With a weaker error term an analogous result was proved before by Baldi and Rinott [3]. It has been noted already by Avram and Bertsimas [2] that this approach should be used to prove central limit theorems concerning P_n.

We start with the proof of Theorem 1. Let $\Pi_n = [X_1, \ldots, X_N]$ be the convex hull of the intersection of K with the Poisson point process $X(n)$ of intensity n. Recall that the volume of K equals one. Set

$$m = \left\lfloor \left(\frac{d_6\, n}{(4d+1)\,\ln n}\right)^{1-\frac{2}{d+1}} \right\rfloor$$

where d_6 is the constant appearing in Lemma 6. According to Lemma 6 we choose m points y_j on the boundary of K. Their Voronoi cells $C(y_j)$ dissect the boundary of K into m parts having approximately the same $(d-1)$-dimensional volume, and each Voronoi cell contains a cap denoted by C_j with volume

$$V(C_j) \geq d_6 m^{-\frac{d+1}{d-1}} = (4d+1)\, n^{-1} \ln n.$$

Let A^m be the event that each C_j contains at least one and at most $3(4d+1)\ln n$ points of $\{X_1, \dots, X_N\}$. The probability that C_j contains no point of the Poisson point process is

$$e^{-nV(C_j)} \le n^{-(4d+1)}.$$

By (7) the probability that C_j contains more than $3(4d+1)\ln n$ points is bounded from above by

$$\frac{3}{3-e}\, e^{-nV(C_j)} \le \frac{3}{3-e}\, n^{-(4d+1)}.$$

Thus we obtain

$$1 \ge \mathbb{P}(A^m) \ge 1 - \sum_{j=1}^{m} c_{15}\, n^{-(4d+1)} = 1 - c_{15}\, mn^{-(4d+1)} \ge 1 - c_{16}\, n^{-4d}.$$

Denote by $\widetilde{\mathbb{P}}$ the conditional probability measure induced by the Poisson point process $X(n)$ given A^m, $\widetilde{\mathbb{P}}(V(\Pi_n) \le x) = \mathbb{P}(V(\Pi_n) \le x \mid A^m)$, and by $\widetilde{\mathbb{E}}$ and $\widetilde{\mathrm{Var}}$ the conditional expectation and variance. The distribution function of $V(\Pi_n)$ remains nearly unchanged under the condition A^m since by (10)

$$\left|\widetilde{\mathbb{P}}(V(\Pi_n) \le x) - \mathbb{P}(V(\Pi_n) \le x)\right| \le O(n^{-4d}).$$

The same holds for the first two moments of $V(K) - V(\Pi_n)$. Because $V(\Pi_n) \le 1$ we have

$$\widetilde{\mathbb{E}}\left(V(\Pi_n)^k\right) - \mathbb{E}(V(\Pi_n)^k) \le \mathbb{E}(V(\Pi_n)^k)\left(\frac{1}{\mathbb{P}(A^m)} - 1\right) = O(n^{-4d}) \quad (23)$$

and

$$\mathbb{E}(V(\Pi_n)^k) - \widetilde{\mathbb{E}}\left(V(\Pi_n)^k\right) \le \mathbb{E}\left(V(\Pi_n)^k\left(1 - I(A^m)\right)\right) = O(n^{-4d}) \quad (24)$$

for $k = 1, 2$. This implies that also the variance is unchanged – up to an error term $O(n^{-4d})$ – under the condition A^m,

$$|\widetilde{\mathrm{Var}}\, V(\Pi_n) - \mathrm{Var}\, V(\Pi_n)| = O(n^{-4d}). \quad (25)$$

We introduce m random variables V_j in the following way: given y_j, $j = 1, \dots, m$ as above, the Voronoi cells $C(y_j)$ of y_j in K dissect K into m parts. Now V_j is the volume difference in each part,

$$V_j = V(C(y_j)) - V(C(y_j) \cap \Pi_n)$$

and

$$V = \sum_{j=1}^{m} V_j = V(K) - V(\Pi_n).$$

Given A^m, the dependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for V_j is obtained if we connect each vertex $j \in \mathcal{V} = \{1, \dots, m\}$ by an edge to all $k \in \mathcal{V}$ with $C(y_k) \cap C(y_j, d_{10} m^{-2/(d-1)}) \neq \emptyset$, where d_{10} is the constant given in Lemma 8. Indeed, if there is no edge in $\mathcal{E}$ connecting j and l then according to Lemma 8 there is no edge of Π_n between vertices of Π_n in $C(y_j)$ and $C(y_l)$. Hence V_j and V_l are independent given A^m. (Recall that Π_n is simplicial with probability one.)

To apply the central limit theorem for dependency graphs it remains to show

$$\max V_i \leq B = c_{17}\, m^{-\frac{d+1}{d-1}} \tag{26}$$

since we already know by Lemma 7 that the maximal degree of $\mathcal{G}$ satisfies

$$D \leq d_8\, (d_{10}^{\frac{1}{2}} + 1)^{d+1}$$

and by (6) and (25)

$$\widetilde{\operatorname{Var}} V = \widetilde{\operatorname{Var}} V(\Pi_n) = \operatorname{Var} V(\Pi_n) + O(n^{-4d}) \geq c_{18}\, n^{-1-\frac{2}{d+1}}.$$

To prove (26) we use that, according to Lemma 8, the Hausdorff distance δ^H between K and Π_n is bounded by $d_9 m^{-2/(d-1)}$, given A^m. It follows that $C(y_i) \backslash \Pi_n$ is contained in a cap at y_i of height

$$4 d_1^{-2} d_5^2 m^{-\frac{2}{d-1}} + d_9\, m^{-\frac{2}{d-1}}\,,$$

the height of the cap $\bar{C}_i$ containing $C(y_i) \cap \partial K$ plus the Hausdorff distance of K and Π_n, and applying Lemma 5 we get

$$V_i \leq V(C(y_i, (4 d_1^{-2} d_5^2 + d_9) m^{-\frac{2}{d-1}}) \leq c_{17}\, m^{-\frac{d+1}{d-1}}$$

which is (26). Combining these estimates proves

$$\left| \widetilde{\mathbb{P}} \left(\frac{V - \widetilde{\mathbb{E}} V}{\sqrt{\widetilde{\operatorname{Var}} V}} \leq x \right) - \Phi(x) \right| \leq c_{19}\, n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+\frac{2}{d+1}} n.$$

We rewrite this central limit theorem to obtain a central limit theorem without the condition A^m. Define $\widetilde{x}$ by

$$\mathbb{E} V + x \sqrt{\operatorname{Var} V} = \widetilde{\mathbb{E}} V + \widetilde{x} \sqrt{\widetilde{\operatorname{Var}} V}.$$

Then by (23) – (25) and by (6) we have $|x - \widetilde{x}| = O(n^{-4d+(d+3)/(2(d+1))}) + |x|\, O(n^{-4d+(d+3)/(d+1)})$. This implies

$$\begin{aligned} F_n(x) &= \mathbb{P}(V \leq \mathbb{E} V + x \sqrt{\operatorname{Var} V}) \\ &= \widetilde{\mathbb{P}}(V \leq \widetilde{\mathbb{E}} V + \widetilde{x} \sqrt{\widetilde{\operatorname{Var}} V}) + O(n^{-4d}) \\ &= \Phi(\widetilde{x}) + O(n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+\frac{2}{d+1}} n) + O(n^{-4d}). \end{aligned}$$

Now $|\Phi(\widetilde{x}) - \Phi(x)| = O(n^{-1})$. For $|x| \leq n$ this follows from $|\Phi(x) - \Phi(\widetilde{x})| \leq |x - \widetilde{x}|$, and for $|x| \geq n$ we have by definition $\widetilde{x} \geq cn$ which implies $|\Phi(\widetilde{x}) - \Phi(x)| \leq \Phi(cn) + \Phi(n)$. Thus in both cases we obtain

$$|F_n(x) - \Phi(x)| = O(n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+\frac{2}{d+1}} n)$$

which is Theorem 1. □

The proof of Theorem 2 is very simlar. We only point out the main differences.

By the upper bound theorem due to McMullen [19] the number of i-dimensional faces $f_i(\Pi_n)$ is bounded by $c_d f_0(\Pi_n)^{\lfloor \frac{d}{2} \rfloor} \leq c_d N^{\frac{d}{2}}$ where N again denotes the number of points of the Poisson point process in K. Hence

$$\widetilde{\mathbb{E}}\left(f_i(\Pi_n)^k\right) - \mathbb{E}(f_i(\Pi_n)^k) \leq c_d^k \mathbb{E}(N^{\frac{dk}{2}}) \left(\frac{1}{\mathbb{P}(A^m)} - 1\right) = O(n^{-3d}) \quad (27)$$

and by Hölder's inequality

$$\mathbb{E}(f_i(\Pi_n)^k) - \widetilde{\mathbb{E}}\left(f_i(\Pi_n)^k\right) \leq \mathbb{E}\left(N^{\frac{dk}{2}}\left(1 - I(A^m)\right)\right) \quad (28)$$

$$\leq \sqrt{\mathbb{E}\left(N^{dk}\right) \mathbb{E}\left((1 - I(A^m))\right)} = O(n^{-d}) \quad (29)$$

for $k = 1, 2$. (For $\alpha > 0$ the expectation of N^α is of order λ^α if N is Poisson distributed with parameter λ.) This implies for the variance under the condition A^m

$$|\widetilde{\mathrm{Var}} f_i(\Pi_n) - \mathrm{Var} f_i(\Pi_n)| = O(n^{-d}). \quad (30)$$

We introduce m random variables g_j in the following way: denote by $\mathcal{F}_i(P)$ the set of i-dimensional faces of a polytope P. Given y_j, $j = 1, \ldots, m$ as above, g_j is the number of i-dimensional faces in the Voronoi cell $C(y_j)$ of each y_j, where each face which is contained in more than one cell is counted in each cell according to the number vertices of the face which are contained in that cell. (Recall that Π_n is simplicial with probability one.)

$$g_j = \sum_{F \in \mathcal{F}_i(\Pi_n)} \frac{1}{i+1} \sum_{v \in \mathcal{F}_0(F)} I(v \in C(y_j))$$

and

$$V = \sum_{j=1}^{m} g_j = f_i(\Pi_n).$$

We use the same dependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for g_j as before. We apply the central limit theorem for dependency graphs using as before that

$$D \leq d_8 \, (d_{10}^{\frac{1}{2}} + 1)^{d+1}$$

and further

$$\max g_j \leq B = c_{20} \ln^{i+1} n. \quad (31)$$

To prove this observe all i-dimensional faces having at least one vertex in C_j have the other vertices either in C_j or in C_l where j and l are connected by an edge in $\mathcal{E}$. Since there are at most D indices l connected to j and each cap C_l contains at most $3(4d+1)\ln n$ points we have

$$g_j \le \binom{3(D+1)(4d+1)\ln n}{i+1} \le (3(D+1)(4d+1)\ln n)^{i+1}$$

which is (31). By (9) and (30)

$$\widetilde{\mathrm{Var}}\, f_i(\Pi_n) \ge c_{21}\, n^{1-\frac{2}{d+1}}.$$

Combining these estimates proves

$$\left| \widetilde{\mathbb{P}} \left(\frac{f_i(\Pi_n) - \widetilde{\mathbb{E}} f_i(\Pi_n)}{\sqrt{\widetilde{\mathrm{Var}}(f_i(\Pi_n))}} \le x \right) - \Phi(x) \right| \le c_{22}\, n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+3i+\frac{2}{d+1}} n.$$

We replace $\widetilde{\mathbb{P}}$ by $\mathbb{P}$

$$\begin{aligned} F_n(x) &= \mathbb{P}(f_i(\Pi_n) \le \mathbb{E} f_i(\Pi_n) + x\sqrt{\mathrm{Var}\, f_i(\Pi_n)}\,) \\ &= \widetilde{\mathbb{P}}(f_i(\Pi_n) \le \widetilde{\mathbb{E}} f_i(\Pi_n) + \widetilde{x}\sqrt{\widetilde{\mathrm{Var}}\, f_i(\Pi_n)}) + O(n^{-4d}) \\ &= \Phi(\widetilde{x}) + O(n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+3i+\frac{2}{d+1}} n) + O(n^{-4d}). \end{aligned}$$

It follows from (27) – (30) that $|x - \widetilde{x}|$ is of order $O(n^{-3d-(d-1)/(2(d+1))}) + |x|\, O(n^{-d-(d-1)/(d+1)})$. The same arguments as before imply

$$|F_n(x) - \Phi(x)| = O(n^{-\frac{1}{2}+\frac{1}{d+1}} \ln^{2+3i+\frac{2}{d+1}} n)$$

which is Theorem 2. □

Remark. The independence of the random variables V_j and g_j – which was made precise by using the dependency graph – does not hold if we replace Π_n by P_n. In the case of a fixed number n of random points in K the number of points contained in one Voronoi cell $C(y_j)$ would effect the distribution of the number of points in any other Voronoi cell $C(y_i)$, and thus V_j and g_j would not be independent of V_i and g_i for all $i \ne j$.

5. Appendix: Geometry of smooth convex sets

In the preceding sections we used some well known facts about smooth convex sets which for convenience of the reader are stated and proved explicitly in this section.

Fix $K \in \mathcal{K}^2_+$. At every boundary point x of K there is a paraboloid $Q(x)$ – given by a quadratic form $b_2^{(x)}$ – osculating ∂K at x. It is essential that these paraboloids approximate the boundary of K uniformly for all $x \in \partial K$:

Choose $\delta > 0$ sufficiently small. Then there exists a $\lambda > 0$ only depending on δ and K, such that for each boundary point x of K the following holds: identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$, x with the origin, and $-e_d$ with the *unit outer normal vector of* K *at* x. *The* λ*-neighborhood* U^λ *of* x *in* ∂K *defined by* $\mathrm{proj}_{\mathbb{R}^{d-1}} U^\lambda = \lambda B^{d-1}$ *can be represented by a convex function* $f^{(x)}(y) \in \mathcal{C}^2$, *i.e.,* $(y, f^{(x)}(y)) \in \partial K$ *for* $y \in \lambda B^{d-1}$. *Furthermore*

$$(1+\delta)^{-1} b_2^{(x)}(y) \le f^{(x)}(y) \le (1+\delta) b_2^{(x)}(y)$$

for $y \in \lambda B^{d-1}$.

Denote by $f_{ij}^{(x)}(0)$ the second partial derivatives of $f^{(x)}$ at the origin. Then

$$b_2^{(x)}(y) := \frac{1}{2} \sum_{i,j} f_{ij}^{(x)}(0) y^i y^j$$

and

$$Q(x) := \left\{ (y,z) | z \ge b_2^{(x)}(y) \right\}.$$

This is well known, a proof is contained, e.g., in [20]. By choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$ the quadratic form $b_2(v)$ can be written as

$$b_2(y) = \frac{1}{2}(k_1 \langle y, e_1 \rangle^2 + \cdots + k_{d-1} \langle y, e_{d-1} \rangle^2)$$

where k_i denote the principal curvatures of ∂K at y. Since for all boundary points of K the principal curvatures k_i are bounded from below and above by positive constants we have

$$c_{23} \|y\|^2 \le b_2(y) \le c_{24} \|y\|^2. \tag{32}$$

We introduce polar coordinates: let $\mathbb{R}^d = (\mathbb{R}^+ \times S^{d-2}) \times \mathbb{R}$ and thus denote by (rv, z) a point in $\mathbb{R}^d$, $r \in \mathbb{R}^+$, $v \in S^{d-2}$, $z \in \mathbb{R}$. For abbreviation write $b_2(\cdot)$ and $f(\cdot)$ instead of $b_2^{(x)}(\cdot)$ and $f^{(x)}(\cdot)$:

$$(1+\delta)^{-1} b_2(v) r^2 \le z = f(rv) \le (1+\delta) b_2(v) r^2.$$

This implies

$$(1+\delta)^{-\frac{1}{2}} b_2(v)^{-\frac{1}{2}} z^{\frac{1}{2}} \le r = r(v,z) \le (1+\delta)^{\frac{1}{2}} b_2(v)^{-\frac{1}{2}} z^{\frac{1}{2}}, \tag{33}$$

where r is the radial function of $K \cap H(e_d, z)$. We now choose λ_0 and thus h_0 sufficiently small such that (33) holds with $\delta = 1$ for all $z \le h_0$. Put $d_1 = (2c_{24})^{-1/2}$ and $d_2 = (h_0 + 2/c_{23})^{1/2}$ where c_{23}, c_{24} are the constants chosen in (32). Let $B(y, r)$ be the ball of radius r and center y.

Lemma 4. *Let* $K \in \mathcal{K}_+^2$ *be given. Then there are constants* d_1, d_2 *such that each for cap* $C^K(x,h)$ *with* $h \le h_0$

$$\partial K \cap B(x, d_1 h^{\frac{1}{2}}) \subset C^K(x,h) \subset B(x, d_2 h^{\frac{1}{2}}).$$

From (33) we obtain estimates for the $(d-1)$-dimensional volume of $K \cap H(e_d, z)$:

$$(1+\delta)^{-\frac{d-1}{2}} \kappa_{d-1} \kappa(u)^{-\frac{1}{2}} z^{\frac{d-1}{2}} \leq V_{d-1}(K \cap H(e_d, z)) \leq (1+\delta)^{\frac{d-1}{2}} \kappa_{d-1} \kappa(u)^{-\frac{1}{2}} z^{\frac{d-1}{2}}$$

where κ_{d-1} is the $(d-1)$-volume of the $(d-1)$-dimensional unit ball. By definition

$$V(C^K(x,h)) = \int_0^h V_{d-1}(K \cap H(e_d, z))\, dz.$$

Put $d_3 = 2^{(d+1)/2} \kappa_{d-1} \kappa(u)^{-1/2} (d+1)^{-1}$.

Lemma 5. *Let $K \in \mathcal{K}^2_+$ be given. There is a constant d_3 such that each for cap $C^K(x,h)$ with $h \leq h_0$*

$$V(C^K(x,h)) \leq d_3 h^{\frac{d+1}{2}}.$$

It is well known that for given m there are points $y_1, \ldots, y_m$ on the boundary of K such that for a suitable r_m the balls $B(y_i, r_m)$, $i = 1, \ldots, m$, are pairwise disjoint, and the union of the balls $B(y_i, 2r_m)$, $i = 1, \ldots, m$, is a covering of ∂K. (If $B(y_i, r_m)$, $i = 1, \ldots, m$, is a maximal packing on ∂K, i.e., there is no point $y \in \partial K$ such that $B(y, r_m)$ is disjoint from all other balls $B(y_i, r_m)$, then each point $y \in \partial K$ has distance at most $2r_m$ from at least one point y_i.) Hence $\sum \kappa_{d-1} r_m^{d-1}$ is approximately the surface area of K. Denote by $C(y_j)$ the Voronoi cell of y_i in K:

$$C(y_i) = \{x \in K : \|x - y_i\| \leq \|x - y_k\| \text{ for all } k \neq i\}.$$

By the above remark the Voronoi cell $C(y_i)$ contains the ball of radius r_m and the Voronoi cell $C(y_i) \cap \partial K$ on the boundary of K is contained in a ball of radius $2r_m$. By Lemma 4 a cap C_j of height $d_2^{-2} r_m^2$ is contained in the Voronoi cell $C(y_j)$, and $C(y_j) \cap \partial K$ is contained in a cap $\bar{C}_j$ of height $4d_1^{-2} r_m^2$ as long as m is sufficiently large, $m \geq m_0$. We summarize this in the following lemma:

Lemma 6. *Let $m \geq m_0$ and $K \in \mathcal{K}^2_+$ be given. Then there are points $y_1, \ldots, y_m \in \partial K$ and caps $C_i = C^K(y_i, h_m)$ and $\bar{C}_i = C^K(y_i, \bar{h}_m)$ with*

$$C_i \subset B(r_m, y_i) \subset C(y_i),$$
$$C(y_i) \cap \partial K \subset B(y_i, 2r_m) \cap \partial K \subset \bar{C}_i,$$

and

$$h_m = d_2^{-2} r_m^2 \text{ and } \bar{h}_m = 4 d_1^{-2} r_m^2.$$

Further there are constants $d_4, \ldots, d_7$ such that

$$d_4\, m^{-\frac{1}{d-1}} \leq r_m \leq d_5\, m^{-\frac{1}{d-1}} \tag{34}$$

and

$$d_6\, m^{-\frac{d+1}{d-1}} \leq V(C_j) \leq d_7\, m^{-\frac{d+1}{d-1}}$$

for all $i = 1, \ldots, m$.

The next lemma estimates the number of Voronoi cells $C(y_i)$ which have nonempty intersection with a cap $C^K(y_j, h)$. Assuming that $h \leq h_1$ is sufficiently small, the Voronoi cell $C(y_i)$ has nonempty intersection with $C^K(y_j, h)$ if $C(y_i) \cap \partial K$ and thus $B(y_i, 2r_m)$ has nonempty intersection with $C^K(y_j, h)$. By Lemma 4, $C^K(y_j, h)$ is contained in a ball $B(y_j, d_2 h^{1/2})$, and thus all for all Voronoi cells having nonempty intersection with $C^K(y_j, h)$ we have

$$C_i \subset [B(y_j, d_2 h^{1/2} + 2r_m) \cap \partial K] \subset C^K(y_j, d_1^{-2}(d_2 h^{1/2} + 2r_m)^2) \qquad (35)$$

for $h \leq h_1, m \geq m_0$. (Recall that $[A]$ denotes the convex hull of the set A.) Summing over all i satisfying (35) we obtain

$$\sum_i d_6 m^{-\frac{d+1}{d-1}} \leq \sum_i V(C_i) \leq V\left(C^K(y_j, d_1^{-2}(d_2 h^{1/2} + 2r_m)^2)\right)$$
$$\leq d_3 \left(d_1^{-2}(d_2 h^{1/2} + 2r_m)^2\right)^{\frac{d+1}{2}} .$$

This proves

Lemma 7. *Let $m \geq m_0$, $K \in \mathcal{K}_+^2$ be given, and choose points y_i, $i = 1, \ldots, m$, according to Lemma 6. Then there is a constant d_8 such that the number of Voronoi cells $C(y_i)$ intersecting the cap $C^K(y_j, h)$ is bounded by $d_8 \left(h^{1/2} m^{1/(d-1)} + 1\right)^{d+1}$.*

The next Lemma deals with the approximation of K by a polytope which is constructed using the Voronoi cells of the points $y_1, \ldots, y_m$. Assume that points $y_1, \ldots, y_m$ are given according to the previous lemma and choose in each cap C_i, $i = 1, \ldots, m$ an arbitrary point x_i. The Hausdorff distance δ^H between K and the convex hull $[x_1, \ldots, x_m]$ is bounded by $16 d_1^{-2} d_5^2 m^{-2/(d-1)}$. Otherwise there would be a facet of $[x_1, \ldots, x_m]$ with larger distance to ∂K, or equivalently, there is a cap $C^K(y, 16 d_1^{-2} d_5^2 m^{-2/(d-1)})$ which contains no point of $x_1, \ldots, x_m$. By Lemma 4 the cap $C^K(y, 16 d_1^{-2} d_5^2 m^{-2/(d-1)})$ contains the intersection of a ball of radius $4 d_5 m^{-1/(d-1)}$ with ∂K. Since by (34) the circumradius of each Voronoi cell is at most $2 d_5 m^{-1/(d-1)}$ this implies that the cap $C^K(y, 16 d_1^{-2} d_5^2 m^{-2/(d-1)})$ contains at least one cap C_j and thus at least one of the points $x_1, \ldots, x_m$. Hence

$$\delta^H(K, [x_1, \ldots, x_m]) \leq 16 d_1^{-2} d_5^2 m^{-\frac{2}{d-1}}. \qquad (36)$$

Let $y \in K$ be an arbitrary point with distance δ to x_j. Since $K \in \mathcal{K}_+^2$ there is a number $R > 0$ depending only on K such that for each boundary point x there is a ball B_R of radius R touching K at x and containing K. It is elementary to see that the midpoint of a segment of length δ in a ball of radius R has distance at least $\frac{\delta^2}{8R}$ to the boundary of B_R. Thus also the segment $[y, x_j]$ has at least distance $\frac{\delta^2}{8R}$ to the boundary of K. By (36) this implies that the line segment $[y, x_j]$ intersects the interior of the convex hull $[x_1, \ldots, x_m]$ if

$$\|x_j - y\| \geq 16\sqrt{R} d_1^{-1} d_5 m^{-\frac{1}{d-1}}. \qquad (37)$$

Assume that y is chosen such that $y \in K \backslash [x_1, \ldots, x_m]$ and y is not contained in $C^K(y_j, \delta^2 m^{-2/(d-1)})$ where δ will be chosen later. By (36) the distance of y to the boundary of K is bounded by $16d_1^{-2}d_5^2m^{-2/(d-1)}$. Hence y is contained in a cap with this height. Denote by $y^{\partial K} \in \partial K$ a point of this cap which is not contained in $C^K(y_j, \delta^2 m^{-2/(d-1)})$. To estimate the distance from x_j to y we use that

$$\|y_j - y^{\partial k}\| \le \|y_j - x_j\| + \|x_j - y\| + \|y - y^{\partial k}\|.$$

By Lemma 4 the distance between y_j and $y^{\partial k}$ is bounded from below by $d_1\delta m^{-1/(d-1)}$, the distance between y and $y^{\partial K}$ is bounded from above by $4d_1^{-1}d_2d_5m^{-1/(d-1)}$, and since $x_j \in C_j$ the distance between y_j and x_j is bounded by $r_m \le d_5 m^{-1/(d-1)}$. Thus

$$\|x_j - y\| \ge \left(d_1\delta - \left(d_5 + 4d_1^{-1}d_2d_5\right)\right) m^{-\frac{1}{d-1}}$$

and choosing δ sufficiently large we see that (37) is satisfied and thus $[x_j, y]$ meets the interior of $[x_1, \dots, x_m]$.

Lemma 8. *Let $m \in \mathbb{N}$, $K \in \mathcal{K}_+^2$, and points y_i, $i = 1, \dots, m$, chosen according to Lemma 6, be given. Choose in each cap C_i an arbitrary point x_i. Then there are constants d_9, d_{10} depending on K such that*

$$\delta^H(K, [x_1, \dots, x_m]) \le d_9 m^{-\frac{2}{d-1}}$$

and such that for $y \in K$ with $y \notin C^K(y_j, d_{10}m^{-2/(d-1)})$ implies that the line segment $[y, x_i]$ intersects the interior of the convex hull $[x_1, \dots, x_m]$.

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