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Reflected symmetric α-stable processes and regional fractional Laplacian

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Abstract. In this paper we investigate the reflected symmetric α-stable processes and their generators. We show that the generators are regional fractional Laplacians on the closed region. In the case of $1 \leq \alpha < 2$ their existence requires that $\frac{\partial u}{\partial n} = 0$ on the boundary. Among other things we obtain the integration by parts formula of the regional fractional Laplacian and the semi-martingale decomposition of the reflected symmetric α-stable processes.

Contents

1. Introduction

Let $0 < \alpha < 2$ and G be a Lipschitz open set of $\mathbb{R}^n$. In this paper we investigate the reflected (symmetric) α-stable process $X = (X_t)_{t \geq 0}$ on $\overline{G}$ which is by definition a Hunt process associated with the following regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\overline{G}, dx)$:

$$\mathcal{E}\big(u, v\big) = \frac{1}{2}\mathcal{A}(n, -\alpha) \int\int_{\overline{G}\times\overline{G}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}}\, dxdy,$$

$$\mathcal{F} = \left\{u \in L^2(\overline{G}),\quad \int\int_{\overline{G}\times\overline{G}} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dxdy < \infty\right\}. \tag{1.1}$$

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Note that when $G = \mathbb{R}^n$, the above process is the usual α-stable process on $\mathbb{R}^n$. In the last few years there has been an increasing interest in the study of α-stable processes which are now widely used in physics (Lévy flights in some of the physics literature), operation research, queuing theory, mathematical finance, risk estimation, and others (see e.g. [2][3][4][18][29][33][37]). Because of the importance of processes in an open set instead of in the whole Euclidean space, in the last few years appeared also number of publications concerning the properties of α-stable processes in a bounded open set (see e.g. [5],[6]-[10],[12]-[16], [23][30][35]). However, due to the difficulties caused by the discontinuity of the sample pathes and the non-local behavior of the generators, the theory of α-stable processes in a bounded open set is still less developed than its continuous counterpart Brownian motion.

To our knowledge the study of the reflected α-stable process $(X_t)_{t\geq 0}$ associated with (1.1) was initiated by K. Bogdan, K. Burdzy and Z.-Q. Chen in their publication [5], where the authors proved that if $0 < \alpha \leq 1$, then the censored α-stable process in G is essentially the same as the reflected α-stable process, while if $1 < \alpha < 2$, then the censored α-stable process in G is identified as a proper subprocess of $(X_t)_{t\geq 0}$ killed upon leaving G (In fact the authors obtained the above result in a more general context of n-set boundary condition). A fundamental result of $(X_t)_{t\geq 0}$ concerning its Feller property and heat kernel estimates was contained in a recent paper [12] by Chen and Kumagai, where the authors proved, again in a more general context, that the reflected α-stable process can be refined to be a Feller process starting from each point $x \in \overline{G}$ and admits a Hölder continuous transition density function $p(t, x, y)$. Furthermore, there are constants $c_2 > c_1 > 0$ such that

$$c_1\big(t^{-n/\alpha} \wedge \frac{t}{|x-y|^{n+\alpha}}\big) < p(t,x,y) < c_2\big(t^{-n/\alpha} \wedge \frac{t}{|x-y|^{n+\alpha}}\big) \tag{1.2}$$

for all $x, y \in \overline{G}$ and $0 < t \leq 1$.

By the results of [5], there are typically three types of α-stable processes with values in an open set G: the subprocess of the $\mathbb{R}^n$-valued α-stable process killed upon leaving G, the censored process in G and the reflected α-stable process on $\overline{G}$. The three types of processes are in such an order that the first mentioned process is identified as a subprocess of the second one killing inside G and the second mentioned process is in turn identified as a subprocess of the last mentioned one killed upon leaving G. In other words, the reflected α-stable process is identified as the original process of all the three typical α-stable processes in G. It implies further more that all the mixture of the three types processes are subprocess of the reflected α-stable processes on $\overline{G}$. For an important class of mixed type processes we mention the α-stable process in G with mixed boundary behavior, which is very useful in the study of boundary value problems related to the regional fractional Laplacian, see our forthcoming paper [22].

In spite of its important role, the theory of reflected α-stable processes is far from mature and adequate. There are many basic questions remaining to be investigated. The purpose of this paper is to investigate two fundamental aspects of reflected α-stable processes. We shall describe the generators and obtain a semi-martingale decomposition of the reflected α-stable processes. To achieve our goal we shall introduce a notion of regional fractional Laplacian $\Delta_G^{\frac{\alpha}{2}}$ and study its analytic properties. In particular we obtain an integration by parts formula for $\Delta_G^{\frac{\alpha}{2}}$ and obtain a necessary and sufficient condition for the existence of $\Delta_G^{\frac{\alpha}{2}}u(x)$ on the boundary points $x \in \partial G$. These results are also of interest by their own.

Let us describe our results in more details. In the next section we introduce the notion of regional fractional Lapalacian $\Delta_G^{\frac{\alpha}{2}}$ in an arbitrary open set G of $\mathbb{R}^n$(Def. 2.1). When $G = \mathbb{R}^n$, $\Delta_G^{\frac{\alpha}{2}}$ coincides with the usual fractional Lapalacian $\Delta^{\frac{\alpha}{2}}$ defined by Fourier transform. While if $G \neq \mathbb{R}^n$ then $\Delta_G^{\frac{\alpha}{2}}$ is different from $\Delta^{\frac{\alpha}{2}}$ and can not be analyzed by the Fourier transform. In this section we shall prove that if u is β-Hölder continuous for some $\beta > \alpha$ in the case of $0 < \alpha < 1$, or if the derivatives of u are all β-Hölder continuous for some $\beta > \alpha - 1$ in the case of $1 \leq \alpha < 2$, then $\Delta_G^{\frac{\alpha}{2}}u(x)$ exists for all $x \in G$ (cf. Proposition 2.2). Moreover, in the case of uniformly β-Hölder continuity (in symbols $u \in C^{\beta}(G)$ or $u \in C^{1+\beta}(G)$ respectively), we give upper bound estimates of $|\Delta_G^{\frac{\alpha}{2}}u|$ (cf. Proposition 2.3). Based on these estimates, we show in Section 3 that $\Delta_G^{\frac{\alpha}{2}}u$ is integrable in a bounded Lipschitz open set and thus obtain an integration by parts formula for $\Delta_G^{\frac{\alpha}{2}}u$ in Lipschitz open set (cf. Theorems 3.3 and 3.4), which plays the role of Green's formula in the Laplacian case. An ingredient in this section is Lemma 3.1 in which we explore some properties of Lipschitz open set in terms of $\rho(x) := dis(x, \partial G)$ (cf. (3.3) and (3.4)), which might be useful elsewhere.

The semi-martingale decomposition of the Feller process $(X_t)_{t\geq 0}$ associated with (1.1) is investigated in Section 4. With the preparation of Sections 2 and 3, and making use of the heat kernel estimates (1.2), we show that if G is a bounded Lipschitz open set, then for any $u \in C^2(\overline{G})$ we have the semi-martingale decomposition of $u(X_t)$:

$$u(X_t) = u(x_0) + M_t + \int_0^t \Delta_G^{\frac{\alpha}{2}}u(X_s)ds, \quad P_{x_0}\ a.s., \quad \forall x_0 \in \overline{G}, \tag{1.3}$$

where $(M_t)_{t\geq 0}$ is a pure jump martingale with the Revuz measure of $< M >_t$ described by the jump measure related to (1.1)(cf. Theorem 4.1). In particular, letting u be the coordinate functions we conclude from (1.3) that $(X_t)_{t\geq 0}$ itself is a semi-martingale and we obtain a semi-martingale decomposition of $(X_t)_{t\geq 0}$(cf. Corollary 4.2), which serves as the Skorohod decomposition of the reflected Brownian motion on a Lipschitz open set. In one dimensional case we have a more concrete semi-martingale decomposition(cf. Corollary 4.3). A main step in proving Theorem 4.1 is Lemma 4.6, which implies that $\Delta_G^{\frac{\alpha}{2}}u(x)dx$ is a difference of two measures in S_{00} and hence we can make use of a strict version of Fukushima's decomposition(cf. Theorem 5.2.5 in [19]) to obtain (1.3).

As an preparation for investigating the generators of (X_t), in Section 5 we invent the notion of regional fractional Laplacian on a closed region $\overline{G} = G \cup \partial G$, denoted by $\Delta_{\overline{G}}^{\frac{\alpha}{2}}$ (cf. Definition 5.1). Special attention is paid to study $\Delta_{\overline{G}}^{\frac{\alpha}{2}}u(z)$ at the point $z \in \partial G$ in the case of $1 \leq \alpha < 2$. We are able to prove that under some mild regularity conditions, $\Delta_G^{\frac{\alpha}{2}} u(x)$ exists on a boundary point $x \in \partial G$ if and only if $\frac{\partial u}{\partial n}(x) = 0$ (cf. Theorem 5.3). Moreover, if $\frac{\partial u}{\partial n} = 0$ in a relatively open subset of ∂G containing z, then $\Delta_G^{\frac{\alpha}{2}} u$ is continuous at $z \in \partial G$ (cf. Theorem 5.4). To prove these results we need to come up with some ingenious trick; see Lemma 5.2 and the arguments of Theorem 5.3 and 5.4 for details. With the above preparations we obtain in Section 6 some quite satisfactory results about the generators of the reflected α-stable process (X_t). Roughly speaking, the Feller generator of (X_t) is nothing but the regional fractional Laplacian on the closed region, thus in the case of $1 \le \alpha < 2$, a smooth function u is in the domain of the Feller generator of $(X_t)_{t \ge 0}$ if and only if $\frac{\partial u}{\partial n} = 0$ on the boundary; see Theorem 6.1 for the accurate statement. The L^p-generators $(A_p^\alpha, \mathcal{D}(A_p^\alpha))$ of (X_t) are also investigated in this section. In the case of $0 < \alpha < \frac{p+1}{p}$, there is no requirement of the boundary behavior for a function to be in the domain $\mathcal{D}(A_p^\alpha)$ of the L^p-generator, while in the case of $\frac{p+1}{p} \le \alpha < 2$, a necessary and sufficient condition for a smooth function u to be in $\mathcal{D}(A_p^\alpha)$ is again that $\frac{\partial u}{\partial n} = 0$ on the boundary. See Theorem 6.3 and 6.4 for detailed statements. We believe that in the case of $1 \le \alpha < 2$ our results concerning the generators of (X_t) are completely new.

In the last two sections we discuss in more detail the regional fractional Laplacian in one dimensional case. In section 7 we consider the situation of $G = (0, 1)$ and $1 < \alpha < 2$. We show that if the limit $F^{2-\alpha} u(z) := \lim_{x \to z} \frac{u'(x)}{(dis(x, \partial G))^{\alpha-2}}$ exists for a function u on G and for $z \in \partial G$, then $F^{2-\alpha} u$ will appear in the integration by parts formula. For details see Theorem 7.5, see also the first paragraph of Section 7 for an intuitive explanation of the result. In Section 8 we prove that the one dimensional regional fractional Laplacian maps $C^{k+2}(G)$ into $C^k(G)$ for any $k \ge 1$ (cf. Theorem 8.1), which implies that $\Delta_G^{\frac{\alpha}{2}} u$ maps smooth functions into smooth functions. We guess that the phenomena suggested by these two sections should be true also in higher dimensional case, but at this stage we can not verify them, except that with some tedious calculations we verified the first order smoothness of $\Delta_G^{\frac{\alpha}{2}} u$ in higher dimensional case (cf. Proposition 8.3). Further investigations are still wanted.

2. Regional fractional Laplacian on G

In this section we assume that $0 < \alpha < 2$ and G is an open set of $\mathbb{R}^n$. Denote by $\mathcal{L}^1 := \mathcal{L}^1(G, \frac{dx}{(1+|x|)^{n+\alpha}})$ all the measurable functions u on G such that

$$\int_G \frac{|u(x)|}{(1 + |x|)^{n+\alpha}} \, dx < \infty. \tag{2.1}$$

For $u \in \mathcal{L}^1$, $x \in G$ and $\varepsilon > 0$, we write

$$\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u(x) = \mathcal{A}(n, -\alpha) \int_{y \in G, |y-x| > \varepsilon} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, dy, \tag{2.2}$$

where $\mathcal{A}(n,-\alpha)=\dfrac{|\alpha|2^{\alpha-1}\Gamma(\frac{n+\alpha}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\frac{\alpha}{2})}$.

Definition 2.1. *Let $u\in\mathcal{L}^1$, we define the regional fractional Laplacian by the formula*

$$\Delta_G^{\frac{\alpha}{2}}u(x)=\lim_{\varepsilon\downarrow 0}\Delta_{G,\varepsilon}^{\frac{\alpha}{2}}u(x),\quad x\in G, \tag{2.3}$$

provided the limit exists.

Remark. When $G=\mathbb{R}^n$, (2.3) coincides with $\Delta^{\frac{\alpha}{2}}$ described in Definition 3.2 of [7], which in turn coincides with the usual fractional Laplacian defined by Fourier transform(see e.g. [1]). In this paper we use the phrase "regional fractional Laplacian" to indicate that $\Delta_G^{\frac{\alpha}{2}}$ relies on the region G and is different from $\Delta^{\frac{\alpha}{2}}$ when $G\neq\mathbb{R}^n$. In [5] $\Delta_G^{\frac{\alpha}{2}}$ is denoted by $A_G^{\frac{\alpha}{2}}$ and is identified as the operator $\Delta^{\frac{\alpha}{2}}+\kappa_G$ for sufficiently regular functions u defined on the whole space which vanish on $G^c:=\mathbb{R}^n\setminus G$, where

$$\kappa_G(x)=\mathcal{A}(n,-\alpha)\int_{G^c}\frac{1}{|x-y|^{n+\alpha}}dy. \tag{2.4}$$

Let u be a function on G and $0<\beta\le 1$. We say that u is β-Hölder continuous on G if

$$\lim_{\delta\downarrow 0}\sup_{y\in G,|y-x|<\delta}\frac{|u(y)-u(x)|}{|x-y|^\beta}<\infty;$$

u is uniformly β-Hölder continuous on G if

$$\sup_{(y,z)\in G\times G,|y-z|<1}\frac{|u(y)-u(z)|}{|z-y|^\beta}<\infty;$$

u is locally uniformly β-Hölder continuous on G if

$$\sup_{(y,z)\in A\times A,|y-z|<1}\frac{|u(y)-u(z)|}{|z-y|^\beta}<\infty$$

for each compact subset $A\subset G$. We shall denote by $u\in C^\beta(G)$ if u is uniformly β-Hölder continuous on G, and by $u\in C^{1+\beta}(G)$ if all the first derivatives of u exist and are uniformly β-Hölder continuous on G.

Proposition 2.2. *Let $u\in\mathcal{L}^1$.*

(i) Let $0<\alpha<1$. If u is β-Hölder continuous on G for some $\beta>\alpha$, then $\Delta_G^{\frac{\alpha}{2}}u(x)$ exists for $x\in G$. If in addition u is locally uniformly β-Hölder continuous on G, then $\Delta_G^{\frac{\alpha}{2}}u$ is continuous on G and $\Delta_{G,\varepsilon}^{\frac{\alpha}{2}}u(x)$ converges to $\Delta_G^{\frac{\alpha}{2}}u(x)$ locally uniformly as $\varepsilon\downarrow 0$.

(ii) Let $1 \le \alpha < 2$. If $u \in C^1(G)$ and all the first derivatives of u are β-Hölder continuous on G for some $\beta > \alpha - 1$, then $\Delta_G^{\frac{\alpha}{2}} u$ exists for $x \in G$. If in addition all the first derivatives of u are locally uniformly β-Hölder continuous on G, then $\Delta_G^{\frac{\alpha}{2}} u$ is continuous and $\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u(x)$ converges to $\Delta_G^{\frac{\alpha}{2}} u(x)$ locally uniformly on G as $\varepsilon \downarrow 0$.

Proof. It is easy to check that if $u \in \mathcal{L}^1$ and $\varepsilon > 0$, then $\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u(x)$ is well defined on G. Moreover, $\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u$ is continuous when u is continuous. In case (i), by the β-Hölder continuity of u, for $x \in G$ we can take $t > 0$ such that

$$M := \sup_{y \in G, |y-x|<t} \frac{|u(y) - u(x)|}{|x - y|^{\beta}} < \infty. \tag{2.5}$$

Let $0 < \alpha < 1$. For $0 < \varepsilon < \delta < t$ we have

$$\begin{aligned}
&|\Delta_{G,\delta}^{\frac{\alpha}{2}} u(x) - \Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u(x)| \\
&\quad = \mathcal{A}(n, -\alpha) \left| \int_{\varepsilon < |y-x| \le \delta, y \in G} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, dy \right| \\
&\quad \le \mathcal{A}(n, -\alpha) \int_{\varepsilon < |y| \le \delta} \frac{M}{|y|^{n+\alpha-\beta}} dy \\
&\quad = M\mathcal{A}(n, -\alpha) \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \int_{\varepsilon}^{\delta} dr \\
&\qquad r^{\beta-\alpha-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} \\
&\quad \le \frac{(2\pi)^n M\mathcal{A}(n, -\alpha)}{\beta - \alpha} (\delta^{\beta-\alpha} - \varepsilon^{\beta-\alpha}),
\end{aligned} \tag{2.6}$$

where (and hence forth) $d\theta_1 \cdots d\theta_{n-1} dr$ refers to the spherical coordinate system. Consequently $\Delta_G^{\frac{\alpha}{2}} u(x)$ exists. If in addition u is locally uniformly β-Hölder continuous for some $\beta > \alpha$, then for any compact set $A \subset G$, we can find $t > 0$ and $M > 0$ such that (2.5) holds for all $x \in A$, and hence by (2.6) $\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u(x)$ converges to $\Delta_G^{\frac{\alpha}{2}} u(x)$ uniformly on A. In case (ii), let us write

$$G_t := \{y \in G : \ dist(x, \partial G) > t\}. \tag{2.7}$$

Then by the Hölder continuity of the derivatives of u, for $x \in G_t$ we can find $0 < s < t$ such that

$$N := \sup_{y \in G, |y-x|<s} \frac{|\nabla u(y) - \nabla u(x)|}{|x - y|^{\beta}} < \infty. \tag{2.8}$$

If the derivatives of u are locally uniformly β-Hölder continuous, then for any compact set $A \subset G_t$, we can find $0 < s < t$ and $N > 0$ such that (2.8) holds for all $x \in A$. Let $1 \le \alpha < 2$, for $0 < \varepsilon < \delta < s$, we have

$$
\begin{aligned}
&|\Delta^{\frac{\alpha}{2}}_{G,\delta}u(x) - \Delta^{\frac{\alpha}{2}}_{G,\varepsilon}u(x)| \\
&\quad = \mathcal{A}(n,-\alpha)\left|\int_{\varepsilon<|y-x|\le\delta, y\in G} \frac{(u(y)-u(x)) - \langle\nabla u(x), y-x\rangle}{|x-y|^{n+\alpha}}\,dy\right| \\
&\quad = \mathcal{A}(n,-\alpha)\left|\int_{\varepsilon<|y-x|\le\delta, y\in G} \frac{\langle\nabla u(\xi(y)) - \nabla u(x), y-x\rangle}{|x-y|^{n+\alpha}}\,dy\right| \\
&\quad \le N\mathcal{A}(n,-\alpha)\int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-2}\int_0^{2\pi} d\theta_{n-1}\int_{\varepsilon}^{\delta} dr \\
&\qquad \cdot r^{\beta-\alpha}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2} \\
&\quad \le \frac{(2\pi)^n N\mathcal{A}(n,-\alpha)}{\beta-\alpha}(\delta^{1+\beta-\alpha} - \varepsilon^{1+\beta-\alpha}),
\end{aligned}
$$

where $\xi(y)$ is determined by the mean value theorem. Thus, we arrive the same conclusion as in case (i). □

For convenience we introduce the following notations.

$$M_\beta(u) := \sup_{x,y\in G}\frac{|u(y)-u(x)|}{|y-x|^\beta} \tag{2.9}$$

$$N_\beta(u) := \sup_{x,y\in G}\sum_{i=1}^n\frac{|\partial_i u(y) - \partial_i u(x)|}{|y-x|^\beta} \tag{2.10}$$

$$\rho(x) := dist(x,\partial G) = \inf\{|y-x| : y\in\partial G\} \tag{2.11}$$

$$d_G := diameter\ of\ G = \sup\{|y-x| : y\in\partial G\} \tag{2.12}$$

$$B(x,\delta) := \{y : |y-x| < \delta\} \tag{2.13}$$

Proposition 2.3. *Let G be a bounded open set.*

(i) Let $0<\alpha<1$. If $u\in C^\beta(G)$ for some $\beta>\alpha$, then $\Delta^{\frac{\alpha}{2}}_G u$ is bounded on G with the following estimate.

$$|\Delta^{\frac{\alpha}{2}}_G u(x)| \le \mathcal{A}(n,-\alpha)(2\pi)^n M_\beta(u)\frac{d_G^{\beta-\alpha}}{\beta-\alpha}. \tag{2.14}$$

(ii) Let $1\le\alpha<2$. If $u\in C^{1+\beta}(G)$ for some $\beta>\alpha-1$, then $\Delta^{\frac{\alpha}{2}}_G u$ is bounded on G_δ ($cf.$(2.7)) for each $\delta>0$. Moreover, we have the following estimates.

$$|\Delta^{\frac{\alpha}{2}}_G u(x)| \le \mathcal{A}(n,-\alpha)(2\pi)^n \left(M_1(u)\ln\frac{d_G}{\rho(x)} + \frac{N_\beta(u)}{\beta}\rho(x)^\beta\right), \quad when\ \alpha=1; \tag{2.15}$$

$$|\Delta^{\frac{\alpha}{2}}_G u(x)| \le \mathcal{A}(n,-\alpha)(2\pi)^n \left(\frac{M_1(u)}{\alpha-1}(\rho(x)^{1-\alpha} - d_G^{1-\alpha}) + \frac{N_\beta(u)}{\beta-\alpha+1}\rho(x)^{\beta-\alpha+1}\right), \quad when\ 1<\alpha<2. \tag{2.16}$$

Proof. We prove only the case (ii) with $1<\alpha<2$, the others can be proved analogously. We have

$$\left|\Delta_G^{\frac{\alpha}{2}}u(x)\right| = \lim_{\varepsilon\downarrow 0}\mathcal{A}(n,-\alpha)\left|\int_{G\cap B(x,\varepsilon)^c}\frac{u(y)-u(x)}{|x-y|^{n+\alpha}}\,dy\right|$$
$$= \lim_{\varepsilon\downarrow 0}\mathcal{A}(n,-\alpha)\bigg|\int_{G\cap B(x,\rho(x))^c}\frac{u(y)-u(x)}{|x-y|^{n+\alpha}}\,dy \qquad (2.17)$$
$$+\int_{B(x,\rho(x))\cap B(x,\varepsilon)^c}\frac{(u(y)-u(x))-\langle\nabla u(x),x-y\rangle}{|x-y|^{n+\alpha}}\,dy\bigg|$$
$$\leq \lim_{\varepsilon\downarrow 0}\mathcal{A}(n,-\alpha)\int_0^{\pi}d\theta_1\cdots\int_0^{\pi}d\theta_{n-2}\int_0^{2\pi}d\theta_{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}$$
$$\cdot\left(\int_{\rho(x)}^{d_G}\frac{M_1(u)}{r^{\alpha}}dr+\int_{\varepsilon}^{\rho(x)}\frac{N_\beta(u)}{r^{\alpha-\beta}}dr\right)$$
$$\leq \mathcal{A}(n,-\alpha)(2\pi)^n\left(\frac{M_1(u)}{\alpha-1}(\rho(x)^{1-\alpha}-d_G^{1-\alpha})+\frac{N_\beta(u)}{\beta-\alpha+1}\rho(x)^{\beta-\alpha+1}\right).$$

□

3. Integration by parts formula

In what follows we assume that G is a bounded Lipschitz open set. By the compactness of ∂G there exist $r_0>0$ and $\eta>0$ such that for each $x\in\partial G$, we can find a Lipschitz function $\Gamma_x:\mathbb{R}^{n-1}\to\mathbb{R}$ with Lipschitz coefficient not greater than η and an orthonormal coordinate system CS_x with which it holds that

$$G\cap B\big(x,r_0\big)=\{y=(y_1,\cdots,y_n):\ y_n>\Gamma_x(\,y_1,\cdots,y_n)\,\}\cap B\big(x,r_0\big). \quad (3.1)$$

We call r_0 the localization radius of G and η the Lipschitz constant of G. In what follows we write $r_1:=\frac{1}{3}r_0$. By (3.1) one can easily check that for each $x\in\partial G$ and $y\in G\cap B(x,r_1)$, it holds that

$$\rho(y)=\inf\{|y-z|:\ z=(z_1,\cdots,z_n)\in B(x,r_0),\ \ z_n=\Gamma_x(z_1,\cdots,z_{n-1})\}. \quad (3.2)$$

Lemma 3.1. *Let G be a bounded Lipschitz open set in $\mathbb{R}^n$, then there exist positive numbers C_1, C_2 such that*

$$\int_{G_s-G_t}dy\leq C_1(t-s),\quad C_2>t>s>0, \quad (3.3)$$

where G_t is defined by (2.7).

Proof. Let $\{\Gamma_x\}$, $\{CS_x\}$, r_0, η be as in the beginning of this subsection and take $C_2=\frac{r_1}{2}$. We shall prove the Lemma in two cases.

Case I: $0<t<(\frac{\sqrt{1+\eta^2}}{\eta}+1)\frac{s}{2}$ and $t<C_2$. Let

$$V_y=\{z:\ z_n\geq y_n,\ (z_1,z_2,\cdots,z_{n-1})=(y_1,y_2,\cdots,y_{n-1})\}\cap B(x,r_1),$$
$$\forall y\in B(x,r_1)\cap\partial G.$$

$$A=\big\{z\in G:\ t\geq dist(z,\partial G)>s\big\},\quad c_\eta=\frac{2(1+\eta^2)}{(1+\eta^2-\eta\sqrt{1+\eta^2})\eta}.$$

We are going to show that

$$|\bar{z} - \bar{\bar{z}}| < \sqrt{1 + c_\eta^2}(t - s), \quad \forall\, \bar{z},\ \bar{\bar{z}} \in V_y \cap A. \tag{3.4}$$

Denote

$$s^* = \inf\{s :\ s = z_n - y_n, z \in V_y \cap A\}, \quad z^* = (y_1, \cdots, y_{n-1}, s^* + y_n), \tag{3.5}$$

It is not hard to see that the proof of (3.4) can be reduced to verify that

$$|\bar{z} - w| > t, \tag{3.6}$$

for $w \in B(x, r_0) \cap \partial G$ and $\bar{z} \in \{z = (z_1, \cdots, z_n) \in V_y : z_n \geq z_n^* + \sqrt{1 + c_\eta^2}(t - s)\}$. Write

$$\begin{aligned} a &= |(w_1 - z_1^*, \cdots, w_{n-1} - z_{n-1}^*)|, \\ \bar{z}^* &= z^* + \left(\frac{(w_1 - z_1^*)\eta s}{a\sqrt{1 + \eta^2}}, \cdots, \frac{(w_{n-1} - z_{n-1}^*)\eta s}{a\sqrt{1 + \eta^2}}, \frac{-s}{\sqrt{1 + \eta^2}} \right). \end{aligned}$$

Then we can check that $|z^* - \bar{z}^*| = s$ and $|(\bar{z}_1^* - z_1^*, \cdots, \bar{z}_{n-1}^* - z_{n-1}^*)| = \frac{\eta s}{\sqrt{1+\eta^2}}$. From (3.1) and (3.2) and the fact that $B(z^*, s) \subset G$ we find

$$\bar{z}_n^* \geq \Gamma(\bar{z}_1^*, \bar{z}_2^*, \cdots, \bar{z}_{n-1}^*). \tag{3.7}$$

When w satisfies $\frac{\eta s}{\sqrt{1+\eta^2}} < |(w_1, w_2, \cdots, w_{n-1}) - (z_1^*, z_2^*, \cdots, z_{n-1}^*)| \leq \frac{s}{2}(\frac{\sqrt{1+\eta^2}}{\eta} + 1)$, we have by Lipschitz property of the boundary and (3.7)

$$\begin{aligned} &\Gamma(w_1, w_2, \cdots, w_{n-1}) \\ &\quad \leq \Gamma(\bar{z}_1^*, \bar{z}_2^*, \cdots, \bar{z}_{n-1}^*) + \eta\left(\frac{s}{2}(\frac{\sqrt{1 + \eta^2}}{\eta} + 1) - \frac{\eta s}{\sqrt{1 + \eta^2}} \right) \\ &\quad \leq \bar{z}_n^* + \frac{1 + \eta\sqrt{1 + \eta^2} - \eta^2}{2\sqrt{1 + \eta^2}} s \\ &\quad \leq z_n^* - \frac{s}{\sqrt{1 + \eta^2}} + \frac{1 + \eta\sqrt{1 + \eta^2} - \eta^2}{2\sqrt{1 + \eta^2}} s \\ &\quad \leq z_n^* - \frac{1 + \eta^2 - \eta\sqrt{1 + \eta^2}}{2\sqrt{1 + \eta^2}} s. \end{aligned} \tag{3.8}$$

When w satisfies $|(w_1, w_2, \cdots, w_{n-1}) - (z_1^*, z_2^*, \cdots, z_{n-1}^*)| \leq \frac{\eta s}{\sqrt{1+\eta^2}}$, noticing that $w \in (B(z^*, |z^* - \bar{z}^*|))^c$ we can check

$$\Gamma(w_1, w_2, \cdots, w_{n-1}) \leq \bar{z}_n^* = z_n^* - \frac{s}{\sqrt{1 + \eta^2}}. \tag{3.9}$$

From (3.8) and (3.9), for w satisfying $|(w_1, w_2, \cdots, w_{n-1}) - (z_1^*, z_2^*, \cdots, z_{n-1}^*)| \leq \frac{s}{2}(\frac{\sqrt{1+\eta^2}}{\eta} + 1)$, we get $\tan(\angle W Z^* \bar{Z} - \frac{\pi}{2}) \geq \frac{(1 + \eta^2 - \eta\sqrt{1 + \eta^2})\eta}{2(1 + \eta^2)}$, hence

$$\cos \angle WZ^*\overline{Z} \le -\frac{1}{\sqrt{1+c_\eta^2}}, \tag{3.10}$$

where W, Z^*, $\overline{Z}$ are the points corresponding to w, z^*, $\overline{z}$ respectively. By the law of cosines and (3.10), we obtain

$$\begin{aligned} |w-\overline{z}|^2 &= |w-z^*|^2 + |z^*-\overline{z}|^2 - 2|w-z^*||z^*-\overline{z}| \cos \angle WZ^*\overline{Z} \\ &\ge s^2 + (1+c_\eta^2)(t-s)^2 + 2(t-s)s > t^2, \end{aligned}$$

for w satisfying $|(w_1, w_2, \cdots, w_{n-1}) - (z_1^*, z_2^*, \cdots, z_{n-1}^*)| \le \frac{s}{2}(\frac{\sqrt{1+\eta^2}}{\eta}+1)$. When w satisfies $|(w_1, w_2, \cdots, w_{n-1}) - (z_1^*, z_2^*, \cdots, z_{n-1}^*)| \ge \frac{s}{2}(\frac{\sqrt{1+\eta^2}}{\eta}+1)$, we can easily check $|w-\overline{z}| > t$ from $t < (\frac{\sqrt{1+\eta^2}}{\eta}+1)\frac{s}{2}$. Therefore we get (3.6).

Let us write δ for $\frac{r_1}{2}$, notice that $\overline{G} - G_\delta \subset \bigcup_{x\in\partial G} B(x, r_1)$, hence by the finite covering principle there exist finitely many points $\{x_i\}_1^{n_0} \in \partial G$ such that $\overline{G} - G_\delta \subset \bigcup_{i=1}^{n_0} B(x_i, r_1)$. Consequently,

$$A = A \cap \bigcup_{i=1}^{n_0} B(x_i, r_1) \tag{3.11}$$

when $s < t < \delta$. From (3.4), we get

$$\int_A dx \le \sum_{i=1}^{n_0} \int_{A\cap B(x_i,r_1)} dx \le n_0(2r_1)^{n-1}\sqrt{1+c_\eta^2}(t-s). \tag{3.12}$$

Thus the lemma is proved in case I.

Case II: $(\frac{\sqrt{1+\eta^2}}{\eta}+1)\frac{s}{2} \le t < C_2$.

By (3.2) and the Lipschitz property of the boundary we have

$$\begin{aligned} &B(x, r_1) \cap \{z = (z_1, \cdots, z_n) : \ z_n - \Gamma_x(z_1, \cdots, z_{n-1}) > \sqrt{1+\eta^2}t \ \} \\ &\qquad \subseteq G_t \cap B(x, r_1), \\ &G_s \cap B(x, r_1) \subseteq B(x, r_1) \cap \{z = (z_1, \cdots, z_n) : \ z_n - \Gamma_x(z_1, \cdots, z_{n-1}) > s \ \}, \end{aligned}$$

therefore

$$\begin{aligned} &A \cap B(x, r_1) = G_s \cap B(x, r_1) - G_t \cap B(x, r_1) \\ &\subseteq \ \{z = (z_1, \cdots, z_n) : \ \sqrt{1+\eta^2}t \ge z_n - \Gamma_x(z_1, \cdots, z_{n-1}) > s \ \}. \end{aligned}$$

Hence by the fact that $(\frac{\sqrt{1+\eta^2}}{\eta}+1)\frac{s}{2} \le t$, we obtain

$$\int_{A\cap B(x,r_1)} dy \le (2r_1)^{n-1}(\sqrt{1+\eta^2}t - s) \le (2r_1)^{n-1}\frac{(\eta-1)^2 + \eta\sqrt{1+\eta^2}}{\sqrt{1+\eta^2}-\eta}(t-s).$$

Applying (3.11), we get

$$\int_A dy \le \sum_{i=1}^{n_0} \int_{A\cap B(x_i,r_1)} dy \le (2r_1)^{n-1} \frac{(\eta-1)^2+\eta\sqrt{1+\eta^2}}{\sqrt{1+\eta^2}-\eta}(t-s),$$

which verifies the lemma in case II. □

Lemma 3.2. *Let G be a bounded Lipschitz open set and g be a nonnegative measurable function on G such that for some constants b_1, b_2 and $1<\beta<2$, it holds that*

$$g(y) \le b_1 \rho(y)^{1-\beta} + b_2.$$

Then $g \in L^1(G, dx)$.

Proof. Let C_1 and C_2 be as in Lemma 3.1. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} \le C_2$. Define for $k \ge N$

$$A_k = \left\{ x \in G : \frac{1}{k+1} < \rho(x) \le \frac{1}{k} \right\}.$$

Writing δ for $\frac{1}{N}$, by Lemma 3.1 we have

$$\int_{G\setminus G_\delta} g(y)dy = \sum_{k\ge N} \int_{A_k} g(y)dy \le \sum_{k\ge N} \frac{C_1}{k(k+1)}(b_1(k+1)^{\beta-1}+b_2) < \infty.$$

On the other hand g is bounded on G_δ. Hence $g \in L^1(G, dx)$. □

Theorem 3.3. *Let G be a bounded Lipschitz open set of $\mathbb{R}^n$. Let $u \in C^\beta(G)$ for some $\beta > \alpha$ when $0<\alpha<1$, and $u \in C^{1+\beta}(G)$ for some $\beta > \alpha - 1$ when $1 \le \alpha < 2$. Then $\Delta_G^{\frac{\alpha}{2}} u \in L^1(G, dx)$. Moreover, if v is a bounded measurable function on G such that*

$$\int\int_{G\times G} \frac{|(u(x)-u(y))(v(x)-v(y))|}{|x-y|^{n+\alpha}}\, dxdy < \infty, \tag{3.13}$$

which is the case e.g. for all bounded v when $0<\alpha<1$ or for all v satisfies $\int\int_{G\times G} \frac{(v(x)-v(y))^2}{|x-y|^{n+\alpha}}\, dxdy < \infty$ when $1 \le \alpha < 2$, then we have the following integration by parts formula.

$$\frac{1}{2}\mathcal{A}(n,-\alpha) \int\int_{G\times G} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\, dxdy = -\int_G v(x)\Delta_G^{\frac{\alpha}{2}} u(x)\, dx. \tag{3.14}$$

Proof. By Proposition 2.3 and Lemma 3.2 we see that $\Delta_G^{\frac{\alpha}{2}} u \in L^1(G, dx)$. Suppose now v satisfies the assumption of the theorem. By the fact that

$$\int_G v(x)\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u(x)\,dx = -\frac{1}{2}\mathcal{A}(n,-\alpha) \int\int_{G\times G} I_{\{|x-y|>\varepsilon\}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy$$

and $\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u$ shares the same estimates stated in Proposition 2.3 (cf. the proof of Proposition 2.3), we conclude easily the integration by parts formula (3.14). □

Remark. If we know that u is in the domain of the generator of (1.1), e.g. $u \in C_c^\infty(G)$, then (3.14) follows directly from the theory of Dirichlet forms. We emphasize that in the above theorem u is in general not in the domain of the generator of (1.1).

For $k \in \mathbb{N}$ we use $C_c^k(\mathbb{R}^n)$ to denote all the functions on $\mathbb{R}^n$ with compact support and having k-th order continuous derivatives. Let $C_c^k(\overline{G})$ be the trace of $C_c^k(\mathbb{R}^n)$ restricted on $\overline{G} := G \cup \partial G$.

Theorem 3.4. *Let G be an open set in $\mathbb{R}^n$ with Lipschitz boundary. Then for $u, v \in C_c^2(\overline{G})$, we have*

$$\frac{1}{2}\mathcal{A}(n,-\alpha)\int\int_{G\times G} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy = -\int_G v(x)\Delta_G^{\frac{\alpha}{2}} u(x)dx.$$

Proof. First it is not hard to show that (3.13) is true in this case. Since the supports of u, v are compact, we can choose a bounded open set $U \subset \mathbb{R}^n$ such that $U \cap G$ is a Lipschitz open set and $U \supset supp[u] \cup supp[v]$. By Theorem 3.3,

$$\begin{aligned}
&\frac{1}{2}\mathcal{A}(n,-\alpha)\int\int_{G\times G} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy \\
&= \frac{1}{2}\mathcal{A}(n,-\alpha)\int\int_{U\cap G\times U\cap G} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy \\
&\quad +\mathcal{A}(n,-\alpha)\int\int_{U\cap G\times(G\cap U^c)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy \\
&= -\int_{U\cap G} v(x)\Delta_U^{\frac{\alpha}{2}} u(x)\,dx - \mathcal{A}(n,-\alpha)\int_{U\cap G} v(dx)\int_{G\cap U^c} \frac{(u(y)-u(x))}{|x-y|^{n+\alpha}}\,dy \\
&= -\int_G v(x)\Delta_G^{\frac{\alpha}{2}} u(x)\,dx
\end{aligned}$$

□

The next Lemma provides us a useful product formula for regional fractional Laplacian.

Lemma 3.5. *Let G be an open set in $\mathbb{R}^n$, u, v be measurable functions on G. Suppose that $uv \in L^1(G, \frac{dx}{(1+|x|)^{n+\alpha}})$, $\Delta_G^{\frac{\alpha}{2}} u$ and $\Delta_G^{\frac{\alpha}{2}} v$ exist, and $\int_G \frac{|(u(y)-u(x))(v(y)-v(x))|}{|x-y|^{n+\alpha}} dy < \infty$, then $\Delta_G^{\frac{\alpha}{2}}(uv)$ exists and*

$$\Delta_G^{\frac{\alpha}{2}}(uv)(x) = v(x)\Delta_G^{\frac{\alpha}{2}} u(x) + u(x)\Delta_G^{\frac{\alpha}{2}} v(x) + \mathcal{A}(n,-\alpha)\int_G \frac{(u(y)-u(x))(v(y)-v(x))}{|x-y|^{n+\alpha}} dy.$$

Proof. By the definition,

$$\begin{aligned}
&\Delta_G^{\frac{\alpha}{2}}(uv)(x) \\
&= \mathcal{A}(n,-\alpha)\lim_{\varepsilon\downarrow 0}\int_G I_{\{|x-y|>\varepsilon\}}\frac{u(y)v(y)-u(x)v(x)}{|x-y|^{n+\alpha}} dy \\
&= \mathcal{A}(n,-\alpha)\lim_{\varepsilon\downarrow 0}\int_G I_{\{|x-y|>\varepsilon\}}\frac{(u(y)-u(x))v(x)}{|x-y|^{n+\alpha}} dy \\
&\quad + \mathcal{A}(n,-\alpha)\lim_{\varepsilon\downarrow 0}\int_G I_{\{|x-y|>\varepsilon\}}\frac{(v(y)-v(x))u(x)}{|x-y|^{n+\alpha}} dy \\
&\quad + \mathcal{A}(n,-\alpha)\lim_{\varepsilon\downarrow 0}\int_G I_{\{|x-y|>\varepsilon\}}\frac{(u(y)-u(x))(v(y)-v(x))}{|x-y|^{n+\alpha}} dy \\
&= v(x)\Delta_G^{\frac{\alpha}{2}} u(x) + u(x)\Delta_G^{\frac{\alpha}{2}} v(x) + \mathcal{A}(n,-\alpha)\int_G \frac{(u(y)-u(x))(v(y)-v(x))}{|x-y|^{n+\alpha}} dy.
\end{aligned}$$

□

4. Semi-martingale decomposition

Let G be a bounded Lipschitz open set and $(X_t)_{t\geq 0}$ be the reflected α-stable process on $\overline{G}$ associated with (1.1). By the result of [12] we assume always that $(X_t)_{t\geq 0}$ has been refined to be a Feller process with continuous transition density function $p(t,x,y)$ satisfying (1.2).

The main goal of this section is to prove theorem 4.1 below.

Theorem 4.1. *Let G be a bounded Lipschitz open set in $\mathbb{R}^n$ and $(X_t)_{t\geq 0}$ be the reflected α-stable process on $\overline{G}$ which has been refined as a Feller process. Suppose that $u \in C(\overline{G})$ and (i) $u \in C^\beta(\overline{G})$ for some $\beta > \alpha$ in the case of $0 < \alpha < 1$, (ii) $u \in C^{1+\beta}(\overline{G})$ for some $\beta > \alpha - 1$ in the case of $1 \leq \alpha < 2$, then we have the decomposition:*

$$u(X_t) = u(x_0) + M_t + \int_0^t \Delta_G^{\frac{\alpha}{2}} u(X_s) ds, \quad a.s.\ P_{x_0}, \quad \forall x_0 \in \overline{G},$$

where $(M_t)_{t\geq 0}$ is a square integrable martingale and the Revuz measure of the sharp bracket process for which is $\left(\mathcal{A}(n,-\alpha)\int_G \frac{(u(z)-u(y))^2}{|z-y|^{n+\alpha}} dz\right) dy$.

The proof of Theorem 4.1 is postponed until later. A direct consequence of the theorem is that $(X_t)_{t\geq 0}$ itself is a semi-martingale. We state it as the following corollary.

Corollary 4.2. *Let G be a bounded Lipschitz open set in* $\mathbb{R}^n$, $(X_t)_{t\geq 0}$ *be the reflected* α*-stable process on* $\overline{G}$ *associated with (1.1). Then* $(X_t)_{t\geq 0}$ *is a semi-martingale and if we write*

$$X_t = (X_t^k)_{k=1}^n,\ x_0 = (x_0^k)_{k=1}^n,$$

then

$$X_t^k = x_0^k + M_t^k + \int_0^t \Delta_G^{\frac{\alpha}{2}} x_k(X_s)ds, \quad a.s.\ P_{x_0},\ k = 1, 2, \cdots, n,\ \forall x_0 \in \overline{G},$$

where $(M_t^k)_{t\geq 0}$ *is a square integrable martingale and the Revuz measure of the sharp bracket process for which is* $\left(\mathcal{A}(n,-\alpha)\int_G \frac{|y_k - z_k|^2}{|z-y|^{n+\alpha}}dz\right)dy$.

In the one dimensional case we have a more concrete decomposition as follows.

Corollary 4.3. *Let* $(X_t)_{t\geq 0}$ *be the reflected* α*-stable process on* $[0, 1]$*, then for all* $x \in [0, 1]$*, we have*

$$\begin{aligned}
X_t &= x + M_t + \mathcal{A}(1,-\alpha)\frac{1}{\alpha-1}\int_0^t \left(X_s^{1-\alpha} - (1-X_s)^{1-\alpha})\right)ds, \\
&\quad a.s.\ \ P_x, \quad when\ \alpha \neq 1; \\
X_t &= x + M_t + \mathcal{A}(1,-1)\int_0^t \log(\frac{1}{X_s} - 1)ds, \\
&\quad a.s.\ \ P_x, \quad when\ \alpha = 1,
\end{aligned}$$

where $(M_t)_{t\geq 0}$ *is a square integrable martingale with the Revuz measure of* $< M >_t$ *given by*

$$\frac{1}{2-\alpha}\mathcal{A}(1,-1)\left(y^{2-\alpha} + (1-y)^{2-\alpha}\right)dy.$$

Proof of Corollary 4.3. By corollary 4.2 and the fact that if $f(x) = x$, then

$$\begin{aligned}
\Delta_{(0,1)}^{\frac{\alpha}{2}} f(x) &= \mathcal{A}(1,-\alpha)\left(\frac{1}{\alpha-1}(x^{1-\alpha} - (1-x)^{1-\alpha})\right), \quad when\ \alpha \neq 1; \\
\Delta_{(0,1)}^{\frac{\alpha}{2}} f(x) &= \mathcal{A}(1,-1)\log(\frac{1}{x} - 1), \quad when\ \alpha = 1.
\end{aligned}$$

□

We now discuss the proof of Theorem 4.1. It relies on a strict version of Fukushima's decomposition, i.e. Theorem 5.2.5 of [19]. For the convenience of the reader we restate it in our context as the following Proposition. For the involved terminologies and notations we refer to [19].

Proposition 4.4. *Let G be a bounded Lipschitz open set in $\mathbb{R}^n$, u be a function on $\overline{G}$ such that*

(1) $u \in \mathcal{F}$, $\mu_{<u>} \in S_{00}$, u is bounded and finely continuous.
(2) there exist $\nu = \nu_1 - \nu_2$ with $\nu_1, \nu_2 \in S_{00}$ and, for a dense subset $\mathcal{L}$ of $\mathcal{F}$,

$$\mathcal{E}(u, v) = \int_{\overline{G}} \widetilde{v}(x)\nu(dx), \quad \forall v \in \mathcal{L}.$$

Then

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad a.s.\ P_x,\ \forall x \in \overline{G},$$

where $M_t^{[u]}$ is a martingale additive functional in the strict sense whose sharp bracket in the strict sense has the Revuz measure $\mu_{<u>}$, and

$$N_t^{[u]} = -A^1 + A^2, \quad a.s.\ P_x,\ \forall x \in \overline{G}.$$

A^1, A^2 are positive continuous additive functionals in the strict sense with Revuz measures ν_1, ν_2 respectively.

The following three lemmas are need to check the conditions of Proposition 4.4. In what follows we assume that G is a bounded Lipschitz open set and shall freely use the notations of Section 3. For $x \in \partial G$ we define

$$U_x = \{y : 0 < dist(y, \partial G) < \delta_2, |y - x| < \delta_1\}, \tag{4.1}$$

where $\delta_1 = \frac{r_1}{4}$, $\delta_2 = \frac{r_1}{4\sqrt{1+\eta^2}}$.

Lemma 4.5. *Let U_x be as above. For $y = (y_1, \cdots, y_n) \in U_x$ we define*

$$\Phi(y_1, \cdots, y_n) = (y_1, \cdots, y_{n-1}, \rho(y)), \quad \forall\, y \in U_x.$$

Then there exists a constant K such that

$$\frac{1}{K}|y - z| \le |\Phi(y) - \Phi(z)| \le K|y - z|, \forall\, y, z \in U_x, \tag{4.2}$$

and consequently

$$\int_{U_x} f(y_1, y_2, \cdots, y_n)dy \le K^n \int_{\Phi(U_x)} f(\Phi^{-1}(y_1, y_2, \cdots, y_n))dy, \tag{4.3}$$

for all positive Borel function f on $\mathbb{R}^n$.

Proof. Let $y, z \in U_x$. it is easy to show that $|\rho(y) - \rho(z)| \le |y - z|$ and consequently $|\Phi(y) - \Phi(z)| \le 2|y - z|$. To prove the left hand side inequality of (4.2), we assume first $|y_n - z_n| \le 2\sqrt{1 + c_\eta^2}|(y_1, \cdots, y_{n-1}) - (z_1, \cdots, z_{n-1})|$, where c_η is the same as in the proof of Lemma 3.1. Since $\Phi(y)$ differs from y only on the n-th coordinate, in this case we have

$$|y - z| \le (2\sqrt{1 + c_\eta^2} + 1)|\Phi(y) - \Phi(z)|. \tag{4.4}$$

Assume now $|y_n - z_n| > 2\sqrt{1+c_\eta^2}|(y_1, y_2, \ldots, y_{n-1}) - (z_1, z_2, \ldots, z_{n-1})|$. Without loss of generality we assume $y_n > z_n$. Let $\overline{y} = (z_1, \cdots, z_{n-1}, y_n)$, by (3.4) we have

$$\begin{aligned}
|\rho(y) - \rho(z)| &\geq |\rho(\overline{y}) - \rho(z)| - |\rho(\overline{y}) - \rho(y)| \\
&\geq \frac{1}{\sqrt{1+c_\eta^2}}|y_n - z_n| - |(y_1, y_2, \ldots, y_{n-1}) - (z_1, z_2, \ldots, z_{n-1})| \\
&> \frac{1}{2\sqrt{1+c_\eta^2}}|y_n - z_n| \geq \frac{1}{4\sqrt{1+c_\eta^2}}|y - z|,
\end{aligned}$$

which together with (4.4) verifies the left hand side inequality of (4.2). Assertion (4.3) is a direct consequence of (4.2). □

Lemma 4.6. *Let G be a bounded Lipschitz open set in $\mathbb{R}^n$, g be a nonnegative continuous function on G such that*

$$\begin{array}{ll}
g(x) < C \cdot \rho(x)^{1-\alpha}, & \textit{when } 1 < \alpha < 2\,; \\
g(x) < C \cdot (|\log \rho(x)| + 1), & \textit{when } \alpha = 1\,; \\
g(x) < C, & \textit{when } \alpha < 1,
\end{array} \tag{4.5}$$

where C is a positive constant. Let $R_\beta g(x) = E_x \int_0^\infty e^{-\beta t} g(X_t)dt$, then $R_\beta g$ is bounded on $\overline{G}$ for any $\beta > 0$.

Proof. The case of $\alpha < 1$ is trivial. We prove only the case $1 < \alpha < 2$, the case of $\alpha = 1$ can be proved similarly. Let $x \in \overline{G}$. We assume first that $\rho(x) < \delta_2$. Let $z \in \partial G$ with $|x - z| = \rho(x)$. Write $\widetilde{x} := \Phi(x)$, without loss of generality we can assume that $\widetilde{x} = (0, \cdots, 0, \rho(x))$. For $0 < t \leq 1$, we have by (1.2)

$$\begin{aligned}
&E_x g(X_t) I_{\{X_t \in U_z\}} \\
&\leq \int_{y \in U_z, |y-x| < t^{\frac{1}{\alpha}}} c_2 t^{-n/\alpha} g(y)dy + \int_{y \in U_z, |y-x| \geq t^{\frac{1}{\alpha}}} \frac{c_2 t}{|x-y|^{n+\alpha}} g(y)dy.
\end{aligned} \tag{4.6}$$

Write $\widetilde{U}_z = \Phi(U_z)$ and apply (4.2) and (4.3), we obtain

$$\begin{aligned}
E_x g(X_t) I_{\{X_t \in U_z\}} \leq K^n &\int_{y \in \widetilde{U}_z, |y - \widetilde{x}| < K t^{\frac{1}{\alpha}}} c_2 t^{-n/\alpha} y_n^{1-\alpha} dy \\
&+ K^n \int_{y \in \widetilde{U}_z, |y - \widetilde{x}| \geq t^{\frac{1}{\alpha}}/K} \frac{c_2 t}{|\widetilde{x} - y|^{n+\alpha}} y_n^{1-\alpha} dy.
\end{aligned} \tag{4.7}$$

Write $d\overline{y} = dy_1 \cdots dy_{n-1}$. For the first term in (4.6) we have

$$\begin{aligned}
&\int_{y\in\widetilde{U}_z, |y-\widetilde{x}|<Kt^{\frac{1}{\alpha}}} t^{-n/\alpha} y_n^{1-\alpha} dy \\
&= \int_{0\vee(\rho(x)-Kt^{\frac{1}{\alpha}})}^{\rho(x)+Kt^{\frac{1}{\alpha}}} dy_n \int_{\mathbb{R}^{n-1}} I_{\{|y-\widetilde{x}|<Kt^{\frac{1}{\alpha}}\}} t^{-n/\alpha} y_n^{1-\alpha} d\overline{y} \\
&\le \int_{0\vee(\rho(x)-Kt^{\frac{1}{\alpha}})}^{\rho(x)+Kt^{\frac{1}{\alpha}}} (2K)^{n-1} t^{-1/\alpha} y_n^{1-\alpha} dy_n \\
&\le \frac{(2K)^{n-1}}{2-\alpha} t^{-1/\alpha} (d_G + Kt^{\frac{1}{\alpha}})^{2-\alpha}. \qquad (4.8)
\end{aligned}$$

For the second term in (4.6) we have

$$\begin{aligned}
&\int_{y\in\widetilde{U}_z, |y-\widetilde{x}|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\
&= \int_{y\in\widetilde{U}_z, |y_n-\rho(x)|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\
&\quad + \int_{y\in\widetilde{U}_z, |y_n-\rho(x)|< t^{\frac{1}{\alpha}}/K, |y-\widetilde{x}|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy. \qquad (4.9)
\end{aligned}$$

Write $\overline{y} = (y_1, y_2, \cdots, y_{n-1})$, $r = |(y_1, y_2, \cdots, y_{n-1})|$ and

$$\begin{aligned}
&(y_1, y_2, \cdots, y_{n-1}) = (r\cos\theta_1, r\sin\theta_1\cos\theta_2, \cdots, r\sin\theta_1\cdots\sin\theta_{n-4}\sin\theta_{n-3}), \\
&\varphi(\theta_1, \cdots, \theta_{n-2}) = \sin^{n-3}\theta_1 \sin^{n-4}\theta_2 \cdots \sin\theta_{n-3}.
\end{aligned}$$

For the first term in (4.8) we have

$$\begin{aligned}
&\int_{y\in\widetilde{U}_z, |y_n-\rho(x)|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\
&\le \left[\int_0^{0\vee(\rho(x)-t^{\frac{1}{\alpha}}/K)} dy_n + \int_{\rho(x)+t^{\frac{1}{\alpha}}/K}^{\delta_1} dy_n\right] \int_{y\in\widetilde{U}_z,} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} d\overline{y} \\
&\le \left[\int_0^{0\vee(\rho(x)-t^{\frac{1}{\alpha}}/K)} dy_n + \int_{\rho(x)+t^{\frac{1}{\alpha}}/K}^{\delta_1} dy_n\right] \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2} \\
&\quad \cdot \int_0^{\delta_1} \varphi(\theta_1, \cdots, \theta_{n-2}) \frac{t\cdot r^{n-2}}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} I_{\{y\in\widetilde{U}_z\}} dr \\
&\le \left[\int_0^{0\vee(\rho(x)-t^{\frac{1}{\alpha}}/K)} dy_n + \int_{\rho(x)+t^{\frac{1}{\alpha}}/K}^{\delta_1} dy_n\right] \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^{t^{\frac{1}{\alpha}}} \varphi(\theta_1, \cdots, \theta_{n-2})\, r^{n-2} t \frac{K^{n+\alpha}}{t^{1+\frac{n}{\alpha}}} y_n^{1-\alpha}\, dr \\
&+ \Big[\int_0^{0 \vee (\rho(x) - t^{\frac{1}{\alpha}}/K)} dy_n + \int_{\rho(x)+t^{\frac{1}{\alpha}}/K}^{\delta_1} dy_n \Big] \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2} \\
&\cdot \int_{t^{\frac{1}{\alpha}}}^{\delta_1} \varphi(\theta_1, \cdots, \theta_{n-2})\, r^{-2-\alpha} t y_n^{1-\alpha}\, dr \\
\le{}& (2\pi)^n \Big[\int_0^{0 \vee (\rho(x) - t^{\frac{1}{\alpha}}/K)} + \int_{\rho(x)+t^{\frac{1}{\alpha}}/K}^{\delta_1} \Big] \cdot \Big(\frac{K^{n+\alpha}}{n-1} + \frac{1}{\alpha+1} \Big) t^{-\frac{1}{\alpha}} y_n^{1-\alpha}\, dy_n \\
\le{}& (2\pi)^n \frac{1}{2-\alpha} \cdot \Big(\frac{K^{n+\alpha}}{n-1} + \frac{1}{\alpha+1} \Big) \delta_1^{2-\alpha} t^{-\frac{1}{\alpha}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{y \in \widetilde{U}_z, |y_n - \rho(x)| < t^{\frac{1}{\alpha}}/K, |y - \widetilde{x}| \ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x} - y|^{n+\alpha}} y_n^{1-\alpha} dy \\
&\le \int_{(\rho(x) - t^{\frac{1}{\alpha}}/K) \vee 0}^{\rho(x)+t^{\frac{1}{\alpha}}/K} dy_n \int_{y \in \widetilde{U}_z, |y - \widetilde{x}| \ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x} - y|^{n+\alpha}} y_n^{1-\alpha} d\overline{y} \\
&\le \int_{(\rho(x) - t^{\frac{1}{\alpha}}/K) \vee 0}^{\rho(x)+t^{\frac{1}{\alpha}}/K} dy_n \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2} \\
&\quad \cdot \int_0^{\delta_1} \varphi(\theta_1, \cdots, \theta_{n-2}) \frac{t \cdot r^{n-2}}{|\widetilde{x} - y|^{n+\alpha}} y_n^{1-\alpha} I_{\{y \in \widetilde{U}_z, |y - \widetilde{x}| \ge t^{\frac{1}{\alpha}}/K\}}\, dr \\
&\le \int_{(\rho(x) - t^{\frac{1}{\alpha}}/2K) \vee 0}^{\rho(x)+t^{\frac{1}{\alpha}}/2K} dy_n \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2} \\
&\quad \cdot \int_{t^{\frac{1}{\alpha}}/K - |y_n - \rho(x)|}^{\delta_1} \varphi(\theta_1, \cdots, \theta_{n-2}) r^{-2-\alpha} t y_n^{1-\alpha}\, dr \\
&\quad + \Big[\int_{(\rho(x) - t^{\frac{1}{\alpha}}/K) \vee 0}^{(\rho(x) - t^{\frac{1}{\alpha}}/2K) \vee 0} dy_n + \int_{\rho(x)+t^{\frac{1}{\alpha}}/2K}^{\rho(x)+t^{\frac{1}{\alpha}}/K} dy_n \Big] \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2} \\
&\quad \cdot \int_0^{t^{\frac{1}{\alpha}}} \varphi(\theta_1, \cdots, \theta_{n-2}) r^{n-2} t \frac{(2K)^{n+\alpha}}{t^{1+\frac{n}{\alpha}}} y_n^{1-\alpha}\, dr \\
&\quad + \Big[\int_{(\rho(x) - t^{\frac{1}{\alpha}}/K) \vee 0}^{(\rho(x) - t^{\frac{1}{\alpha}}/2K) \vee 0} dy_n + \int_{\rho(x)+t^{\frac{1}{\alpha}}/2K}^{\rho(x)+t^{\frac{1}{\alpha}}/K} dy_n \Big] \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2} \\
&\quad \cdot \int_{t^{\frac{1}{\alpha}}}^{\delta_1} \varphi(\theta_1, \cdots, \theta_{n-2}) r^{-2-\alpha} t y_n^{1-\alpha}\, dr \\
&\le (2\pi)^n \int_{(\rho(x) - t^{\frac{1}{\alpha}}/2K) \vee 0}^{\rho(x)+t^{\frac{1}{\alpha}}/2K} \frac{(2K)^{1+\alpha}}{\alpha+1} t^{-\frac{1}{\alpha}} y_n^{1-\alpha} dy_n
\end{aligned}
$$

$$+(2\pi)^n\left[\int_{(\rho(x)-t^{\frac{1}{\alpha}}/K)\vee 0}^{(\rho(x)-t^{\frac{1}{\alpha}}/2K)\vee 0}+\int_{\rho(x)+t^{\frac{1}{\alpha}}/2K}^{\rho(x)+t^{\frac{1}{\alpha}}/K}\right]\left(\frac{(2K)^{n+\alpha}}{n-1}+\frac{1}{\alpha+1}\right)t^{-\frac{1}{\alpha}}y_n^{1-\alpha}dy_n$$
$$\leq (2\pi)^n\frac{1}{2-\alpha}\cdot\left(\frac{(2K)^{1+\alpha}}{\alpha+1}+\frac{(2K)^{n+\alpha}}{n-1}+\frac{1}{\alpha+1}\right)\delta_1^{2-\alpha}t^{-\frac{1}{\alpha}}. \tag{4.10}$$

From (4.6) to (4.9) we conclude that there exists $c_3>0$ such that

$$E_x g(X_t)I_{\{X_t\in U_z\}}\leq c_3t^{-\frac{1}{\alpha}}.$$

By condition (1.2) there exist $c_4>0$ such that

$$E_x g(X_t)I_{\{X_t\notin U_z\}}\leq c_4.$$

Hence

$$E_x g(X_t)=E_x g(X_t)I_{\{X_t\in U_z\}}+E_x g(X_t)I_{\{X_t\notin U_z\}}\leq c_3t^{-\frac{1}{\alpha}}+c_4. \tag{4.11}$$

We now assume that $\rho(x)\geq\delta_2$. By (1.2), writing δ for $\frac{\delta_2}{2}$, we have

$$\begin{aligned}E_x g(X_t)&=\int_{\overline{G}-G_\delta}p(t,x,y)g(y)dy+\int_{G_\delta}p(t,x,y)g(y)dy\\ &\leq\int_{\overline{G}-G_\delta}c_2t\delta^{-n-\alpha}g(y)dy+C\delta^{\alpha-1}.\end{aligned} \tag{4.12}$$

By the same methods used in Lemma 3.2, we see that there exist $c_5, c_6>0$ satisfying

$$\int_{\overline{G}-G_\delta}c_2t\delta^{-n-\alpha}g(y)dy\leq c_5t+c_6. \tag{4.13}$$

By (4.11) and (4.12) we obtain

$$E_x g(X_t)\leq C\delta^{\alpha-1}+c_5t+c_6. \tag{4.14}$$

By (4.10) and (4.13) we conclude that there exist c_7, c_8 such that for all $x\in\overline{G}$,

$$E_x g(X_t)\leq c_7t^{-\frac{1}{\alpha}}+c_8,\quad\forall 0<t\leq 1. \tag{4.15}$$

For $\beta>0$ and $x\in\overline{G}$, applying (4.14) we have

$$\begin{aligned}R_\beta g(x)&=\int_0^\infty e^{-\beta t}P_tg(x)dt\\ &\leq\int_0^1(c_7t^{-\frac{1}{\alpha}}+c_8)dt+\int_1^\infty e^{-\beta t}P_{t-1}P_1g dt\\ &=\frac{\alpha}{\alpha-1}c_7+c_8+\frac{1}{\beta}(c_7+c_8),\end{aligned}$$

hence $R_1 g$ is bounded on $\overline{G}$. □

Recall that a positive Radon measure on $\overline{G}$ is said to be of finite energy integral, denoted by $\mu \in S_0$, if and only if there exists for each $\alpha > 0$ an element $U_\alpha \mu \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(U_\alpha \mu, v) = \int_{\overline{G}} v(x)\mu(dx), \forall v \in \mathcal{F} \cap C(\overline{G}).$$

(cf. [19] Section 2.2). Following [19] , we denote by S_{00} all the measures $\mu \in S_0$ such that $\mu(\overline{G}) < \infty$ and $R_1\mu$ is bounded on $\overline{G}$.

Lemma 4.7. *Let g be as in Lemma 4.6. Define $\mu(dx) = I_G(x)g(x)dx$. Then $\mu \in S_{00}$.*

Proof. By Lemma 3.2 we have $\mu(\overline{G}) < \infty$. Write $R_\alpha \mu = R_\alpha g$. By the above Lemma we need only to check that $\mu \in S_0$ and $R_1\mu = U_1\mu$. To this end we notice that by (5.1.14) and (5.1.12) of Theorem 5.1.3 in [19] and Lemma 5.1.5 in [19], for any α-excessive function u, we have

$$\lim_{\beta\to\infty} \beta(u, R_\alpha\mu - \beta R_{\beta+\alpha}R_\alpha\mu) = \lim_{\beta\to\infty} \beta(u, R_{\beta+\alpha}\mu) = \int_{\overline{G}} u(x)\mu(dx).$$

In particular, taking $u = R_\alpha\mu$, we see that $R_\alpha\mu \in \mathcal{F}$ and $\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_{\overline{G}} v(x)\mu(dx)$ for all α-excessive functions $u \in \mathcal{F}$, and hence for all $u \in \mathcal{F} \cap C(\overline{G})$. Therefore $\mu \in S_0$ and $R_1\mu = U_1\mu$. □

Proof of Theorem 4.1. Let $u \in C^2(\overline{G})$. Then by the theory of Dirichlet forms the energy measure of u is (cf. [19], Section 5.3):

$$\mu_{<u>}(dy) = \left(\mathcal{A}(n, -\alpha)\int_G \frac{(u(z) - u(y))^2}{|z - y|^{n+\alpha}}dz\right)dy.$$

Clearly $\mu_{<u>} \in S_{00}$ since $\mathcal{A}(n, -\alpha)\int_G \frac{(u(z) - u(y))^2}{|z - y|^{n+\alpha}}dz$ is bounded. By the integration by parts formula obtained in Section 3 we get

$$\begin{aligned}\mathcal{E}(u, v) &= -\int_G v(y)\Delta_G^{\frac{\alpha}{2}}u(y)dy \\ &= -\int_G v(y)I_{\{y:\Delta_G^{\frac{\alpha}{2}}u(y)<0\}}\Delta_G^{\frac{\alpha}{2}}u(y)dy \\ &\quad -\int_G v(y)I_{\{y:\Delta_G^{\frac{\alpha}{2}}u(y)>0\}}\Delta_G^{\frac{\alpha}{2}}u(y)dy, \quad \forall v \in C^\infty(\overline{G}).\end{aligned}$$

Let

$$\nu_1(dy) = -I_{\{y:\Delta_G^{\frac{\alpha}{2}}u(y)<0\}}\Delta_G^{\frac{\alpha}{2}}u(y)dy,$$

$$\nu_2(dy) = I_{\{y:\Delta_G^{\frac{\alpha}{2}}u(y)>0\}}\Delta_G^{\frac{\alpha}{2}}u(y)dy.$$

From Proposition 2.3 and Lemma 4.7 we can see $\nu_1, \nu_2 \in S_{00}$. Hence the theorem is true by Proposition 4.4. □

Corollary 4.8. *Let G be an open set in* $\mathbb{R}^n$ *with Lipschitz boundary,* $u \in C_c^2(\overline{G})$, $(X_t)_{t\geq 0}$ *be the reflected* α*-stable process on* $\overline{G}$*, then the conclusion in theorem 4.1 is true.*

Proof. For $u \in C_c^2(\overline{G})$, it is easy to see $\mathcal{A}(n, -\alpha)\int_G \frac{(u(z)-u(y))^2}{|z-y|^{n+\alpha}}dz$ is bounded and belongs to $L^2(G)$. Hence $\mu_{<u>}(dy) \in S_{00}$. Let $R > 0$ be such that $supp\,[u] \subset B(0, \frac{R}{2})$. Noticing that $\Delta_G^{\frac{\alpha}{2}} u$ is bounded on $G \cap B(0, R)^c$ and belongs to $L^2(G \cap B(0, R)^c)$, we can prove $\nu_1, \nu_2 \in S_{00}$ by the same method of Proposition 4.7. Therefore we can prove the theorem in the same way as Theorem 4.1. □

5. Fractional Laplacian on closed regions

For the purpose of studying the generators of reflected α-stable processes, we need to investigate the regional fractional Laplacian on a closed region. Let G be an open set of $\mathbb{R}^n$, $\overline{G} := G \cup \partial G$.

Definition 5.1. *Let* $u \in \mathcal{L}^1(\overline{G}, \frac{dx}{(1+|x|^{n+\alpha})})$ *and* $0 < \alpha < 2$*, we define the regional fractional Laplacian* $\Delta_{\overline{G}}^{\frac{\alpha}{2}}$ *by the formula*

$$\Delta_{\overline{G}}^{\frac{\alpha}{2}} u(x) = \lim_{\varepsilon \downarrow 0} \Delta_{\overline{G},\varepsilon}^{\frac{\alpha}{2}} u(x), \quad x \in \overline{G},$$

provided the limit exists. Here and henceforth $\Delta_{\overline{G},\varepsilon}^{\frac{\alpha}{2}} u(x)$ *is defined by the same formula as (2.2) with G replaced by* $\overline{G}$*.*

Clearly if x is an inner point of $\overline{G}$, then $\Delta_{\overline{G}}^{\frac{\alpha}{2}} u(x) = \Delta_G^{\frac{\alpha}{2}} u(x)$ and all the previous discussions are available in this case. Therefore in what follows we shall concentrate to the case of $x \in \partial G$. The next lemma will be crucial for our analysis. In what follows let us write $\mathbb{R}_+^n = \{z = (z_1, \cdots, z_n) : \ z_n \geq 0\}$. $\langle z, v\rangle$ will denote the usual inner product in $\mathbb{R}^n$. $m(dz)$ denotes the area measure on the sphere $\{z \in \mathbb{R}^n : \ |z| = 1\}$.

Lemma 5.2. *Let* $\psi(x)$ *be a nonnegative nondecreasing measurable function on* $[0, 1]$ *such that* $\int_0^1 \psi(s)ds > 0$*. Let* $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ *with* $|v| = 1$*. Then we have the following assertions.*

$$\int_{|z|=1} I_{\mathbb{R}_+^n} \psi(z_n)\langle z, v\rangle m(dz) = 0, \quad if\ v_n = 0; \tag{5.1}$$

$$\int_{|z|=1} I_{\mathbb{R}_+^n} \psi(z_n)\langle z, v\rangle m(dz) > 0, \quad if\ v_n > 0; \tag{5.2}$$

$$\int_{|z|=1} I_{\mathbb{R}_+^n} \psi(z_n)\langle z, v\rangle m(dz) < 0, \quad if\ v_n < 0. \tag{5.3}$$

Proof. We prove first (5.1). By taking an orthonormal transform in $\mathbb{R}^{n-1}$, we may assume that $v = (1, 0, \cdots, 0)$. Employing spherical system for $z = (z_1, z_2, \cdots, z_n)$ with $|z| = 1$ we set

$$z = (\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1}, \cdots, \sin\theta_1 \cos\theta_2, \cos\theta_1).$$

Write $\varphi(\theta_1, \cdots, \theta_{n-1}) = \psi(\cos\theta_1) \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}$, we have when $n \geq 3$,

$$\begin{aligned}
&\int_{|z|=1} I_{\mathbb{R}^n_+} \psi(z_n)\langle z, v\rangle m(dz) \\
&\quad = \int_0^{\frac{\pi}{2}} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \cdot \varphi(\theta_1, \cdots, \theta_{n-1})\langle z, v\rangle \\
&\quad = \int_0^{\frac{\pi}{2}} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-2} \int_0^{\pi} d\theta_{n-1} \cdot \varphi(\theta_1, \cdots, \theta_{n-1}) \\
&\qquad \cdot\Big(\sin\theta_1 \sin\theta_2 \cdots \sin(2\pi - \theta_{n-1}) + \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} \Big) \\
&\quad = 0
\end{aligned}$$

When $n = 2$ we get the same conclusion with a simpler calculation. Thus (5.1) is verified. Next we prove (5.2). Again by taking an orthonormal transform in $\mathbb{R}^{n-1}$ we may assume that $v = (\cos\theta, 0, \cdots, 0, \sin\theta)$ for some $0 < \theta \leq \frac{\pi}{2}$. Let

$$\begin{aligned}
&v_0 = (1, 0, \cdots, 0), \\
&\Pi_t = \{y = (y_1, y_2, \cdots, y_n) : y_n = r\sin t,\ \ y_1 = r\cos t, r \in \mathbb{R}\}, \\
&\Sigma_t = \{y = (y_1, y_2, \cdots, y_n) : \langle y, \left(\cos\left(t + \frac{\pi}{2}\right), 0, \cdots, 0, \sin\left(t + \frac{\pi}{2}\right)\right)\rangle \geq 0\}.
\end{aligned}$$

For $y \in \Sigma_\theta \cap \mathbb{R}^n_+$ with $|y| > 0$, there exists $\theta^* \in [0, \pi - \theta)$ and $r > 0$ such that

$$y_1 = r\cos(\theta + \theta^*), \quad y_n = r\sin(\theta + \theta^*),$$

hence

$$\langle v, y\rangle = r\cos(\theta + \theta^*)\cos\theta + r\sin(\theta + \theta^*)\sin\theta = r\cos\theta^* > y_1 = \langle v_0, y\rangle.$$

Notice that $(0, \cdots, 0, 1)$ is a inner point of $\Sigma_\theta \cap \mathbb{R}^n_+$ and $\int_{1-\varepsilon}^1 \psi(s)ds > 0$ for any $\varepsilon > 0$, we have

$$\int_{|z|=1} \psi(z_n)\langle z, v\rangle I_{\Sigma_\theta \cap \mathbb{R}^n_+} m(dz) > \int_{|z|=1} \psi(z_n)\langle z, v_0\rangle I_{\Sigma_\theta \cap \mathbb{R}^n_+} m(dz). \quad (5.4)$$

For $y \in \mathbb{R}^n$, let $\Phi(y)$ be its symmetric point with respect to the plane $\Pi_{\frac{\theta}{2}}$. More precisely, $\Phi(y) = y^* - (y - y^*)$, where y^* is the unique point in $\Pi_{\frac{\theta}{2}}$ satisfying $\langle y - y^*, z\rangle = 0$ for all $z \in \Pi_{\frac{\theta}{2}}$. By the symmetricity property one can check that

$$\langle y, v\rangle = \langle \Phi(y), v_0\rangle \ \textit{for all}\ y \in \mathbb{R}^n_+ \backslash \Sigma_\theta, \quad (5.5)$$

and $\Phi(\Sigma_{\frac{\theta}{2}}\backslash\Sigma_\theta) = \mathbb{R}^n_+\backslash\Sigma_{\frac{\theta}{2}}$. One can also check that for $z \in \Sigma_{\frac{\theta}{2}}\backslash\Sigma_\theta$ it holds that $\langle z, v\rangle \geq \langle\Phi(z), v\rangle$ and $z_n \geq \Phi(z)_n$. Consequently by the nondecreasing property of ψ, for $z \in \Sigma_{\frac{\theta}{2}}\backslash\Sigma_\theta$ we have

$$\begin{aligned}
&\psi(z_n)\langle z, v\rangle + \psi(\Phi(z)_n)\langle\Phi(z), v\rangle \\
&= \Big(\psi(\Phi(z)_n) - \psi(z_n)\Big)\Big(\langle\Phi(z), v\rangle - \langle z, v\rangle\Big) + \psi(z_n)\langle\Phi(z), v\rangle \\
&\quad +\psi(\Phi(z)_n)\langle z, v\rangle \\
&\geq \psi(z_n)\langle\Phi(z), v\rangle + \psi(\Phi(z)_n)\langle z, v\rangle. \qquad (5.6)
\end{aligned}$$

Applying (5.5) and (5.6), we have

$$\begin{aligned}
&\int_{|z|=1}\psi(z_n)\langle z, v\rangle I_{\mathbb{R}^n_+\backslash\Sigma_\theta} m(dz) \\
&= \int_{|z|=1}\psi(z_n)\langle z, v\rangle I_{\mathbb{R}^n_+\backslash\Sigma_{\frac{\theta}{2}}} m(dz) + \int_{|z|=1}\psi(z_n)\langle z, v\rangle I_{\Sigma_{\frac{\theta}{2}}\backslash\Sigma_\theta} m(dz) \\
&= \int_{|z|=1}\Big(\psi(z_n)\langle z, v\rangle + \psi(\Phi(z)_n)\langle\Phi(z), v\rangle\Big) I_{\Sigma_{\frac{\theta}{2}}\backslash\Sigma_\theta} m(dz) \\
&\geq \int_{|z|=1}\Big(\psi(z_n)\langle\Phi(z), v\rangle + \psi(\Phi(z)_n)\langle z, v\rangle\Big) I_{\Sigma_{\frac{\theta}{2}}\backslash\Sigma_\theta} m(dz) \\
&= \int_{|z|=1}\psi(z_n)\langle z, v_0\rangle I_{\mathbb{R}^n_+\backslash\Sigma_\theta} m(dz). \qquad (5.7)
\end{aligned}$$

Combing (5.5), (5.7) and (5.1),

$$\begin{aligned}
&\int_{|z|=1} I_{\mathbb{R}^n_+}\psi(z_n)\langle z, v\rangle m(dz) \\
&= \int_{|z|=1}\psi(z_n)\langle z, v\rangle I_{\Sigma_\theta\backslash\mathbb{R}^n_+} m(dz) + \int_{|z|=1}\psi(z_n)\langle z, v\rangle I_{\mathbb{R}^n_+\backslash\Sigma_\theta} m(dz) \\
&> \int_{|z|=1} I_{\mathbb{R}^n_+}\psi(z_n)\langle z, v_0\rangle m(dz) = 0,
\end{aligned}$$

which proves (5.2). (5.3) can be proved similarly. □

In what follows for $x=(x_1, \cdots, x_n)$ we write $x=(\widetilde{x}, x_n)$ with $\widetilde{x}=(x_1, \cdots, x_{n-1})$. Let $\beta > 0$. Recall that an open set G in $\mathbb{R}^n$ is said to have $C^{1,\beta}$ boundary (G is a $C^{1,\beta}$ open set in abbreviation) if G is a Lipschitz open set and for each $x \in \partial G$ the function Γ_x specified in (3.1) is differentiable and

$$\limsup_{\widetilde{y}\downarrow\widetilde{x}} \frac{|\nabla\Gamma(\widetilde{y}) - \nabla\Gamma(\widetilde{x})|}{|\widetilde{y} - \widetilde{x}|^\beta} := C < \infty. \qquad (5.8)$$

Theorem 5.3. *(i) Let $0 < \alpha < 1$ and G be a Lipschitz open set in $\mathbb{R}^n$. If $u \in C^\beta(\overline{G})$ for some $\beta > \alpha$(i.e.u is the restriction on $\overline{G}$ of some $\widetilde{u} \in C^\beta(U)$ for some open set $U \supset \overline{G}$). Then $\Delta^{\frac{\alpha}{2}}_G u(x)$ exists for all $x \in \overline{G}$ and $\Delta^{\frac{\alpha}{2}}_G u(x)$ is continuous on $\overline{G}$.*

(ii) *Let* $1 \le \alpha < 2$, G *be a bounded* $C^{1,\beta}$ *open set for some* $\beta > \alpha - 1$, *and* $u \in C^2(\overline{G})$. *Then* $\Delta_G^{\frac{\alpha}{2}} u(x)$ *exists for* $x \in \partial G$ *if and only if* $\frac{\partial u}{\partial n}(x) = 0$ ($\frac{\partial u}{\partial n}$: *derivative in the inward normal direction*).

Proof. Assertion (i) can be proved following the argument of Proposition 2.2 (i). We prove only the assertion (ii) with $1 < \alpha < 2$. The case of $\alpha = 1$ can be proved similarly. Let $x \in \partial G$. Without loss of generality we may take coordinate system CS_x such that x is the origin and the inward normal unit vector of G at x is $(0, 0, \cdots, 0, 1)$. Let $\widetilde{u} \in C_c^2(\mathbb{R}^n)$ be an extension of u to $\mathbb{R}^n$. Set

$$N = \sup_{y \in \mathbb{R}^n} |\nabla \widetilde{u}(y)|, \quad M = \sup_{y \in \mathbb{R}^n} \sum_{i=1}^n \sum_{j=1}^n |(\partial_i \partial_j \widetilde{u}(y))|. \tag{5.9}$$

Sufficiency: By the definition,

$$\begin{aligned} &|\Delta_{G,\varepsilon_1}^{\frac{\alpha}{2}} u(x) - \Delta_{G,\varepsilon_2}^{\frac{\alpha}{2}} u(x)| \\ &\quad \le \mathcal{A}(n, -\alpha) \left| \int_{\varepsilon_2 < |y-x| < \varepsilon_1} I_{\mathbb{R}_+^n} \frac{\widetilde{u}(y) - \widetilde{u}(x)}{|x-y|^{n+\alpha}}\, dy \right| \\ &\qquad + \mathcal{A}(n, -\alpha) \int_{\mathbb{R}_+^n \triangle G} I_{\{\varepsilon_2 < |y-x| < \varepsilon_1\}} \frac{|\widetilde{u}(y) - \widetilde{u}(x)|}{|x-y|^{n+\alpha}}\, dy. \end{aligned} \tag{5.10}$$

Applying spherical coordinate system at point x we have

$$\begin{aligned} &\int_{\varepsilon_2 < |y-x| < \varepsilon_1, \mathbb{R}_+^n} \frac{\langle \nabla \widetilde{u}(x), \frac{y-x}{|y-x|} \rangle |y-x|}{|x-y|^{n+\alpha}}\, dy \\ &= \int_0^{\frac{\pi}{2}} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} \\ &\quad \int_{\varepsilon_2}^{\varepsilon_1} \frac{\langle \nabla \widetilde{u}(x), \frac{y-x}{|y-x|} \rangle}{r^\alpha} dr \\ &= \frac{\varepsilon_2^{1-\alpha} - \varepsilon_1^{1-\alpha}}{\alpha - 1} \int_{|z|=1} I_{\mathbb{R}_+^n} \psi(z_n) \langle z, \nabla u(x) \rangle m(dz), \end{aligned}$$

where $\psi(x) \equiv 1$. Notice that the n-th coordinate of $\nabla u(x)$ is zero, applying Lemma 5.2 we see the above integral is zero. So we have

$$\begin{aligned} &\left| \int_{\varepsilon_2 < |y-x| < \varepsilon_1, \mathbb{R}_+^n} \frac{\widetilde{u}(y) - \widetilde{u}(x)}{|x-y|^{n+\alpha}}\, dy \right| \\ &\quad = \left| \int_{\varepsilon_2 < |y-x| < \varepsilon_1, \mathbb{R}_+^n} \frac{\widetilde{u}(y) - \widetilde{u}(x) - \langle \nabla \widetilde{u}(x), \frac{y-x}{|y-x|} \rangle |y-x|}{|x-y|^{n+\alpha}}\, dy \right| \\ &\quad \le \frac{(2\pi)^n M}{2-\alpha} \varepsilon_1^{2-\alpha}. \end{aligned} \tag{5.11}$$

On the other hand, notice that $\nabla\Gamma(\widetilde{x}) = 0$ by our choice of the coordinate system CS_x. Therefore by (5.8), with $\xi(\widetilde{y})$ being determined by the mean value theorem, we have

$$|\Gamma_x(\widetilde{y})| = |\nabla\Gamma_x(\xi(\widetilde{y})), \widetilde{y} - \widetilde{x})\rangle| \le C|\widetilde{y} - \widetilde{x}|^{1+\beta}.$$

Thus, when ε_1 is small enough we get

$$\begin{aligned}
&\int_{\{\mathbb{R}^n_+\}\triangle G} I_{\{\varepsilon_2<|y-x|<\varepsilon_1\}} \frac{|\widetilde{u}(y) - \widetilde{u}(x)|}{|x-y|^{n+\alpha}}\, dy \\
&\quad\le N \int_{|\widetilde{y}-\widetilde{x}|<\varepsilon_1} d\widetilde{y}\Big(\int_0^{\Gamma_x(\widetilde{y})} I_{\{\Gamma_x(\widetilde{y})\ge 0\}} \frac{dy_n}{|x-y|^{n+\alpha-1}} \\
&\qquad + \int_{\Gamma_x(\widetilde{y})}^0 I_{\{\Gamma_x(\widetilde{y})<0\}} \frac{dy_n}{|x-y|^{n+\alpha-1}}\Big) \\
&\quad\le \frac{NC(2\pi)^{n-1}}{1+\beta-\alpha}\varepsilon_1^{1+\beta-\alpha}.
\end{aligned}\tag{5.12}$$

Combing (5.10)–(5.12) we obtain the existence of $\Delta_G^{\frac{\alpha}{2}} u(x)$.

Necessity: When $\frac{\partial u}{\partial x}(x) \neq 0$, by Lemma 5.2, we have $F = |\int_{|z|=1} I_{\mathbb{R}^n_+}\psi(z_n) \langle z, \nabla u(x)\rangle m(dz)| > 0$ for $\psi(x) \equiv 1$. Hence by (5.11) and (5.12) we obtain

$$\begin{aligned}
&\left|\int_{\varepsilon_2<|y-x|<\varepsilon_1} I_{\overline{G}} \frac{u(y) - u(x)}{|x-y|^{n+\alpha}}\, dy\right| \\
&\ge -\left|\int_{\varepsilon_2<|y-x|<\varepsilon_1} I_{\mathbb{R}^n_+} \frac{\widetilde{u}(y) - \widetilde{u}(x) - \langle\nabla\widetilde{u}(x), \frac{y-x}{|y-x|}\rangle|y-x|}{|x-y|^{n+\alpha}}\, dy\right| \\
&\quad + \left|\int_{\varepsilon_2<|y-x|<\varepsilon_1} I_{\mathbb{R}^n_+} \frac{\langle\nabla\widetilde{u}(x), \frac{y-x}{|y-x|}\rangle|y-x|}{|x-y|^{n+\alpha}}\, dy\right| \\
&\quad - \int_{\mathbb{R}^n_+\triangle G} I_{\{\varepsilon_2<|y-x|<\varepsilon_1\}} \frac{|\widetilde{u}(y) - \widetilde{u}(x)|}{|x-y|^{n+\alpha}}\, dy \\
&\ge F\frac{\varepsilon_2^{1-\alpha} - \varepsilon_1^{1-\alpha}}{\alpha - 1} - \frac{(2\pi)^n M}{2-\alpha}\varepsilon_1^{2-\alpha} - \frac{NC(2\pi)^n}{\beta-\alpha+1}\varepsilon_1^{1+\beta-\alpha}.
\end{aligned}$$

Therefore $\Delta_{G,\varepsilon}^{\frac{\alpha}{2}} u(x)$ does not converge since $\varepsilon_2^{1-\alpha} \to \infty$. □

Theorem 5.4. *In the situation of Theorem 5.3 (ii), let $z \in \partial G$.*

(i) If $\frac{\partial u}{\partial n} = 0$ in a relatively open subset of ∂G containing z, then $\Delta_G^{\frac{\alpha}{2}} u$ is continuous at z.

(ii) If $\frac{\partial u}{\partial n}(z) > 0$, then $\lim_{x\in G, x\to z} \Delta_G^{\frac{\alpha}{2}} u(x) = \infty$.

(iii) If $\frac{\partial u}{\partial n}(z) < 0$, $\lim_{x\in G, x\to z} \Delta_G^{\frac{\alpha}{2}} u(x) = -\infty$.

Proof. Again we prove only the case of $1 < \alpha < 2$. Suppose that $\frac{\partial u}{\partial n}(x) = 0$ on a relatively open subset of ∂G containing z, then from the sufficient part of the above proof we can find $\delta > 0$ such that

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in B(z,\delta) \cap \partial G} |\mathcal{A}(n, -\alpha) \int_{y \in \overline{G}, \varepsilon_2 < |y-x| < \varepsilon_1} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, dy| = 0. \quad (5.13)$$

On the other hand by the continuity of u on $\overline{G}$, one can easily check that for each fixed $\delta > 0$, $\Delta^{\frac{\alpha}{2}}_{\overline{G},\delta} u$ is uniformly continuous on $\overline{G}$. Therefore in order to prove assertion (i) of this theorem we need only to check that we can find $\delta_1 > 0$ such that

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in G \cap B(z,\delta_1)} |\mathcal{A}(n, -\alpha) \int_{y \in \overline{G}, \varepsilon_2 < |y-x| < \varepsilon_1} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, dy| = 0. \quad (5.14)$$

Let us take $0 < \delta_1 < \delta \wedge \frac{r_0}{6}$, where δ is specified by (5.13) and r_0 is specified by (3.1). For $x \in B(z, \delta_1) \cap G$, take a point $x_0 \in \partial G$ such that

$$|x - x_0| = dis(x, \partial G) := \rho(x).$$

In what follows we shall freely use the notations of Theorem 5.3 with x replaced by x_0. In particular $\widetilde{u} \in C_c^2(\mathbb{R}^n)$ is an extension of u to $\mathbb{R}^n$ and CS_{x_0} is a coordinate system with the same property as in Theorem 5.3. Note that by our choice of CS_{x_0} and the fact that $|x - x_0| = \rho(x)$, the coordinate of x must be $(0, \cdots, 0, \rho(x))$. In the case of $0 < \varepsilon_2 < \varepsilon_1 \leq \rho(x) \wedge \delta_1$, we have

$$\begin{aligned}
&\mathcal{A}(n, -\alpha) \left| \int_{y \in \overline{G}, \varepsilon_2 < |y-x| \leq \varepsilon_1} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, dy \right| \\
&\quad \leq \mathcal{A}(n, -\alpha) \left| \int_{y \in \overline{G}, \varepsilon_2 < |y-x| \leq \varepsilon_1} \frac{u(y) - u(x) - \langle \nabla u(x), y - x \rangle}{|x - y|^{n+\alpha}} \, dy \right| \\
&\quad \leq \mathcal{A}(n, -\alpha) \frac{(2\pi)^n M}{2 - \alpha} \varepsilon_1^{\beta - \alpha} \qquad (5.15)
\end{aligned}$$

with M being specified by (5.9). We now assume that $\rho(x) < \varepsilon_1 \leq \delta_1$, and $\varepsilon_2 = \rho(x)$. Set

$$\begin{aligned}
A_1 &= (\overline{G} \bigtriangleup \mathbb{R}^n_+) \cap \{y \in \mathbb{R}^n : \varepsilon_2 < |y - x| \leq \varepsilon_1\}, \\
A_2 &= \{y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+ : y_n \leq 2\rho(x), \varepsilon_2 < |y - x| \leq \varepsilon_1\}, \\
A_3 &= \{y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+ : y_n > 2\rho(x), \varepsilon_2 < |y - x| \leq \varepsilon_1\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left| \int_{y \in \overline{G}, \varepsilon_2 < |y-x| < \varepsilon_1} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, dy \right| \\
&\quad \leq \int_{A_1} \frac{|\widetilde{u}(y) - \widetilde{u}(x)|}{|x - y|^{n+\alpha}} \, dy + \left| \int_{A_2} \frac{\widetilde{u}(y) - \widetilde{u}(x)}{|x - y|^{n+\alpha}} \, dy \right| + \left| \int_{A_3} \frac{\widetilde{u}(y) - \widetilde{u}(x)}{|x - y|^{n+\alpha}} \, dy \right|.
\end{aligned}$$

By the argument used in proving (5.12) we can check that

$$\int_{A_1} \frac{|\widetilde{u}(y)-\widetilde{u}(x)|}{|x-y|^{n+\alpha}}\,dy \le \frac{CN(2\pi)^{n-1}}{1+\beta-\alpha}\varepsilon_1^{1+\beta-\alpha}. \tag{5.16}$$

Since A_2 is symmetric with respect to the point x, we have

$$\begin{aligned}
&\left|\int_{A_2} \frac{\widetilde{u}(y)-\widetilde{u}(x)}{|x-y|^{n+\alpha}}\,dy\right| \\
&\quad\le \left|\int_{A_2} \frac{\widetilde{u}(y)-\widetilde{u}(x)-\langle\nabla\widetilde{u}(x),y-x\rangle}{|x-y|^{n+\alpha}}\,dy\right| \\
&\quad\le \int_{A_2} \frac{M}{|x-y|^{n+\alpha-2}}\,dy \le \frac{M(2\pi)^n}{2-\alpha}\varepsilon_1^{2-\alpha}.
\end{aligned} \tag{5.17}$$

Next we estimate the integral on A_3.

$$\begin{aligned}
&\int_{A_3} \frac{\widetilde{u}(y)-\widetilde{u}(x)}{|x-y|^{n+\alpha}}\,dy \\
&= \int_{A_3} \frac{\widetilde{u}(y)-\widetilde{u}(x)-\langle\nabla\widetilde{u}(x),y-x\rangle}{|x-y|^{n+\alpha}}\,dy + \int_{A_3} \frac{\langle\nabla\widetilde{u}(x)-\nabla\widetilde{u}(x_0),y-x\rangle}{|x-y|^{n+\alpha}}\,dy \\
&\quad + \int_{A_3} \frac{\langle\nabla\widetilde{u}(x_0),y-x\rangle}{|x-y|^{n+\alpha}}\,dy
\end{aligned} \tag{5.18}$$

Similarly to (5.17), we can show that the first and the second terms of (5.18) are bounded by $\frac{1}{2-\alpha}(2\pi)^n M\varepsilon_1^{2-\alpha}$. Employing the spherical coordinate system centered at x for the last term of (5.18) we obtain

$$\int_{A_3} \frac{\langle\nabla\widetilde{u}(x_0),y-x\rangle}{|x-y|^{n+\alpha}}\,dy = \int_{\rho(x)}^{\varepsilon_1} F(r)\frac{r^{n-1}}{r^{n+\alpha-1}}dr, \tag{5.19}$$

where

$$F(r) = \int_{|y|=1} I_{\mathbb{R}^n_+}|\nabla\widetilde{u}(x_0)|I_{\{ry_n>\rho(x)\}}\langle\frac{\nabla\widetilde{u}(x_0)}{|\nabla\widetilde{u}(x_0)|},y\rangle m(dy).$$

Applying Lemma 5.2, for $r>\rho(x)$ we have $F(r)\equiv 0$ provided $\frac{\partial u}{\partial n}(x_0)=0$, which together with (5.15)-(5.19) implies (5.14), proving (i). On the other hand, if $\frac{\partial u}{\partial n}(z)>0$, then $\frac{\partial u}{\partial n}(x_0)>\frac{1}{2}\frac{\partial u}{\partial n}(z)>0$ when $|z-x_0|$ is small enough. Thus Lemma 5.2 implies $F(r)>0$ for $r>\rho(x)$. Moreover, $F(r)$ is an increasing function of r, hence the right hand side of (5.19) tends to $+\infty$ when $x\to z$, which together with what we have proved implies the assertion (ii). Similarly $\frac{\partial u}{\partial n}(x_0)<0$ implies the assertion (iii). □

Remark. (i) By the above proof we see that if $\frac{\partial u}{\partial n}(z)>0$ for $z\in\partial G$, then there exist positive constants C_z and δ_z such that

$$\Delta_G^{\frac{\alpha}{2}}u(x)\ge C_z\rho(x)^{1-\alpha},\quad \forall x\in G\cap B(z,\delta_z). \tag{5.20}$$

(ii) It is interesting to note a certain similarity of the condition on $\frac{\partial u}{\partial n}$ in Theorem 5.4 and the discussion in [26] where fractional derivatives are investigated as generators of Feller semigroups when working in the half-space. For $0 < \alpha < 1$ denote by D_R^α the Riemann-Liouville fractional derivative of order α on $\mathbb{R}_+$ and by D_C^α the Caputo fractional derivative of order α on R_+. For f sufficiently smooth it holds that

$$D_R^\alpha f(x) = D_C^\alpha f(x) + f(0)\frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \ \forall x > 0.$$

For $x \to 0$ one can show that if $D_C^\alpha f(x)$ is well behaved and tends to 0 then $D_R^\alpha f(x)$ will diverge unless $f(0) = 0$. This leads to the conclusion (see [26]) that $-D_C^\alpha$ with only certain domain can be extended to a generator of a Feller semigroup. In particular $D_C^\alpha f(0) = 0$ is a necessary condition in case of reflection. The full analogy becomes apparent when considering operators of type $-p(x, D_x) - D_C^\alpha$ where $p(x, D_x)$ acts on functions defined for $x \in \mathbb{R}^n$ and D_C^α acts on functions defined for $x_{n+1} \in \mathbb{R}_+$, hence when the boundary value problems for $-p(x, D_x) - D_C^\alpha$ on $\mathbb{R}^n \times \mathbb{R}_+$ are considered, compare also [25] and [28].

6. Generators of reflected α-stable processes

Let G be a bounded Lipschitz open set in $\mathbb{R}^n$. For $0 < \alpha < 2$, let $(X_t)_{t\geq 0}$ be the reflected α-stable process on $\overline{G}$ which has been refined as a Feller process. Denote by $P_t f(x) = E_x f(X_t)$ whenever it makes sense. Then $(P_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on the Banach space $C(\overline{G})$ equipped with uniform norm. We denote by A_F^α with domain $\mathcal{D}(A_F^\alpha)$ its generator on $C(\overline{G})$.

Theorem 6.1. *(i) Let $0 < \alpha < 1$ and G be a bounded Lipschitz open set in $\mathbb{R}^n$. If $u \in C^\beta(\overline{G})$ for some $\beta > \alpha$, then $u \in \mathcal{D}(A_F^\alpha)$.*

(ii) Let $1 \leq \alpha < 2$ and G be a bounded Lipschitz open set in $\mathbb{R}^n$ with $C^{1,\beta}$ boundary for some $\beta > \alpha - 1$. Then a function $u \in C^2(\overline{G})$ belongs to $\mathcal{D}(A_F^\alpha)$ if and only if $\frac{\partial u}{\partial n}(x) = 0$ for all $x \in \partial G$. In the above two cases if $u \in \mathcal{D}(A_F^\alpha)$ then $A_F^\alpha u = \Delta_G^{\frac{\alpha}{2}} u$.

Proof. In both the above two cases by Theorem 4.1 we have the following semimartingale decomposition

$$u(X_t) = u(x_0) + M_t + \int_0^t \Delta_G^{\frac{\alpha}{2}} u(X_s)ds. \quad a.s.\ P_{x_0}, \ \forall x_0 \in \overline{G},$$

Therefore for all $x \in \overline{G}$ we have

$$\frac{P_t u(x) - u(x)}{t} = E_x \frac{1}{t}\int_0^t \Delta_G^{\frac{\alpha}{2}} u(X_s)ds. \tag{6.1}$$

In case (i) or in case (ii) with $\frac{\partial u}{\partial n} = 0$ on ∂G, by theorem 5.3 (i) or Theorem 5.4 we know that $\Delta_G^{\frac{\alpha}{2}} u$ is continuous on $\overline{G}$. Therefore we have

$$\lim_{t \downarrow 0} \frac{1}{t}(P_t u(x) - u(x)) = \Delta_G^{\frac{\alpha}{2}} u(x), \ \forall x \in \overline{G}. \tag{6.2}$$

Since $\overline{G}$ is a compact metric space and $(X_t)_{t \geq 0}$ is a right continuous Feller process, hence (6.2) implies further that

$$\lim_{t \downarrow 0} \sup_{x \in \overline{G}} |\frac{1}{t}(P_t u(x) - u(x)) - \Delta_G^{\frac{\alpha}{2}} u(x)| = 0. \tag{6.3}$$

In case (ii) with $\frac{\partial u}{\partial n}(x) > 0$(resp. $\frac{\partial u}{\partial n}(x) < 0$) for some point $x \in \partial G$, then by (6.1) and Theorem 5.4 we have $\lim_{t \downarrow 0} \frac{1}{t}(P_t u(x) - u(x)) = \infty$ (reap. $= -\infty$). Therefore u does not belong to $\mathcal{D}(A_F^\alpha)$. ◻

Remark. In the above theorem if $u \in \mathcal{D}(A_F^\alpha)$, we have also Dynkin formula:

$$\lim_{r \downarrow 0} \frac{Eu(X_\tau)}{E_x \tau} = \Delta_G^{\frac{\alpha}{2}} u(x), \quad \forall x \in \overline{G}. \tag{6.4}$$

where τ is the first leaving time from $B(x, r)$.

Note that $(P_t)_{t \geq 0}$ is extended to a strongly continuous contraction semigroup on $L^p(\overline{G}, dx)$ for each $p \geq 1$. When there is no risk of confusion we denote again by $(P_t)_{t \geq 0}$ the L^p semigroup. We denote by A_p^α with domain $\mathcal{D}(A_p^\alpha)$ the corresponding L^p-generators. To investigate the L^p-generators we need an estimate of $P_t g$ stated in the following lemma which may also have interest by its own.

Lemma 6.2. *Let function $g(x)$ be as in Lemma 4.6, then there exists a constant C such that*

$$E_x g(X_t) < C(1 + g(x)), \ \forall x \in G, t > 0. \tag{6.5}$$

Proof. The case of $\alpha < 1$ is trivial. We prove first the case of $1 < \alpha < 2$. Employing the notations of Lemmas 4.5 and 4.6, for $0 < t \leq 1$ and $x \in G$, we have by (4.6)

$$\begin{aligned} &E_x g(X_t) I_{\{X_t \in U_z\}} \\ &\leq K^n \int_{y \in \widetilde{U}_z, |y - \widetilde{x}| < K t^{\frac{1}{\alpha}}} c_2 t^{-n/\alpha} y_n^{1-\alpha} dy \\ &\quad + K^n \int_{y \in \widetilde{U}_z, |y - \widetilde{x}| \geq t^{\frac{1}{\alpha}}/K} \frac{c_2 t}{|\widetilde{x} - y|^{n+\alpha}} y_n^{1-\alpha} dy. \end{aligned} \tag{6.6}$$

To estimate the first term of the right hand side of (6.6), when $K t^{\frac{1}{\alpha}} < \frac{1}{2}\rho(x)$ we have

$$\begin{aligned} \int_{|y - \widetilde{x}| < K t^{\frac{1}{\alpha}}} t^{-n/\alpha} y_n^{1-\alpha} dy &\leq 2^{\alpha - 1} \int_{|y - \widetilde{x}| < K t^{\frac{1}{\alpha}}} t^{-n/\alpha} (\frac{\rho(x)}{2})^{1-\alpha} dy \\ &\leq 2^{\alpha-1}(2K)^n \rho(x)^{1-\alpha}. \end{aligned} \tag{6.7}$$

While for $Kt^{\frac{1}{\alpha}} \geq \frac{1}{2}\rho(x)$ we have

$$\begin{aligned}
&\int_{y\in\widetilde{U}_z,|y-\widetilde{x}|<Kt^{\frac{1}{\alpha}}} t^{-n/\alpha} y_n^{1-\alpha} dy \\
&\quad\leq \int_{0\vee(\rho(x)-Kt^{\frac{1}{\alpha}})}^{\rho(x)} 2^{n-1}K^{n-1}t^{-1/\alpha} y_n^{1-\alpha} dy_n \\
&\qquad + \int_{\rho(x)}^{\rho(x)+Kt^{\frac{1}{\alpha}}} 2^{n-1}K^{n-1}t^{-1/\alpha} y_n^{1-\alpha} dy_n \\
&\quad\leq \frac{1}{2-\alpha} 2^{n-1}K^{n-1}t^{-1/\alpha}\rho(x)^{2-\alpha} + 2^nK^n\rho(x)^{1-\alpha} \\
&\quad\leq \frac{1}{2-\alpha} 2^nK^n\rho(x)^{1-\alpha} + 2^nK^n\rho(x)^{1-\alpha} \\
&\quad\leq \frac{3-\alpha}{2-\alpha} 2^nK^n\rho(x)^{1-\alpha}. \qquad (6.8)
\end{aligned}$$

For the last term of (6.6), when $t^{\frac{1}{\alpha}}/K \geq \frac{1}{2}\rho(x)$ we have

$$\begin{aligned}
&\int_{y\in\widetilde{U}_z,|y-\widetilde{x}|\geq t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\
&\leq \int_0^{\frac{\rho(x)}{2}} dy_n \int_{y\in\widetilde{U}_z,|y-\widetilde{x}|\geq t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\
&\quad + \int_{y\in\widetilde{U}_z,|y-\widetilde{x}|\geq t^{\frac{1}{\alpha}}/K, y_n\geq\frac{\rho(x)}{2}} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\
&\leq \int_0^{\frac{\rho(x)}{2}} y_n^{1-\alpha} dy_n \int_0^\infty I_{\{y\in\widetilde{U}_z,|y-\widetilde{x}|\geq t^{\frac{1}{\alpha}}/K\}} \frac{(2\pi)^n t}{|y-\widetilde{x}|^{2+\alpha}} dr \\
&\quad + \frac{2^{\alpha-1}(2\pi)^nK^\alpha}{\alpha}\rho(x)^{1-\alpha} \\
&\leq \int_0^{\frac{\rho(x)}{2}} y_n^{1-\alpha} dy_n \int_{(\frac{t^{\frac{1}{\alpha}}}{K}-\rho(x))\vee 0}^{\delta_1} \frac{(2\pi)^n t}{(r^2+\frac{\rho(x)^2}{4})^{\frac{2+\alpha}{2}}} dr + \frac{2^{\alpha-1}(2\pi)^nK^\alpha}{\alpha}\rho(x)^{1-\alpha} \\
&\leq \int_0^{\frac{\rho(x)}{2}} y_n^{1-\alpha} dy_n \int_{(\frac{t^{\frac{1}{\alpha}}}{K\rho(x)}-1)\vee 0}^{\infty} \frac{(2\pi)^n t\rho(x)^{-1-\alpha}}{(r^2+\frac{1}{4})^{\frac{2+\alpha}{2}}} dr + \frac{2^{\alpha-1}(2\pi)^nK^\alpha}{\alpha}\rho(x)^{1-\alpha} \\
&\leq (2\pi)^n t\rho(x)^{-1-\alpha}\left(((\frac{t^{\frac{1}{\alpha}}}{K\rho(x)}-1)\vee 0)^{-1-\alpha}\wedge M\right) \\
&\quad \int_0^{\rho(x)} y_n^{1-\alpha} dy_n + \frac{2^{\alpha-1}(2\pi)^nK^\alpha}{\alpha}\rho(x)^{1-\alpha} \\
&\leq \frac{(2\pi)^n}{2-\alpha} t\rho(x)^{1-2\alpha}\left(((\frac{t^{\frac{1}{\alpha}}}{K\rho(x)}-1)\vee 0)^{-1-\alpha}\wedge M\right) \\
&\quad + \frac{2^{\alpha-1}(2\pi)^nK^\alpha}{\alpha}\rho(x)^{1-\alpha}, \qquad (6.9)
\end{aligned}$$

where $M = \int_0^\infty \frac{1}{(r^2+\frac{1}{4})^{\frac{2+\alpha}{2}}} dr$. Next we estimate the first term in the right hand side of (6.9). When $t^{\frac{1}{\alpha}} > 2K\rho(x)$, we have

$$\begin{aligned} &t\rho(x)^{1-2\alpha}\left(((\frac{t^{\frac{1}{\alpha}}}{K\rho(x)} - 1) \vee 0)^{-1-\alpha} \wedge M\right) \\ &\le t\rho(x)^{1-2\alpha}\left(\frac{t^{\frac{1}{\alpha}}}{2K\rho(x)}\right)^{-1-\alpha} \\ &\le \rho(x)^{2-\alpha} t^{\frac{-1}{\alpha}} (2K)^{1+\alpha} \\ &\le \rho(x)^{1-\alpha}(2K)^{\alpha}. \end{aligned} \tag{6.10}$$

When $t^{\frac{1}{\alpha}} \le 2K\rho(x)$, we have

$$\begin{aligned} &t\rho(x)^{1-2\alpha}\left(((\frac{t^{\frac{1}{\alpha}}}{K\rho(x)} - 1) \vee 0)^{-1-\alpha} \wedge M\right) \\ &\le \rho(x)^{1-\alpha}(2K)^{\alpha} M. \end{aligned} \tag{6.11}$$

From (6.9),(6.10) and (6.11), when $t^{\frac{1}{\alpha}}/K \ge \frac{1}{2}\rho(x)$, we have

$$\begin{aligned} &\int_{y\in\widetilde{U}_z, |y-\widetilde{x}|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\ &\le \frac{(2\pi)^n}{2-\alpha}(2K)^{\alpha}(M+1)\rho(x)^{1-\alpha} + \frac{2^{\alpha-1}(2\pi)^n K^{\alpha}}{\alpha}\rho(x)^{1-\alpha}. \end{aligned} \tag{6.12}$$

At last we estimate the last term in (6.6) when $t^{\frac{1}{\alpha}}/K < \frac{1}{2}\rho(x)$.

$$\begin{aligned} &\int_{y\in\widetilde{U}_z, |y-\widetilde{x}|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\ &\le \int_{y\in\widetilde{U}_z, \frac{\rho(x)}{2}\ge|y-\widetilde{x}|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\ &\quad + \int_{y\in\widetilde{U}_z, |y-\widetilde{x}|\ge\frac{\rho(x)}{2}} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy. \end{aligned} \tag{6.13}$$

For the first term of the above right hand side,

$$\begin{aligned} &\int_{y\in\widetilde{U}_z, \frac{\rho(x)}{2}\ge|y-\widetilde{x}|\ge t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy \\ &\le 2^{\alpha-1}(2\pi)^n\rho(x)^{1-\alpha}\int_{t^{\frac{1}{\alpha}}/K}^{\infty} \frac{t}{r^{1+\alpha}} dr \\ &\le 2^{\alpha-1}(2\pi)^n K^{\alpha}\rho(x)^{1-\alpha}. \end{aligned} \tag{6.14}$$

Notice that when t $Kt^{\frac{1}{\alpha}} < \frac{1}{2}\rho(x)$, the last term in (6.13) is less than the right hand side of (6.12), therefore by (6.13) and (6.14),

$$\int_{y\in\widetilde{U}_z,|y-\widetilde{x}|\geq t^{\frac{1}{\alpha}}/K} \frac{t}{|\widetilde{x}-y|^{n+\alpha}} y_n^{1-\alpha} dy$$
$$\leq 2^{\alpha-1}(2\pi)^n K^\alpha \rho(x)^{1-\alpha} + \frac{(2\pi)^n}{2-\alpha}(2K)^\alpha (M+1)\rho(x)^{1-\alpha}$$
$$+\frac{2^{\alpha-1}(2\pi)^n K^\alpha}{\alpha}\rho(x)^{1-\alpha}. \tag{6.15}$$

From (6.6),(6.7),(6.8),(6.12),(6.15) we see the Lemma is true when $1 < \alpha < 2$ and $0 < t \leq 1$. When $1 < \alpha < 2$ and $t > 1$, by the Markov property and the estimate (4.14) we see the lemma is true.

When $\alpha = 1$ we can prove the lemma similarly as above by using the fact that

$$\int_0^{\rho(x)} |\log y_n| dy_n = \int_0^{\rho(x)} dy_n \int_{y_n}^1 \frac{1}{s} ds$$
$$= \int_0^{\rho(x)} ds \int_0^s \frac{1}{s} dy_n + \int_{\rho(x)}^1 ds \int_0^{\rho(x)} \frac{1}{s} dy_n$$
$$= \rho(x) + \rho(x)|\log \rho(x)|$$

for $0 < \rho(x) < 1$. □

Theorem 6.3. *Let G be a bounded Lipschitz open set in $\mathbb{R}^n$.*

(i) If $0 < \alpha < 1$ and $u \in C^\beta(G)$ for some $\beta > \alpha$, then $u \in \mathcal{D}(A_p^\alpha)$ and $\Delta_G^{\frac{\alpha}{2}} u$ is a version of $A_p^\alpha u$ for any $p \geq 1$.

(ii) Let $p \geq 1$. If $1 \leq \alpha < \dfrac{p+1}{p}$ and $u \in C^{1+\beta}(G)$ for some $\beta > \alpha - 1$, then $u \in \mathcal{D}(A_p^\alpha)$ and $\Delta_G^{\frac{\alpha}{2}} u$ is a version of $A_p^\alpha u$.

Proof. Note that the value of a function g on the boundary ∂G is negligible in $L^p(\overline{G}, dx)$. Let g be a function as in Lemma 4.6, by (1.2) it is ready to check that

$$\lim_{t\to 0} P_t g(x) = g(x), \quad \forall x \in G. \tag{6.16}$$

Suppose now u is a function satisfying the conditions stated in the theorem. we define

$$A_t = \int_0^t \Delta_G^{\frac{\alpha}{2}} u(X_s) ds, \quad t \geq 0. \tag{6.17}$$

By (6.16), the dominated convergence theorem and the integration by parts formula (3.14), we can check that $(A_t)_{t\geq 0}$ satisfies Theorem 5.2.4 (iv) of [19] and hence we have Fukushima decomposition:

$$u(X_t) = u(x_0) + M_t + \textstyle\int_0^t \Delta_G^{\frac{\alpha}{2}} u(X_s) ds, \quad P_{x_0} - a.s. \;\; q.e.\, x_0 \in \overline{G}, \tag{6.18}$$

where M_t is a uniformly integrable martingale. Employing again (6.16), Lemma 6.2 and the dominated convergence theorem we conclude from (6.18) that

$$\lim_{t\downarrow 0}\int_{\overline{G}}\left|\frac{1}{t}(P_t u - u) - \Delta_G^{\frac{\alpha}{2}}u\right|^p dx = 0.$$

□

Remark. Let $(L, \mathcal{D}(L))$ be the generator of the Dirichlet form (1.1). The above theorem implies that if G is a bounded Lipschitz open set and $0 < \alpha < \frac{3}{2}$, then $C^2(\overline{G}) \subset \mathcal{D}(L)$ and for $u \in C^2(\overline{G})$ we have $Lu = \Delta_G^{\frac{\alpha}{2}}u$ a.e.

Theorem 6.4. *Let $p > 1$, $\dfrac{p+1}{p} \leq \alpha < 2$ and G be a bounded $C^{1,\beta}$ open set for some $\beta > \alpha - 1$. Then a function $u \in C^2(\overline{G})$ belongs to $\mathcal{D}(A_p^\alpha)$ if and only if $\frac{\partial u}{\partial n}(x) = 0$ for all $x \in \partial G$. If $u \in \mathcal{D}(A_p^\alpha)$ then $\Delta_G^{\frac{\alpha}{2}}u$ is a version of $A_p^\alpha u$.*

Proof. Let $u \in C^2(\overline{G})$. Suppose that $\frac{\partial u}{\partial n}(x) = 0$ for all $x \in \partial G$. Then the conclusion of the theorem follows directly from (6.3). Suppose that $\frac{\partial u}{\partial n}(z) \neq 0$, say $\frac{\partial u}{\partial n}(z) > 0$ for some $z \in \partial G$. Then by (5.20) and the condition $\dfrac{p+1}{p} \leq \alpha$ we can check that $\Delta_G^{\frac{\alpha}{2}}u$ is not in $L^p(\overline{G}, dx)$. Therefore by (6.1) we see that u is not in $\mathcal{D}(A_p^\alpha)$. □

7. Effect of the boundary values

In this section we go back to discuss further the integration by parts formula in one dimensional case. Comparing with the Green's formula for usual Laplacian or integration by parts formula for usual first order derivatives, one may wonder why there is no boundary value term in our integration by parts formula (3.14). We think the reason is that $\Delta_G^{\frac{\alpha}{2}}$ is non-local and the values of $\Delta_G^{\frac{\alpha}{2}}u(x)$, no matter how far $dis(x, \partial G)$ is, is always effected by the values of the function and its first order derivatives near the boundary. Consequently if the function and its first order derivatives are uniformly bounded on the region, which is the case we imposed in Theorem 3.3, then the integrals in (3.14) have accumulated enough the effect of the values near the boundary and hence there is no extra boundary value term appearing in the formula. However, if the first order derivatives of the function is not uniformly bounded, then either $\Delta_G^{\frac{\alpha}{2}}u$ does not exist, or in the integration by parts formula we must add a term reflecting the singularity of the first order derivatives near the boundary. To illustrate the effect of the boundary values we investigate in more detail the integration by parts formula on the one dimensional region (0,1).

Definition 7.1. *For $u \in C^1(0,1)$ and $0 < \gamma < 1$, we define the γ boundary operator F^γ of u by:*

$$F^\gamma u(0) = \lim_{t\downarrow 0} u'(t)t^\gamma, \quad F^\gamma u(1) = \lim_{t\uparrow 1} u'(t)(1-t)^\gamma$$

whenever the limits exist.

Lemma 7.2. *Let* $1 < \alpha < 2$ *and* $u, v \in C^1(0,1)$. *If* $F^{2-\alpha}u$ *and* $F^{2-\alpha}v$ *exist, then*

$$\int\int_{(0,1)\times(0,1)} \frac{|(u(x)-u(y))(v(x)-v(y))|}{|x-y|^{1+\alpha}}\, dxdy < \infty. \tag{7.1}$$

Proof. By the Cauchy inequality we need only to check the case of $u = v$. Assume first that $u(x) = x^{\alpha-1}$. Employing the equality

$$|a^\beta - b^\beta| \le \frac{|a-b|}{|a^{1-\beta} + b^{1-\beta}|}, \quad \forall\, 0 < \beta < 1,\ a, b > 0,$$

we get

$$\begin{aligned}
&\int\int_{(0,1)\times(0,1)} \frac{(x^{\alpha-1} - y^{\alpha-1})^2}{|x-y|^{1+\alpha}}\, dxdy \\
&\le \int\int_{(0,1)\times(0,1)} \frac{1}{(x^{2-\alpha} + y^{2-\alpha})^2 |x-y|^{\alpha-1}}\, dxdy \\
&\le \int_0^1 dx \int_0^1 \frac{1}{|x-y|^{3-\alpha}} dy \\
&\le \frac{1}{2-\alpha} \int_0^1 \left(x^{\alpha-2} + (1-x)^{\alpha-2} \right) dx \\
&< \infty.
\end{aligned} \tag{7.2}$$

We assume next that $\lim_{t\downarrow 0} u'(t) t^{2-\alpha} = C$ and u' is bounded on $(1/2, 1)$. In this case there exists a constant C' such that $|u'(x)| \le C' x^{\alpha-2}$, $\forall x \in (0,1)$. Therefore

$$\begin{aligned}
&\int\int_{(0,1)\times(0,1)} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\, dxdy \\
&= \int\int_{(0,1)\times(0,1)} \frac{1}{|x-y|^{1+\alpha}} dxdy \left(\int_x^y u'(s) ds \right)^2 \\
&\le \int\int_{(0,1)\times(0,1)} \frac{C'}{|x-y|^{1+\alpha}} dxdy \left(\int_x^y s^{\alpha-2} ds \right)^2 \\
&= \frac{1}{\alpha-1} \int\int_{(0,1)\times(0,1)} \frac{C'(x^{\alpha-1} - y^{\alpha-1})^2}{|x-y|^{1+\alpha}}\, dxdy \\
&< \infty.
\end{aligned}$$

The general case can be derived from the above facts. □

Lemma 7.3. *Let* $1 < \alpha < 2$, $u(x) = x^{\alpha-1}$, *then*

$$\Delta_{(0,1)}^{\frac{\alpha}{2}} u(x) = \frac{x^{\alpha-1}}{\alpha(1-x)^\alpha} + \frac{1}{\alpha x} - \frac{1}{\alpha x(1-x)^\alpha}. \tag{7.3}$$

Proof. It is not hard to see

$$\int_{x+\varepsilon}^{1} \frac{x^{\alpha-1}}{|y-x|^{1+\alpha}}dy = \frac{x^{\alpha-1}}{\alpha}\left(\frac{1}{\varepsilon^\alpha} - \frac{1}{(1-x)^\alpha}\right),$$
$$\int_{0}^{x-\varepsilon} \frac{x^{\alpha-1}}{|y-x|^{1+\alpha}}dy = \frac{x^{\alpha-1}}{\alpha}\left(\frac{1}{\varepsilon^\alpha} - \frac{1}{x^\alpha}\right).$$

For the first integral we let $t = \frac{y^{\alpha-1}}{(y-x)^{\alpha-1}}$, then $y = \dfrac{xt^{\frac{1}{\alpha-1}}}{t^{\frac{1}{\alpha-1}}-1}$, $dy = \dfrac{-xt^{\frac{2-\alpha}{\alpha-1}}}{(\alpha-1)(t^{\frac{1}{\alpha-1}}-1)^2}dt$. Thus

$$\int_{x+\varepsilon}^{1} \frac{y^{\alpha-1}}{|y-x|^{1+\alpha}}dy = \int_{\frac{1}{(1-x)^{\alpha-1}}}^{\frac{(x+\varepsilon)^{\alpha-1}}{\varepsilon^{\alpha-1}}} \frac{t^{\frac{1}{\alpha-1}}}{(\alpha-1)x}dt$$
$$= \frac{1}{\alpha x}\left(\frac{(x+\varepsilon)^\alpha}{\varepsilon^\alpha} - \frac{1}{(1-x)^\alpha}\right).$$

For the second integral we let $t = \frac{y^{\alpha-1}}{(x-y)^{\alpha-1}}$, then $y = \dfrac{xt^{\frac{1}{\alpha-1}}}{t^{\frac{1}{\alpha-1}}+1}$, $dy = \dfrac{xt^{\frac{2-\alpha}{\alpha-1}}}{(\alpha-1)(t^{\frac{1}{\alpha-1}}+1)^2}dt$. Thus

$$\int_{0}^{x-\varepsilon} \frac{y^{\alpha-1}}{|y-x|^{1+\alpha}}dy = \int_{0}^{\frac{(x-\varepsilon)^{\alpha-1}}{\varepsilon^{\alpha-1}}} \frac{t^{\frac{1}{\alpha-1}}}{(\alpha-1)x}dt = \frac{1}{\alpha x}\frac{(x-\varepsilon)^\alpha}{\varepsilon^\alpha}.$$

Therefore

$$\Delta^{\frac{\alpha}{2}}_{(0,1),\varepsilon}u(x) = \frac{1}{\alpha x}\left(\frac{(x+\varepsilon)^\alpha + (x-\varepsilon)^\alpha}{\varepsilon^\alpha} - \frac{2x^\alpha}{\varepsilon^\alpha}\right)$$
$$+\frac{x^{\alpha-1}}{\alpha(1-x)^\alpha} + \frac{1}{\alpha x} - \frac{1}{\alpha x(1-x)^\alpha}.$$

Since $\lim_{\varepsilon\downarrow 0} \frac{1}{\alpha x}\left(\frac{(x+\varepsilon)^\alpha + (x-\varepsilon)^\alpha}{\varepsilon^\alpha} - \frac{2x^\alpha}{\varepsilon^\alpha}\right) = 0$, we get (7.3). □

Remark. Let $a > 0$. Using the same argument as above we can get

$$\Delta^{\frac{\alpha}{2}}_{(0,a)}u(x) = \frac{x^{\alpha-1}}{\alpha(a-x)^\alpha} + \frac{1}{\alpha x} - \frac{a^\alpha}{\alpha x(a-x)^\alpha}.$$

Letting $a \to \infty$, we get $\Delta^{\frac{\alpha}{2}}_{(0,\infty)}u(x) = 0$, which gives another way to show that u is a harmonic function on $(0,\infty)$(see [5]).

Proposition 7.4. *Let* $1 < \alpha < 2$ *and* u *be a function on* $[0,1]$*. Suppose that* $\frac{u(x)}{x^{\alpha-1}} \in C^{1+\beta}[0,1]$ *and* $\frac{u(x)}{(1-x)^{\alpha-1}} \in C^{1+\beta}[0,1]$ *for some* $\beta > \alpha - 1$*, then* $\Delta^{\frac{\alpha}{2}}_{(0,1)}u \in L^1(0,1)$.

Proof. By Lemma 7.3 we can prove that $\Delta^{\frac{\alpha}{2}}_{(0,1)} x^{\alpha-1}$ is bounded, so the conclusion follows from Lemma 3.2 and Lemma 3.5. □

Theorem 7.5. *Let* $1 < \alpha < 2$ *and* $u, v \in C^1(0,1)$. *Assume that* u' *is* β*- Hölder continuous on* $(0,1)$ *for some* $\beta > \alpha - 1$ *and* $F^{2-\alpha}u$, $F^{2-\alpha}v$ *exist on the boundary, then*

$$\begin{aligned}\frac{1}{2}\mathcal{A}(1,-\alpha)\int\int_{(0,1)\times(0,1)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dxdy \\ = C_\alpha v F^{2-\alpha}u\Big|_0^1 - \lim_{\delta\downarrow 0}\int_\delta^{1-\delta} v(x)\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x)\,dx,\end{aligned} \tag{7.4}$$

where

$$\begin{aligned}C_\alpha = \frac{\mathcal{A}(1,-\alpha)}{\alpha(\alpha-1)}\Bigg(\int_0^1 \frac{1}{s^{2-\alpha}|1-s|^{\alpha-1}} - \frac{1}{\alpha-1} \\ + \lim_{\delta\downarrow 0}\int_\delta^1\left(\frac{1}{s^{2-\alpha}|s-\delta|^{\alpha-1}} - \frac{1}{s}\right)ds\Bigg).\end{aligned}$$

In particular $C_{\frac{3}{2}} = \frac{4}{3}(\pi - 2 + \ln 4)$.

Proof. By proposition 2.1 we see that $\Delta^{\frac{\alpha}{2}}_{(0,1)}u$ exists and $\Delta^{\frac{\alpha}{2}}_{(0,1),\varepsilon}u(x)$ converges to $\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x)$ uniformly on $[\delta, 1-\delta]$ for each $\delta > 0$. Without loss of generality we assume $F^{2-\alpha}u(0) = F^{2-\alpha}u(1) = 1$.

For δ and δ' satisfying $0 < \delta' < \delta < 1/2$, we put

$$\begin{aligned}G &= \{(x,y) : |x-y| < \delta'\} \cup \{(x,y) \in (0,1)\times(0,1)\ : 0 < x < \delta, 0 < y < \delta\} \\ &\quad \cup \{(x,y) : 1-\delta < x < 1, 1-\delta < y < 1\}, \\ G_1 &= G^c \cap \{(x,y) : 0 < x < \delta, 0 < y < 1)\}, \\ G_2 &= G^c \cap \{(x,y) : \delta \le x < 1-\delta, 0 < y < 1)\}, \\ G_3 &= G^c \cap \{(x,y) : 1-\delta \le x < 1, 0 < y < 1)\},\end{aligned}$$

By the symmetricity of $G^c := (0,1)\times(0,1)\backslash G$ in x and y, we have

$$\begin{aligned}\int\int_{G^c} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dxdy \\ = \int\int_{G^c} 2\frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}\,dxdy \\ = \int\int_{G_1\cup G_2\cup G_3} 2\frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}\,dxdy.\end{aligned} \tag{7.5}$$

First we estimate $\int_0^\delta \int_\delta^1 \frac{(u(y)-u(x))v(x)}{|x-y|^{1+\alpha}}\, dxdy$.

$$
\begin{aligned}
&\int_0^\delta \int_\delta^1 \frac{(u(y)-u(x))v(x)}{|x-y|^{1+\alpha}}\, dxdy \\
&\quad = \int_0^\delta dx \int_\delta^1 \frac{v(x)}{|x-y|^{1+\alpha}} dy \int_x^y u'(s)ds \\
&\quad = \int_0^\delta dx \int_x^\delta ds \int_\delta^1 \frac{v(x)}{|x-y|^{1+\alpha}} u'(s)dy \\
&\qquad + \int_0^\delta dx \int_\delta^1 ds \int_s^1 \frac{v(x)}{|x-y|^{1+\alpha}} u'(s)dy \\
&\quad = \frac{1}{\alpha}\int_0^\delta dx \int_x^\delta \left(\frac{v(x)u'(s)}{|\delta-x|^\alpha} - \frac{v(x)u'(s)}{|1-x|^\alpha}\right) ds \\
&\qquad + \frac{1}{\alpha}\int_0^\delta dx \int_\delta^1 \left(\frac{v(x)u'(s)}{|s-x|^\alpha} - \frac{v(x)u'(s)}{|1-x|^\alpha}\right) ds \\
&\quad = \frac{1}{\alpha}\int_0^\delta ds \int_0^s \frac{v(x)u'(s)}{|\delta-x|^\alpha} dx + \frac{1}{\alpha}\int_\delta^1 ds \int_0^\delta \frac{v(x)u'(s)}{|s-x|^\alpha} dx + I_1 \\
&\quad = \frac{1}{\alpha(\alpha-1)}\int_0^\delta \left(\frac{v(0)u'(s)}{|\delta-s|^{\alpha-1}} - \frac{v(0)u'(s)}{\delta^{\alpha-1}}\right) ds \\
&\qquad + \frac{1}{\alpha(\alpha-1)}\int_\delta^1 \left(\frac{v(0)u'(s)}{|s-\delta|^{\alpha-1}} - \frac{v(0)u'(s)}{s^{\alpha-1}}\right) ds + I_1 + I_2 \\
&\quad = \frac{1}{\alpha(\alpha-1)}\int_0^\delta \left(\frac{v(0)}{s^{2-\alpha}|\delta-s|^{\alpha-1}} - \frac{v(0)}{s^{2-\alpha}\delta^{\alpha-1}}\right) ds \\
&\qquad + \frac{1}{\alpha(\alpha-1)}\int_\delta^1 \left(\frac{v(0)}{s^{2-\alpha}|s-\delta|^{\alpha-1}} - \frac{v(0)}{s}\right) ds + I_1 + I_2 + I_3 \\
&\quad = \frac{v(0)}{\alpha(\alpha-1)}\left(\int_0^1 \frac{1}{s^{2-\alpha}|1-s|^{\alpha-1}} - \frac{1}{\alpha-1}\right. \\
&\qquad \left. + \int_\delta^1 \left(\frac{1}{s^{2-\alpha}|s-\delta|^{\alpha-1}} - \frac{1}{s}\right) ds \right) + I_1 + I_2 + I_3,
\end{aligned}
\tag{7.6}
$$

where

$$
\begin{aligned}
I_1 &= -\frac{1}{\alpha}\left(\int_0^\delta dx \int_x^\delta \frac{v(x)u'(s)}{|1-x|^\alpha} ds + \int_0^\delta dx \int_\delta^1 \frac{v(x)u'(s)}{|1-x|^\alpha} ds\right), \\
I_2 &= \frac{1}{\alpha}\left(\int_0^\delta ds \int_0^s \frac{(v(x)-v(0))u'(s)}{|\delta-x|^\alpha} dx + \int_\delta^1 ds \int_0^\delta \frac{(v(x)-v(0))u'(s)}{|s-x|^\alpha} dx\right), \\
I_3 &= \frac{1}{\alpha(\alpha-1)}\int_0^\delta \left(\frac{v(0)}{|\delta-s|^{\alpha-1}} - \frac{v(0)}{\delta^{\alpha-1}}\right)\left(u'(s) - \frac{1}{s^{2-\alpha}}\right) ds \\
&\quad + \frac{1}{\alpha(\alpha-1)}\int_\delta^1 \left(\frac{v(0)}{|s-\delta|^{\alpha-1}} - \frac{v(0)}{s^{\alpha-1}}\right)\left(u'(s) - \frac{1}{s^{2-\alpha}}\right) ds.
\end{aligned}
$$

We claim that the third term in the bracket of (7.6) has limit when $\delta \downarrow 0$. To this end we notice that the function

$$f(\delta) = \int_\delta^1 \left(\frac{1}{s^{2-\alpha}|s-\delta|^{\alpha-1}} - \frac{1}{s}\right) ds$$

is increasing as $\delta \downarrow 0$, so we need only to show $f(\delta)$ is bounded.

$$\begin{aligned}
f(\delta) &= \sum_{k=1}^{[\delta^{-1}]} \int_{k\delta}^{(k+1)\delta} \left(\frac{1}{s^{2-\alpha}|s-\delta|^{\alpha-1}} - \frac{1}{s}\right) ds \\
&\leq \frac{1}{2-\alpha} + \sum_{k=2}^{[\delta^{-1}]} \int_{k\delta}^{(k+1)\delta} \frac{1}{s^{2-\alpha}} \left(\frac{1}{|s-\delta|^{\alpha-1}} - \frac{1}{s^{\alpha-1}}\right) ds \\
&\leq \frac{1}{2-\alpha} + \frac{1}{\alpha-1} \sum_{k=2}^{[\delta^{-1}]} \left(\frac{1}{(k+1)^{1-\alpha}} - \frac{1}{k^{1-\alpha}}\right)\left(\frac{1}{(k-1)^{\alpha-1}} - \frac{1}{(k+1)^{\alpha-1}}\right) \\
&\leq \frac{1}{2-\alpha} + (\alpha-1) \sum_{k=2}^{[\delta^{-1}]} \frac{k^{\alpha-2}}{(k-1)^\alpha} \\
&< \frac{1}{2-\alpha} + (\alpha-1)2^\alpha. \qquad (7.7)
\end{aligned}$$

Since v is continuous and $\lim_{t\downarrow 0} u'(t)t^{2-\alpha} = 1$, we have

$$\lim_{x\downarrow 0}(v(x) - v(0)) = 0, \quad u'(s) - \frac{1}{s^{2-\alpha}} = o(\frac{1}{s^{2-\alpha}}).$$

Hence we can prove I_1, I_2, I_3 are finite and

$$\lim_{\delta\downarrow 0} I_1 = \lim_{\delta\downarrow 0} I_1 = \lim_{\delta\downarrow 0} I_3 = 0.$$

Therefore,

$$\begin{aligned}
&\lim_{\delta\downarrow 0} \int_0^\delta \int_\delta^1 \frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}\, dxdy \\
&= \frac{-v(0)}{\alpha(\alpha-1)}\left(\int_0^1 \frac{1}{s^{2-\alpha}|1-s|^{\alpha-1}} - \frac{1}{\alpha-1}\right. \\
&\quad \left. + \lim_{\delta\downarrow 0} \int_\delta^1 \left(\frac{1}{s^{2-\alpha}|s-\delta|^{\alpha-1}} - \frac{1}{s}\right) ds\right). \qquad (7.8)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{\delta\downarrow 0} \int_{1-\delta}^1 \int_0^{1-\delta} \frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}\, dxdy \\
&= \frac{v(1)}{\alpha(\alpha-1)}\left(\int_0^1 \frac{1}{s^{2-\alpha}|1-s|^{\alpha-1}} - \frac{1}{\alpha-1}\right. \\
&\quad \left. + \lim_{\delta\downarrow 0} \int_\delta^1 \left(\frac{1}{s^{2-\alpha}|s-\delta|^{\alpha-1}} - \frac{1}{s}\right) ds\right). \qquad (7.9)
\end{aligned}$$

Next, by the dominated convergence theorem we have

$$\lim_{\delta'\downarrow 0}\int_{G_1}\frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}dxdy=\int_0^{\delta}\int_{\delta}^{1}\frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}dxdy. \quad (7.10)$$

Similarly,

$$\lim_{\delta'\downarrow 0}\int_{G_3}\frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}dxdy=\int_{1-\delta}^{\delta}\int_{0}^{1-\delta}\frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}dxdy. \quad (7.11)$$

Because $\Delta^{\frac{\alpha}{2}}_{(0,1),\varepsilon}u(x)$ converges to $\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x)$ uniformly on $[\delta, 1-\delta]$, we have

$$\begin{aligned}&\lim_{\delta'\downarrow 0}\mathcal{A}(1,-\alpha)\int\int_{G_2}\frac{(u(x)-u(y))v(x)}{|x-y|^{1+\alpha}}\,dxdy\\&=-\int_{\delta}^{1-\delta}v(x)\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x)dx. \quad (7.12)\end{aligned}$$

By the fact that $\lim_{\delta\downarrow 0}\int\int_{G}dxdy=0$ and Lemma 7.2 we have

$$\begin{aligned}&\frac{1}{2}\mathcal{A}(1,-\alpha)\int\int_{(0,1)\times(0,1)}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dxdy\\&=\frac{1}{2}\mathcal{A}(1,-\alpha)\lim_{\delta\downarrow 0}\lim_{\delta'\downarrow 0}\int_{G^c}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dxdy. \quad (7.13)\end{aligned}$$

Combing (7.5),(7.7),(7.8),(7.9),(7.10),(7.11), we get

$$\begin{aligned}&\frac{1}{2}\mathcal{A}(1,-\alpha)\int\int_{(0,1)\times(0,1)}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dxdy\\&=C_\alpha vF^{2-\alpha}u\Big|_0^1-\lim_{\delta\downarrow 0}\int_{\delta}^{1-\delta}v(x)\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x)\,dx,\end{aligned}$$

which completes the proof. □

Corollary 7.6. *In the situation of Theorem 7.5, if in addition we assume that* $\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x)\in L^1(0,1)$*, then*

$$\begin{aligned}&\frac{1}{2}\mathcal{A}(1,-\alpha)\int\int_{(0,1)\times(0,1)}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dxdy\\&=C_\alpha vF^{2-\alpha}u\Big|_0^1-\int_0^1 v(x)\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x)\,dx. \quad (7.14)\end{aligned}$$

In particular, if u satisfies the conditions in Proposition 7.4, then (7.14) is true.

Proof. The conclusion follows from Proposition 7.4 and Theorem 7.5. □

Using the same method of theorem 3.4, we can get the results in the case of the half line.

Proposition 7.7. *Let* $0 < \alpha < 1$. *If* u *is* β*- hölder continuous on* $(0, \infty)$ *for some* $\beta > \alpha$ *and supp*$[u]$ *is compact,* $v \in C_b(0, \infty)$*, then*

$$\frac{1}{2}\mathcal{A}(1, -\alpha) \int\int_{[0,\infty)\times[0,\infty)} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+\alpha}}\, dxdy$$
$$= -C_\alpha v F^{2-\alpha} u(0) - \int_{[0,\infty)} v(x)\Delta^{\frac{\alpha}{2}}_{(0,\infty)} u(x)\, dx.$$

Let $1 \leq \alpha < 2$. *If* $u \in C^{1+\beta}(0, \infty)$ *for some* $\beta > \alpha - 1$, *supp*$[u]$ *is compact,* $v \in C_b^1(0, \infty)$, *then the equality above is true.*

A simple but interesting application of the above discussion is the semi-martingale decomposition below, where the additive functionals A^1 and A^2 share some feature of local times as in the Brownian motion case.

Theorem 7.8. *Let* $1 < \alpha < 2$. *If a function* u *on* $[0, 1]$ *satisfies the conditions in Proposition 7.4, then*

$$u(X_t) = u(x_0) + M_t - F^{2-\alpha} u(1) A_t^1 + F^{2-\alpha} u(0) A_t^2$$
$$+ \int_0^t \Delta^{\frac{\alpha}{2}}_{(0,1)} u(X_s) ds, \quad a.s.\ P_{x_0},\ \forall x_0 \in [0, 1].$$

Where A^1, A^2 *are positive continuous additive functionals in the strict sense with Revuz measure* $\nu_1 = C_\alpha \delta_{\{1\}}$, $\nu_2 = C_\alpha \delta_{\{0\}}$ *respectively.* $(M_t)_{t\geq 0}$ *is a square integrable martingale and the Revuz measure of the corresponding sharp bracket process is* $\left(\mathcal{A}(n, -\alpha) \int_0^1 \frac{(u(z) - u(y))^2}{|z - y|^{n+\alpha}} dz\right) dy$.

Proof. By (1.2) we can easily check that in the one dimensional case there are constants B_1 and B_2 such that $p(t, x, y) \leq B_1 + B_2 t^{\frac{-1}{\alpha}}$ for all $t > 0$ and $x, y \in [0, 1]$. From this fact we see that any finite Borel measure μ on $[0, 1]$ is in S_{00}. Therefore we have $\nu_1, \nu_2 \in S_{00}$.

On the other hand by Theorem 7.5 we have the integration by parts formula:

$$\frac{1}{2}\mathcal{A}(1, -\alpha) \int\int_{(0,1)\times(0,1)} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+\alpha}}\, dxdy$$
$$= C_\alpha v F^{2-\alpha} u\Big|_0^1 - \int_0^1 v(x)\Delta^{\frac{\alpha}{2}}_{(0,1)} u(x)\, dx, \quad \forall v \in C^\infty[0, 1].$$

Consequently the conditions in Proposition 4.4 are satisfied and the desired assertion follows. □

8. Smoothness of the regional fractional Laplacian

It is well known that the usual Laplacian is smooth in the sense that it maps C^{k+2} functions into C^k functions. In this section we shall prove that the regional fractional Laplacian share the same feature at least in the one dimensional case.

Theorem 8.1. *Let G be an open set in $\mathbb{R}$ and $k \in \mathbb{N}$. Suppose that $u \in L^1(G, \frac{1}{(1+|x|)^{1+\alpha}})$ and (i) $u^{(k)}$ is locally uniformly β-Hölder continuous on G for some $\beta > \alpha$ in the case of $0 < \alpha < 1$, (ii) $u^{(k+1)}$ is locally uniformly β-Hölder continuous on G for some $\beta > \alpha - 1$ in the case of $1 \le \alpha < 2$, then $\Delta_G^{\frac{\alpha}{2}} u \in C^k(G)$ and*

$$\frac{d}{dx^k}\Delta_G^{\frac{\alpha}{2}}u(x) = \frac{\mathcal{A}(1,-\alpha)}{2}\left(\varphi^{(k)}(x) + \lim_{\varepsilon\downarrow 0}\int_{\delta>|y-x|>\varepsilon}\frac{u^{(k)}(y)-u^{(k)}(x)}{|x-y|^{1+\alpha}}dy\right)$$

when $\delta < \rho(x)$, where

$$\varphi(x) = \int_{|y-x|>\delta, y\in G}\frac{u(y)-u(x)}{|x-y|^{1+\alpha}}dy.$$

Proof. By the assumption we know that $\Delta_G^{\frac{\alpha}{2}}u$ exist. We have

$$\begin{aligned}
&\frac{d}{dx}\left(\int_{x+\varepsilon}^{x+\delta}\frac{u(y)-u(x)}{|x-y|^{1+\alpha}}dy\right)\\
&\quad= \frac{u(x+\delta)-u(x)}{\delta^{1+\alpha}} - \frac{u(x+\varepsilon)-u(x)}{\varepsilon^{1+\alpha}} + \int_{x+\varepsilon}^{x+\delta}\frac{-u'(x)}{|x-y|^{1+\alpha}}dy\\
&\qquad+(1+\alpha)\int_{x+\varepsilon}^{x+\delta}\frac{u(y)-u(x)}{|x-y|^{2+\alpha}}dy\\
&\quad= \int_{x+\varepsilon}^{x+\delta}\frac{u'(y)-u'(x)}{|x-y|^{1+\alpha}}dy. \qquad (8.1)
\end{aligned}$$

Similarly,

$$\frac{d}{dx}\left(\int_{x-\delta}^{x-\varepsilon}\frac{u(y)-u(x)}{|x-y|^{1+\alpha}}dy\right) = \int_{x-\delta}^{x-\varepsilon}\frac{u'(y)}{|x-y|^{1+\alpha}}dy. \qquad (8.2)$$

By (8.1) and (8.2) we have

$$\frac{d}{dx}\left(\Delta_{G,\varepsilon}^{\frac{\alpha}{2}}u\right)=\int_{\delta>|y-x|>\varepsilon}\frac{u'(y)-u'(x)}{|x-y|^{1+\alpha}}dy+\frac{d}{dx}\left(\int_{|y-x|>\delta, y\in G}\frac{u(y)-u(x)}{|x-y|^{1+\alpha}}dy\right).$$

With the same argument as in the proof of Proposition 2.2 we can show that $\frac{d}{dx}(\Delta_{G,\varepsilon}^{\frac{\alpha}{2}}u)(x)$ converges uniformly on any compact subset of G. Therefore $\Delta_G^{\frac{\alpha}{2}}u(x)$ is differentiable.

In the same way we have

$$\frac{d}{dx}\left(\int_{\delta>|y-x|>\varepsilon}\frac{u'(y)-u'(x)}{|x-y|^{1+\alpha}}dy\right) = \int_{\delta>|y-x|>\varepsilon}\frac{u''(y)-u''(x)}{|x-y|^{1+\alpha}}dy.$$

Thus the desired conclusion is obtained by induction. □

Example. When $G = (0,1)$, we can get the following explicit formula.

$$\frac{d}{dx}\Delta^{\frac{\alpha}{2}}_{(0,1)}u(x) = -\frac{u(1)-u(x)}{(1-x)^{1+\alpha}} + \frac{u(0)-u(x)}{x^{1+\alpha}} + \Delta^{\frac{\alpha}{2}}_{(0,1)}u'(x).$$

We conjecture that in the higher dimensional case we might have the same smooth property of $\Delta^{\frac{\alpha}{2}}_{G}$. But because of the complex calculation, at this stage we can only prove the first order differentiability. We need the following lemma.

Lemma 8.2. *Let G be an open set of $\mathbb{R}^n$ and u be a integrable function on G. Suppose that u is continuous in an open neighborhood U of $x_0 \in G$ and $dist(x_0, \partial U) > r > 0$. Then the function $\varphi(x) = \int_{y\in G, |y-x|>r} u(y)dy$ is differentiable at point x_0 and*

$$\frac{\partial\varphi}{\partial x_i}(x_0) = \int_{|y-x_0|=r} u(y)\frac{x_i - y_i}{|x-y|}m(dy),$$

where m(dy) is the surface Lebesgue measure.

Proof. By the Green's formula. □

Proposition 8.3. *Let G be an open set in $\mathbb{R}^n$. Suppose that $u(x) \in L^1(G, \frac{1}{(1+|x|)^{(n+\alpha)}})$ and (i) all the derivatives of u is locally uniformly β-Hölder continuous on G for some $\beta > \alpha$ in the case of $0 < \alpha < 1$, (ii) all the second order derivatives of u is locally uniformly β-Hölder continuous on G for some $\beta > \alpha - 1$ in the case of $1 \le \alpha < 2$, then $\Delta^{\frac{\alpha}{2}}_{G}u \in C^1(G)$.*

Proof. Since $\Delta^{\frac{\alpha}{2}}_{G}u(x) = \lim_{\varepsilon\downarrow 0}\Delta^{\frac{\alpha}{2}}_{G,\varepsilon}u(x)$, we need only to prove that the derivation of $\Delta^{\frac{\alpha}{2}}_{G,\varepsilon}u(x)$ converges to a continuous function uniformly on each compact set of G.

Let $\delta < d_x$, by Lemma 8.2 we have

$$\begin{aligned}
&\frac{2}{\mathcal{A}(n,-\alpha)}\frac{\partial}{\partial x_i}\Delta^{\frac{\alpha}{2}}_{G,\varepsilon}u(x) \\
&\quad = -\int_{y\in G,|y-x|>\varepsilon}\frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy - (n+\alpha)\int_{y\in G,|y-x|>\varepsilon} \\
&\qquad \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy + \int_{|y-x|=\varepsilon}\frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy) \\
&\quad = -\int_{y\in G,|y-x|>\varepsilon}\frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy + \int_{|y-x|=\varepsilon}\frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy) \\
&\qquad -(n+\alpha)\int_{y\in G,d_G>|y-x|>\delta}\frac{(u(x)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy \\
&\qquad -(n+\alpha)\int_{\varepsilon}^{\delta}dr\int_{|y-x|=r}\frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}m(dy)
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{y\in G,|y-x|>\varepsilon} \frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy + \int_{y\in G,|y-x|=\delta} \frac{(u(y)-u(x)(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy)\\
&\quad -(n+\alpha)\int_{y\in G,d_G>|y-x|>\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy\\
&\quad -\int_\varepsilon^\delta dr\ \cdot r^{-(n+\alpha)}\frac{d}{dr}\int_{|y-x|=1} r^{n-1}(u(x+r(y-x))-u(x))(x_i-y_i)m(dy)\\
&= -\int_{y\in G,|y-x|>\varepsilon} \frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy + \int_{y\in G,|y-x|=\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy)\\
&\quad -(n+\alpha)\int_{y\in G,d_G>|y-x|>\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy\\
&\quad -\int_\varepsilon^\delta dr\ \cdot r^{-(\alpha+1)}\int_{|y-x|=1} \langle\nabla u(x+r(y-x)), y-x\rangle\cdot(x_i-y_i)m(dy)\\
&\quad -(n-1)\int_\varepsilon^\delta dr\cdot r^{-(\alpha+2)}\int_{|y-x|=1}(u(x+r(y-x))-u(x))(x_i-y_i)m(dy)\\
&= -\int_{y\in G,|y-x|>\varepsilon} \frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy + \int_{y\in G,|y-x|=\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy)\\
&\quad -(n+\alpha)\int_{y\in G,d_G>|y-x|>\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy\\
&\quad -\int_{\varepsilon<|y-x|<\delta} \frac{\langle\nabla u(y)), y-x\rangle\cdot(x_i-y_i)}{|x-y|^{(n+\alpha+2)}}dy\\
&\quad -(n-1)\int_{\varepsilon<|y-x|<\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{(n+\alpha+2)}}dy\\
&= -\int_{y\in G,|y-x|>\delta} \frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy - \int_{y\in G,\delta>|y-x|>\varepsilon} \frac{n\partial_i u(x)(x_i-y_i)^2}{|x-y|^{n+\alpha+2}}dy \qquad (8.3)\\
&\quad + \int_{y\in G,|y-x|=\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy) - (n+\alpha)\int_{y\in G,d_G>|y-x|>\delta}\\
&\quad \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy - \int_{\varepsilon<|y-x|<\delta} \frac{\langle\nabla u(y)), y-x\rangle\cdot(x_i-y_i)}{|x-y|^{(n+\alpha+2)}}dy\\
&\quad -(n-1)\int_{\varepsilon<|y-x|<\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{(n+\alpha+2)}}dy\\
&= -\int_{y\in G,|y-x|>\delta} \frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy + \int_{y\in G,|y-x|=\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy)\\
&\quad -(n+\alpha)\int_{y\in G,d_G>|y-x|>\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy\\
&\quad -\int_{\varepsilon<|y-x|<\delta} \frac{\sum_{j\neq i}\partial_j u(y)(y_j-x_j)(x_i-y_i)}{|x-y|^{(n+\alpha+2)}}dy
\end{aligned}
$$

$$+\int_{\varepsilon<|y-x|<\delta} \frac{(\partial_i u(y)-\partial_i u(x))(x_i-y_i)^2}{|x-y|^{(n+\alpha+2)}}dy$$
$$-(n-1)\int_{\varepsilon<|y-x|<\delta} \frac{(u(y)-u(x)-\partial_i u(x)(x_i-y_i))(x_i-y_i)}{|x-y|^{(n+\alpha+2)}}dy$$
$$=-\int_{y\in G,|y-x|>\delta} \frac{\partial_i u(x)}{|x-y|^{n+\alpha}}dy+\int_{y\in G,|y-x|=\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+1}}m(dy)$$
$$-(n+\alpha)\int_{y\in G,d_G>|y-x|>\delta} \frac{(u(y)-u(x))(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy$$
$$-\int_{\varepsilon<|y-x|<\delta} \frac{\sum_{j=1}^{n}(\partial_j u(y)-\partial_j u(x))(y_j-x_j)(x_i-y_i)}{|x-y|^{(n+\alpha+2)}}dy$$
$$-(n-1)\int_{\varepsilon<|y-x|<\delta} \frac{\left(u(y)-u(x)-\sum_{j=1}^{n}\partial_j u(x)(y_j-x_j)\right)(x_i-y_i)}{|x-y|^{n+\alpha+2}}$$
$$+(n-1)\frac{\left(\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\partial_j\partial_k u(x)(y_k-x_k)(y_j-x_j)\right)(x_i-y_i)}{|x-y|^{n+\alpha+2}}dy, \qquad (8.4)$$

where in the steps of (8.3) and (8.4) we used orthonormal transform.

By (8.4) we know that $\frac{\partial}{\partial x_i}(\Delta_{G,\varepsilon}^{\frac{\alpha}{2}}u)(x)$ converges uniformly on any compact subset of G, therefore $\Delta_G^{\frac{\alpha}{2}}u(x)$ is differentiable. □

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