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A functional central limit theorem for diffusions on periodic submanifolds of $\mathbb{R}^N$

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Abstract. We prove a functional central limit theorem for diffusions on periodic submanifolds of $\mathbb{R}^N$. The proof is an adaptation of a method presented in [BenLioPap] and [Bha] for proving functional central limit theorems for diffusions with periodic drift vector-fields. We then apply the central limit theorem in order to obtain a recurrence and a transience criterion for periodic diffusions. Other fields of applications could be heat-kernel estimates, similar to the ones obtained in [Lot].

1. Introduction

Let M be a closed, connected sub-manifold of $\mathbb{R}^N$. We assume that there exists a lattice $\Lambda \subset \mathbb{R}^N$ such that the translation by elements of Λ maps M into M. Furthermore, we assume that M/Λ is compact and has an orientation. As a sub-manifold of the Riemannian manifold $\mathbb{R}^N$, M carries a natural Riemannian metric. The associated Riemannian volume-measure will be denoted with v_0. Let L be a periodic, elliptic differential operator of second order defined on $C^2(M)$. The operator L generates a diffusion-process X on M (see [Hsu] p.24). For every function $g \in C^\infty(M)$

$$M_t^g := g(X_t) - g(X_0) - \int_0^t Lg(X_s)ds$$

is a local martingale with respect to a suitable filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. As in [Hsu] we assume the filtration to be complete and right continuous. Since the generator L is periodic, the diffusion process X on M can be naturally identified with a diffusion X^Λ on M/Λ. Since L is elliptic, it generates a strongly continuous contraction semigroup $t \mapsto e^{tL}$ on $C(M)$ and it has positive fundamental solutions $p : \mathbb{R}^+ \times M \times M \to \mathbb{R}^+$ with respect to v_0. In local coordinates L has the following expression

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$$L = \frac{1}{2}\sum_{i,j}^{d} a^{ij}\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{d} b^i \partial_{x_i}$$

with smooth coefficients a^{ij}, b^i. Furthermore, one defines for $h, g \in C^\infty(M)$

$$\Gamma(h, g) = L(hg) - hLg - gLh.$$

In local coordinates Γ takes the following form

$$\Gamma(h, g) = \sum_{i,j}^{d} a^{ij}(\partial_{x_i} h)(\partial_{x_j} g).$$

Since M/Λ is compact, there exists an invariant probability measure μ for X^Λ on M/Λ, and two constants $C, \lambda > 0$ such that for all periodic $g \in L^\infty(M, v_0)$ with $\int_{M/\Lambda} g d\mu = 0$ one has

$$\|e^{tL} g\|_\infty \leq C\|g\|_\infty e^{-t\lambda}$$

for all $t \geq 0$ (see [BenLioPap] p.365). The positive real value λ gives a spectral gap for the restriction of L to

$$L^2_p(M, \mu) := \left\{ g : M \to \mathbb{R} \text{ periodic; } \int_{M/\Lambda} g^2 d\mu < \infty \right\}.$$

Therefore, the resolvent in zero of L is defined on the orthogonal complement of the constant functions in $L^2_p(M, \mu)$. This implies that for all $g \in L^2_p(M, \mu)$ with $\int_{M/\Lambda} g d\mu = 0$ the Poisson-problem $L\psi = g$ has a solution in $\psi \in L^2_p(M, \mu)$. By elliptic regularity ψ is in $C^\infty(M)$ if $g \in C^\infty(M)$. For $1 \leq \alpha \leq N$ the restriction of the coordinate functions $k^\alpha : \mathbb{R}^N \to \mathbb{R}; x \mapsto x^\alpha$ to M will be denoted by f^α. We note that $Lf^\alpha \in L^2_p(M, \mu) \cap C^\infty(M)$. Therefore, the Poisson-problem

$$L\psi^\alpha = Lf^\alpha - \int_{M/\Lambda} Lf^\alpha d\mu$$

has a solution in $L^2_p(M, \mu) \cap C^\infty(M)$. We denote by $\overline{\mathrm{Lf}}$ the vector in $\mathbb{R}^N$ with components $\overline{Lf}^\alpha := \int_{M/\Lambda} Lf^\alpha d\mu$ for $1 \leq \alpha \leq N$. Since M is a sub-manifold of $\mathbb{R}^N$ the diffusion X can also be interpreted as a semi-martingale in $\mathbb{R}^N$ (see [Hsu] p.21). Therefore, X can be viewed as a random variable taking its values in the space of càdlàg functions $D_{\mathbb{R}^N}([0, \infty[)$ with the usual Skorohod topology (see [EthKur] p.118). We want to show that the distributions of the rescaled semi-martingales

$$X^{(n)}_t := n^{-1/2}(X_{nt} - X_0 - nt\overline{\mathrm{Lf}})$$

converge in the weak sense to the distribution of a Gaussian martingale with independent increments. For a given positive semi-definite, symmetric $N \times N$-matrix Σ, there exists a unique Gaussian $\mathcal{F}_t$-martingale W^Σ on $\mathbb{R}^N$ starting in zero with covariation process $\langle W^\Sigma, W^\Sigma\rangle_t = t\Sigma$ for all $t \geq 0$ and $\mathbb{P}$-a.s. (see [EthKur] p.338). By Levy's characterization theorem the projections of W^Σ onto one dimensional subspaces of $\mathbb{R}^N$ are multiples of Brownian motions (see [RevYor] p.141).

2. Results and proofs

Theorem 1. *The sequence $X^{(n)}$ converges to W^{Σ} in distribution with*

$$\Sigma_{\alpha\beta} := \int_{M/\Lambda} \Gamma(f^{\alpha} - \psi^{\alpha}, f^{\beta} - \psi^{\beta}) d\mu.$$

Proof. We first prove, that $X^{(n)}$ is an asymptotic martingale. One has

$$d\psi^{\alpha}(X_t) = dM_t^{\psi^{\alpha}} + L\psi^{\alpha}(X_t)dt = dM_t^{\psi^{\alpha}} + Lf^{\alpha}(X_t)dt - \overline{Lf}^{\alpha} dt$$

and therefore,

$$\begin{aligned} f^{\alpha}(X_t) - f^{\alpha}(X_0) - t\overline{Lf}^{\alpha} &= M_t^{f^{\alpha}} + \int_0^t (Lf^{\alpha}(X_s) - \overline{Lf}^{\alpha}) ds \\ &= M_t^{f^{\alpha}} - M_t^{\psi^{\alpha}} - \psi^{\alpha}(X_t) + \psi^{\alpha}(X_0). \end{aligned}$$

Now, since ψ is bounded, one has $\mathbb{P}$-a.s.

$$n^{-1/2}(f^{\alpha}(X_{nt}) - f^{\alpha}(X_0) - nt\overline{Lf}^{\alpha}) \simeq M_t^{(n),\alpha} := n^{-1/2}(M_{nt}^{f^{\alpha}} - M_{nt}^{\psi^{\alpha}}),$$

where $M^{(n)}$ is a $\mathcal{F}_t$-martingale. Therefore, $X^{(n)}$ is an asymptotic martingale. Now, one can apply the central limit theorem for martingales to $M^{(n)}$. In order to do so, we have to show that the covariation-process of $M^{(n)}$ converges in $L^2(\Omega, \mathbb{P})$ to the covariation process of W^{Σ}. For $M^{\alpha} := M^{f^{\alpha}} - M^{\psi^{\alpha}}$ one has (see [Hsu] p.30)

$$\begin{aligned} \langle M^{\alpha}, M^{\beta} \rangle_t &= \langle M^{f^{\alpha}} - M^{\psi^{\alpha}}, M^{f^{\beta}} - M^{\psi^{\beta}} \rangle_t \\ &= \int_0^t \Gamma(f^{\alpha} - \psi^{\alpha}, f^{\beta} - \psi^{\beta})(X_s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle M^{(n),\alpha}, M^{(n),\beta} \rangle_t &= n^{-1} \langle M^{\alpha}, M^{\beta} \rangle_{nt} \\ &= \int_0^t \Gamma(f^{\alpha} - \psi^{\alpha}, f^{\beta} - \psi^{\beta})(X_{ns}) ds. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbb{E}\left[\left(\langle M^{(n),\alpha}, M^{(n),\beta} \rangle_t - t\Sigma_{\alpha\beta}\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_0^t \left(\Gamma(f^{\alpha} - \psi^{\alpha}, f^{\beta} - \psi^{\beta})(X_{ns}) - \Sigma_{\alpha\beta}\right) ds\right)^2\right] \\ &= \mathbb{E}\left[\int_0^t \int_0^t R(X_{ns}) R(X_{nr}) ds dr\right] \\ &= 2 \int_0^t \int_0^s \mathbb{E}\left[R(X_{ns}) R(X_{nr})\right] dr ds, \end{aligned}$$

where

$$R(x) := \Gamma(f^\alpha - \psi^\alpha, f^\beta - \psi^\beta)(x) - \Sigma_{\alpha\beta}.$$

Now, by the Markov-property of the diffusion-process X one has

$$\begin{aligned}
&\mathbb{E}\left[R(X_{ns})R(X_{nr})\right] = \mathbb{E}\left[R(X_{nr})\mathbb{E}\left[R(X_{ns})|\mathcal{F}_{nr}\right]\right] \\
&= \mathbb{E}\left[R(X_{nr})\left(e^{n(s-r)L}R\right)(X_{nr})\right] \le \mathbb{E}\left[|R(X_{nr})|\right] Ce^{-n(s-r)\lambda}\|R\|_{\sup} \\
&\le C\|R\|^2_{\sup} e^{-n(s-r)\lambda}.
\end{aligned}$$

Therefore, one has for $n \to \infty$

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle M^{(n),\alpha}, M^{(n),\beta}\rangle_t - t\Sigma_{\alpha\beta}\right)^2\right] \le 2C\|R\|^2_{\sup}\int_0^t\int_0^s e^{-n(s-r)\lambda} dr ds \\
&= \frac{2C\|R\|^2_{\sup}}{n\lambda}\int_0^t \left(1 - e^{-ns\lambda}\right) ds \le \frac{2C\|R\|^2_{\sup}}{n\lambda} t \longrightarrow 0.
\end{aligned}$$

Now the result follows from the central limit theorem for martingales (see [EthKur] p.339). □

The following result was proved in [BenLioPap] and [Bat]:

Corollary 1. *Let X be a diffusion on $\mathbb{R}^N$ with periodic generator*

$$L = \sum_{i,j}^{N} a^{ij}\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{N} b^i\partial_{x_i}.$$

Then $X^{(n)}$ converges in distribution to W^Σ with

$$\Sigma_{\alpha\beta} = \int_{\mathbb{R}^N/\Lambda}\sum_{i,j}^{N}(\delta_{i\alpha} - \partial_{x_i}\psi^\alpha)a^{ij}(\delta_{j\beta} - \partial_{x_j}\psi^\beta)d\mu,$$

where ψ^α is the periodic solution to

$$L\psi^\alpha = b^\alpha - \int_{\mathbb{R}^N/\Lambda} b^\alpha d\mu$$

with $\int_{M/\Lambda}\psi^\alpha d\mu = 0$.

Proof. One has $Lf^\alpha = b^\alpha$ and $\partial_{x_i} f^\alpha = \delta_{i\alpha}$. □

Now, let M be a manifold with a periodic Riemannian metric g. Let ∇ be its associated Levi-Civita connection and Δ the resulting Laplace-Beltrami operator. The process generated by $L = \Delta$ is called Brownian motion on M.

Corollary 2. *Let X be Brownian motion on M. Then $X^{(n)}$ converges in distribution to W^Σ, where*

$$\Sigma_{\alpha\beta} = \frac{1}{v(M/\Lambda)}\int_{M/\Lambda} g(\nabla(f^\alpha - \psi^\alpha), \nabla(f^\beta - \psi^\beta))dv,$$

where v is the Riemannian volume on M/Λ, and ψ^α is the periodic solution to

$$\Delta\psi^\alpha = \Delta f^\alpha$$

with $\int_{M/\Lambda}\psi^\alpha dv = 0$.

Proof. This follows, since for all $f, h \in \mathcal{C}^\infty(M)$ one has $\Gamma(f,h) = g(\nabla f, \nabla h)$ (see [Hsu] p.80). Furthermore, $\int_{M/\Lambda} \Delta f^\alpha dv = 0$ follows from the divergence theorem and the fact that ∇f^α is periodic. □

We call a Riemannian manifold $M \subset \mathbb{R}^N$ harmonic, iff the restriction of the identity on $\mathbb{R}^N$ to M is harmonic.

Corollary 3. *Let M be a periodic, harmonic submanifold of $\mathbb{R}^N$ and let X be Brownian motion on M. Then $X^{(n)}$ converges in distribution to W^Σ with*

$$\Sigma_{\alpha\beta} = \frac{1}{v(M/\Lambda)} \int_{M/\Lambda} g(\nabla f^\alpha, \nabla f^\beta) dv,$$

where v is the Riemannian volume on M/Λ.

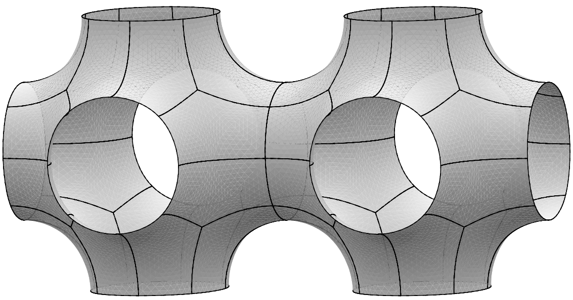
Proof. Since M is harmonic, one has $\Delta f^\alpha = 0$ for $1 \leq \alpha \leq N$ (see [Aub] p.350). Therefore, $\psi^\alpha = 0$ in the previous corollary. □

Minimal submanifolds M of $\mathbb{R}^N$ with the metric obtained by restricting the metric on $\mathbb{R}^N$ to the tangential space of M are harmonic manifolds (see [Jos] p.394). Periodic minimal surfaces are relevant in material science and crystallography, where they are used to describe boundary layers.

Example 1. The triply periodic Schwarz P-surface (see [Kar]) illustrated in the figure below is a minimal surface in $\mathbb{R}^3$. A computation using the periodicity of the manifold and the fact that the surface intersects the boundary ∂Q of the unit-cube Q perpendicularly shows

$$\int_{M/\Lambda} g(\nabla f^\alpha, \nabla f^\beta) dv_0 = \int_{M \cap \partial Q} f^\beta \nu(f^\alpha) dl = \int_{M \cap Q \cap \{x^\beta = 1\}} \nu(f^\alpha) dl,$$

where dl denotes the differential of the arc length on the intersection of M with ∂Q and ν denotes the outward pointing unit-vectorfield on $\partial(M \cap Q)$. Since $\nu(f^\alpha)$ vanishes on $M \cap \{x^\beta = 1\}$ for $\alpha \neq \beta$ the resulting diffusion coefficient for the limiting Brownian motion is a diagonal matrix. Further, since $\nu(f^\alpha) = 1$ on $M \cap \{x^\alpha = 1\}$, the diagonal entries are the length of the shortest closed geodesics on M divided by the volume of M/Λ.



The figure shows two cells of the triply periodic Schwarz P-surface. Note that the manifold intersects the boundary of the unit-cube perpendicularly. The figure was produced with Surface Evolver software.

3. Applications

We now want to link the recurrence resp. transience of X to the recurrence resp. transience of W^{Σ}. Since, all the involved processes are continuous, we can restrict our attention to $C_{\mathbb{R}^N}([0,\infty[)$ with uniform convergence on compact sets (see [EthKur] p.153). For a given Borel-set $U \subset \mathbb{R}^N$ we define the set of recurrent paths

$$R(U) := \left\{\omega \in C_{\mathbb{R}^N}([0,\infty[); \forall m \in \mathbb{N}, \exists t \geq m \text{ such that } \omega(t) \in U\right\}.$$

We note that X is recurrent on M, if and only if $\mathbb{P}_X(R(U)) = 1$ for all open sets $U \subset \mathbb{R}^N$ intersecting M. For a Borel-set $U \subset \mathbb{R}^N$ we define the set of transient paths

$$T(U) := \left\{\omega \in C_{\mathbb{R}^N}([0,\infty[); \exists s \geq 0 \text{ such that } \omega(t) \notin U,\ \forall t \geq s\right\},$$

and note that X is transient on M, if and only if $\mathbb{P}_X(T(U)) = 1$ for all open bounded sets $U \subset \mathbb{R}^N$. The set $T(U)$ is the complement of the set $R(U)$ in $C_{\mathbb{R}^N}([0,\infty[)$. Furthermore, the set $R(U)$ is open in $C_{\mathbb{R}^N}([0,\infty[)$, if U is open in $\mathbb{R}^N$. In order to apply the central limit theorem from the previous section, we need the following lemma.

Lemma 1. *Let Y be a diffusion process on a sub-manifold M of $\mathbb{R}^N$ and let $U, V \subset M$ be such that*

$$\delta := \inf_{x \in U} \mathbb{P}(Y_{1/2} \in V | Y_0 = x) > 0.$$

Then one has $\mathbb{P}_Y(R(U) \backslash R(V)) = 0$.

Proof. Let $\tau(0) = 0$ and $\tau(n) := \inf\{t \geq \max(n, \tau(n-1)+1); Y_t \in U\}$. Let $t \mapsto T_t$ be the semigroup associated to Y on $L^{\infty}(M,\nu)$. Then, one has by the strong Markov property (see [Hsu] p.31) for all $l \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}\left[\prod_{m=l}^{k} \mathbf{1}_{V^c}(Y_{\tau(m)+1/2})\right] &= \mathbb{E}\left[\prod_{m=l}^{k-1} \mathbf{1}_{V^c}(Y_{\tau(m)+1/2}) \mathbb{E}\left[\mathbf{1}_{V^c}(Y_{\tau(k)+1/2}) | \mathcal{F}_{\tau(k)}\right]\right] \\
&= \mathbb{E}\left[\prod_{m=l}^{k-1} \mathbf{1}_{V^c}(Y_{\tau(m)+1/2}) T_{1/2} \mathbf{1}_{V^c}(Y_{\tau(k)})\right] \\
&\leq (1-\delta)\mathbb{E}\left[\prod_{m=l}^{k-1} \mathbf{1}_{V^c}(Y_{\tau(m)+1/2})\right] \\
&\leq (1-\delta)^{k-l} \longrightarrow 0 \quad \text{when} \quad k \to \infty.
\end{aligned}$$

Therefore, one has with $C_m := \{Y_{\tau(m)+1/2} \in V\}$ for all $l \in \mathbb{N}$

$$\mathbb{P}_Y\left(R(U) \cap \bigcap_{m=l}^{k} C_m^c\right) \leq \mathbb{E}\left[\prod_{m=l}^{k} \mathbf{1}_{V^c}(Y_{\tau(m)+1/2})\right] \longrightarrow 0.$$

From this follows since $\limsup C_m \subset R(V)$

$$\mathbb{P}_Y\left(R(U)\cap R(V)^c\right) \le \mathbb{P}_Y\left(R(U)\cap\bigcup_{l\in\mathbb{N}}\bigcap_{m\ge l} C_m^c\right) = 0.$$

□

Lemma 2. *Let Σ be positive definite. For all bounded open sets $U \subset \mathbb{R}^N$ one has $\mathbb{P}_{W^\Sigma}(\partial R(U)) = 0$.*

Proof. We have $\partial R(U) \subset R(U_1)$, where $U_1 := \{x \in \mathbb{R}^N;\, \mathrm{dist}(x, U) < 1\}$. Further, W^Σ is a diffusion process on $\mathbb{R}^N$ and

$$\delta := \inf_{x\in U_1} \mathbb{P}(W^\Sigma_{1/2} \in U | W^\Sigma_0 = x) > 0,$$

since Σ is positive definite. Therefore, one can apply the previous lemma. Since $R(U)$ is open, it follows that

$$\mathbb{P}_{W^\Sigma}(\partial R(U)) = \mathbb{P}_{W^\Sigma}(\partial R(U)\backslash R(U)) \le \mathbb{P}_{W^\Sigma}(R(U_1)\backslash R(U)) = 0.$$

□

Theorem 2. *If* $\overline{\mathrm{Lf}} \neq 0$ *then X is transient on M.*

Proof. We have to show that for every ball $B_r(0)$ intersecting M one has $\mathbb{P}_X(T(B_r(0))) = 1$. Let $c \in \mathbb{R}^N$ with $c \neq 0$, we define the set of paths leaving the family of shifted half-spaces

$$H_{c,t} := \left\{x \in \mathbb{R}^N;\, \langle c, x\rangle > |c|^2 t\right\},$$

by

$$S(c) := \left\{\omega \in C_{\mathbb{R}^N}([0,\infty[);\, \exists s \ge 0 \text{ such that } \omega(t) \notin H_{c,t},\ \forall t \ge s\right\}.$$

Now, by the reflection principle for Brownian motion (see [RevYor] p.100) one has for all $k \in \mathbb{N}$

$$\mathbb{P}(\exists t \ge k,\ W_t^\Sigma \in H_{c,t}) \le \sum_{m=k}^\infty \mathbb{P}(\exists t \in [m, m+1[,\ W_t^\Sigma \in H_{c,m})$$
$$\le \sum_{m=k}^\infty \mathbb{P}\left(\sup_{0\le t\le m+1} B_t > |c|m\right) \le \sum_{m=k}^\infty 2\mathbb{P}\left(B_{m+1} > |c|m\right) < \infty,$$

where B is the Brownian motion obtained by projecting W to the one dimensional sub-space of $\mathbb{R}^N$ defined by $\mathbb{R}c$. This implies for $k \to \infty$

$$\mathbb{P}(W_t^\Sigma \notin H_{c,t},\ \forall t \ge k) = 1 - \mathbb{P}(\exists t \ge k,\ W_t^\Sigma \in H_{c,t}) \longrightarrow 1.$$

From this follows, $\mathbb{P}_{W^\Sigma}(S(c)) = 1$ for all $c \in \mathbb{R}^N$ with $c \neq 0$, which implies that $\mathbb{P}_{W^\Sigma}(\partial S(c)) = 0$ since $\partial S(c) \subset S(c/2)^c$. Then, by the central limit theorem it follows for all $r > 0$ and large n

$$\begin{aligned}\mathbb{P}_X(T(B_r(0))) &= \mathbb{P}(\exists s \geq 0 \text{ with } X_{nt} \notin B_r(0),\ \forall t \geq s)\\ &= \mathbb{P}(\exists s \geq 0 \text{ with } X_t^{(n)} \notin B_{r/\sqrt{n}}(0) - \sqrt{n}t\overline{\mathrm{Lf}},\ \forall t \geq s)\\ &\geq \mathbb{P}(\exists s \geq 0 \text{ with } X_t^{(n)} \notin H_{-\overline{\mathrm{Lf}},t},\ \forall t \geq s)\\ &= \mathbb{P}_{X^{(n)}}(S(-\overline{\mathrm{Lf}})) \longrightarrow \mathbb{P}_{W^\Sigma}(S(-\overline{\mathrm{Lf}})) = 1.\end{aligned}$$

This proves the theorem. □

Theorem 3. *Under the assumptions* $\overline{\mathrm{Lf}} = 0$ *and* Σ *positive definite, the diffusion* X *is recurrent on* M*, if and only if* W^Σ *is recurrent on* $\mathbb{R}^N$.

Proof. First we note that $\mathrm{Span}(\Lambda) = \mathbb{R}^N$, since Σ is positive definite. By the previous lemma, one has $\mathbb{P}_{W^\Sigma}(\partial R(B)) = 0$ for all balls B in $\mathbb{R}^N$. It follows from the central limit theorem that

$$\lim_{n\to\infty} \mathbb{P}_{X^{(n)}}(R(B)) = \mathbb{P}_{W^\Sigma}(R(B)).$$

Now, recurrence of X on M implies that $\mathbb{P}_X(R(U)) = 1$ for all open sets U intersecting M. Therefore, $\mathbb{P}_{X^{(n)}}(R(B)) = 1$, if $\sqrt{n}B$ intersects M, which is always fulfilled for large $n \in \mathbb{N}$, since $\mathrm{Span}(\Lambda) = \mathbb{R}^N$. This implies that $\mathbb{P}_{W^\Sigma}(R(B)) = 1$ for all balls $B \subset \mathbb{R}^N$. This proves the recurrence of W^Σ. Now, let's assume that W^Σ is recurrent. This implies that

$$\lim_{n\to\infty} \mathbb{P}_X(R(B_{\sqrt{n}}(0))) = \lim_{n\to\infty} \mathbb{P}_{X^{(n)}}(R(B_1(0))) = \mathbb{P}_{W^\Sigma}(R(B_1(0))) = 1.$$

For all $\epsilon > 0$ there is a $r \in \mathbb{N}$ such that $\mathbb{P}_X(R(B_r(0))) \geq 1 - \epsilon$. Now, since L is elliptic and M connected for a given open set $U \subset M$, one has

$$\delta := \inf_{x \in B_r(0) \cap M} \int_U p(1/2, x, y) d\nu(y) > 0.$$

Furthermore, X satisfies the strong Markov property (see [Hsu] p.31). Therefore, the first lemma implies $\mathbb{P}_X\,(R(B_r(0)) \setminus R(U)) = 0$. It follows, $\mathbb{P}_X(R(U)) \geq \mathbb{P}_X(R(B_r(0))) \geq 1 - \epsilon$. Letting ϵ go to zero proves the theorem. □

Theorem 4. *Under the assumption that* $\overline{\mathrm{Lf}} = 0$ *and* Σ *positive definite, the diffusion* X *is transient on* M*, if and only if* W^Σ *is transient on* $\mathbb{R}^N$.

Proof. Since $\partial T(U) = \partial R(U)$, we know that $\mathbb{P}_{W^\Sigma}(\partial T(U)) = 0$. Therefore, the central limit theorem gives

$$\lim_{n\to\infty} \mathbb{P}_{X^{(n)}}(T(U)) = \mathbb{P}_{W^\Sigma}(T(U)).$$

Now, if X is transient on M, then $\mathbb{P}_X(T(U)) = 1$ for all bounded $U \subset \mathbb{R}^N$. From this follows $\mathbb{P}_{X^{(n)}}(T(U)) = 1$. This implies $\mathbb{P}_{W^\Sigma}(T(U)) = 1$ for all bounded

$U \subset \mathbb{R}^N$ and therefore the transience of W^Σ. Now, let's assume that W^Σ is transient on $\mathbb{R}^N$. We then have

$$\mathbb{P}_X(T(B_r(0))) \geq \mathbb{P}_{X^{(n)}}(T(B_r(0))) \longrightarrow \mathbb{P}_{W^\Sigma}(T(B_r(0))) = 1.$$

Therefore, we have $\mathbb{P}_X(T(B_r(0))) = 1$ for all $r > 0$. This means that X is transient on M. □

Example 2. Let M be a periodic submanifold of $\mathbb{R}^N$ with periodic Riemannian metric g. Let $L := \Delta + B$, where B is a divergence-free periodic vectorfield on M. In this situation the invariant measure is the Riemannian volume v on M/Λ. It follows from the divergence theorem that

$$\overline{Lf}^\alpha = \frac{1}{v(M/\Lambda)} \int_{M/\Lambda} Bf^\alpha dv = \frac{1}{v(M/\Lambda)} \int_{M/\Lambda} g(B, \nabla f^\alpha) dv.$$

Therefore, $\overline{Lf}^\alpha$ can be interpreted as a measure for the drift-vectorfield B in the x^α-direction.

Example 3. Let again M be a periodic submanifold of $\mathbb{R}^N$ with periodic Riemannian metric g. Let ρ be a smooth, positive function such that the restriction of ρ to M/Λ is a probability density with respect to the Riemannian volume dv. Then the ρdv is the invariant measure for the Diffusion on M/Λ generated by the operator $L := \frac{1}{2}\Delta - \frac{1}{2\rho}(\nabla\rho)$. It follows that

$$\overline{Lf}^\alpha = \frac{1}{2}\int_{M/\Lambda} \rho \Delta f^\alpha dv - \frac{1}{2}\int_{M/\Lambda} g(\nabla f^\alpha, \nabla\rho) dv = \int_{M/\Lambda} \rho\Delta f^\alpha dv.$$

In this situation $\overline{Lf}^\alpha > 0$ indicates that the process stays more time at locations where Δf^α is positive. This results in a drift of the limiting Brownian motion in the x^α-direction.

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