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Stabilization by noise for a class of stochastic reaction-diffusion equations

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Abstract. We prove uniqueness, ergodicity and strongly mixing property of the invariant measure for a class of stochastic reaction-diffusion equations with multiplicative noise, in which the diffusion term in front of the noise may vanish and the deterministic part of the equation is not necessary asymptotically stable. To this purpose, we show that the L^1-norm of the difference of two solutions starting from any two different initial data converges $\mathbb{P}$-a.s. to zero, as time goes to infinity.

1. Introduction

We are here interested in the study of the ergodic properties of a class of stochastic reaction-diffusion equations perturbed by a multiplicative noise, where the diffusion coefficient in front of the noise may be degenerate and the deterministic part is not asymptotically stable. Namely, we want to show that if such a class of equations admits an invariant measure, then such invariant measure is unique, ergodic and strongly mixing.

The toy-model we have in mind (see next section for the general setting) is the following stochastic reaction-diffusion equation in a bounded interval $[0, L]$

$$\frac{\partial u}{\partial t} = \Delta u - \lambda_1 u^{2n+1} + \lambda u + \lambda_2 + g(u)\frac{\partial^2 w}{\partial t \partial \xi}, \quad u(0) = x \in C[0, L], \qquad (1.1)$$

endowed with some boundary conditions. Here $\lambda_1 \in [0, \infty)$, $n \in \mathbb{N}^\star$ and $\lambda_2 \in \mathbb{R}$, g is a strictly monotone Lipschitz-continuous function, possibly vanishing at some point, such that for any $R > 0$

$$|g(t) - g(s)| \geq \mu_R\, |s - t|, \quad s, t \in [-R, R],$$

for some positive constant μ_R (take for example $g \in C^1(\mathbb{R})$ such that $|g'(t)| > 0$, for any $t \in \mathbb{R}$) and λ is any real constant such that

$$\begin{cases} \lambda_1 > 0 \Longrightarrow \lambda < \mu_R^2/2L, \quad \text{for any } R > 0, \\ \lambda_1 = 0 \text{ and } \inf\limits_{R>0} \mu_R > 0 \Longrightarrow \lambda < \inf\limits_{R>0} \mu_R^2/2L \end{cases}$$

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(for all details and comments on these conditions -see the statement of Theorem 2.1- we refer to Remark 2.2).

This means that here neither it is possible to prove any smoothing effect of the transition semigroup associated with equation (1.1) (and hence apply the Doob theorem which in turn implies the uniqueness of the invariant measure), nor the stochastic term can be just regarded as a perturbation of a deterministic equation which *already* shows a stable behaviour. Thus, we are considering a situation in which, without any non-degeneracy assumption on the noise, the stochastic part plays a crucial role in stabilizing an equation which is not necessarily asymptotically stable, trying to extend in some sense to an infinite dimensional setting some results on stabilization by noise proved in finite dimension (see e.g. [1], [2], [18] and [19]; see also [3], [4], [12] and [15] for some results in infinite dimension).

In the case of PDEs perturbed by a noisy term of multiplicative type, the proof of uniqueness of the invariant measure can be quite delicate. If the space dimension $d = 1$ and the multiplication diffusion term g is bounded from below by a positive constant, it is possible to prove the uniqueness of the invariant measure through the Doob theorem, by showing that the associated transition semigroup is strong Feller and irreducible (see [10] for all details). But as soon as one goes from dimension $d = 1$ to dimension $d > 1$, since the noise has to be colored in space in order to have function-valued solutions, it is no more clear how to get strong Feller property by means of the Bismut-Elworthy formula even in the case of non-degenerate diffusion coefficients (see [10] and also [5] for some generalizations).

When the space dimension d is bigger than 1 and/or the diffusion coefficient g is not bounded from below, one can try to show that the equation is asymptotically stable, that is one can show that the difference of the laws of two solutions $u^x(t)$ and $u^y(t)$, starting from any two initial data x and y, converges to zero, as time t goes to infinity. But, while in the case of additive noise this is not difficult to prove if the coefficients of the deterministic part are sufficiently dissipative, in the case of multiplicative noise this can be difficult.

In [21], by using semigroups techniques, Sowers is able to prove mean square convergence to zero of the $C[0, L]$-norm of $u^x(t) - u^y(t)$ for stochastic reaction diffusion equations in dimension $d = 1$, having Lipschitz-continuous coefficients, under the assumptions that the deterministic part is asymptotically stable, the diffusion term is bounded from above and below and a parameter sufficiently small is put in front of the noise.

In [17], by using a coupling method, comparison arguments and martingale representation, Mueller extends Sowers results and, removing the small parameter in front of the stochastic term, shows that the $L^1(0, L)$-norm of $u^x(t) - u^y(t)$ converges to zero, $\mathbb{P}$-a.s. But in order to apply the coupling method Mueller has to assume that the diffusion g is bounded both from above and below and hence in the case he considers the strong Feller property and irreducibility hold, as well. However, it is important to stress that what is really interesting in Mueller's paper [17] are the techniques he uses, some of them have been largely used also in the present paper.

Concerning the coupling method, we recall that it has been used also recently by several other authors (see [11], [16] and [13] and references quoted therein), to prove exponential mixing properties of some PDEs perturbed by a degenerate noise of additive type.

In [6] we have proved the existence of an invariant measure for a general class of stochastic reaction-diffusion systems with multiplicative noise in any space dimension, having polynomially growing reaction terms and diffusion coefficients which are unbounded both from above and from below. But unfortunately it has not been possible to prove that such an invariant measure is unique, without assuming a strong enough dissipativity of the reaction term (to this purpose see also [8] and [7]).

The aim of this paper is exactly to see, starting from these examples, when it is possible to have uniqueness. For the moment we can only treat the one-dimensional case, but in comparison with the works by Sowers and Mueller we are already able to remove any conditions of boundedness from above and below for the diffusion g, of Lipschitz-continuity for the non-linearity f and of asymptotic stability for the deterministic part. However this is only the starting point: actually, in the sequel it will be interesting to see what happens in the case of space dimension $d > 1$, with colored noise, and in the case of systems, for which it is not possible to use comparison arguments.

2. Assumptions and statement of the main result

The class of equations we are considering is

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \mathcal{A}u(t,\xi) + f(\xi, u(t,\xi)) + g(\xi, u(t,\xi))\,\dfrac{\partial^2 w}{\partial t \partial \xi}(t,\xi), \quad t \geq 0, \;\; \xi \in [0,L], \\ u(0,\xi) = x(\xi), \quad \xi \in [0,L], \qquad \mathcal{B}u(t,0) = \mathcal{B}u(t,L) = 0, \quad t \geq 0. \end{cases} \tag{2.1}$$

Here $\mathcal{A}$ is a second order uniformly-elliptic operator given in divergence form, that is

$$\mathcal{A}h = (ah')', \qquad h \in C^2[0,L],$$

for some $a \in C^1[0,L]$ such that $a(\xi) \geq \epsilon > 0$, for any $\xi \in [0,L]$. Moreover

$$\mathcal{B}h = \alpha\left(h' - \beta\right) + (1-\alpha)h,$$

for some $\beta \in \mathbb{R}$ and $\alpha = 0, 1$. Notice that, instead of the Neumann and the Dirichlet conditions above, we could also consider periodic boundary conditions $u(t,0) = u(t,L)$ and $\partial u/\partial\xi(t,0) = \partial u/\partial \xi(t,L)$, for $t \geq 0$ (and the treatment of this case would be even easier).

The reaction term f fulfills the following conditions.

Hypotheses 1. *The function $f : [0,L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover*

1. *there exist $m \geq 1$ and $c \geq 0$ such that*

$$\sup_{\xi \in [0,L]} |f(\xi,\sigma)| \leq c\,(1 + |\sigma|^m), \qquad \sigma \in \mathbb{R}; \tag{2.2}$$

2. *for any* $\xi \in [0, L]$ *and* $\sigma, \rho \in \mathbb{R}$

$$f(\xi, \sigma) - f(\xi, \rho) = \lambda(\xi, \sigma, \rho)(\sigma - \rho),$$

for some locally bounded function $\lambda : [0, L] \times \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\sup_{\substack{\xi \in [0,L] \\ \sigma, \rho \in \mathbb{R}}} \lambda(\xi, \sigma, \rho) =: \lambda < +\infty; \tag{2.3}$$

3. *if* $m > 1$*, there exist* $a > 0$ *and* $c \geq 0$ *such that for any* $\xi \in [0, L]$ *and* $\sigma, \rho \in \mathbb{R}$

$$(f(\xi, \sigma + \rho) - f(\xi, \sigma))\rho \leq -a\,|\rho|^{m+1} + c\left(1 + |\sigma|^{m+1}\right),$$

where m is the constant in (2.2).

In particular, from the second condition above we have that $f(\xi, \cdot)$ is locally Lipschitz-continuous, uniformly with respect to $\xi \in [0, L]$, and the mapping

$$\sigma \in \mathbb{R} \mapsto f_\lambda(\xi, \sigma) := f(\xi, \sigma) - \lambda\,\sigma \in \mathbb{R} \tag{2.4}$$

is non-increasing, for any fixed $\xi \in [0, L]$.

In the case $m = 1$, any continuous function f such that $f(\xi, \cdot)$ has linear growth, belongs to $C^1(\mathbb{R})$ and

$$\frac{\partial f}{\partial \sigma}(\xi, \sigma) \leq \lambda, \quad (\xi, \sigma) \in [0, L] \times \mathbb{R},$$

fulfills Hypothesis 1. In the case $m > 1$, the example we have in mind is

$$f(\xi, \sigma) := f_1(\xi, \sigma) + f_2(\xi, \sigma),$$

where $f_1 : [0, L] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping such that $f_1(\xi, \cdot)$ is locally Lipschitz-continuous with linear growth, uniformly with respect to $\xi \in [0, L]$, and

$$f_2(\xi, \sigma) = -c(\xi)\sigma^{2n+1} + \sum_{j=1}^{2n} c_j(\xi)\sigma^j,$$

for some $n \in \mathbb{N}^\star$ with $2n + 1 = m$ and some continuous coefficients c, c_j : $[0, L] \to \mathbb{R}$ such that

$$\inf_{\xi \in [0,L]} c(\xi) =: c_0 > 0.$$

We want to stress here that the constant λ in (2.3) can be taken positive (see condition (2.11) below) and no relation is assumed between λ and $\mathcal{A}$. This means that we are not assuming any asymptotic stability for the deterministic part of equation (2.1)

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, \xi) = \mathcal{A}u(t, \xi) + f(\xi, u(t, \xi)), \quad t \geq 0, \quad \xi \in [0, L], \\ u(0, \xi) = x(\xi), \quad \xi \in [0, L], \qquad \mathcal{B}u(t, 0) = \mathcal{B}(t, L) = 0, \quad t \geq 0. \end{cases}$$

Concerning the diffusion coefficient g in front of the noise we assume the following conditions.

Hypotheses 2. *The function* $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ *is continuous. Moreover*

1. *the mapping* $g(\xi, \cdot)$ *is Lipschitz continuous, uniformly with respect to* $\xi \in [0, L]$*, that is there exists* $L > 0$ *such that*

$$\sup_{\xi \in [0,L]} |g(\xi,\sigma) - g(\xi,\rho)| \leq L\,|\sigma - \rho|, \qquad \sigma, \rho \in \mathbb{R}; \tag{2.5}$$

2. *for any* $R > 0$ *there exists* $\mu_R > 0$ *such that*

$$\inf_{\xi \in [0,L]} |g(\xi,\sigma) - g(\xi,\rho)| \geq \mu_R\,|\sigma - \rho|, \qquad \sigma, \rho \in [-R, R]. \tag{2.6}$$

Clearly, any function g such that $g(\xi, \cdot) \in C^1(\mathbb{R})$ and $|\partial g/\partial\sigma\,(\xi,\sigma)| > 0$, for any $(\xi, \sigma) \in [0, L] \times \mathbb{R}$, fulfills condition (2.6). Condition (2.6) implies that the mapping $\sigma \mapsto g(\xi, \sigma)$ is either strictly increasing or decreasing. This means that $g(\xi, \cdot)$ may vanish, but if it vanishes has only one zero. Moreover, if we define

$$\mu := \inf_{R>0} \mu_R,$$

we have that

$$\inf_{\xi \in [0,L]} \left| \frac{\partial g}{\partial \sigma}(\xi,\sigma) \right| \geq \mu.$$

In particular, when $\mu = 0$ we allow $\partial g(\xi,\sigma)/\partial\xi$ to go to zero, as $|\sigma|$ goes to infinity, and hence $g(\xi, \cdot)$ can be taken also bounded.

It is important to stress that, as we are not assuming no-degeneracy of the diffusion coefficient g in front of the noise, in order to prove the uniqueness of the invariant measure we cannot proceed by showing that the transition semigroup associated with equation (2.1) is strongly Feller. Actually, one possible key ingredient in the proof of the strong Feller property is the Bismut-Elworthy formula, and in order to give a meaning to it we have to assume

$$\mathbb{E} \int_0^t \left| g^{-1}(\cdot, u^x(s))\, D_x u^x(s) h \right|^2_{L^2(0,L)} ds < \infty,$$

where $D_x u^x(s)$ is the derivative of $u^x(s)$ with respect to x, along the direction h (for all details see [5, Chapter 6]).

Finally, the stochastic perturbation $\partial^2 w/\partial t \partial\xi$ has to be interpreted as the formal derivative of a Brownian sheet w on $[0, \infty) \times [0, L]$ defined on some underlying complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Given the Brownian sheet w, stochastic integration against $\partial^2 w/\partial t \partial\xi$ follows in the classical way (see [9] and [22]).

In [6] it has been proved that under Hypothesis 1 and Hypothesis 2-1. equation (2.1) admits a unique adapted mild solution u^x in $L^p(\Omega; C([0, T]; C[0, L]))$, for any initial datum $x \in C[0, L]$, for any $T > 0$ and any $p \geq 1$ (the case $m = 1$ was clearly already known in the previous literature). This means that there exists a unique adapted process u^x having continuous trajectories which satisfies

$$u^x(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(u^x(s))\, ds + \int_0^t e^{(t-s)A} G(u^x(s))\, dw(s),$$

where e^{tA} is the semigroup generated by the realization A in $C[0, L]$ of the differential operator $\mathcal{A}$, endowed with the boundary conditions $\mathcal{B}$, F and G are the composition operators associated respectively with f and g, that is

$$F(x)(\xi) := f(\xi, x(\xi)), \quad G(x)(\xi) = g(\xi, x(\xi)), \quad \xi \in [0, L],$$

and finally

$$w(t) = \sum_{k=1}^{\infty} e_k \beta_k(t) := \sum_{k=1}^{\infty} e_k \int_0^t \int_0^L e_k(\xi) w(d\xi, ds), \tag{2.7}$$

where $\{e_k\}$ is any complete orthonormal basis in $L^2(0, L)$.

In [6] we have also proved that in the case $m > 1$

$$\mathbb{E} \sup_{t \geq 0} |u^x(t)|^p_{C[0,L]} \leq c_p \left(1 + |x|^p_{C[0,L]}\right) \tag{2.8}$$

and

$$\sup_{t \geq t_0} \mathbb{E}\, |u^x(t)|_{C^{\theta_\star}[0,L]} < \infty,$$

for some $\theta_\star > 0$ and for any $t_0 > 0$. In particular, the family of measures $\{\mathcal{L}(u^x(t))\}_{t \geq t_0}$ is tight in $C([0, L], \mathcal{B}(C[0, L]))$ and hence, thanks to the Krylov-Bogoliubov theorem, the transition semigroup associated with equation (2.1) and defined by

$$P_t \varphi(x) = \mathbb{E}\, \varphi(u^x(t)), \quad x \in C[0, L], \quad t \geq 0, \tag{2.9}$$

for any φ belonging to $B_b(C[0, L])$, the space of Borel and bounded functions on $C[0, L]$ with values in $\mathbb{R}$, admits an invariant measure μ (as a matter of fact in [6] we have studied the more general case of systems with general boundary conditions in bounded domains of $\mathbb{R}^d$, with $d \geq 1$).

Our aim here is to show that such an invariant measure μ is unique (and in particular ergodic) and strongly mixing. To this purpose, with an approach under many respects similar to that used by Mueller in [17] we show that the following convergence result holds.

Theorem 2.1. *Assume that Hypotheses 1 and 2 hold. Moreover*

1. *if* $m = 1$, *assume that* $\mathcal{B}h = h' - \beta$ *and*

$$\inf_{R>0} \mu_R > 0, \quad \lambda < \inf_{R>0} \mu_R^2 / 2L; \tag{2.10}$$

2. *if* $m > 1$, *assume that*

$$\lambda < \mu_R^2 / 2L, \ \textit{for any } R > 0, \tag{2.11}$$

where m, λ *and* μ_R *are the constants introduced respectively in* (2.2), (2.3) *and* (2.6). *Then for any* $x, y \in C[0, L]$ *we have*

$$\lim_{t \to +\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} = 0, \qquad \mathbb{P} - a.s.$$

Remark 2.2. 1. It is important to stress that conditions (2.10) and (2.11) are in some sense sharp. Actually, if we consider the one-dimensional problem

$$du = (\lambda u + \lambda_2)\, dt + \mu u\, dB_t, \qquad u(0) = x \in \mathbb{R},$$

where B_t is a standard Brownian motion, then

$$u^x(t) - u^y(t) = (x - y) \exp\left(\left(\lambda - \frac{\mu^2}{2}\right) t\right) \exp(\mu B_t), \qquad t \geq 0,$$

so that $u^x(t) - u^y(t)$ converges to zero $\mathbb{P}$-a.s., as t goes to infinity, iff $\lambda < \mu^2/2$. On the other hand, if instead of $\mathbb{P}$-almost sure convergence we would like to have mean-wise convergence, since we have

$$\mathbb{E}\left(u^x(t) - u^y(t)\right) = (x - y) \exp(\lambda t),$$

we should require the stronger condition $\lambda < 0$, which does not take into account of the *strength* μ of the noise.

2. Note that the two requirements (2.10) and (2.11) on λ do not both coincides with

$$\lambda < \inf_{R>0} \mu_R^2/2L.$$

Actually in the case $m = 1$ condition (2.10) implies that

$$\inf_{R>0} \mu_R = 0 \Longrightarrow \lambda < 0.$$

On the contrary, if $m > 1$ condition (2.11) implies that

$$\begin{cases} \exists \min\limits_{R>0} \mu_R \Longrightarrow \lambda < \dfrac{1}{2L} \min\limits_{R>0} \mu_R^2 = \dfrac{1}{2L} \inf\limits_{R>0} \mu_R^2, \\ \nexists \min\limits_{R>0} \mu_R \Longrightarrow \lambda \leq \dfrac{1}{2L} \inf\limits_{R>0} \mu_R^2, \end{cases}$$

so that, since $\mu_R > 0$ for any $R > 0$, we have

$$\inf_{R>0} \mu_R = 0 \Longrightarrow \lambda \leq 0.$$

In particular, unlike in the case $m = 1$, if $m > 1$ and g exhibits a growth less than linear we can also take $\lambda = 0$.

As we will show later on, Theorem 2.1 implies that for any φ in $C_b(C[0, L])$, the space of uniformly continuous and bounded functions defined on $C[0, L]$, it holds

$$\lim_{t \to +\infty} P_t\varphi(x) - P_t\varphi(y) = 0, \qquad x, y \in C[0, L]. \tag{2.12}$$

Therefore, if there exists an invariant measure for P_t, it is unique (and hence ergodic) and in addition it is *strongly mixing*, that is for any $\varphi \in L^2(C[0, L], \mu)$ it satisfies

$$\lim_{t \to \infty} P_t\varphi = \int_{C[0,L]} \varphi(x)\, d\mu(x), \qquad \text{in } L^2(C[0, L], \mu),$$

(for general results on ergodicity and strongly mixing property see [10, Chapter 3]).

3. A comparison result

Here we give a proof of a comparison result for the solution u^x of equation (2.1). An analogous result for equations with locally Lipschitz coefficients having linear growth can be found for example in [14, Theorem 2.1]. Here, for the sake of completeness, we give a self-contained proof which adapts also to the case we are considering, in which the reaction term has polynomial growth.

Theorem 3.1. *For any $x, y \in C[0, L]$ such that $x \leq y$ we have*

$$\mathbb{P}\left(u^x(t) \leq u^y(t), \ \ t \geq 0\right) = 1. \tag{3.1}$$

Proof. Step 1. For any $n \in \mathbb{N}$ we define

$$w^n(t) := P_n w(t) = \sum_{k=1}^{n} e_k \beta_k(t), \quad t \geq 0,$$

where $\{e_k\}$ is a complete orthonormal basis in $L^2(0, L)$ which diagonalizes A, P_n is the projection operator of $L^2(0, L)$ onto span$\{e_1, \dots, e_n\}$ and $\beta_k(t)$ is defined as in (2.7). For each $n \in \mathbb{N}$ we consider the approximating problem

$$du(t) = [Au(t) + F(u(t))]\,dt + G(u(t))\,dw^n(t), \quad u(0) = x, \tag{3.2}$$

and denote by u_n^x its solution. Notice that $u_n^x \in L^2(\Omega; L^2(0, T; H^1(0, L)))$, for any $T > 0$, and if we set $\rho_n(t) := u_n^x(t) - u_n^y(t)$ we have that ρ_n solves the problem

$$\begin{cases} d\rho(t) = [\hat{A}\rho(t) + F(u_n^x(t)) - F(u_n^y(t))]\,dt + [G(u_n^x(t)) - G(u_n^y(t))]\,dw^n(t), \\ \rho(0) = x - y, \end{cases}$$

where $\hat{A}$ is the realization of the operator $\mathcal{A}$ endowed either with homogeneous Neumann or with Dirichlet boundary conditions.

Now, proceeding as in [14, Theorem 2.1], we can show that if $x \leq y$ and if $f(\xi, \cdot)$ is Lipschitz-continuous, uniformly with respect to $\xi \in [0, L]$, with Lipschitz constant M, then

$$\mathbb{E}|\rho_n^+(t)|^2_{L^2(0,L)} = \mathbb{E}\int_0^L \left(\rho_n^+(t,\xi)\right)^2 d\xi \leq (2M + cnL^2)\int_0^t \mathbb{E}\left|\rho_n^+(s)\right|^2_{L^2(0,L)} ds,$$

where L is the Lipschitz constant of $g(\xi, \cdot)$ and c is some positive constant. This implies that

$$\mathbb{E}\,|\rho_n^+(t)|^2_{L^2(0,L)} = 0, \quad t \geq 0,$$

and hence, as $\rho_n : [0, \infty) \times [0, L] \to \mathbb{R}$ is continuous, $\mathbb{P}$-a.s., it is immediate to conclude that

$$\mathbb{P}\left(u_n^x(t) \leq u_n^y(t), \ \ t \geq 0\right) = \mathbb{P}\left(\rho_n^+(t) = 0, \ \ t \geq 0\right) = 1. \tag{3.3}$$

Step 2. We show that for any $T>0$, $p\geq 1$ and $x\in C[0,L]$

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{(t,\xi)\in[0,T]\times[0,L]}|u_n^x(t,\xi)-u^x(t,\xi)|^p\right)=0. \tag{3.4}$$

According to (3.3) this in particular implies that (3.1) holds when $f(\xi,\cdot)$ is Lipschitz continuous.

We have

$$\begin{aligned}u^x(t)-u_n^x(t) &= \int_0^t e^{(t-s)\hat{A}}\left[F(u^x(s))-F(u_n^x(s))\right]ds\\ &\quad+\int_0^t e^{(t-s)\hat{A}}\left[G(u^x(s))-G(u_n^x(s))\right]dw^n(s)\\ &\quad+\int_0^t e^{(t-s)\hat{A}}G(u^x(s))(I-P_n)\,dw(s)\\ &=: I_1^n(t)+I_2^n(t)+I_3^n(t).\end{aligned}$$

We estimate each term. For any $t\in[0,T]$ we have

$$\begin{aligned}|I_1^n(t)|^p_{C[0,L]} &\leq \sup_{s\in[0,t]}\|e^{s\hat{A}}\|^p_{\mathcal{L}(C[0,L])}\left(\int_0^t\left|F(u^x(s))-F(u_n^x(s))\right|_{C[0,L]}ds\right)^p\\ &\leq \sup_{s\in[0,T]}\|e^{s\hat{A}}\|^p_{\mathcal{L}(C[0,L])}M^pT^{p-1}\int_0^t\left|u^x(s)-u_n^x(s)\right|^p_{C[0,L]}ds.\end{aligned} \tag{3.5}$$

Next, due to [6, Theorem 4.2] there exists $p_\star\geq 1$ such that for any $p\geq p_\star$

$$\mathbb{E}\sup_{s\in[0,t]}|I_2^n(t)|^p_{C[0,L]}\leq c_p(T)\int_0^t\mathbb{E}\sup_{r\in[0,s]}|u^x(r)-u_n^x(r)|^p_{C[0,L]}ds,$$

for some continuous increasing function $c_p(t)$ independent of n and vanishing at $t=0$. Finally, for any $(t,\xi)\in[0,\infty)\times[0,L]$ we have

$$I_3^n(t,\xi)=\sum_{k=n+1}^{\infty}\int_0^t e^{(t-s)\hat{A}}\left[G(u^x(s))e_k\right](\xi)\,d\beta_k(s)$$

and proceeding as in [6, proof of Theorem 4.2] we obtain

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T]}|I_3^n(t)|^p_{C[0,L]}=0. \tag{3.6}$$

Now, collecting all terms for any $t\in[0,T]$ we get

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}|u^x(s)-u_n^x(s)|^p_{C[0,L]} &\leq c'_p(T)\int_0^t\mathbb{E}\sup_{r\in[0,s]}|u^x(r)-u_n^x(r)|^p_{C[0,L]}ds\\ &\quad+\mathbb{E}\sup_{t\in[0,T]}|I_3^n(t)|^p_{C[0,L]},\end{aligned}$$

so that, according to the Gronwall lemma and to (3.6) we can conclude that

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}|u^x(t)-u^x_n(t)|^p_{C[0,L]}\\&\quad\leq \exp(c'_p(T)\,T)\,\mathbb{E}\sup_{t\in[0,T]}|I^n_3(t)|^p_{C[0,L]}\to 0,\quad n\to\infty.\end{aligned}$$

This means that there exits $\{n_k\}\uparrow+\infty$ such that

$$\mathbb{P}\left(\lim_{k\to\infty}\sup_{t\in[0,T]}|u^x(t)-u^x_{n_k}(t)|_{C[0,L]}=\lim_{k\to\infty}\sup_{t\in[0,T]}|u^y(t)-u^y_{n_k}(t)|_{C[0,L]}=0\right)=1.$$

Thus, since due to what we have proved at Step 1 for any n_k

$$\mathbb{P}\left(u^x_{n_k}(t)\leq u^y_{n_k}(t),\quad t\geq 0\right)=1,$$

we can conclude that if $f(\xi,\cdot)$ is Lipschitz continuous then (3.1) holds.

Step 3. Assume that $f(\xi,\cdot)$ is only locally Lipschitz continuous. If (3.1) does not hold, then there exists $\bar{T}>0$ such that

$$\mathbb{P}\left(\sup_{(t,\xi)\in[0,\bar{T}]\times[0,L]}u^x(t,\xi)-u^y(t,\xi)>0\right)>0.$$

Since u^x and u^y belong to $L^p(\Omega;C([0,T];C[0,L]))$, for any $T>0$, we have

$$\mathbb{P}\left(\bigcup_{R=1}^{\infty}\left\{\sup_{t\in[0,\bar{T}]}|u^x(t)|_{C[0,L]}\vee\sup_{t\in[0,\bar{T}]}|u^y(t)|_{C[0,L]}\leq R\right\}\right)=1.$$

Hence, there exists $\bar{R}>0$ such that

$$\mathbb{P}\left(\sup_{\substack{t\in[0,\bar{T}]\\ \xi\in[0,L]}}u^x(t,\xi)-u^y(t,\xi)>0,\quad\sup_{t\in[0,\bar{T}]}|u^x(t)|_{C[0,L]}\vee\sup_{t\in[0,\bar{T}]}|u^y(t)|_{C[0,L]}\leq\bar{R}\right)>0.\tag{3.7}$$

But this is not possible. Indeed, if

$$\sup_{t\in[0,\bar{T}]}|u^x(t)|_{C[0,L]}\vee\sup_{t\in[0,\bar{T}]}|u^y(t)|_{C[0,L]}\leq\bar{R},$$

then for any $t\in[0,\bar{T}]$ we have that $u^x(t)$ and $u^y(t)$ coincide with the solutions $u^x_{\bar{R}}(t)$ and $u^y_{\bar{R}}(t)$ (starting respectively from x and y) of the equation

$$du(t)=\left[Au(t)+F_{\bar{R}}(u(t))\right]dt+G(u(t))\,dw(t),$$

where

$$F_{\bar{R}}(x)(\xi)=f_{\bar{R}}(\xi,x(\xi)),\quad\xi\in[0,L],$$

and $f_{\bar{R}}(\xi, \cdot)$ is a mapping which is Lipschitz continuous, uniformly with respect to $\xi \in [0, L]$, and which coincides with $f(\xi, \cdot)$ on $[-\bar{R}, \bar{R}]$. Thus, according to what proved at Step 2

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,\bar{T}]} u^x(t) - u^y(t) > 0, \quad \sup_{t\in[0,\bar{T}]} |u^x(t)|_{C[0,L]} \vee \sup_{t\in[0,\bar{T}]} |u^y(t)|_{C[0,L]} \leq \bar{R}\right)\\
&\quad = \mathbb{P}\left(\sup_{t\in[0,\bar{T}]} u^x_{\bar{R}}(t) - u^y_{\bar{R}}(t) > 0, \quad \sup_{t\in[0,\bar{T}]} |u^x(t)|_{C[0,L]} \vee \sup_{t\in[0,\bar{T}]} |u^y(t)|_{C[0,L]} \leq \bar{R}\right)\\
&\quad \leq \mathbb{P}\left(\sup_{t\in[0,\bar{T}]} u^x_{\bar{R}}(t) - u^y_{\bar{R}}(t) > 0\right) = 0,
\end{aligned}$$

which contradicts (3.7). □

4. Proof of Theorem 2.1

In the proof we distinguish the case of Neumann and the case of Dirichlet bondary conditions.

4.1. The case of Neumann boundary conditions

For any $x, y \in C[0, L]$ we define $\rho(t) := u^x(t) - u^y(t)$. Clearly $\rho(t)$ is the unique mild solution to the equation

$$\begin{cases} d\rho(t) = \left[\hat{A}\rho(t) + F(u^x(t)) - F(u^y(t))\right] dt + \left[G(u^x(t)) - G(u^y(t))\right] dw(t),\\ \rho(0) = x - y, \end{cases} \tag{4.1}$$

where $\hat{A}$ is the realization in $C[0, L]$ of the differential operator $\mathcal{A}$ endowed with the homogeneous Neumann boundary conditions. As well known this is equivalent to the fact that

$$\begin{aligned}
&\int_0^L [\rho(t,\xi)\varphi(t,\xi) - \rho(0,\xi)\varphi(0,\xi)]\, d\xi\\
&\quad = \int_0^t \int_0^L \rho(s,\xi)\left(\frac{\partial\varphi}{\partial t} + \mathcal{A}\varphi\right)(s,\xi)\, d\xi\, ds\\
&\qquad + \int_0^t \int_0^L \left[f(\xi, u^x(s,\xi)) - f(\xi, u^y(s,\xi))\right]\varphi(s,\xi)\, d\xi\, ds\\
&\qquad + \int_0^t \int_0^L \left[g(\xi, u^x(s,\xi)) - g(\xi, u^y(s,\xi))\right]\varphi(s,\xi)\, w(ds, d\xi),
\end{aligned} \tag{4.2}$$

for any $\varphi \in C^{1,2}([0,\infty) \times [0, L])$ such that

$$\frac{\partial\varphi}{\partial\xi}(t, 0) = \frac{\partial\varphi}{\partial\xi}(t, L) = 0, \quad t \geq 0,$$

(for a proof see [9] and [22, Chapter 3]). Thus, if we take $\varphi \equiv 1$ in the formula above, we can conclude that

$$\begin{aligned}
\int_0^L \rho(t,\xi)\,d\xi &= \int_0^L (x-y)(\xi)\,d\xi \\
&\quad + \int_0^t \int_0^L \left[f(\xi, u^x(s,\xi)) - f(\xi, u^y(s,\xi))\right] d\xi\,ds \\
&\quad + \int_0^t \int_0^L \left[g(\xi, u^x(s,\xi)) - g(\xi, u^y(s,\xi))\right] w(ds, d\xi) \\
&= \int_0^L (x-y)(\xi)\,d\xi + \int_0^t \int_0^L \left[f_\lambda(\xi, u^x(s,\xi)) - f_\lambda(\xi, u^y(s,\xi))\right] d\xi\,ds \\
&\quad + \lambda \int_0^t \int_0^L \rho(s,\xi)\,d\xi\,ds \\
&\quad + \int_0^t \int_0^L \left[g(\xi, u^x(s,\xi)) - g(\xi, u^y(s,\xi))\right] w(ds, d\xi),
\end{aligned} \tag{4.3}$$

where f_λ is the function defined in (2.4) by

$$f_\lambda(\xi,\sigma) = f(\xi,\sigma) - \lambda\sigma, \qquad (\xi,\sigma) \in [0,L] \times \mathbb{R}.$$

In what follows for any $t \geq 0$ we shall set

$$\begin{aligned}
Z(t) &:= \int_0^L \rho(t,\xi)\,d\xi, \\
D(t) &:= \int_0^L \left[f_\lambda(\xi, u^x(t,\xi)) - f_\lambda(\xi, u^y(t,\xi))\right] d\xi, \\
M(t) &:= \int_0^t \int_0^L \left[g(\xi, u^x(s,\xi)) - g(\xi, u^y(s,\xi))\right] w(ds, d\xi).
\end{aligned} \tag{4.4}$$

Then (4.3) can be written more concisely as

$$Z(t) = \langle x - y \rangle + \lambda \int_0^t Z(s)\,ds + \int_0^t D(s)\,ds + M(t), \quad t \geq 0, \tag{4.5}$$

with the usual notation

$$\langle x - y \rangle = \int_0^L (x-y)(\xi)\,d\xi.$$

Lemma 4.1. *For any $t \geq 0$, let us define*

$$\mathcal{G}_t := \sigma\left(\int_0^\infty \int_0^L v(s,\xi) w(ds,d\xi),\ v \in L^2([0,\infty) \times [0,L]),\right.$$
$$\left. \operatorname{supp} v \subseteq [0,t] \times [0,L]\right).$$

Then, under Hypotheses 1 and 2, for any $x, y \in C[0, L]$ such that $x \geq y$ the process $M(t)$ defined in (4.4) *is a $\{\mathcal{G}_t\}_{t\geq 0}$ martingale. Moreover, there exists an adapted process $U(t)$ such that*

$$\frac{d\,\langle M\rangle_t}{dt} = Z^2(t)U(t), \quad t \geq 0, \tag{4.6}$$

and

$$\mathbb{P}\left(\frac{1}{L}\inf_{R>0}\mu_R^2 \leq U(t) < \infty,\ \ t \geq 0\right) = 1. \tag{4.7}$$

Proof. The process $M(t)$ is clearly a $\{\mathcal{G}_t\}_{t\geq 0}$ martingale and

$$\langle M\rangle_t = \int_0^t \int_0^L \left[g(\xi, u^x(s,\xi)) - g(\xi, u^y(s,\xi))\right]^2 d\xi ds.$$

Hence, according to (2.6) if

$$|u^x(t)|_{C[0,L]} \vee |u^y(t)|_{C[0,L]} \leq R,$$

we have

$$\begin{aligned}\frac{d\,\langle M\rangle_t}{dt} &= \int_0^L \left[g(\xi, u^x(t,\xi)) - g(\xi, u^y(t,\xi))\right]^2 d\xi\\ &\geq \mu_R^2 \int_0^L \left[u^x(t,\xi) - u^y(t,\xi)\right]^2 d\xi.\end{aligned}$$

Now, due to (2.8) for any $x, y \in C([0, L])$ we have

$$\mathbb{P}\left(\sup_{t\geq 0}|u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0}|u^y(t)|_{C[0,L]} < \infty\right) = 1,$$

then $\mathbb{P}$-a.s. for any $t \geq 0$ we have

$$\begin{aligned}\frac{d\,\langle M\rangle_t}{dt} &\geq \inf_{R>0}\mu_R^2 \int_0^L \rho^2(t,\xi)\,d\xi\\ &\geq \frac{1}{L}\inf_{R>0}\mu_R^2\left(\int_0^L \rho(t,\xi)\,d\xi\right)^2 = \frac{1}{L}\inf_{R>0}\mu_R^2\, Z^2(t).\end{aligned}$$

On the other side, due to (2.5) we have

$$\frac{d\,\langle M\rangle_t}{dt} \leq L^2 \int_0^L \rho^2(t,\xi)\,d\xi = L^2\,|\rho(t)|^2_{L^2(0,L)}. \tag{4.8}$$

In particular, since from Theorem 3.1 we have that $\rho(t) \geq 0$, for any $t \geq 0$, $\mathbb{P}$-a.s., we obtain

$$Z(t) = \int_0^L \rho(t,\xi)\,d\xi = |\rho(t)|_{L^1(0,L)} = 0 \Longrightarrow \frac{d\,\langle M\rangle_t}{dt} = 0.$$

Therefore, if we set

$$U(t) := \begin{cases} Z^{-2}(t)\,\dfrac{d\,\langle M\rangle_t}{dt} & \text{if } Z(t) \neq 0, \\ \dfrac{1}{L}\,\inf\limits_{R>0} \mu_R^2 & \text{if } Z(t) = 0, \end{cases}$$

we have that $U(t)$ is an adapted process which fulfills (4.6) and such that

$$\mathbb{P}\left(U(t) \geq \frac{1}{L}\,\inf_{R>0} \mu_R^2,\ t \geq 0\right) = 1.$$

Moreover, thanks to (4.8) we have

$$\{U(t) = +\infty,\ \text{for some } t \geq 0\} \subseteq \left\{\frac{d\,\langle M\rangle_t}{dt} = +\infty,\ \text{for some } t \geq 0\right\}$$

$$\subseteq \left\{|\rho(t)|_{L^2(0,L)} = +\infty,\ \text{for some } t \geq 0\right\}$$

and then, since $\rho \in L^p(\Omega; C([0,T]; C[0,L]))$, for any $T > 0$, we have

$$\mathbb{P}\,(U(t) = +\infty,\ \text{for some } t \geq 0) = 0.$$

□

Once we have constructed the process $U(t)$, $t \geq 0$, we define

$$V(t) := \int_0^t U(s)\,ds, \quad t \geq 0.$$

According to (4.7), if $\inf_{R>0} \mu_R > 0$ we have that V is a strictly increasing, continuous and adapted process with

$$V(0) = 0, \quad \lim_{t\to\infty} V(t) = +\infty, \quad \mathbb{P} - \text{a.s.}$$

Then the process

$$T(t) := \inf\,\{s \geq 0,\ V(s) > t\} = V^{-1}(t) \quad t \geq 0,$$

defines a random time change. In particular, if f is a bounded measurable function defined on $[a,b] \subseteq [0,\infty)$ we have

$$\int_a^b f(s)U(s)\,ds = \int_a^b f(s)dV(s) = \int_{V(a)}^{V(b)} f(T(t))\,dt. \tag{4.9}$$

Next, we define

$$X(t) := Z(T(t)), \quad t \geq 0.$$

Our aim is proving that $X(t)$ converges to zero $\mathbb{P}$-a.s., as t goes to infinity.

Lemma 4.2. *Assume that Hypotheses 1 and 2 hold and assume that*

$$\inf_{R>0} \mu_R > 0.$$

Moreover, fix any $x, y \in C[0, L]$ *such that* $x \geq y$. *Then there exists a Brownian motion* $B(t)$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

$$X(t) = \langle x - y\rangle + \lambda \int_0^t \frac{X(s)}{U(T(s))}\, ds + \int_0^t \frac{D(T(s))}{U(T(s))}\, ds + \int_0^t X(s)\, dB(s), \quad t \geq 0. \tag{4.10}$$

Proof. If we substitute t with $T(t)$ in (4.5) we have

$$X(t) = Z(T(t)) = \langle x - y\rangle + \lambda \int_0^{T(t)} Z(s)\, ds + \int_0^{T(t)} D(s)\, ds + M(T(t)).$$

If we set $s = T(r)$, we have $r = T^{-1}(s) = V(s)$ and hence $dr = dV(s) = U(s)\, ds = U(T(r))\, ds$. This implies that

$$\int_0^{T(t)} Z(s)\, ds = \int_0^t \frac{Z(T(r))}{U(T(r))}\, dr = \int_0^t \frac{X(r)}{U(T(r))}\, dr,$$

and analogously

$$\int_0^{T(t)} D(s)\, ds = \int_0^t \frac{D(T(s))}{U(T(s))}\, ds.$$

Thus, in order to conclude the proof it remains to show that there exists some Brownian motion $B(t)$ such that

$$M(T(t)) = \int_0^t X(s)\, dB(s).$$

Thanks to Lemma 4.1 we have

$$\langle M\rangle_t = \int_0^t Z^2(s) U(s)\, ds, \quad t \geq 0,$$

and then, according to (4.9), we get

$$\langle M\rangle_{T(t)} = \int_0^{T(t)} Z^2(s) U(s)\, ds = \int_0^{V(T(t))} Z^2(T(s))\, ds = \int_0^t Z^2(T(s))\, ds, \quad t \geq 0. \tag{4.11}$$

Now, if we define $N(t) := M(T(t))$, for any $t \geq 0$, as proved for example in [20, Proposition V.1.5] we have that $N(t)$ is a $\{\mathcal{G}_{T(t)}\}_{t\geq 0}$ martingale and

$$\langle N\rangle_t = \langle M\rangle_{T(t)}, \quad t \geq 0.$$

Hence, due to (4.11) we get that $d\langle N\rangle_t$ is a.s. equivalent to the Lebesgue measure, with non-negative density $Z^2(T(t))$, that is

$$d\langle N\rangle_t = Z^2(T(t))\,dt, \quad t \geq 0.$$

In particular, there exists a Brownian motion $B(t)$ such that

$$M(T(t)) = N(t) = \int_0^t Z(T(s))\,dB(s) = \int_0^t X(s)\,dB(s), \quad t \geq 0,$$

(for a proof see for example [20, Proposition V.3.8 and following remarks]). This clearly concludes the proof of the present Lemma. □

Proof of Theorem 2.1. Step 1. Assume first that there exists $\bar{R} > 0$ such that

$$\mu_{\bar{R}} = \inf_{R>0} \mu_R > 0.$$

As shown in Theorem 3.1, if $x \geq y$ we have that $u^x(t) \geq u^y(t)$, for any $t \geq 0$, $\mathbb{P}$-a.s. Then, recalling that the mapping $f_\lambda(\xi, \cdot)$ introduced in (2.4) is non-increasing, we have that the process $D(t)$ defined in (4.4) is non-positive, $\mathbb{P}$-a.s., so that

$$\int_0^t \frac{D(T(s))}{U(T(s))}\,ds \leq 0, \quad t \geq 0, \quad \mathbb{P}-\text{a.s.}.$$

Moreover, according to (4.7) we have that

$$\frac{1}{U(t)} \leq \frac{L}{\mu_{\bar{R}}^2}, \quad t \geq 0, \quad \mathbb{P}-\text{a.s.}$$

so that from (4.10) we get

$$X(t) \leq \langle x - y\rangle + \frac{\lambda L}{\mu_{\bar{R}}^2}\int_0^t X(s)\,ds + \int_0^t X(s)\,dB(s).$$

By comparison this yields

$$Z(T(t)) = X(t) \leq \langle x - y\rangle \exp\left(\left(\frac{\lambda L}{\mu_{\bar{R}}^2} - \frac{1}{2}\right)t\right)\exp(B(t)), \quad t \geq 0, \quad \mathbb{P}-\text{a.s.}$$

and this implies that if $\lambda L/\mu_{\bar{R}}^2 < 1/2$ and $x \geq y$

$$\lim_{t\to+\infty}\int_0^L \left(u^x(t,\xi) - u^y(t,\xi)\right)\,d\xi = \lim_{t\to+\infty} Z(t)$$
$$= \lim_{t\to+\infty} Z(T(t)) = 0, \quad \mathbb{P}-\text{a.s.}$$

Thus, since for any $x, y \in C[0, L]$ we have

$$|u^x(t) - u^y(t)|_{L^1(0,L)} \leq |u^x(t) - u^{x\wedge y}(t)|_{L^1(0,L)} + |u^{x\wedge y}(t) - u^y(t)|_{L^1(0,L)},$$

we can conclude the proof of Theorem 2.1 in the case that the infimum of μ_R is attained at some $\bar{R}$.

Step 2. Assume that there not exists $\min_{R>0} \mu_R$ and that the constant m in (2.2) is strictly greater than 1. If

$$\mathbb{P}\left(\limsup_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} > 0 \right) > 0,$$

due to (2.8) we have that there exists $\bar{R} > 0$ such that

$$\mathbb{P}\left(\limsup_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} > 0,\ \sup_{t\geq 0} |u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0} |u^y(t)|_{C[0,L]} \leq \bar{R} \right) > 0. \tag{4.12}$$

We show that this leads us to a contradiction. Without any loss of generality we can assume that the mapping $\sigma \mapsto g(\xi, \sigma)$ is increasing (the case of $g(\xi, \cdot)$ decreasing can be treated analogously) and that $\mu_{R_1} \leq \mu_{R_2}$ if $R_1 > R_2$. If we define

$$g_{\bar{R}}(\xi, \sigma) := \begin{cases} g(\xi, \sigma) & \text{if } \sigma \in [-\bar{R}, \bar{R}], \\ \mu_{\bar{R}}(\sigma - \bar{R}) + g(\xi, \bar{R}) & \text{if } \sigma > \bar{R}, \\ \mu_{\bar{R}}(\sigma + \bar{R}) + g(\xi, -\bar{R}) & \text{if } \sigma < -\bar{R}, \end{cases}$$

it is immediate to check that $g_{\bar{R}}$ fulfills Hypothesis 2 and

$$\inf_{\xi\in[0,L]} |g_{\bar{R}}(\xi, \sigma) - g_{\bar{R}}(\xi, \rho)| \geq \mu_{\bar{R}}\, |\sigma - \rho|, \quad \sigma, \rho \in \mathbb{R}.$$

According to what we have just proved, if we denote by $u^x_{\bar{R}}$ and $u^y_{\bar{R}}$ the solutions of equation (2.1) corresponding to the diffusion coefficient $g_{\bar{R}}$, with initial data x and y respectively, we have that

$$\mathbb{P}\left(\lim_{t\to+\infty} |u^x_{\bar{R}}(t) - u^y_{\bar{R}}(t)|_{L^1(0,L)} = 0 \right) = 1. \tag{4.13}$$

On the other side, if

$$\sup_{t\geq 0} |u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0} |u^y(t)|_{C[0,L]} \leq \bar{R},$$

we have

$$u^x(t) = u^x_{\bar{R}}(t), \quad u^y(t) = u^y_{\bar{R}}(t), \quad t \geq 0,$$

and then due to (4.13) we get

$$\begin{aligned} &\mathbb{P}\left(\limsup_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} > 0,\ \sup_{t\geq 0} |u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0} |u^y(t)|_{C[0,L]} \leq \bar{R} \right) \\ &= \mathbb{P}\left(\limsup_{t\to+\infty} |u^x_{\bar{R}}(t) - u^y_{\bar{R}}(t)|_{L^1(0,L)} > 0,\ \sup_{t\geq 0} |u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0} |u^y(t)|_{C[0,L]} \leq \bar{R} \right) \\ &\leq \mathbb{P}\left(\limsup_{t\to+\infty} |u^x_{\bar{R}}(t) - u^y_{\bar{R}}(t)|_{L^1(0,L)} > 0 \right) = 0. \end{aligned}$$

But this contradicts (4.12).

Step 3. Assume that there not exists $\min_{R>0}\mu_R$, the constant m in (2.2) is 1 and $\inf_{R>0}\mu_R > 0$. Proceeding as in Step 1, due to (4.7) we have

$$Z(T(t)) \leq \langle x-y\rangle \exp\left(\left(\frac{\lambda L}{\inf_{R>0}\mu_R^2}-\frac{1}{2}\right)t\right)\exp(B(t)),\quad t\geq 0,\quad \mathbb{P}-\text{a.s.}$$

and then we can conclude as in Step 1, if $\lambda < \inf_{R>0}\mu_R^2/2L$. □

4.2. The case of Dirichlet boundary conditions

In the previous subsection we have proved Theorem 2.1 in the case of inhomogeneous Neumann boundary conditions. Here we show that Theorem 2.1 is still valid in the case of Dirichlet boundary conditions, if the constant m introduced in (2.2) is strictly greater than 1.

If, as in subsection 4.1, we set $\rho(t) := u^x(t) - u^y(t)$, $t \geq 0$, we have that ρ satisfies equation (4.1), where in this case $\hat{A}$ is the realization of the operator $\mathcal{A}$ in $C[0, L]$, endowed with Dirichlet boundary conditions. Thus, for any $\varphi \in C^{1,2}([0,+\infty)\times[0, L])$ such that $\varphi(t, 0) = \varphi(t, L) = 0$, we have that (4.2) holds.

Now, we need the following preliminary result.

Lemma 4.3. *For any* $0 < \delta < L/2$ *there exists a non-negative function* $\psi_\delta \in C^2[0, L]$ *such that*

$$\mathcal{A}\psi_\delta \leq 0,\quad \psi_\delta(0) = \psi_\delta(L) = 0,\quad \psi_\delta \equiv 1,\quad \textit{on } [\delta, L-\delta]. \tag{4.14}$$

Proof. Notice that it is sufficient to show that for each $0 < \delta < L/2$ there exists $\psi_{\delta,1} \in C^2[0, L/2]$ such that $\mathcal{A}\psi_{\delta,1} \leq 0$, $\psi_{\delta,1}(0) = 0$ and $\psi_{\delta,1} \equiv 1$ on $[\delta, L/2]$. Actually, by repeating the same arguments we find another function $\psi_{\delta,2} \in C^2([L/2, L])$ vanishing at L and fulfilling $\mathcal{A}\psi_{\delta,2} \leq 0$ and $\psi_{\delta,2} \equiv 1$ on $[L/2, L-\delta]$ and then the function ψ_δ is obtained by gluing together $\psi_{\delta,1}$ and $\psi_{\delta,2}$.

If we define

$$f_\delta(\xi) := \frac{6}{\delta^3}\xi\,(\xi-\delta)\left(\int_0^\xi \frac{1}{a(\eta)}\,d\eta\right)^{-1},\qquad \xi \in (0,\delta]$$

and set $f_\delta(0) = -6\delta^2/a(0)$ and $f_\delta(\xi) \equiv 0$ on $[\delta, L/2]$, it is immediate to check that f_δ is a continuous non-positive mapping on $[0, L/2]$ and

$$\int_0^\delta f_\delta(\xi)\int_0^\xi \frac{1}{a(\eta)}\,d\eta\,d\xi = -1. \tag{4.15}$$

Next, we define

$$\begin{aligned}\psi_\delta(\xi) := &\left(1-\int_0^\delta \frac{g(\eta)}{a(\eta)}\,d\eta\right)\left(\int_0^\delta \frac{1}{a(\eta)}\,d\eta\right)^{-1}\int_0^\xi \frac{1}{a(\eta)}\,d\eta\\ &+\int_0^\xi \frac{g(\eta)}{a(\eta)}\,d\eta,\quad \xi\in[0,\delta],\end{aligned}$$

where

$$g(\xi) := \int_0^\xi f_\delta(\eta)\,d\eta.$$

Note that $\psi_\delta(0) = 0$ and since $\psi_\delta(\delta) = 1$ we can extend ψ_δ as a continuous mapping on $[0, L/2]$ by setting $\psi_\delta(\xi) \equiv 1$ on $[\delta, L/2]$. Moreover $\psi_\delta \in C^2([0,\delta))$ and $\mathcal{A}\psi_\delta = f_\delta$ on $[0,\delta)$. This means that it remains to show that

$$\lim_{\xi\to\delta^-} \psi_\delta'(\xi) = \lim_{\xi\to\delta^-} \psi_\delta''(\xi) = 0.$$

We have

$$\psi_\delta'(\xi) = \left(1 - \int_0^\delta \frac{g(\eta)}{a(\eta)}\,d\eta\right)\left(\int_0^\delta \frac{1}{a(\eta)}\,d\eta\right)^{-1} \frac{1}{a(\xi)} + \frac{g(\xi)}{a(\xi)},$$

and then

$$\lim_{\xi\to\delta^-} \psi_\delta'(\xi) = \frac{1}{a(\delta)}\left[\left(1 - \int_0^\delta \frac{g(\eta)}{a(\eta)}\,d\eta\right)\left(\int_0^\delta \frac{1}{a(\eta)}\,d\eta\right)^{-1} + g(\delta)\right].$$

Therefore, $\lim_{\xi\to\delta^-} \psi_\delta'(\xi) = 0$ if and only if

$$\int_0^\delta f_\delta(\eta)\,d\eta = g(\delta) = -\left(1 - \int_0^\delta \frac{g(\eta)}{a(\eta)}\,d\eta\right)\left(\int_0^\delta \frac{1}{a(\eta)}\,d\eta\right)^{-1},$$

that is if and only if

$$g(\delta)\int_0^\delta \frac{1}{a(\eta)}\,d\eta = \int_0^\delta \frac{g(\eta)}{a(\eta)}\,d\eta - 1.$$

Since

$$\int_0^\delta \frac{g(\eta)}{a(\eta)}\,d\eta = g(\delta)\int_0^\delta \frac{1}{a(\eta)}\,d\eta - \int_0^\delta f_\delta(\eta)\int_0^\eta \frac{1}{a(\rho)}\,d\rho\,d\eta,$$

it follows that $\lim_{\xi\to\delta^-} \psi_\delta'(\xi) = 0$ if and only if

$$\int_0^\delta f_\delta(\eta)\int_0^\eta \frac{1}{a(\rho)}\,d\rho\,d\eta = -1,$$

which is (4.15).

Concerning the second derivative, we have

$$\psi_\delta''(\xi) = -\frac{a'(\xi)}{a(\xi)}\,\psi_\delta'(\xi) + \frac{g'(\xi)}{a(\xi)}.$$

Thus, as $\lim_{\xi\to\delta^-} \psi_\delta'(\xi) = 0$, we have $\lim_{\xi\to\delta^-} \psi_\delta''(\xi) = 0$ if and only if

$$\frac{g'(\delta)}{a(\delta)} = \frac{f_\delta(\delta)}{a(\delta)} = 0,$$

which is clearly satisfied, as $f_\delta(\delta) = 0$. □

Thus, if for any $n \in \mathbb{N}$ we set

$$\varphi_n(t,\xi) := \psi_{1/n}(\xi), \quad (t,\xi) \in [0,+\infty) \times [0,L],$$

from (4.2) we obtain

$$\begin{aligned}\int_0^L \rho(t,\xi)\varphi_n(\xi)\,d\xi &= \int_0^L (x-y)(\xi)\varphi_n(\xi)\,d\xi + \int_0^t\int_0^L \rho(s,\xi)\mathcal{A}\varphi_n(\xi)\,d\xi\,ds \\ &+\lambda \int_0^t\int_0^L \rho(s,\xi)\varphi_n(\xi)\,d\xi ds \\ &+ \int_0^t\int_0^L \left[f_\lambda(\xi,u^x(s,\xi)) - f_\lambda(\xi,u^y(s,\xi))\right]\varphi_n(\xi)\,d\xi\,ds \\ &+\int_0^t\int_0^L \left[g(\xi,u^x(s,\xi)) - g(\xi,u^y(s,\xi))\right]\varphi_n(\xi)\,w(ds,d\xi).\end{aligned}$$

If we assume $x \geq y$, thanks to Theorem 3.1 we have that $\rho(t,\xi) \geq 0$, $\mathbb{P}$-a.s., and hence, recalling that $\mathcal{A}\varphi_n \leq 0$, we have

$$\begin{aligned}\int_0^L \rho(t,\xi)\varphi_n(\xi)\,d\xi &\leq \int_0^L (x-y)(\xi)\varphi_n(\xi)\,d\xi + \lambda\int_0^t\int_0^L \rho(s,\xi)\varphi_n(\xi)\,d\xi ds \\ &+ \int_0^t\int_0^L \left[f_\lambda(\xi,u^x(s,\xi)) - f_\lambda(\xi,u^y(s,\xi))\right]\varphi_n(\xi)\,d\xi\,ds \\ &+\int_0^t\int_0^L \left[g(\xi,u^x(s,\xi)) - g(\xi,u^y(s,\xi))\right]\varphi_n(\xi)w(ds,d\xi).\end{aligned}$$

Therefore, if we define as in (4.3)

$$\begin{aligned}Z_n(t) &:= \int_0^L \rho(t,\xi)\varphi_n(\xi)\,d\xi \\ D_n(t) &:= \int_0^L \left[f_\lambda(\xi,u^x(t,\xi)) - f_\lambda(\xi,u^y(t,\xi))\right]\varphi_n(\xi)\,d\xi \\ M_n(t) &:= \int_0^t\int_0^L \left[g(\xi,u^x(s,\xi)) - g(\xi,u^y(s,\xi))\right]\varphi_n(\xi)\,w(ds,d\xi),\end{aligned}$$

we have

$$Z_n(t) \leq \langle (x-y)\varphi_n\rangle + \lambda\int_0^t Z_n(s)\,ds + \int_0^t D_n(s)\,ds + M_n(t), \quad t \geq 0.$$

Now, it is immediate to check that all arguments used in the proof of Lemma 4.1 adapts to the martingale $M_n(t)$, so that we can conclude that there exists an adapted process $U_n(t)$ such that

$$\frac{d\,\langle M_n\rangle}{dt} = Z_n^2(t)U_n(t), \quad t \geq 0,$$

and

$$\mathbb{P}\left(\frac{1}{L}\inf_{R>0}\mu_R^2 \leq U_n(t) < \infty,\ t \geq 0\right) = 1.$$

Hence, proceeding as in the case of Neumann boundary conditions studied in the previous subsection, we can conclude that for any $x, y \in C[0, L]$ and for any $n \in \mathbb{N}$ we have

$$\lim_{t\to+\infty} |(u^x(t) - u^y(t))\varphi_n|_{L^1(0,L)} = 0, \quad \mathbb{P}-\text{a.s.}, \tag{4.16}$$

where φ_n is a non-negative function in $C^2[0, L]$ fulfilling (4.14).

Next, assume that for $x \geq y$

$$\mathbb{P}\left(\limsup_{t\to+\infty}\left|u^x(t) - u^y(t)\right|_{L^1(0,L)} > 0\right) > 0.$$

As we are assuming $m > 1$, we have that (2.8) holds and then for any $x \in C[0, L]$

$$\mathbb{P}\left(\sup_{t\geq 0}|u^x(t)|_{C[0,L]} < \infty\right) = 1.$$

This implies that there exists some $\bar{R} > 0$ such that

$$\mathbb{P}\left(\limsup_{t\to+\infty}|u^x(t) - u^y(t)|_{L^1(0,L)} > 0,\ \sup_{t\geq 0}|u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0}|u^y(t)|_{C[0,L]} \leq \bar{R}\right) > 0. \tag{4.17}$$

If

$$\sup_{t\geq 0}|u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0}|u^y(t)|_{C[0,L]} \leq \bar{R},$$

for any $t \geq 0$ we have

$$|(u^x(t) - u^y(t))(1-\varphi_n)|_{L^1(0,L)} \leq 2\bar{R}\,|1-\varphi_n|_{L^1(0,L)} \to 0, \quad \text{as } n \to \infty.$$

Then for $\epsilon > 0$ fixed we can find $n_\epsilon \in \mathbb{N}$ such that

$$\sup_{t\geq 0}|u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0}|u^y(t)|_{C[0,L]} \leq \bar{R}$$
$$\Longrightarrow \sup_{t\geq 0}|(u^x(t) - u^y(t))(1-\varphi_{n_\epsilon})|_{L^1(0,L)} \leq \epsilon.$$

This allows to conclude that $u^x(t) - u^y(t)$ converges to zero in $L^1(0, L)$ norm. Actually, if

$$\sup_{t\geq 0}|u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0}|u^y(t)|_{C[0,L]} \leq \bar{R},$$

we have for any $t \geq 0$

$$|u^x(t) - u^y(t)|_{L^1(0,L)} \leq \epsilon + |(u^x(t) - u^y(t))\varphi_{n_\epsilon}|_{L^1(0,L)}$$

and hence, thanks to (4.16), due to the arbitrariness of $\epsilon > 0$ we have

$$\lim_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} = 0.$$

But this contradicts (4.17) as

$$\mathbb{P}\left(\limsup_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} > 0, \ \sup_{t\geq 0} |u^x(t)|_{C[0,L]} \vee \sup_{t\geq 0} |u^y(t)|_{C[0,L]} \leq \bar{R}\right)$$
$$\leq \mathbb{P}\left(\limsup_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} > 0, \ \lim_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} = 0\right) = 0.$$

We have thus proved that Theorem 2.1 holds also in the case of Dirichlet boundary conditions, if $m > 1$.

5. Conclusion

Let P_t be the transition semigroup associated to equation (2.1) and defined as in (2.9) by

$$P_t\varphi(x) := \mathbb{E}\varphi(u^x(t)), \qquad t \geq 0,$$

for any $x \in C[0, L]$ and any $\varphi \in C_b(C[0, L])$, the Banach space of uniformly continuous and bounded functions defined on $C[0, L]$ with values in $\mathbb{R}$. Our aim here is to show how from Theorem 2.1 it is possible to deduce that for any $x, y \in C[0, L]$ and any $\varphi \in C_b(C[0, L])$

$$\lim_{t\to+\infty} P_t\varphi(x) - P_t\varphi(y) = 0. \tag{5.1}$$

Clearly, if $\varphi \in C_b(L^1(0, L))$ from Theorem 2.1 we obtain (5.1) immediately. Next lemma shows how to pass from $C_b(L^1(0, L))$ to the larger space $C_b(C[0, L])$.

Lemma 5.1. *For any $\varphi \in C_b(C[0, L])$ there exists a sequence $\{\varphi_n\} \subset C_b(L^1(0, L))$ such that*

$$\begin{cases} \lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x), \quad x \in C[0, L], \\ \|\varphi_n\|_{C_b(L^1(0,L))} \leq \|\varphi\|_{C_b(C[0,L])}. \end{cases}$$

Proof. If $x \in L^1(0, L)$, we can extend it continuously as a $L^1(0, \infty)$-function in such a way that if $x \in C[0, L]$ then its extension is in $C[0, +\infty)$. Let Px denote this extension. For any $n \in \mathbb{N}$ and $x \in L^1(0, L)$ we define

$$x_n(\xi) := n \int_\xi^{\xi+\frac{1}{n}} Px(\eta)\, d\eta, \qquad \xi \in [0, L].$$

Clearly for any $n \in \mathbb{N}$ we have that $x_n \in C[0, L]$ and if $x \in C[0, L]$ we have

$$\lim_{n\to\infty} x_n = x, \quad \text{in } C[0, L].$$

Now, for any $\varphi \in C_b(C[0,L])$ we set

$$\varphi_n(x) := \varphi(x_n), \qquad x \in L^1(0,L).$$

For any $x \in L^1(0,L)$ we have

$$|\varphi_n(x)| = |\varphi(x_n)| \leq \sup_{x \in C[0,L]} |\varphi(x)| = \|\varphi\|_{C_b(C[0,L])},$$

so that

$$\sup_{x \in L^1(0,L)} |\varphi_n(x)| \leq \sup_{x \in C[0,L]} |\varphi(x)|.$$

Moreover, for any $x, y \in L^1(0,L)$ we have

$$x_n(\xi) - y_n(\xi) = n \int_\xi^{\xi+\frac{1}{n}} (Px(\eta) - Py(\eta))\, d\eta$$

and then

$$|x_n(\xi) - y_n(\xi)| \leq n \int_\xi^{\xi+\frac{1}{n}} |Px(\eta) - Py(\eta)|\, d\eta \leq n\, |Px - Py|_{L^1(0,\infty)}. \tag{5.2}$$

This implies that φ_n is uniformly continuous on $L^1(0,L)$. Indeed, for any $\epsilon > 0$ we fix $\delta_\epsilon > 0$ such that

$$|z_1 - z_2|_{C[0,L]} \leq \delta_\epsilon \Longrightarrow |\varphi(z_1) - \varphi(z_2)| \leq \epsilon.$$

Now, as $P : L^1(0,L) \to L^1(0,\infty)$ is continuous there exists $\rho_\epsilon > 0$ such that

$$|x - y|_{L^1(0,L)} \leq \rho_\epsilon \Longrightarrow |Px - Py|_{L^1(0,\infty)} \leq \frac{\delta_\epsilon}{n}.$$

Then, due to (5.2) we have

$$|x - y|_{L^1(0,L)} \leq \rho_\epsilon \Longrightarrow |x_n - y_n|_{C[0,L]} \leq n|Px - Py|_{L^1(0,\infty))} \leq \delta_\epsilon,$$

and then

$$|\varphi_n(x) - \varphi_n(y)| = |\varphi(x_n) - \varphi(y_n)| \leq \epsilon.$$

Finally, if $x \in C[0,L]$ it is immediate to check that the sequence $\{x_n\}$ converges to x in $C[0,L]$ and then $\varphi_n(x)$ converges to $\varphi(x)$. □

From the previous lemma, if $\varphi \in C_b(C[0,L])$ and if $\{\varphi_n\} \subset C_b(L^1(0,L))$ is a sequence as in Lemma 5.1, by the dominated convergence theorem we have that

$$\lim_{n\to\infty} P_t\varphi_n(x) = P_t\varphi(x), \qquad t \geq 0,$$

for any fixed $x \in C[0,L]$. This clearly implies that (5.1) holds for any $\varphi \in C_b(C[0,L])$. Hence, by standard arguments we can conclude that the following theorem holds

Theorem 5.2. *Under the same conditions of Theorem 2.1, if equation* (2.1) *admits an invariant measure (supported on* $C[0,L]$*), such an invariant measure is unique, ergodic and strongly mixing.*

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