Probab. Theory Relat. Fields 133, 175–189 (2005)
Digital Object Identifier (DOI) 10.1007/s00440-004-0420-5

Dénes Petz · Júlia Réffy

Large deviation for the empirical eigenvalue density of truncated Haar unitary matrices

Received: 29 October 2002 / Revised version: 30 November 2004 /
Published online: 6 June 2005 –

Abstract. Let U_m be an $m \times m$ Haar unitary matrix and $U_{[m,n]}$ be its $n \times n$ truncation. In this paper the large deviation is proven for the empirical eigenvalue density of $U_{[m,n]}$ as $m/n \to \lambda$ and $n \to \infty$. The rate function and the limit distribution are given explicitly. $U_{[m,n]}$ is the random matrix model of quq, where u is a Haar unitary in a finite von Neumann algebra, q is a certain projection and they are free. The limit distribution coincides with the Brown measure of the operator quq.

1 Introduction

Although the asymptotics of the eigenvalue density of different random matrices has been widely studied since the pioneering work of Wigner [18], the first large deviation theorem for the empirical eigenvalue density of self-adjoint Gaussian random matrices was proven by Ben Arous and Guionnet much later [1]. After the publication of their work, several similar theorems were obtained for different kind of random matrices. In particular, Haar distributed unitaries were discussed by Hiai and Petz [11] and the monograph [10] contains more information about similar results (see also [13, 2]). Free probability theory has inspired non-commutative large deviation results for random matrices recently, see [7], for example.

The aim of this article is to prove the large deviation theorem for the empirical eigenvalue density of truncated Haar unitary random matrices, and to determine the limit measure. Let U be an $m \times m$ Haar distributed unitary matrix. By truncating $m - n$ bottom rows and $m - n$ last columns, we get an $n \times n$ matrix. The truncated matrix is a contraction, hence the eigenvalues are in the unit disc. Our aim is to study the asymptotics of the empirical eigenvalue density when $n \to \infty$ and $m/n \to \lambda$. The truncated Haar unitaries appeared in the works [19, 5]. Since our random matrix model is unitarily invariant, the limiting eigenvalue density is rotation invariant in the complex plane. It turns out that the limiting density is supported on the disc of radius $1/\sqrt{\lambda}$. In this paper the large deviation result is

D. Petz, J. Réffy: Department for Mathematical Analysis, Budapest University of Technology and Economics 1521 Budapest XI., Hungary. e-mail: petz@math.bme.hu; reffyj@math.bme.hu

Mathematics Subject Classification (2000): 60F10 (15A52, 46L53, 60F05)

Key words or phrases: Random matrices – Joint eigenvalue distribution – Haar unitary – Truncated Haar unitary – Large deviation – Rate function – Free probability – Random matrix model

established and the exact form of the rate function is given. The large deviation implies the weak convergence of the empirical eigenvalue density of the truncated unitaries with probability one.

The paper is organized as follows. Section 2 contains some preliminaries about potential theory and large deviations. The large deviation result is stated in Section 3. Section 4 contains the proof of our main result. In Section 5 of the paper we make a connection to free probability theory. The truncated Haar unitaries form a random matrix model for the non-commutative random variable quq, where u is an appropriate unitary, q is a projection and they are assumed to be free. We observe that the limiting eigenvalue density coincides with the Brown measure of the operator quq. Our paper is based on the joint eigenvalue density of truncated unitaries. In the Appendix we sketch the derivation of this formula following the original paper [19].

2 Preliminaries

In this section we review the setting of large deviation for the empirical eigenvalue density of random matrices and collect some useful concepts and results from potential theory.

Assume that $T_n(\omega)$ is a random $n \times n$ matrix with complex eigenvalues $\zeta_1(\omega), \dots, \zeta_n(\omega)$. (If we want, we can fix an ordering of the eigenvalues, for example, regarding their absolute values and phases, but that is not necessary.) The *empirical eigenvalue density* of $T_n(\omega)$ is the random atomic measure

$$P_n(\omega) := \frac{\delta(\zeta_1(\omega)) + \cdots + \delta(\zeta_n(\omega))}{n},$$

where $\delta(z)$ denotes the Dirac measure supported on $\{z\} \subset \mathbb{C}$. Therefore P_n is a random measure, or a measure-valued random variable.

Let us recall the definition of the *large deviation principle* [6]. Let (P_n) be a sequence of measures on a topological space X. The large deviation principle holds with *rate function* $I : \mathcal{M}(X) \to \mathbb{R}^+ \cup \{+\infty\}$ in the scale n^{-2} if

$$\liminf_{n\to\infty} \frac{1}{n^2} \log P_n(G) \geq -\inf_{x\in G} I(x)$$

for all open set $G \subset X$, and

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P_n(F) \leq -\inf_{x\in F} I(x)$$

for all closed set $F \subset X$.

Let $U(m)$ be an $m \times m$ Haar distributed unitary matrix. By truncating $m-n$ bottom rows and $m-n$ last columns, we get a $n \times n$ matrix $U_{[m,n]}$. The truncated matrix $U_{[m,n]}$ is not a unitary but its operator norm is at most 1. Hence the eigenvalues $\zeta_1, \zeta_2, \dots, \zeta_n$ lie in the disc $\mathcal{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. The relevant topological space is $\mathcal{M}(\mathcal{D})$, the space of probability measures on $\mathcal{D}$. Note that this space is a compact metrizable space with respect to the weak convergence of measures. Let

$P_{[m,n]}$ be the empirical eigenvalue density of $U_{[m,n]}$. Hence $P_{[m,n]}$ may be regarded as a measure on $\mathcal{M}(\mathcal{D})$.

We are going to benefit from the fact that the joint probability density of the eigenvalues of $U_{[m,n]}$ is

$$\frac{1}{C_{[m,n]}} \prod_{1\le i<j\le n} |\zeta_i - \zeta_j|^2 \prod_{i=1}^{n} (1 - |\zeta_i|^2)^{m-n-1}$$

according to [19], see also the Appendix. The normalizing constant

$$C_{[m,n]} = \pi^n n! \prod_{j=0}^{n-1} \binom{m-n+j-1}{j}^{-1} \frac{1}{m-n+j} \tag{1}$$

was obtained in [14].

Next we recall some definitions and theorems of potential theory [15]. For a signed measure ν on $\mathcal{D}$

$$\Sigma(\nu) := \iint_{\mathcal{D}^2} \log|z-w|\, d\nu(z)\, d\nu(w)$$

is the negative logarithmic energy of ν. Since

$$\Sigma(\nu) = \inf_{\alpha<0} \iint_{\mathcal{D}^2} \max(\log|z-w|, \alpha)\, d\nu(z)\, d\nu(w),$$

this functional is upper semi-continuous. We want to show its concavity.

The following lemma is strongly related to the properties of the logarithmic kernel $K(z,w) = \log|z-w|$ (cf. Theorem 1.16 in [12]).

Lemma 2.1 *Let ν be a compactly supported signed measure on $\mathbb{C}$ such that $\nu(\mathbb{C}) = 0$. Then $\Sigma(\nu) \le 0$, and $\Sigma(\nu) = 0$ if and only if $\nu = 0$.*

From this lemma we can deduce strictly concavity of the functional Σ. First we prove that

$$\Sigma\left(\frac{\mu_1+\mu_2}{2}\right) \ge \frac{\Sigma(\mu_1)+\Sigma(\mu_2)}{2}, \tag{2}$$

for all $\mu_1, \mu_2 \in \mathcal{M}(\mathcal{D})$, moreover the equality holds if and only if $\mu_1 = \mu_2$. For this, apply Lemma 2.1 for the signed measure $\nu = \mu_1 - \mu_2$. The strict midpoint concavity (2) implies strict concavity by well-known arguments.

Let $K \subset \mathbb{C}$ be a compact subset of the complex plane, and $\mathcal{M}(K)$ be the collection of all probability measures with support in K. The *logarithmic energy* $E(\mu)$ of a $\mu \in \mathcal{M}(K)$ is defined as

$$E(\mu) := \iint_{K^2} \log\frac{1}{|z-w|}\, d\mu(z)\, d\mu(w),$$

and the *energy* V of K by

$$V := \inf\{E(\mu) : \mu \in \mathcal{M}(K)\}.$$

The quantity

$$\mathrm{cap}(K) := e^{-V}$$

is called the *logarithmic capacity* of K. The *logarithmic potential* of $\mu \in \mathcal{M}(K)$ is the function

$$U^{\mu} := \int_K \log \frac{1}{|z-w|}\, d\mu(w)$$

defined on K.

Let $K \subset \mathbb{C}$ be a closed set, and $Q : K \to (-\infty, \infty]$ be a lower semi-continuous function which is finite on a set of positive capacity. The integral

$$I_Q(\mu) := \iint_{K^2} \log \frac{1}{|z-w|}\, d\mu(z)\, d\mu(w) + 2\int_K Q(z)\, d\mu(z)$$

is called *weighted energy*.

The following result tells about the minimizer of the weighted potential (cf. Theorem I.3.3 in [15]).

Proposition 2.2 *Let Q as above. Assume that $\sigma \in \mathcal{M}(K)$ has compact support, $E(\sigma) < \infty$ and*

$$U^{\sigma}(z) + Q(z)$$

coincides with a constant F on the support of σ and is at least as large as F on K. Then σ is the unique measure in $\mathcal{M}(K)$ such that

$$I_Q(\sigma) = \inf_{\mu \in \mathcal{M}(K)} I_Q(\mu),$$

i.e., σ is the so-called equilibrium measure associated with Q.

The following lemma is the specialization of Proposition 2.2 to a radially symmetric function $Q : \mathcal{D} \to (-\infty, \infty]$, i. e., $Q(z) = Q(|z|)$. We assume that Q is differentiable on (0, 1) with absolute continuous derivative bounded below, moreover $rQ'(r)$ increasing on (0, 1) and

$$\lim_{r \to 1} rQ'(r) = \infty.$$

Let $r_0 \geq 0$ be the smallest number for which $Q'(r) > 0$ for all $r > r_0$, and we set R_0 be the smallest solution of $R_0 Q'(R_0) = 1$. Clearly $0 \leq r_0 < R_0 < 1$.

Lemma 2.3 *If the above conditions hold, them the functional I_Q attains its minimum at a unique measure μ_Q supported on the annulus*

$$S_Q = \{z : r_0 \leq |z| \leq R_0\},$$

and the density of μ_Q is given by

$$d\mu_Q(z) = \frac{1}{2\pi}(rQ'(r))'\, dr\, d\varphi, \qquad z = re^{\mathrm{i}\varphi}.$$

Proof The proof is similar to the one of Theorem IV.6.1 in [15]. Using the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{|z - re^{\mathrm{i}\varphi}|}\, d\varphi = \begin{cases} -\log r, & \text{if } |z| \le r \\ -\log |z|, & \text{if } |z| > r, \end{cases}$$

we get that

$$\begin{aligned} U^{\mu}(z) &= \frac{1}{2\pi}\int_{r_0}^{R_0} (rQ'(r))' \int_0^{2\pi} \log \frac{1}{|z - re^{\mathrm{i}\varphi}|}\, d\varphi\, dr \\ &= Q(R_0) - \log R_0 - Q(z), \end{aligned}$$

for $z \in S_Q$, since $r_0 = 0$ or $Q'(r_0) = 0$. We have

$$U^{\mu}(z) + Q(z) = Q(R_0) - \log R_0,$$

which is clearly a constant.

Next we check that $U^{\mu}(z) + Q(z) \ge Q(R_0) - \log R_0$ for $|z| < r_0$ and for $|z| > R_0$. So μ_Q satisfies conditions of Theorem 2.2 and it must be the unique minimizer. □

3 The large deviation theorem

Our large deviation theorem for truncated Haar unitaries is the following.

Theorem 3.1 *Let $U_{[m,n]}$ be the $n \times n$ truncation of an $m \times m$ Haar unitary random matrix and let $1 < \lambda < \infty$. If $m/n \to \lambda$ as $n \to \infty$, then the sequence of empirical eigenvalue densities $P_n = P_{[m,n]}$ satisfies the large deviation principle in the scale $1/n^2$ with rate function*

$$I(\mu) := -\iint_{\mathcal{D}^2} \log |z - w|\, d\mu(z)\, d\mu(w) - (\lambda - 1)\int_{\mathcal{D}} \log(1 - |z|^2)\, d\mu(z) + B,$$

for $\mu \in \mathcal{M}(\mathcal{D})$, where

$$B := -\frac{\lambda^2 \log \lambda}{2} + \frac{\lambda^2 \log(\lambda - 1)}{2} - \frac{\log(\lambda - 1)}{2} + \frac{\lambda - 1}{2}.$$

Furthermore, there exists a unique $\mu_0 \in \mathcal{M}(\mathcal{D})$ given by the density

$$d\mu_0(z) = \frac{(\lambda - 1)r}{\pi\left(1 - r^2\right)^2}\, dr\, d\varphi, \quad z = re^{\mathrm{i}\varphi}$$

on $\{z : |z| \le 1/\sqrt{\lambda}\}$ such that $I(\mu_0) = 0$.

Set

$$F(z, w) := -\log |z - w| - \frac{\lambda - 1}{2}\left(\log(1 - |z|^2) + \log(1 - |w|^2)\right),$$

and

$$F_\alpha(z, w) := \min(F(z, w), \alpha),$$

for $\alpha > 0$. Since $F_\alpha(z, w)$ is bounded and continuous

$$\mu \in \mathcal{M}(\mathcal{D}) \mapsto \iint_{\mathcal{D}^2} F_\alpha(z, w)\, d\mu(z)\, d\mu(w).$$

is continuous in the weak* topology, when the support of μ is restricted to a compact set. The functional I is written as

$$\begin{aligned} I(\mu) &= \iint_{\mathcal{D}^2} F(z, w)\, d\mu(z)\, d\mu(w) + B \\ &= \sup_{\alpha>0} \iint_{\mathcal{D}^2} F_\alpha(z, w)\, d\mu(z)\, d\mu(w) + B\,, \end{aligned}$$

hence I is lower semi-continuous.

We can write I in the form

$$I(\mu) = -\Sigma(\mu) - (\lambda - 1) \int_{\mathcal{D}} \log(1 - |z|^2)\, d\mu(z) + B.$$

Here the first part $-\Sigma(\mu)$ is strictly convex (as it was established in the previous section) and the second part is affine in μ. Therefore I is a strictly convex functional.

If X is compact and $\mathcal{A}$ is a base for the topology, then the large deviation principle is equivalent to the following conditions (Theorem 4.1.18 in [6]):

$$-I(x) = \inf_{x\in G, G\in\mathcal{A}} \left\{ \limsup_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\} = \inf_{x\in G, G\in\mathcal{A}} \left\{ \liminf_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\}$$

for all $x \in X$. We apply this result in the case $X = \mathcal{M}(\mathcal{D})$, and we choose

$$\left\{ \mu' \in \mathcal{M}(\mathcal{D}) : \left| \int_{\mathcal{D}} z^{k_1} \bar{z}^{k_2}\, d\mu'(z) - \int_{\mathcal{D}} z^{k_1} \bar{z}^{k_2}\, d\mu(z) \right| < \varepsilon \text{ for } k_1 + k_2 \le m \right\}.$$

to be $G(\mu; m, \varepsilon)$. For $\mu \in \mathcal{M}(\mathcal{D})$ the sets $G(\mu; m, \varepsilon)$ form a neighborhood base of μ for the weak* topology of $\mathcal{M}(\mathcal{D})$, where $m \in \mathbb{N}$ and $\varepsilon > 0$. To obtain the theorem, we have to prove that

$$\begin{aligned} -I(\mu) &\ge \inf_G \left\{ \limsup_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\}, \\ -I(\mu) &\le \inf_G \left\{ \liminf_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\}, \end{aligned}$$

where G runs over neighborhoods of μ.

The large deviation theorem implies the almost sure weak convergence.

Theorem 3.2 *Let $U_{[m,n]}$, P_n and μ_0 as in Theorem 3.1. Then*

$$P_n(\omega) \xrightarrow{n\to\infty} \mu_0$$

weakly with probability 1.

The proof is standard, one benefits from the compactness of the level sets of the rate function and the Borel-Cantelli lemma is used, see [6].

4 Proof of the large deviation

In this section we prove Theorem 3.1. Our method is based on the explicit form of the joint eigenvalue density.

First we compute the limit of the normalizing constant $C_{[m,n]}$ given in (1).

$$
\begin{aligned}
B &=: \lim_{n\to\infty} \frac{1}{n^2} \log C_{[m,n]} \\
&= -\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n-1} \log \binom{m-n+j-1}{j} \\
&= -\lim_{n\to\infty} \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{n-1-i}{n-1} \log \frac{m-n-1+i}{i}.
\end{aligned}
$$

Here the limit of a Riemannian sum can be recognized and this gives an integral:

$$
\begin{aligned}
B &= -\int_0^1 (1-x)\log\left(\frac{\lambda-1+x}{x}\right) dx \\
&= -\frac{\lambda^2\log\lambda}{2} + \frac{\lambda^2\log(\lambda-1)}{2} - \frac{\log(\lambda-1)}{2} + \frac{\lambda-1}{2}.
\end{aligned}
$$

The lower and upper estimates are stated in the form of lemmas.

Lemma 4.1 *For every $\mu \in \mathcal{M}(\mathcal{D})$,*

$$
\inf_G \left\{ \limsup_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\} \le -\iint_{\mathcal{D}^2} F(z,w)\, d\mu(z)\, d\mu(w) - B
$$

where G runs over a neighborhood base of μ.

This is the easier estimate, one can follow the proof of the earlier large deviation theorems, see [1, 2, 10].

Lemma 4.2 *For every $\mu \in \mathcal{M}(\mathcal{D})$,*

$$
\inf_G \left\{ \liminf_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\} \ge -\iint_{\mathcal{D}^2} F(z,w)\, d\mu(z)\, d\mu(w) - B,
$$

where G runs over a neighborhood base of μ.

Proof If

$$
\iint_{\mathcal{D}^2} F(z,w)\, d\mu(z)\, d\mu(w)
$$

is infinite, then we have a trivial case. Therefore we may assume that this double integral is finite.

Since $F(z,w)$ is bounded from below, we have

$$
\iint_{\mathcal{D}^2} F(z,w)\, d\mu(z)\, d\mu(w) = \lim_{k\to\infty} \iint_{\mathcal{D}^2} F(z,w)\, d\mu_k(z)\, d\mu_k(w)
$$

with the conditional measure

$$\mu_k(B) = \frac{\mu(B \cap \mathcal{D}_k)}{\mu(\mathcal{D}_k)},$$

for all Borel set B, where

$$\mathcal{D}_k := \left\{ z : |z| \le 1 - \frac{1}{k} \right\}.$$

So it suffices to assume, that the support of μ is contained in $\mathcal{D}_k$ for some $k \in \mathbb{N}$.

Next we reguralize the measure μ. For any $1/k(k+1) > \varepsilon > 0$, let φ_ε be a nonnegative C^∞-function supported in the disc $\{z : |z| < \varepsilon\}$ such that

$$\int_{\mathcal{D}} \varphi_\varepsilon(z)\, dz = 1,$$

and $\varphi_\varepsilon * \mu$ be the convolution of μ with φ_ε. This means that $\varphi_\varepsilon * \mu$ has the density

$$\int_{\mathcal{D}} \varphi_\varepsilon(z - w)\, d\mu(w)$$

on $\mathcal{D}_{k+1}$. Thanks to concavity and upper semi-continuity of Σ restricted on probability measures with uniformly bounded supports, it is easy to see that

$$\Sigma(\varphi_\varepsilon * \mu) \ge \Sigma(\mu).$$

Also

$$\lim_{\varepsilon \to 0} \int_{\mathcal{D}} \log{(1 - |z|)^2}\; d(\varphi_\varepsilon * \mu)(z) = \int_{\mathcal{D}} \log\left(1 - |z|^2\right)\, d\mu(z),$$

since $\log\left(1 - |z|^2\right)$ is bounded on $\mathcal{D}_{k+1}$. Hence we may assume that μ has a continuous density f on the unit disc $\mathcal{D}$, and $\delta \le f(z) \le \delta^{-1}$ for some $\delta > 0$.

We want to partition the disc into annuli of equal measure. Let $k = [\sqrt{n}]$, and choose

$$0 = r_0^{(n)} \le r_1^{(n)} \le \cdots \le r_{k-1}^{(n)} \le r_k^{(n)} = 1,$$

such that

$$\mu\left(\left\{ z = re^{\mathrm{i}\varphi} : r \in [r_{i-1}^{(n)}, r_i^{(n)}] \right\}\right) = \frac{1}{k} \quad \text{for} \quad 1 \le i \le k.$$

Note that

$$k^2 \le n \le k(k+2),$$

and there exists a sequence $l_1, \ldots, l_k$ such that $k \le l_i \le k+2$, for $1 \le i \le k$, and $\sum_{i=1}^k l_i = n$. Now we partition radially. For fixed i let

$$0 = \varphi_0^{(n)} \le \varphi_1^{(n)} \le \cdots \le \varphi_{l_i - 1}^{(n)} \le \varphi_{l_i}^{(n)} = 2\pi,$$

such that

$$\mu\left(\left\{z = re^{\mathrm{i}\varphi} : r \in [r_{i-1}^{(n)}, r_i^{(n)}], \varphi \in [\varphi_{j-1}^{(n)}, \varphi_j^{(n)}]\right\}\right) = \frac{1}{kl_i} \quad \text{for} \quad 1 \le j \le l_i.$$

In this way we divided $\mathcal{D}$ into n pieces $S_1^{(n)}, \dots, S_n^{(n)}$. Here

$$\frac{\delta(1-\varepsilon_n)}{n} \le \frac{\delta}{kl_i} = \int_{S_i^{(n)}} dz \le \frac{1}{k^2\delta} \le \frac{1+\varepsilon_n'}{n\delta}, \tag{3}$$

where $\varepsilon_n = 2/(\sqrt{n}+2) \to 0$ and $\varepsilon_n' = 1/(\sqrt{n}-1) \to 0$ as $n \to \infty$. We can suppose that

$$\lim_{n\to\infty}\left(\max_{1\le i\le n} \operatorname{diam}\left(S_i^{(n)}\right)\right) = 0. \tag{4}$$

In each part $S_i^{(n)}$ we take a smaller one $D_i^{(n)}$, similarly to $S_i^{(n)}$ by dividing the radial and phase intervals above into three equal parts, and selecting the middle ones, so that

$$\frac{\delta(1-\varepsilon_n)}{9n} \le \int_{D_i^{(n)}} dz \le \frac{1+\varepsilon_n'}{9n\delta}. \tag{5}$$

We set

$$\Delta_n := \left\{(\zeta_1, \dots, \zeta_n) : \zeta_i \in D_i^{(n)}, 1 \le i \le n\right\}.$$

For any neighborhood G of μ

$$\Delta_n \subset \{\zeta \in \mathcal{D}^n : \mu_\zeta \in G\}$$

for every n large enough. Then

$$\begin{aligned}
&P_n(G) \ge \overline{\nu}_n(\Delta_n)\\
&= \frac{1}{Z_n}\int\cdots\int_{\Delta_n} \exp\left((n-1)\sum_{i=1}^n(\lambda-1)\log\left(1-|\zeta_i|^2\right)\right)\\
&\qquad\qquad\qquad\qquad\times \prod_{1\le i<j\le n}|\zeta_i-\zeta_j|^2\, d\zeta_1\dots d\zeta_n\\
&\ge \frac{1}{Z_n}\left(\frac{\delta(1-\varepsilon_n)}{9n}\right)^{n^2} \exp\left((n-1)(\lambda-1)\sum_{i=1}^n \min_{\zeta\in D_i^{(n)}} \log\left(1-|\zeta|^2\right)\right)\\
&\qquad\qquad\qquad\qquad\times \prod_{1\le i<j\le n}\left(\min_{\zeta\in D_i^{(n)},\eta\in D_j^{(n)}} |\zeta-\eta|^2\right).
\end{aligned}$$

Here for the first part we establish

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(n-1)(\lambda-1)}{n^2} \sum_{i=1}^{n} \min_{\zeta\in D_i^{(n)}} \log\left(1-|\zeta|^2\right) \\
&= \lim_{n\to\infty} \frac{\lambda-1}{n} \sum_{i=1}^{n} \min_{\zeta\in D_i^{(n)}} \log\left(1-|\zeta|^2\right) \\
&= (\lambda-1) \int_{\mathcal{D}} \log\left(1-|\zeta|^2\right) f(\zeta)\, d\zeta,
\end{aligned}$$

because of (4) and verify

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{2}{n^2} \sum_{1\le i<j\le n} \log\left(\min_{\zeta\in D_i^{(n)},\eta\in D_j^{(n)}} |\zeta-\eta|\right) \\
&\ge \iint_{\mathcal{D}^2} f(\zeta) f(\eta) \log|\zeta-\eta|\, d\zeta\, d\eta. \qquad (6)
\end{aligned}$$

for the second part. We have

$$\begin{aligned}
&\iint_{\mathcal{D}^2} f(\zeta) f(\eta) \log|\zeta-\eta|\, d\zeta\, d\eta \\
&\le 2 \sum_{1\le i<j\le n} \int_{S_i^{(n)}} \int_{S_j^{(n)}} f(\zeta) f(\eta) \log|\zeta-\eta|\, d\zeta\, d\eta \\
&\le 2 \sum_{1\le i<j\le n} \log\left(\max_{\zeta\in S_i^{(n)},\eta\in S_j^{(n)}} |\zeta-\eta|\right) \int_{S_i^{(n)}} f(\zeta)\, d\zeta \int_{S_j^{(n)}} f(\eta)\, d\eta \\
&\le \frac{2(1+\varepsilon_n)^2}{n^2} \sum_{i<j} \log\left(\max_{\zeta\in S_i^{(n)},\eta\in S_j^{(n)}} |\zeta-\eta|\right).
\end{aligned}$$

Since the construction of $S_i^{(n)}$ and $D_i^{(n)}$ yields

$$\lim_{n\to\infty} \frac{2(1+\varepsilon_n)^2}{n^2} \sum_{1\le i<j\le n} \log\left(\frac{\max_{\zeta\in S_i^{(n)},\eta\in S_j^{(n)}} |\zeta-\eta|}{\min_{\zeta\in D_i^{(n)},\eta\in D_j^{(n)}} |\zeta-\eta|}\right) = 0,$$

we obtain (6). □

The last step is to minimize I. Now we apply Lemma 2.3 for

$$Q(z) := -\frac{\lambda-1}{2} \log\left(1-|z|^2\right)$$

on $\mathcal{D}$. This function satisfies the conditions of the lemma. Hence the support of the limit measure μ_0 is the disc

$$S_\lambda = \left\{z : |z| \le \frac{1}{\sqrt{\lambda}}\right\},$$

and the density is given by

$$d\mu_0 = \frac{1}{\pi}(rQ'(r))'\,dr\,d\varphi = \frac{1}{\pi}\frac{(\lambda-1)r}{\left(1-r^2\right)^2}\,dr\,d\varphi, \quad z = re^{i\varphi}.$$

For this μ_0 again by [15]

$$\begin{aligned}
I(\mu_0) &= \frac{1}{2}Q\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{1}{2}\log\lambda + \frac{1}{2}\int_{S_\lambda} Q(z)d\mu_0(z) + B \\
&= -\frac{\lambda-1}{2}\log(\lambda-1) + \frac{1}{2\lambda}\log\lambda - \frac{(\lambda-1)^2}{2\pi}\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{\lambda}}}\frac{r\log(1-r^2)}{(1-r^2)^2}\,dr\,d\varphi \\
&= -\frac{\lambda-1}{2}\log(\lambda-1) + \frac{1}{2\lambda}\log\lambda - \frac{\lambda-1}{2}\left(\lambda\log\left(\frac{\lambda-1}{\lambda}\right)+1\right) + B \\
&= 0.
\end{aligned}$$

The uniqueness of μ_0 satisfying $I(\mu_0) = 0$ follows from the strict convexity of I.

5 Some connection to free probability

Let Q_m be an $m \times m$ projection matrix of rank n, and let U_m be an $m \times m$ Haar unitary. Then the matrix $Q_m U_m Q_m$ has the same non-zero eigenvalues as $U_{[m,n]}$, but it has $m - n$ zero eigenvalues. The large deviation result for $U_{[m,n]}$ is easily modified to have the following.

Theorem 5.1 *Let $1 < \lambda < \infty$ and Q_m, U_m as above. If $m/n \to \lambda$ as $n \to \infty$, then the sequence of empirical eigenvalue densities $Q_m U_m Q_m$ satisfies the large deviation principle in the scale $1/n^2$ with rate function*

$$\tilde{I}(\tilde{\mu}) := \begin{cases} I(\mu), & \text{if} \quad \tilde{\mu} = (1-\lambda^{-1})\delta_0 + \lambda^{-1}\mu, \\ +\infty, & \text{otherwise} \end{cases}$$

Furthermore, the measure

$$\tilde{\mu}_0 = (1-\lambda^{-1})\delta_0 + \lambda^{-1}\mu_0$$

is the unique minimizer of $\tilde{I}$, and $\tilde{I}(\tilde{\mu}_0) = 0$.

Now let $\mathcal{M}$ be a von Neumann algebra and τ be a faithful normal trace on $\mathcal{M}$. The pair $(\mathcal{M}, \tau)$ is often called a *non-commutative probability space*. A unitary $u \in \mathcal{M}$ is called a *Haar unitary* if $\tau(u^k) = 0$ for every non-zero integer k. Let $q \in \mathcal{M}$ be a projection such that $\tau(q) = \lambda$. If u and q are *free* (see [10] or [17] for more details about free probability), then the above (U_m, Q_m) is a random matrix model of the pair (u, q). This means that

$$\frac{1}{m}E\left(\mathrm{Tr}\,\mathcal{P}(U_m, U_m^*, Q_m)\right) \to \tau\left(\mathcal{P}(u, u^*, q)\right)$$

for any polynomial $\mathcal{P}$ of three non-commuting indeterminants. This statement is a particular case of Voiculescu's fundamental result about *asymptotic freeness* ([16], or Theorem 4.3.5 on p. 154 in [10]).

For an element a of the von Neumann algebra $\mathcal{M}$, the *Fuglede-Kadison* determinant can be defined by:

$$\Delta(a) := \lim_{\varepsilon \to +0} \exp \tau \left(\log(a^* a + \varepsilon I)^{1/2} \right).$$

It was shown by L.G. Brown in 1983 that the function

$$\lambda \mapsto \frac{1}{2\pi} \log \Delta(a - \lambda I)$$

is subharmonic and its Laplacian (taken in the distribution sense) is a probability measure μ_a concentrated on the spectrum of a [4]. This measure is called the *Brown measure* and it is a sort of extension of the spectral multiplicity measure of normal operators:

$$\tau(g(a)) = \int_{\mathbb{C}} g(z)\, d\mu_a(z) \tag{7}$$

for any function g on $\mathbb{C}$ that is analytic in a domain containing the spectrum of a. The Brown measure is computed for quite a few examples in the paper [3].

Let u be a Haar unitary, and $q = q^* = q^2$ be free from u. Then uq is a so-called *R-diagonal* operator and its Brown measure is rotation invariant in the complex plane. According to [8] the Brown measure has an atom of mass $1 - \lambda^{-1}$ at zero and the absolute continuous part has a density

$$\frac{(\lambda - 1) r}{\pi \lambda \left(1 - r^2\right)^2}\, dr\, d\varphi \qquad (z = re^{\mathrm{i}\varphi})$$

on $\{z : |z| \leq 1/\sqrt{\lambda}\}$. We just observe that this measure coincides with the limiting measure in our large deviation theorem. In the moment we cannot deduce the Brown measure from the large deviation result but it is definitely worthwhile to study the relation.

Appendix

Let U_m be an $m \times m$ Haar unitary matrix and write it in the block-matrix form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an $n \times n$, B is $n \times (m-n)$, C is $(m-n) \times n$ and D is an $(m-n) \times (m-n)$ matrix. The space of $n \times n$ (complex) matrices is easily identified with $\mathbb{R}^{2n^2}$ and the push forward of the usual Lebesgue measure is denoted by λ_n. It was obtained in [5] that for $m \geq 2n$, the distribution measure of the $n \times n$ matrix A is absolute continuous with respect to λ_n and the density is

$$C(n, m) \det(1 - A^* A)^{m-2n} \mathbf{1}_{\|A\| \leq 1} d\lambda_n(A). \tag{8}$$

To determine the joint distribution of the eigenvalues $\zeta_1, \zeta_2, \dots, \zeta_n$ of A, we need only the matrices A and C, and by a unitary transformation we transform A to an upper triangular form

$$\begin{pmatrix} \zeta_1 & \Delta_{1,2} & \Delta_{1,3} & \dots & \Delta_{1,n} \\ 0 & \zeta_2 & \Delta_{2,3} & \dots & \Delta_{2,n} \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \zeta_n \\ C_1 & C_2 & C_3 & \dots & C_n \end{pmatrix}, \tag{9}$$

where $C_1, C_2, \dots, C_n$ are the column vectors of the matrix C. It is well-known that the Jacobian of this transformation is a multiple of

$$\prod_{1 \le i < j \le n} |\zeta_i - \zeta_j|^2.$$

Note that the columns of the matrix (9) are normalized and pairwise orthogonal. Following the idea of [19], we integrate out the variables $\Delta_{1,i}, \Delta_{2,i}, \dots, \Delta_{i-1,i}, C_i$, $i \le n$.

One can construct $(n-m) \times (n-m)$ matrices $X^{(i)}$ such that

$$\Delta_{ij} = \frac{1}{\overline{\zeta}_i} C_i^* X^{(i)} C_j. \tag{10}$$

We have $X^{(1)} = I$ and

$$X^{(i)} = I + \sum_{k<i} X^{(k)} \frac{C_k C_k^*}{|\zeta_k|^2} X^{(k)}.$$

Since

$$C_i^* C_i + \sum_{k<i} \overline{\Delta}_{ki} \Delta_{ki} = C_i^* X^{(i)} C_i,$$

the vectors C_i satisfy the equations

$$C_i^* X^{(i)} C_i = 1 - |\zeta_i|^2. \tag{11}$$

Geometrically, the point $(C_{1i}, \dots, C_{m-n,i})$ lies in the ellipsoid given by $X^{(i)}$. To compute the volume of this ellipsoid it is enough to know the determinant of $X^{(i)}$ and this is obtained from the above recursion:

$$\det X^{(i)} = \frac{\det X^{(i-1)}}{|\zeta_{i-1}|^2} = \prod_{j<i} \frac{1}{|\zeta_j|^2}.$$

After this preparation we move to integration. First we integrate with respect to the last column $\Delta_{1,n}, \Delta_{2,n}, \dots, \Delta_{n-1,n} C_n$. For fixed $\Delta_{1,n} \dots \Delta_{n-1,n}$ the distribution of $C_{1,n}, \dots, C_{m-n-1,n}$ is uniform on the set

$$|C_{1,n}|^2 + \dots + |C_{m-n-1,n}|^2 \le 1 - |\zeta_n|^2 - |\Delta_{1,n}|^2 \dots |\Delta_{n-1,n}|^2,$$

i.e. inside the ellipsoid defined by (11). The volume of this $m-n-1$ dimensional complex ellipsoid is

$$\frac{(1-|\zeta_n|^2)^{m-n-1}}{\det X^{(n)}} = (1-|\zeta_n|^2)^{m-n-1}\prod_{i<n}|\zeta_i|^2,$$

Integration out of $\Delta_{i,n}$ gives a factor $|\zeta_i|^{-2}$ from (10) and all together we obtain a factor $(1-|\zeta_n|^2)^{m-n-1}$ from the last column. The same procedure may be applied to the other columns.

Acknowledgements. The authors thank to Fumio Hiai, Ofer Zeitouni and the referee for useful remarks. The work has been partially supported by Hungarian OTKA No. T032662 .

References

1. Ben Arous, G., Guionnet, A.: Large deviation for Wigner's law and Voiculescu's non-commutative entropy, Probab. Theory Relat. Fields **108**, 517–542 (1997)
2. Ben Arous, G., Zeitouni, O.: Large deviations from the circular law, ESAIM: Probability and Statistics **2**, 123–134 (1998)
3. Biane, P., Lehner, F.: Computation of some examples of Brown's spectral measure in free probability, Colloq. Math. **90**, 181–211 (2001)
4. Brown, L.G.: Lidskiǐ's theorem in the type II case, in *Geometric methods in operator algebras (Kyoto 1983)*, Longman Sci. Tech., Harlow, 1–35 1986
5. Collins, M.B.: Intégrales matricielles et Probabilités Non-Commutatives, Ph.D. thesis, University of Paris 6, 2003
6. Dembo, A., Zeitouni, O.: *Large Deviations Techniques and Applications*, Jones and Bartlett, Boston, 1993
7. Guionnet, A.: Large deviations upper bounds and central limit theorems for non-commutative functionals of Gaussian large random matrices, Ann. Inst. H. Poincaré Probab. Statist. **38**, 341–384 (2002)
8. Haagerup, U., Larsen, F.: Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras, J. Funct. Anal. **176**, 331–367 (2000)
9. Hiai, F., Petz, D.: Eigenvalue density of the Wishart matrix and large deviations, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1**, 633–646 (1998)
10. Hiai, F., Petz, D.: *The Semicircle Law, Free Random Variables and Entropy*, American Mathematical Society, 2000
11. Hiai, F., Petz, D.: A large deviation theorem for the empirical eigenvalue distribution of random unitary matrices, Ann. Inst. Henri Poincaré, Probabilités et Statistiques **36**, 71–85 (2000)
12. Landkof, N.S.: *Foundations of the Modern Potential Theory*, Spinger, Berlin - Heidelberg – New York, 1972
13. Petz, D., Hiai, F.: Logarithmic energy as entropy functional, in *Advances in Differential Equations and Mathematical Physics* (eds: E. Carlen, E.M. Harrell, M. Loss), Contemporary Math. **217**, 205–221 (1998)
14. Petz, D., Réffy, J.: Asymptotics of large Haar unitary matrices, to be published in Periodica Math. Hungar.
15. Saff, E.B., Totik, V.: *Logarithmic potentials with external fields*, Springer, New York, 1997

16. Voiculescu, D.V.: Limit laws for random matrices and free products, Invent. Math. **104**, 201–220 (1991)
17. Voiculescu, D.V., Dykema, K.J., Nica, A.: *Free random variables*, American Mathematical Society, 1992
18. Wigner, E.P.: On the distribution of the roots of certain symmetric matrices, Ann. of Math. **67**, 325–327 (1958)
19. Życzkowski, K., Sommers, H-J.: Truncation of random unitary matrices, J. Phys. A: Math. Gen. **33**, 2045–2057 (2000)