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Path regularity for Feller semigroups via Gaussian kernel estimates and generalizations to arbitrary semigroups on C_0

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Abstract. Given $\gamma \in (-1, 1)$, we present a dyadic growth condition $(C^{\gamma}_{\delta-\mathrm{loc}})$ on the finite dimensional distributions of operator semigroups on $C_0(E)$ which - for $\gamma > 0$ and Feller semigroups - assures that the corresponding Feller process has paths in local Hölder spaces and in weighted Besov spaces of order γ. We show that, for operator semigroups satisfying Gaussian kernel estimates of order $m > 1$, condition $(C^{\gamma}_{\delta-\mathrm{loc}})$ holds for all $\gamma < \frac{2}{m} - 1$, and even for all $\gamma < \frac{1}{m}$ in the case of Feller semigroups. Such Gaussian kernel estimates are typical for Feller semigroups on fractals of walk dimension m and for semigroups generated by elliptic operators on $\mathbb{R}^D$ of order $m \geq D$.

Contents

0. Introduction

Let $X = (X_t)_{t\in[0,1]}$ be a stochastic process with values in a complete metric space (E, d) and with sample space $(\Omega, \mathcal{F}, \mathbb{P})$. In order to assure the existence of a modification of X with Hölder-continuous paths, it suffices to show by the Kolmogorov-Čentsov Theorem that X is $\frac{1}{m}$-Hölder-continuous in $L_p(\Omega, \mathbb{P})$ for some $m > 1$, $p \in (m, \infty)$, i.e. $\mathbb{E}\, d(X_{t_1}, X_{t_2})^p \leq C|t_1 - t_2|^{p/m}$ for all $0 \leq t_1 < t_2 \leq 1$. The main step in the proof of this theorem is to verify, given $\gamma \in (0, \frac{1}{m} - \frac{1}{p})$, the

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following property of the X_t for dyadic times $t \in D_n := \{ j2^{-n}\,;\ j = 0, \ldots, 2^n\}$:

$$\sum_{n=0}^{\infty} \mathbb{P}_{(X_t)_{t\in D_n}} [\,\mathrm{Var}_{\infty}^{(2^n)} \geq 2^{-\gamma n}\,] \quad < \quad \infty\,. \tag{1}$$

Here we use the following notation for all $N \in \mathbb{N}$:

$$\mathrm{Var}_{\infty}^{(N)}\ :\ \mathrm{E}^{N+1} \to \mathbb{R}_+\,,\ (x_0, x_1, \ldots, x_N) \mapsto \max_{1\leq j\leq N} d(x_{j-1}, x_j)\,.$$

Indeed, then $\mathcal{N} := \limsup_{n\to\infty}\{\,(X_t)_{t\in D_n} \in [\mathrm{Var}_{\infty}^{(2^n)} \geq 2^{-\gamma n}]\,\}$ is a $\mathbb{P}$-nullset by the Borel-Cantelli lemma. For all $\omega \in \Omega\setminus\mathcal{N}$, the path $t \mapsto X_t(\omega)$ is γ-Hölder-continuous when restricted to the dyadic points $D := \{ j2^{-n}\,;\ n \in \mathbb{N},\ j = 0, \ldots, 2^n\}$ of [0, 1]. Hence, by density of D in [0, 1] and completeness of E, it is straightforward to construct a modification of X with γ-Hölder-continuous paths.

Now suppose that, with respect to a family of probability measures $(\mathbb{P}^x)_{x\in E}$, our process X is a Feller process. If we denote by $(T_t)_{t\in\mathbb{R}_+}$ the associated Feller semigroup on $C_0(E)$ (extended to $B_b(E)$), then

$$\mathbb{P}^x_{(X_{t_1},\ldots,X_{t_n})} \ =\ Q^x_{(t_1,\ldots,t_n)} \ \text{ for all }\ x \in E\,,\ 0 \leq t_1 < t_2 < \ldots < t_n \leq 1\,, \tag{2}$$

where the finite dimensional distributions $Q^x_{(t_1,\ldots,t_n)} : \mathcal{B}(E^n) \to [0, 1]$ are given by

$$Q^x_{(t_1,\ldots,t_n)}\,(B_1 \times \ldots \times B_n) \quad := \quad T_{t_1}\Big(\chi_{B_1}\, T_{t_2-t_1}\,\chi_{B_2}\ \ldots\ T_{t_n-t_{n-1}}\,\chi_{B_n}\Big)(x) \tag{3}$$

for $B_1, \ldots, B_n \in \mathcal{B}(E)$ and extension to $\mathcal{B}(E^n)$, the Borel σ-field. Then, in view of (2), property (1) for $\mathbb{P}^x$ instead of $\mathbb{P}$ reads as

$$\sum_{n=0}^{\infty} |Q^x_{D_n}|\ [\,\mathrm{Var}_{\infty}^{(2^n)} \geq 2^{-\gamma n}\,] \quad < \quad \infty\,. \tag{4}$$

Property (4) can be investigated for arbitrary operator semigroups $(T_t)_{t\in\mathbb{R}_+}$ on $C_0(E)$ (extended to $B_b(E)$); observe that (3) then yields complex measures $Q^x_{(t_1,\ldots,t_n)} : \mathcal{B}(E^n) \to \mathbb{C}$. By what was said before, property (4) can be seen as a generalized path regularity of $(T_t)_{t\in\mathbb{R}_+}$.

Our assumption on $(T_t)_{t\in\mathbb{R}_+}$ is the following 'Gaussian' estimate of order $m > 1$ for the kernels $k_t : E \times \mathcal{B}(E) \to \mathbb{C}$ of the operators T_t (see Proposition 1.3):

$$(GE_m) \qquad |k_t(x, \cdot)|\,(B(x, \lambda t^{1/m})^c) \ \leq\ g_m(\lambda) \ \text{ for all }\ x \in E\,,\ t \in (0, 1]\,,\ \lambda \geq 0\,.$$

Here and in the whole paper, we denote $B(x, r) := \{y \in E; d(x, y) < r\}$ and we use for $m > 1$ the Gaussian decay function of order m

$$g_m : \mathbb{R}_+ \to \mathbb{R}_+\,,\ \lambda \mapsto C_0 \exp\Big(-b\lambda^{\frac{m}{m-1}}\Big)\,,$$

where $b > 0$, $C_0 > 1$ are fixed constants. Such Gaussian kernel estimates are typical for Feller semigroups on Sierpinski carpets of walk dimension m [BB] and for semigroups generated by elliptic operators on $\mathbb{R}^D$ of order $m \geq D$ [D] [AT]. In both cases, one even has pointwise Gaussian estimates; see Section 3 for these and other examples.

We show that (4) holds for all $\gamma < \frac{2}{m} - 1$, and even for all $\gamma < \frac{1}{m}$ in the case of contractive semigroups (in particular Feller semigroups).

We present a modification of (4) which - for $\gamma \in (0,1)$ and Feller semigroups - assures that the corresponding Feller process has paths in local Hölder spaces (with control of the local Hölder constant !) and thus in weighted Besov spaces of order γ over $\mathbb{R}_+$. We show that this modification holds for the same γ-ranges as before.

Weighted Besov regularity of $\mathbb{R}^D$-valued Feller processes $(X_t)_{t\in\mathbb{R}_+}$ has been proved in [S3] under polynomial decay assumptions on the maximal process $\sup_{s\le t} d(X_s, x)$ of the type

$$\mathbb{P}^x\,[\,\exists s \le t : X_s \in B(x,r)^c\,] \;\le\; C\,t\,r^{-m} \quad \text{for all } x \in \mathbb{R}^D,\ r,t>0\,.$$

This estimate is typical for stable Lévy processes of order $m \in (0,2)$ [Pr] and has been generalized in [S2] to Feller processes generated by pseudo-differential operators whose symbol $p : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{C}$ satisfies

$$|p(x,\xi)| \;\le\; C\,r^{-m} \quad \text{for all } x \in \mathbb{R}^D\,,\ r>0\,,\ \xi \in B(0,r^{-1})\,.$$

Notice that, for the particular case of Feller semigroups (i.e. $|k_t(x,\cdot)| = \mathbb{P}^x\,[\,X_t \in \cdot\,]$), our condition (GE_m) can be rewritten in the following probabilistic form of Gaussian decay for the process $d(X_t,x)$:

$$(GE_m^{\mathbb{P}}) \qquad \mathbb{P}^x\,[\,X_t \in B(x,r)^c\,] \;\le\; g_m(t^{-\frac{1}{m}}\,r) \quad \text{for all } x \in E\,,\ t \in (0,1]\,,\ r>0\,.$$

Notice also that non-trivial examples of semigroups satisfying (GE_m) for some $m \in (1,2)$ do not seem to be known.

Pointwise Gaussian estimates for semigroups generated by elliptic operators on $\mathbb{R}^D$ of order $m \ge D$ and generalized Gaussian estimates for the case $m < D$ are the crucial tool to extend these semigroups and some of their operator properties from L_2 (where they are initially given by form techniques) to L_p. We mention the properties of having maximal regularity [CD] [BK1], an H^∞ functional calculus [DM] [BK2] and spectral multipliers [DOS] [B].

The main intention of this paper is to point out that the widely studied property of Gaussian kernel decay implies - even for non-Feller semigroups ! - suitable Kolmogorov-Čentsov-type conditions such as (4) which

- for Feller semigroups, have the probabilistic interpretation of Hölder and Besov regularity of the Feller process (as shown by the path regularity results proved in this paper),
- for non-Feller semigroups, have no 'probabilistic' interpretation so far.

For the verification of (4), Gaussian decay is crucial in the case of non-Feller semigroups but can be weakened to polynomial decay (under a loss of γ-range) in the case of Feller semigroups. This has been suggested to the author by the referee.

1. Semigroups of operators and complex kernels

In this section we collect some results on semigroups of operators and kernels which are (partly) well-known for the sub-Markovian case and have to be extended to the complex case.

Let (E,d) be a separable locally compact metric space and $\mathcal{B}(E)$ its Borel σ-field. By $B_b(E)$ we denote the space of bounded Borel functions $f : E \to \mathbb{C}$ with the supremum norm $\|\cdot\|_\infty$. By $C_0(E)$ we denote the space of continuous functions $f : E \to \mathbb{C}$ vanishing at infinity, with the supremum norm as well. Finally, $\mathcal{M}(E)$ denotes the space of complex (regular) Borel measures $\mu : \mathcal{B}(E) \to \mathbb{C}$ with the total variation norm. Let (E', d') be another separable locally compact metric space.

Definition. *(a) A mapping $k : E \times \mathcal{B}(E') \to \mathbb{C}$ is called a kernel if*

(1) $\{k(x,\cdot); x \in E\}$ is a bounded subset of $\mathcal{M}(E')$.

(2) $k(\cdot, B) \in B_b(E)$ for all $B \in \mathcal{B}(E')$.

In the case $E = E'$ we call k a kernel on E.

(b) A family $(k_t)_{t\in\mathbb{R}_+}$ of kernels on E is called a semigroup of kernels on E if

$$k_{s+t}(x,B) = \int_E k_t(y,B)\, k_s(x,dy) \quad \textit{for all } \; s,t \in \mathbb{R}_+ \,,\; x \in E \,,\; B \in \mathcal{B}(E) \,. \tag{5}$$

The notation introduced in the following lemma will be used in the whole paper.

Lemma 1.1. *If $k : E \times \mathcal{B}(E') \to \mathbb{C}$ is a kernel then so is*

$$|k| : E \times \mathcal{B}(E') \to \mathbb{R}_+ \,,\; (x,B) \mapsto |k(x,\cdot)|(B) \,.$$

Proof. Obviously, $\{|k|(x,\cdot); x \in E\}$ is a bounded subset of $\mathcal{M}(E')$. Hence it suffices to show that

$$M := \{ B \in \mathcal{B}(E') \,;\; |k|(\cdot, B) \in B_b(E) \} \supset \{ B \in \mathcal{B}(E') \,;\; B \text{ open} \} \,. \tag{6}$$

Indeed, since the $|k|(x,\cdot)$ are finite positive measures, the collection M is a monotone class. Hence (6) implies $M = \mathcal{B}(E')$ by the Monotone Class Theorem [DeM, Thm. p. 22]. By the Isomorphism Theorem for standard Borel spaces [Pa] we can assume $E' = \mathbb{R}$.

For the proof of (6) for $E' = \mathbb{R}$, it suffices to show $|k|(\cdot, I) \in B_b(E)$ for all open intervals $I \subset \mathbb{R}$. Indeed, for all open $B \subset \mathbb{R}$, there exists a sequence of disjoint open intervals (I_n) such that $B = \cup_n I_n$, hence we obtain

$$|k|(\cdot, B) = \sum_n |k|(\cdot, I_n) \in B_b(E) \,.$$

So we consider $I = (a,b)$ with $-\infty \le a < b \le \infty$. But then

$$|k|(\cdot,(a,b)) \quad = \quad \sup_{\substack{a<t_0<\ldots<t_n<b \\ t_0,\ldots,t_n\in\mathbb{Q}}} \sum_{j=1}^n |k(\cdot,(t_{j-1},t_j])| \quad \in \quad B_b(E)$$

as the sup over a countable set of measurable functions. Here we used the one-to-one correspondence $\nu \mapsto F_\nu$ between $\mathcal{M}(\mathbb{R})$ and the set of all right-continuous functions on $\mathbb{R}$ of finite variation that vanish at $-\infty$ given by

$$F_\nu : \mathbb{R} \to \mathbb{C} \,,\; x \mapsto \nu((-\infty, x]) \,.$$

Recall that, by right-continuity of F_ν, we have

$$|\nu|((a,b)) = \mathrm{Var}_1^{(a,b)}(F_\nu) \overset{\text{by def.}}{=} \sup_{a<t_0<\ldots<t_n<b} \sum_{j=1}^{n} |F_\nu(t_j) - F_\nu(t_{j-1})|$$

$$= \sup_{\substack{a<t_0<\ldots<t_n<b\\ t_0,\cdots,t_n\in\mathbb{Q}}} \sum_{j=1}^{n} |\nu((t_{j-1}, t_j])| \quad . \qquad \square$$

The next lemma will be crucial for the construction of 'finite dimensional distributions' (in the complex case).

Lemma 1.2. *Let $k : E \times \mathcal{B}(E) \to \mathbb{C}$ and $k' : E \times \mathcal{B}(E') \to \mathbb{C}$ be kernels. Then there exists a unique kernel $\widetilde{k} : E \times \mathcal{B}(E \times E') \to \mathbb{C}$ such that*

$$\widetilde{k}(x, B \times B') = \int_B k'(y, B')\, k(x, dy) \quad \textit{for all} \;\; x \in E\,,\; B \in \mathcal{B}(E)\,,\; B' \in \mathcal{B}(E')\,.$$

Proof. At first we consider the standard case where k and k' are positive. For all $x \in E$, the mapping

$$\beta_x : \mathcal{B}(E) \times \mathcal{B}(E') \to \mathbb{R}_+\,,\; (B, B') \mapsto \int_B k'(y, B')\, k(x, dy)$$

has the properties $\beta(B, \cdot) \in \mathcal{M}(E')$ and $\beta(\cdot, B') \in \mathcal{M}(E)$ for all $B \in \mathcal{B}(E)\,,\; B' \in \mathcal{B}(E')$. By the extension theorem for positive bimeasures (e.g. [DeM, p. 129]), there exists $\nu_x \in \mathcal{M}(E \times E')$ such that $\nu_x(B \times B') = \beta_x(B, B')$ for all $B \in \mathcal{B}(E)\,,\; B' \in \mathcal{B}(E')$. We define

$$\widetilde{k} : E \times \mathcal{B}(E \times E') \to \mathbb{R}_+\,,\; (x, A) \mapsto \nu_x(A)\;.$$

Obviously, $\{\, \widetilde{k}(x, \cdot)\,;\; x \in E \,\}$ is a bounded subset of $\mathcal{M}(E \times E')$. Hence it remains to show that

$$M \;:=\; \{\, A \in \mathcal{B}(E \times E')\,;\; \widetilde{k}(\cdot, A) \in B_b(E) \,\} \;=\; \mathcal{B}(E \times E')\;. \tag{7}$$

But $M \supset \mathcal{B}(E) \times \mathcal{B}(E')$ since $\widetilde{k}(\cdot, B \times B') = \int_E \chi_B(y) k(y, B')\, k(\cdot, dy)$ and $\int_E f(y)\, k(\cdot, dy) \in B_b(E)$ for all $f \in B_b(E)$ [this is clear for all step functions f, and these are $\|\cdot\|_\infty$-dense in $B_b(E)$]. Moreover, M is a montone class which is seen as in the proof of Lemma 1.1. Hence (7) follows by the Monotone Class Theorem.

Now we consider the general case. With k also $\Re k$ and $\Im k$ are kernels, thus $|\Re k|$ and $|\Im k|$ are kernels by Lemma 1.1, hence $(\Re k)_\pm = \frac{1}{2}(|\Re k| \pm \Re k)$ and $(\Im k)_\pm = \frac{1}{2}(|\Im k| \pm \Im k)$ are (positive) kernels. By applying this argument also to k', we obtain positive kernels $k_l\,,\; k'_j$ such that

$$k \;=\; k_1 - k_2 + i(k_3 - k_4) \qquad \text{and} \qquad k' \;=\; k'_1 - k'_2 + i(k'_3 - k'_4)\;.$$

By the first case, there exist kernels $\widetilde{k}_{jl} : E \times \mathcal{B}(E \times E') \to \mathbb{R}_+$ such that

$$\widetilde{k}_{jl}(x, B \times B') \;=\; \int_B k'_j(y, B')\, k_l(x, dy)\ .$$

Moreover, there exist $z_{jl} \in \mathbb{C}$ such that ($|z_{jl}| = 1$ and)

$$\int_B k'(y, B')\, k(x, dy) \;=\; \sum_{j,l=1}^{4} z_{jl} \int_B k'_j(y, B')\, k_l(x, dy)$$

for all $x \in E$, $B \in \mathcal{B}(E)$, $B' \in \mathcal{B}(E')$. Then the desired kernel is

$$\widetilde{k} : E \times \mathcal{B}(E \times E')\,,\ (x, A) \mapsto \sum_{j,l=1}^{4} z_{jl}\, \widetilde{k}_{jl}(x, A). \qquad \square$$

The next characterization follows from an application of the Riesz Representation Theorem $C_0(E)^* = \mathcal{M}(E)$ [C,Thm. III.5.7] and the Monotone Class Theorem; it is well-known, at least in the sub-Markovian case. Since in the standard proof applies also in our complex case, we do not give a proof here.

Proposition 1.3. *(a) Let $(T_t)_{t\in\mathbb{R}_+}$ be an operator semigroup on $C_0(E)$. Then there exists a semigroup of kernels $(k_t)_{t\in\mathbb{R}_+}$ on E such that for all $f \in C_0(E)$*

$$T_t f(x) \;=\; \int_E f(y)\, k_t(x, dy) \quad \textit{for all}\ \ t \in \mathbb{R}_+\,,\ x \in E\ . \tag{8}$$

(b) Let $(k_t)_{t\in\mathbb{R}_+}$ be a semigroup of kernels on E. Then (8)*, employed for all $f \in B_b(E)$, yields an operator semigroup $(T_t)_{t\in\mathbb{R}_+}$ on $B_b(E)$.*

In view of the preceding characterization, we recall the fundamental notation/relation of Feller semigroups (which are sometimes also called Feller-Dynkin semigroups) and Feller processes; see e.g. [EK, § 4.2] .

Remark 1.4. (a) Let $(T_t)_{t\in\mathbb{R}_+}$ be a Feller semigroup on $C_0(E)$, i.e. a positive contractive C_0 semigroup. Let $(k_t)_{t\in\mathbb{R}_+}$ be the associated semigroup of kernels on E. If $(T_t)_{t\in\mathbb{R}_+}$ has the conservation property (i.e. $T_t 1 = 1$ for all $t \in \mathbb{R}_+$) then there exists an E-valued process $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in E}$ such that

$$k_t(x, \cdot) \;=\; \mathbb{P}^x[\, X_t \in \cdot\,]\ \ \text{for all}\ \ t \in \mathbb{R}_+\,,\ x \in E\ . \tag{9}$$

(b) An E-valued process $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in E}$ is called a Feller process if (9) yields a semigroup of kernels $(k_t)_{t\in\mathbb{R}_+}$ on E whose associated operator semigroup $(T_t)_{t\in\mathbb{R}_+}$ [restricted to $C_0(E)$] is a Feller semigroup on $C_0(E)$.

Note that $T_t 1$ is well-defined by Proposition 1.3. The strong continuity of a semigroup $(T_t)_{t\in\mathbb{R}_+}$ on $C_0(E)$ can be characterized (under a natural kernel assumption) by a weakening of the conservation property which seems to be new.

Proposition 1.5. *Let $(T_t)_{t\in\mathbb{R}_+}$ be a bounded semigroup on $C_0(E)$ such that $h^c(r,t) \to 0$ for $t \to 0$ and all $r > 0$. Then the following are equivalent:*
(a) $(T_t)_{t\in\mathbb{R}_+}$ is strongly continuous.
(b) $T_t 1 \to 1$ locally uniformly for $t \to 0$.

Here and in the following, we use for all $r, t \in \mathbb{R}_+$ the notation

$$h^c(r,t) \ := \ \sup_{x\in E} |k_t|(x, B(x,r)^c) \ ,$$

where $|k_t|(x,\cdot) := |k_t(x,\cdot)|$ is the variation of the measure $k_t(x,\cdot)$. Note that $h^c(0,t) = \| T_t \|$ for all $t \in \mathbb{R}_+$.

Proof. (a)⇒(b) Let $K \subset E$ be compact. We have to show that $\| T_t 1 - 1 \|_{L_\infty(K)} \to 0$ for $t \to 0$. There exists $f \in C_0(E)$ such that $r := d(K, [f \neq 1]) > 0$. Then

$$\begin{aligned} \|T_t 1 - 1\|_{L_\infty(K)} &= \|T_t(1-f) + T_t f - f\|_{L_\infty(K)} \quad [\, f = 1 \text{ on } K] \\ &\le (1 + \| f \|_\infty)\, h^c(r,t) + \|T_t f - f\|_\infty \\ &\to 0 \ \text{ for } \ t \to 0 \ \text{ by hypothesis and (a)} \ . \end{aligned}$$

(b)⇒(a) Let $f \in C_0(E)$. We have to show $\| T_t f - f \|_\infty \to 0$ for $t \to 0$. We can assume $K :=$supp(f) to be compact. Thus, given $\varepsilon > 0$, there exists $r > 0$ such that

$$|f(x) - f(y)| \sup_{t\in\mathbb{R}_+} \| T_t \| \ \le \ \varepsilon \quad \text{for all } \ x \in E\ ,\ y \in B(x,r)\, .$$

We deduce for all $t \in \mathbb{R}_+$, $x \in E$:

$$|T_t f(x) - f(x)| \ = \ \left| \int_E (f(y) - f(x))\, k_t(x,dy) \ + \ f(x)(T_t 1(x) - 1) \right|$$

$$\le \varepsilon \ + \ 2\| f \|_\infty h^c(r,t) \ + \ \| f \|_\infty \| T_t 1 - 1 \|_{L_\infty(K)}$$

$$\le 3\varepsilon \ \text{ for all } t > 0 \text{ small enough by hypothesis and (b)}\, .$$ □

In many applications, the semigroup $(T_t)_{t\in\mathbb{R}_+}$ under consideration is initially given on $L_2(E,\mu)$ by form techniques and consists of integral operators. Under the following natural conditions on the integral kernels, $(T_t)_{t\in\mathbb{R}_+}$ acts also on $C_0(E)$.

Proposition 1.6. *Suppose that all balls in E are relatively compact. Let $\mathcal{A} \supset \mathcal{B}(E)$ be a σ-field over E and $\mu : \mathcal{A} \to [0,\infty]$ a σ-finite measure. Let $(T_t)_{t\in\mathbb{R}_+}$ be a C_0 semi- group on $L_2 := L_2(E,\mathcal{A},\mu)$ such that, for all $t > 0$, there exists $p_t \in C(E^2)$ with*

$$\int_E |p_t(x,y)|\, d\mu(y) \ \le \ C_t \quad \textit{for all} \ \ x \in E\ ,$$

$$T_t f(x) \ = \ \int_E p_t(x,y) f(y)\, d\mu(y) \quad \textit{for all} \ \ f \in L_2 \cap L_\infty\, ,\ \mu\textit{-a.e. } x \in E\ ,$$

$$\sup_{x\in E} \int_{B(x,r)^c} |p_t(x,y)|\, d\mu(y) \quad \to \quad 0 \quad for \quad r \to \infty\,.$$

Then $(T_t)_{t\in\mathbb{R}_+}$ acts as an operator semigroup on $C_0(E)$.

Proof. We have to show that $(T_t)_{t\in\mathbb{R}_+}$ has the Feller property $T_t(C_0(E)) \subset C_0(E)$ for all $t \in \mathbb{R}_+$. Since $\|T_t f\|_\infty \leq C_t \|f\|_\infty$ for all $f \in B_b(E)$ by hypothesis, it suffices to show $T_t(C_c(E)) \subset C_0(E)$, where $C_c(E)$ is the space of continuous functions $f : E \to \mathbb{C}$ of compact support. So let $f \in C_c(E)$. Then $T_t f$ is continuous by dominated convergence, hence it remains to show that $M := [\,|T_t f| \geq \varepsilon\,]$ is contained in a compact set. But $\overline{B(K,r)}$ is compact for all $r > 0$ and $K :=\mathrm{supp}(f)$ by hypothesis on E, and

$$M \subset \{x \in E\,;\, \int_K |p_t(x,y)|\, d\mu(y) \geq \tfrac{\varepsilon}{\|f\|_\infty}\} \subset \overline{B(K,r)}$$

provided $\int_{B(x,r)^c} |p_t(x,y)|\, d\mu(y) < \frac{\varepsilon}{\|f\|_\infty}$ for all $x \in E$. □

2. Path regularity criteria for Feller semigroups and generalizations to arbitrary semigroups

This section contains the main results of this paper. Let (E, d) be a locally compact complete metric space. In the rest of the paper, we denote, given an operator semigroup $(T_t)_{t\in\mathbb{R}_+}$ on $C_0(E)$, also by $(T_t)_{t\in\mathbb{R}_+}$ its natural extension to $B_b(E)$ and by $(k_t)_{t\in\mathbb{R}_+}$ its associated semigroup of kernels on E; see Proposition 1.3.

The following result introduces the main objects of this paper, i.e. the 'finite dimensional distributions' associated to an arbitrary operator semigroup on $C_0(E)$.

Proposition 2.1. *Let $(T_t)_{t\in\mathbb{R}_+}$ be an operator semigroup on $C_0(E)$.*

(a) For all $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \ldots < t_n$, there exists a unique kernel $Q_{(t_1,\ldots,t_n)} : E \times \mathcal{B}(E^n) \to \mathbb{C}$ such that for all $B_1, \ldots, B_n \in \mathcal{B}(E)$

$$Q_{(t_1,\ldots,t_n)}\,(B_1 \times \ldots \times B_n) \quad = \quad T_{t_1}\Big(\chi_{B_1}\, T_{t_2-t_1}\chi_{B_2}\, \ldots\, T_{t_n-t_{n-1}}\chi_{B_n}\Big)\,.$$

(b) For all $N \in \mathbb{N}$, $r, t_0 \in \mathbb{R}_+$, $t > 0$, the following estimate holds:

$$\Big\|\, |Q_{t_0+(0,t,\ldots,Nt)}|\,[\,\mathrm{Var}_\infty^{(N)} \geq \mathrm{r}\,]\,\Big\|_\infty \quad \leq \quad \mathrm{N}\, \|\mathrm{T}_{t_0}\|\, \|\mathrm{T_t}\|^{\mathrm{N}-1}\, \mathrm{h^c(r,t)}\,.$$

Here and in the following, we often write $Q_{(t_1,\ldots,t_n)}(B) := Q_{(t_1,\ldots,t_n)}(\cdot, B) \in B_b(E)$, $|Q_{(t_1,\ldots,t_n)}|\,[\,\mathrm{Var}_\infty^{(n-1)} \geq \mathrm{r}\,] := |\mathrm{Q}_{(t_1,\ldots,t_n)}|(\cdot, [\,\mathrm{Var}_\infty^{(n-1)} \geq \mathrm{r}\,]) \in \mathrm{B_b(E)}$ (in the notation of Lemma 1.1) and $Q^x_{(t_1,\ldots,t_n)} := Q_{(t_1,\ldots,t_n)}(x,\cdot) \in \mathcal{M}(E^n)$.

Proof. (a) The case $n = 1$ being clear for $Q_{(t)} := k_t$ by Proposition 1.3, we take $n > 1$. By induction hypotheses and Lemma 1.2, there exists a kernel $Q_{(t_1,\ldots,t_n)} : E \times \mathcal{B}(E^n) \to \mathbb{C}$ such that for all $B_1, \ldots, B_n \in \mathcal{B}(E)$, $x \in E$:

$$\begin{aligned} Q_{(t_1,\ldots,t_n)}(x, B_1 \times \ldots \times B_n) &= \int_{B_1} Q_{(t_2-t_1,\ldots,t_n-t_1)}(y, B_2 \times \ldots \times B_n)\, Q_{(t_1)}(x, dy) \\ &= T_{t_1}\Big(\chi_{B_1}\, T_{t_2-t_1}\chi_{B_2}\, \ldots\, T_{t_n-t_{n-1}}\chi_{B_n}\Big)(x)\,, \end{aligned}$$

where we used in the second step that $\int_E f(y)\, Q_{(t_1)}(x,dy) = T_{t_1} f(x)$ for all $f \in B_b(E)$ by Proposition 1.3.
(b) For the sets $M_j := \{ (x_0, \dots, x_N) \in E^{N+1}\, ;\ x_j \in B(x_{j-1}, r)^c \}$, we have

$$[\,\mathrm{Var}_\infty^{(N)} \geq \mathrm{r}\,] \quad = \quad \mathrm{M}_1 \cup \dots \cup \mathrm{M}_\mathrm{N} \,.$$

Moreover, from (a) we deduce for all $B \in \mathcal{B}(E^{N+1})$:

$$\begin{aligned}&|Q_{t_0+(0,t,\dots,Nt)}|(x, B)\\ &\quad\leq \int_E \cdots \int_E \chi_B(x_0, \dots, x_N)\, |k_t|(x_{N-1}, dx_N) \dots |k_{t_0}|(x, dx_0)\,.\end{aligned}$$

Hence we obtain for $j = 1, \dots, N$ the following estimate yielding the assertion:

$$\begin{aligned}&|Q_{t_0+(0,t,\dots,Nt)}|\,(x, M_j)\\ &\leq \int_E \cdots \int_E \int_{B(x_{j-1},r)^c} \int_E \cdots \int_E |k_t|(x_{N-1}, dx_N) \dots |k_{t_0}|(x, dx_0)\\ &\leq \|T_{t_0}\|\ \|T_t\|^{N-1}\, h^c(r,t).\end{aligned}$$

□

2.1. Hölder-regularity on [0, 1]

We begin with a reformulation of the the Kolmogorov-Čentsov Theorem (see e.g. [KS, Thm. II.2.8]) as discussed in our introduction. For the sake of completeness, we give a proof in Section 4.

Theorem A. *For all $n \in \mathbb{N}$ and $0 \leq t_1 < \dots < t_n \leq 1$, let $\mu_{(t_1,\dots,t_n)} : \mathcal{B}(E^n) \to \mathbb{R}_+$ be a measure. Suppose that*

$$(C^\gamma) \qquad \sum_{n=0}^{\infty} \mu_{D_n}[\,\mathrm{Var}_\infty^{(2^n)} \geq 2^{-\gamma n}\,] \quad < \quad \infty$$

$$\|\,\mathrm{Var}_\infty^{(1)}\,\|_{\mathrm{L_p}(\mu_{(\mathrm{t_1,t_2})})} \to 0 \quad \textit{whenever} \quad 0 < \mathrm{t}_2 - \mathrm{t}_1 \to 0 \tag{10}$$

for some $\gamma \in \mathbb{R}$ and $p \in (0, \infty)$. If $\gamma \in (0, 1)$ and $\mu_{(t_1,\dots,t_n)} = \mathbb{P}_{(X_{t_1},\dots,X_{t_n})}$ for some E-valued process $(\Omega, \mathbb{P}, (X_t)_{t\in[0,1]})$, then there exists a γ-Hölder-continuous modification $(\widetilde{X}_t)_{t\in[0,1]}$.

The parts (a), (b) of the following result show that the above conditions (C^γ), (10) which - for positive γ and in a probabilistic setting - yield a γ-Hölder-continuous process, are implied - for (if $m \geq 2$) negative γ and in our general setting - by Gaussian kernel estimates of order m as in (GE_m).

Proposition 2.2. *Let $(T_t)_{t\in\mathbb{R}_+}$ be an operator semigroup on $C_0(E)$. Suppose that there exists $m > 1$ such that (GE_m) holds.*

(a) For all $\gamma < \frac{2}{m} - 1$, the following estimate holds:

$$\sum_{n=0}^{\infty} \Big\|\, |Q_{D_n}|\,[\,\mathrm{Var}_\infty^{(2^n)} \geq 2^{-\gamma n}\,]\,\Big\|_\infty \quad < \quad \infty\,.$$

(b) For all $p \in (0,\infty)$, *the following estimate holds:*

$$\|\,\mathrm{Var}_\infty^{(1)}\,\|_{\mathrm{L_p}(|\mathrm{Q}^{\mathrm{x}}_{(\mathrm{t_1},\mathrm{t_2})}|)} \quad \leq \quad \mathrm{C}|\mathrm{t_1}-\mathrm{t_2}|^{1/\mathrm{m}} \quad \textit{for all} \;\; 0 \leq \mathrm{t_1} < \mathrm{t_2} \leq 1\,,\ \mathrm{x} \in \mathrm{E}\,.$$

(c) If $(T_t)_{t\in\mathbb{R}_+}$ *is contractive then we have for all* $\gamma < \frac{1}{m}$:

$$\sum_{n=0}^{\infty} \Big\|\, |Q_{D_n}|\,[\,\mathrm{Var}_\infty^{(2^\mathrm{n})} \geq 2^{-\gamma \mathrm{n}}\,]\,\Big\|_\infty \quad < \quad \infty\,.$$

(d) (*Case* $\gamma = \frac{1}{m}$) *If* $(T_t)_{t\in\mathbb{R}_+}$ *is contractive then we have for all* $\varepsilon > 0$:

$$\sum_{n=0}^{\infty} \Big\|\, |Q_{D_n}|\,[\,\mathrm{Var}_\infty^{(2^\mathrm{n})} \geq (1+\varepsilon)2^{-\mathrm{n}/\mathrm{m}}\,(\mathrm{b}^{-1}\log(2^\mathrm{n}))^{\frac{\mathrm{m}-1}{\mathrm{m}}}\,]\,\Big\|_\infty \quad < \quad \infty\,.$$

As an application of Proposition 2.1 the proof of this result is relatively short.

Proof. For each $n \in \mathbb{N}$, we apply Proposition 2.1(b) for $N = 2^n$, $t_0 = 0$ and $t = 2^{-n}$. Since, by hypothesis (GE_m), the inequality $h^c(r,t) \leq g_m(t^{-1/m}r)$ holds for all $r \geq 0$, $t \in (0,1]$, we have for all sequences (r_n) in $\mathbb{R}_+$:

$$\sum_{n=0}^{\infty} \Big\|\, |Q_{D_n}|\,[\,\mathrm{Var}_\infty^{(2^\mathrm{n})} \geq \mathrm{r_n}\,]\,\Big\|_\infty \quad \leq \quad \sum_{\mathrm{n}=0}^{\infty} 2^\mathrm{n}\,\|\mathrm{T_0}\|\,\|\mathrm{T}_{2^{-\mathrm{n}}}\|^{2^\mathrm{n}-1}\,\mathrm{h^c}(\mathrm{r_n},2^{-\mathrm{n}})$$

$$\leq \|T_0\| \sum_{n=0}^{\infty} 2^n\, g_m(2^{n/m} r_n)\,\|T_{2^{-n}}\|^{2^n-1} \quad =: \quad \|T_0\|\,S\,.$$

For the case $r_n = 2^{-\gamma n}$ with $\gamma < \frac{2}{m} - 1$, we can estimate S by

$$S \;\leq\; \sum_{n=0}^{\infty} 2^n\, g_m(\,(2^n)^{\frac{1}{m}-\gamma}\,)\,C_0^{2^n-1} \;=\; \sum_{n=0}^{\infty} 2^n\,\exp\Big(-b(2^n)^{\frac{1-\gamma m}{m-1}}\Big)\,C_0^{2^n} \;<\; \infty \tag{11}$$

since $\frac{1-\gamma m}{m-1} > 1$. This proves (a). For the proof of (c) and (d) we assume $(T_t)_{t\in\mathbb{R}_+}$ to be contractive. Moreover, we fix $\gamma \leq \frac{1}{m}$ and put

$$r_n \;:=\; \begin{cases} 2^{-\gamma n} & \gamma < \frac{1}{m} \\ (1+\varepsilon)2^{-n/m}\,(b^{-1}\log(2^n))^{\frac{m-1}{m}} & \gamma = \frac{1}{m} \end{cases}\,.$$

Then S is finite because

$$S = \begin{cases} \sum_{n=0}^{\infty} 2^n\, g_m(\,(2^n)^{\frac{1}{m}-\gamma}\,) \;<\; \infty & \gamma < \frac{1}{m} \\ \sum_{n=0}^{\infty} 2^n\, C_0\, 2^{-n(1+\varepsilon)^{\frac{m}{m-1}}} < \;\infty & \gamma = \frac{1}{m} \end{cases}\,.$$

(b) is shown as a more general statement in Proposition 2.5(b). □

Parts (a) and (c) of Proposition 2.2 are the announced generalizations of γ-Hölder path continuity to a non-probabilistic setting, part (d) somehow is a corresponding (partial) generalization of Levy's modulus of continuity which says for any standard Brownian motion $(X_t)_{t\in[0,1]}$ on $\mathbb{R}$:

$$\limsup_{s\searrow 0} \max_{\substack{t_2-t_1\le s\\ 0\le t_1<t_2\le 1}} w(s)^{-1}|X_t - X_s| \;=\; 1 \quad \text{almost surely}\,,$$

where $w(s) := s^{1/2}\,(2\log(\frac{1}{s}))^{1/2}$. The main step in the (more difficult) upper estimate "lim sup... $\le$ 1 almost surely" is to show for all $\varepsilon, \theta \in [0, 1)$ such that $(1+\varepsilon)^2 > \frac{1+\theta}{1-\theta}$ (see e.g. [KS, p. 115/116])

$$\sum_{n=1}^{\infty} \mathbb{P}_{(X_t)_{t\in D_n}}\Big(\{x \in \mathbb{R}^{2^n+1};\ \max_{\substack{j-i\le 2^{n\theta}\\ 0\le i<j\le 2^n}} w((j-i)2^{-n})^{-1}|x_i - x_j| \ge 1+\varepsilon\}\Big) \;<\; \infty\,.$$

In part (d) of Proposition 2.2, applied for $T_t = e^{t\Delta}$, $E = \mathbb{R}$, $m = 2$, $b = \frac{1}{2}$, we consider only the set corresponding to the case $\theta = 0$:

$$\{x \in \mathbb{R}^{2^n+1};\ \max_{1\le j\le 2^n} w(2^{-n})^{-1}\,|x_{j-1} - x_j| \ge 1+\varepsilon\}$$

$$=\ [\,\mathrm{Var}_{\infty}^{(2^n)} \ge (1+\varepsilon)2^{-n/2}\,(2\log(2^n))^{\frac{1}{2}}\,]\,.$$

Corollary 2.3. *Let* $(\Omega, \mathbb{P}^x, (X_t)_{t\in[0,1]})_{x\in E}$ *be an* E*-valued Feller process (restricted to* $[0, 1]$*). Suppose that there exists* $m > 1$ *such that* $(GE_m^{\mathbb{P}})$ *holds. Then we have for all* $x \in E$ *and* $\gamma \in (0, \frac{1}{m})$:

$$X \;\in\; C^{\gamma}([0,1]; E) \quad \mathbb{P}^x\textit{-almost surely.}$$

Proof. The hypotheses of Proposition 2.2(b),(c) are satisfied for the semigroup $(T_t)_{t\in\mathbb{R}_+}$ associated to our Feller process since $|k_t(x,\cdot)| = \mathbb{P}^x\,[\,X_t \in \cdot\,]$. Hence we can apply Theorem A. □

In particular, we reobtain the well-known fact that Brownian motion on $\mathbb{R}^D$ is locally γ-Hölder-continuous for all $\gamma \in (0, \frac{1}{2})$. Stronger results are given now.

2.2. Local Hölder-regularity on $\mathbb{R}_+$

It is well-known that Brownian motion on $\mathbb{R}^D$ is not globally Hölder-continuous but only locally. In order to measure this localness, we fix for the whole section a stricly decreasing continuous function $\delta : \mathbb{R}_+ \to \mathbb{R}_{>0}$ with $\delta(\infty) = 0$.

Definition. *Let* $\gamma \in (0, 1)$*. A function* $f : \mathbb{R}_+ \to E$ *is called* δ*-locally* γ*-Hölder-continuous if*

$$\|\, f \,\|_{C^{\gamma}_{\delta-\mathrm{loc}}(\mathbb{R}_+;E)} \quad := \quad \sup_{\substack{s,t\in\mathbb{R}_+\\ 0<t-s\le\delta(s)}} d(f(s), f(t))\,|t-s|^{-\gamma} \quad < \quad \infty\,.$$

In this case, we write $f \in C^{\gamma}_{\delta-\mathrm{loc}}(\mathbb{R}_+; E)$.

The next result is a version of Theorem A for processes $(X_t)_{t\in\mathbb{R}_+}$ to have a δ-locally γ-Hölder-continuous modification; its proof will be given in Section 4. We need the following δ-dependent sets of dyadics:

$$\widetilde{D}_n \ := \ \{ \, j2^{-n} \, ; \ j = 0, \ldots , N_n \, \} \quad , \text{ where } \quad N_n := \lceil \delta^{-1}(2^{-(n+1)})2^n \rceil + 2^n \ .$$

Theorem 2.4. *For all $n \in \mathbb{N}$ and $0 \le t_1 < \ldots < t_n < \infty$, let $\mu_{(t_1,\ldots,t_n)} : \mathcal{B}(E^n) \to \mathbb{R}_+$ be a measure. Suppose that*

$$(C^{\gamma}_{\delta-\mathrm{loc}}) \qquad\qquad \sum_{n=0}^{\infty} \mu_{\widetilde{D}_n} [\, \mathrm{Var}^{(\mathrm{N_n})}_{\infty} \ge 2^{-\gamma \mathrm{n}} \,] \quad < \quad \infty$$

$$\| \, \mathrm{Var}^{(1)}_{\infty} \, \|_{\mathrm{L_p}(\mu_{(\mathrm{t_1},\mathrm{t_2})})} \to 0 \quad \textit{whenever} \quad 0 < \mathrm{t_2} - \mathrm{t_1} \to 0 \tag{12}$$

for some $\gamma \in \mathbb{R}$ and $p \in (0,\infty)$. If $\gamma \in (0,1)$ and $\mu_{(t_1,\ldots,t_n)} = \mathbb{P}_{(X_{t_1},\ldots,X_{t_n})}$ for some E-valued process $(\Omega, \mathbb{P}, (X_t)_{t\in\mathbb{R}_+})$, then there exists a δ-locally γ-Hölder-continuous modification $(\widetilde{X}_t)_{t\in\mathbb{R}_+}$ satisfying $\sup_{T\ge 1} T^{-1}\,\delta(2T)^{1-\gamma}$ $\| \, \widetilde{X} \, \|_{Osc([0,T];E)} < \infty$.

Here we denote for a function $f : [a,b] \to E$ its oscillation by $\| \, f \, \|_{Osc([a,b];E)} := \sup\{ \, d(f(s), f(t)) \, ; \ s,t \in [a,b] \, \}$. The parts (a), (b) of the following result show that the above conditions $(C^{\gamma}_{\delta-\mathrm{loc}})$ and (12) which - for positive γ and in a probabilistic setting - yield a δ-locally γ-Hölder-continuous process, are implied - for (if $m \ge 2$) negative γ and in our general setting - by the Gaussian kernel estimate (GE_m).

Proposition 2.5. *Let $(T_t)_{t\in\mathbb{R}_+}$ be an operator semigroup on $C_0(E)$. Suppose that there exists $m > 1$ such that (GE_m) holds.*

(a) Let $\gamma < \frac{2}{m} - 1$ be such that

$$\delta(t) \ \le \ a(t - C)^{-\frac{m-1}{2-(\gamma+1)m}} \quad \textit{for all} \ \ t \in \mathbb{R}_+ \ \textit{large enough}$$

and some constants $a < \frac{1}{2}\left(\frac{b}{\log(C_0)}\right)^{\frac{m-1}{2-(\gamma+1)m}}$, $C > 0$. Then

$$\sum_{n=0}^{\infty} \left\| \, |Q_{\widetilde{D}_n}| \, [\, \mathrm{Var}^{(\mathrm{N_n})}_{\infty} \ge 2^{-\gamma \mathrm{n}} \,] \, \right\|_{\infty} \quad < \quad \infty \ .$$

(b) For all $p \in (0,\infty)$, the following estimate holds:

$$\| \, \mathrm{Var}^{(1)}_{\infty} \, \|_{\mathrm{L_p}(|\mathrm{Q}^{\mathrm{x}}_{(\mathrm{t_1},\mathrm{t_2})}|)} \ \le \ \mathrm{C}^{\mathrm{t_1}+1} \, |\mathrm{t_1} - \mathrm{t_2}|^{1/\mathrm{m}} \quad \textit{whenever} \ \ 0 < \mathrm{t_2} - \mathrm{t_1} \le 1 \, , \ \mathrm{x} \in \mathrm{E} \ .$$

(c) Let $(T_t)_{t\in\mathbb{R}_+}$ be contractive and $\gamma < \frac{1}{m}$ such that

$$\delta(t) \ \le \ a \log(t/C)^{-\frac{m-1}{1-\gamma m}} \quad \textit{for all} \ \ t \in \mathbb{R}_+ \ \textit{large enough} \tag{13}$$

and some constants $a < \frac{1}{2}\, b^{\frac{m-1}{1-\gamma m}}$, $C > 0$. Then

$$\sum_{n=0}^{\infty} \left\| \, |Q_{\widetilde{D}_n}| \, [\, \mathrm{Var}^{(\mathrm{N_n})}_{\infty} \ge 2^{-\gamma \mathrm{n}} \,] \, \right\|_{\infty} \quad < \quad \infty \ .$$

Proof. (a) By hypothesis on γ, we have $\alpha := \frac{1-\gamma m}{m-1} > 1$. By hypotheses on δ and a, we have for all $s \in \mathbb{R}_+$ large enough:

$$\delta^{-1}(\tfrac{1}{2s}) \;\le\; (2a)^{\alpha-1} s^{\alpha-1} \quad \text{and} \quad (2a)^{\alpha-1} \;<\; \frac{b}{\log(C_0)} \;.$$

Since $N_n = \lceil \delta^{-1}(2^{-(n+1)})2^n \rceil + 2^n \le (2a)^{\alpha-1}(2^n)^\alpha + 1 + 2^n$ there exist $\tilde b < b$, $\tilde C > 0$ such that $N_n\, C_0^{N_n} \le \tilde C e^{\tilde b (2^n)^\alpha}$ for all $n \in \mathbb{N}$. Now we deduce the assertion as follows via Proposition 2.1(b) and hypothesis (GE_m) :

$$\sum_{n=0}^\infty \Big\| \, |Q_{\widetilde D_n}| \,[\, \mathrm{Var}_\infty^{(\mathrm{N_n})} \ge 2^{-\gamma \mathrm{n}} \,] \Big\|_\infty \;\le\; \sum_{\mathrm{n}=0}^\infty \mathrm{N_n} \, \|\mathrm{T_0}\| \, \|\mathrm{T_{2^{-n}}}\|^{\mathrm{N_n}-1} \, \mathrm{h^c}(2^{-\gamma \mathrm{n}}, 2^{-\mathrm{n}})$$

$$\le \|T_0\| \sum_{n=0}^\infty N_n \, C_0^{N_n} \, e^{-b(2^n)^\alpha} \quad < \quad \infty \,. \tag{14}$$

(c) By hypothesis on γ, we have $\alpha := \frac{1-\gamma m}{m-1} > 0$. By hypotheses on δ and a, we have for all $s \in \mathbb{R}_+$ large enough:

$$\delta^{-1}(\tfrac{1}{2s}) \;\le\; C\, e^{(2a)^\alpha s^\alpha} \quad \text{and} \quad (2a)^\alpha \;<\; b \;.$$

Since $N_n = \lceil \delta^{-1}(2^{-(n+1)})2^n \rceil + 2^n$ there exist $\tilde b < b$, $\tilde C > 0$ such that $N_n \le \tilde C e^{\tilde b (2^n)^\alpha}$ for all $n \in \mathbb{N}$. Now we deduce the assertion as follows via Proposition 2.1(b), contractivity and hypothesis (GE_m) :

$$\sum_{n=0}^\infty \Big\| \, |Q_{\widetilde D_n}| \,[\, \mathrm{Var}_\infty^{(\mathrm{N_n})} \ge 2^{-\gamma \mathrm{n}} \,] \Big\|_\infty \;\le\; \sum_{\mathrm{n}=0}^\infty \mathrm{N_n} \, \|\mathrm{T_0}\| \, \|\mathrm{T_{2^{-n}}}\|^{\mathrm{N_n}-1} \, \mathrm{h^c}(2^{-\gamma \mathrm{n}}, 2^{-\mathrm{n}})$$

$$\le \sum_{n=0}^\infty N_n \, C_0 \, e^{-b(2^n)^\alpha} \quad < \quad \infty \,.$$

(b) Denote $A(x,l,r) := B(x,(l+1)r)^c \setminus B(x,lr)$ and $r_t := t^{1/m}$. We have for all $x_1 \in E$, using (GE_m) in the third step:

$$\int_E d(x_1,x_2)^p \, |k_{t_2-t_1}|(x_1,dx_2)$$

$$\le \sum_{l=0}^\infty \int_{A(x_1,l,r_{t_2-t_1})} [(l+1) r_{t_2-t_1}]^p \, |k_{t_2-t_1}|(x_1,dx_2)$$

$$\le r_{t_2-t_1}^p \sum_{l=0}^\infty (l+1)^p \int_{B(x_1,l r_{t_2-t_1})^c} |k_{t_2-t_1}|(x_1,dx_2)$$

$$\le r_{t_2-t_1}^p \sum_{l=0}^\infty (l+1)^p \, g_m(l) \quad = \quad r_{t_2-t_1}^p C \;.$$

From this estimate the assertion is direct since $\|T_t\| \le C_0^{\lceil t \rceil}$ for all $t > 0$ (by (GE_m) and induction) and

$$\| \operatorname{Var}_\infty^{(1)} \|^{\mathrm{p}}_{\mathrm{L_p}(|\mathrm{Q}^{\mathrm{x}}_{(\mathrm{t}_1,\mathrm{t}_2)}|)} = \int_E \int_E d(x_1, x_2)^p \, |k_{t_2 - t_1}|(x_1, dx_2) \, |k_{t_1}|(x, dx_1)$$

$$\le \| T_{t_1} \| \sup_{x_1 \in E} \int_E d(x_1, x_2)^p \, |k_{t_2 - t_1}|(x_1, dx_2). \qquad \square$$

Since (13) holds for the function δ considered in the following corollary, its proof is completely analogous to the proof of Corollary 2.3 and therefore omitted.

Corollary 2.6. *Let* $(\Omega, \mathbb{P}^x, (X_t)_{t \in \mathbb{R}_+})_{x \in E}$ *be an* E*-valued Feller process. Suppose that there exists* $m > 1$ *such that* $(GE_m^{\mathbb{P}})$ *holds. Then we have for all* $x \in E$, $\gamma \in (0, \frac{1}{m})$, $\alpha > \frac{m-1}{1-\gamma m}$ *and* $\delta(t) := \log(2+t)^{-\alpha}$:

$$X \in C^\gamma_{\delta-\mathrm{loc}}(\mathbb{R}_+; E) \ \ \textit{and} \ \ \sup_{T \ge 1} T^{-1} \, \delta(2T)^{1-\gamma} \, \| X \|_{Osc([0,T];E)} < \infty$$

$\mathbb{P}^x$*-almost surely.*

2.3. Weighted Besov-regularity on $\mathbb{R}_+$

We will use the following standard notation for Besov spaces. For $f : \mathbb{R}_+ \to E$ and $h \in \mathbb{R}$, let $|\Delta_h f| : \mathbb{R}_+ \to \mathbb{R}_+, t \mapsto d(f((t+h)_+), f(t))$. Given $\gamma, p \in (0, \infty)$ and a weight $\phi : \mathbb{R}_+ \to (0, \infty)$, we write $f \in \dot{B}^\gamma_{p\infty}(\mathbb{R}_+, \phi; E)$ if

$$\| f \|_{\dot{B}^\gamma_{p\infty}(\mathbb{R}_+, \phi; E)} \quad := \quad \sup_{0 < r \le \frac{1}{2}} \ r^{-\gamma} \sup_{|h| \le r} \| \, |\Delta_h f| \, \|_{L_p(\mathbb{R}_+, \phi)} \quad < \quad \infty .$$

Here $L_p(\mathbb{R}_+, \phi)$ denotes the weighted L_p-space, i.e. $\|g\|_{L_p(\mathbb{R}_+, \phi)} := \|g\phi\|_{L_p(\mathbb{R}_+)}$. Recall that, with the analogous definitions for functions on $\mathbb{R}$, one has if $\gamma \in (0, 1)$, $E = \mathbb{R}^D$ and $\phi : \mathbb{R} \to (0, \infty)$ is an 'admissible' weight in the sense of [ST, § 5.1]

$$\| f \|_{\dot{B}^\gamma_{p\infty}(\mathbb{R}, \phi; \mathbb{R}^D)} \quad \sim \quad \| f\phi \|_{\dot{B}^\gamma_{p\infty}(\mathbb{R}; \mathbb{R}^D)} \ ,$$

$$\| f \|_{B^\gamma_{p\infty}(\mathbb{R}, \phi; \mathbb{R}^D)} \quad \sim \quad \| f \|_{\dot{B}^\gamma_{p\infty}(\mathbb{R}, \phi; \mathbb{R}^D)} \ + \ \| f \|_{L_p(\mathbb{R}, \phi; \mathbb{R}^D)} \ , \tag{15}$$

where $\dot{B}^\gamma_{p\infty}(\mathbb{R}; \mathbb{R}^D)$ and $B^\gamma_{p\infty}(\mathbb{R}; \mathbb{R}^D)$ are the classical homogeneous and inhomogeneous Besov spaces (without weights) as defined e.g. in [Pe, § 3], and $B^\gamma_{p\infty}(\mathbb{R}, \phi; \mathbb{R}^D)$ is the weighted counterpart as defined e.g. in [ST, § 5.1]. See Schilling's paper [S3, § 3] for a summary of these facts on (inhomogeneous) Besov spaces.

As in the preceding section, let $\delta : \mathbb{R}_+ \to \mathbb{R}_{>0}$ be a stricly decreasing continuous function with $\delta(\infty) = 0$. Obviously, the property $f \in C^\gamma_{\delta-\mathrm{loc}}(\mathbb{R}_+; E)$ is stronger than the property $f \in \dot{B}^\gamma_{p\infty}(\mathbb{R}_+, \phi; E)$ provided the weight ϕ has a sufficient decay with respect to $\frac{1}{\delta}$. In the following, we use the notation $\langle t \rangle := \sqrt{1+t^2}$.

Lemma 2.7. *Let $\gamma \in (0,1)$ and $f : \mathbb{R}_+ \to E$ be δ-locally γ-Hölder-continuous. We have for all $p \in (0,\infty)$ and weights $\phi : \mathbb{R}_+ \to (0,\infty)$:*

$$\| f \|_{\dot{B}^{\gamma}_{p\infty}(\mathbb{R}_+,\phi;E)} \le (1+\delta(0)) \, \| \tfrac{\phi}{\delta(\cdot+\frac{1}{2})} \|_{L_p(\mathbb{R}_+)} \, \| f \|_{C^{\gamma}_{\delta-loc}(\mathbb{R}_+;E)} \,,$$

$$\| d(f(0), f) \|_{L_p(\mathbb{R}_+,\phi)} \le \| \tfrac{\langle\cdot\rangle\,\phi}{\delta(2\langle\cdot\rangle)^{1-\gamma}} \|_{L_p(\mathbb{R}_+)} \sup_{T\ge 1} T^{-1}\delta(2T)^{1-\gamma} \, \| f \|_{Osc([0,T];E)} \,.$$

Proof. Denote the supremum in the second assertion by S and let $t_0 \in \mathbb{R}_+$, $r \in (0, \frac{1}{2}]$. We have $d(f(0), f(t_0)) \le \| f \|_{Osc([0,\langle t_0\rangle];E)} \le S \frac{\langle t_0\rangle}{\delta(2\langle t_0\rangle)^{1-\gamma}}$ and by Lemma 4.2 below:

$$\begin{aligned}
& r^{-\gamma} \sup_{|h|\le r} d(f((t_0+h)_+), f(t_0)) \\
& \le \sup_{s_0\in[(t_0-r)_+,t_0]} \; \sup_{\substack{s,t\in[s_0,s_0+r]\\ 0<t-s\le r}} d(f(s), f(t))\, |t-s|^{-\gamma} \\
& \le \sup_{s_0\in[(t_0-r)_+,t_0]} (2\tfrac{r}{\delta(s_0+r)} + 1) \, \|f\|_{C^{\gamma}_{\delta-loc}(\mathbb{R}_+;E)} \\
& \le \tfrac{1+\delta(0)}{\delta(t_0+\frac{1}{2})} \, \|f\|_{C^{\gamma}_{\delta-loc}(\mathbb{R}_+;E)} \,.
\end{aligned}$$

Now the assertions follow by integrating these two estimates with respect to t_0 . □

The next result follows from Corollary 2.6 and Lemma 2.7 .

Corollary 2.8. *Let $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in E}$ be an E-valued Feller process. Suppose that there exists $m > 1$ such that $(GE^{\mathbb{P}}_m)$ holds. Then we have for all $x \in E$, $\gamma \in (0, \frac{1}{m})$, $p \in (0,\infty)$ and $\mu > \frac{1}{p}$:*

$$X \in \dot{B}^{\gamma}_{p\infty}(\mathbb{R}_+, \langle\cdot\rangle^{-\mu}; E) \;\; \textit{and} \;\; d(x, X) \in L_p(\mathbb{R}_+, \langle\cdot\rangle^{-\mu-1}) \quad \mathbb{P}^x\textit{-almost surely,}$$

in particular $X_{\cdot\vee 0} \in B^{\gamma}_{p\infty}(\mathbb{R}, \langle\cdot\rangle^{-\mu-1}; \mathbb{R}^D)$ $\mathbb{P}^x$-almost surely if $E = \mathbb{R}^D$ by (15).

By Lemma 2.7, the above conditions $(C^{\gamma}_{\delta-\mathrm{loc}})$ and (12) yield - for positive γ and Feller semigroups - a process $X \in \dot{B}^{\gamma}_{p\infty}(\mathbb{R}_+, \phi; E)$ provided $\frac{\phi}{\delta(\cdot+\frac{1}{2})} \in L_p(\mathbb{R}_+)$. And they are implied - for (if $m \ge 2$) negative γ and general semigroups - by the Gaussian kernel estimate (GE_m) .

In the following result, we give a condition (B^{γ}) which - for positive γ and Feller semigroups - yields a process $X \in \dot{B}^{\gamma}_{p\infty}(\mathbb{R}_+, \phi; E)$ under the weaker condition $\phi \in L_p(\mathbb{R}_+)$; the proof is given in Section 4. On the other hand, for general semigroups satisfying (GE_m), we do not obtain (B^{γ}) for any $\gamma \in \mathbb{R}$; see (16) below.

Theorem 2.9. *For all $n \in \mathbb{N}$ and $0 \le t_1 < \ldots < t_n < \infty$, let $\mu_{(t_1,\ldots,t_n)} : \mathcal{B}(E^n) \to \mathbb{R}_+$ be a measure. Suppose that*

$$(B^{\gamma}) \qquad \sum_{n=0}^{\infty} 2^{\gamma n} \, \| \mathrm{Var}^{(2^n)}_{\infty} \|_{\mathrm{L_q}(\mu_{\mathrm{t+D_n}})} \;\le\; \mathrm{C}_1 \;\; \textit{for all} \;\; \mathrm{t} \in \mathbb{R}_+$$

for some $\gamma \in \mathbb{R}$ *and* $q \in [1,\infty)$. *If* $\gamma > 0$ *and* $\mu_{(t_1,\dots,t_n)} = \mathbb{P}_{(X_{t_1},\dots,X_{t_n})}$ *for some right-continuous* E*-valued process* $(\Omega, \mathbb{P}, (X_t)_{t\in\mathbb{R}_+})$*, then we have for all* $p \in (0,q]$ *and* $0 < \phi \in L_p(\mathbb{R}_+)$:

$$\| X \|_{L_q(\Omega;\dot{B}^{\gamma}_{p\infty}(\mathbb{R}_+,\phi;E))} \quad \le \quad 2^{1+\gamma} C_1 \| \phi \|_{L_p(\mathbb{R}_+)} \,.$$

In particular, we have $X \in \dot{B}^{\gamma}_{p\infty}(\mathbb{R}_+,\phi;E)$ *almost surely.*

Proposition 2.10. *Let* $(T_t)_{t\in\mathbb{R}_+}$ *be a contractive operator semigroup on* $C_0(E)$. *Suppose that there exists* $m > 1$ *such that* (GE_m) *holds. Then we have for all* $q \in (0,\infty)$ *and* $\gamma < \frac{1}{m} - \frac{1}{q}$:

$$\sum_{n=0}^{\infty} 2^{\gamma n} \| \mathrm{Var}_{\infty}^{(2^n)} \|_{L_q(|Q^x_{t+D_n}|)} \le C \textit{ for all } x \in E,\ t \in \mathbb{R}_+ \,.$$

Proof. Let $q \in (0,\infty)$. We have for all $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ and $x \in E$:

$$\| \mathrm{Var}_{\infty}^{(2^n)} \|^q_{L_q(|Q^x_{t+D_n}|)} \quad = \quad \int_0^\infty |Q^x_{t+D_n}| \,[\, \mathrm{Var}_{\infty}^{(2^n)} > r^{1/q} \,]\, dr$$

$$\le \int_0^\infty 2^n \, \|T_t\| \, \|T_{2^{-n}}\|^{2^n-1} \, h^c(r^{1/q}, 2^{-n}) \, dr \quad [\text{ by Proposition 2.1(b) }]$$

$$\le 2^n \int_0^\infty g_m(2^{n/m} r^{1/q}) \, dr \qquad [\text{ by contractivity and } (GE_m) \,]$$

$$= q \int_0^\infty g_m(s) \, ds \;\; 2^{n(1-q/m)} \quad = \quad C_1^q \, 2^{n(1-q/m)} \,.$$

This implies for all $\gamma < \frac{1}{m} - \frac{1}{q}$ and $x \in E$, $t \in \mathbb{R}_+$:

$$\sum_{n=0}^{\infty} 2^{\gamma n} \| \mathrm{Var}_{\infty}^{(2^n)} \|_{L_q(|Q^x_{t+D_n}|)} \quad \le \quad C_1 \sum_{n=0}^{\infty} 2^{n(\gamma+1/q-1/m)} \quad = \quad C_2(\gamma). \qquad \square$$

For non-contractive semigroups, our proof does not work for any $\gamma \in \mathbb{R}$:

$$\sum_{n=0}^{\infty} 2^{\gamma n} \| \mathrm{Var}_{\infty}^{(2^n)} \|_{L_q(|Q^x_{t+D_n}|)} \le C_1 \|T_t\|^{1/q} \sum_{n=0}^{\infty} C_0^{(2^n-1)/q} \, 2^{n(\gamma+1/q-1/m)} = \infty \,. \tag{16}$$

Corollary 2.11. *Let* $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in E}$ *be an* E*-valued Feller process. Suppose that there exist constants* $m, C_0 > 1$, $b > 0$ *such that*

$$\mathbb{P}^x [\, X_t \in B(x,r)^c \,] \quad \le \quad C_0 \exp\Big(-b \, t^{-\frac{1}{m-1}} \, r^{\frac{m}{m-1}} \Big)$$

for all $x \in E$, $t \in (0,1]$, $r > 0$. *Then we have for all* $x \in E$, $p \in (0,\infty)$, $0 < \phi \in L_p(\mathbb{R}_+)$ *and* $\gamma \in (0, \frac{1}{m})$:

$$\dot{X} \in B^{\gamma}_{p\infty}(\mathbb{R}_+, \phi; E) \quad \mathbb{P}^x\text{-almost surely.}$$

Proof. Choose $q \in [p \vee 1, \infty)$ such that $\gamma < \frac{1}{m} - \frac{1}{q}$. The hypotheses of Proposition 2.10 are satisfied for the semigroup $(T_t)_{t\in\mathbb{R}_+}$ associated to our Feller process. Hence we can apply Theorem 2.9. □

Criteria for $\mathbb{R}^D$-valued Feller processes to have paths in Besov spaces can be found e.g. in works of Ciesielski, Kerkyacharian and Roynette [CKR], Herren [H] and Schilling [S1][S3] . The most general global criterion is Schilling's main result in [S3, Thm. 4.2] which can be summarized as follows.

Theorem B. *Let* $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in\mathbb{R}^D}$ *be an* $\mathbb{R}^D$*-valued Feller process. Suppose that there exist constants* $C, \alpha, \beta > 0$ *such that*

$$\mathbb{P}^x\,[\,\exists s \le t : X_s \in B(x,r)^c\,] \quad \le \quad \begin{cases} Ctr^{-\alpha} & r \ge 1 \\ Ctr^{-\beta} & r \le 1 \end{cases}$$

for all $x \in \mathbb{R}^D$, $r, t > 0$. *Then we have for all* $x \in \mathbb{R}^D$, $p \in (0,\infty)$, $\mu > \frac{1}{p} + \frac{1}{\alpha}$ *and* $\gamma \in (0, \frac{1}{p} \wedge \frac{1}{\beta})$:

$$X_{\cdot\vee 0} \in B^{\gamma}_{p\infty}(\mathbb{R}, \langle\cdot\rangle^{-\mu}; \mathbb{R}^D) \quad \mathbb{P}^x\text{-almost surely.}$$

We make the following observation. In order to assure that the Feller process X_t in our Corollaries 2.8 and 2.11 possesses paths in Besov spaces, the process $d(X_t, x)$ is supposed to have Gaussian decay (uniformly in x), whereas in Schilling's Theorem B for a similar conclusion the maximal process $\sup_{s\le t} d(X_s, x)$ is supposed to have polynomial decay (uniformly in x). An improvement of our results requiring only polynomial decay of the process $d(X_t, x)$ (uniformly in x) will be given in the next section.

While examples of Feller processes for which our Corollaries 2.8 and 2.11 are applicable will be given in the next section, we want to mention that, for Feller processes generated by an extension of a pseudo-differential operator, the hypothesis of Theorem B follows from suitable growth conditions on its symbol. This can be seen from the following result which is due to Schilling [S2, Lemma 4.1] and generalizes an earlier result of Pruitt [Pr] for Lévy processes. For all details on pseudo-differential operators, we refer to [S2][S3] and the references given there.

Theorem C. *Let* $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in\mathbb{R}^D}$ *be a Feller process generated by an extension of a pseudo-differential operator with a symbol* $p : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{C}$ *such that*

$$|p(x,\xi)| \le C(1+|\xi|^2) \text{ for all } x, \xi \in \mathbb{R}^D$$

and $p(x,\cdot)$ *is given by a Lévy-Khinchine formula for all* $x \in \mathbb{R}^D$. *Then*

$$\mathbb{P}^x\,[\,\exists s \le t : X_s \in B(x,r)^c\,] \le Ct \sup_{\substack{y\in B(x,2r)\\ \xi\in B(0,r^{-1})}} |p(y,\xi)| \text{ for all } x \in \mathbb{R}^D,\ r, t > 0 .$$

Therefore, in the situation of Theorem C, the hypothesis of Theorem B always holds for $\beta = 2$ ($r \leq 1$) and can be expressed in terms of the following generalizations of the Blumenthal-Getoor index [BG] of the symbol $p(x, \xi)$ introduced in [S2] :

$$\beta_0 := \sup \Big\{ \lambda \geq 0 \, ; \, \limsup_{r \to \infty} r^\lambda \sup_{y \in \mathbb{R}^D} \sup_{\xi \in B(0, r^{-1})} |p(y, \xi)| \; = \; 0 \Big\}$$

$$\beta_\infty := \inf \Big\{ \lambda > 0 \, ; \, \limsup_{r \to 0} r^\lambda \sup_{y \in \mathbb{R}^D} \sup_{\xi \in B(0, r^{-1})} |p(y, \xi)| \; = \; 0 \Big\}$$

2.4. Polynomial decay in the case of Feller semigroups

For the verification of the Kolmogorov-Čentsov-type conditions (C^γ) and ($C^\gamma_{\delta-\mathrm{loc}}$) in the preceding subsections, the hypothesis of Gaussian kernel decay was crucial; see the estimates (11) and (14). In the case of Feller semigroups, this hypothesis can be weakened to polynomial decay. The price to pay is that we obtain (C^γ) and ($C^\gamma_{\delta-\mathrm{loc}}$) only for a smaller γ-range.

In combination with our path regularity results [which are based on (C^γ) and ($C^\gamma_{\delta-\mathrm{loc}}$)], we thus obtain path regularity criteria for Feller processes requiring only polynomial decay of the process $d(X_t, x)$.

In this sense, the following result improves Corollary 2.3 on Hölder regularity on [0, 1]. Similar improvements of Corollary 2.6 (local Hölder-regularity on $\mathbb{R}_+$) and Corollary 2.8 (weighted Besov-regularity on $\mathbb{R}_+$) can be obtained analogously and are therefore omitted.

Corollary 2.12. *Let* $(\Omega, \mathbb{P}^x, (X_t)_{t\in[0,1]})_{x\in E}$ *be an* E*-valued Feller process* (*restricted to* [0, 1]). *Suppose that there exist* $m > 0$ *and* $\alpha > 1$ *such that* $m\alpha > 1$ *and*

$$\mathbb{P}^x \, [\, X_t \in B(x, r)^c \,] \; \leq \; C \, (t r^{-m})^\alpha \quad \textit{for all} \;\; x \in E \, , \; t \in (0, 1] \, , \; r > 0 \, .$$

Then we have for all $x \in E$ *and* $\gamma \in (0, \frac{1}{m}(1 - \frac{1}{\alpha}) \wedge 1)$:

$$X \; \in \; C^\gamma([0, 1]; E) \quad \mathbb{P}^x\textit{-almost surely.}$$

Proof. As before, we will apply Theorem A and denote by $(T_t)_{t\in\mathbb{R}_+}$ the Feller semigroup associated to our Feller process. Arguing as at the beginning of the proof of Proposition 2.2 yields

$$\sum_{n=0}^\infty \Big\| \, Q_{D_n} \, [\, \mathrm{Var}_\infty^{(2^n)} \geq 2^{-\gamma n} \,] \, \Big\|_\infty \leq \sum_{n=0}^\infty 2^n \sup_{x \in E} \mathbb{P}^x \, [\, X_{2^{-n}} \in B(x, 2^{-\gamma n})^c \,]$$
$$\leq C \sum_{n=0}^\infty 2^{n(1-\alpha(1-\gamma m))} \quad < \quad \infty$$

provided $\gamma < \frac{1}{m}(1 - \frac{1}{\alpha})$. Now let $p \in (0, m\alpha - 1)$ and $\varepsilon \in (0, \frac{\alpha}{m\alpha - p})$. Denoting $r_t := t^\varepsilon$, we obtain $\| \mathrm{Var}_\infty^{(1)} \|_{L_p(Q^x_{(t_1,t_2)})} \le C'(t_2 - t_1)^{\varepsilon(p-m\alpha)+\alpha}$ exactly as in the proof of Proposition 2.5 since

$$\int_E d(x_1, x_2)^p \, |k_{t_2-t_1}|(x_1, dx_2) \le r^p_{t_2-t_1} \sum_{l=0}^\infty (l+1)^p \, \mathbb{P}^{x_1} [\, X_{t_2-t_1} \in B(x_1, r_{t_2-t_1})^c \,]$$

$$\le C\, r^{p-m\alpha}_{t_2-t_1} \, (t_2 - t_1)^\alpha \sum_{l=0}^\infty (l+1)^p \, l^{-m\alpha} \quad = \quad C' \, (t_2 - t_1)^{\varepsilon(p-m\alpha)+\alpha} \, .$$

Since $\varepsilon(p - m\alpha) + \alpha > 0$, our Theorem A yields the assertion. □

3. The Gaussian kernel estimate (GE_m)

In this section, we give some examples of spaces E and operator semigroups $(T_t)_{t\in\mathbb{R}_+}$ on $C_0(E)$ satisfying the main hypothesis of our results, i.e. the Gaussian kernel estimate (GE_m). Since in our examples the kernels $k_t : E \times \mathcal{B}(E) \to \mathbb{C}$ have densities $p_t : E \times E \to \mathbb{C}$ [with respect to a given reference measure μ] satisfying pointwise Gaussian estimates of some order m, we want to make the obvious remark that the latter imply (GE_m) provided μ is doubling.

Remark 3.1. Let (E, d, μ) be a metric measure space of dimension $D \ge 0$, i.e.

$$\mu(B(x, \lambda r)) \quad \le \quad C_1 \lambda^D \, \mu(B(x, r)) \ \text{ for all } \ x \in E \, , \ r > 0 \, , \ \lambda \ge 1 \, .$$

Let $p \in C(E^2)$ satisfy for some $r > 0$

$$|p(x, y)| \le \mu(B(x, r))^{-1} \, g(\tfrac{d(x,y)}{r}) \quad \text{for all } \ x, y \in E \, ,$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function. Then

$$\int_{B(x,\lambda r)^c} |p(x, y)| \, d\mu(y) \le C_1 \sum_{k=0}^\infty (k + \lambda + 1)^D \, g(k + \lambda) \ \text{ for all } \ x \in E \, , \ \lambda \ge 0 \, .$$

Proof. Denoting $A(x, k, r) := B(x, (k+1)r)^c \setminus B(x, kr)$, we have

$$\int_{B(x,\lambda r)^c} |p(x, y)| \, d\mu(y) \le \mu(B(x, r))^{-1} \sum_{k=0}^\infty \int_{A(x,k+\lambda,r)} g(\tfrac{d(x,y)}{r}) \, d\mu(y)$$

$$\le \mu(B(x, r))^{-1} \sum_{k=0}^\infty \mu(B(x, (k + \lambda + 1)r)) \; g(k + \lambda)$$

$$\le C_1 \sum_{k=0}^\infty (k + \lambda + 1)^D \, g(k + \lambda).$$

□

3.1. Feller semigroups

The examples in this first part are given in terms of Feller processes $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in E}$ satisfying $(GE_m^{\mathbb{P}})$. Then the associated Feller semigroup $(T_t)_{t\in\mathbb{R}_+}$ on $C_0(E)$ satisfies (GE_m) since $|k_t(x,\cdot)| = \mathbb{P}^x\,[\,X_t \in \cdot\,]$.

Example 1. Diffusions on fractals of walk dimension m
This is due to Barlow and Bass [BB] . We consider a class of fractal subsets of $\mathbb{R}^D$ $(D \geq 2)$ formed by the following generalization of the construction of the Cantor ternary set. Let $F_0 = [0,1]^D$ and $l_F \in \mathbb{N}_{\geq 3}$. Divide F_0 into l_F^D equal subcubes, remove a symmetric pattern of subcubes from F_0 and call what remains F_1 . Now repeat this procedure: divide each subcube of F_1 into l_f^D equal parts, remove the same symmetric pattern from each and call what remains F_2 . Continuing in this way we obtain a decreasing sequence (F_n) of closed subsets of $[0,1]^D$. We consider the so-called unbounded (generalized) Sierpinski carpet $E := \bigcup_{n\in\mathbb{N}} l_F^n F$, where $F := \bigcap_{n\in\mathbb{N}} F_n$, equipped with the Euclidean metric. The following result is proved in [BB, Thm. 5.7 , Prop. 5.8] .

Theorem. *There exist a nondegenerate E-valued Feller process* $(\Omega, \mathbb{P}^x, (X_t)_{t\in\mathbb{R}_+})_{x\in E}$ *and $m \geq 2$ such that for all $x \in E$, $t, \lambda \in \mathbb{R}_+$*

$$\mathbb{P}^x\,[\,X_t \in B(x, \lambda t^{1/m})^c\,] \quad \leq \quad g_m(\lambda)\ .$$

Since $(X_t)_{t\in\mathbb{R}_+}$ is locally isotropic, it is called a Brownian motion on E. The parameter m is called the walk dimension. Notice that the transition densities $p_t(x,y)$ [with respect to Hausdorff measure] even satisfy pointwise Gaussian (upper and lower) estimates of order m [BB, Thm. 1.3] .

Example 2. Brownian motion on Riemannian manifolds
Let M be a geodesically complete Riemannian manifold, equipped with the geodesic distance d and the Riemannian measure μ. Assume that μ is doubling, that the heat semigroup $(T_t)_{t\in\mathbb{R}_+}$ has the conservation property and that the heat kernel $p_t \in C(M^2)$ satisfies

$$p_t(x,x) \;\leq\; C\,\mu(B(x,\sqrt{t}))^{-1} \;\text{ for all }\; x \in M\,,\; t > 0\,.$$

Then, by a result of Grigor'yan [G], the heat kernel satisfies even pointwise Gaussian estimates of order 2 :

$$p_t(x,y) \;\leq\; \mu(B(x,\sqrt{t}))^{-1}\, g_2\big(\tfrac{d(x,y)}{\sqrt{t}}\big) \;\text{ for all }\; x,y \in M\,,\; t > 0\ .$$

Hence, due to Proposition 1.6 (observe that all balls in M are relatively compact by geodesical completeness) and Remark 3.1, $(T_t)_{t\in\mathbb{R}_+}$ is a positive contractive semigroup on $C_0(M)$. Moreover, $(T_t)_{t\in\mathbb{R}_+}$ is strongly continuous on $C_0(M)$ by Proposition 1.5 and Remark 3.1. In other words, $(T_t)_{t\in\mathbb{R}_+}$ is a Feller semigroup with the conservation property, and the corresponding M-valued Feller process (cf. Remark 1.4) is called a Brownian motion on M. Finally, (GE_2) holds again by Remark 3.1, hence $(GE_2^{\mathbb{P}})$ holds as well.

3.2. Arbitrary semigroups

In this second part we give two classes of non-Feller semigroups satisfying (GE_m).

Example 3. Elliptic operators on $\mathbb{R}^D$ of order $m \geq D$
These operators A are given by forms $\mathfrak{a} : H^k(\mathbb{R}^D) \times H^k(\mathbb{R}^D) \to \mathbb{C}$ of the type

$$\mathfrak{a}(u, v) = \int_{\mathbb{R}^D} \sum_{|\alpha|=|\beta|=k} a_{\alpha,\beta}\, \partial^\alpha u\, \overline{\partial^\beta v}\, dx \quad ,$$

where we assume $a_{\alpha,\beta} \in L_\infty(\mathbb{R}^D; \mathbb{C})$ for all α, β and Garding's inequality

$$Re\, \mathfrak{a}(u, u) \quad \geq \quad \varepsilon\, \|\nabla^k u\|_2^2 \ \text{ for all } \ u \in H^k(\mathbb{R}^D)\ ,$$

for some $\varepsilon > 0$ and $\|\nabla^k u\|_2^2 := \sum_{|\alpha|=k} \|\partial^\alpha u\|_2^2$. Then $\mathfrak{a}$ is a closed sectorial form, and the associated operator A on $L_2(\mathbb{R}^D)$ is given by

$$u \in D(A) \text{ and } Au = g \quad \Longleftrightarrow \quad u \in H^k(\mathbb{R}^D) \text{ and } \langle g, v\rangle = \mathfrak{a}(u, v) \text{ for all } v \in H^k.$$

This operator A is the generator of a C_0 semigroup $(T_t)_{t\in\mathbb{R}_+}$ on $L_2(\mathbb{R}^D)$. Suppose that $m := 2k \geq D$. Then, by results of Davies [D, § 6] and Auscher&Tchamitchian [AT, § 1.7], for all $t > 0$, there exists $p_t \in C(\mathbb{R}^D \times \mathbb{R}^D)$ such that

$$T_t f(x) \;=\; \int_E p_t(x, y) f(y)\, d\mu(y) \quad \text{for all } \ f \in L_2 \cap L_\infty\,,\ \text{a.e. } x \in E \ ,\tag{17}$$

$$|p_t(x, y)| \;\leq\; t^{-D/m}\, g_m\!\left(\tfrac{d(x,y)}{t^{1/m}}\right) \quad \text{for all } \ x, y \in \mathbb{R}^D\ .\tag{18}$$

Hence, by Remark 3.1 and Proposition 1.6, $(T_t)_{t\in\mathbb{R}_+}$ acts as an operator semigroup on $C_0(\mathbb{R}^D)$ satisfying (GE_m). It is well-known that $(T_t)_{t\in\mathbb{R}_+}$ is Markovian if $m = 2$ and the coefficients $a_{\alpha,\beta}$ are real-valued.

Example 4. Schrödinger operators with singular potentials on $\mathbb{R}^D$
Let $A = \Delta - V$ be a Schrödinger operator on $L_2(\mathbb{R}^D)$, $D \geq 3$, where we assume the $\mathbb{R}$-valued potential V to be in the 'global Kato class'. Then A is the generator of a C_0 semigroup $(T_t)_{t\in\mathbb{R}_+}$ on $L_2(\mathbb{R}^D)$, and for all $t > 0$, there exists $p_t \in C(\mathbb{R}^D \times \mathbb{R}^D)$ such that (17) and (18) hold for $m = 2$. Hence, again by Remark 3.1 and Proposition 1.6, $(T_t)_{t\in\mathbb{R}_+}$ acts as an operator semigroup on $C_0(\mathbb{R}^D)$ satisfying (GE_2). If V_- is unbounded then, for no $\omega \in \mathbb{R}_+$, the semigroup $(e^{-t\omega} T_t)_{t\in\mathbb{R}_+}$ is contractive on $C_0(\mathbb{R}^D)$. The reference for this example is Simon's paper [Si].

4. Proofs of the path regularity criteria

For the proof of the standard Kolmogorov-Čentsov Theorem on Hölder-regularity of processes $(X_t)_{t\in[0,1]}$ and for the proof of our version on δ-local Hölder-regularity of processes $(X_t)_{t\in\mathbb{R}_+}$ (see Theorems A and 2.4), we need a criterion for functions $f : D \to E$ to have a Hölder-continuous extension to $\overline{D} = [0, 1]$ which takes into account only the behaviour of f on the finite sets D_n. The following lemma can be extracted from the usual proof of the Kolmogorov-Čentsov Theorem as given e.g. in [KS, p. 54].

Lemma 4.1. *Let $f : D \to E$ be a function and $M_n := \max\{ d(f(s), f(t))\,;\ s, t \in D_n, t - s = 2^{-n} \}$. Let $\gamma > 0$ and $m \in \mathbb{N}$. Then*

$$\sup_{\substack{s,t\in D\\ |t-s|\le 2^{-m}}} d(f(s), f(t)) \le 2 \sum_{n=m}^{\infty} M_n \ .$$

$$\sup_{\substack{s,t\in D\\ 0<|t-s|\le 2^{-m}}} d(f(s), f(t))\, |t-s|^{-\gamma} \le 2^{1+\gamma} \sup_{j\ge m} 2^{\gamma j} \sum_{n=j}^{\infty} M_n \quad =: \quad S \ .$$

Moreover, if $S < \infty$ then f has a continuous extension to $\overline{D} = [0, 1]$ satisfying

$$\sup_{\substack{s,t\in[0,1]\\ 0<|t-s|\le 2^{-m}}} d(f(s), f(t))\, |t-s|^{-\gamma} \le S \ .$$

Proof. For the proof of the first assertion, it suffices to show for all $j \ge m$:

$$\max_{\substack{s,t\in D_j\\ |t-s|\le 2^{-m}}} d(f(s), f(t)) \le 2 \sum_{n=m}^{j} M_n \ .$$

This is obvious for $j = m$ since $s, t \in D_m$ and $|t - s| \le 2^{-m}$ implies $|t - s| \in \{0, 2^{-m}\}$. So let $j > m$ and $s, t \in D_j$ with $|t - s| \le 2^{-m}$. We can assume $s \le t$ and set $s' := \min\{u \in D_{j-1}\,;\, u \ge s\}$, $t' := \max\{u \in D_{j-1}\,;\, u \le t\}$. Then $|t' - s'| \le 2^{-m}$ and $s' - s, t - t' \in \{0, 2^{-j}\}$, hence

$$\begin{aligned} d(f(s), f(t)) &\le d(f(s), f(s')) + d(f(s'), f(t')) + d(f(t'), f(t)) \\ &\le M_j + 2\sum_{n=m}^{j-1} M_n + M_j = 2\sum_{n=m}^{j} M_n \ . \end{aligned}$$

The second assertion is seen as follows by using the first assertion:

$$\begin{aligned} \sup_{\substack{s,t\in D\\ 0<|t-s|\le 2^{-m}}} d(f(s), f(t))\, |t-s|^{-\gamma} &= \sup_{j\ge m} \sup_{\substack{s,t\in D\\ 2^{-(j+1)}<|t-s|\le 2^{-j}}} d(f(s), f(t))\, |t-s|^{-\gamma} \\ &\le 2^{1+\gamma} \sup_{j\ge m} 2^{\gamma j} \sum_{n=j}^{\infty} M_n \ . \end{aligned}$$

Now assume $S < \infty$. Given $t \in [0, 1]$ and $D \ni t_n \to t$, the second assertion shows that $(f(t_n))$ is a Cauchy sequence in E and that $f(t) := \lim_n f(t_n)$ yields a (well-defined) extension to $[0, 1]$. Moreover, if $D \ni t_n \to t$ and $D \ni s_n \to s$, $s \ne t$, then one obtains $d(f(s), f(t)) \le S\,|t - s|^{\gamma}$ from the second assertion and

$$d(f(s), f(t)) \quad \le \quad d(f(s), f(s_n)) + d(f(s_n), f(t_n)) + d(f(t_n), f(t)). \square$$

The following lemma is a simple induction over $\lceil \frac{|t-s|}{\delta} \rceil$.

Lemma 4.2. *Let $f : [a,b] \to E$ be a function and $\delta > 0$, $\gamma \geq 0$, $\lambda \geq 1$. Then*

$$\sup_{\substack{s,t\in[a,b]\\0<|t-s|\leq\lambda\delta}} d(f(s), f(t))\, |t-s|^{-\gamma} \leq (2[\lambda \wedge \tfrac{b-a}{\delta}] + 1) \sup_{\substack{s,t\in[a,b]\\0<|t-s|\leq\delta}} d(f(s), f(t))\, |t-s|^{-\gamma} .$$

The next result follows directly from Lemma 4.1 and Lemma 4.2 .

Corollary 4.3. *Let $f : D \to E$ be a function and $M_n := \max\{ d(f(s), f(t))$; $s,t \in D_n, t-s = 2^{-n}\}$. Let $\gamma \in (0,1)$, $C_1 > 0$ and $m \in \mathbb{N}$ such that*

$$M_n \leq C_1 2^{-\gamma n} \text{ for all } n \geq m .$$

Then f has a continuous extension to $\overline{D} = [0,1]$ satisfying

$$\sup_{\substack{s,t\in[0,1]\\0<|t-s|\leq 2^{-m}}} d(f(s), f(t))\, |t-s|^{-\gamma} \leq C_1 \frac{2^{1+\gamma}}{1-2^{-\gamma}} .$$

In particular, f is γ-Hölder-continuous on $[0,1]$ and

$$2^{\gamma m} \| f \|_{Osc([0,1];E)} \vee \| f \|_{C^{\gamma}([0,1];E)} \leq C_1 \frac{2^{1+\gamma}}{1-2^{-\gamma}} (2^{m+1}+1) .$$

Now we give a similar criterion for functions on $D^{\infty} := \{j2^{-n};\, j,n \in \mathbb{N}\}$ to have a δ-locally γ-Hölder-continuous extension to $\overline{D^{\infty}} = \mathbb{R}_+$.

Corollary 4.4. *Let $f : D^{\infty} \to E$ be a function and $M_n := \max\{ d(f(s), f(t))$; $s,t \in \widetilde{D}_n, t-s = 2^{-n}\}$, where $\widetilde{D}_n := \{j2^{-n};\, j = 0,\dots,N_n\}$, $N_n := \lceil \delta^{-1}(2^{-(n+1)})2^n \rceil + 2^n$ and $\delta \in C(\mathbb{R}_+, \mathbb{R}_{>0})$ is strictly decreasing with $\delta(\infty) = 0$. Let $\gamma \in (0,1)$, $C > 0$ and $m \in \mathbb{N}$ such that*

$$M_n \leq C\, 2^{-\gamma n} \text{ for all } n \geq m .$$

Then f has a δ-locally γ-Hölder-continuous extension to $\overline{D^{\infty}} = \mathbb{R}_+$ satisfying

$$\sup_{T\geq 1} T^{-1}\, \delta(2T)^{1-\gamma} \| f \|_{Osc([0,T];E)} < \infty .$$

Proof. For all $n \in \mathbb{N}$, denote $D^{\infty}_n := \{j2^{-n};\, j \in \mathbb{N}\}$. The idea of the proof is to apply Corollary 4.3 for all $a \in D^{\infty}_m$ to

$$f_a : D \to E ,\ t \mapsto f(a+t) \quad \text{and} \quad m_a := \lfloor \log_2(\delta(a)^{-1}) \rfloor \vee m .$$

We have $a + D_n \subset \widetilde{D}_n$ for all $a \in D^{\infty}_m, n \geq m_a$ since $D^{\infty}_m + D_n \subset D^{\infty}_n$ and

$$a+1 \leq \delta^{-1}(2^{-(n+1)}) + 1 \leq N_n 2^{-n}$$

by defintion of m_a and N_n. We deduce for all $a \in D_m^\infty$ and $n \geq m_a$:

$$\begin{aligned} M_n(f_a) &:= \max\{\, d(f_a(s), f_a(t))\,;\; s,t \in D_n\,,\; t-s = 2^{-n}\,\} \\ &= \max\{\, d(f(s), f(t))\,;\; s,t \in a + D_n\,,\; t-s = 2^{-n}\,\} \\ &\leq M_n \quad \leq \quad C\, 2^{-\gamma n} \;\text{ by hypothesis}\,. \end{aligned}$$

Hence, by Corollary 4.3, each f_a has a continuous extension to [0, 1] satisfying

$$\sup_{\substack{s,t\in[0,1] \\ 0<|t-s|\leq 2^{-m_a}}} d(f_a(s), f_a(t))\, |t-s|^{-\gamma} \quad \leq \quad C\, \frac{2^{1+\gamma}}{1-2^{-\gamma}} \quad =: \quad C_1\,.$$

In other words, f has a continuous extension to $\mathbb{R}_+$ satisfying

$$\sup_{\substack{s,t\in[a,a+1] \\ 0<|t-s|\leq 2^{-m_a}}} d(f(s), f(t))\, |t-s|^{-\gamma} \quad \leq \quad C_1 \quad \text{for all } \; a \in D_m^\infty\,. \tag{19}$$

Now we obtain $f \in C^\gamma_{\delta-loc}(\mathbb{R}_+; E)$ as follows. Let $s, t \in \mathbb{R}_+$ such that $0 < t - s \leq \delta(s)$. By using Lemma 4.2, we can assume $t - s \leq 2^{-m} \wedge \delta(s)$. There exists $a \in D_m^\infty \cap [s-2^{-m}, s]$. Then $s, t \in [a, a+1]$ since $a \leq s \leq t = s+(t-s) \leq a + 2^{-m} + 2^{-m} \leq a + 1$ and

$$0 \;<\; |t-s| \;\leq\; 2^{-m} \wedge \delta(s) \;\leq\; 2^{-m} \wedge \delta(a) \;\leq\; 2^{-m_a}\,.$$

Hence $d(f(s), f(t))\, |t-s|^{-\gamma} \leq C_1$ by (19) and $f \in C^\gamma_{\delta-loc}(\mathbb{R}_+; E)$ is proved. For the proof of the second assertion, it suffices to show for all $k \in \mathbb{N}$:

$$\|\, f\,\|_{Osc([0,2^k];E)} \quad \leq \quad 3C_1\, 2^k\, (2^m \vee \delta(2^k)^{-1})^{1-\gamma}\,. \tag{20}$$

Indeed, then we obtain for all $T \geq 1$, letting $k := \lfloor \log_2 T \rfloor$:

$$\begin{aligned} \|\, f\,\|_{Osc([0,T];E)} &\leq\; 3C_1\, 2^{k+1}\, (2^m \vee \delta(2^{k+1})^{-1})^{1-\gamma} \\ &\leq\; 6C_1\, T\, (2^m \vee \delta(2T)^{-1})^{1-\gamma}\,. \end{aligned}$$

Similarly as before, line (20) is proved by applying Corollary 4.3 for all $k \in \mathbb{N}$ to

$$f_k : D \to E\;,\; t \mapsto f(2^k t) \quad \text{ and } \quad m_k := k + (\lfloor \log_2(\delta(2^k)^{-1}) \rfloor \vee m)\,.$$

We have $2^k D_n \subset \widetilde{D}_{n-k}$ for all $k \in \mathbb{N}$, $n \geq m_k$ since $2^k D_n \subset D_{n-k}^\infty$ and

$$2^k \;\leq\; \delta^{-1}(2^{-(n-k+1)}) \;\leq\; N_{n-k}\, 2^{k-n}$$

by defintion of m_k and N_{k-n}. We deduce for all $k \in \mathbb{N}$ and $n \geq m_k$:

$$\begin{aligned} M_n(f_k) &:= \max\{\, d(f_k(s), f_k(t))\,;\; s,t \in D_n\,,\; t-s = 2^{-n}\,\} \\ &= \max\{\, d(f(s), f(t))\,;\; s,t \in 2^k D_n\,,\; t-s = 2^{k-n}\,\} \\ &\leq M_{n-k} \quad \leq \quad C\, 2^{\gamma(k-n)} \;\text{ by hypothesis}\,. \end{aligned}$$

Hence, by Corollary 4.3, we obtain (20) as follows:

$$\| f \|_{Osc([0,2^k];E)} = \| f_k \|_{Osc([0,1];E)} \le 2^{-\gamma m_k} \, 2^{\gamma k} C_1 \, (2^{m_k+1} + 1)$$

$$\le 3C_1 \, 2^k \, (2^m \vee \delta(2^k)^{-1})^{1-\gamma}. \qquad \square$$

Proof of Theorem A. Denoting $M_n := \max\{d(X_s, X_t)\,;\ s, t \in D_n, t - s = 2^{-n}\}$, we obtain from the hypotheses

$$\sum_{n=0}^{\infty} \mathbb{P}[\, M_n \ge 2^{-\gamma n}\,] = \sum_{n=0}^{\infty} \mu_{D_n}[\,\mathrm{Var}_{\infty}^{(2^n)} \ge 2^{-\gamma n}\,] \quad < \quad \infty\,.$$

Hence, by the Borel-Cantelli lemma, we have $\mathbb{P}(\mathcal{N}) = 0$ for $\mathcal{N} := \limsup_{n\to\infty} [\, M_n \ge 2^{-\gamma n}\,]$. For all $\omega \in \mathcal{N}^c$, there exists $m(\omega) \in \mathbb{N}$ such that $M_n(\omega) \le 2^{-\gamma n}$ for all $n \ge m(\omega)$. Thus, by Corollary 4.3, the restriction of $X.(\omega)$ to D has a γ-Hölder continuous extension $\widetilde{X}.(\omega)$ to $[0, 1]$.

It remains to show that $(\widetilde{X}_t)$ is a modification of (X_t). Let $t \in [0, 1]$ and $D \ni t_n \to t,\ t_n \ne t$. Then $X_t = \widetilde{X}_t$ a.s. follows from $d(X_{t_n}, \widetilde{X}_t) \to 0$ on $\mathcal{N}^c$ and

$$\mathbb{E}(\, d(X_{t_n}, X_t)^p\,) = \| \,\mathrm{Var}_{\infty}^{(1)}\, \|^{\mathrm{p}}_{\mathrm{L_p}(\mu_{(t_n \wedge t, t_n \vee t)})} \quad \to \quad 0. \qquad \square$$

Proof of Theorem 2.4. Denoting $M_n := \max\{d(X_s, X_t)\,;\ s, t \in \widetilde{D}_n, t - s = 2^{-n}\}$, we obtain from the hypotheses

$$\sum_{n=0}^{\infty} \mathbb{P}[\, M_n \ge 2^{-\gamma n}\,] = \sum_{n=0}^{\infty} \mu_{\widetilde{D}_n}[\,\mathrm{Var}_{\infty}^{(N_n)} \ge 2^{-\gamma n}\,] \quad < \quad \infty\,.$$

Hence, by the Borel-Cantelli lemma, we have $\mathbb{P}(\mathcal{N}) = 0$ for $\mathcal{N} := \limsup_{n\to\infty} [\, M_n \ge 2^{-\gamma n}\,]$. For all $\omega \in \mathcal{N}^c$, there exists $m(\omega) \in \mathbb{N}$ such that $M_n(\omega) \le 2^{-\gamma n}$ for all $n \ge m(\omega)$. Thus, by Corollary 4.4, the restriction of $X.(\omega)$ to D^∞ has a δ-locally γ-Hölder-continuous extension $\widetilde{X}.(\omega)$ to $\mathbb{R}_+$ satisfying

$$\sup_{T\ge 1}\ T^{-1}\, \delta(2T)^{1-\gamma}\, \| \,\widetilde{X}.(\omega)\, \|_{Osc([0,T];E)} < \infty\,.$$

The proof that $(\widetilde{X}_t)$ is a modification of (X_t) is as before. $\square$

Proof of Theorem 2.9. Let $t_o \in \mathbb{R}_+$. Denoting $M_{t_o,n} := \max\{d(X_s, X_t)\,;\ s, t \in t_o + D_n, t - s = 2^{-n}\}$, we have by right-continuity of X and Lemma 4.1(b):

$$M_{t_o} := \sup_{\substack{s,t \in t_o+[0,1] \\ s\ne t}} d(X_s, X_t)\, |t-s|^{-\gamma} \quad = \quad \sup_{\substack{s,t \in t_o+D \\ s\ne t}} d(X_s, X_t)\, |t-s|^{-\gamma}$$

$$\le\ 2 \sup_{m\in\mathbb{N}} 2^{\gamma(m+1)} \sum_{n=m}^{\infty} M_{t_o,n} \quad \le \quad 2^{1+\gamma} \sum_{n=0}^{\infty} 2^{\gamma n}\, M_{t_o,n}\,.$$

Since $\mathbb{E}(M_{t_o,n}^q) = \| \,\mathrm{Var}_{\infty}^{(2^n)}\, \|^{\mathrm{q}}_{\mathrm{L_q}(\mu_{t_o+D_n})}$, we deduce

$$\mathbb{E}\Big(M_{t_o}^q\Big)^{\frac{1}{q}} \quad \le \quad 2^{1+\gamma} \sum_{n=0}^{\infty} 2^{\gamma n}\, \mathbb{E}\Big(M_{t_o,n}^q\Big)^{\frac{1}{q}} \quad \le \quad 2^{1+\gamma} C_1\,.$$

This yields, writing $\psi(r) := r^{-\gamma}$ and $L_\infty(\psi) := L_\infty((0, \frac{1}{2}], \psi)$:

$$\mathbb{E}\Big(\| r \mapsto \sup_{|h|\le r} |\Delta_h X_{t_o}| \|^q_{L_\infty(\psi)} \Big) = \mathbb{E}\Big(\sup_{0<r\le\frac{1}{2}} r^{-\gamma q} \sup_{|h|\le r} d(X_{t_o+h}, X_{t_o})^q \Big)$$

$$\le \mathbb{E}\Big(\sup_{\substack{s,t\in(t_o-\frac{1}{2})_+ +[0,1] \\ s\neq t}} d(X_s, X_t)^q \, |t-s|^{-\gamma q} \Big)$$

$$= \mathbb{E}\Big(M^q_{(t_o-\frac{1}{2})_+} \Big) \quad \le \quad 2^{(1+\gamma)q}\, C_1^q \,.$$

We extend $(X_t)_{t\in\mathbb{R}_+}$ to $\mathbb{R}$ by setting $X_{-t} := X_0$ for all $t \in \mathbb{R}_+$. Denoting $L_p(\phi) = L_p(\mathbb{R}, \phi)$, one obtains the assertion as follows:

$$\mathbb{E}\Big(\Big\| \sup_{|h|\le\cdot} \| \, |\Delta_h X| \, \|_{L_p(\phi)} \Big\|^q_{L_\infty(\psi)} \Big)$$

$$\le \| \phi \|_p^{q-p} \, \mathbb{E}\Big(\int_{\mathbb{R}} \| \sup_{|h|\le\cdot} |\Delta_h X_t| \, \|^q_{L_\infty(\psi)} \, \phi(t)^p \, dt \Big)$$

$$= \| \phi \|_p^{q-p} \int_{\mathbb{R}} \mathbb{E}\Big(\| \sup_{|h|\le\cdot} |\Delta_h X_t| \, \|^p_{L_\infty(\psi)} \Big) \phi(t)^p \, dt$$

$$\le 2^{(1+\gamma)q}\, C_1^q \, \| \phi \|_p^q. \qquad \square$$

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