Probab. Theory Relat. Fields 133, 145–174 (2005)
Digital Object Identifier (DOI) 10.1007/s00440-004-0413-4

Krzysztof Burdzy · Leonid Mytnik

Super-Brownian motion with reflecting historical paths. II. Convergence of approximations

Received: 9 September 2002 / Revised version: 31 October 2004 /
Published online: 10 February 2005 –

Abstract. We prove that the sequence of finite reflecting branching Brownian motion forests defined by Burdzy and Le Gall ([1]) converges in probability to the "super-Brownian motion with reflecting historical paths." This solves an open problem posed in [1], where only tightness was proved for the sequence of approximations. Several results on path behavior were proved in [1] for all subsequential limits–they obviously hold for the unique limit found in the present paper.

1. Introduction

The goal of this paper is to complete the main stage of a research project started by Burdzy and Le Gall in [1]. The authors of that paper set out to define and study a "super-Brownian motion with reflecting historical paths." They constructed a sequence of branching particle systems which they believed converged to a limit representing, in the intuitive sense, the process named above. However, the main result of [1] proves only tightness for the sequence of approximations. This will be remedied in this paper—we will prove the convergence. Moreover, we will construct a sequence of approximations which converges not only in the sense of distribution but also in probability. This will allow us to complete the definition of a "super-Brownian motion with reflecting historical paths" in the sense that our main result will identify a single probability distribution on an appropriate space. A rigorous statement of our main result is given as Theorem 2.1 in Section 2—it requires a fair amount of notation.

We would like to mention that some path properties have been already established for the super-Brownian motion with reflecting historical paths in [1]. The properties have been proved for every subsequential limit of the approximating

K. Burdzy: Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98115-4350, USA. e-mail: burdzy@math.washington.edu

L. Mytnik: Faculty of Industrial Engineering and Management, Technion – Israel Institute of Technology, Haifa 32000, Israel. e-mail: leonid@ie.technion.ac.il

Mathematics Subject Classification (2000): Primary 60H15, Secondary 35R60

Key words or phrases: Super-Brownian motion – Reflecting paths – Brownian snake – Martingale problem

Supported in part by NSF Grant DMS-0071486, Israel Science Foundation Grants 12/98 and 116/01 - 10.0, and the U.S.-Israel Binational Science Foundation (grant No. 2000065).

sequence of branching particle systems, so those theorems obviously apply to the process constructed in this paper.

Our approach is closely related to and heavily dependent on techniques developed in [1], and that paper in turn uses many tools presented in [7], in particular the "Brownian snake" originally introduced by Le Gall. For an introduction to the theory of superprocesses see, for example, [4].

We will now describe our main ideas. Just as in [1], we start with a sequence of branching particle systems approximating the historical super-Brownian motion. The accuracy of the approximation is determined by a parameter $\epsilon > 0$. The sequence of approximations is consistent in the sense that ϵ-particle system may be obtained from a δ-particle system by pruning some branches, for any $\delta < \epsilon$. Then paths of every ϵ-particle system are relabelled so that the paths in the new system are reflecting. The problem with this approach is that the reflecting systems are no longer consistent, i.e., after relabelling, the distance (in an appropriate metric) between reflecting ϵ-particle system and reflecting δ-particle system may be large, at least for some ω. To tackle this problem, we will first consider two reflecting super-Brownian motions. In this model, particles of one system reflect with the particles of the other system, and since the model is non-historical, reflections between particles of the same system are irrelevant to the evolution of two reflecting super-Brownian motions. Intuitively speaking, the "number" of reflections is much smaller in this model than in the fully reflecting historical super-Brownian motion. This allows us to prove convergence of appropriate finite particle system approximations in this model. The result generalizes to any finite number of reflecting super-Brownian motions (again, in the non-historical setting reflections between the particles of the same particle system are irrelevant). Finally, the super-Brownian motion with reflecting historical paths is obtained as the limit of families of reflecting historical Brownian motions, with larger and larger numbers of particle systems in the families.

Section 2 contains the construction of a finite particle system and the statements of our main results. Section 3 is a review of some relevant facts on local times and excursions. Section 4 presents a construction of a pair ("coupling") of finite particle systems reflecting with each other. It also contains the proof of convergence of such approximations. The main result of this paper is proved in Section 5.

Notation. We will adopt the following notation conventions.
Let E be a Polish space.
$M_F(E)$ (M_F) — finite measures on E (on $\mathbb{R}$).
$D_E[a, b)$ (D_E) — Skorohod space of cadlag E-valued paths on $[a, b)$ (on $[0, \infty)$).
$C_E[a, b)$ (C_E) — the space of continuous E-valued paths on $[a, b)$ (on $[0, \infty)$).
$C_u(E)$ — the space of bounded, real-valued, uniformly continuous functions on E.
$B(E)$ — the space of bounded Borel measurable real-valued functions on E.
$\langle \mu, f\rangle = \int f d\mu$ for any measure μ and function f.

We will try to use as much as possible of the notation from [1] to help the reader follow our arguments, as we will often refer to that paper.

2. Finite branching particle systems and the statement of main results

We will be brief in our presentation of super-Brownian motion, historical processes, finite particle systems, Brownian snake, etc. The reader is asked to consult [1] and [7] for more details.

First we are going to introduce the historical super-Brownian motion based on Le Gall's Brownian snake construction. Consider $\mu \in M_F$ and assume, to simplify the proofs, that it has a compact support and that it is absolutely continuous with respect to the Lebesgue measure. Let $\beta/2$ be the reflecting Brownian motion, i.e., $\{\beta_s, s \geq 0\} \stackrel{\text{dist}}{=} \{2|B_s|, s \geq 0\}$, stopped at time $\tau = \inf\{t : L_t^0 = \langle \mu, 1 \rangle\}$, where B_s is the standard Brownian motion with $B_0 = 0$ and $(L_s^x, x \geq 0, s \geq 0)$ denotes the jointly continuous family of local times of β normalized in such a way that, for every nonnegative Borel function ϕ on $\mathbb{R}_+$ and $t > 0$,

$$\int_0^t \phi(\beta_s)\, ds = \int_{\mathbb{R}_+} \phi(x) L_t^x\, dx.$$

Let $\{W_s\,, s \geq 0\}$ be the Brownian snake driven by the process β and such that

$$\mu = \int_0^\tau dL_s^0\, \delta_{W_s(0)}\,. \tag{2.1}$$

Recall that for any fixed $s \geq 0$, $\{W_s(t), t \geq 0\}$ is a Brownian motion stopped at time β_s. In usual definitions of the Brownian snake, $W_s(0)$ is a constant. The Brownian snake satisfying (2.1) can be obtained as the limit of processes W^ϵ — discrete snakes constructed in Section 2.4 of [1]. The existence of such a process W is stated at the beginning of Section 5.3 of [1]. We can also obtain W by concatenating a Poisson point process of Brownian snake excursions with intensity $\int \mu(dy)\mathbb{N}_y$, in the notation of Theorem IV.4 of Le Gall [7] (see that theorem for more details). The historical super-Brownian motion connected to W is defined in Section 5.3 of [1] via the formula

$$Y_t = \int_0^\tau dL_s^t\, \delta_{W_s}\,.$$

The corresponding M_F-valued process (super-Brownian motion) is defined by

$$X_t = \int_0^\tau dL_s^t \delta_{W_s(t)}\,.$$

Next we will define finite branching particle systems approximating the historical super-Brownian motion. Fix an arbitrary $\epsilon > 0$. For any $s \geq 0$, $t \geq 2\epsilon$, let $W_s^{t-2\epsilon}$ be the path of the Brownian snake with index s stopped at time $t - 2\epsilon$. Let

$$H_t^\epsilon \equiv \int_0^\tau dL_s^t\, \delta_{W_s^{t-2\epsilon}}\,, \quad t \geq 2\epsilon.$$

In other words, H_t^ϵ is the family of trajectories of those particles whose descendants are alive at time t, but the paths in H_t^ϵ are stopped at time $t - 2\epsilon$. The measure

H_t^ϵ is purely atomic. Note that, by definition, any atom $y(\cdot)$ of H_t^ϵ is a path whose values are defined for *all* times, and

$$y(s) = y(t - 2\epsilon), \quad \forall s \geq t - 2\epsilon. \tag{2.2}$$

Next we redefine the masses of these atoms—we give mass ϵ to each atom of H_t^ϵ for any $t \geq 2\epsilon$. The resulting measure-valued process will be denoted by $\{Y_t^\epsilon, t \geq 2\epsilon\}$. Note that in the original superprocess setting, for any $t \geq 2\epsilon$, Y_t^ϵ records positions and historical paths (up to time $t - 2\epsilon$) of the particles having descendants at time t. On the other hand, Y^ϵ represents binary branching historical Brownian motions with branching rate ϵ^{-1} such that, for any $t \geq 2\epsilon$, Y_t^ϵ records the position of this system at time $t - 2\epsilon$. The difference of 2ϵ between the time of the original superprocess and the time of corresponding ϵ-particle system is counterintuitive but it is actually meant to simplify some proofs in the last part of the paper. We will frequently use the following "convention" in verbal descriptions of the process Y^ϵ and analogous processes introduced later on.

Convention 1. For any $t \geq 2\epsilon$, Y_t^ϵ records historical paths of the particles of ϵ-system which are alive at time $t - 2\epsilon$. Sometimes, with a little abuse of notation, we will also identify the measure-valued process with the particle system whose evolution it records.

The initial positions of the particles are distributed according to the Poisson measure on $\mathbb{R}$ with intensity $\epsilon^{-1}\mu$ (see Proposition 3.5 of [2]). The M_F-valued process corresponding to Y^ϵ is defined by

$$\langle X_t^\epsilon, \phi \rangle = \int \phi(y(t)) Y_t^\epsilon(dy), \quad \forall \phi \in C(\mathbb{R}),$$

where $y(t) = y(t - 2\epsilon)$, according to (2.2).

It was explained in Section 2.2 of [1] how any finite branching particle system can be coded as a marked forest consisting of the set $\mathcal{T}^\epsilon$ of edges (i.e., particles) which is a subset of

$$\{1, 2, \dots\} \times \bigcup_{n=0}^{\infty} \{1, 2\}^n,$$

and the family $(l_u^\epsilon, u \in \mathcal{T}^\epsilon)$ of lengths of edges (i.e., lifetimes of particles). Let $(\mathcal{T}^\epsilon, (l_u^\epsilon, u \in \mathcal{T}^\epsilon))$ be a marked forest representing the genealogical structure of Y^ϵ and let $(\beta^\epsilon, s \in [0, \tau^\epsilon])$ be the random walk corresponding to this marked forest (see Section 2.2 of [1]). Note that the spatial positions of the particles are irrelevant for the construction of β^ϵ—what matters is β itself. By (2.3) of [1] we get

$$\lim_{\epsilon \downarrow 0} \beta^\epsilon = \beta, \text{ a.s.} \tag{2.3}$$

We can use the process β^ϵ and a collection of historical paths of the particles to construct a convenient representation of the ϵ-branching Brownian motions known as discrete snake. For any $s \in [0, \tau^\epsilon]$ we can associate with s a unique edge in $\mathcal{T}^\epsilon$. Then we let $W_s^\epsilon(\cdot)$ to be the historical path of the particle in the ϵ-system that corresponds to this edge and is stopped at time β^ϵ_s. In other words, $W^\epsilon_s(\cdot)$ records the historical path up to time β^ϵ_s of one of the particles which is alive at time β^ϵ_s. For more details on the construction of W^ϵ see Section 2.4 of [1]. Now let

$$L^{\epsilon,t}_u = L^{\epsilon,t}_u(\beta^\epsilon) = \epsilon\mathrm{Card}\{r \in [0,u) : \beta^\epsilon_r = t \text{ and } \beta^\epsilon_v > t, \\ \text{for } v \in (r, r+\delta], \text{ for some } \delta > 0\}. \tag{2.4}$$

In other words, $\epsilon^{-1}L^{\epsilon,t}_s$ is the number of upcrossings of β^ϵ above level t before time s. The definition of the process $\{W^\epsilon_s,\ s \in [0,\tau^\epsilon]\}$ immediately implies that (see Section 2.5 of [1] and recall our 2ϵ shift),

$$Y^\epsilon_{t+2\epsilon} = \int_0^{\tau^\epsilon} dL^{\epsilon,t}_s \delta_{W^\epsilon_s}. \tag{2.5}$$

This gives us a representation of the process Y^ϵ in terms of the Brownian snake.

It easily follows from the definition of Y^ϵ and Theorem 3.10 of [2] that as $\epsilon \to 0$,

$$Y^\epsilon_t \to Y_t, \quad \text{in } M_F(C), \quad P-\text{a.s.}, \ \forall t > 0,$$

and

$$Y^\epsilon_\cdot \to Y_\cdot, \quad \text{in } D_{M_F(C)},$$

in probability (see Remark 2 after Theorem 3.10 of [2]). Also

$$X^\epsilon_\cdot \to X_\cdot, \quad \text{in } D_{M_F},$$

in probability.

We will now discuss reflecting binary branching Brownian motions. The closed support of a measure ν on $\mathbb{R}$ will be denoted $\mathrm{supp}(\nu)$, i.e., $\mathrm{supp}(\nu)$ is the smallest closed set A such that $\nu(A^c) = 0$. Recall μ and τ from the beginning of this section. Fix arbitrary positive measures μ_1 and μ_2 which satisfy the following assumptions:

1. $\mu_1 + \mu_2 = \mu$.
2. The support of μ_1 lies to the left of the support of μ_2, that is, for any $x_i \in \mathrm{supp}(\mu_i),\ i = 1, 2$, we have $x_1 \le x_2$.

Let

$$\tau_1 = \inf\{t : L^0_t = \langle \mu_1, 1\rangle\},$$
$$Y^1_t = \int_0^{\tau_1} dL^t_s \delta_{W_s}, \qquad Y^2_t = \int_{\tau_1}^{\tau} dL^t_s \delta_{W_s}.$$

Then Y^1 and Y^2 are two historical super-Brownian motions starting at μ_1 and μ_2 respectively. We define their approximations $Y^{1,\epsilon}$ and $Y^{2,\epsilon}$ in the same way as Y^ϵ was defined for the process Y. Note that

$$Y = Y^1 + Y^2,$$
$$Y^\epsilon = Y^{1,\epsilon} + Y^{2,\epsilon}.$$

The process $\{Y_t^{i,\epsilon}, t \geq 2\epsilon\}$ represents the historical branching particle system with the initial positions of particles distributed according to the Poisson random measure with intensity $\mu^i/\epsilon,\ i = 1, 2$. Let $\beta^{i,\epsilon}$ be the corresponding random walk defined in the same way as β^ϵ was defined for Y^ϵ. By (2.3) and our construction we get

$$\lim_{\epsilon\downarrow 0} \beta^{i,\epsilon} = \beta^i,\ \text{a.s.},\quad i = 1, 2, \tag{2.6}$$

where $\beta_s^1 = \beta_{s\wedge\tau_1}$, $\beta_s^2 = \beta_{(\tau_1+s)\wedge\tau}$.

Arguing as in Section 3 of [1], we relabel the paths of Y^ϵ to obtain a reflecting branching particle system $\widetilde{Y}^\epsilon$. For any $t \geq 2\epsilon$, if $\widetilde{w}$ and $\widetilde{w}'$ are atoms of $\widetilde{Y}_t^\epsilon$ then we either have $\widetilde{w}(r) \geq \widetilde{w}'(r)$ for all $r \in [0, t]$ or $\widetilde{w}(r) \leq \widetilde{w}'(r)$ for all $r \in [0, t]$. Define

$$\widetilde{Y}_t^{i,\epsilon}(dy) = \mathbf{1}_{\{y(0)\in \mathrm{supp}(\mu_i)\}} \widetilde{Y}_t^\epsilon(dy),\quad \forall t \geq 2\epsilon,\ i = 1, 2.$$

That is, the branching particle system $\widetilde{Y}^{i,\epsilon}$ (recall Convention 1) represents the subsystem of $\widetilde{Y}^\epsilon$ consisting of those trees in $\widetilde{Y}^\epsilon$ which start at time 0 from the same points as $Y^{i,\epsilon}$. Hence,

$$\widetilde{Y}^\epsilon \equiv \widetilde{Y}^{1,\epsilon} + \widetilde{Y}^{2,\epsilon}.$$

It follows from the definition that the paths inside of each of $\widetilde{Y}^{i,\epsilon}, i = 1, 2$, are reflecting. The corresponding M_F-valued processes are defined by

$$\langle \widetilde{X}_t^{i,\epsilon}, \phi\rangle = \int \phi(y(t))\widetilde{Y}_t^{i,\epsilon}(dy),\quad \forall \phi \in C(\mathbb{R}),\quad i = 1, 2.$$

The aim of this paper is to prove the following result.

Theorem 2.1. *There exists a process* $\widetilde{Y} \in C_{M_F(C)}[0, \infty)$ *such that*

$$\widetilde{Y}^\epsilon \to \widetilde{Y},\quad \text{as } \epsilon \downarrow 0,\quad \text{in } D_{M_F(C)}[0, \infty),$$

in probability.

The crucial step in the proof will be the following theorem.

Theorem 2.2. *There exists a process* $\left(\widetilde{X}^1, \widetilde{X}^2\right) \in C_{M_F\times M_F}[0, \infty)$ *such that*

$$\left(\widetilde{X}^{1,\epsilon}, \widetilde{X}^{2,\epsilon}\right) \to \left(\widetilde{X}^1, \widetilde{X}^2\right),\quad \text{as } \epsilon \downarrow 0,\quad \text{in } D_{M_F\times M_F}[0, \infty),$$

in probability.

3. Review of local times and excursions

Fix an arbitrary $a > 0$. Recall that β has the distribution of the twice of reflecting Brownian motion, $\beta_0 = 0$, and L_t^x is the family of local times of β. Here we will assume that β is stopped at time

$$\tau = \tau_a = \inf\{t : L_t^0 = a\}.$$

For any $s \geq 0$ and $\nu > 0$ let

$$\eta_1^\nu(s) = \begin{cases} \sup\{t : L_\tau^s - L_t^s \geq \nu\}, & L_\tau^s \geq \nu, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_2^\nu(s) = \begin{cases} \inf\{t : L_t^s \geq \nu\}, & L_\tau^s \geq \nu, \\ \tau, & \text{otherwise}, \end{cases}$$

$$L^{s,t,j}(\beta, \nu) = \begin{cases} L_\tau^t - L_{\eta_1^\nu(s)}^t, & j = 1, \\ L_{\eta_2^\nu(s)}^t, & j = 2, \end{cases}$$

$$L^{s,j}(\beta, \nu) = \sup_{t \geq s} L^{s,t,j}(\beta, \nu), \quad j = 1, 2,$$

$$L^j(\beta, \nu) = \sup_{s \geq 0} L^{s,j}(\beta, \nu), \quad j = 1, 2.$$

In other words, $L^1(\beta, \nu)$ is the maximum local time accumulated at any level $t \geq s$, over the interval $[\eta_1^\nu(s), \infty)$, for any $s \geq 0$, and $L^2(\beta, \nu)$ has the similar meaning, with $[\eta_1^\nu(s), \infty)$ replaced by $[0, \eta_2^\nu(s))$.

For any $s \geq 0$, denote by (a_i^s, b_i^s), $i = 1, 2, \ldots$ the excursion intervals of β above level s, where the ordering of this countably infinite set of intervals is arbitrary, for example, it may be the ordering according to the decreasing length. For any i denote by e_i^s the corresponding excursion

$$e_i^s(u) = \beta_{(a_i^s + u) \wedge b_i^s} - s, \quad u \geq 0.$$

For $t > s \geq 0$ let $L_i^{s,t}(\beta)$ be the local time of excursion e_i^s at level t. Note that

$$L_\tau^t(\beta) = \sum_i L_i^{s,t}(\beta).$$

Let $I_s^{\nu,1}$ be the set of indices of excursions originating at level s in the interval $[\eta_1^\nu(s), \infty)$. More precisely

$$I_s^{\nu,1} = \{i : a_i^s \geq \eta_1^\nu(s)\}.$$

Analogously

$$I_s^{\nu,2} = \{i : b_i^s \leq \eta_2^\nu(s)\}.$$

It is easy to check that, for $s < t$, $L^{s,t,j}(\beta, \nu)$ satisfies

$$L^{s,t,j}(\beta, \nu) = \sum_{i \in I_s^{\nu,j}} L_i^{s,t}(\beta), \quad j = 1, 2.$$

In the following, P^a will refer to the law of β starting at 0 and stopped at τ_a. If there is no ambiguity, we will suppress the dependence on β in the notation.

Lemma 3.1. *For any $a > 0$ and arbitrary small $p, \overline{\alpha} > 0$ there exists $\nu^* = \nu^*(p, \overline{\alpha}) > 0$ such that*

$$\sup_{i=1,2} \sup_{b \le a} P^b \left(L^i(\nu) \ge \overline{\alpha} \right) \le p/8, \quad \forall \nu \le \nu^*.$$

Proof. The random variables $L^1(\nu)$ and $L^2(\nu)$ have the same distribution, by the invariance of the reflected Brownian motion β under time reversal at the stopping time τ. Hence, it will suffice to prove the lemma for $i = 2$ only.

The quantity $P^b \left(L^2(\nu) \ge \overline{\alpha} \right)$ is a non-decreasing function of b so it is enough to prove that for some $\nu^* > 0$ and all $\nu \le \nu^*$ we have $P^a \left(L^2(\nu) \ge \overline{\alpha} \right) \le p/8$. The function $\nu \to L^2(\nu)$ is non-decreasing so it will suffice to show that $\lim_{\nu \to 0} L^2(\nu) = 0$, P^a-a.s. Suppose that $\lim_{\nu \to 0} L^2(\nu) \ne 0$ with positive probability. We will show that this assumption leads to a contradiction. Fix an ω such that $\lim_{k \to \infty} L^2(1/k) = c > 0$ and find sequences of levels $\{t_k\}$ and $\{s_k\}$ such that $t_k > s_k$ and $L^{s_k, t_k, 2}(1/k) \ge c$ for all k. This implies that for each k, there exist r_k, u_k and q_k such that $r_k < u_k < q_k \le \eta_2^{1/k}(s_k)$, $\beta(r_k) = \beta(q_k) = s_k$, $\beta(u_k) = t_k$ and $L_{q_k}^{t_k} \ge c/2$. By compactness, we may assume that $s_k \to s_\infty$, $t_k \to t_\infty$, $r_k \to r_\infty$, $u_k \to u_\infty$ and $q_k \to q_\infty$. Recall that L_t^x is jointly continuous in t and x, a.s. Hence,

$$L_{q_\infty}^{s_\infty} \le \liminf_{k \to \infty} L_{q_k}^{s_k} \le \liminf_{k \to \infty} L_{\eta_2^{1/k}(s_k)}^{s_k} \le \liminf_{k \to \infty} 1/k = 0.$$

For the same reason,

$$L_{q_\infty}^{t_\infty} \ge \limsup_{k \to \infty} L_{q_k}^{t_k} \ge c/2,$$

so $t_\infty > s_\infty$. Note that $u_\infty \le q_\infty$ and so $L_{u_\infty}^{s_\infty} \le L_{q_\infty}^{s_\infty} = 0$. Let T_x denote the hitting time of x by β. By the Ray-Knight theorem, simultaneously for all rational x and y such that $s_\infty < x < y < t_\infty$, we have $L_{T_y}^{s_\infty} \ge \inf_{z \in [0,x]} L_{T_y}^z > 0$, a.s. By the continuity of β, $\beta(u_\infty) = t_\infty$, so $T_y \le u_\infty$ for $y < t_\infty$. Hence, $L_{u_\infty}^{s_\infty} \ge L_{T_y}^{s_\infty} > 0$, contradicting our earlier assertion that $L_{u_\infty}^{s_\infty} = 0$. □

Recall the process β^ϵ, a continuous time random walk corresponding to β and introduced in Section 2 (see also Section 2.2 of [1]). Also recall the process $\{L_u^{\epsilon,t},\ t \ge 0, u \ge 0\}$ defined in (2.4). Then, for any $s \ge 0$ and $\nu > 0$ let

$$\eta_1^{\epsilon,\nu}(s) = \begin{cases} \sup\{t : L_{\tau^\epsilon}^{\epsilon,s} - L_t^{\epsilon,s} \ge \nu\}, & L_{\tau^\epsilon}^{\epsilon,s} \ge \nu, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_2^{\epsilon,\nu}(s) = \begin{cases} \inf\{t : L_t^{\epsilon,s} \ge \nu\}, & L_{\tau^\epsilon}^{\epsilon,s} \ge \nu, \\ \tau^\epsilon, & \text{otherwise}. \end{cases}$$

For any $s \geq 0$, denote by $(a_i^{s,\epsilon}, b_i^{s,\epsilon})$, $i = 1, 2, \dots$ the excursion intervals of β^ϵ above level s, and denote by $e_i^{s,\epsilon}$ the corresponding excursions. For $t \geq s \geq 0$ let $L_i^{s,t,\epsilon}(\beta^\epsilon)$ be the rescaled number of upcrossings of excursion $e_i^{s,\epsilon}$ above level t. Let

$$I_s^{\nu,1,\epsilon} = \{i : a_i^{s,\epsilon} \geq \eta_1^{\epsilon,\nu}(s)\}.$$

Analogously

$$I_s^{\nu,2,\epsilon} = \{i : b_i^{s,\epsilon} \leq \eta_2^{\epsilon,\nu}(s)\}.$$

Denote

$$L^{j,\epsilon}(\beta^\epsilon, \nu) = \sup_{t \geq s \geq 0} \sum_{i \in I_s^{\nu,j,\epsilon}} L_i^{s,t,\epsilon}(\beta^\epsilon, \nu), \quad j = 1, 2.$$

Lemma 3.2. *Let β^ϵ, β be as above. Let $p, \overline{\alpha}, \nu^*$ be as in the previous lemma. Then there exists ϵ_1 sufficiently small such that for all $\epsilon \leq \epsilon_1$ and $\nu \leq \nu^*/2$,*

$$\sup_{i=1,2} \sup_{b \leq a} P^b \left(L^{i,\epsilon}(\beta^\epsilon, \nu) \geq 2\overline{\alpha} \right) \leq p/4.$$

Remark 3.3. It is legitimate to use notation P^b in the above lemma since β^ϵ is a function of β.

Proof of Lemma 3.2. We will prove the lemma only in the case $i = 2$. The case $i = 1$ can be treated similarly. Since $P^b \left(L^{i,\epsilon}(\beta^\epsilon, \nu) \geq 2\overline{\alpha} \right)$ is a non-decreasing function of b and ν, it is enough to prove that

$$P^a \left(L^{2,\epsilon}(\beta^\epsilon, \nu^*/2) \geq 2\overline{\alpha} \right) \leq p/4,$$

for all $\epsilon < \epsilon_1$.

By Lemma 2.1 of [1],

$$\lim_{\epsilon \downarrow 0} \sup_{s \geq 0, t \geq 0} \left| L_s^{\epsilon,t} - L_s^t \right| = 0, \quad P^a - \text{a.s.}$$

Hence, we may fix $\epsilon_1 < \nu^*/8$ sufficiently small, such that for any $\epsilon < \epsilon_1$,

$$P^a \left(\sup_{s \geq 0, t \geq 0} \left| L_s^{\epsilon,t} - L_s^t \right| \leq \bar{\nu}/4 \right) \geq 1 - p/8, \tag{3.1}$$

where $\bar{\nu} \equiv \min(\nu^*/2, \overline{\alpha})$. Let

$$A^\epsilon \equiv \left\{ \omega : \sup_{s \geq 0, t \geq 0} \left| L_s^{\epsilon,t} - L_s^t \right| \leq \bar{\nu}/4 \right\}.$$

We claim that

$$L^t_{\eta_2^{\epsilon,\nu^*/2}(s)} \leq L^t_{\eta_2^{\nu^*}(s)}, \quad \forall t \geq s \geq 0, \forall \epsilon \leq \epsilon_1, \omega \in A^\epsilon. \tag{3.2}$$

We will prove (3.2) in two steps. First let s be an arbitrary time such that $L^s_\tau < \nu^*$. Then, since the process is stopped at time τ, we get

$$\begin{aligned} L^t_{\eta_2^{\epsilon,\nu^*/2}(s)} &\le L^t_\tau \\ &= L^t_{\eta_2^{\nu^*}(s)}, \quad \forall s: \ L^s_\tau < \nu^*, \end{aligned} \tag{3.3}$$

where the second line follows from the definition of $\eta_2^{\nu^*}(s)$. Define

$$\mathcal{J} = \{s \ge 0 : L^s_\tau \ge \nu^*\}.$$

As a second step in the proof of (3.2) we need to prove the following:

$$\eta_2^{\epsilon,\nu^*/2}(s) \le \eta_2^{\nu^*}(s), \ \forall s \in \mathcal{J}, \forall \epsilon \le \epsilon_1, \omega \in A^\epsilon. \tag{3.4}$$

We will prove (3.4) by contradiction. Suppose, there exist $\epsilon < \epsilon_1, \omega \in A^\epsilon$ and $s \ge 0$, such that

$$\eta_2^{\epsilon,\nu^*/2}(s) > \eta_2^{\nu^*}(s).$$

Then on A^ϵ we get

$$\begin{aligned} L^{\epsilon,s}_{\eta_2^{\epsilon,\nu^*/2}(s)} &\ge L^{\epsilon,s}_{\eta_2^{\nu^*}(s)} \\ &\ge L^s_{\eta_2^{\nu^*}(s)} - \bar{\nu}/4 \\ &\ge \nu^* - \nu^*/8 \\ &= 7\nu^*/8, \end{aligned}$$

where in the third inequality we use the identity $L^s_{\eta_2^{\nu^*}(s)} = \nu^*$ which is trivial for $s \in \mathcal{J}$. Now recall, that by the definition, $L^{\epsilon,s}_{\eta_2^{\epsilon,\nu^*/2}(s)} \le \nu^*/2+\epsilon$, and since $\epsilon < \nu^*/8$ we get

$$5\nu^*/8 \ge L^{\epsilon,s}_{\eta_2^{\epsilon,\nu^*/2}(s)} \ge 7\nu^*/8$$

which is the required contradiction. Now (3.4) and (3.3) imply (3.2).

Let $A \equiv \left\{\omega : L^2(\nu^*) \le \bar{\alpha}\right\}$ and fix an arbitrary $\epsilon \le \epsilon_1$. It follows from (3.2) that for any $\omega \in A \cap A^\epsilon$,

$$\begin{aligned} L^{\epsilon,t}_{\eta_2^{\epsilon,\nu^*/2}(s)} &\le L^t_{\eta_2^{\epsilon,\nu^*/2}(s)} + \bar{\nu}/4 \\ &\le L^t_{\eta_2^{\nu^*}(s)} + \bar{\nu}/4 \\ &\le \bar{\alpha} + \bar{\alpha}/4 \\ &\le 2\bar{\alpha}, \quad \forall t \ge s \ge 0. \end{aligned}$$

In other words, $A^\epsilon \cap A \subset \{L^{2,\epsilon}(\beta^\epsilon, \nu^*/2) \le 2\overline{\alpha}\}$. By Lemma 3.1 and (3.1), $P^a(A^\epsilon \cap A) \ge 1 - p/4$, and we are done. □

Let X^ϵ be the M_F-valued process constructed in Section 2 and set

$$B(x,r) \equiv \{y \in \mathbb{R} : |y-x| < r\}.$$

Lemma 3.4. *Fix ν, p arbitrary small.*

(i) *There exist $\alpha = \alpha(\nu, p), \epsilon_2 = \epsilon_2(\nu, p) > 0$, such that*

$$P\left(\sup_{t\geq 2\epsilon, x\in\mathbb{R}} X_t^{\epsilon}([x, x+\alpha]) \leq \nu/2\right) \geq 1 - p/4, \quad \forall \epsilon \leq \epsilon_2.$$

(ii) *For any $\overline{\alpha}$ arbitrary small, there exists $\epsilon_3 = \epsilon_3(\overline{\alpha}, p)$ such that*

$$P\left(\sup_{t\geq 2\max(\epsilon',\epsilon'')}\ \sup_{x\in\mathbb{R}, r\geq 0} \left|X_t^{\epsilon'}(B(x,r)) - X_t^{\epsilon''}(B(x,r))\right| \geq \overline{\alpha}\right)$$
$$\leq p/2, \quad \forall \epsilon', \epsilon'' \leq \epsilon_3.$$

Proof. Recall from Section 2 that $X^{\epsilon} \to X$, in D_{M_F}, in probability, as $\epsilon \downarrow 0$. For each $t \geq 0$, define the distribution functions of the measures X_t and X_t^{ϵ}:

$$F_t(x) \equiv X_t((-\infty, x]), \quad x \in \mathbb{R},$$
$$F_t^{\epsilon}(x) \equiv X_t^{\epsilon}((-\infty, x]), \quad x \in \mathbb{R}.$$

It is well known (see e.g. [6], [12]) that $X_t(dx)$ is absolutely continuous with respect to the Lebesgue measure for every t, and its density is jointly continuos in (t, x). Hence $F_t(x)$ is also jointly continuous in (t, x). By Theorems X.10, X.11 of [3] we obtain that for each $t > 0$,

$$\sup_x \left|F_t^{\epsilon}(x) - F_t(x)\right| \to 0, \quad \text{as } \epsilon \downarrow 0,$$

in probability. Now we can use the fact that convergence in the Skorohod space to a continuous limit is equivalent to uniform convergence on compacts (see e.g. Lemma 3.10.1 of [5]) to get

$$\sup_{t\leq T} \sup_{x\in\mathbb{R}} \left|F_t^{\epsilon}(x) - F_t(x)\right| \to 0, \quad \text{as } \epsilon \downarrow 0,$$

in probability, for any $T > 0$. From the above convergence and the fact that with probability 1, $X_t = 0$ for all t sufficiently large, we get

$$\sup_{t\geq 2\epsilon}\ \sup_{x\in\mathbb{R}, r\geq 0} \left|X_t^{\epsilon}(B(x,r)) - X_t(B(x,r))\right| \to 0, \quad \text{as } \epsilon \downarrow 0, \tag{3.5}$$

in probability.

The lemma follows easily from (3.5). □

Lemma 3.5.

(a) $\lim_{\delta\downarrow 0} \sup_{s,t} |W_s(t+\delta) - W_s(t)| = 0, \quad P-\text{a.s.},$

(b) $\lim_{\delta\downarrow 0}\left(\limsup_{\epsilon\downarrow 0} P\left(\sup_{s\geq 2\epsilon}\ \sup_{|t-t'|\leq\delta}\ \sup_{\widetilde{y}\in \mathrm{supp}(\widetilde{Y}_s^{\epsilon})} \left|\widetilde{y}(t) - \widetilde{y}(t')\right| > \eta\right)\right) = 0, \quad \forall \eta > 0. \quad P-\text{a.s.}$

Proof. **(a)** is a well known fact, see, for example, (2.7) of [1]. For **(b)** see Theorem 4.1 of [1]. □

The last lemma easily implies the following.

Corollary 3.6. *For any $\alpha, p > 0$, there exists $\epsilon_4 = \epsilon_4(\alpha, p) > 0$ such that*

$$(\mathbf{a})\quad P\left(\sup_{s\geq 0, t\geq 2\epsilon}\sup_{u\leq 2\epsilon}|W_s(t-2\epsilon)-W_s(t-u)|\geq \alpha/4\right)\leq p/8,\quad \forall \epsilon\leq\epsilon_4,$$

$$(\mathbf{b})\quad P\left(\sup_{s\geq 2\epsilon}\sup_{|t-t'|\leq 2\epsilon}\sup_{\widetilde{y}\in \mathrm{supp}(\widetilde{Y}_s^\epsilon)}\left|\widetilde{y}(t)-\widetilde{y}(t')\right| > \alpha/4\right)\leq p/8,\quad \forall\epsilon\leq\epsilon_4\,.$$

4. Construction of a pair of reflecting super-Brownian motions

Recall from Section 2 that $(\mathcal{T}^\epsilon, (l_u^\epsilon, u \in \mathcal{T}^\epsilon))$ is the marked genealogical forest corresponding to the ϵ-branching particle system Y_t^ϵ. Each element $u \in \mathcal{T}^\epsilon$ corresponds to a particle with the lifetime $l_u^\epsilon = \zeta_u^\epsilon - \xi_u^\epsilon$, where ζ_u^ϵ and ξ_u^ϵ are the death and birth times of the particle u respectively. The spatial motion of u is assumed to be a continuous function $f_u : [\xi_u^\epsilon, \zeta_u^\epsilon] \to \mathbb{R}$ and, moreover, $f_{u'}(\xi_{u'}^\epsilon) = f_u(\zeta_u^\epsilon)$ if u' is an offspring of u. Of course, in this paper, $\{f_u, u \in \mathcal{T}^\epsilon\}$ are Brownian paths. The historical path of u is the continuous function $w_u : [0, \zeta_u^\epsilon) \to \mathbb{R}$ such that for every $t \in [0, \zeta_u^\epsilon)$, $w_u(t)$ is the position at time t of the ancestor of u alive at that time. Let $l^\epsilon(\mathcal{T}^\epsilon) = \zeta^\epsilon(\mathcal{T}^\epsilon)$ be the lifetime of the ϵ-particle system Y_t^ϵ. For $\epsilon, t > 0$, Y_t^ϵ records the paths of (some) individuals up to time $t - 2\epsilon$. Hence,

$$\zeta^\epsilon(\mathcal{T}^\epsilon) = \left(\sup_{s\leq\tau}\beta_s - 2\epsilon\right)_+ .$$

We will need a truncation operator T_u acting on the forest. We define $T_u\mathcal{T}^\delta$ to be the subtree of $\mathcal{T}^\delta$ starting from the particle u.

We will use the "erasure of branches" idea of Neveu [8] to construct appropriately related ϵ- and δ-branching particle systems for any $\delta < \epsilon$. These in turn will be used to construct a δ-particle system with reflection from an ϵ-particle system with reflection.

Fix arbitrary $\epsilon > \delta > 0$. First we will just consider ϵ- and δ-marked branching forests corresponding to the particle systems without reflection—we will ignore the spatial motion of the particles. The following is the essence of Neveu's construction, but it can be also deduced from our historical process description. To pass from the δ-branching forest to the ϵ-branching forest one should erase each edge with no offspring (leaf) of the δ-forest from its endpoint to a point on the branch located at the distance $2(\epsilon - \delta)$ from the endpoint towards the root of the corresponding tree. If the length of that branch is more than $2(\epsilon - \delta)$ we cut it off by exactly $2(\epsilon - \delta)$. If the length of that edge is less than $2(\epsilon - \delta)$ we erase it completely and we proceed to the parent edge only when the neighboring edge (recall that the branching is binary) is also completely erased. The edges that have not been erased are then relabelled (edges with null lifetimes are excluded—this may change some marks) and this defines a marked ϵ-forest. More precisely, if $u \in \mathcal{T}^\delta$ satisfies

$$l^\delta(T_u\mathcal{T}^\delta) > 2(\epsilon - \delta) \tag{4.1}$$

then u belongs to the ϵ-forest after erasure and relabelling but it may have a different lifetime. Otherwise this particle and all its descendants are completely erased. For any $u \in \mathcal{T}^\delta$ define

$$l_u^{*,\delta} \equiv \min\left(l_u^\delta, (l^\delta(T_u\mathcal{T}^\delta) - 2(\epsilon - \delta))_+\right)$$

and

$$\mathcal{U}^1 \equiv \left\{u \in \mathcal{T}^\delta : l_u^{*,\delta} > 0,\ l_u^{*,\delta} = l^\delta(T_u\mathcal{T}^\delta) - 2(\epsilon - \delta)\right\}.$$

Note that $l_u^{*,\delta}$ depends on ϵ but we will suppress this dependence in the notation.

After relabelling, for any particle $u \in \mathcal{U}^1$ there is a particle $v_u \in \mathcal{T}^\epsilon$ with death time $\zeta_{v_u}^\epsilon = \xi_u^\delta + l_u^{*,\delta}$. Now recall (see Neveu-Pitman [9], [10]) that each death time of a particle without offspring $v_u \in \mathcal{T}^\epsilon$ corresponds to a local 2ϵ-maximum of the Brownian motion β. The branch $u \in \mathcal{U}^1$ is associated with a unique excursion e_u^δ of β on an interval (a_u, b_u), such that $\beta(a_u) = \beta(b_u) = \xi_u^\delta + l_u^{*,\delta}$ and

$$\begin{aligned}\sup_{a_u \le s \le b_u} \beta(s) &= \xi_u^\delta + l_u^{*,\delta} + \sup_{0 \le s \le b_u - a_u} e_u^\delta(s)\\ &= \xi_u^\delta + l_u^{*,\delta} + 2\epsilon.\end{aligned}$$

For $u, v \in \mathcal{T}^\delta$, let $v < u$ mean that v is an ancestor of u, and

$$\begin{aligned}\mathcal{U}^2 \equiv &\left\{u \in \mathcal{T}^\delta : \xi_u^\delta > 0,\ l_u^{*,\delta} = 0,\ l_v^{*,\delta} = l_v^\delta,\ \forall v < u\right\}\\ &\bigcup \left\{u \in \mathcal{T}^\delta : l_u^{*,\delta} = 0,\ \xi_u^\delta = 0\right\}.\end{aligned}$$

The first subset in the definition of $\mathcal{U}^2$ consists of the particles which are completely erased up to the parent level, but the subtree corresponding to their cousins is not completely erased, and hence their parents are not affected by the erasure. The second subset in the definition of $\mathcal{U}^2$ consists of the particles which are born at time zero and completely erased up to time zero (so, they do not have parents).

Again for any particle $u \in \mathcal{U}^2$ there is a subtree $T_u\mathcal{T}^\delta$ with life duration $l^\delta(T_u\mathcal{T}^\delta) < 2(\epsilon - \delta)$. Hence there is a particle $v_u \in \mathcal{T}^\delta$ with the death time $\xi_u^\delta + l^\delta(T_u\mathcal{T}^\delta)$. Then again (by Neveu-Pitman [9], [10]) there is a unique local 2δ-maximum of the Brownian motion β corresponding to the death time $\xi_u^\delta + l^\delta(T_u\mathcal{T}^\delta)$ of the particle v_u. Let e_u^δ be the unique excursion of β on an interval (a_u, b_u) corresponding to this local maximum, such that $\beta(a_u) = \beta(b_u) = \xi_u^\delta$ and

$$\begin{aligned}\sup_{a_u \le s \le b_u} \beta(s) &= \xi_u^\delta + \sup_{0 \le s \le b_u - a_u} e_u^\delta(s)\\ &= \xi_u^\delta + l^\delta(T_u\mathcal{T}^\delta) + 2\delta.\end{aligned}$$

Let $M = \left|\mathcal{U}^1 \cup \mathcal{U}^2\right|$ be the total number of elements in $\mathcal{U}^1 \cup \mathcal{U}^2$ and let

$$s_i = \beta(a_{u_i}),\quad i = 1, \ldots M,$$

be the "erasure levels" corresponding to the elements u_i of $\mathcal{U}^1 \cup \mathcal{U}^2$.

Recall from Section 3.1 of [1] that for any $\mathcal{T}^\epsilon$ there exists a forest $\widetilde{\mathcal{T}}^\epsilon$ representing the reflecting particle system. The historical paths $\widetilde{w}_{\widetilde{v}}$ are defined for $\widetilde{v} \in \widetilde{\mathcal{T}}^\epsilon$ in the same way as w_u are defined for $u \in \mathcal{T}^\epsilon$. Similarly, $\widetilde{\xi}_{\widetilde{v}}^\epsilon$ and $\widetilde{\zeta}_{\widetilde{v}}^\epsilon$ denote the times of birth and death of $\widetilde{v}$.

Lemma 4.1. *For every $i = 1, \dots, M$, with $s_i > 0$, there exists a unique $\widetilde{v}_i \in \widetilde{\mathcal{T}}^\epsilon$ such that $\widetilde{\xi}^\epsilon_{\widetilde{v}_i} < s_i \le \widetilde{\zeta}^\epsilon_{\widetilde{v}_i}$ and*

$$\widetilde{w}_{\widetilde{v}_i}(s_i) = W_{a_{u_i}}(s_i).$$

The proof of the lemma is elementary but tedious so it is left to the reader.

Next we will redefine the Brownian snake W on excursion intervals (a_{u_i}, b_{u_i}). For any $s \in (a_{u_i}, b_{u_i})$ let

$$\widehat{W}^{i,\delta}_s(t) = \begin{cases} W_s(t), & 0 \le s_i \le t \le \beta_s, \\ \widetilde{w}_{\widetilde{v}_i}(t), & s_i > 0, t \le s_i, \end{cases} \tag{4.2}$$

and let $\widehat{W}^{i,\delta,t}_s$ be the path of $\widehat{W}^{i,\delta}_s$ stopped at time t. For any $s \in (a_{u_i}, b_{u_i})$ one can think of the path $\widehat{W}^{i,\delta}_s$ as a path of a particle which up to time s_i follows the path of ϵ-reflecting particle $\widetilde{v}_i$ from Lemma 4.1 (if $s_i > 0$), and on the time interval $[s_i, \beta_s]$ follows the path of a particle from "non-reflecting" system (this path is encoded in the Brownian snake path in (4.2)).

For any $i = 1, \dots, M$, let

$$\vec{H}^{i,\delta}_t \equiv \mathbf{1}_{\{t \ge s_i + 2\delta\}} \int_{a_{u_i}}^{b_{u_i}} \delta_{\widehat{W}^{i,\delta,t-2\delta}_s} L^t(ds).$$

Denote by (a^i_k, b^i_k), $k = 1, 2, \dots$ the excursion intervals of β on (a_{u_i}, b_{u_i}) starting from the level $t - 2\delta$ and reaching the level t. We will denote the number of such excursions by $\vec{M}^{i,\delta}_t$ and for each $k = 1, \dots, \vec{M}^{i,\delta}_t$ choose arbitrary $s^i_k \in (a^i_k, b^i_k)$ such that $\beta(s^i_k) = t$. Trivially, $\widehat{W}^{i,\delta,t-2\delta}_s = \widehat{W}^{i,\delta,t-2\delta}_{s^i_k}$ for all $s \in (a^i_k, b^i_k)$, $k = 1, \dots, \vec{M}^{i,\delta}_t$, and hence, we can write

$$\begin{aligned} \vec{H}^{i,\delta}_t &= \mathbf{1}_{\{t \ge s_i + 2\delta\}} \sum_{k=1}^{\vec{M}^{i,\delta}_t} \int_{a^i_k}^{b^i_k} \delta_{\widehat{W}^{i,\delta,t-2\delta}_s} L^t(ds) \\ &= \sum_{k=1}^{\vec{M}^{i,\delta}_t} \left(L^t(b^i_k) - L^t(a^i_k) \right) \delta_{\widehat{W}^{i,\delta,t-2\delta}_{s^i_k}} . \end{aligned} \tag{4.3}$$

Hence, by the previous discussion we see that the atoms of $\vec{H}^{i,\delta}_t$ (whenever $t \ge s_i + 2\delta$ and $\vec{H}^{i,\delta}_t(1) > 0$) record the historical paths of the particles which up to time s_i coincide with the path of $\widetilde{v}_i$. On the time interval $[s_i, t - 2\delta]$ each of these particles follows a path of a δ-particle which is a descendant of u_i (or u_i itself) and which survived up to time $t - 2\delta$. In what follows we will call all the particles whose evolution is recorded by $\sum_{i=1}^M \vec{H}^{i,\delta}_t$ — "extra" non-reflecting δ-particles (however keep in mind that they are non-reflecting only on the time intervals $[s_i, t - 2\delta]$!).

Now we are ready to define the "historical" process representing the δ-particle system built on the top of the ϵ-reflecting particle system by adding some extra branches without reflection. Let

$$\widehat{H}^\delta_t = \widetilde{Y}^\epsilon_{t+2(\epsilon-\delta)} + \sum_{i=1}^M \vec{H}^{i,\delta}_t , \quad t \ge 2\delta. \tag{4.4}$$

Next we give mass δ^{-1} to each atom of $\widehat{H}^{\delta}_t$; this defines a historical measure valued process $\widehat{Y}^{\delta}_t$. The "historical" process $\widehat{Y}^{\delta}$ records the evolution of the δ-particle system with a special recipe for reflection. For any $t \geq 2\delta$, $\widehat{Y}^{\delta}_t$ records the evolution *up to time* $t - 2\delta$ of two types of particles (recall our Convention 1):

(i) particles corresponding to the atoms of $\widetilde{Y}^{\epsilon}_{t+2(\epsilon-\delta)}$ — those are ϵ-reflecting particles which are alive at time $t - 2\delta$;
(ii) "extra" non-reflecting δ-particles (see discussion below (4.3)).

Note that the particles in (ii) do not reflect with each other and do not reflect with particles in (i) on the time intervals $[s_i, t - 2\delta]$.

Let $\widehat{X}^{\delta}_t$ be the measure valued processes corresponding to $\widehat{Y}^{\delta}_t$ in the same way as X^{ϵ}_t corresponds to Y^{ϵ}_t. Then it is easy to check that

$$\widehat{X}^{\delta}_t = X^{\delta}_t, \quad t \geq 2\delta. \tag{4.5}$$

Define

$$\widehat{Y}^{i,\delta}_t(d\omega) = \mathbf{1}_{\{\omega(0)\in \mathrm{supp}(\mu_i)\}} \widehat{Y}^{\delta}_t(d\omega), \quad i = 1, 2.$$

Assumption. In what follows we fix $0 < p, \overline{\alpha} < 1$ arbitrary small and $a = \langle \mu, 1\rangle$. Then we choose $\nu^* \leq \overline{\alpha}$ as in Lemma 3.1. For those $p, \overline{\alpha}, \nu^*$ we choose ϵ_1 as in Lemma 3.2. Then we choose $\alpha = \alpha(\nu^*, p)$, $\epsilon_2 = \epsilon_2(\nu^*, p)$ and $\epsilon_3 = \epsilon_3(\overline{\alpha}, p)$ as in Lemma 3.4. Finally we choose $\epsilon_4 = \epsilon_4(\alpha, p)$ as in Corollary 3.6. Now fix some $\epsilon \leq \min(\epsilon_i, i = 1, 2, 3, 4)$.

It is easy to see from our construction that if $\langle \widetilde{Y}^{i,\epsilon}_t, 1\rangle > 0$ then $\langle \widehat{Y}^{i,\delta}_t, 1\rangle > 0$, $i = 1, 2$. Define

$$\sigma^{i,\epsilon} \equiv \inf\{t \geq 2\epsilon : \langle \widetilde{Y}^{i,\epsilon}_t, 1\rangle = 0\}.$$

Let $\widehat{X}^{i,\delta}_t$ and $\widetilde{X}^{i,\epsilon}_t$ be the measure valued processes corresponding to $\widehat{Y}^{i,\delta}_t$ and $\widetilde{Y}^{i,\epsilon}_t$ in the same way as X^{ϵ}_t corresponds to Y^{ϵ}_t. For $t < \sigma^{1,\epsilon}$ let $\widehat{r}^{\delta}(t)$ (resp. $\widetilde{r}^{\epsilon}(t)$) be the right boundary of $\mathrm{supp}(\widehat{X}^{1,\delta}_t)$ (resp. $\mathrm{supp}(\widetilde{X}^{1,\epsilon}_t)$) and for $t < \sigma^{2,\epsilon}$ let $\widehat{l}^{\delta}(t)$ (resp. $\widetilde{l}^{\epsilon}(t)$) be the left boundary of $\mathrm{supp}(\widehat{X}^{2,\delta}_t)$ (resp. $\mathrm{supp}(\widetilde{X}^{2,\epsilon}_t)$).

Lemma 4.2. *For every* $\delta \leq \epsilon$,

$$P\left(\sup_{2\epsilon\leq t<\sigma^{1,\epsilon}} \left|\widehat{r}^{\delta}(t) - \widetilde{r}^{\epsilon}(t)\right| \leq \alpha/4\right) \geq 1 - p/4,$$

$$P\left(\sup_{2\epsilon\leq t<\sigma^{2,\epsilon}} \left|\widehat{l}^{\delta}(t) - \widetilde{l}^{\epsilon}(t)\right| \leq \alpha/4\right) \geq 1 - p/4,$$

$$P\left(\sup_{2\epsilon\leq t<\sigma^{1,\epsilon}\wedge\sigma^{2,\epsilon}} (\widehat{r}^{\delta}(t) - \widehat{l}^{\delta}(t)) \leq \alpha/2\right) \geq 1 - p/2.$$

Proof. By (4.4), the quantity $\sup_{2\epsilon\leq t<\sigma^{1,\epsilon}} \left|\widehat{r}^{\delta}(t) - \widetilde{r}^{\epsilon}(t)\right|$ is bounded by the maximum of oscillations of the paths in the support of $\widetilde{Y}^{\epsilon}_t, t \geq 2\epsilon$ over time intervals of length $2(\epsilon-\delta)$ (described in Corollary 3.6 (b)), and the maximum of oscillations of extra paths added to the reflecting ϵ-branching system. The oscillations of the extra paths are bounded by oscillations of the Brownian snake paths (again over time intervals of length $2(\epsilon-\delta)$) described in Corollary 3.6 (a). Hence, the first inequality follows from Corollary 3.6. The second inequality is analogous, and the third one follows from the first two and the fact that, by construction, $\widetilde{l}^{\epsilon}(t) \geq \widetilde{r}^{\epsilon}(t)$. □

For each $i=1,2$, we relabel the paths of $\widehat{Y}_t^{i,\delta}$ to create a new "historical" process $\overline{Y}_t^{i,\delta}$ whose paths are reflecting from each other. Note that the paths are reflecting within each $\overline{Y}_t^{i,\delta}$, but the paths of $\overline{Y}_t^{1,\delta}$ are not reflecting from those in $\overline{Y}_t^{2,\delta}$. The corresponding measure-valued processes will be denoted $\overline{X}_t^{i,\delta}$. Obviously $\overline{X}_t^{i,\delta} = \widehat{X}_t^{i,\delta}$, $i=1,2$. Let $\overline{r}^{\delta}(t)$ be the right boundary of $\operatorname{supp}(\overline{X}_t^{1,\delta})$ and $\overline{l}^{\delta}(t)$ be the left boundary of $\operatorname{supp}(\overline{X}_t^{2,\delta})$. Since $\overline{X}_t^{i,\delta} = \widehat{X}_t^{i,\delta}$, we immediately have

$$\overline{r}^{\delta}(t) = \widehat{r}^{\delta}(t), \qquad \overline{l}^{\delta}(t) = \widehat{l}^{\delta}(t), \tag{4.6}$$

and Lemma 4.2 implies that

$$P\left(\sup_{2\epsilon \leq t < \sigma^{1,\epsilon} \wedge \sigma^{2,\epsilon}} (\overline{r}^{\delta}(t) - \overline{l}^{\delta}(t)) \leq \alpha/2\right) \geq 1 - p/2, \quad \forall \delta \leq \epsilon. \tag{4.7}$$

Let $(\overline{\mathcal{T}}^{i,\delta}, (\bar{l}_u, u \in \overline{\mathcal{T}}^{i,\delta}))$ be the marked tree corresponding to the genealogical structure of $\overline{Y}^{i,\delta}$, and $\overline{\beta}^{i,\delta}$, $i=1,2$, be the corresponding random walks. Note that by independence of the motion and the branching, the total mass of $\overline{X}^{i,\delta}$ is the critical Galton-Watson branching process with the rate of branching δ^{-1}, and $\overline{\beta}^{i,\delta}$ has the same distribution as $\beta^{i,\delta}$, $i=1,2$. We use (2.6) to get

$$\overline{\beta}^{i,\delta} \to \beta^i, \quad i=1,2,$$

in distribution. By Lemma 3.2,

$$\sup_{i=1,2} P\left(\overline{L}^{i,\delta}(\overline{\beta}^{i,\delta}, \nu^*/2) \geq 2\overline{\alpha}\right) \leq p/4, \quad \forall \delta \leq \epsilon, \tag{4.8}$$

where $\overline{L}^{i,\delta}$ is defined relative to $\overline{\beta}^{i,\delta}$ in the same way as $L^{i,\epsilon}$ was defined relative to β^{ϵ} in Section 3. In fact, recalling that $\overline{Y}^{i,\delta}$, $i=1,2$, records evolution of reflecting particle system, one can easily check that in this case $\overline{L}^{i,\delta}$, $i=1,2$, satisfies the following inequalities

$$\overline{L}^{1,\delta}(\overline{\beta}^{1,\delta}, \nu^*/2) \geq \sup_{t \geq 2\delta} \overline{Y}_t^{1,\delta}\left(y : \overline{X}_s^{1,\delta}([y(s),\infty)) \leq \nu^*/2, \text{ for some } s \leq t\right), \tag{4.9}$$

$$\overline{L}^{2,\delta}(\overline{\beta}^{2,\delta}, \nu^*/2) \geq \sup_{t \geq 2\delta} \overline{Y}_t^{2,\delta}\left(y : \overline{X}_s^{2,\delta}((-\infty,y(s)]) \leq \nu^*/2, \text{ for some } s \leq t\right). \tag{4.10}$$

In other words if $z_s^1 = \inf\{x \in \operatorname{supp}(\overline{X}_s^{1,\delta}) : \overline{X}_s^{1,\delta}([x,\infty)) \leq \nu^*/2\}$, then the quantity $\overline{L}^{1,\delta}(\overline{\beta}^{1,\delta}, \nu^*/2)$ is greater than or equal to the maximum mass in the reflecting particle system recorded by $\overline{Y}^{1,\delta}$ that may "descend" from the particles in $[z_s, \infty)$ for any $s \geq 0$. One can give an analogous interpretation to (4.10).

Let

$$
\begin{aligned}
\mathbf{T}^i &\equiv \left\{t \geq 2\epsilon : \langle \widetilde{X}_t^{i,\epsilon}, 1\rangle \geq 4\overline{\alpha}\right\}, \quad i = 1, 2, \\
\overline{\eta}_1^{\overline{\alpha}}(t) &= \begin{cases} \sup\{x : \int_x^\infty \widetilde{X}_t^{1,\epsilon}(dy) \geq 4\overline{\alpha}\}, & \forall t \in \mathbf{T}^1, \\ -\infty, & \forall t \notin \mathbf{T}^1, \end{cases} \\
\overline{\eta}_2^{\overline{\alpha}}(t) &= \begin{cases} \inf\{x : \int_{-\infty}^x \widetilde{X}_t^{2,\epsilon}(dy) \geq 4\overline{\alpha}\}, & \forall t \in \mathbf{T}^2, \\ \infty, & \forall t \notin \mathbf{T}^2, \end{cases} \\
\Gamma^{1,\delta,\epsilon} &= \left\{\omega : \overline{X}_t^{2,\delta}\left((-\infty, \widetilde{r}^\epsilon(t) - \alpha/2)\right) = 0, \ \forall t < \sigma^{1,\epsilon}\right\} \\
&\quad \cap \left\{\omega : \sup_{t\geq 2\epsilon} \sup_{x\in\mathbb{R}, r\geq 0} \left|X_t^\epsilon(B(x,r)) - X_t^\delta(B(x,r))\right| \leq \overline{\alpha}\right\} \\
&\quad \cap \left\{\omega : X_t^\epsilon\left([\widetilde{r}^\epsilon(t) - \alpha/2, \widetilde{r}^\epsilon(t)]\right) \leq \nu^*/2, \ \forall t \in \mathbf{T}^1\right\} \\
&\equiv \Gamma_1^{1,\delta,\epsilon} \cap \Gamma_2^{1,\delta,\epsilon} \cap \Gamma_3^{1,\delta,\epsilon}, \qquad (4.11) \\
\Gamma^{2,\delta,\epsilon} &= \left\{\omega : \overline{X}_t^{1,\delta}\left((\widetilde{l}^\epsilon(t) + \alpha/2, \infty)\right) = 0, \ \forall t < \sigma^{2,\epsilon}\right\} \\
&\quad \cap \left\{\omega : \sup_{t\geq 2\epsilon} \sup_{x\in\mathbb{R}, r\geq 0} \left|X_t^\epsilon(B(x,r)) - X_t^\delta(B(x,r))\right| \leq \overline{\alpha}\right\} \\
&\quad \cap \left\{\omega : X_t^\epsilon\left([\widetilde{l}^\epsilon(t), \widetilde{l}^\epsilon(t) + \alpha/2]\right) \leq \nu^*/2, \ \forall t \in \mathbf{T}^2\right\} \\
&\equiv \Gamma_1^{2,\delta,\epsilon} \cap \Gamma_2^{2,\delta,\epsilon} \cap \Gamma_3^{2,\delta,\epsilon}. \qquad (4.12)
\end{aligned}
$$

Note that

$$X_t^\epsilon = \widetilde{X}_t^\epsilon = \widetilde{X}_t^{1,\epsilon} + \widetilde{X}_t^{2,\epsilon}, t \geq 2\epsilon. \qquad (4.13)$$

Hence it follows easily from the definition of $\overline{\eta}_1^{\overline{\alpha}}(t)$ and the assumption $\nu^* \leq \overline{\alpha}$, that for any $\omega \in \Gamma^{1,\delta,\epsilon}$

$$\widetilde{X}_t^{1,\epsilon}\left([\overline{\eta}_1^{\overline{\alpha}}(t), \widetilde{r}^\epsilon(t)]\right) > \widetilde{X}_t^{1,\epsilon}\left([\widetilde{r}^\epsilon(t) - \alpha/2, \widetilde{r}^\epsilon(t)]\right), \quad \forall t \in \mathbf{T}^1.$$

Therefore

$$\overline{\eta}_1^{\overline{\alpha}}(t) < \widetilde{r}^\epsilon(t) - \alpha/2, \quad \forall t \in \mathbf{T}^1, \ \forall \omega \in \Gamma^{1,\delta,\epsilon}. \qquad (4.14)$$

In the same way we get

$$\overline{\eta}_2^{\overline{\alpha}}(t) > \widetilde{l}^\epsilon(t) + \alpha/2, \quad \forall t \in \mathbf{T}^2, \ \forall \omega \in \Gamma^{2,\delta,\epsilon}. \qquad (4.15)$$

By Lemma 3.4 we obtain for $i = 1, 2$,

$$P\left(\left(\Gamma_2^{i,\delta,\epsilon}\right)^c\right) + P\left(\left(\Gamma_3^{i,\delta,\epsilon}\right)^c\right) \leq p/2 + p/4. \qquad (4.16)$$

Moreover,

$$\begin{aligned}\left(\Gamma_1^{1,\delta,\epsilon}\right)^c &= \left\{\overline{X}_t^{2,\delta}\left((-\infty,\widetilde{r}^\epsilon(t)-\alpha/2)\right)>0,\ \text{for some } t\in[2\epsilon,\sigma^{1,\epsilon})\right\}\\ &\subset\left\{\overline{l}^\delta(t)<\widetilde{r}^\epsilon(t)-\alpha/2\ \text{ for some } t\in[2\epsilon,\sigma^{1,\epsilon})\right\}\\ &\subset\left\{\widehat{l}^\delta(t)<\widetilde{l}^\epsilon(t)-\alpha/2\ \text{ for some } t\in[2\epsilon,\sigma^{1,\epsilon})\right\},\end{aligned}$$

where the second inclusion follows by (4.6) and the inequality $\widetilde{l}^\epsilon(t)\ge\widetilde{r}^\epsilon(t)$. Now apply Lemma 4.2 to get

$$P\left(\left(\Gamma_1^{1,\delta,\epsilon}\right)^c\right)\le p/4. \tag{4.17}$$

Similarly we obtain

$$P\left(\left(\Gamma_1^{2,\delta,\epsilon}\right)^c\right)\le p/4. \tag{4.18}$$

Combine (4.16), (4.17), (4.18) to get

$$\begin{aligned}P\left(\Gamma^{i,\delta,\epsilon}\right)&\ge 1-p/4-p/2-p/4 \\ &=1-p.\end{aligned}\tag{4.19}$$

Let $\overline{X}_t^\delta=\overline{X}_t^{1,\delta}+\overline{X}_t^{2,\delta}$. Since $\overline{X}_t^\delta=\widehat{X}_t^\delta$, $t\ge 2\delta$, we get by (4.5) that

$$\overline{X}_t^\delta=X_t^\delta,\quad t\ge 2\delta. \tag{4.20}$$

It follows from (4.11) and (4.20) that for any $\omega\in\Gamma^{1,\delta,\epsilon}$ and $t\in\mathbf{T}^1$,

$$\begin{aligned}\overline{X}_t^{1,\delta}\left([\overline{\eta}_1^{\overline{\alpha}}(t),\infty)\right)&\ge\overline{X}_t^{1,\delta}\left([\overline{\eta}_1^{\overline{\alpha}}(t),\widetilde{r}^\epsilon(t)-\alpha/2)\right)\\ &=\overline{X}_t^{\delta}\left([\overline{\eta}_1^{\overline{\alpha}}(t),\widetilde{r}^\epsilon(t)-\alpha/2)\right)\\ &=X_t^{\delta}\left([\overline{\eta}_1^{\overline{\alpha}}(t),\widetilde{r}^\epsilon(t)-\alpha/2)\right)\\ &\ge X_t^{\epsilon}\left([\overline{\eta}_1^{\overline{\alpha}}(t),\widetilde{r}^\epsilon(t)-\alpha/2)\right)-\overline{\alpha}\\ &=X_t^{\epsilon}\left([\overline{\eta}_1^{\overline{\alpha}}(t),\widetilde{r}^\epsilon(t)]\right)-X_t^\epsilon\left([\widetilde{r}^\epsilon(t)-\alpha/2,\widetilde{r}^\epsilon(t)]\right)-\overline{\alpha}.\end{aligned}\tag{4.21}$$

Again by (4.11)

$$X_t^\epsilon\left([\widetilde{r}^\epsilon(t)-\alpha/2,\widetilde{r}^\epsilon(t)]\right)\le\nu^*/2,\quad\forall\omega\in\Gamma^{1,\delta,\epsilon},\ t\in\mathbf{T}^1. \tag{4.22}$$

Use (4.13) to obtain

$$\begin{aligned}X_t^\epsilon\left([\overline{\eta}_1^{\overline{\alpha}}(t),\widetilde{r}^\epsilon(t)]\right)&\ge\widetilde{X}_t^{1,\epsilon}\left([\overline{\eta}_1^{\overline{\alpha}}(t),\widetilde{r}^\epsilon(t)]\right)\\ &\ge 4\overline{\alpha},\ t\in\mathbf{T}^1,\end{aligned}\tag{4.23}$$

where the second inequality follows by the definitions of $\widetilde{r}^{\epsilon}(\cdot)$ and $\overline{\eta}_1^{\overline{\alpha}}(\cdot)$. Recall that $\nu^* \le \overline{\alpha}$, and combine (4.21), (4.22), (4.23) to get that for any $\omega \in \Gamma^{1,\delta,\epsilon}$ and $t \in \mathbf{T}^1$

$$\overline{X}_t^{1,\delta}\left([\overline{\eta}_1^{\overline{\alpha}}(t), \infty)\right) \ge 2\overline{\alpha}. \tag{4.24}$$

In a similar way we get that for any $\omega \in \Gamma^{2,\delta,\epsilon}$ and $t \in \mathbf{T}^2$,

$$\overline{X}_t^{2,\delta}\left((-\infty, \overline{\eta}_2^{\overline{\alpha}}(t)]\right) \ge 2\overline{\alpha}.$$

We proceed to do one more (final!) relabelling of the particles. We relabel the particles of $\mathrm{supp}(\overline{Y}^{1,\delta}) \cup \mathrm{supp}(\overline{Y}^{2,\delta})$ in such a way that all paths are reflecting from each other. We group the relabelled paths into two families $\widetilde{Y}^{1,\delta}$ and $\widetilde{Y}^{2,\delta}$ so that the roots of trees in $\widetilde{Y}^{i,\delta}$ are the same as in $\overline{Y}^{i,\delta}$. Note that we are using the same notation $\widetilde{Y}^{i,\delta}$ as in Section 2. This is no coincidence—the pair of processes just defined is the same as $(\widetilde{Y}^{1,\delta}, \widetilde{Y}^{2,\delta})$ of Section 2, since the intermediate labelling scheme of $\overline{Y}^{i,\delta}$ has no effect on the final result.

Define $\widetilde{S}^{i,\delta}(t)$ (resp. $\overline{S}^{i,\delta}(t)$) to be the collection of atoms of $\widetilde{Y}_t^{i,\delta}$ (resp. $\overline{Y}_t^{i,\delta}$), $i = 1, 2$. Also let

$$\widetilde{S}^{1,\delta,\overline{\alpha}}(t) = \left\{\widetilde{y}(\cdot) \in \widetilde{S}^{1,\delta}(t) : \ \widetilde{y}(t) \le \overline{\eta}_1^{\overline{\alpha}}(t)\right\},$$
$$\widetilde{S}^{2,\delta,\overline{\alpha}}(t) = \left\{\widetilde{y}(\cdot) \in \widetilde{S}^{2,\delta}(t) : \ \widetilde{y}(t) \ge \overline{\eta}_2^{\overline{\alpha}}(t)\right\},$$
$$\overline{S}^{1,\delta,\overline{\alpha}}(t) = \left\{\overline{y}(\cdot) \in \overline{S}^{1,\delta}(t) : \ \overline{y}(t) \le \overline{\eta}_1^{\overline{\alpha}}(t)\right\},$$
$$\overline{S}^{2,\delta,\overline{\alpha}}(t) = \left\{\overline{y}(\cdot) \in \overline{S}^{2,\delta}(t) : \ \overline{y}(t) \ge \overline{\eta}_2^{\overline{\alpha}}(t)\right\},$$

$$A^{1,\delta} \equiv \left\{\forall t \in \mathbf{T}^1, \forall y \in \overline{S}^{1,\delta,\overline{\alpha}}(t) : \ y(s) \le \widetilde{r}^{\epsilon}(s) - 3\alpha/4, \forall s \le t\right\},$$
$$A^{2,\delta} \equiv \left\{\forall t \in \mathbf{T}^2, \forall y \in \overline{S}^{2,\delta,\overline{\alpha}}(t) : \ y(s) \ge \widetilde{l}^{\epsilon}(s) + 3\alpha/4, \forall s \le t\right\}.$$

Lemma 4.3. *We have for* $i = 1, 2$,

$$P\left(A^{i,\delta} \cap \Gamma^{i,\delta,\epsilon}\right) \ge 1 - \frac{7}{4}p.$$

Proof.

$$\begin{aligned}
&P\left(\Gamma^{1,\delta,\epsilon} \cap \left(A^{1,\delta}\right)^c\right)\\
&= P\left(\Gamma^{1,\delta,\epsilon} \cap \left\{\exists t \in \mathbf{T}^1, y \in \overline{S}^{1,\delta,\overline{\alpha}}(t) \ : y(s) > \widetilde{r}^{\epsilon}(s) - 3\alpha/4, \text{ for some } s \le t\right\}\right)\\
&\le P\left(\Gamma^{1,\delta,\epsilon} \cap \left\{\exists t \in \mathbf{T}^1, y \in \overline{S}^{1,\delta,\overline{\alpha}}(t) \ : y(s) > \overline{r}^{\delta}(s) - \alpha, \text{ for some } s \le t\right\}\right) + p/4\\
&\le P\Bigg(\Gamma^{1,\delta,\epsilon} \cap \left\{\exists t \in \mathbf{T}^1, y \in \overline{S}^{1,\delta,\overline{\alpha}}(t) : y(s) > \overline{r}^{\delta}(s) - \alpha, \text{ for some } s \le t\right\}\\
&\qquad \cap \left\{\overline{X}_u^{1,\delta}\left([\overline{r}^{\delta}(u) - \alpha, \overline{r}^{\delta}(u)]\right) \le \nu^*/2, \ \forall u \ge 2\delta\right\}\Bigg) + p/2\\
&\equiv P\left(\Gamma^{1,\delta,\epsilon} \cap D^{1,\delta} \cap D^{2,\delta}\right) + p/2.
\end{aligned} \tag{4.25}$$

The first inequality follows by Lemma 4.2 and (4.6), the second by Lemma 3.4. Now suppose that $D^{1,\delta}$ holds, that is there exists

$$t \in \mathbf{T}^1, y \in \overline{S}^{1,\delta,\overline{\alpha}}(t): \; y(s) > \overline{r}^{\delta}(s) - \alpha, \text{ for some } s \le t.$$

For such t we have on $\Gamma^{1,\delta,\epsilon}$:

$$\begin{aligned}\overline{Y}_t^{1,\delta}\big(y: \; y(s) > \overline{r}^{\delta}(s) - \alpha, \text{ for some } s \le t\big) &\ge \overline{Y}_t^{1,\delta}\left(y: \; y(t) \ge \overline{\eta}_1^{\overline{\alpha}}(t)\right)\\ &= \overline{X}_t^{1,\delta}\left([\overline{\eta}_1^{\overline{\alpha}}(t), \infty)\right)\\ &\ge 2\overline{\alpha}, \end{aligned} \tag{4.26}$$

where the first inequality follows by definition of $\overline{S}^{1,\delta,\overline{\alpha}}(t)$ and the last one by (4.24). If in addition we assume that $D^{2,\delta}$ holds we easily get from (4.26) that (for the same t)

$$\overline{Y}_t^{1,\delta}\left(y: \; \overline{X}_s^{1,\delta}([y(s), \infty)) \le \nu^*/2, \text{ for some } s \le t\right) \ge 2\overline{\alpha}, \tag{4.27}$$

This and (4.9) immediately imply that

$$\Gamma^{1,\delta,\epsilon} \cap D^{1,\delta} \cap D^{2,\delta} \subset \left\{\overline{L}^{1,\delta}\left(\overline{\beta}^{1,\delta}, \nu^*/2\right) \ge 2\overline{\alpha}\right\}. \tag{4.28}$$

Hence by (4.25) and (4.8) we obtain

$$\begin{aligned}P\left(\Gamma^{1,\delta,\epsilon} \cap \left(A^{1,\delta}\right)^c\right) &\le P\left(\overline{L}^{1,\delta}\left(\overline{\beta}^{1,\delta}, \nu^*/2\right) \ge 2\overline{\alpha}\right) + p/2\\ &\le p/4 + p/2 = 3p/4.\end{aligned}$$

The inequality $P\left(\Gamma^{2,\delta,\epsilon} \cap \left(A^{2,\delta}\right)^c\right) \le 3p/4$ follows along the same lines. This and (4.19) yield the desired result. □

Lemma 4.4. *For any* $\omega \in \bigcap_{i=1,2}(\Gamma^{i,\delta,\epsilon} \cap A^{i,\delta})$,

$$\widetilde{X}_t^{1,\delta}\left([\overline{\eta}_2^{\overline{\alpha}}(t), \infty)\right) = 0, \quad \forall t \ge 2\epsilon \tag{4.29}$$

$$\widetilde{X}_t^{2,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) = 0, \quad \forall t \ge 2\epsilon. \tag{4.30}$$

Proof. Let $\omega \in \bigcap_{i=1,2}(\Gamma^{i,\delta,\epsilon} \cap A^{i,\delta})$. Fix an arbitrary $t \in \mathbf{T}^1$ and $y \in \overline{S}^{1,\delta,\overline{\alpha}}(t)$. Recall that all the paths inside $\overline{S}^{1,\delta,\overline{\alpha}}(t)$ are reflecting. Hence y is not crossed by any path from any family $\overline{S}^{1,\delta,\overline{\alpha}}(s)$, for any $s \le t$. If after the final relabelling $y \notin \widetilde{S}^{1,\delta,\overline{\alpha}}(t)$, it means that y is intersected by a path in the family $\bigcup_{s \le t} \overline{S}^{2,\delta,\overline{\alpha}}(s)$. This combined with the first line in the definition of $\Gamma^{1,\delta,\epsilon}$ shows that

$$\exists s: \; y(s) \ge \widetilde{r}^{\epsilon}(s) - \alpha/2.$$

This however contradicts the fact that $\omega \in A^{1,\delta}$. Therefore

$$\overline{S}^{1,\delta,\overline{\alpha}}(t) \subset \widetilde{S}^{1,\delta,\overline{\alpha}}(t), \qquad \forall t \in \mathbf{T}^1, \omega \in \bigcap_{i=1,2}(\Gamma^{i,\delta,\epsilon} \cap A^{i,\delta}).$$

Hence for any $t \in \mathbf{T}^1$ we get

$$\overline{X}_t^{1,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) \le \widetilde{X}_t^{1,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) \le X_t^{\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right), \quad (4.31)$$

where the last inequality follows by

$$X_t^{\delta} = \widetilde{X}_t^{\delta} = \widetilde{X}_t^{1,\delta} + \widetilde{X}_t^{2,\delta}, t \ge 2\delta. \quad (4.32)$$

(Note that that (4.32) is just (4.13) with $\epsilon = \delta$.) On the other hand, (4.14) and the first line in the definition of $\Gamma^{1,\delta,\epsilon}$ imply that $\overline{X}_t^{2,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) = 0$. From this and (4.20) we obtain

$$\begin{aligned}\overline{X}_t^{1,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) &= \overline{X}_t^{\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right)\\ &= X_t^{\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right). \end{aligned} \quad (4.33)$$

By (4.31) and (4.33) we obtain

$$\widetilde{X}_t^{1,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) = X_t^{\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right). \quad (4.34)$$

Then the above equation together with (4.32) immediately imply (4.30) for $t \in \mathbf{T}^1$. For $t \notin \mathbf{T}^1$, (4.30) follows immediately by definition of $\overline{\eta}_1^{\overline{\alpha}}(t)$.

Analogously we get

$$\overline{S}^{2,\delta,\overline{\alpha}}(t) \subset \widetilde{S}^{2,\delta,\overline{\alpha}}(t), \qquad \forall t \in \mathbf{T}^2, \omega \in \bigcap_{i=1,2}(\Gamma^{i,\delta,\epsilon} \cap A^{i,\delta}),$$

and derive (4.29) along the same lines as we derived (4.30). □

Lemma 4.5.

$$P\left(\sup_{t\ge 2\epsilon}\left|\langle \widetilde{X}_t^{i,\epsilon}, 1\rangle - \langle \widetilde{X}_t^{i,\delta}, 1\rangle\right| \ge 14\overline{\alpha}\right) \le \frac{14}{4}p, \qquad i = 1, 2.$$

Proof. Consider the case $i = 1$. Assume that $\omega \in \bigcap_{i=1,2}\left(\Gamma^{i,\delta,\epsilon} \cap A^{i,\delta}\right)$ and fix arbitrary $t \ge 2\epsilon$. Then

$$\begin{aligned}\left|\langle \widetilde{X}_t^{1,\epsilon}, 1\rangle - \langle \widetilde{X}_t^{1,\delta}, 1\rangle\right| \le{}& \left|\widetilde{X}_t^{1,\epsilon}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) - \widetilde{X}_t^{1,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right)\right|\\ &+ \left|\widetilde{X}_t^{1,\epsilon}\left((\overline{\eta}_1^{\overline{\alpha}}(t), \infty)\right)\right| + \left|\widetilde{X}_t^{1,\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t), \infty)\right)\right|.\end{aligned} \quad (4.35)$$

By the definition of $\overline{\eta}_1^{\overline{\alpha}}(t)$ and Lemma 4.4 we get

$$\begin{aligned}&\left|\widetilde{X}_t^{1,\epsilon}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) - \widetilde{X}_t^{1,\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right)\right|\\ &\le \left|\widetilde{X}_t^{\epsilon}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) - \widetilde{X}_t^{\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right)\right|\\ &= \left|X_t^{\epsilon}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right) - X_t^{\delta}\left((-\infty, \overline{\eta}_1^{\overline{\alpha}}(t)]\right)\right|\\ &\le \overline{\alpha}\end{aligned} \quad (4.36)$$

where the equality follows by (4.13), (4.32), and the last inequality follows by the second line in the definition of $\Gamma^{1,\delta,\epsilon}$.

We will bound the second term on the right hand side of (4.35). By the definition of $\overline{\eta}_1^{\overline{\alpha}}(t)$ we get

$$\widetilde{X}_t^{1,\epsilon}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\infty)\right) \le 4\overline{\alpha}. \tag{4.37}$$

Consider the last term on the right hand side of (4.35). First by Lemma 4.4 we get

$$\widetilde{X}_t^{1,\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\infty)\right) = \widetilde{X}_t^{1,\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\overline{\eta}_2^{\overline{\alpha}}(t))\right). \tag{4.38}$$

Then we obtain

$$\begin{aligned}\widetilde{X}_t^{1,\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\overline{\eta}_2^{\overline{\alpha}}(t))\right) &\le X_t^{\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\overline{\eta}_2^{\overline{\alpha}}(t))\right)\\ &\le \overline{\alpha} + X_t^{\epsilon}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\overline{\eta}_2^{\overline{\alpha}}(t))\right)\\ &\le 9\overline{\alpha},\end{aligned}$$

where the first inequality follows by (4.32), the second one follows by the second line of the definition of $\Gamma^{1,\delta,\epsilon}$, and the last one by the definition of $\overline{\eta}_i^{\overline{\alpha}}(t)$, $i=1,2$. Hence from (4.38) we get

$$\widetilde{X}_t^{1,\delta}\left((\overline{\eta}_1^{\overline{\alpha}}(t),\infty)\right) \le 9\overline{\alpha}. \tag{4.39}$$

Combine (4.35), (4.36), (4.37), (4.39) to get

$$\left|\langle \widetilde{X}_t^{1,\epsilon},1\rangle - \langle \widetilde{X}_t^{1,\delta},1\rangle\right| \le 14\overline{\alpha},$$

A similar inequality holds for $i=2$. The result follows now from Lemma 4.3. □

Proof of Theorem 2.2. Let

$$\begin{aligned}\widetilde{F}_t^{i,\epsilon}(x) &= \widetilde{X}_t^{i,\epsilon}\left((-\infty,x]\right), \quad i=1,2,\\ F_t^{\epsilon}(x) &= X_t^{\epsilon}\left((-\infty,x]\right),\end{aligned}$$

be the distribution functions of $\widetilde{X}_t^{i,\epsilon}$ and X_t^{ϵ} (respectively) defined for every $t\ge 0$. Since the support of $\widetilde{X}_t^{1,\epsilon}$ lies to the left of the support of $\widetilde{X}_t^{2,\epsilon}$ and $X^{\epsilon} = \widetilde{X}^{1,\epsilon} + \widetilde{X}^{2,\epsilon}$ we get

$$\widetilde{F}_t^{1,\epsilon}(x) = \min\left(F_t^{\epsilon}(x), \langle \widetilde{X}_t^{1,\epsilon},1\rangle\right), \tag{4.40}$$

$$\widetilde{F}_t^{2,\epsilon}(x) = \max\left(F_t^{\epsilon}(x) - \langle \widetilde{X}_t^{1,\epsilon},1\rangle,\ 0\right). \tag{4.41}$$

By Lemma 4.5, $\{(\langle \widetilde{X}^{1,\epsilon},1\rangle, \langle \widetilde{X}^{2,\epsilon},1\rangle), \epsilon\le 1\}$ is a Cauchy family, in the sense of convergence in probability. Moreover $X^{\epsilon} = \widetilde{X}^{1,\epsilon} + \widetilde{X}^{2,\epsilon}$ converges in D_{M_F} to $X\in C_{M_F}$, in probability. This, and (4.40), (4.41) imply that $\widetilde{F}^{i,\epsilon}$, $i=1,2$, converge uniformly on the compacts of $\mathbb{R}_+\times\mathbb{R}$, in probability, and hence $(\widetilde{X}^{1,\epsilon}, \widetilde{X}^{2,\epsilon})$ converges in $D_{M_F\times M_F}$ in probability, as $\epsilon\downarrow 0$. □

Next we are going to extend the above result to define reflecting super-Brownian motions starting at any time $s \geq 0$. Fix an arbitrary $s \geq 0$, and let $\widetilde{X}_s^{1,s}$ and $\widetilde{X}_s^{2,s}$ be any random measures in M_F such that

1. $\widetilde{X}_s^{1,s} + \widetilde{X}_s^{2,s} = X_s$.
2. If $\langle \widetilde{X}_s^{1,s}, 1\rangle > 0$, $\langle \widetilde{X}_s^{2,s}, 1\rangle > 0$ then the support of $\widetilde{X}_s^{1,s}$ lies to the left of the support of $\widetilde{X}_s^{2,s}$, that is for any $x_i \in \mathrm{supp}(\widetilde{X}_s^{i,s})$, $i = 1, 2$, we have $x_1 \leq x_2$.
3. $(\widetilde{X}_s^{1,s}, \widetilde{X}_s^{2,s}) \in \mathcal{F}_s{}^X := \sigma\{X_t\,, t \leq s\}$.

We will consider truncations of all superprocesses, historical processes with and without reflection, etc. to the time interval $[s, \infty)$ (the related ϵ-approximations of superprocesses and historical processes will be truncated to the time interval $[s + 2\epsilon, \infty)$). Let $\{Y_t^{s,\epsilon}, t \geq s + 2\epsilon\}$ be the natural truncation of $\{Y_t^{\epsilon}, t \geq 2\epsilon\}$, and note that it is the same as the approximating branching system constructed on the basis of the historical process $\{Y_t, t \geq 0\}$ truncated to the interval $[s, \infty)$. In other words, one obtains the same process $\{Y_t^{s,\epsilon}, t \geq s + 2\epsilon\}$, no matter which operation is performed first on $\{Y_t, t \geq 0\}$ — truncation to the interval $[s, \infty)$ or passing to the semi-discrete approximation. Let $\{(\widetilde{X}_t^{1,s,\epsilon}, \widetilde{X}_t^{2,s,\epsilon}),\ t \geq s + 2\epsilon\}$ be a pair of approximating reflecting super-Brownian motions defined as in Section 2 but this time relative to $\{Y_t^{s,\epsilon}, t \geq s + 2\epsilon\}$. The processes start at time $t = s + 2\epsilon$ at

$$\widetilde{X}_{s+2\epsilon}^{i,s,\epsilon}(dx) = \mathbf{1}_{\{x \in \mathrm{supp}(\widetilde{X}_s^{i,s})\}} X_{s+2\epsilon}^{\epsilon}(dx), \quad i = 1, 2.$$

Proposition 4.6. *There exists a pair of processes* $\{(\widetilde{X}_t^{1,s}, \widetilde{X}_t^{2,s}),\ t \geq s\}$ *in* $C_{M_F \times M_F}[s, \infty)$ *("reflecting super-Brownian motions") such that*

$$(\widetilde{X}^{1,s,\epsilon}, \widetilde{X}^{2,s,\epsilon}) \to (\widetilde{X}^{1,s}, \widetilde{X}^{2,s}),$$

as $\epsilon \downarrow 0$, *in probability in* $D_{M_F \times M_F}[s, \infty)$.

Proof. Note that X_s is absolutely continuous with respect to Lebesgue measure P-a.s.. Also, it is well-known (see e.g. Corollary III.1.7 of [11]) that the range of X is compact, P-a.s.. Hence X_s satisfies the same assumptions as $X_0 = \mu$ before. Moreover, P-a.s., $\widetilde{X}_s^{1,s}, \widetilde{X}_s^{2,s}$ also satisfy the same assumptions as μ_1 , μ_2. Therefore, for any $\overline{\alpha} > 0$, for P-a.s. ω,

$$\lim_{\epsilon,\delta\downarrow 0, \epsilon>\delta} P\left(\sup_{t\geq s+2\epsilon} \left|\langle \widetilde{X}_t^{i,s,\epsilon}, 1\rangle - \langle \widetilde{X}_t^{i,s,\delta}, 1\rangle\right| \geq \overline{\alpha} \,\middle|\, \widetilde{X}_s^{1,s}, \widetilde{X}_s^{2,s}\right)(\omega) = 0, \quad i = 1, 2,$$

by Lemma 4.5. Hence, by dominated convergence we obtain that

$$\lim_{\epsilon,\delta\downarrow 0, \epsilon>\delta} P\left(\sup_{t\geq s+2\epsilon} \left|\langle \widetilde{X}_t^{i,s,\epsilon}, 1\rangle - \langle \widetilde{X}_t^{i,s,\delta}, 1\rangle\right| \geq \overline{\alpha}\right) = 0, \qquad i = 1, 2,$$

and then we can proceed as in the proof of Theorem 2.2. □

We will generalize the last result even further by passing from pairs to families of reflecting super-Brownian motions. For a real $s \geq 0$ and integers $i \in \mathbb{Z}$, $n \geq 1$, let $\mu^{i,s,n} \in M_F$ be the truncation of X_s to $[i2^{-n}, (i+1)2^{-n})$, i.e.,

$$\mu^{i,s,n}(A) = X_s(A \cap [i2^{-n}, (i+1)2^{-n})), \quad A \subset \mathbb{R}.$$

For a fixed $s \geq 0$ and $\epsilon > 0$, let $\{(\widetilde{X}_t^{i,s,\epsilon,n}, \ t \geq s + 2\epsilon, i \in \mathbb{Z}\}$ be the family of approximating reflecting super-Brownian motions constructed from $\{Y_t^{s,\epsilon}, t \geq s + 2\epsilon\}$ in a way similar to that in Section 2, with different processes starting at time $t = s + 2\epsilon$ from

$$\widetilde{X}_{s+2\epsilon}^{i,s,\epsilon,n}(dx) = \mathbf{1}_{\{x \in \mathrm{supp}(\mu^{i,s,n})\}} X_{s+2\epsilon}^{\epsilon}(dx), \quad i \in \mathbb{Z}.$$

Let $J_s^\epsilon \equiv \left\{ i : \langle \widetilde{X}_{s+2\epsilon}^{i,s,\epsilon,n}, 1 \rangle > 0 \right\}$, and $\left| J_s^\epsilon \right|$ be the total number of elements in J_s^ϵ. As we have mentioned at the beginning of the proof of Proposition 4.6, the range of X is compact, P-a.s., and hence

$$\sup_{0<\epsilon\leq 1} \sup_{s\geq 0} \left| J_s^\epsilon \right| < \infty, \quad P - \text{a.s.}. \tag{4.42}$$

Theorem 4.7. *Fix an $s \geq 0$. For each $n \geq 1$ there exists a family of reflecting super-Brownian motions $\{\widetilde{X}_t^{i,s,n}, t \geq s, \ i \in \mathbb{Z}\}$ such that*

- **(a)** $\widetilde{X}_s^{i,s,n} = \mu^{i,s,n}, \ \forall i \in \mathbb{Z}$.
- **(b)** $\lim_{\epsilon \downarrow 0} \left(\widetilde{X}^{i,s,\epsilon,n}\right)_{i \in \mathbb{Z}} = \left(\widetilde{X}^{i,s,n}\right)_{i\in\mathbb{Z}}$, *in* $\left(D_{M_F}[s,\infty)\right)^\infty$, *in probability.*
- **(c)** *For any $t \geq s, i \in \mathbb{Z}$, such that $\langle \widetilde{X}_t^{i,s,n}, 1\rangle > 0$ and $\langle \widetilde{X}_t^{i+1,s,n}, 1\rangle > 0$, the support of $\widetilde{X}_t^{i,s,n}$ lies to the left of the support of $\widetilde{X}_t^{i+1,s,n}$.*

Proof. Fix arbitrary $s \geq 0$ and $n \geq 1$. For each $j \in \mathbb{Z}$ define

$$\begin{aligned}
\widetilde{X}_t^{1,j,s,\epsilon} &\equiv \sum_{i:i\leq j} \widetilde{X}_t^{i,s,\epsilon,n}, \quad t \geq s + 2\epsilon, \\
\widetilde{X}_t^{2,j,s,\epsilon} &\equiv \sum_{i:i\geq j+1} \widetilde{X}_t^{i,s,\epsilon,n}, \quad t \geq s + 2\epsilon.
\end{aligned}$$

By Proposition 4.6

$$(\widetilde{X}^{1,j,s,\epsilon}, \widetilde{X}^{2,j,s,\epsilon}) \to (\widetilde{X}^{1,j,s}, \widetilde{X}^{2,j,s}),$$

as $\epsilon \downarrow 0$, in probability in $D_{M_F \times M_F}[s,\infty)$ for every $j \in \mathbb{Z}$. Hence for any $K \geq 1$,

$$(\widetilde{X}^{1,j,s,\epsilon}, \widetilde{X}^{2,j,s,\epsilon})_{j\in\{-K,\dots,K\}} \to (\widetilde{X}^{1,j,s}, \widetilde{X}^{2,j,s})_{j\in\{-K,\dots,K\}},$$

as $\epsilon \downarrow 0$, in probability in $(D_{M_F\times M_F}[s,\infty))^{2K+1}$. Now recall that

$$\widetilde{X}_t^{j,s,\epsilon,n} \equiv \widetilde{X}_t^{1,j,s,\epsilon} - \widetilde{X}_t^{1,j-1,s,\epsilon}, \quad t \geq s + 2\epsilon.$$

Hence for any $K \geq 1$,

$$(\widetilde{X}^{j,s,\epsilon,n})_{j\in\{-K+1,\dots,K\}} \to (\widetilde{X}^{j,s,n})_{j\in\{-K+1,\dots,K\}},$$

as $\epsilon \downarrow 0$, in $(D_{M_F}[s,\infty))^{2K}$, in probability. Now recall that, by (4.42) the number of non identically zero processes $\widetilde{X}^{j,s,\epsilon,n}$, $j \in \mathbb{Z}$, is finite, P-a.s. Therefore we immediately get

$$\lim_{\epsilon\downarrow 0}\left(\widetilde{X}^{i,s,\epsilon,n}\right)_{i\in\mathbb{Z}} = \left(\widetilde{X}^{i,s,n}\right)_{i\in\mathbb{Z}} \text{ in } \left(D_{M_F}[s,\infty)\right)^{\infty},$$

in probability, and the result follows. □

5. Proof of Theorem 2.1

Recall our conventions from the "Notation" section at the end of the Introduction. For any $x \in \mathbb{R}$ define $[x]_n = \frac{[x2^n]}{2^n}$. For any $k, n \geq 1$ consider the following family of functions:

$$D^{k,n} = \left\{\psi \in B\left(\mathbb{R}^k\right) : \ \psi(x_1, \dots, x_k) = \psi([x_1]_n, \dots, [x_k]_n),\ \forall (x_1, \dots, x_k) \in \mathbb{R}^k\right\}.$$

In words, $D^{k,n}$ consists of functions which are constant on rectangles (or cubes) of the form $[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}) \times \cdots \times [\frac{i_k}{2^n}, \frac{i_k+1}{2^n})$. Next we define some families of functions on $C_{\mathbb{R}}[0,\infty)$:

$$\begin{aligned} D_t &= \Big\{\psi \in C(C_{\mathbb{R}}[0,\infty)) : \ \psi(y) = \overline{\psi}(y(t_1), \dots, y(t_k)) \\ &\qquad \text{for some } \overline{\psi} \in C_u(\mathbb{R}^k), \quad 0 \leq t_1 < \cdots < t_k < t,\ k \geq 1\Big\}, \\ D_t^{k,n} &= \Big\{\psi \in B(C_{\mathbb{R}}[0,\infty)) : \ \psi(y) = \overline{\psi}(y(t_1), \dots, y(t_k)) \\ &\qquad \text{for some } \overline{\psi} \in D^{k,n}, \quad 0 \leq t_1 < \cdots < t_k < t\Big\}. \end{aligned}$$

The following two auxiliary lemmas will help us complete the proof of Theorem 2.1.

Lemma 5.1. *Fix arbitrary $t > 0$. Let $\{Z^\epsilon,\ 0 < \epsilon \leq 1\}$ be a family of $M_F(C[0,t])$-valued random variables, whose laws are tight in the space of all probability measures on $M_F(C[0,t])$. Suppose for each $k, n \geq 1$ and $\psi \in D_t^{k,n}$ there exists a random variable Z^ψ such that $\langle Z^\epsilon, \psi\rangle \to Z^\psi$*, as $\epsilon \downarrow 0$, *in probability. Then there exists a random measure $Z \in M_F(C[0,t])$ such that*

$$Z^\epsilon \to Z\,, \text{as } \epsilon \downarrow 0,$$

in $M_F(C[0,t])$, in probability.

Lemma 5.2. *Let $\{\widetilde{Y}^\epsilon,\ 0 < \epsilon \leq 1\}$ be a family of processes in $D_{M_F(C)}[0,\infty)$ whose laws are tight in the space of all probability measures on $D_{M_F(C)}[0,\infty)$, and any limit law is supported by $C_{M_F(C)}[0,\infty)$. If for each $t > 0$, $\widetilde{Y}_t^\epsilon$ converges in $M_F(C)$, in probability, then there exists a process $\widetilde{Y} \in C_{M_F(C)}[0,\infty)$ such that*

$$\widetilde{Y}^\epsilon \to \widetilde{Y}, \quad \text{as } \epsilon \downarrow 0,$$

in $D_{M_F(C)}[0,\infty)$, in probability.

Proof of Lemma 5.1. We adopt the argument which is used for the proof of Lemma A.5 of [2]. Let d be a metric on $M_F(C[0,t])$. It will suffice to show that $\{Z^\epsilon\}$ is a Cauchy sequence in the metric d, in probability. We will argue by contradiction. Suppose that $\{Z^\epsilon\}$ is not Cauchy, that is, there exist $\eta > 0$, $M > 0$, $\{\epsilon_m, \delta_m, m \geq 1\}$ such that $\epsilon_m, \delta_m \downarrow 0$, as $m \to \infty$, and

$$P\left(d(Z^{\epsilon_m}, Z^{\delta_m}) \geq \eta\right) \geq \eta, \ \forall m \geq M.$$

By our assumptions we can choose a subsequence $(Z^{\epsilon'_m}, Z^{\delta'_m})$ which converges in $M_F(C[0,t]) \times M_F(C[0,t])$ in law. Let (Z', Z'') be its limit point defined possibly on another probability space. Now note that the set of functions $\{D_t^{k,n}, k, n \geq 1\}$ is dense in D_t in the uniform topology. Hence we easily see that,

$$\lim_{m\to\infty} P\left(\left|\langle Z^{\epsilon'_m}, \psi\rangle - \langle Z^{\delta'_m}, \psi\rangle\right| \geq \gamma\right) = 0, \quad \forall \psi \in D_t, \ \gamma > 0.$$

Since the functions in D_t are continuous on $M_F(C[0,t])$ and D_t separates measures in $M_F(C[0,t])$ we immediately get that

$$Z' = Z'', \quad P - \text{a.s.}.$$

But this implies that

$$\lim_{m\to\infty} P\left(d(Z^{\epsilon'_m}, Z^{\delta'_m}) \geq \eta\right) = 0,$$

yielding the contradiction and we are done. □

Proof of Lemma 5.2. We again adopt the argument which is used for the proof of Lemma A.5 of [2]: some changes are required, therefore we give the proof here.

Let d be a metric on $M_F(C)$ and $\vec{d}$ be a metric on $D_{M_F(C)}$. It is enough to show that $\{\widetilde{Y}^\epsilon\}$ is a Cauchy sequence in $\vec{d}$ in probability. We will argue by contradiction. Suppose that $\{\widetilde{Y}^\epsilon\}$ is not Cauchy, i.e., there exist $\eta > 0$, $M > 0$, $\{\epsilon_m, \delta_m, m \geq 1\}$ such that $\epsilon_m, \delta_m \downarrow 0$, as $m \to \infty$, and

$$P\left(\vec{d}(\widetilde{Y}^{\epsilon_m}, \widetilde{Y}^{\delta_m}) \geq \eta\right) \geq \eta, \ \forall m \geq M.$$

By our assumptions we can choose a subsequence $(\widetilde{Y}^{\epsilon'_m}, \widetilde{Y}^{\delta'_m})$ which converges in $D_{M_F(C)} \times D_{M_F(C)}$ in law. Let $(\widetilde{Y}', \widetilde{Y}'') \in C_{M_F(C)} \times C_{M_F(C)}$ be its limit point defined possibly on another probability space. On the other hand by our assumption that $\widetilde{Y}_t^\epsilon$ converges in $M_F(C)$, in probability, for each $t > 0$, we get

$$P\left(d(\widetilde{Y}_t^{\epsilon'_m}, \widetilde{Y}_t^{\delta'_m}) \geq \eta\right) \to 0, \ \forall t \geq 0.$$

Hence,

$$\widetilde{Y}'_t = \widetilde{Y}''_t, P - a.s., \quad \forall t \geq 0.$$

Since $(\widetilde{Y}', \widetilde{Y}'') \in C_{M_F(C)} \times C_{M_F(C)}$, we immediately obtain

$$\widetilde{Y}'_t = \widetilde{Y}''_t, \quad \forall t \geq 0, \ P - a.s. \tag{5.1}$$

Let $\widetilde{Y} = \widetilde{Y}'_t = \widetilde{Y}''_t$. Then we have

$$(\widetilde{Y}^{\epsilon'_m}, \widetilde{Y}^{\delta'_m}) \to (\widetilde{Y}, \widetilde{Y})$$

in law. However this means that

$$P\left(\vec{d}(\widetilde{Y}^{\epsilon'_m}, \widetilde{Y}^{\delta'_m}) \geq \eta\right) \to 0,$$

yielding the contradiction and we are done. □

Fix arbitrary $t > 0$, $n \geq 1$, and $0 \leq s_1 < \cdots < s_K < t$. Let $\{\widetilde{X}_s^{i,s_k,n}, s \geq s_k, i \in \mathbb{Z}, k = 1, \ldots, K\}$ be the family of reflecting super-Brownian motions constructed in Theorem 4.7. Let $\widetilde{l}^{k,i,n,t}$ ($\widetilde{r}^{k,i,n,t}$) be the left (right) endpoint of the support of $\widetilde{X}_t^{i,s_k,n}$ defined whenever $\langle \widetilde{X}_t^{i,s_k,n}, 1\rangle > 0$. Note that we always have $\widetilde{r}^{k,i,n,t} \leq \widetilde{l}^{k,i+1,n,t}$ but not necessarily $\widetilde{r}^{k,i,n,t} = \widetilde{l}^{k,i+1,n,t}$. Define

$$I_k \equiv \{i \in \mathbb{Z} : \langle \widetilde{X}_t^{i,s_k,n}, 1\rangle > 0\}$$

and let $|I_k|$ be the total number of elements in I_k. Let $M_{n,t}$ be the set of all real numbers $\widetilde{l}^{k,i,n,t}$ and $\widetilde{r}^{k,i,n,t}$ for $k = 1, 2, \ldots, K$, $i \in I_k$.

Lemma 5.3. *With probability* 1*, only a finite number of processes in the family* $\{\widetilde{X}^{i,s_k,n}(s), s \geq s_k,\ i \in \mathbb{Z}, k = 1, \ldots, K\}$ *survive up to time* $t > s_K$*, that is,*

$$\sum_{k=1}^{K} |I_k| < \infty, \qquad P - \text{a.s.}$$

Proof. As we have mentioned in the proof of Proposition 4.6 (see also Corollary III.1.4 of [11]), the support of X_t is compact for any $t \geq 0$, P-a.s.; note that for $t = 0$ this is the consequence of our assumption on $X_0 = \mu$. Therefore for any time s_k, $k = 1, \ldots, K$,

$$|J_k| \equiv \left|\left\{i : \langle \widetilde{X}_{s_k}^{i,s_k,n}, 1\rangle > 0\right\}\right| < \infty, \quad P - a.s.$$

Clearly, by construction of our processes $I_k \subset J_k$, $k = 1, \ldots, K$, and the result follows immediately. □

Proposition 5.4. *Let* $\widetilde{Y}^\epsilon$ *be as in Theorem 2.1. For any* $K, n \geq 1$, $t > 0$, $\psi \in D_t^{K,n}$,

$$P\left(\left|\langle \widetilde{Y}_t^{\epsilon_1}, \psi\rangle - \langle \widetilde{Y}_t^{\epsilon_2}, \psi\rangle\right| > \eta\right) \to 0, \quad as\ \ \epsilon_1, \epsilon_2 \downarrow 0,\ \forall \eta > 0. \tag{5.2}$$

Proof. Fix an arbitrary $\overline{\psi} \in D^{K,n}$, and let

$$\psi(y) = \overline{\psi}(y(s_1), \ldots, y(s_K)). \tag{5.3}$$

Since $n \geq 1$, $s_1 < \cdots < s_K$ were arbitrary it is enough to show (5.2) for this ψ. Fix η, $p > 0$ arbitrary small. Let $A(\gamma) = \{x \in \mathbb{R} : \text{dist}(x, M_{n,t}) > \gamma\}$. Recall that X_t has a continuous density and hence, by Lemma 5.3, we can find $\gamma > 0$ so small that

$$P(X_t(A(\gamma)^c) > \eta/(4\,\|\psi\|_\infty)) < p.$$

We will fix such γ till the end of the proof of the proposition. By Theorem 4.7 and reflection properties of $\widetilde{X}^{i,s_k,\epsilon,n}$ we immediately get that for each $k = 1, \dots, K$ and $i \in \mathbb{Z}$,

$$\lim_{\epsilon\downarrow 0} P\left(\widetilde{X}_t^{j,s_k,\epsilon,n}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)) > 0 \,\Big|\, \langle \widetilde{X}_t^{i,s_k,n}, 1\rangle > 0\right) = 0, \quad \forall j \neq i.$$

Apply Lemma 5.3 to show that there exists $\epsilon^* > 0$ so small that for $\epsilon < \epsilon^*$

$$P\Big(\widetilde{X}_t^{j,s_k,\epsilon,n}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)) > 0, \ \text{ for some } j \neq i, i \in I_k, k = 1, \dots K\Big) < p/2.$$

Hence for any $\epsilon_1, \epsilon_2 < \epsilon^*$, we can define $\Gamma^{\epsilon_1,\epsilon_2} \subset \Omega$ such that $P(\Gamma^{\epsilon_1,\epsilon_2}) \geq 1 - p$ and for any $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$

$$\widetilde{X}_t^{j,s_k,\epsilon_m,n}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)) = 0, \quad \forall\ j \neq i, i \in I_k, k = 1, \dots K, m = 1, 2. \tag{5.4}$$

In other words for any $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$, we have

$$X_t^{\epsilon_m}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)) = \widetilde{X}_t^{i,s_k,\epsilon_m,n}((\widetilde{l}^{k,i,n,t}+\gamma, \widetilde{r}^{k,i,n,t}-\gamma)),$$
$$\forall i \in I_k, \quad k = 1, \dots K, m = 1, 2.$$

Fix an arbitrary $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$, and any $\widetilde{y}^1, \widetilde{y}^2 \in \mathrm{supp}(\widetilde{Y}_t^{\epsilon_1}) \cup \mathrm{supp}(\widetilde{Y}_t^{\epsilon_2})$ such that for every $k = 1, \dots, K$, and some $i_k \in I_k$, we have $\widetilde{y}^1(t), \widetilde{y}^2(t) \in (\widetilde{l}^{k,i_k,n,t} + \gamma, \widetilde{r}^{k,i_k,n,t} - \gamma)$. Such choice of $\widetilde{y}^1(t), \widetilde{y}^2(t)$ and (5.4) imply that

$$[\widetilde{y}^1(s_k)]_n = [\widetilde{y}^2(s_k)]_n = \frac{i_k}{2^n}, \quad \forall k = 1, \dots, K. \tag{5.5}$$

Since ψ is given by (5.3), it follows from (5.5) that

$$\psi(\widetilde{y}^1) = \psi(\widetilde{y}^2), \tag{5.6}$$

for above $\widetilde{y}^1, \widetilde{y}^2$.

Now we can represent the set $A(\gamma)$ as follows:

$$A(\gamma) = \bigcup_{l=1}^{\infty} A^l(\gamma),$$

where $A^l(\gamma)$ are open connected intervals such that $A^l(\gamma) \cap A^m(\gamma) = \emptyset$, for all $l \neq m$. In fact, by Lemma 5.3, there is only a finite number of non-empty sets $A^l(\gamma)$. Now, for any $l \geq 1$ such that $X_t(A^l(\gamma)) > 0$, and $k = 1, \dots, K$, there exists a unique $j(l, k)$ such that

$$A^l(\gamma) \subset (\widetilde{l}^{k,j(l,k),n,t} + \gamma, \widetilde{r}^{k,j(l,k),n,t} - \gamma). \tag{5.7}$$

By (5.6) and (5.7) we get that ψ is constant on each of the sets

$$\{\widetilde{y} \in \mathrm{supp}(\widetilde{Y}_t^{\epsilon_1}) \cup \mathrm{supp}(\widetilde{Y}_t^{\epsilon_2}) : \ \ \widetilde{y}(t) \in A^l(\gamma)\}, \ l \geq 1.$$

Hence for any $l \geq 1$ we can define

$$\psi_l \equiv \begin{cases} \psi(\widetilde{y}), & \text{for } \{\widetilde{y} \in \mathrm{supp}(\widetilde{Y}_t^{\epsilon_1}) \cup \mathrm{supp}(\widetilde{Y}_t^{\epsilon_2}) : \ \widetilde{y}(t) \in A^l(\gamma)\}, \text{ if } X_t(A^l(\gamma)) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and we have

$$\int \psi(\widetilde{y})\mathbf{1}_{\{\widetilde{y}(t)\in A^l(\gamma)\}}\widetilde{Y}_t^{\epsilon_m}(d\widetilde{y}) = X_t^{\epsilon_m}(A^l(\gamma))\psi_l\,, \quad \forall l \geq 1,\ m = 1, 2,$$

for any $\omega \in \Gamma^{\epsilon_1,\epsilon_2}$. By combining all our estimates, we obtain

$$\begin{aligned}
&P\left(\left|\langle \widetilde{Y}_t^{\epsilon_1}, \psi\rangle - \langle \widetilde{Y}_t^{\epsilon_2}, \psi\rangle\right| > \eta\right) \\
&\leq P\left(\left\{\left|\sum_{l=1}^{\infty}(X_t^{\epsilon_1}(A^l(\gamma)) - X_t^{\epsilon_2}(A^l(\gamma)))\psi_l\right| > \eta/2\right\} \bigcap \Gamma^{\epsilon_1,\epsilon_2}\right) + p \\
&\quad + P\left((X_t^{\epsilon_1}(A(\gamma)^c) + X_t^{\epsilon_2}(A(\gamma)^c))\,\|\psi\|_\infty > \eta/2\right) \\
&\to p + P\left((X_t(A(\gamma)^c) > \eta/(4\,\|\psi\|_\infty)\right) \qquad (\text{as } \epsilon_1, \epsilon_2 \downarrow 0) \\
&\leq 2p,
\end{aligned}$$

and since p was arbitrary the proof is complete. □

Proof of Lemma 2.1. It follows immediately from Proposition 5.4 that for any K, $n \geq 1, t > 0, \psi \in D_t^{K,n}$, $\langle \widetilde{Y}_t^\epsilon, \psi\rangle$ converges in probability. By Theorem 1.1 of [1], any limit law of $\widetilde{Y}_t^\epsilon$ belongs to $C_{M_F(C)}[0,\infty)$. Hence by Lemmas 5.1 and 5.2 we see that there exists a process $\widetilde{Y} \in C_{M_F(C)}[0,\infty)$ such that

$$\widetilde{Y}^\epsilon \to \widetilde{Y}, \quad \text{as } \epsilon \downarrow 0,$$

in $D_{M_F(C)}[0,\infty)$, in probability, and the proof is complete. □

Acknowledgements. The first author would like to express his gratitude for the opportunity to visit Technion (Israel) and Institut Mittag-Leffler (Sweden) where this research was partially done. The second author is grateful to Jean-François Le Gall for several helpful discussions of the problem. We thank the referee for a very careful reading of the original manuscript and many suggestions for improvement.

References

1. Burdzy, K., Le Gall, J.F.: Super-Brownian motion with reflecting historical paths. Probab. Theory Relat. Fields **121**, 447–491 (2001)
2. Dawson, D. Perkins, E.: Historical Processes. Memoirs of the American Mathematical Society 454, 1991
3. Doob, J.L.: Measure theory. Springer-Verlag, 1993
4. Etheridge, A.M.: An Introduction to Superprocesses. University Lecture Series, 20. American Mathematical Society, Providence, RI, 2000
5. Ethier, S.N., Kurtz, T.G.: Markov Process: Characterization and Convergence. John Wiley and Sons, New York, 1986
6. Konno, N., Shiga, T.: Stochastic partial differential equations for some measure-valued diffusions. Probab. Theory Relat. Fields **79**, 201–225 (1988)
7. Le Gall, J.F.: Spatial Branching Processes, Random Snakes and Partial Differential Equations. Birkhäuser, ETH Zürich, 1999
8. Neveu, J.: Erasing a branching tree. Supplement to Adv. Applied Probability 1986, pp. 101–108

9. Neveu, J., Pitman, J.: The branching process in a Brownian excursion. Séminaire de Probabilités XXIII. Lecture Notes in Mathematics **1372**, 248–257 (1989)
10. Neveu, J., Pitman, J.: Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. Séminaire de Probabilités XXIII. Lecture Notes in Mathematics **1372**, 239–247 (1989)
11. Perkins, E.: Dawson-Watanabe Superprocesses and Measure-valued Diffusions. Lectures on probability theory and statistics (Saint-Flour, 1999), Lecture Notes in Mathematics **1781**, 125–329 (2002)
12. Reimers, M.: One-dimensional stochastic partial differential equations and the branching measure diffusion. Probab. Theory Relat. Fields **81**, 319–340 (1989).