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Laplace's method for iterated complex Brownian integrals

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Abstract. A kind of Laplace's method is developped for iterated stochastic integrals where integrators are complex standard Brownian motions. Then it is used to extend properties of Bougerol and Jeulin's path transform in the random case when simple representations of complex semisimple Lie algebras are not supposed to be minuscule.

1. Introduction

1.1. Background

The Gaussian Unitary Ensemble (G.U.E.) is the probability measure

$$Z_d{}^{-1} \exp\Big(-\mathrm{tr}(M^2)/2\Big)\,\mathrm{d}M$$

on the linear space of $d \times d$ Hermitian matrices where $\mathrm{d}M$ is the Lebesgue measure and Z_d a normalization constant. Baryshnikov [3] and Gravner, Tracy & Widom [7] have provided a path representation of the largest eigenvalue λ_1 of the G.U.E.:

Theorem 1.1. *Let $W_1, \dots, W_d$ be independent standard real Brownian motions. Then λ_1 has the same distribution as*

$$\max_{1\geqslant t_1\geqslant\cdots\geqslant t_{d-1}\geqslant 0} W_1(1) - W_1(t_1) + W_2(t_1) + \cdots - W_{d-1}(t_{d-1}) + W_d(t_{d-1}).$$

Using a geometric approach, Bougerol & Jeulin [6] have given a similar formula for all the eigenvalues. Let us recall the general set–up: consider a complex reductive Lie algebra $\mathfrak{g}$, that is $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ is semisimple and $\mathfrak{g}_2$ is the center. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition, $\mathfrak{a}^+$ a Weyl chamber of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ ($\mathfrak{a}^+ = \mathfrak{a}_1^+ + \mathfrak{g}_2$ where $\mathfrak{a}_1^+$ is a Weyl chamber in $\mathfrak{g}_1$). We equip $\mathfrak{p}$ with an Euclidean structure such that $\mathfrak{p} \cap \mathfrak{g}_1$ and $\mathfrak{p} \cap \mathfrak{g}_2$ are orthogonal and

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its restriction to $\mathfrak{p} \cap \mathfrak{g}_1$ is given by the Killing form. For any $X \in \mathfrak{p}$, we define $\mathrm{rad}(X) \in \overline{\mathfrak{a}^+}$ as the only element in the closure $\overline{\mathfrak{a}^+}$ of $\mathfrak{a}^+$ such that

$$X \in \mathrm{Ad}(K)\mathrm{rad}(X),$$

where K is the compact group with Lie algebra $\mathfrak{k}$. Let $C_0([0,1],\mathfrak{a})$ (resp. $C_0([0,1],\overline{\mathfrak{a}^+})$) be the set of continuous paths $w : [0,1] \to \mathfrak{a}$ (resp. $w : [0,1] \to \overline{\mathfrak{a}^+}$) such that $w(0) = 0$. Bougerol & Jeulin have introduced a continuous projection $\mathcal{T} : C_0([0,1],\mathfrak{a}) \to C_0([0,1],\overline{\mathfrak{a}^+})$ and they have proved for several cases the following property:

Property 1.2. *Let Z be the Euclidean Brownian motion on $\mathfrak{p}$ and W the Euclidean Brownian motion on $\mathfrak{a}$. Then the process* $\mathrm{rad}(Z)$ *has the same law as* $\mathcal{T}W$.

Actually, Bougerol & Jeulin [6] proved that property 1.2 holds when $\mathfrak{g}_1 = \mathfrak{sl}(d+1,\mathbb{C})$, and it was enough in order to give a path representation of the eigenvalues of the G.U.E., by embedding the G.U.E. in the Euclidean Brownian motion Z.

When $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, then $\mathfrak{a} = \mathbb{R}$, $\overline{\mathfrak{a}^+} = \mathbb{R}_+$ and

$$(\mathcal{T}w)(t) = w(t) - 2 \min_{t \geqslant s \geqslant 0} w(s). \tag{1}$$

This path transform occurs in Littelmann [9] in connection with representation theory of complex groups, and in Pitman [12], where it is shown that when W is the usual Brownian motion, then $\mathcal{T}W$ is a three–dimensional Bessel process. Such an elementary transformation (1) is called Pitman transform. O'Connell & Yor [10] have extended Pitman's theorem by a completely different approach of the eigenvalues of the G.U.E.: starting from Burke's reversibility property of queues, they have proposed a path transformation which can be seen as iteration of elementary Pitman transforms. More recently, Biane, Bougerol & O'Connell [5] have extended results of both [6] and [10] by a completely different proof, valid for all root systems.

1.2. Organization of the paper and main results

The aim of this paper is to extend property 1.2 to any complex reductive Lie algebra. Since a reductive Lie algebra is the orthogonal sum of an Euclidean space and a semisimple Lie algebra, it is enough to consider the semisimple case (see [6] for details on the reductive case and path representation of the G.U.E. eigenvalues). To this end, we first develop a kind of Laplace's method for iterated stochastic integrals with respect to complex standard Brownian motions. This method provides a bypass to the criterion based on minuscule representations that is proved and used in [6]. Our approach works only in the complex case, since we need the invariance in law of complex brownian motion under rotations. Here is the result that justifies the title:

Theorem 1.3. *Let d denote a positive integer, S the set $\{1,\dots,d\}$ and $\beta^1,\dots,\beta^d$ complex independent Brownian motions, defined on the same filtered probability space.*

Let $g_1, \dots, g_d : [0,1] \to \mathbb{R}$ *be continuous functions and* $r \in \mathbb{N}^*$*. For* $p \in \{1, \dots, r\}$, $\alpha \in S^p$*, let* $(u_n^{p,\alpha})_{n\in\mathbb{N}^*}$ *be a sequence of nonzero complex numbers such that*

$$\lim_{n\to\infty} \frac{1}{n} \log |u_n^{p,\alpha}| = 0,$$

and $\left(\lambda_\alpha^p\right)_{\alpha\in S^p}$ *a family of complex numbers. Assume that one of the* λ_α^r*'s is nonzero and put*

$$X_n = \sum_{p=1}^{r} \sum_{\alpha\in S^p} \lambda_\alpha^p u_n^{p,\alpha} \int_0^1 \left(\int_0^{t_1} \dots \left(\int_0^{t_{p-1}} \mathrm{e}^{n g_{\alpha_p}(t_p)} \mathrm{d}\beta_{t_p}^{\alpha_p}\right) \dots \mathrm{e}^{n g_{\alpha_2}(t_2)} \mathrm{d}\beta_{t_2}^{\alpha_2}\right) \times \mathrm{e}^{n g_{\alpha_1}(t_1)} \mathrm{d}\beta_{t_1}^{\alpha_1}.$$

Then we have, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \log |X_n| = \max_{1\leqslant p\leqslant r} \max_{\substack{\alpha\in S^p \\ \lambda_\alpha^p\neq 0}} \max_{\Delta_p} g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_p}.$$

Then it is quite easy to derive the path representation in the random case. Indeed let us denote by G a connected Lie group with Lie algebra $\mathfrak{g}$, by K the connected Lie subgroup with Lie algebra $\mathfrak{k}$, and by P the Riemannian symmetric space G/K and remember that

$$\mathrm{Exp} : \mathfrak{p} \to \mathrm{P}, \ X \mapsto \exp(X).K$$

is a diffeomorphism. We use finite irreducible representations π of $\mathfrak{g}$ and iterated stochastic integrals appear as the entries of the matrix representations of the process $\pi\left(\mathrm{rad}\left(\mathrm{Exp}^{-1}\left(B^\tau\right)\right)\right)$, where B^τ is the Doob h–transform of the Brownian motion B on P with respect to a ground state τ on P chosen in such a way that, in law,

$$\mathrm{rad}\,(Z) \sim \mathrm{rad}\left(\mathrm{Exp}^{-1}\left(B^\tau\right)\right).$$

By making use of the scaling invariance of B, it is then enticing to use Laplace's method to compute $\mathrm{rad}\left(\mathrm{Exp}^{-1}\left(B^\tau\right)\right)$ and particularly to prove that the lower bound always holds, in paragraph 2.2 and 3.3 (this was conjectured in [6]). Therefore we may state in the conclusion that property 1.2 is valid for any complex semisimple Lie algebra $\mathfrak{g}$:

Theorem 1.4. *For any complex semisimple Lie algebra* $\mathfrak{g}$*, the processes* $\mathrm{rad}(Z)$ *and* $\mathcal{T}W$ *have the same law.*

In the conclusion, we also state some deterministic properties of the path transform $\mathcal{T}$ as consequences of random properties, and particularly the fact that this transform takes its values in $C_0([0,1], \overline{\mathfrak{a}^+})$, which is not obvious in view of the very definition of $\mathcal{T}$ (see paragraph 3.5). This is another originality since $\mathcal{T}$ is introduced in [6] by means of deterministic dynamical systems on P. Relating to this, we do not refer explicitely to ground state processes, but we use stochastic exponential to introduce the process B^τ as the solution to a kind of stochastic dynamical system.

We end with the example of exceptionnal Lie algebra G_2 that was missing from [6], because fundamental representations of such a Lie algebra are not minuscule. For a similar reason, it is also possible to treat the case of the last fundamental representation of a Lie algebra of type B_l (which is not minuscule) in the same way as the others.

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2. Laplace's method

In this section, let d denote a positive integer, S the set $\{1, \dots, d\}$ and $\beta^1, \dots, \beta^d$ complex independent Brownian motions, defined on the same filtered probability space. Given $r \in \mathbb{N}^*$, let Δ_r denote the set $\big\{(t_1, \dots, t_r) \in \mathbb{R}^r \;\big|\; 1 \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_r \geqslant 0\big\}$.

2.1. Deterministic Laplace's method and other lemmas

Theorem 2.1. *Let K be the closure of a relatively compact non–empty open subset of $\mathbb{R}^d$, $f, g : K \to \mathbb{R}$ be continuous functions, where $f \geqslant 0$, and M be the maximum of g. Assuming that there exists $x_0 \in K$ such that $g(x_0) = M$ and $f(x_0) > 0$, we have*

$$\lim_{\varepsilon \to 0} \varepsilon \log \int_K \mathrm{e}^{\frac{1}{\varepsilon} g(x)} f(x)\,\mathrm{d}x = M.$$

Proof. Let $\eta > 0$. Since f and g are continuous, there exists $\rho > 0$ such that

$$M - \eta \leqslant g(x) \leqslant M \quad \text{and} \quad f(x) > \frac{f(x_0)}{2}, \text{ for any } x \in B(x_0, \rho) \cap K.$$

Let us write $B = B(x_0, \rho) \cap K$. By hypothesis on the nature of K, the Lebesgue measure $\mathrm{vol}(B)$ of B is positive. We have

$$\int_K f(x)\mathrm{e}^{\frac{1}{\varepsilon} g(x)}\mathrm{d}x \geqslant \int_B f(x)\mathrm{e}^{\frac{1}{\varepsilon} g(x)}\mathrm{d}x \geqslant \frac{f(x_0)}{2}\mathrm{e}^{\frac{1}{\varepsilon}(M-\eta)}\mathrm{vol}(B),$$

hence

$$\liminf_{\varepsilon \to 0} \varepsilon \log \int_K f(x)\mathrm{e}^{\frac{1}{\varepsilon} g(x)}\mathrm{d}x \geqslant M - \eta.$$

On the other hand,

$$\int_K f(x)\mathrm{e}^{\frac{1}{\varepsilon} g(x)}\mathrm{d}x \leqslant \mathrm{vol}(K)\mathrm{e}^{\frac{1}{\varepsilon} M} \max_K f,$$

hence

$$\limsup_{\varepsilon \to 0} \varepsilon \log \int_K f(x)\mathrm{e}^{\frac{1}{\varepsilon} g(x)}\mathrm{d}x \leqslant M.$$

Finally

$$M - \eta \leqslant \liminf_{\varepsilon \to 0} \varepsilon \log \int_K f(x)\mathrm{e}^{\frac{1}{\varepsilon} g(x)}\mathrm{d}x \leqslant \limsup_{\varepsilon \to 0} \varepsilon \log \int_K f(x)\mathrm{e}^{\frac{1}{\varepsilon} g(x)}\mathrm{d}x \leqslant M,$$

from what we deduce the result by making $\eta \to 0$. □

The next lemma is a particular case of lemma 4.4 in [6].

Lemma 2.2. *Let $r \in \mathbb{N}^*$, $U_1, \ldots, U_r$ be standard independent Gaussian random variables. Given a family $(c_k)_{1 \leqslant k \leqslant r}$ of nonzero complex numbers, we have*

$$\mathbb{P}\left(\left|\sum_{k=1}^{r} c_k U_k\right| \leqslant \rho \max_{1 \leqslant k \leqslant r} |c_k|\right) \leqslant \rho, \quad \rho > 0.$$

The next one is quite easy to prove:

Lemma 2.3. *Let U and Θ be random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, U taking its values in a measurable space $\left(\tilde{\Omega}, \tilde{\mathcal{A}}\right)$ and Θ taking its values in $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, endowed with its Borelian σ–algebra. If, for any $\varphi \in \mathbb{R}$, the random variables $(U, \Theta + \varphi)$ and (U, Θ) have the same law, then U and Θ are independent and Θ has uniform law on $\mathbb{T}$.*

The following result is interesting in its own right and will be very useful in the next section. In the proof we will denote by $\mathfrak{U}$ the uniform probability on $[0, 2\pi]$ and by $\mathrm{D}(\xi, \nu)$ the closed disc in $\mathbb{C}$ with center $\xi \in \mathbb{C}$ and radius $\nu > 0$. We will also use the following inequality:

$$\mathfrak{U}\left(\left\{\mathrm{e}^{-i\theta} \in \mathrm{D}(\xi, \nu)\right\}\right) \leqslant \pi\nu, \qquad \xi \in \mathbb{C},\ \nu \in (0, 1). \tag{2}$$

Indeed, given $\nu \in (0, 1)$, we easily see that $\mathfrak{U}\left(\left\{\mathrm{e}^{-i\theta} \in \mathrm{D}(\xi, \nu)\right\}\right)$ only depends on $|\xi|$ and is maximum whenever $|\xi| = \sqrt{1 - \nu^2}$. Thus we may suppose that $\xi = -\sqrt{1 - \nu^2}$ and we have

$$\mathfrak{U}\left(\left\{\mathrm{e}^{-i\theta} \in \mathrm{D}(\xi, \nu)\right\}\right) \leqslant 2 \arcsin(\nu), \qquad \nu \in (0, 1),$$

as is easily deduced from figure 1, and (2) follows from inequality $\arcsin(x) \leqslant \frac{\pi}{2} x,\ x \in [0, 1]$.

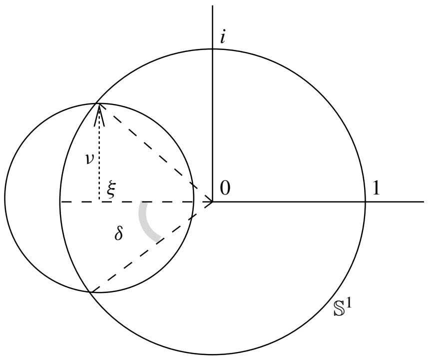

Fig. 1. $\sin\delta = \nu, \quad \nu \in [0, 1]$

Lemma 2.4. *Let f be a trigonometric polynomial*

$$f(\theta) = \sum_{k=-1}^{N} c_k \mathrm{e}^{ik\theta},$$

with fixed degree $N \in \mathbb{N}^*$ and coeffficients $c_{-1}, c_0, c_1, c_2, \ldots, c_N \in \mathbb{C}$. Assuming $c_0 = 1$, there exists a constant $C_N > 0$ only depending on N (and not on the c_k's) such that,

$$\int_{[0,2\pi]} \mathbf{1}_{\{|f(\theta)|\leqslant\rho\}} \mathrm{d}\theta \;\leqslant\; C_N \rho^{\frac{1}{N+1}}, \quad \rho \in (0,1].$$

Proof. Let us write f as

$$f(\theta) = c_{-1}\mathrm{e}^{-i\theta} + 1 + c_1\mathrm{e}^{i\theta} + c_2\mathrm{e}^{2i\theta} + \cdots + c_N\mathrm{e}^{iN\theta} = \mathrm{e}^{iN\theta} P\left(\mathrm{e}^{-i\theta}\right),$$

where $P(z) = c_{-1}z^{N+1} + z^N + c_1 z^{N-1} + c_2 z^{N-2} + \cdots + c_N,\ z \in \mathbb{C}$.

- If $c_{-1} \neq 0$, let $\xi_1, \ldots, \xi_{N+1}$ be the roots of P, and S their sum, in such a way that

$$\frac{1}{c_{-1}} = -S, \tag{3}$$

$$\text{and} \quad |f(\theta)| \leqslant \rho \quad \Leftrightarrow \quad \prod_{j=1}^{N+1} \left|\mathrm{e}^{-i\theta} - \xi_j\right| \leqslant \frac{\rho}{|c_{-1}|}.$$

When $|c_{-1}| \geqslant \frac{1}{2(N+1)}$, we have

$$|f(\theta)| \leqslant \rho \;\Rightarrow\; \mathrm{e}^{-i\theta} \in \bigcup_{j=1}^{N+1} \mathrm{D}\left(\xi_j; (2(N+1)\rho)^{\frac{1}{N+1}}\right).$$

Combined with (2), this yields

$$\mathfrak{U}(\{|f(\theta)| \leqslant \rho\}) \leqslant 2^{\frac{1}{N+1}}\pi(N+1)^{1+\frac{1}{N+1}}\rho^{\frac{1}{N+1}}, \qquad 0 < \rho < \frac{1}{2(N+1)}. \tag{4}$$

When $0 < |c_{-1}| < \frac{1}{2(N+1)}$, there exists $i_0 \in \{1, \ldots, N+1\}$ such that $|\xi_{i_0}| - 1 > \frac{1}{2(N+1)}|S|$. Indeed, if we suppose the converse, then $|S| \leqslant (N+1)\left(1 + \frac{1}{2(N+1)}|S|\right)$, and by (3), this leads to contradict $|c_{-1}| < \frac{1}{2(N+1)}$. We may suppose that $i_0 = 1$ and then we obtain, when $|z| = 1$ and $|P(z)| \leqslant \rho$,

$$\begin{aligned}\frac{\rho}{|c_{-1}|} &> |z - \xi_1| \prod_{j=2}^{N+1} \left|z - \xi_j\right| \geqslant (|\xi_1| - 1) \prod_{j=2}^{N+1} \left|z - \xi_j\right| \\ &> \frac{1}{2(N+1)} \frac{1}{|c_{-1}|} \prod_{j=2}^{N+1} \left|z - \xi_j\right|,\end{aligned}$$

$$\text{thus} \quad \prod_{j=2}^{N+1} \left|z - \xi_j\right| < 2(N+1)\rho \quad \text{and} \quad z \in \bigcup_{j=2}^{N+1} \mathrm{D}\left(\xi_j; (2(N+1)\rho)^{\frac{1}{N}}\right).$$

By (2), this gives

$$\mathcal{U}\left(\{|f(\theta)| \leqslant \rho\}\right) \leqslant 2^{\frac{1}{N}} \pi N (N+1)^{\frac{1}{N}} \rho^{\frac{1}{N}}, \qquad 0 < \rho < \frac{1}{2(N+1)}. \tag{5}$$

- If $c_{-1} = 0$, let $\xi_1, \ldots, \xi_N$ be the roots of P. Then

$$|f(\theta)| \leqslant \rho \quad \Rightarrow \quad \prod_{j=1}^{N} \left|\mathrm{e}^{-i\theta} - \xi_j\right| \leqslant \rho, \quad \textit{i.e.} \quad \mathrm{e}^{-i\theta} \in \bigcup_{j=1}^{N} \mathrm{D}\left(\xi_j, \rho^{\frac{1}{N}}\right),$$

thus by (2),

$$\mathcal{U}\left(\{|f(\theta)| \leqslant \rho\}\right) \leqslant N\pi\rho^{\frac{1}{N}}, \qquad \rho \in (0, 1]. \tag{6}$$

- From (4), (5), (6) and the case when $\rho \geqslant \frac{1}{2(N+1)}$, and by taking

$$C_N = \max\left\{2^{\frac{1}{N+1}}\pi(N+1)^{1+\frac{1}{N+1}},\ 2^{\frac{1}{N}}\pi N(N+1)^{\frac{1}{N}},\ N\pi,\ 2(N+1)\right\},$$

we get the result. $\square$

2.2. Lower bound

Theorem 2.5. *Let $\varphi_1, \ldots, \varphi_d : \mathbb{R}_+ \to \mathbb{R}_+^*$ be continuous functions, and $r \in \mathbb{N}^*$. For any $p \in \{1, \ldots, r\}$, let $\left(\lambda_\alpha^p\right)_{\alpha \in S^p}$ be a family of complex numbers. Fix $t > 0$, $q \in \{1, \ldots, r\}$ and $T_0 = t > T_1 > T_2 > \cdots > T_{q-1} > 0 = T_q$, and assume that one of the λ_α^q, $\alpha \in S^q$, is nonzero. Put, for $k \in \{1, \ldots, d\}$,*

$$M_\cdot^k = \int_0^\cdot \varphi_k(s)\mathrm{d}\beta_s^k,$$

for $\alpha = (\alpha_1, \ldots, \alpha_p) \in S^p$,

$$X^{p,\alpha} = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_1}^{\alpha_1},$$

and

$$X = \sum_{p=1}^{r} \sum_{\alpha \in S^p} \lambda_\alpha^p X^{p,\alpha}.$$

Then there exist constants $\nu \in \mathbb{N}^*$, $\nu \geqslant r$ and $C > 0$, only depending on r and not on the φ_k's, such that

$$\mathbb{P}\left(|X| \leqslant \rho \max_{\alpha \in S^q} \left|\lambda_\alpha^q\right| \prod_{k=1}^{q} \left(\int_{T_k}^{T_{k-1}} \varphi_{\alpha_k}(s)^2 \mathrm{d}s\right)^{1/2}\right) \leqslant C\rho^{1/\nu}, \qquad \rho \in (0, 1].$$

Proof. For $p \geqslant 1$ and $\alpha \in S^p$, let

$$\mu_\alpha^p = \lambda_\alpha^p \prod_{k=1}^{p} \left(\int_{T_k}^{T_{k-1}} \varphi_{\alpha_k}(s)^2 \mathrm{d}s \right)^{1/2}.$$

Proof of Theorem 2.5 will be by induction on $r \geqslant 1$. If $r = 1$, we have also $q = 1$ and X may be written as

$$X = \sum_{k=1}^{d} \lambda_k \left(\int_0^t \varphi_k(s)^2 \mathrm{d}s \right)^{\frac{1}{2}} \frac{M_t^k}{\left(\int_0^t \varphi_k(s)^2 \mathrm{d}s \right)^{\frac{1}{2}}},$$

with coefficients $\lambda_k \in \mathbb{C}$. By a straightforward application of lemma 2.2, we have

$$\forall \rho > 0 \qquad \mathbb{P}\left(|X| \leqslant \rho \max_{1 \leqslant k \leqslant d} |\lambda_k| \left(\int_0^t \varphi_k(s)^2 \mathrm{d}s \right)^{1/2} \right) \leqslant \rho.$$

- Let us now consider the case when $r \geqslant 2$, assuming the result is true for $r - 1$. The random variable X may be written as

$$\begin{aligned} X = {} & X^{1,1} + V^1 \\ & + X^{2,1} + V^2 + X^{2,2} \\ & + X^{3,1} + V^3 + X^{3,2} + X^{3,3} \\ & \vdots \\ & + X^{r,1} + V^r + X^{r,2} + X^{r,3} + \cdots + X^{r,r}, \end{aligned}$$

where

$$X^{1,1} = \sum_{\alpha \in S^1} \lambda_\alpha^1 \int_0^{T_1} \mathrm{d}M_{t_1}^{\alpha_1}, \qquad V^1 = \sum_{\alpha \in S^1} \lambda_\alpha^1 \int_{T_1}^{t} \mathrm{d}M_{t_1}^{\alpha_1},$$

and for any $p \in \{2, \ldots, r\}$,

$$\begin{aligned} X^{p,1} &= \sum_{\alpha \in S^p} \lambda_\alpha^p \int_0^{T_1} \int_0^{t_1} \cdots \int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_1}^{\alpha_1} \\ V^p &= \sum_{\alpha \in S^p} \lambda_\alpha^p \left(\int_{T_1}^{t} \mathrm{d}M_{t_1}^{\alpha_1} \right) \left(\int_0^{T_1} \int_0^{t_2} \cdots \int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_2}^{\alpha_2} \right) \\ X^{p,2} &= \sum_{\alpha \in S^p} \lambda_\alpha^p \int_{T_1}^{t} \int_{T_1}^{t_1} \int_0^{T_1} \int_0^{t_3} \cdots \int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_1}^{\alpha_1} \\ & \vdots \\ X^{p,p} &= \sum_{\alpha \in S^p} \lambda_\alpha^p \int_{T_1}^{t} \int_{T_1}^{t_1} \cdots \int_{T_1}^{t_{p-2}} \int_{T_1}^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_1}^{\alpha_1}. \end{aligned}$$

This decomposition of X is obtained by chain rules. Indeed, for $p \in \{1, \ldots, r\}$, we divide $\Delta_p = \left\{(t_1, \ldots, t_p) \in \mathbb{R} \mid 1 \geqslant t_1 \geqslant \cdots \geqslant t_p \geqslant 0\right\}$ into $p+1$ smaller integration domains, each of these being associated to one of the position of T_1 w.r.t. the variables $t_1, \ldots, t_p$. In particular, we stand out the random variable V^p by taking into account that M^{α_1} has independent increments, which makes possible to express

$$\int_{T_1}^{t} \int_0^{T_1} \int_0^{t_2} \cdots \int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_2}^{\alpha_2} \mathrm{d}M_{t_1}^{\alpha_1}$$

as the product

$$\left(\int_{T_1}^{t} \mathrm{d}M_{t_1}^{\alpha_1}\right)\left(\int_0^{T_1} \int_0^{t_2} \cdots \int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_2}^{\alpha_2}\right).$$

Let $V = V^1 + V^2 + \cdots + V^r$. As a first step, we establish inequality (9) below which entails that V almost surely never vanishes. For $p \in \{1, \ldots, r\}$ and $\alpha \in S^p$, let

$$U_\alpha = \frac{\int_{T_1}^{t} \mathrm{d}M_{t_1}^{\alpha_1}}{\left(\int_{T_1}^{t} {\varphi_{\alpha_1}}^2\right)^{1/2}}, \qquad Y_\alpha^p = \int_0^{T_1} \int_0^{t_2} \cdots \int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p} \ldots \mathrm{d}M_{t_2}^{\alpha_2},$$

$$\tilde{\lambda}_\alpha^p = \lambda_\alpha^p \left(\int_{T_1}^{t} {\varphi_{\alpha_1}}^2\right)^{1/2},$$

in such a way that

$$V = \sum_{p=1}^{r} \sum_{\alpha \in S^p} \tilde{\lambda}_\alpha^p U_\alpha Y_\alpha^p.$$

We choose $\alpha^\circ \in S^q$ such that $|\mu_{\alpha^\circ}^q| = \max\{|\mu_\alpha^q|,\ \alpha \in S^q\}$. We have

$$V = \left(\sum_{p=1}^{r} \sum_{\substack{\alpha \in S^p \\ \alpha_1 = \alpha_1^\circ}} \tilde{\lambda}_\alpha^p Y_\alpha^p\right) U_{\alpha^\circ} + \sum_{p=1}^{r} \sum_{\substack{\alpha \in S^p \\ \alpha_1 \neq \alpha_1^\circ}} \tilde{\lambda}_\alpha^p U_\alpha Y_\alpha^p.$$

This can be written as $V = AU_{\alpha^\circ} + B$, where $U_{\alpha^\circ} \sim \mathcal{N}_{\mathbb{R}^2}(0, \mathrm{Id})$ is independent of the random variables A and B. Denoting by $\mathbb{P}_{(A,B)}$ the law of the $\mathbb{C}^2$–valued r.v. (A, B), we have

$$\mathbb{P}\left(|V| \leqslant \rho \left|\mu_{\alpha^\circ}^q\right|\right) = \int_{\mathbb{C}^2} \mathbb{P}\left(|aU_{\alpha^\circ} + b| \leqslant \rho \left|\mu_{\alpha^\circ}^q\right|\right) \mathrm{d}\mathbb{P}_{(A,B)}(a, b). \tag{7}$$

By translation properties[1] of $\mathcal{N}_{\mathbb{R}^2}(0, \mathrm{Id})$, the right hand side of (7) is smaller than

$$\int_{\mathbb{C}^2} \mathbb{P}\left(|aU_{\alpha^\circ}| \leqslant \rho \left|\mu_{\alpha^\circ}^q\right|\right) \mathrm{d}\mathbb{P}_{(A,B)}(a, b) = \mathbb{P}\left(|AU_{\alpha^\circ}| \leqslant \rho \left|\mu_{\alpha^\circ}^q\right|\right).$$

[1] If U is a standard complex Gaussian r.v. then

$$\mathbb{P}\left(|aU + b| \leqslant \gamma\right) \leqslant \mathbb{P}\left(|aU| \leqslant \gamma\right), \quad a \in \mathbb{C}^*, b \in \mathbb{C}, \gamma > 0.$$

Hence we have

$$\mathbb{P}\left(|V| \leqslant \rho|\mu^q_{\alpha^\circ}|\right) \leqslant \mathbb{P}\left(|A| \leqslant \rho|\mu^q_{\alpha^\circ}|\,|U_{\alpha^\circ}|^{-1}\right). \tag{8}$$

Notice that

$$\max_{\substack{\alpha\in S^q\\ \alpha_1=\alpha_1^\circ}} \left\{\left|\tilde{\lambda}^q_\alpha\right| \prod_{k=2}^{q} \left(\int_{T_k}^{T_{k-1}} \varphi_{\alpha_k}{}^2\right)^{1/2}\right\} = |\mu^q_{\alpha^\circ}|.$$

and that

$$A = \sum_{p=1}^{r} \sum_{\substack{\alpha\in S^p\\ \alpha_1=\alpha_1^\circ}} \tilde{\lambda}^p_\alpha \int_0^{T_1}\int_0^{t_2}\cdots\int_0^{t_{p-1}} \mathrm{d}M^{\alpha_p}_{t_p}\ldots \mathrm{d}M^{\alpha_2}_{t_2}.$$

By induction hypothesis, there exist $K > 0$ and $\nu' \geqslant r-1$, only depending on r, such that

$$\mathbb{P}\left(|A| \leqslant \tau\left|\mu^q_{\alpha^\circ}\right|\right) \leqslant K\tau^{1/\nu'}, \quad \tau\in(0,1].$$

From the independence of A and U_{α°, we get by (8)

$$\mathbb{P}\left(|V| \leqslant \rho|\mu^q_{\alpha^\circ}|\right) \leqslant K\,\mathbb{E}\left[|U_{\alpha^\circ}|^{-1/\nu'}\right]\rho^{1/\nu'}.$$

Since $U_{\alpha^\circ} \sim \mathcal{N}_{\mathbb{R}^2}(0,\mathrm{Id})$, we know that $\mathbb{E}\left[|U_{\alpha^\circ}|^{-1/\nu'}\right]$ is a finite positive constant, independent of the φ_k's. Let $C' = K\,\mathbb{E}\left[|U_{\alpha^\circ}|^{-1/\nu'}\right]$. We get

$$\mathbb{P}\left(|V| \leqslant \rho \max_{\alpha\in S^q}\left\{|\lambda^p_\alpha| \prod_{k=1}^{q}\left(\int_{T_k}^{T_{k-1}} \varphi_{\alpha_k}{}^2\right)^{1/2}\right\}\right) \leqslant C'\rho^{1/\nu'}, \quad \rho\in(0,1]. \tag{9}$$

As a consequence, V almost surely never vanishes.

- Let us introduce some notations:

$$Z^{p,j} = \frac{X^{p,j}}{V}, \quad p\in\{1,\ldots,r\},\ j\in\{1,\ldots,p\}$$
$$Z_j = \sum_{p=j}^{r} Z^{p,j}, \quad j\in\{1,\ldots,r\},$$

in such a way that

$$X = V\left(1+\sum_{p=1}^{r}\sum_{j=1}^{p} Z^{p,j}\right) = V\left(1+\sum_{j=1}^{r} Z_j\right).$$

We may now use the invariance in law of complex Brownian motion under rotations: if β is a standard complex Brownian motion, the same holds for the process $\tilde{\beta}$ defined by

$$\tilde{\beta}_s = \begin{cases} \beta_s & \text{when } 0 \leqslant s \leqslant T_1 \\ \beta_{T_1} + \mathrm{e}^{i\varphi}(\beta_s - \beta_{T_1}) & \text{when } s \geqslant T_1 \end{cases},$$

with some fixed $\varphi \in \mathbb{R}$. This is a consequence of Paul Lévy's characterization theorem (see [13], IV, §3). Let us introduce the functional

$$\mathrm{T}(\beta_1, \ldots, \beta_d) = \begin{pmatrix} X^{1,1} & V^1 & & & \\ X^{2,1} & V^2 & X^{2,2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ X^{r,1} & V^r & X^{r,2} & \ldots & X^{r,r} \end{pmatrix}$$

of the Brownian motion $(\beta_1, \ldots, \beta_d)$ in $\mathbb{C}^d$. Then $\mathrm{T}(\tilde{\beta}_1, \ldots, \tilde{\beta}_d)$ has same law as $\mathrm{T}(\beta_1, \ldots, \beta_d)$, that is to say[2] that

$$\mathrm{T}(\tilde{\beta}_1, \ldots, \tilde{\beta}_d) = \begin{pmatrix} X^{1,1} & \mathrm{e}^{i\varphi}V^1 & & & \\ X^{2,1} & \mathrm{e}^{i\varphi}V^2 & \mathrm{e}^{2i\varphi}X^{2,2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ X^{r,1} & \mathrm{e}^{i\varphi}V^r & \mathrm{e}^{2i\varphi}X^{r,2} & \ldots & \mathrm{e}^{ri\varphi}X^{r,r} \end{pmatrix}$$

has the same law as $\mathrm{T}(\beta_1, \ldots, \beta_d)$, for any $\varphi \in \mathbb{R}$. We deduce the identity in law

$$(Z_1, Z_2, Z_3, \ldots, Z_r) \sim \left(\mathrm{e}^{-i\varphi}Z_1, \mathrm{e}^{i\varphi}Z_2, \mathrm{e}^{2i\varphi}Z_3, \ldots, \mathrm{e}^{(r-1)i\varphi}Z_r\right).$$

Writing $Z_1 = R_1\mathrm{e}^{i[\Theta_1-\Theta_2]}$, $Z_2 = R_2\mathrm{e}^{i\Theta_2}$, $Z_3 = R_3\mathrm{e}^{i[\Theta_3+2\Theta_2]}, \ldots, Z_r = R_r\mathrm{e}^{i[\Theta_r+(r-1)\Theta_2]}$, we obtain, by lemma 2.3, that the r.v. Θ_2 is independent of $(R_1, R_2, \ldots, R_r, \Theta_1, \Theta_3, \ldots, \Theta_r)$ and has uniform law on $\mathbb{T}$. Hence, for any $\rho \in (0, 1]$,

$$\begin{aligned} &\mathbb{P}\left(|1 + Z_1 + Z_2 + \cdots + Z_r| \leqslant \rho\right) \\ &= \mathbb{E}\left[\mathbf{1}_{\left\{\left|1+R_1\mathrm{e}^{i[\Theta_1-\Theta_2]}+R_2\mathrm{e}^{i\Theta_2}+R_3\mathrm{e}^{i[\Theta_3+2\Theta_2]}+\cdots+R_r\mathrm{e}^{i[\Theta_r+(r-1)\Theta_2]}\right|\leqslant\rho\right\}}\right] \\ &= \mathbb{E}\left[\frac{1}{2\pi}\int_{[0,2\pi]}\mathbf{1}_{\left\{\left|1+R_1\mathrm{e}^{i[\Theta_1-\theta]}+R_2\mathrm{e}^{i\theta}+R_3\mathrm{e}^{i[\Theta_3+2\theta]}+\cdots+R_r\mathrm{e}^{i[\Theta_r+(r-1)\theta]}\right|\leqslant\rho\right\}}\mathrm{d}\theta\right] \end{aligned} \quad (10)$$

Given $(\rho_1, \rho_2, \ldots, \rho_r, \theta_1, \theta_3, \ldots, \theta_r)$ in $\left(\mathbb{R}_+^*\right)^r \times \mathbb{T}^{r-1}$, let us consider the trigonometric polynomial

$$f(\theta) = 1 + \rho_1\mathrm{e}^{i(\theta_1-\theta)} + \rho_2\mathrm{e}^{i\theta} + \rho_3\mathrm{e}^{i(\theta_3+2\theta)} + \cdots + \rho_r\mathrm{e}^{i(\theta_r+(r-1)\theta)}.$$

[2] What changes in the script is that $\mathrm{d}\beta^j$ is replaced by $\mathrm{e}^{i\varphi}\mathrm{d}\beta^j$ in each integral $\int_{T_1}^{t_q}$ which appears in the expressions of the $X^{p,k}$'s and V^p's.

By lemma 2.4, there exist $C'' > 0$ and $\nu'' \in \mathbb{N}^*$ only depending on r, such that

$$\frac{1}{2\pi}\int_{[0,2\pi]} \mathbf{1}_{\{|f(\theta)|\leqslant\rho\}}\mathrm{d}\theta \leqslant C''\rho^{1/\nu''}, \quad \rho\in(0,1].$$

By (10), we get

$$\mathbb{P}\left(|1+Z_1+\cdots+Z_r|\leqslant\rho\right)\leqslant C''\rho^{1/\nu''}, \quad \rho\in(0,1]. \tag{11}$$

- From (9) et (11), we deduce

$$\begin{aligned}\mathbb{P}\left(|X|\leqslant\rho|\mu_{\alpha^\circ}|\right) &\leqslant \mathbb{P}\left(|V|\leqslant\sqrt{\rho}|\mu_{\alpha^\circ}|\right)+\mathbb{P}\left(|1+Z_1+\cdots+Z_r|\leqslant\sqrt{\rho}\right)\\ &\leqslant C'\rho^{1/2\nu'}+C''\rho^{1/2\nu''}\\ &\leqslant C\rho^{1/\nu}, \quad \rho\in(0,1]\end{aligned}$$

where $C=\max\{2C',2C''\}$ and $\nu=\max\{2\nu',2\nu''\}\geqslant 2(r-1)\geqslant r$. □

Remark. With the notations of Theorem 2.5, the random variable X almost surely never vanishes.

Corollary 2.6. *Let $g_1,\ldots,g_d:[0,1]\to\mathbb{R}$ be continuous functions and $r\in\mathbb{N}^*$. For $p\in\{1,\ldots,r\}$, $\alpha\in S^p$, let $(u_n^{p,\alpha})_{n\in\mathbb{N}^*}$ be a sequence of nonzero complex numbers such that*

$$\lim_{n\to\infty}\frac{1}{n}\log|u_n^{p,\alpha}|=0,$$

and $\left(\lambda_\alpha^p\right)_{\alpha\in S^p}$ a family of complex numbers. Assume that one of the λ_α^r's is nonzero and put

$$X_n=\sum_{p=1}^r\sum_{\alpha\in S^p}\lambda_\alpha^p u_n^{p,\alpha}\int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{p-1}}\mathrm{e}^{ng_{\alpha_p}(t_p)}\mathrm{d}\beta_{t_p}^{\alpha_p}\right)\ldots\mathrm{e}^{ng_{\alpha_2}(t_2)}\mathrm{d}\beta_{t_2}^{\alpha_2}\right)$$
$$\times\mathrm{e}^{ng_{\alpha_1}(t_1)}\mathrm{d}\beta_{t_1}^{\alpha_1}.$$

Then we have, almost surely,

$$\liminf_{n\to\infty}\frac{1}{n}\log|X_n|\geqslant\max_{1\leqslant p\leqslant r}\max_{\substack{\alpha\in S^p\\ \lambda_\alpha^p\neq 0}}\max_{\Delta_p} g_{\alpha_1}\oplus\cdots\oplus g_{\alpha_p}.$$

Proof. • We will use the convention $\max\emptyset=-\infty$. Let us introduce some notations:

$$M_t^{k,n}=\int_0^t\mathrm{e}^{ng_k(s)}\mathrm{d}\beta_s^k, \quad k\in\{1,\ldots,d\},$$

and for $\alpha\in S^p$, $p\in\{1,\ldots,r\}$,

$$X_n^\alpha=\int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{p-1}}\mathrm{d}M_{t_p}^{\alpha_p,n}\right)\ldots\mathrm{d}M_{t_2}^{\alpha_2,n}\right)\mathrm{d}M_{t_1}^{\alpha_1,n}, \quad \alpha\in S^p,\ p\in\{1,\ldots,r\},$$

in such a way that

$$X_n = \sum_{p=1}^{r} \sum_{\alpha \in S^p} \lambda_\alpha^p u_n^{p,\alpha} X_n^\alpha .$$

We set, for $q \in \{1, \ldots, r\}$ and $T_0 = 1 > T_1 > T_2 > \cdots > T_{q-1} > 0 = T_q$,

$$\mathcal{M}_n^q = \max_{\alpha \in S^q} \left\{ |\lambda_\alpha^q u_n^{q,\alpha}| \prod_{k=1}^{q} \left(\int_{T_k}^{T_{k-1}} \mathrm{e}^{2ng_{\alpha_k}} \right)^{1/2} \right\} .$$

• Let $q \in \{1, \ldots, r\}$, $T_0 = 1 > T_1 > T_2 > \cdots > T_{q-1} > 0 = T_q$. By Theorem 2.5,

$$\mathbb{P}\left(|X_n| \leqslant \rho \mathcal{M}_n^q\right) \leqslant C\rho^{\frac{1}{\nu}}, \quad \rho \in (0, 1]$$

thus

$$\mathbb{P}\left(|X_n| \leqslant \frac{1}{n^{2\nu}} \mathcal{M}_n^q\right) \leqslant C\frac{1}{n^2}.$$

By Borel–Cantelli's lemma, we get

$$\mathbb{P}\left(\liminf_{n\to\infty} \left\{|X_n| \geqslant \frac{1}{n^{2\nu}} \mathcal{M}_n^q\right\}\right) = 1.$$

Hence, almost surely,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log |X_n| &\geqslant \liminf_{n\to\infty} \frac{1}{n} \log \left(\frac{1}{n^{2\nu}} \max_{\alpha \in S^q} \left\{ |\lambda_\alpha^q u_n^{q,\alpha}| \prod_{k=1}^{q} \left(\int_{T_k}^{T_{k-1}} \mathrm{e}^{2ng_{\alpha_k}} \right)^{1/2} \right\} \right) \\
&\geqslant \max_{\substack{\alpha \in S^q \\ \lambda_\alpha^q \neq 0}} \liminf_{n\to\infty} \left(\frac{1}{n} \log |\lambda_\alpha^q u_n^{q,\alpha}| + \frac{1}{n} \log \prod_{k=1}^{q} \left(\int_{T_k}^{T_{k-1}} \mathrm{e}^{2ng_{\alpha_k}} \right)^{1/2} \right) \\
&\geqslant \max_{\substack{\alpha \in S^q \\ \lambda_\alpha^q \neq 0}} \quad \max \left\{ g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_q}(s) \;\middle|\; s \in \prod_{1 \leqslant k \leqslant q} [T_k, T_{k-1}] \right\} .
\end{aligned}$$

(we made use of deterministic Laplace's method for simple integrals, see Theorem 2.1).

• Let us define, for $m \in \mathbb{N}^*$,

$$K_m = \bigcup_{\substack{(k_1, \ldots, k_q) \in \mathbb{N}^* \\ m = k_0 > k_1 > \cdots > k_q > 0}} \left\{ \left(t_1, \ldots, t_q\right) \in \mathbb{R}^q \;\middle|\; t_j \in \left[\frac{k_j}{m}, \frac{k_{j-1}}{m} \right] \right\} .$$

Each of the elementary sets that compose K_m is included in one of the form[3] $\prod_{1\leqslant k\leqslant q}[T_k, T_{k-1}]$, where $T_0 = 1 > T_1 > T_2 > \cdots > T_{q-1} > 0 = T_q$. By the first part of the proof, we get

$$\liminf_{n\to\infty} \frac{1}{n} \log |X_n| \geqslant \max_{\substack{\alpha\in S^q\\ \lambda_\alpha^r\neq 0}} \quad \max_{K_m} \quad g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_q}.$$

Since the right hand side converges to

$$\max_{\substack{\alpha\in S^q\\ \lambda_\alpha^q\neq 0}} \quad \max_{\Delta_q} \quad g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_q},$$

as $m \to \infty$, we get, for any $q \in \{1, \ldots, r\}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log |X_n| \geqslant \max_{\substack{\alpha\in S^q\\ \lambda_\alpha^q\neq 0}} \quad \max_{\Delta_q} \quad g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_q},$$

hence the result. □

2.3. Upper bound

Theorem 2.7. *Let* $g_1, \ldots, g_d : [0, 1] \to \mathbb{R}$ *be continuous functions and* p *be a positive integer. Let*

$$M_t^{k,n} = \int_0^t \mathrm{e}^{ng_k(s)}\mathrm{d}\beta_s^k, \quad k \in \{1, \ldots, d\},\ n \in \mathbb{N}^*.$$

and

$$X_n^\alpha = \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{p-1}} \mathrm{d}M_{t_p}^{\alpha_p,n}\right) \ldots \mathrm{d}M_{t_2}^{\alpha_2,n}\right) \mathrm{d}M_{t_1}^{\alpha_1,n}, \quad n \in \mathbb{N}^*,\ \alpha \in S^p.$$

Then, almost surely

$$\lim_{n\to\infty} \frac{1}{n} \log |X_n^\alpha| = \max_{\Delta_p}\ g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_p}.$$

Proof. By corollary 2.6, it suffices to prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log |X_n^\alpha| \leqslant \max_{\Delta_p}\ g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_p}.$$

Let us denote by $\mathcal{M}_\alpha$ the right hand side in the former inequality and put $g_\alpha = g_{\alpha_1} \oplus \cdots \oplus g_{\alpha_p}$. We have

$$\mathbb{E}\left[\left|X_n^\alpha\right|^2\right] = 2\int_0^1 \int_0^{s_1} \cdots \int_0^{s_{p-1}} \mathrm{e}^{2ng_\alpha(s_1,\ldots,s_p)}\mathrm{d}s_p \ldots \mathrm{d}s_1 \leqslant \frac{2}{p!}\mathrm{e}^{2n\mathcal{M}_\alpha}.$$

[3] Note that if $C = \prod_{1\leqslant i\leqslant q}[S_i, S_i'] \subset \Delta_q$, then

$$C \subset [S_1, 1] \times [S_2, S_1] \times \cdots \times [S_{q-1}, S_{q-2}] \times [0, S_{q-1}] \quad \left(\subset \Delta_q\right).$$

By Markov inequality with order 2, we have for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{1}{n}\log|X_n^\alpha| > \mathcal{M}_\alpha + \varepsilon\right) = \mathbb{P}\left(|X_n^\alpha| > \mathrm{e}^{n(\mathcal{M}_\alpha+\varepsilon)}\right)$$
$$\leqslant \frac{2}{p!}\mathrm{e}^{2n\mathcal{M}_\alpha}\mathrm{e}^{-2n(\mathcal{M}_\alpha+\varepsilon)} = \frac{2}{p!}\mathrm{e}^{-2n\varepsilon},$$

thus by Borel–Cantelli's lemma, we get

$$\mathbb{P}\left(\limsup_{n\to\infty}\left\{\frac{1}{n}\log|X_n^\alpha| > \mathcal{M}_\alpha + \varepsilon\right\}\right) = 0,$$

and this holds for any $\varepsilon > 0$, which entails

$$\limsup_{n\to\infty}\frac{1}{n}\log|X_n^\alpha| \leqslant \mathcal{M}_\alpha. \qquad \square$$

Notice that given r sequences $(x_n^{(1)}), \ldots, (x_n^{(r)})$ of nonzero complex numbers such that

$$\lim_{n\to\infty}\frac{1}{n}\log|x_n^{(k)}| = m_k \in \mathbb{R},$$

we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(|x_n^{(1)} + \cdots + x_n^{(r)}|\right) \leqslant \max_{1\leqslant k\leqslant r} m_k.$$

Combining this with corollary 2.6 and theorem 2.7, we easily obtain theorem 1.3.

Remark. As the reader could have noticed, the proof of theorem 2.5 is based on the fact that integrating Brownian motions are complex and on their invariance in law under rotations. It is possible to prove some interesting partial results (that look like theorem 2.7) in the case when Brownian motions are real, but not as general as theorem 1.3. The proofs are very different.

3. Path representation of the eigenvalues in the random case

3.1. Recalls and notations relative to semisimple Lie algebras and symmetric spaces

3.1.1. Notations

We consider a simply connected semisimple complex Lie group G, with Lie algebra $\mathfrak{g}$. As usual, the adjoint representations are denoted by Ad and ad. Given $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$, let θ be the associated Cartan involution of $\mathfrak{g}$ and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Denoting by J the complex structure of $\mathfrak{g}$, $\mathfrak{h} = \mathfrak{a} + J\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Σ denote the root system of $(\mathfrak{g}, \mathfrak{a})$, and, for each $\alpha \in \Sigma$,

$$\mathfrak{g}_\alpha = \left\{X \in \mathfrak{g} \;\middle|\; \forall H \in \mathfrak{a},\ [H, X] = \alpha(H)X\right\}.$$

By choosing a Weyl chamber $\mathfrak{a}^+$, we determine the set Σ_+ of positive roots of $(\mathfrak{g}, \mathfrak{a})$. Let S be the set of simple positive roots, which is a basis of $\mathfrak{a}^*$. Let us note

$$\overline{\mathfrak{n}} = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}_{-\alpha},$$

and A, N, K the connected Lie subgroups of G with respective Lie algebras $\mathfrak{a}$, $\overline{\mathfrak{n}}$, $\mathfrak{k}$. For each $\alpha \in \Sigma$, we choose a nonzero y_α in the subspace $\mathfrak{g}_{-\alpha}$ (which has dimension 1 over $\mathbb{C}$). Define Q as the scalar product on $\mathfrak{a} \oplus \overline{\mathfrak{n}}$

$$Q(X_1 + Y_1, X_2 + Y_2) = \kappa(X_1, X_2) - \frac{1}{2}\kappa(Y_1, \theta Y_2), \quad X_1, X_2 \in \mathfrak{a},\ Y_1, Y_2 \in \overline{\mathfrak{n}},$$

(κ denote the Killing form of the real Lie algebra $\mathfrak{g}$).

The closed Weyl chamber $\overline{\mathfrak{a}^+}$ is a fundamental domain with respect to the adjoint representation of K on $\mathfrak{p}$: given $X \in \mathfrak{p}$, $\mathrm{rad}(X)$ is defined as

$$\overline{\mathfrak{a}^+} \cap \mathrm{Ad}(K)X = \{\mathrm{rad}(X)\}.$$

Let us denote by P the symmetric space of the non–compact type[4] G/K. The map

$$\mathrm{Exp} : \mathfrak{p} \to \mathrm{P},\ X \mapsto \exp(X).K$$

is a diffeomorphism and $\overline{A^+} = \mathrm{Exp}\left(\overline{\mathfrak{a}^+}\right)$ is a fundamental domain with respect to the action of K on P, and given $x \in \mathrm{P}$, $\mathrm{Rad}(x) \in \mathfrak{p}$ is defined by

$$\overline{A^+} \cap K.x = \{\mathrm{Exp}(\mathrm{Rad}(x))\}.$$

Moreover, we have $\mathrm{Rad}\,(\mathrm{Exp}(X)) = \mathrm{rad}(X)$ and the map rad is continuous.

When $\mathfrak{e}$ is a subset of $\mathfrak{g}$ which contains 0, we denote by $C_0\big([0, 1], \mathfrak{e}\big)$ the set of continuous functions from $[0, 1]$ into $\mathfrak{e}$ that vanish at 0. It is endowed with the topology of uniform convergence.

Let π be an irreducible finite representation of $\mathfrak{g}$, with space V ($\dim V < \infty$) and $\mathcal{P}(\pi)$ be the set of weights of π. Denote by Λ_π the highest weight of π and $\tilde{\Lambda}_\pi$ the smallest one. Each $\mu \in \mathcal{P}(\pi)$ can be written in a unique way as $\mu = \Lambda_\pi - \sum_{\alpha \in S} k_\alpha \alpha$, where $k_\alpha \in \mathbb{N}$, hence we can set

$$p(\mu) = \sum_{\alpha \in S} k_\alpha \quad \text{and} \quad r = \max\left\{p(\mu)\ \middle|\ \mu \in \mathcal{P}(\pi)\right\} = p(\tilde{\Lambda}_\pi).$$

Let $\mathcal{D}(\pi) = \left\{ (\alpha_1, \ldots, \alpha_r) \in S^r\ \middle|\ \pi(y_{\alpha_1}) \ldots \pi(y_{\alpha_r}) \neq 0\right\}$ and, given $w \in C_0\big([0, 1], \mathfrak{a}\big)$,

$$\big(\mathcal{T}_\pi w\big)(t) = \max_{(\alpha_1, \ldots, \alpha_r) \in \mathcal{D}(\pi)}\ \max_{t \geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant 0} \Lambda_\pi\big(w(t)\big) - \sum_{k=1}^{r} \alpha_k\big(w(t_k)\big), \quad t \in [0, 1].$$

[4] Remember that the map $f : AN \to \mathrm{P},\ g \mapsto gK$ is a diffeomorphism and Q is chosen in such a way that $\mathrm{d}f(e) : \mathfrak{a} \oplus \mathfrak{n} \to \mathfrak{p}$ is an isometry.

3.1.2. Refinements

We describe the notion of refinement in lemma 3.1 below in exactly the same way as in [6] where it is introduced. When $\mu \in \mathcal{P}(\pi)$, let $V_\mu = \{\, v \in V \;\big|\; \pi(h).v = \mu(h)v \quad \forall h \in \mathfrak{h} \,\}$. Then we have

$$V = \bigoplus_{\mu \in \mathcal{P}(\pi)} V_\mu \ , \tag{12}$$

and for any $\alpha \in \Sigma, \quad \pi\big(\mathfrak{g}_\alpha\big) V_\mu \subset V_{\mu+\alpha}$. We choose a basis $(e_i)_{1 \le i \le d}$ adapted to the decomposition (12) in the following way:

- e_1 is primitive in V, that is to say that either e_1 spans V as a $\mathfrak{g}$–module or $e_1 \in V_{\Lambda_\pi} \backslash \{0\}$.
- $e_i = \pi(y_{\alpha_k}) \ldots \pi(y_{\alpha_1}) e_1 \ \big(\in V_{\Lambda_\pi - \alpha_1 - \cdots - \alpha_k} \big)$ where $\alpha_1, \ldots, \alpha_k \in S$.
- this basis is ordered in such a way that $i \geqslant j \ \Rightarrow \ p(\mu_i) \geqslant p(\mu_j)$.

Remark. We have $V_{\tilde{\Lambda}_\pi} = \mathbb{C} e_d$, because $V_{\tilde{\Lambda}_\pi}$ is one–dimensional (over $\mathbb{C}$).

Lemma 3.1. *Let* $\sigma_1, \ldots, \sigma_m \in \Sigma_+$ *and* $\mu \in \mathcal{P}(\pi)$ *such that*

$$\pi(y_{\sigma_m}) \ldots \pi(y_{\sigma_1}) \ V_\mu \neq \{0\}.$$

There exist $(\alpha_1, \ldots \alpha_r) \in \mathcal{D}(\pi)$ *and* $1 \leqslant i_1 \leqslant \cdots \leqslant i_m \leqslant i_{m+1} \leqslant r$ *with the following property*

$$\mu = \Lambda_\pi - \sum_{j=1}^{i_1 - 1} \alpha_j \ , \ \sigma_1 = \sum_{j=i_1}^{i_2 - 1} \alpha_j \ , \ldots, \ \sigma_m = \sum_{j=i_m}^{i_{m+1}-1} \alpha_j$$
$$\textit{and } \mu - \sum_{n=1}^{m} \sigma_n \ - \tilde{\Lambda}_\pi = \sum_{j=i_{m+1}}^{r} \alpha_j .$$

We say that $(\alpha_1, \ldots \alpha_r)$ *refines* $(\mu, \sigma_1, \ldots, \sigma_m)$.

Let $\tilde{\pi}$ be the representation of G such that $\mathrm{d}\tilde{\pi}(e) = \pi$. Since K is compact, $\tilde{\pi}(K)$ is a compact subgroup of $\mathrm{GL}(V)$. It is well known that consequently there exists an hermitian product $(.\,,.)$ over V which is $\tilde{\pi}(K)$–invariant (see [14], footnote in the proof of theorem 4.11.7, p 345–346). On the other hand, any $g \in G$ can be written as $g = k_1 \exp(H) k_2$, where $k_1, k_2 \in K$, $H \in \overline{\mathfrak{a}^+}$, and $\mathrm{Rad}(g) = H$ is unique (this is the Cartan decomposition of G). With these notations, we have:

Lemma 3.2. *The subspaces* V_μ, $\mu \in \mathcal{P}(\pi)$ *are orthogonal to each other. Moreover, given* $g \in G$, *we have*

$$\log \|\tilde{\pi}(g)\| = \Lambda_\pi(\mathrm{Rad}(g)).$$

3.2. Expression with coordinates

For each $\alpha \in \Sigma_+$, we choose $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $Q(y_\alpha, y_\alpha) = 1$ and then the standard Euclidean Brownian motion $(\beta_t)_{t \geqslant 0}$ on $\mathfrak{a} \oplus \overline{\mathfrak{n}}$ can be written as

$$\beta_t = W_t + \sum_{\alpha \in \Sigma_+} \beta_t^\alpha y_\alpha,$$

where W is the standard Brownian motion on $\mathfrak{a}$ and the β^α's are complex standard Brownian motions independent of each other, independent of W.

Given $\varepsilon > 0$, the process $B^{\tau,\varepsilon}$ on the solvable group AN, is introduced in [6] as the stochastic exponential $\mathcal{E}\left(\frac{1}{\varepsilon}\beta\right)$ of $\frac{1}{\varepsilon}\beta$ (see [2] or [8]) since it is given by the Stratonovitch stochastic differential equation

$$\begin{cases} \varepsilon \delta B_t^{\tau,\varepsilon} = B_t^{\tau,\varepsilon} \delta \beta_t \\ B_0^{\tau,\varepsilon} = e \end{cases}.$$

Let $X_t^\varepsilon = \tilde{\pi}\left(B_t^{\tau,\varepsilon}\right)$. Then X^ε is the solution of the Stratonovitch SDE on the Lie algebra $\mathcal{L}(V)$ of endomorphisms on V

$$\begin{cases} \varepsilon \delta X_t^\varepsilon = X_t^\varepsilon \delta \tilde{\beta}_t \\ X_0^\varepsilon = e \end{cases}. \tag{13}$$

where $\tilde{\beta}_t = \pi(\beta_t)$. In other words, X^ε is the stochastic exponential of $\frac{1}{\varepsilon}\tilde{\beta}$.

We identify $\mathfrak{gl}(V)$ and $\mathfrak{gl}(d, \mathbb{C})$ by the choice of a basis $(e_i)_{1 \leqslant i \leqslant d}$ of V adapted to the decomposition (12). Matrix representations of elements of $\mathfrak{gl}(V)$ will be relative to this fixed basis. Equation (13) turns into

$$\begin{cases} \varepsilon \left(X_t^\varepsilon\right)_{i,j} = \sum_{k=1}^d \int_0^t \left(X_t^\varepsilon\right)_{i,k} \delta\left(\tilde{\beta}_t\right)_{k,j}, \\ \left(X_0^\varepsilon\right)_{i,j} = \delta_{i,j} \end{cases} \qquad i, j \in \{1, \ldots, d\}. \tag{14}$$

The semimartingale $\frac{1}{\varepsilon}\tilde{\beta}$ takes its values in the Lie algebra of lower triangular matrices and its diagonal is $\frac{1}{\varepsilon}\mathrm{diag}(\mu_1(W_s), \ldots, \mu_d(W_s))$. Hence, $X^\varepsilon = \mathcal{E}\left(\frac{1}{\varepsilon}\tilde{\beta}\right)$ is a semimartingale with values in the Lie group of invertible lower triangular matrices (see [8]) and

$$\left(X_t^\varepsilon\right)_{i,i} = \frac{1}{\varepsilon} \int_0^t \left(X_s^\varepsilon\right)_{i,i} \delta(\mu_i(W_s)), \quad i \in \{1, \ldots, d\}.$$

It is well known that

$$\left(X_s^\varepsilon\right)_{i,i} = \exp\left(\frac{1}{\varepsilon}\mu_i(W_s)\right), \quad s \geqslant 0,$$

and for $i > j$, we have

$$\begin{aligned}\varepsilon\left(X_t^{\varepsilon}\right)_{i,j} &= \int_0^t \left(X_s^{\varepsilon}\right)_{i,j}\delta\left(\tilde{\beta}_s\right)_{j,j} + \sum_{k=j+1}^{i}\int_0^t \left(X_s^{\varepsilon}\right)_{i,k}\delta\left(\tilde{\beta}_s\right)_{k,j}\\ &= \int_0^t \left(X_s^{\varepsilon}\right)_{i,j}\delta\left(\mu_j(W_s)\right) + \sum_{k=j+1}^{i}\int_0^t \left(X_s^{\varepsilon}\right)_{i,k}\delta\left(\tilde{\beta}_s\right)_{k,j},\end{aligned}$$

thus

$$\delta\left(\mathrm{e}^{-\frac{1}{\varepsilon}\mu_j(W_t)}\varepsilon\left(X_t^{\varepsilon}\right)_{i,j}\right) = \mathrm{e}^{-\frac{1}{\varepsilon}\mu_j(W_t)}\sum_{k=j+1}^{i}\left(X_s^{\varepsilon}\right)_{i,k}\delta\left(\tilde{\beta}_s\right)_{k,j}.$$

Given $i \in \{2, \ldots, d\}$, we begin by computing $\left(X_t^{\varepsilon}\right)_{i,i}$ then we compute $\left(X_t^{\varepsilon}\right)_{i,i-1}$, $\left(X_t^{\varepsilon}\right)_{i,i-2}, \ldots, \left(X_t^{\varepsilon}\right)_{i,j}$ by successive stochastic integrations. In this way, we get the entry $(X^{\varepsilon})_{i,j}$ as a sum of iterated stochastic integrals:

$$\begin{aligned}\left(X_t^{\varepsilon}\right)_{i,j} = &\sum_{1\leqslant m\leqslant i-j}\frac{1}{\varepsilon^m}\sum_{j<l_1<\cdots<l_{m-1}}\\ &\times\int_{t>t_1>\cdots>t_m>0}\mathrm{e}^{\frac{1}{\varepsilon}\left[\mu_j(W_t-W_{t_1})+\mu_{l_1}(W_{t_1}-W_{t_2})+\cdots+\mu_{l_{m-1}}(W_{t_{m-1}}-W_{t_m})+\mu_i(W_{t_m})\right]}\\ &\times\delta\left(\tilde{\beta}_{t_m}\right)_{i,l_{m-1}}\delta\left(\tilde{\beta}_{t_{m-1}}\right)_{l_{m-1},l_{m-2}}\ldots\delta\left(\tilde{\beta}_{t_1}\right)_{l_1,j}.\end{aligned} \tag{15}$$

Let $j \in \{1, \ldots, d\}$, $t \in [0, 1]$ and $\alpha \in \Sigma_+$. We have

$$\tilde{\beta}_t.e_j = \pi(\beta_t).e_j = \sum_{i=j+1}^{d}\left(\tilde{\beta}_t\right)_{i,j}e_i.$$

We project this relation on $V_{\mu_j-\alpha}$:

$$\beta_t^{\alpha}\pi(y_\alpha).e_j = \sum_{\substack{i>j\\ \mu_j-\mu_i=\alpha}}\left(\tilde{\beta}_t\right)_{i,j}e_i.$$

By induction on m, we obtain, given $t_1, \ldots, t_m \in [0, 1]$ and $\alpha_1, \ldots, \alpha_m \in \Sigma_+$,

$$\beta_{t_m}^{\alpha_m}\pi(y_{\alpha_m})\ldots\beta_{t_1}^{\alpha_1}\pi(y_{\alpha_1}).e_j = \sum_{\substack{i_m>\cdots>i_1>j=i_0\\ \mu_{i_{k-1}}-\mu_{i_k}=\alpha_k}}\left(\tilde{\beta}_{t_m}\right)_{i_m,i_{m-1}}\cdots\left(\tilde{\beta}_{t_1}\right)_{i_1,j}e_{i_m}.$$

Thus we get by (15)

$$\left(X_t^\varepsilon\right)_{i,j} = \sum_{1\leqslant m\leqslant i-j} \frac{1}{\varepsilon^m} \sum_{j<l_1<\cdots<l_{m-1}<i}$$
$$\times \int_{t>t_1>\cdots>t_m>0} \mathrm{e}^{-\frac{1}{\varepsilon}\left[\mu_j(W_t-W_{t_1})+\mu_{l_1}(W_{t_1}-W_{t_2})+\cdots+\mu_{l_{m-1}}(W_{t_{m-1}}-W_{t_m})+\mu_i(W_{t_m})\right]}$$
$$\times \sum_{\substack{(\alpha_1,\ldots,\alpha_m)\in\Sigma_+^m \\ \mu_{l_{k-1}}-\mu_{l_k}=\alpha_k}} \left[\pi\left(y_{\alpha_m}\right)\ldots\pi\left(y_{\alpha_1}\right)\right]_{i,j} \delta\beta_{t_m}^{\alpha_m}\ldots\delta\beta_{t_1}^{\alpha_1}.$$

Finally, for $i > j$,

$$\left(X_t^\varepsilon\right)_{i,j} = \sum_{\substack{1\leqslant m\leqslant i-j \\ \alpha_1,\ldots,\alpha_m\in\Sigma_+}} \frac{1}{\varepsilon^m}\left[\pi\left(y_{\alpha_m}\right)\ldots\pi\left(y_{\alpha_1}\right)\right]_{i,j} \mathrm{e}^{\frac{1}{\varepsilon}\mu_j(W_t)}$$
$$\times \int_{t>t_1>\cdots>t_m>0} \mathrm{e}^{-\frac{1}{\varepsilon}\left[\sum_k \alpha_k(W_{t_k})\right]}\delta\beta_{t_m}^{\alpha_m}\ldots\delta\beta_{t_1}^{\alpha_1}. \tag{16}$$

Since W is independent of the β^α's and since the Brownian motions β^α are complex, there is no need to make a distinction between Itô and Stratonovitch integrals in (16) and then δ will be replaced by d.

3.3. *Lower bound*

We now assume that $\varepsilon = \frac{1}{n}, n \in \mathbb{N}^*$, we fix $t > 0$ and we put $X_n(t) = X_t^{\frac{1}{n}}$. By (16),

$$(X_n(t))_{d,1} = \sum_{\substack{1\leqslant m\leqslant d-1 \\ \alpha_1,\ldots,\alpha_m\in\Sigma_+}} n^m \left[\pi\left(y_{\alpha_m}\right)\ldots\pi\left(y_{\alpha_1}\right)\right]_{d,1} \mathrm{e}^{n\Lambda_\pi(W_t)}$$
$$\times \int_0^t \int_0^{t_1}\cdots\int_0^{t_{m-1}} \mathrm{e}^{-n\sigma_m(W_{t_m})}\mathrm{d}\beta_{t_m}^{\alpha_m}\ldots \mathrm{e}^{-n\sigma_1(W_{t_1})}\mathrm{d}\beta_{t_1}^{\alpha_1}. \tag{17}$$

By Laplace's method (theorem 1.3), we have, almost surely,

$$\lim_{n\to\infty}\frac{1}{n}\log\left|(X_n(t))_{d,1}\right| = \max_{\substack{\sigma_1,\ldots,\sigma_r\in\Sigma_+ \\ \left[\pi(y_{\sigma_r})\ldots\pi(y_{\sigma_1})\right]_{d,1}\neq 0}} \max_{t\geqslant t_1\geqslant\cdots\geqslant t_r\geqslant 0} \Lambda_\pi(W_t)$$
$$-\sigma_r(W_{t_r}) - \cdots - \sigma_1(W_{t_1}),$$

taking into account that the process W is independent of the processes $\beta^\alpha, \alpha \in \Sigma_+$. Let[5] $\sigma_1^\circ, \ldots, \sigma_r^\circ \in \Sigma_+$ be such that $\left[\pi(y_{\sigma_r^\circ})\ldots\pi(y_{\sigma_1^\circ})\right]_{d,1} \neq 0$. Let us use a refinement $(\alpha_1, \ldots, \alpha_r)$ of $\left(\Lambda_\pi, \sigma_1^\circ, \ldots, \sigma_r^\circ\right)$(see lemma 3.1): we have

[5] Such roots do exist; see the very definitions of basis $(e_1, \ldots, e_d)$ and integer $r(\pi)$ in sections 3.1.2 and 3.1.1.

$$\sigma_1^\circ = \sum_{j=1}^{i_2} \alpha_j, \ldots, \sigma_r^\circ = \sum_{j=i_r}^{i_{r+1}-1} \alpha_j, \quad \text{with } i_{m+1} = r+1, i_1 = 1,$$

and necessarily[6] $\left(\sigma_1^\circ, \ldots, \sigma_r^\circ\right) = (\alpha_1, \ldots, \alpha_r) \in \mathcal{D}(\pi)$. Thus we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \left|X_n(t)_{d,1}\right| = & \max_{(\alpha_1,\ldots,\alpha_r)\in\mathcal{D}(\pi)} \max_{t\geqslant t_1\geqslant\cdots\geqslant t_r\geqslant 0} \Lambda_\pi(W_t) \\ & -\alpha_r(W_{t_r}) - \cdots - \alpha_1(W_{t_1}),\end{aligned}$$

that is

$$\lim_{n\to\infty} \frac{1}{n} \log \left|X_n(t)_{d,1}\right| = (\mathcal{T}_\pi W)(t), \quad \text{almost surely.}$$

Hence

$$\liminf_{n\to\infty} \frac{1}{n} \log \max_{1\leqslant i,j\leqslant d} \left|X_n(t)_{i,j}\right| \geqslant \lim_{n\to\infty} \frac{1}{n} \log \left|X_n(t)_{d,1}\right| = (\mathcal{T}_\pi W)(t).$$

Since norms on $\mathfrak{gl}(V)$ are equivalent, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \|X_n(t)\| \geqslant (\mathcal{T}_\pi W)(t), \tag{18}$$

almost surely and for any irreducible representation π of $\mathfrak{g}$.

3.4. Upper bound

Let $i, j \in \{1, \ldots, d\}$, $i > j$, such that there exist $m \in \{1, \ldots, i-j\}$ and $\left(\sigma_1^\circ, \ldots, \sigma_m^\circ\right) \in \Sigma_+^m$ satisfying $\left[\pi(y_{\sigma_m^\circ}) \ldots \pi(y_{\sigma_1^\circ})\right]_{i,j} \neq 0$; the random variable $\left(X_t^\varepsilon\right)_{i,j}$ almost surely never vanishes and by Laplace's method (theorem 1.3), we know that, almost surely,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log \left|X_n(t)_{i,j}\right| \leqslant & \max_{\substack{1\leqslant p\leqslant i-j \\ \sigma_1,\ldots,\sigma_p\in\Sigma_+ \\ \left[\pi(y_{\sigma_p})\ldots\pi(y_{\sigma_1})\right]_{i,j}\neq 0}} \max_{t\geqslant t_1\geqslant\cdots\geqslant t_p\geqslant 0} \mu_j(W_t) \\ & -\sigma_p(W_{t_p}) - \cdots - \sigma_1(W_{t_1}).\end{aligned}$$

By making use of a refinement of $\left(\mu_j, \sigma_1, \ldots, \sigma_p\right)$, $p \in \{1, \ldots, i-j\}$ (see lemma 3.1), we get

$$\begin{aligned}& \max_{t\geqslant t_1\geqslant\cdots\geqslant t_p\geqslant 0} \mu_j(W_t) - \sigma_p(W_{t_p}) - \cdots - \sigma_1(W_{t_1}) \\ & \leqslant \max_{(\alpha_1,\ldots,\alpha_r)\in\mathcal{D}(\pi)} \max_{t\geqslant t_1\geqslant\cdots\geqslant t_r\geqslant 0} \Lambda_\pi(W_t) \\ & \quad -\alpha_r(W_{t_r}) - \cdots - \alpha_1(W_{t_1}) = (\mathcal{T}_\pi W)_t\,.\end{aligned}$$

[6] What is pertinent here is that such positive roots $\sigma_1^0, \ldots, \sigma_r^0$ turn out to be simple roots.

Hence, almost surely,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left(\max_{1\leqslant i,j\leqslant d} \left| X_n(t)_{i,j} \right| \right) \leqslant (\mathcal{T}_\pi W)_t \,,$$

and since norms on $\mathfrak{gl}(V)$ are equivalent, we have also

$$\limsup_{n\to\infty} \frac{1}{n} \log \|X_n(t)\| \leqslant (\mathcal{T}_\pi W)_t \,. \tag{19}$$

Theorem 3.3. *Given any finite irreducible representation π of $\mathfrak{g}$,*

$$\lim_{n\to\infty} \frac{1}{n} \Lambda_\pi \left(\mathrm{Rad} \left(B_t^{\tau,\frac{1}{n}} \right) \right) = (\mathcal{T}_\pi W)_t \,, \quad \textit{almost surely.}$$

Proof. This is a consequence of inequalities (18) and (19), and of lemma 2.5 in [6] that is reproduced here in lemma 3.2. □

3.5. Conclusion

For $\alpha \in S$, let us denote by H_α the vector in $\mathfrak{a}$ that represents, via κ, the linear form $\frac{2\alpha}{\langle \alpha,\alpha\rangle}$. The fundamental weights are the elements of the dual basis $(\varpi_\alpha)_{\alpha\in S}$ of $(H_\alpha)_{\alpha\in S}$. For $\alpha \in S$, let us denote by π_α the fundamental simple representation of $\mathfrak{g}$, with highest weight ϖ_α. By theorem 3.3,

$$\lim_{n\to\infty} \frac{1}{n} \varpi_\alpha \left(\mathrm{Rad} \left(B_t^{\tau,\frac{1}{n}} \right) \right) = \left(\mathcal{T}_{\pi_\alpha} W \right)_t \,, \tag{20}$$

almost surely and for any $\alpha \in S$. This motives us to define $\mathcal{T} : C_0 ([0, 1], \mathfrak{a}) \to C_0 ([0, 1], \mathfrak{a})$ by

$$\begin{aligned} \left(\mathcal{T} w\right)(t) = \sum_{\alpha\in S} \left(\mathcal{T}_{\pi_\alpha} w\right)(t) H_\alpha &= w(t) \\ &\quad - \sum_{\alpha\in S} U_t(\alpha, w)\, H_\alpha, \qquad w \in C_0([0, 1]),\ t \in [0, 1], \end{aligned}$$

where

$$U_t(\alpha, w) = \min_{(\alpha_1,\ldots,\alpha_r)\in\mathcal{D}(\pi_\alpha)} \ \min_{t\geqslant t_1\geqslant\cdots\geqslant t_r\geqslant 0} \sum_{k=1}^{r} \alpha_k \left(w(t_k) \right) .$$

Proposition 3.4. *The path transform $\mathcal{T}$ has the following properties:*

1. *The sequence* $\left(\frac{1}{n} \mathrm{Rad} \left(B_t^{\tau,\frac{1}{n}} \right) \right)_{n\geqslant 1}$ *converges almost surely to* $(\mathcal{T} W)_t$.
2. *The map $\mathcal{T}$ is continuous.*
3. *The map $\mathcal{T}$ takes its values in* $C_0 \left([0, 1], \overline{\mathfrak{a}^+} \right)$.
4. *Given any finite irreducible representation π of $\mathfrak{g}$, and any $w \in C_0([0, 1], \mathfrak{a})$,*

$$\Lambda_\pi \left(\mathcal{T} w \right) = \left(\mathcal{T}_\pi w \right) .$$

Proof. From (20), we easily get 1). For 2), it suffices to prove that for any $\alpha \in S$,

$$\mathcal{T}_{\pi_\alpha} : C_0\left([0,1], \mathfrak{a}\right) \to \mathcal{C}([0,1])$$

is continuous. Let π be an irreducible representation of $\mathfrak{g}$, $r = r(\pi)$ and (w_l) a sequence on $C_0\left([0,1], \mathfrak{a}\right)$, that converges uniformly to w. Fix $(\alpha_1, \ldots \alpha_r) \in \mathcal{D}(\pi)$ and define

$$\mathcal{Q} : C_0\left([0,1], \mathfrak{a}\right) \to \mathcal{C}([0,1])$$

by

$$(\mathcal{Q}v)(t) = \min_{t \geqslant t_1 \geqslant \cdots \geqslant t_r \geqslant 0} \sum_{k=1}^{r} \alpha_k\left(v(t_k)\right).$$

Since $\mathcal{Q}w_l$ is nonincreasing for any $l \in \mathbb{N}$ and $(\mathcal{Q}w_l)$ tends to $\mathcal{Q}w$, it is well known that the convergence is uniform on $[0,1]$. From this we deduce that $\mathcal{T}_\pi$ is continuous.

We endow $C_0([0,1], \mathfrak{a})$ with its Borel σ–algebra and with the Wiener measure $\mathcal{W}$, and consider W as the canonical process on this Wiener space. The process $\mathcal{T}W$ is continuous, so by 1), its trajectories are $\mathcal{W}$–almost surely in $\overline{\mathfrak{a}^+}$. There exists a borelian set $\mathcal{A}$ in $C_0\left([0,1], \mathfrak{a}\right)$, such that $\mathcal{W}(\mathcal{A}) = 1$ and that for any $w \in \mathcal{A}$, $\mathcal{T}w \in C_0\left([0,1], \overline{\mathfrak{a}^+}\right)$. This borelian set is dense in $C_0\left([0,1], \mathfrak{a}\right)$, and then we get 3), because $\mathcal{T}$ is continuous. We obtain 4) by similar arguments and by theorem 3.3. □

The following result gives us a link between radial part of Brownian motion in $\mathfrak{p}$ and $\mathcal{T}$–transform of Brownian motion on $\mathfrak{a}$, via the processes $\varepsilon B_t^{\tau,\varepsilon}$ on the solvable group AN. This is proposition 3.2 of [6]. It appears in [1] that $\mathrm{rad}(Z)$ is actually the so–called Brownian motion in the Weyl chamber $\mathfrak{a}^+$ (see also [4]).

Proposition 3.5. *Let Z be the Euclidian Brownian motion on $\mathfrak{p}$, starting at 0. The processes $\frac{1}{n}\mathrm{Rad}\left(B^{\tau,\frac{1}{n}}\right)$ and $\mathrm{rad}(Z)$ have the same law.*

According to theorem 3.3 and theorem 3.5, given $t \geqslant 0$, the random variables $\mathrm{rad}(Z_t)$ and $(\mathcal{T}W)_t$ have the same law. The almost sure convergence in theorem 3.3 enables us to say quite more.

Corollary 3.6 (and theorem 1.4)**.** *The processes $\mathrm{rad}(Z)$ and $\mathcal{T}W$ have the same law.*

Proof. By proposition 3.4, 1), for any $t_1, \ldots, t_q \in [0,1]$, the sequence of random variables

$$\left(\frac{1}{n}\mathrm{Rad}\left(B_{t_1}^{\tau,1/n}\right), \ldots, \frac{1}{n}\mathrm{Rad}\left(B_{t_q}^{\tau,1/n}\right)\right)_{n \geqslant 0}$$

tends almost surely to $\left((\mathcal{T}W)_{t_1}, \ldots, (\mathcal{T}W)_{t_q}\right)$ thus by theorem 3.5, $\big(\mathrm{rad}(Z_{t_1}), \ldots,$ $\mathrm{rad}(Z_{t_q})\big)$ and $\left((\mathcal{T}W)_{t_1}, \ldots, (\mathcal{T}W)_{t_q}\right)$ have the same law. We conclude by monotone class theorem, since $\mathrm{rad}(Z)$ et $\mathcal{T}W$ are continuous processes and the Borel σ–algebra of $\mathcal{C}([0,1])$ is spanned by the projections

$$\mathrm{pr}_t : \mathcal{C}([0,1]) \to \mathfrak{a}, \; f \mapsto f(t). \qquad \square$$

3.6. Example: Lie algebra of type G_2

We are interested in the case when $\mathfrak{g}$ is a Lie algebra of type G_2, that is a semisimple Lie algebra over $\mathbb{C}$ whose root system is isomorphic to the one represented in figure 2, where $S = \{\alpha, \beta\}$, $\|\beta\| = \sqrt{2}\,\|\alpha\|$, $\frac{\langle\alpha,\beta\rangle}{\|\alpha\|\|\beta\|} = -\frac{\sqrt{3}}{2}$. Notice that the Weyl chamber is a convex cone with angle $\frac{\pi}{6}$ (figure 2).

From the root system itself, the weights of the fundamental representations are easy to describe (see [11]) p. 295). The fundamental weights of G_2 are given by $\Lambda_{\pi_1} = 2\alpha + \beta$ and $\Lambda_{\pi_2} = 3\alpha + 2\beta$, as π_1 and π_2 denote the associated simple fundamental representations. We have on one hand

$$\mathcal{P}(\pi_1) = \{0, \pm\alpha, \pm(2\alpha+\beta), \pm(\alpha+\beta)\},$$

and since $\tilde{\Lambda}_{\pi_1} = -\Lambda_{\pi_1}$, $r(\pi_1) = 6$. Then

$$\mathcal{D}(\pi_1) = \{(\alpha, \beta, \alpha, \alpha, \beta, \alpha)\}.$$

On the other hand, π_2 is the adjoint representation and

$$\mathcal{P}(\pi_2) = \mathcal{R} \cup \{0\}.$$

Since $\tilde{\Lambda}_{\pi_2} = -\Lambda_{\pi_2}$, we have $r(\pi_2) = 10$ and

$$\begin{aligned}\mathcal{D}(\pi_2) = \{&(\beta,\alpha,\alpha,\beta,\alpha,\beta,\alpha,\alpha,\alpha,\beta), (\beta,\alpha,\alpha,\beta,\alpha,\beta,\alpha,\alpha,\beta,\alpha),\\ &(\beta,\alpha,\alpha,\alpha,\beta,\beta,\alpha,\alpha,\alpha,\beta), (\beta,\alpha,\alpha,\alpha,\beta,\beta,\alpha,\alpha,\beta,\alpha)\}.\end{aligned}$$

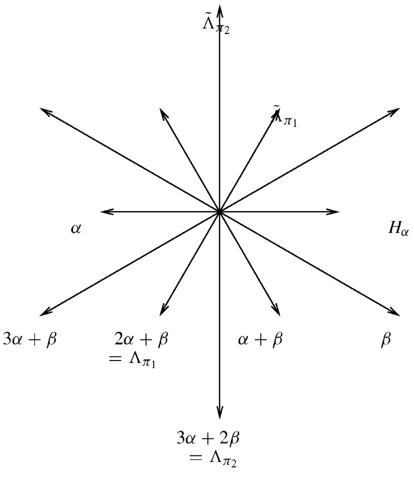

Fig. 2

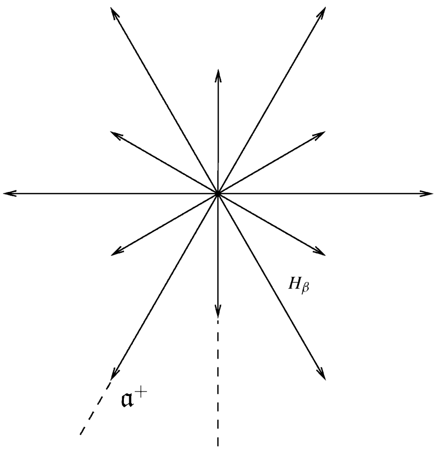

Fig. 3

Hence corollary 3.6 holds, despite π_1 and π_2 are not minuscule since 0 is a weight for these representations.

Remark. It is also possible to treat the case of the last fundamental representation π of a Lie algebra of type B_l (which is not minuscule) in the same way as the others. The computation of $\mathcal{D}(\pi)$ and $r(\pi)$ is perhaps more tricky in contrast to the minuscule case or the rank–two Lie algebra case.

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