Probab. Theory Relat. Fields 132, 579–612 (2005)
Digital Object Identifier (DOI) 10.1007/s00440-004-0406-3

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Limit theorems for sums of random exponentials

Received: 19 February 2004 / Revised version: 23 October 2004 /
Published online: 10 February 2005 –

Abstract. We study limiting distributions of exponential sums $S_N(t) = \sum_{i=1}^N e^{tX_i}$ as $t \to \infty$, $N \to \infty$, where (X_i) are i.i.d. random variables. Two cases are considered: (A) $\operatorname{ess\,sup} X_i = 0$ and (B) $\operatorname{ess\,sup} X_i = \infty$. We assume that the function $h(x) = -\log \mathsf{P}\{X_i > x\}$ (case B) or $h(x) = -\log \mathsf{P}\{X_i > -1/x\}$ (case A) is regularly varying at ∞ with index $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). The appropriate growth scale of N relative to t is of the form $e^{\lambda H_0(t)}$ $(0 < \lambda < \infty)$, where the rate function $H_0(t)$ is a certain asymptotic version of the function $H(t) = \log \mathsf{E}[e^{tX_i}]$ (case B) or $H(t) = -\log \mathsf{E}[e^{tX_i}]$ (case A). We have found two critical points, $\lambda_1 < \lambda_2$, below which the Law of Large Numbers and the Central Limit Theorem, respectively, break down. For $0 < \lambda < \lambda_2$, under the slightly stronger condition of normalized regular variation of h we prove that the limit laws are stable, with characteristic exponent $\alpha = \alpha(\varrho, \lambda) \in (0, 2)$ and skewness parameter $\beta \equiv 1$.

1. Introduction

1.1. The problem

In this work, we are concerned with partial sums of exponentials of the form

$$S_N(t) = \sum_{i=1}^{N} e^{tX_i}, \tag{1.1}$$

where (X_i) is a sequence of independent identically distributed random variables and both t and N tend to infinity. Our goal is to study the limiting distribution of $S_N(t)$ and to explore possible 'phase transitions' due to various rates of growth of the parameters t and N.

In such analysis, two cases are naturally distinguished according to whether X_i are bounded above (*case A*) or unbounded above (*case B*). In the former case,

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*Research supported in part by the DFG grants 436 RUS 113/534 and 436 RUS 113/722.

Mathematics Subject Classification (2000): 60G50, 60F05, 60E07

Key words or phrases: Sums of independent random variables – Random exponentials – Regular variation – Exponential Tauberian theorems – Central limit theorem – Weak limit theorems – Infinitely divisible distributions – Stable laws

without loss of generality we may and will assume that the upper edge of the support of X_i is zero, $\operatorname{ess\,sup} X_i = 0$.

One can expect that the results will depend on the structure of the upper distribution tail of X_i. In this paper, we focus on the class of distributions with the upper tail of the *Weibull/Fréchet* form

$$\mathsf{P}\{X_i > x\} \approx \begin{cases} \exp(-cx^{\varrho}) & \text{as } x \to +\infty \quad \text{(case B)}, \\ \exp(-c(-x)^{-\varrho}) & \text{as } x \to 0- \quad \text{(case A)}, \end{cases} \tag{1.2}$$

where $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). More precisely, we will be assuming that the function $\log \mathsf{P}\{X_i > x\}$ is regularly varying in a vicinity of $\operatorname{ess\,sup} X_i$ with index $\varrho \in (1, \infty)$ (case B) or $-\varrho \in (-\infty, 0)$ (case A). For example, a normal distribution is contained in this class (case B, $\varrho = 2$).

1.2. Motivation

1.2.1. Topics in Probability. One motivation for this study is quite abstract and purely probabilistic. In fact, such a setting provides a natural tool to interpolate between the classical limit theorems concerning the bulk of the sample, i.e. the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT), on the one hand, and limit theorems for extreme values, on the other hand. It is clear that the asymptotic behaviour of $S_N(t)$ is largely determined by the relationship between the parameters t and N. If, for instance, one lets N tend to infinity with t fixed or growing very slowly, then, under appropriate (exponential) moment conditions, the usual LLN and CLT should be valid. In contrast, if the growth rate of N is small enough as compared to t, then the asymptotic behaviour of the sum $S_N(t)$ is dominated by its maximal term. We will see that when both t and N tend to infinity, a rich intermediate picture emerges made up of various limit regimes.

In this connection, let us mention a recent paper by Schlather [16] who studied the asymptotics of the l_p-norms of samples of positive i.i.d. random variables, $\|Y_{1n}\|_p = \left(\sum_{i=1}^n Y_i^p\right)^{1/p}$, where the norm order $p = p(n)$ grows together with the sample size n. The link with our setting becomes clear if one puts $Y_i = e^{X_i}$. Qualitatively speaking, in [16] it was demonstrated that under a suitable parametrization of the functional relation between p and n, there is a 'homotopy' for the limit distributions of $\|Y_{1n}\|_p$ extending from the CLT to a limit law for extreme values. The situation where $p = p(n) \to \infty$ as $n \to \infty$ arises if the random variables Y_i are bounded above and, in the sense of extreme value theory, belong to the domain of attraction of the Weibull distribution $\Psi_\alpha(x) = \exp\left(-(-x)^\alpha\right)$ ($\alpha > 0$, $x < 0$) [16, Theorem 2.3].

Application of our work to the limit distribution of l_p-norms is discussed in [4]. Let us point out that our results are complementary to [16], since for random variables X_i with the Weibull/Fréchet tails (1.2) the distribution of the maximum of $e^{X_1}, \dots, e^{X_n}$ can be shown to converge to the Gumbel (double exponential) distribution $\Lambda(x) = \exp\left(-e^{-x}\right)$, $x \in \mathbb{R}$ (see [4]). Note that in the case of attraction to Λ, [16, Theorem 2.4] gives only a partial result for exponential random variables.

1.2.2. Branching populations. The second motivation (in fact, the most important one) is related to long-term dynamics in random media. In the simplest situation, exponential sums emerge as the (quenched) mean population size of a colony of non-interacting branching processes with random branching rates. Indeed, consider N branching processes $Z_i(t)$ driven by branching rates $X_i = X_i(\omega)$ $(i = 1, \dots, N)$. More precisely, for a fixed (*quenched*) environment ω, each $Z_i(t)$ is a Markov continuous-time branching process such that during time $dt \to 0$, with probability $|X_i| dt$ a particle may split into two (if $X_i > 0$) or die (if $X_i < 0$). Note that the function $m_i(t) := \mathsf{E}^{\omega}[Z_i(t)]$ satisfies the differential equation $m_i' = X_i m_i$ (see [2, p. 108]). Assuming that $Z_i(0) = 1$ we obtain $m_i(t) = e^{tX_i}$, and hence the total quenched mean population size is given by the sum (1.1).

In more interesting and realistic situations, there is spatial motion of particles and hence interaction between individual populations. We believe that the problem of long-term dynamics for such systems can be essentially reduced, in each particular case, to sums involving random exponentials, and therefore various asymptotic regimes that we establish in the present paper will provide a basic building block for the understanding of new dynamical phase transitions for branching processes in random media. In general, such exponential sums may contain random weights, thus having the form $S_N(t) = \sum_{i=1}^{N} Y_i \, e^{tX_i}$. Here, the parameter N will characterize the spatial span of the initial population, while the random variables X_i and Y_i represent the local (spectral) characteristics of the quenched branching process, according to the mechanisms of dynamical randomness in the medium. Typically, the weights (Y_i) are expected to be mutually independent when conditioned on the (X_i). These more difficult questions, including a more general type of the abstract problem, will be addressed elsewhere.

1.2.3. Random Energy Model. A completely different example is provided by the Random Energy Model (REM) introduced by Derrida [7] as a simplified version of the Sherrington–Kirkpatrick model of spin glass. The REM describes a system of size n with 2^n energy levels $E_i = \sqrt{n}\, X_i$ $(i = 1, \dots, 2^n)$, where (X_i) are i.i.d. random variables with standard normal distribution. Thermodynamics of the system is determined by the partition function $\mathcal{Z}_n(\beta) := \sum_{i=1}^{2^n} e^{\beta \sqrt{n}\, X_i}$, which exemplifies the exponential sum (1.1) with $N = 2^n$, $t = \beta\sqrt{n}$.

The free energy for the REM, first obtained in [7] using heuristic arguments, is given by

$$F(\beta) := \lim_{n\to\infty} \frac{\log \mathcal{Z}_n(\beta)}{n} = \begin{cases} \beta^2/2 + \beta_{\mathrm{c}}^2/2 & \text{if } \ 0 < \beta \le \beta_{\mathrm{c}}, \\ \beta\beta_{\mathrm{c}} & \text{if } \ \beta \ge \beta_{\mathrm{c}}, \end{cases} \tag{1.3}$$

where $\beta_{\mathrm{c}} = \sqrt{2\log 2}$. Eisele [8] and Olivieri and Picco [13] have rigorously derived the limit (1.3) (in probability and a.s.) and also extended this result to the case where X_i have the Weibull-type tail (1.2) (case B).[1]

[1] Distributions considered in [8, 13] are subject to the condition $x^{-\varrho} h(x) \to \text{const} > 0$ as $x \to +\infty$, where $h(x) = -\log \mathsf{P}\{X_i > x\}$ and $1 < \varrho < \infty$ (see [8, Theorem 2.3]), which is more restrictive than our assumption of regular variation of $h(\cdot)$.

Recently, a detailed analysis of the limit laws for $\mathcal{Z}_n(\beta)$ in the Gaussian case has been accomplished by Bovier et al. [6]. In particular, is has been shown that in addition to the phase transition at the critical point β_c, manifested as the LLN breakdown for $\beta > \beta_c$, within the region $\beta < \beta_c$ there is a second phase transition at $\tilde{\beta}_c = \sqrt{\log 2/2} = \frac{1}{2}\beta_c$, in that for $\beta > \tilde{\beta}_c$ the fluctuations of $\mathcal{Z}_n(\beta)$ become non-Gaussian. In the present work, we extend these results to the class of distributions with Weibull/Fréchet-type tails of the form (1.2). As compared to the paper [6] which proceeded from extreme value theory, we use methods of theory of summation of independent random variables. This general and powerful approach simplifies the proofs and in particular reveals that non-Gaussian limit laws are in fact stable.[2]

1.2.4. Risk theory. Finally, let us point out one application related to insurance. A basic quantity in risk theory is the aggregate claim amount $Y(t) := \sum_{i=1}^{N(t)} U_i$, where (U_i) is a sequence of i.i.d. claim sizes and $N(t)$ is a claim counting process independent of (U_i) [15, Sect. 5.1.4]. A common problem is to estimate the moment generating function $m_U(s) := \mathsf{E}[e^{sU_i}]$, in particular for large s. Such a question arises, for example, in connection with the Lundberg bounds for the tail distribution of $Y(t)$.[3] The Lundberg bounds are constructed using the root s^* of the equation $m_U(s) = 1/p > 1$ (see [15, p. 125]), where the parameter p has the meaning of the claim arrival rate. Hence, the case $p \to 0$ (and therefore $s^* \to \infty$) corresponds to the practically important situation of small ‘claim load’.

The statistical method for estimating the unknown solution s^* can be based on the empirical moment generating function $\hat{m}_U(s) := n^{-1} \sum_{i=1}^{n} e^{sU_i}$ (cf. (1.1)). A natural estimator $\tilde{s}$ defined by the equation $\hat{m}_Y(\tilde{s}) = 1/p$ has nice asymptotic properties including a.s.-consistency and asymptotic normality, providing $1/p$ is fixed or bounded [15, p. 130]. However, the asymptotic behaviour of $\tilde{s}$ when both n and $1/p$ are large does not seem to have been addressed so far.

2. Statement of the main results

2.1. Regularity and scaling

Denote $\omega_X := \operatorname{ess\,sup} X \equiv \sup\{x : \mathsf{P}(X > x) > 0\}$. Therefore, cases A and B mentioned in Section 1.1 correspond to $\omega_X = 0$ and $\omega_X = +\infty$, respectively. In view of the above interpretation of the problem in terms of branching populations (see Section 1.2.2), this labelling can be mnemonically associated with *annihilation* (case A) and *branching* (case B).

Let us make the following notational convention that will allow us to consider both cases A and B simultaneously.

[2] Some applications of our results to the REM are discussed in [4].

[3] Similar questions are of interest in other areas such as queueing theory (the equilibrium waiting time in $M/G/1$ queue, see [1, p. 269, 281]) and storage models (a dam process, see [1, Ch. XIII, § 3, 4]).

*Notation***.** In the symbols $\pm$, $\mp$, $\gtrless$ and the like, the *upper sign always refers to case B*, whereas the *lower sign corresponds to case A*. The notation $a^{\pm}$ stands for the power $a^{\pm 1}$.

Assume that $\mathsf{P}\{X < \omega_X\} = 1$, that is, X is finite with probability 1 (case B) or there is no atom at point $\omega_X = 0$ (case A). Consider the *log-tail distribution function*

$$h(x) := \begin{cases} -\log \mathsf{P}\{X > x\}, & x \in \mathbb{R} \quad \text{(case B)}, \\ -\log \mathsf{P}\{X > -1/x\}, & x > 0 \quad \text{(case A)}. \end{cases} \tag{2.1}$$

Clearly, in both cases $h(\cdot)$ is non-negative, non-decreasing, and right-continuous; it takes finite values in its domain and $h(x) \to +\infty$ as $x \to +\infty$. According to the above $\pm$-convention, the upper tail of the distribution of X can be written down in a united manner as

$$\mathsf{P}\{X > x\} = \exp\{-h(\pm x^{\pm})\}, \qquad x < \omega_X. \tag{2.2}$$

We will be working under the assumption that *h is regularly varying at infinity with index ϱ* (we write $h \in R_\varrho$), where $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). That is, for any constant $\kappa > 0$ we have $h(\kappa x)/h(x) \to \kappa^{\varrho}$ as $x \to +\infty$.

It follows that the *cumulant generating function*

$$H(t) := \pm \log \mathsf{E}[e^{tX}], \qquad t \geq 0, \tag{2.3}$$

is well defined; furthermore, it is non-decreasing and $H(t) \to +\infty$ as $t \to +\infty$. The link between the asymptotics of the functions h and H at infinity is characterized by the fundamental Kasahara–de Bruijn exponential Tauberian theorem (see Lemma 3.1 below). In particular, $h \in R_\varrho$ if and only if $H \in R_{\varrho'}$, where

$$\varrho' := \frac{\varrho}{\varrho \mp 1}\,. \tag{2.4}$$

Recalling that $1 < \varrho < \infty$ in case B and $0 < \varrho < \infty$ in case A, we get

$$1 < \varrho' < \infty \quad \text{(case B)}, \qquad 0 < \varrho' < 1 \quad \text{(case A)}. \tag{2.5}$$

According to (2.3), the expected value of the sum $S_N(t)$ is given by

$$\mathsf{E}[S_N(t)] = \sum_{i=1}^{N} \mathsf{E}[e^{tX_i}] = N e^{\pm H(t)},$$

suggesting that the function $H(t)$ sets up an appropriate (exponential) scale of the form $e^{\lambda H(t)}$ for the number of terms $N = N(t)$. However, the suitable rate function is not $H(t)$, but rather its particular asymptotic version $H_0(t) \sim H(t)$ provided by the Kasahara–de Bruijn Tauberian theorem.[4]

[4] This makes no difference in the 'crude' Theorems 2.1 and 2.2 below, but is crucial for the more delicate Theorems 2.3, 2.4 and 2.5.

The following two values appear to be critical with respect to the scale $\lambda H_0(t)$,

$$\lambda_1 := \frac{\varrho'}{\varrho}, \qquad \lambda_2 := 2^{\varrho'}\frac{\varrho'}{\varrho}, \tag{2.6}$$

in that the LLN and CLT break down below λ_1 and λ_2, respectively. Let us also introduce the parameter

$$\alpha \equiv \alpha(\varrho, \lambda) := \left(\frac{\varrho\lambda}{\varrho'}\right)^{1/\varrho'}, \tag{2.7}$$

which will be shown to play the role of characteristic exponent in the limit laws and hence provides their natural parametrization. In particular, note that the critical values of α corresponding to λ_1, λ_2 are given by $\alpha_1 = 1$, $\alpha_2 = 2$, respectively.

Below the critical point λ_2, the behaviour of the sum $S_N(t)$ becomes more sensitive to subtle details of the upper tail's structure. However, enough control is gained via imposing a slightly stronger condition on regularity of the log-tail distribution function h—that of *normalized regular variation*, $h \in NR_\varrho$ (see [5, p. 15]). This condition will be discussed in detail in Section 5.1. One of equivalent definitions is that for any $\varepsilon > 0$, the function $h(x)/x^{\varrho-\varepsilon}$ is ultimately increasing, while $h(x)/x^{\varrho+\varepsilon}$ is ultimately decreasing (see Lemma 5.2 below).

Under this assumption, the relationship between the functions h and H_0 can be characterized explicitly (see Section 5.1). Here we note that $H_0(t)$ can be found (for all t large enough) as the unique solution of the equation $\varrho' H_0 = \varrho h((\varrho' H_0/t)^{\pm})$ (Lemma 5.5).

2.2. *Statement of the main theorems*

We proceed to state our results. The first two theorems assert that $S_N(t)$ satisfies the Law of Large Numbers and the Central Limit Theorem in their conventional form provided that the number of terms N in $S_N(t)$ grows fast enough relative to t (roughly speaking, $N \gg e^{\lambda_1 H_0(t)}$ for LLN or $N \gg e^{\lambda_2 H_0(t)}$ for CLT). Denote

$$\lambda := \liminf_{t\to\infty} \frac{\log N}{H_0(t)}. \tag{2.8}$$

Theorem 2.1. *Suppose that $h \in R_\varrho$ and $\lambda > \lambda_1$. Then*

$$\frac{S_N(t)}{\mathsf{E}[S_N(t)]} \xrightarrow{p} 1 \qquad (t \to \infty).$$

Theorem 2.2. *Suppose that $h \in R_\varrho$ and $\lambda > \lambda_2$. Then*

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{\sqrt{\mathsf{Var}[S_N(t)]}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty).$$

For our further theorems, we need to specify the growth rate of N more precisely.

Scaling Assumption. The number $N = N(t)$ of terms in the sum $S_N(t)$ satisfies the condition

$$\lim_{t\to\infty} N\, e^{-\lambda H_0(t)} = 1, \tag{2.9}$$

where λ is a parameter such that $0 < \lambda < \infty$.

Let $\mu = \mu(t)$ be the (unique) solution of the equation

$$h\big((\mu H_0(t)/t)^{\pm}\big) = \frac{\lambda\varrho}{\varrho'}\, h\big((\varrho' H_0(t)/t)^{\pm}\big).$$

One can show (see Lemma 5.8 below) that

$$\lim_{t\to\infty} \mu(t) = \frac{\varrho\lambda}{\alpha}. \tag{2.10}$$

Let us also set

$$\eta_1(t) := \frac{\mu(t) H_0(t)}{t}. \tag{2.11}$$

We are now in a position to state one of our main results.

Theorem 2.3. *Assume that $h \in NR_\varrho$ and the scaling condition (2.9) is fulfilled. Let $0 < \lambda < \lambda_2$ and set*

$$B(t) := e^{\pm\mu(t) H_0(t)}, \tag{2.12}$$

$$A(t) := \begin{cases} \mathsf{E}[S_N(t)] & (\lambda_1 < \lambda < \lambda_2), \\ N B_1(t) & (\lambda = \lambda_1), \\ 0 & (0 < \lambda < \lambda_1), \end{cases} \tag{2.13}$$

where $B_1(t)$ is a truncated exponential moment,

$$B_1(t) := \mathsf{E}\big[e^{tX}\mathbf{1}_{\{X \le \pm\eta_1(t)\}}\big]. \tag{2.14}$$

Then, as $t \to \infty$,

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} \mathcal{F}_\alpha, \tag{2.15}$$

where $\mathcal{F}_\alpha$ is a stable law with exponent $\alpha \in (0, 2)$ defined in (2.7) and skewness parameter $\beta = 1$. The characteristic function of the law $\mathcal{F}_\alpha$ is given by

$$\phi_\alpha(u) = \begin{cases} \exp\left\{-\Gamma(1-\alpha)|u|^\alpha \exp\left(-\dfrac{i\pi\alpha}{2}\operatorname{sgn} u\right)\right\} & (\alpha \ne 1) \\ \exp\left\{iu(1-\gamma) - \dfrac{\pi}{2}|u|\left(1 + i\operatorname{sgn} u \cdot \dfrac{2}{\pi}\log|u|\right)\right\} & (\alpha = 1) \end{cases} \tag{2.16}$$

where $\gamma = 0.5772\ldots$ is the Euler constant.

Remark. For $1 < \alpha < 2$, we use an analytic continuation of the gamma function in (2.16), $\Gamma(1-\alpha) = \Gamma(2-\alpha)/(1-\alpha)$.

Let us now describe what happens at the critical points. In fact, the Law of Large Numbers and the Central Limit Theorem prove to be valid at λ_1 and λ_2, respectively; however the normalizing constants now require some truncation.

Theorem 2.4. *Under the hypotheses of Theorem 2.3, let $\lambda = \lambda_1$. Then*

$$\frac{S_N(t)}{NB_1(t)} \xrightarrow{p} 1 \qquad (t \to \infty), \tag{2.17}$$

where $B_1(t)$ is given by (2.14).

Theorem 2.5. *Under the hypotheses of Theorem 2.3, let $\lambda = \lambda_2$. Then*

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{\sqrt{NB_2(t)}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty),$$

where $B_2(t)$ is a truncated exponential moment of 'second order',

$$B_2(t) := \mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X \le \pm\eta_1(t)\}}\big]. \tag{2.18}$$

Remark. Note that by (2.9), we have $B(t) \sim N^{\pm\mu(t)/\lambda}$ as $t \to \infty$. In particular, (2.10) implies that in case B the normalization function $B(t)$ amounts to N raised to the power $\mu(t)/\lambda \sim \varrho/\alpha > 1/\alpha$. This should be compared to classical results in the i.i.d. case (see [12, p. 37, 46]), where the normalization is essentially of the form $N^{1/\alpha}$. As we see, in case B the sum $S_N(t)$ has a limiting stable distribution by virtue of a non-classical (heavier) normalization. As for case A, we have $B(t) \sim N^{-\mu(t)/\lambda} \to 0$, which has no analogies in classical theory.

Overall, it may seem surprising that i.i.d. random variables having finite exponential moments (or even bounded above as in case A) can be in the domain of attraction of a stable law, reproducing under various scalings the conventional picture of classical theory (but with non-classical normalization). It is also quite striking that the two apparently different cases A and B have so much in common and lead to the same limiting distributions. These results suggest that stable distributions as the limit laws for sums of i.i.d. random variables, possess greater universality than it used to be believed, and may appear as limits for various classes of parametric transformations of the form $Y_i(t) = F(X_i, t)$, where (X_i) is an i.i.d. sequence satisfying appropriate conditions on the upper distribution tail. We intend to explore this issue in greater detail in the future.

The remaining part of the paper is laid out as follows. In Section 3 we specify our regularity assumption on the distribution tail of the random variables X_i and formulate the Tauberian theorem of Kasahara–de Bruijn. In Section 4 we prove the LLN above λ_1 (Theorem 2.1) and the CLT above λ_2 (Theorem 2.2). In Section 5, the condition of normalized regular variation of the function h is discussed. Section 6 is devoted to the proof of Theorem 2.3 ($0 < \lambda < \lambda_2$). First, we demonstrate convergence to an infinitely divisible law (Theorem 6.1), which is then reduced to a canonical stable form (Theorem 6.2). In Section 7 we prove the LLN at $\lambda = \lambda_1$ (Theorem 2.4) and the CLT at $\lambda = \lambda_2$ (Theorem 2.5). The Appendix contains the proof of Lemma 5.13 about the asymptotics of truncated exponential moments.

3. Preliminaries

3.1. Regularity

Let us start by making precise our basic assumption on the regularity of the log-tail distribution function h defined in (2.1).

Regularity Assumption. The function h is *regularly varying at* ∞ *with index* ϱ (we write $h \in R_\varrho$), such that $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). That is, for every constant $\kappa > 0$

$$\lim_{x\to\infty} \frac{h(\kappa x)}{h(x)} = \kappa^{\varrho}. \tag{3.1}$$

It is known that $h \in R_\varrho$ if and only if h admits the *Karamata representation*

$$h(x) = c(x) \exp\left\{\int_a^x \frac{\varrho + \varepsilon(u)}{u}\, du\right\} \qquad (x \geq a) \tag{3.2}$$

for some $a > 0$, where $c(\cdot)$, $\varepsilon(\cdot)$ are measurable functions and $c(x) \to c_0 > 0$, $\varepsilon(x) \to 0$ as $x \to \infty$ [5, p. 21].

The following result, known as the *Uniform Convergence Theorem* (UCT) [5, p. 22], significantly extends the definition of regular variation.

Lemma 3.1 (UCT). *If* $h \in R_\varrho$ *with* $\varrho > 0$ *then (3.1) holds uniformly in* κ *on each interval* $(0, b]$.

3.2. Exponential Tauberian theorems

Recall that the *generalized inverse* of a function f is defined by $f^{\leftarrow}(y) := \inf\{x : f(x) > y\}$ [5, p. 28]. The next result shows that the generalized inverse inherits the property of regular variation and, quite naturally, is an 'asymptotic inverse' (cf. [5, p. 28]).

Lemma 3.2. *If* $f \in R_\varrho$ *with* $\varrho > 0$*, then there exists* $g \in R_{1/\varrho}$ *such that*

$$g(f(x)) \sim f(g(x)) \sim x \qquad (x \to \infty).$$

Such g *is unique to within asymptotic equivalence, and one version is* $f^{\leftarrow}$.

For $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A), we define the 'conjugate' index ϱ' by the formula (2.4). Rearranging (2.4), we obtain the useful identities

$$\frac{\varrho'}{\varrho} = \pm(\varrho' - 1), \qquad \frac{\varrho}{\varrho'} = \varrho \mp 1. \tag{3.3}$$

We are now in a position to formulate the exponential Tauberian theorems of Kasahara and de Bruijn (see [5, Theorems 4.12.7, 4.12.9]), which play the fundamental role in our analysis. We will state both theorems in a unified way and in terms convenient for our purposes.

Lemma 3.3 (Kasahara–de Bruijn's exponential Tauberian theorem). *Let h be the log-tail distribution function (2.2) and H the corresponding cumulant generating function (2.3). Suppose that $\varphi \in R_{1/\varrho}$ and put*

$$\psi(u) := u\,\varphi(u)^{\mp} \in R_{1/\varrho'}. \tag{3.4}$$

Then

$$h(x) \sim \frac{1}{\varrho}\,\varphi^{\leftarrow}(x) \quad (x \to \infty) \quad \Longleftrightarrow \quad H(t) \sim \frac{1}{\varrho'}\,\psi^{\leftarrow}(t) \quad (t \to \infty). \tag{3.5}$$

In particular, $h \in R_\varrho$ if and only if $H \in R_{\varrho'}$.

Let us point out that the function

$$H_0(t) := \frac{1}{\varrho'}\,\psi^{\leftarrow}(t) \sim H(t), \tag{3.6}$$

appearing in (3.5), is the rate function H_0 mentioned above in Section 2.1.

3.3. Some elementary inequalities

The following inequalities will be useful (see [11, Theorem 41, p. 39]): *Let $a > 0$, $b > 0$ and $a \neq b$, then*

$$p\,a^{p-1}(a-b) < a^p - b^p < p\,b^{p-1}(a-b) \qquad (0 < p < 1), \tag{3.7}$$

$$p\,b^{p-1}(a-b) < a^p - b^p < p\,a^{p-1}(a-b) \qquad (p < 0 \;\; or \;\; p > 1). \tag{3.8}$$

Let us also record a technical lemma.

Lemma 3.4. *Consider the function*

$$v_\lambda(x) := \lambda(x-1) \mp (x^{\varrho'} - x), \qquad x \geq 1. \tag{3.9}$$

If $\lambda > \lambda_1$ then there exists $x_0 > 1$ such that $v_\lambda(x) > 0$ for all $x \in (1, x_0)$.

Proof. By (2.6) and (3.3), we have $\lambda_1 = \varrho'/\varrho = \pm(\varrho' - 1)$. Note that $v_\lambda(1) = 0$ and $v'_\lambda(x) = \lambda \mp (\varrho' x^{\varrho'-1} - 1)$, so that $v'_\lambda(1) = \lambda \mp (\varrho' - 1) = \lambda - \lambda_1 > 0$, according to the hypothesis of the lemma. Therefore, Taylor's formula yields $v_\lambda(x) = (x-1)\big(v'_\lambda(1) + o(1)\big) > 0$ for all $x > 1$ sufficiently close to 1. $\square$

4. Limit theorems above the critical points

In this section, the parameter λ is defined by (2.8). We also recall that λ_1 and λ_2 are given by (2.6).

4.1. Proof of Theorem 2.1

Let us set

$$S_N^*(t) := \frac{S_N(t)}{\mathsf{E}[S_N(t)]} = \frac{1}{N}\sum_{i=1}^{N} e^{tX_i \mp H(t)},$$

so one has to prove that $S_N^*(t) \xrightarrow{p} 1$ as $t \to \infty$. To this end, it suffices to show that $\lim_{t\to\infty} \mathsf{E}|S_N^*(t) - 1|^r = 0$ for some $r > 1$.

By von Bahr–Esseen's inequality [3, Theorem 2], for any $r \in [1, 2]$ we have

$$\mathsf{E}|S_N^* - 1|^r \le 2N^{1-r}\,\mathsf{E}|e^{tX\mp H(t)} - 1|^r \le 2N^{1-r}\,\mathsf{E}|e^{tX\mp H(t)} + 1|^r.$$

Applying the elementary inequality $(x+1)^r \le 2^{r-1}(x^r + 1)$ $(x > 0,\ r \ge 1)$, which follows easily from Jensen's inequality, we further obtain

$$\mathsf{E}|S_N^* - 1|^r \le 2^r N^{1-r} e^{\pm H(rt)\mp rH(t)} + 2^r N^{1-r}. \tag{4.1}$$

Since $H \in R_{\varrho'}$ and $H(t) \sim H_0(t)$ (see (3.6)), we get, using (2.8),

$$\liminf_{t\to\infty}\left[\frac{(r-1)\log N}{H(t)} \mp \frac{H(rt)}{H(t)} \pm r\right] = \lambda(r-1) \mp r^{\varrho'} \pm r \equiv v_\lambda(r).$$

By Lemma 3.4 we can choose $r > 1$ such that $v_\lambda(r) > 0$, which implies that the right-hand side of (4.1) is bounded by $e^{-cH(t)} = o(1)$ as $t \to \infty$.

4.2. Proof of Theorem 2.2

Denote

$$\sigma(t)^2 := \mathsf{Var}[e^{tX}] = \mathsf{E}[e^{2tX}] - \left(\mathsf{E}[e^{tX}]\right)^2 = e^{\pm H(2t)} - e^{\pm 2H(t)}. \tag{4.2}$$

Lemma 4.1. *As $t \to \infty$,*

$$\sigma(t)^2 = e^{\pm H(2t)}(1 + o(1)) \qquad \textit{and} \qquad e^{\pm 2H(t)} = \sigma(t)^2\, o(1). \tag{4.3}$$

Proof. In view of (4.2) it suffices to prove the first statement. Note that

$$e^{\mp H(2t)}\sigma(t)^2 = 1 - e^{\mp H(2t)\pm 2H(t)}. \tag{4.4}$$

Using that $H \in R_{\varrho'}$ we obtain

$$\lim_{t\to\infty}\left(\pm\frac{H(2t)}{H(t)} \mp 2\right) = \pm(2^{\varrho'} - 2) = |2^{\varrho'} - 2| > 0$$

(see (2.5)). Hence, the exponential term on the right-hand side of (4.4) vanishes as $t \to \infty$, and (4.3) follows. □

The following lemma is a variation of Chebyshev's inequality.

Lemma 4.2. *Let $Y \ge 0$ be a random variable. Then for any $\tau > 0$ and all $k \le m$*

$$\mathsf{E}\Big[Y^k \mathbf{1}_{\{Y>\tau\}}\Big] \le \tau^{k-m}\,\mathsf{E}\big[Y^m\big]. \tag{4.5}$$

Proof. Similarly to the usual proof of Chebyshev's inequality, we write

$$\mathsf{E}\big[Y^m\big] \ge \mathsf{E}\big[Y^m \mathbf{1}_{\{Y>\tau\}}\big] = \mathsf{E}\Big[Y^{m-k}\cdot Y^k \mathbf{1}_{\{Y>\tau\}}\Big] \ge \tau^{m-k}\,\mathsf{E}\Big[Y^k \mathbf{1}_{\{Y>\tau\}}\Big],$$

whence (4.5) follows. □

Proof of Theorem 2.2. In view of Lemma 4.1, the statement of the theorem may be rewritten as follows:

$$\frac{S_N(t) - N\,e^{\pm H(t)}}{N^{1/2}\,e^{\pm H(2t)/2}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t\to\infty). \tag{4.6}$$

Denote

$$Y_i \equiv Y_i(t) := \frac{e^{tX_i}}{N^{1/2}\,e^{\pm H(2t)/2}}. \tag{4.7}$$

1) According to the Central Limit Theorem for independent summands (see, e.g., [14, Theorem 18, p. 95]), we firstly need to check that for all $\tau > 0$

$$\sum_{i=1}^N \mathsf{P}\{Y_i(t) > \tau\} = N\,\mathsf{P}\{Y(t) > \tau\} \to 0 \qquad (t\to\infty).$$

Assuming that $r > 1$, let us apply Chebyshev's inequality (of order $2r$) and recall the definition (4.7) to obtain

$$N\,\mathsf{P}\{Y>\tau\} \le N\tau^{-2r}\,\mathsf{E}[Y^{2r}] = N^{1-r}\tau^{-2r}e^{\pm H(2rt)\mp rH(2t)}. \tag{4.8}$$

Using that $H \in R_{\varrho'}$ and $H(t)\sim H_0(t)$ as $t\to\infty$, we find

$$\begin{aligned}\liminf_{t\to\infty}\left[\frac{(r-1)\log N}{H(t)} \mp \frac{H(2rt)}{H(t)} \pm \frac{rH(2t)}{H(t)}\right] &= \lambda(r-1) \mp (2r)^{\varrho'} \pm r2^{\varrho'}\\ &= 2^{\varrho'}\Big(2^{-\varrho'}\lambda(r-1) \mp (r^{\varrho'}-r)\Big) \equiv 2^{\varrho'}v_{\lambda'}(r),\end{aligned}$$

where $\lambda' := 2^{-\varrho'}\lambda$ and the function $v_\lambda(\cdot)$ is defined in (3.9). By the theorem's hypothesis, $\lambda' > 2^{-\varrho'}\lambda_2 = \lambda_1$ and hence, by Lemma 3.4, $v_{\lambda'}(r) > 0$ for a suitable $r > 1$. Therefore, the right-hand part of (4.8) tends to zero as $t\to\infty$.

2) Next, we have to verify that for every $\tau > 0$, as $t\to\infty$,

$$\sum_{i=1}^N \left\{\mathsf{E}\Big[Y_i^2\mathbf{1}_{\{Y_i\le\tau\}}\Big] - \Big(\mathsf{E}\big[Y_i\mathbf{1}_{\{Y_i\le\tau\}}\big]\Big)^2\right\} = N\,\mathsf{Var}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] \to 1. \tag{4.9}$$

By Lemma 4.1, $\mathsf{Var}[Y]\sim 1/N$, so the condition (4.9) can be rewritten in the form

$$N\,\mathsf{Var}[Y] - N\,\mathsf{Var}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] \to 0.$$

Expanding the variances, the left-hand side is represented as

$$N\,\mathsf{E}\Big[Y^2\mathbf{1}_{\{Y>\tau\}}\Big] - N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y>\tau\}}\big]\,\mathsf{E}\big[Y\big(1+\mathbf{1}_{\{Y\le\tau\}}\big)\big]. \tag{4.10}$$

Applying Lemma 4.2 to the first term in (4.10) (with $k = 2,\ m = 2r > 2$) yields

$$N\,\mathsf{E}\Big[Y^2\,\mathbf{1}_{\{Y>\tau\}}\Big] \le N\tau^{-2(r-1)}\,\mathsf{E}[Y^{2r}] = o(1), \tag{4.11}$$

as shown in the first part of the proof. The second term in (4.10) is bounded by $2N(\mathsf{E}[Y])^2 = 2e^{\mp H(2t)\pm 2H(t)}$, which is $o(1)$ by Lemma 4.1. Hence, (4.10) vanishes as $t \to \infty$, and (4.9) follows.

3) Finally, we need to show that

$$\sum_{i=1}^{N}\mathsf{E}[Y_i] - \sum_{i=1}^{N}\mathsf{E}\big[Y_i\,\mathbf{1}_{\{Y_i\le\tau\}}\big] = N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y>\tau\}}\big] \to 0.$$

Indeed, applying Lemma 4.2 with $k = 1$, $m = 2r$ $(r > 1)$, we obtain the estimate

$$N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y>\tau\}}\big] \le N\tau^{1-2r}\,\mathsf{E}[Y^{2r}] = o(1)$$

(see (4.8), (4.11)), and the proof is complete. □

5. Normalized regularity and the Basic Identity

5.1. Normalized regular variation

From now on we impose the following

Normalized Regularity Assumption. The log-tail distribution function h is *normalized regularly varying* at infinity, $h \in NR_\varrho$ (with $1 < \varrho < \infty$ in case B and $0 < \varrho < \infty$ in case A), that is, it can be represented in the form

$$h(x) = c\,\exp\left\{\int_a^x \frac{\varrho + \varepsilon(u)}{u}\,du\right\} \qquad (x \ge a), \tag{5.1}$$

where $c = \text{const} > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$ (see [5, p. 15]). That is to say, the function $c(\cdot)$ in the Karamata representation (3.2) is now required to be a constant.

More insight into the property of normalized regular variation is given by the following lemma (cf. [5, p. 15]).

Lemma 5.1. *Let h be a positive (measurable) function. Then $h \in NR_\varrho$ if and only if h is differentiable (a.e.) and*

$$\frac{xh'(x)}{h(x)} \to \varrho \qquad (x \to \infty). \tag{5.2}$$

Another important characterization of normalized regularly varying functions is provided by the following lemma (see [5, Theorem 1.5.5]).

Lemma 5.2. *A positive (measurable) function h is normalized regularly varying with index ϱ, i.e. $h \in NR_\varrho$, if and only if for every $\varepsilon > 0$ the function $h(x)/x^{\varrho-\varepsilon}$ is ultimately increasing and the function $h(x)/x^{\varrho+\varepsilon}$ is ultimately decreasing.*

The next lemma yields a useful integral representation of normalized regularly varying functions.

Lemma 5.3. *A function $h \in NR_\varrho$ can be written in the form*

$$h(x) = h(a) + \int_a^x \frac{h(u)}{u}\,(\varrho + \varepsilon(u))\,du \qquad (x \ge a), \tag{5.3}$$

where $\varepsilon(x) \to 0$ as $x \to \infty$.

Proof. Consider the function

$$D(x) := h(x) - h(a) - \int_a^x \frac{h(u)}{u}\,(\varrho + \varepsilon(u))\,du.$$

Obviously, $D(a) = 0$. Representation (5.1) implies that h is absolutely continuous, hence the derivative h' exists (a.e.) and

$$h'(x) = c\,\exp\left\{\int_a^x \frac{\varrho + \varepsilon(u)}{u}\,du\right\} \cdot \frac{\varrho + \varepsilon(x)}{x} = \frac{h(x)\,(\varrho + \varepsilon(x))}{x}. \tag{5.4}$$

Therefore, $D(\cdot)$ is absolutely continuous as well and we have (a.e.)

$$D'(x) = h'(x) - \frac{h(x)}{x}\,(\varrho + \varepsilon(x)) = 0.$$

Hence, $D(x) \equiv 0$ and (5.3) follows. □

The following lemma can be viewed as a refinement of the UCT (Lemma 3.1) for the case of normalized regular variation.

Lemma 5.4. *If $h \in NR_\varrho$ ($\varrho > 0$) then, uniformly in κ on each interval $[\kappa_0, \kappa_1] \subset (0, \infty)$,*

$$\frac{h(\kappa x) - h(x)}{h(x)} = (\kappa^\varrho - 1)\,(1 + o(1)) \qquad (x \to \infty).$$

Proof. Suppose for definiteness that $\kappa \ge 1$ (the case $0 < \kappa \le 1$ is considered similarly). Using the representation (5.3), after the substitution $u = xy$ we have

$$\frac{h(\kappa x) - h(x)}{h(x)} = \int_1^\kappa \frac{h(xy)}{h(x)\,y}\,(\varrho + \varepsilon(xy))\,dy. \tag{5.5}$$

The UCT (Lemma 3.1) implies that the function under the integral sign converges to $\varrho\, y^{\varrho - 1}$ uniformly on $[1, \kappa_1]$ as $x \to \infty$. Therefore, the integral in (5.5) converges, uniformly in $\kappa \in [1, \kappa_1]$, to $\int_1^\kappa \varrho\, y^{\varrho-1}\,dy = \kappa^\varrho - 1$. □

5.2. Basic Identity

Let us now re-examine the application of the Kasahara–de Bruijn Tauberian theorem (Lemma 3.3) to our situation. Note that the function $\varrho h(x)$ is continuous and, by Lemma 5.2, ultimately strictly increasing, hence its ordinary inverse $\varphi(t) := (\varrho h)^{-1}(t)$ is well defined and strictly increasing for all t large enough. In turn, for all x large enough we have

$$\varphi^{-1}(x) = \varrho h(x). \tag{5.6}$$

It then follows that the function $\psi(t)$ defined by (3.4) is ultimately strictly increasing as well. For suppose $s < t$, then the required inequality $\psi(s) < \psi(t)$ is equivalent to $s\varphi(s)^{\mp} < t\varphi(t)^{\mp}$, or

$$\varphi^{-1}(x)\, x^{\mp} < \varphi^{-1}(y)\, y^{\mp}, \tag{5.7}$$

where $x := \varphi(s)$, $y := \varphi(t)$ and $x < y$. Using (5.6), inequality (5.7) can be rewritten as $h(x)\, x^{-\varrho+\varepsilon} < h(y)\, y^{-\varrho+\varepsilon}$ with $\varepsilon := \varrho \mp 1 > 0$, and the latter holds by Lemma 5.2.

Consequently, the inverse function ψ^{-1} exists and is ultimately increasing. Therefore, formula (3.6) is reduced to

$$\psi^{-1}(t) = \varrho' H_0(t). \tag{5.8}$$

For the sake of notational convenience, let us introduce the function

$$s(t) := \left(\frac{\varrho' H_0(t)}{t}\right)^{\pm}, \qquad t > 0. \tag{5.9}$$

Since $H_0 \in R_{\varrho'}$, we have $s(t) \in R_{\pm(\varrho'-1)} = R_{|\varrho'-1|}$ and hence $s(t) \to \infty$ as $t \to \infty$.

We are now in a position to characterize explicitly the link arising between the regularly varying functions h and H_0 through the Tauberian correspondence. Remarkably, due to normalized regular variation of h, such a relationship has the form of an exact equation, rather than just an asymptotic relation.

Lemma 5.5. *For all t large enough, the functions h and H_0 satisfy the equation*

$$\varrho' H_0(t) \equiv \varrho h(s(t)). \tag{5.10}$$

Remark. Remembering that $s(\cdot)$ is expressed through H_0 (see (5.9)), the identity (5.10) can be viewed as a functional equation for H_0.

Proof of Lemma 5.5. Let us apply ψ to (5.8) and use relation (3.4) to obtain

$$t = \psi\big(\varrho' H_0(t)\big) = \varrho' H_0(t)\, \varphi\big(\varrho' H_0(t)\big)^{\mp},$$

that is,

$$\varphi\big(\varrho' H_0(t)\big) = \left(\frac{\varrho' H_0(t)}{t}\right)^{\pm} \equiv s(t).$$

Hence, using (5.6) we get $\varrho' H_0(t) = \varphi^{-1}(s(t)) = \varrho h(s(t))$. $\square$

In order to rewrite equation (5.10) in a form suitable for us (to be called 'Basic Identity'), we need to make some technical preparations. Recall that α is defined in (2.7). Conversely, using (3.3) λ is expressed in either of the two forms

$$\lambda = \frac{\varrho'\alpha^{\varrho'}}{\varrho} \equiv \pm(\varrho'-1)\,\alpha^{\varrho'}. \tag{5.11}$$

Lemma 5.6. *For large enough s, there exists the unique root $\tilde{\mu}(s)$ of the equation*

$$h\big((\tilde{\mu}/\varrho')^{\pm}s\big) = \alpha^{\varrho'}h(s), \tag{5.12}$$

given by the formula

$$\tilde{\mu}(s)^{\pm} = \frac{(\varrho')^{\pm}}{s}\,h^{-1}\big(\alpha^{\varrho'}h(s)\big). \tag{5.13}$$

In particular, if $\alpha = 1$ then $\tilde{\mu}(s) \equiv \varrho'$.

Proof. Recall that h is normalized regularly varying and (absolutely) continuous (see (5.1)). Therefore, by Lemma 5.2 it is strictly increasing in some $[a,\infty)$, so the (usual) inverse h^{-1} exists and is defined on $[h(a),\infty)$. Hence, the equation (5.12) can be solved to yield formula (5.13), which is well defined for all s large enough. The case $\alpha = 1$ follows easily. □

Lemma 5.7. *The function $\tilde{\mu}(\cdot)$ defined in Lemma 5.6 is ultimately bounded above and below, and furthermore, for all s large enough*

$$\min\{1,\alpha^{\varrho'/2\varrho}\} \le \left(\frac{\tilde{\mu}(s)}{\varrho'}\right)^{\pm} \le \max\{1,\alpha^{2\varrho'/\varrho}\}. \tag{5.14}$$

Proof. If $\alpha \le 1$ then, due to monotonicity of the function h^{-1},

$$\frac{1}{s}\,h^{-1}\big(\alpha^{\varrho'}h(s)\big) \le \frac{1}{s}\,h^{-1}(h(s)) = 1. \tag{5.15}$$

In the case $\alpha > 1$, we note that for every $\kappa > 1$ and all s large enough

$$\kappa\, h(s) \le h(\kappa^{2/\varrho}s), \tag{5.16}$$

because $h \in R_\varrho$ and hence $\lim_{s\to\infty} h(\kappa^{2/\varrho}s)/h(s) = \kappa^2 > \kappa$. Applying inequality (5.16) with $\kappa = \alpha^{\varrho'} > 1$, we get

$$\frac{1}{s}\,h^{-1}\big(\alpha^{\varrho'}h(s)\big) \le \frac{1}{s}\,h^{-1}\big(h(\alpha^{2\varrho'/\varrho}s)\big) = \alpha^{2\varrho'/\varrho}. \tag{5.17}$$

Combining (5.15) and (5.17) and using (5.13), the upper bound (5.14) follows.

Similarly, for $\alpha \ge 1$ we obtain

$$\frac{1}{s}\,h^{-1}\big(\alpha^{\varrho'}h(s)\big) \ge \frac{1}{s}\,h^{-1}(h(s)) = 1,$$

whereas for $\alpha < 1$

$$\frac{1}{s}\,h^{-1}\big(\alpha^{\varrho'}h(s)\big) \ge \frac{1}{s}\,h^{-1}\big(h(\alpha^{\varrho'/2\varrho}s)\big) = \alpha^{\varrho'/2\varrho},$$

which is consistent with the lower bound in (5.14). □

Lemma 5.8. *The function $\tilde{\mu}(s)$ has a finite limit as $s \to \infty$, given by*

$$\lim_{s\to\infty} \tilde{\mu}(s) = \varrho'\alpha^{\varrho'-1}. \tag{5.18}$$

Proof. Since $\tilde{\mu}(\cdot)$ is bounded (see Lemma 5.7), the UCT (Lemma 3.1) implies

$$h\big((\tilde{\mu}(s)/\varrho')^{\pm} s\big) \sim \left(\frac{\tilde{\mu}(s)}{\varrho'}\right)^{\pm\varrho} h(s) \qquad (s \to \infty).$$

Comparing this with equation (5.12), we obtain

$$\left(\frac{\tilde{\mu}(s)}{\varrho'}\right)^{\pm\varrho} \sim \alpha^{\varrho'} \qquad (s \to \infty),$$

whence it follows that the limit (5.18) exists and is given by

$$\lim_{s\to\infty} \tilde{\mu}(s) = \varrho'\alpha^{\pm\varrho'/\varrho} = \varrho'\alpha^{\varrho'-1},$$

in view of the first of the identities (3.3). □

Let us define the function

$$\mu(t) := (\tilde{\mu} \circ s)(t) = \tilde{\mu}(s(t)), \tag{5.19}$$

where $s(t)$ is given by (5.9). From the definition of $\tilde{\mu}(s)$ (see Lemma 5.6), it is clear that for all t large enough the function $\mu(t)$ satisfies the equation

$$h\big((\mu(t)/\varrho')^{\pm} s(t)\big) = \alpha^{\varrho'} h(s(t)). \tag{5.20}$$

Since $s(t) \to \infty$, Lemma 5.8 implies that

$$\lim_{t\to\infty} \mu(t) = \varrho'\alpha^{\varrho'-1}. \tag{5.21}$$

For $\tau > 0$, denote

$$\eta_\tau(t) := \frac{\mu(t)H_0(t) \pm \log\tau}{t}. \tag{5.22}$$

In particular, for $\tau = 1$

$$\eta_1(t) = \frac{\mu(t)H_0(t)}{t} = \frac{\mu(t)}{\varrho'}\, s(t)^{\pm} \tag{5.23}$$

(see (5.9)). From equations (5.23) and (5.21) it follows

$$\eta_1(t)^{\pm} = \big(\mu(t)/\varrho'\big)^{\pm} s(t) \to \infty \qquad (t \to \infty). \tag{5.24}$$

Furthermore, it is easy to see that

$$\frac{\eta_\tau(t)}{\eta_1(t)} = 1 \pm \frac{\log\tau}{\mu(t)H_0(t)} \to 1 \qquad (t \to \infty). \tag{5.25}$$

Hence, using (5.21) we obtain

$$\frac{t\,\eta_\tau(t)}{H_0(t)} = \frac{\eta_\tau(t)}{\eta_1(t)}\,\mu(t) \to \varrho'\alpha^{\varrho'-1} \qquad (t \to \infty). \tag{5.26}$$

The following lemma will play a crucial role in our analysis.

Lemma 5.9 (Basic Identity). *For all t large enough,*

$$h\big(\eta_1(t)^{\pm}\big) \equiv \lambda H_0(t). \tag{5.27}$$

Proof. From (5.23) and (5.20) it follows

$$h\big(\eta_1(t)^{\pm}\big) = h\big((\mu(t)/\varrho')^{\pm}s(t)\big) = \alpha^{\varrho'}h(s(t)).$$

By Lemma 5.5 and relation (5.11), this coincides with $\lambda H_0(t)$. □

5.3. Implications of the Basic Identity

In this section, we prove three useful lemmas concerning the asymptotics of various 'perturbations' of the function $h(\eta_1(t)^{\pm})$. Of particular importance for further calculations will be Lemma 5.12.

Lemma 5.10. *Let $g(\cdot)$ be such that $tg(t)/H_0(t) \to 0$ as $t \to \infty$. Set $\tilde{\eta}_{\tau,y}(t) := \eta_\tau(t) \mp yg(t)$. Then for each $\tau > 0$ uniformly in y on every finite interval $[y_0, y_1]$*

$$\lim_{t\to\infty} \frac{h(\tilde{\eta}_{\tau,y}(t)^{\pm})}{t\,\tilde{\eta}_{\tau,y}(t)} = \frac{\alpha}{\varrho}. \tag{5.28}$$

In particular, for $g \equiv 0$ one has

$$\lim_{t\to\infty} \frac{h(\eta_\tau(t)^{\pm})}{t\,\eta_\tau(t)} = \frac{\alpha}{\varrho}. \tag{5.29}$$

Proof. Relation (5.26) implies that, uniformly in $y \in [y_0, y_1]$,

$$\kappa_y(t) := \frac{\tilde{\eta}_{\tau,y}(t)}{\eta_1(t)} = 1 \pm \frac{\log\tau}{t\eta_1(t)} \mp \frac{ytg(t)}{H_0(t)} \cdot \frac{H_0(t)}{t\eta_1(t)} \to 1 \qquad (t \to \infty).$$

Therefore, by the UCT (Lemma 3.1), uniformly in y on any finite interval $[y_0, y_1]$

$$h(\tilde{\eta}^{\pm}_{\tau,y}) = h(\kappa_y^{\pm}\eta_1^{\pm}) \sim \kappa_y^{\pm\varrho}h(\eta_1^{\pm}) \sim h(\eta_1^{\pm}).$$

Hence, taking into account Lemma 5.9 and the limit (5.21), we obtain

$$\frac{h(\tilde{\eta}^{\pm}_{\tau,y})}{t\tilde{\eta}_{\tau,y}} \sim \frac{h(\eta^{\pm}_{\tau})}{t\eta_\tau} = \frac{\lambda H_0(t)}{t\eta_1} = \frac{\lambda}{\mu(t)} \to \frac{\lambda}{\varrho'\alpha^{\varrho'-1}} = \frac{\alpha}{\varrho},$$

in view of formula (2.7). □

Lemma 5.11. *Under the conditions of Lemma 5.10, for each $\tau > 0$*

$$\lim_{t\to\infty} \frac{h(\eta_\tau(t)^{\pm}) - h(\tilde{\eta}_{\tau,y}(t)^{\pm})}{tg(t)} = \alpha y,$$

uniformly in y on every finite interval $[y_0, y_1]$.

Proof. Similarly to the proof of Lemma 5.10 we get

$$\kappa_y(t) := \frac{\tilde{\eta}_{\tau,y}(t)}{\eta_\tau(t)} = 1 \mp \frac{ytg(t)}{H_0(t)} \cdot \frac{H_0(t)}{t\eta_\tau(t)} \to 1 \qquad (t \to \infty)$$

uniformly in $y \in [y_0, y_1]$. Therefore, for all large enough t the function $\kappa_y(t)$ is uniformly bounded, $0 < \kappa_0 \le \kappa_y(t) \le \kappa_1 < \infty$. Applying Lemma 5.4 we have

$$h(\eta_\tau^\pm) - h(\tilde{\eta}_{\tau,y}^\pm) \sim -h(\eta_\tau^\pm)(\kappa_y^{\pm\varrho} - 1) \qquad (t \to \infty). \tag{5.30}$$

Furthermore,

$$\kappa_y^{\pm\varrho} - 1 = \left(1 \mp \frac{yg(t)}{\eta_\tau(t)}\right)^{\pm\varrho} - 1 \sim -\frac{\varrho y g(t)}{\eta_\tau(t)}.$$

Substituting this into (5.30) and using the limit (5.29), we finally obtain

$$h(\eta_\tau^\pm) - h(\tilde{\eta}_{\tau,y}^\pm) \sim h(\eta_\tau^\pm)\frac{\varrho y g(t)}{\eta_\tau(t)} \sim \alpha y t g(t),$$

and the lemma follows. □

Lemma 5.12. *For each* $\tau > 0$

$$\lim_{t\to\infty} \left[h(\eta_\tau(t)^\pm) - h(\eta_1(t)^\pm)\right] = \alpha \log \tau.$$

Proof. Apply Lemma 5.11 with $y = \log\tau,\ g(t) = 1/t$. □

5.4. Asymptotics of truncated exponential moments

The goal of this section is to establish some general estimates for truncated exponential moments, which will be instrumental later on. Recall that the parameter $\alpha > 0$ is defined in (2.7).

Lemma 5.13. *If* $\tau > 0$ *is a fixed number then*
(i) *for each* $p > \alpha$,

$$\lim_{t\to\infty} e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\, \mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X\le\pm\eta_\tau\}}\Big] = \frac{\alpha}{p-\alpha};$$

(ii) *for each* $p < \alpha$,

$$\lim_{t\to\infty} e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\, \mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X>\pm\eta_\tau\}}\Big] = \frac{\alpha}{\alpha-p}.$$

The proof of this lemma is deferred to the Appendix.
In the case $p = \alpha$ not covered by Lemma 5.13, we prove one crude estimate that will nevertheless be sufficient for our purposes below.

Lemma 5.14. *For $\alpha > 0$, denote*

$$B_\alpha(t) := \mathsf{E}\Big[e^{\alpha t X}\mathbf{1}_{\{X \le \pm\eta_1\}}\Big], \tag{5.31}$$

where $\eta_1(t)$ is defined in (5.23). Then

$$b_\alpha(t) := e^{\mp\alpha t \eta_1 + h(\eta_1^\pm)} B_\alpha(t) \to +\infty \qquad (t \to \infty). \tag{5.32}$$

Proof. Set $\tilde\eta_1(t) := \eta_\tau(t) \mp g(t),\ g(t) := t^{-1+\varrho'/2}$. Integration by parts yields

$$\begin{aligned}\mathsf{E}\Big[e^{\alpha t X}\mathbf{1}_{\{X \le \pm\eta_1\}}\Big] &\ge \mathsf{E}\Big[e^{\alpha t X}\mathbf{1}_{\{\pm\tilde\eta_1 < X \le \pm\eta_1\}}\Big] = \int_{\pm\tilde\eta_1}^{\pm\eta_1} e^{\alpha t x}\, d\big(1 - e^{-h(\pm x^\pm)}\big)\\ &= -\int_{\pm\tilde\eta_1}^{\pm\eta_1} e^{\alpha t x}\, d\big(e^{-h(\pm x^\pm)}\big) \ge -e^{\pm\alpha t\eta_1 - h(\eta_1^\pm)} + \alpha t \int_{\pm\tilde\eta_1}^{\pm\eta_1} e^{\alpha t x - h(\pm x^\pm)}\, dx.\end{aligned} \tag{5.33}$$

Making here the substitution $\pm x = \eta_1(t) \mp y g(t) =: \tilde\eta_{1,y}(t)$, we obtain

$$b_\alpha(t) \ge -1 + \alpha t g(t) \int_0^1 e^{-\alpha t g(t) y + h(\eta_1^\pm) - h(\tilde\eta_{1,y}^\pm)}\, dy. \tag{5.34}$$

By Lemma 5.11, $h(\eta_1^\pm) - h(\tilde\eta_{1,y}^\pm) = \alpha t g(t) y(1 + o(1))$, uniformly in $y \in [0, 1]$. So for any $\delta > 0$ and all large enough t we have $h(\eta_1^\pm) - h(\tilde\eta_{1,y}^\pm) \ge \alpha t g(t) y(1-\delta)$. Returning to (5.34) we get

$$b_\alpha(t) \ge -1 + \alpha t g(t) \int_0^1 e^{-\alpha t g(t)\delta y}\, dy = -1 + \frac{1}{\delta}\Big(1 - e^{-\alpha t g(t)\delta}\Big),$$

hence $\liminf_{t\to\infty} b_\alpha(t) \ge (1/\delta) - 1$. Since the number $\delta > 0$ can be chosen arbitrarily small, it follows that $\liminf_{t\to\infty} b_\alpha(t) = +\infty$, as claimed. □

The next lemma provides some additional information in the case $p = \alpha$.

Lemma 5.15. *For any $\tau > 0$*

$$\lim_{t\to\infty} e^{\mp\alpha t\eta_1 + h(\eta_1^\pm)}\, \mathsf{E}\Big[e^{\alpha t X}\big(\mathbf{1}_{\{X \le \pm\eta_\tau\}} - \mathbf{1}_{\{X \le \pm\eta_1\}}\big)\Big] = \alpha \log \tau. \tag{5.35}$$

Proof. Let us assume for definiteness that $\tau \ge 1$, so that $\pm\eta_\tau(t) \ge \pm\eta_1(t)$. Integrating by parts and using the substitution $x = \pm\eta_1(t) + (1/t)\log y$, we obtain

$$\begin{aligned}&e^{\mp\alpha t\eta_1 + h(\eta_1^\pm)}\, \mathsf{E}\Big[e^{\alpha t X}\mathbf{1}_{\{\pm\eta_1 < X \le \pm\eta_\tau\}}\Big]\\ &\qquad = 1 - e^{\alpha\log\tau + h(\eta_1^\pm) - h(\eta_\tau^\pm)} + \alpha \int_1^\tau e^{\alpha \log y + h(\eta_1^\pm) - h(\eta_y^\pm)}\, \frac{dy}{y}.\end{aligned} \tag{5.36}$$

By Lemma 5.12 we have $h(\eta_1^\pm) - h(\eta_y^\pm) \to -\alpha \log y$ as $t \to \infty$, uniformly in $1 \le y \le \tau$, and in particular $h(\eta_1^\pm) - h(\eta_\tau^\pm) \to -\alpha\log\tau$. It is then easy to see that the right-hand side of (5.36) tends to $\alpha\log\tau$ as $t \to \infty$. □

For convenience of reference, we record here some further estimates for truncated moments of the random variable e^{tX} under a certain normalization adopted in this section. Namely, consider the random variables

$$Y_i \equiv Y_i(t) := \frac{e^{tX_i}}{(NB_\alpha(t))^{1/\alpha}}, \tag{5.37}$$

where N is subject to the scaling assumption (2.9) and B_α is defined in (5.31). For $\alpha > 0$ and $\tau > 0$ denote

$$\tilde{\eta}_{\alpha,\tau}(t) := \pm\frac{\log(NB_\alpha(t))}{\alpha t} \pm \frac{\log\tau}{t}. \tag{5.38}$$

From (5.37) it is seen that the inequality $Y(t) > \tau$ is equivalent to $X > \pm\tilde{\eta}_{\alpha,\tau}(t)$.

Recalling representation (5.32) and using the Basic Identity (5.27), we obtain

$$NB_\alpha(t) \sim e^{\lambda H_0(t) \pm \alpha t\eta_1 - h(\eta_1^{\pm})}\, b_\alpha(t) = e^{\pm\alpha t\eta_1}\, b_\alpha(t). \tag{5.39}$$

Therefore, formula (5.38) implies

$$\tilde{\eta}_{\alpha,\tau}(t) = \eta_1(t) \pm \frac{\log b_\alpha(t)}{\alpha t} + \frac{O(1)}{t}, \tag{5.40}$$

whence it follows that for all sufficiently large t

$$\pm\tilde{\eta}_{\alpha,\tau}(t) > \pm\eta_1(t). \tag{5.41}$$

Lemma 5.16. *For any p such that $0 \le p < \alpha$ and each $\tau > 0$*

$$\lim_{t\to\infty} N\,\mathsf{E}\big[Y(t)^p\,\mathbf{1}_{\{Y(t)>\tau\}}\big] = 0. \tag{5.42}$$

In particular, for $p = 0$ this yields

$$\lim_{t\to\infty} N\,\mathsf{P}\{Y(t) > \tau\} = 0. \tag{5.43}$$

Proof. From (5.37), (5.38) and (5.41) we obtain

$$\mathsf{E}\big[Y^p\,\mathbf{1}_{\{Y>\tau\}}\big] \le (NB_\alpha)^{-p/\alpha}\ \mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X>\pm\eta_1\}}\Big].$$

Using Lemma 5.13(ii) and relations (2.9), (5.39), (5.27) and (5.32), we get

$$\begin{aligned} \frac{N}{(NB_\alpha)^{p/\alpha}}\ \mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X>\pm\eta_1\}}\Big] &\sim \frac{e^{\lambda H_0(t)}}{e^{\pm pt\eta_1}\, b_\alpha^{p/\alpha}} \cdot \frac{\alpha}{\alpha - p}\, e^{\pm pt\eta_1 - h(\eta_1^{\pm})} \\ &= \frac{\alpha}{\alpha - p}\, b_\alpha^{-p/\alpha} = o(1). \end{aligned} \tag{5.44}$$

Thus, relation (5.42) is proved. □

Denote

$$y_\alpha \equiv y_\alpha(t) := \frac{e^{\pm t\eta_1(t)}}{(NB_\alpha(t))^{1/\alpha}}, \tag{5.45}$$

so that $Y > y_\alpha$ if and only if $X > \pm\eta_1$. From (5.39) it follows that $y_\alpha(t) \sim b_\alpha(t)^{-1/\alpha} \to 0$.

Lemma 5.17. *Suppose that* $p > 0$. *Then for any* $\tau > 0$

$$\lim_{t\to\infty} N\,\mathsf{E}\big[Y(t)^p\,\mathbf{1}_{\{y_\alpha(t)<Y(t)\le\tau\}}\big] = 0. \tag{5.46}$$

Proof. Pick a number q such that $0 < q < \min\{\alpha, p\}$. Applying Chebyshev's inequality (4.5), we can write

$$\begin{aligned} N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{y_\alpha<Y\le\tau\}}\big] &\le N\tau^{p-q}\,\mathsf{E}\big[Y^q\,\mathbf{1}_{\{y_\alpha<Y\}}\big] \\ &= \frac{N\tau^{p-q}}{(NB_\alpha)^{q/\alpha}}\,\mathsf{E}\Big[e^{qtX}\,\mathbf{1}_{\{X>\pm\eta_1\}}\Big], \end{aligned}$$

and the latter expression is $o(1)$ as shown above (see (5.44)). □

Lemma 5.18. *Suppose that* $p > \alpha > 0$. *Then for any* $\tau > 0$

$$\lim_{t\to\infty} N\,\mathsf{E}\big[Y(t)^p\,\mathbf{1}_{\{Y(t)\le\tau\}}\big] = 0. \tag{5.47}$$

Proof. Let us write

$$N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{Y\le\tau\}}\big] = N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{Y\le y_\alpha\}}\big] + N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{y_\alpha<Y\le\tau\}}\big]. \tag{5.48}$$

Applying Lemma 5.13(i), one can show, similarly to (5.44), that the first term on the right-hand side of (5.48) is asymptotically equivalent to

$$\frac{e^{\lambda H_0(t)}}{e^{\pm pt\eta_1(t)}b_\alpha(t)^{p/\alpha}}\cdot\frac{\alpha}{p-\alpha}\,e^{\pm pt\eta_1 - h(\eta_1^\pm)} = \frac{\alpha}{p-\alpha}\,b_\alpha(t)^{-p/\alpha} = o(1),$$

while the second term on the right of (5.48) is $o(1)$ by Lemma 5.17. □

6. Limit theorems below λ_2

6.1. Convergence to an infinitely divisible law

Denote

$$Y_i \equiv Y_i(t) := \frac{e^{tX_i}}{B(t)}, \tag{6.1}$$

where $B(t)$ is defined in (2.12). According to classical theorems on weak convergence of sums of independent random variables (see [14, p. 80–82]), in order that the sum

$$S_N^*(t) := \sum_{i=1}^N Y_i(t) - A^*(t)$$

converges in distribution to an infinitely divisible law with characteristic function

$$\phi(u) = \exp\left\{iau - \frac{\sigma^2 u^2}{2} + \int_{|x|>0} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) dL(x)\right\}, \tag{6.2}$$

it is necessary and sufficient that the following three conditions be fulfilled:

1) In all points of its continuity, the function $L(\cdot)$ satisfies

$$L(x) = \begin{cases} \lim_{t\to\infty} N\,\mathsf{P}\{Y \le x\} & (x < 0), \\ -\lim_{t\to\infty} N\,\mathsf{P}\{Y > x\} & (x > 0). \end{cases} \tag{6.3}$$

2) The constant σ^2 is given by

$$\sigma^2 = \lim_{\tau\to 0+} \limsup_{t\to\infty} N\,\mathsf{Var}\big[Y\,\mathbf{1}_{\{Y\le\tau\}}\big] = \lim_{\tau\to 0+} \liminf_{t\to\infty} N\,\mathsf{Var}\big[Y\,\mathbf{1}_{\{Y\le\tau\}}\big]. \tag{6.4}$$

3) For each $\tau > 0$, the constant a satisfies the identity

$$\lim_{t\to\infty} \big\{N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\le\tau\}}\big] - A^*(t)\big\} = a + \int_0^\tau \frac{x^3}{1+x^2}\,dL(x) - \int_\tau^\infty \frac{x}{1+x^2}\,dL(x). \tag{6.5}$$

As the first step towards the proof of Theorem 2.3, we establish convergence to an infinitely divisible law.

Theorem 6.1. *Suppose that* $0 < \lambda < \lambda_2$. *Then*

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $B(t)$ *and* $A(t)$ *are defined in (2.12) and (2.13), respectively, and* $\mathcal{F}_\alpha$ *is an infinitely divisible law with characteristic function*

$$\phi_\alpha(u) = \exp\left\{iau + \alpha\int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{dx}{x^{\alpha+1}}\right\}, \tag{6.6}$$

where the constant a *is given by*

$$a = \begin{cases} \dfrac{\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & (\alpha \ne 1), \\ 0 & (\alpha = 1). \end{cases} \tag{6.7}$$

6.2. Proof of Theorem 6.1

The proof is broken down into steps according to formulas (6.3), (6.4) and (6.5).

Proposition 6.1. *The function L defined in (6.3) is given by*

$$L(x) = \begin{cases} 0, & x < 0, \\ -x^{-\alpha}, & x > 0. \end{cases} \tag{6.8}$$

Proof. Since $Y \geq 0$, it is clear that $L(x) \equiv 0$ for $x < 0$. Henceforth, assume that $x > 0$. Using (6.1), (2.12) and (2.9), we obtain

$$N\,\mathsf{P}\{Y(t) > x\} = N\,\mathsf{P}\big\{X > \pm\eta_x(t)\big\} \sim e^{\lambda H_0(t) - h(\eta_x(t)^{\pm})} \qquad (t \to \infty),$$

where $\eta_x(t)$ is defined in (5.22). Furthermore, by Lemmas 5.9 and 5.12

$$\lambda H_0(t) - h(\eta_x(t)^{\pm}) = h(\eta_1(t)^{\pm}) - h(\eta_x(t)^{\pm}) \to -\alpha \log x \qquad (t \to \infty),$$

so from (6.3) we get $L(x) = -e^{-\alpha \log x} = -x^{-\alpha}$. □

Proposition 6.2. *For σ^2 defined in (6.4), for all $\alpha \in (0, 2)$ we have $\sigma^2 \equiv 0$.*

Proof. Since $0 \leq \mathsf{Var}\big[Y\,\mathbf{1}_{\{Y \leq \tau\}}\big] \leq \mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\big]$, it suffices to prove that

$$\lim_{\tau \to 0+} \lim_{t \to \infty} N\,\mathsf{E}\Big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\Big] = 0.$$

Recalling (6.1) and (2.12) and using condition (2.9), we have

$$N\,\mathsf{E}\Big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\Big] \sim e^{(\lambda \mp 2\mu(t))H_0(t)}\,\mathsf{E}\Big[e^{2tX}\mathbf{1}_{\{X \leq \pm\eta_\tau\}}\Big] \qquad (t \to \infty). \tag{6.9}$$

Application of Lemma 5.13(i) with $p = 2$ and $0 < \alpha < 2$ yields

$$\mathsf{E}\Big[e^{2tX}\mathbf{1}_{\{X \leq \pm\eta_\tau\}}\Big] \sim \frac{\alpha}{2-\alpha}\,e^{\pm 2t\eta_\tau - h(\eta_\tau^{\pm})} \qquad (t \to \infty).$$

Returning to (6.9) and recalling relation (5.22), we conclude that

$$\begin{aligned} N\,\mathsf{E}\Big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\Big] &\sim \frac{\alpha}{2-\alpha}\,e^{\lambda H_0(t) \mp 2\mu(t)H_0(t) \pm 2t\eta_\tau - h(\eta_\tau^{\pm})} \\ &= \frac{\alpha}{2-\alpha}\,e^{\lambda H_0(t) - h(\eta_\tau^{\pm}) + 2\log\tau} \to \frac{\alpha}{2-\alpha}\,e^{(2-\alpha)\log\tau} = \frac{\alpha}{2-\alpha}\,\tau^{2-\alpha}, \end{aligned}$$

where we have also used Lemmas 5.9 and 5.12. Letting now $\tau \to 0+$, we see that $\tau^{2-\alpha} \to 0$, since $2 - \alpha > 0$. □

Proposition 6.3. *Set $A^*(t) := A(t)/B(t)$, where $B(t)$ and $A(t)$ are given by (2.12) and (2.13), respectively. Then the limit*

$$D_\alpha(\tau) := \lim_{t \to \infty} \big\{N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y \leq \tau\}}\big] - A^*(t)\big\} \tag{6.10}$$

exists for all $\alpha \in (0, 2)$ and is given by

$$D_\alpha(\tau) = \begin{cases} \dfrac{\alpha}{1-\alpha}\,\tau^{1-\alpha} & (\alpha \neq 1), \\ \log\tau & (\alpha = 1). \end{cases} \tag{6.11}$$

Proof. Using expressions (6.1), (2.12) and recalling (5.22) we obtain

$$N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big]=Ne^{\mp\mu(t)H_0(t)}\,\mathsf{E}\Big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_\tau\}}\Big]. \tag{6.12}$$

1) Let $0<\alpha<1$, then $A^*=0$. Lemma 5.13(i) with $p=1$ yields

$$\mathsf{E}\Big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_\tau\}}\Big]\sim\frac{\alpha}{1-\alpha}\,e^{\pm t\eta_\tau-h(\eta_\tau^\pm)}\qquad(t\to\infty).$$

Hence, on account of the scaling condition (2.9) the right-hand side of (6.12) is asymptotically equivalent to

$$\frac{\alpha}{1-\alpha}\,e^{\lambda H_0(t)\mp\mu(t)H_0(t)\pm t\eta_\tau-h(\eta_\tau^\pm)}=\frac{\alpha}{1-\alpha}\,e^{\log\tau+\lambda H_0(t)-h(\eta_\tau^\pm)}.$$

Finally, using the Basic Identity (5.27) and Lemma 5.12, we get

$$\log\tau+\lambda H_0(t)-h(\eta_\tau^\pm)\to(1-\alpha)\log\tau\qquad(t\to\infty), \tag{6.13}$$

and (6.11) follows.

2) Let $1<\alpha<2$. Using (6.12), (2.9), (2.12) and (2.13), we obtain

$$\begin{aligned}N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big]-A^*(t)&=-N\,e^{\mp\mu(t)H_0(t)}\,\mathsf{E}\Big[e^{tX}\mathbf{1}_{\{X>\pm\eta_\tau\}}\Big]\\&\sim-\frac{\alpha}{\alpha-1}\,e^{\log\tau+\lambda H_0(t)-h(\eta_\tau^\pm)},\end{aligned}$$

where we used Lemma 5.13(ii) with $p=1$. Applying (6.13) we arrive at (6.11).

3) Let $\alpha=1$. Similarly as above, we obtain using Lemmas 5.15 and 5.9:

$$\begin{aligned}&N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big]-A^*(t)\\&\qquad=Ne^{\mp\mu(t)H_0(t)}\Big(\mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_\tau\}}\big]-\mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\big]\Big)\\&\qquad\sim e^{\lambda H_0(t)\mp\mu(t)H_0(t)}\cdot e^{\pm t\eta_1-h(\eta_1^\pm)}\log\tau\\&\qquad=\log\tau,\end{aligned}$$

and the proof is complete. □

Proposition 6.4. *The parameter a defined in (6.7) satisfies the identity (6.5) with* $L(\cdot)$ *specified by (6.8), that is,*

$$D_\alpha(\tau)=a+\alpha\int_0^\tau\frac{x^{2-\alpha}}{1+x^2}\,dx-\alpha\int_\tau^\infty\frac{x^{-\alpha}}{1+x^2}\,dx\qquad(\tau>0), \tag{6.14}$$

where $D_\alpha(\tau)$ *is given by (6.11).*

Proof. 1) Let $0 < \alpha < 1$. Observe that

$$\int_0^\tau \frac{x^{2-\alpha}}{1+x^2}\,dx = \frac{1}{1-\alpha}\,\tau^{1-\alpha} - \int_0^\tau \frac{x^{-\alpha}}{1+x^2}\,dx.$$

Due to (6.7) and (6.11), equation (6.14) amounts to

$$\int_0^\infty \frac{x^{-\alpha}}{1+x^2}\,dx = \frac{\pi}{2\cos\frac{\alpha\pi}{2}}, \tag{6.15}$$

which is true by [9, #3.241(2)].

2) For $1 < \alpha < 2$, we note that

$$\int_\tau^\infty \frac{x^{-\alpha}}{1+x^2}\,dx = \frac{\tau^{1-\alpha}}{\alpha-1} - \int_\tau^\infty \frac{x^{2-\alpha}}{1+x^2}\,dx,$$

and hence, in view of (6.11) and (6.7), equation (6.14) is reduced to

$$\frac{\pi}{2\cos\frac{\alpha\pi}{2}} + \int_0^\infty \frac{x^{2-\alpha}}{1+x^2}\,dx = 0, \tag{6.16}$$

which again follows from [9, #3.241(2)].

3) Finally, for $\alpha = 1$ equation (6.14) takes the form

$$\log\tau = \int_0^\tau \frac{x}{1+x^2}\,dx - \int_\tau^\infty \frac{1}{(1+x^2)x}\,dx. \tag{6.17}$$

The integrals on the right of (6.17) are easily evaluated to yield

$$\frac{1}{2}\log(1+x^2)\Big|_0^\tau - \frac{1}{2}\log\frac{x^2}{1+x^2}\Big|_\tau^\infty = \log\tau,$$

and this completes the proof of Proposition 6.4. □

Proof of Theorem 6.1. Gathering the results of Propositions 6.1, 6.2, 6.3 and 6.4, which identify the ingredients of the limit characteristic function ϕ_α, we conclude that Theorem 6.1 is true. □

6.3. Stability of the limit law

In this section, we show that the infinitely divisible law $\mathcal{F}_\alpha$ with characteristic function (6.6) is in fact stable.

Theorem 6.2. *The characteristic function ϕ_α determined by Theorem 6.1 corresponds to a stable probability law with exponent $\alpha \in (0, 2)$ and skewness parameter $\beta = 1$, and can be represented in the canonical form (2.16).*

Remark. Formula (6.8) and Proposition 6.2 imply that ϕ_α corresponds to a stable law (see [12, Theorem 2.2.1]). We give a direct proof of this fact by reducing ϕ_α to the canonical form (2.16), which allows us to identify explicitly all the parameters.

Proof of Theorem 6.2. According to general theory[5] (see, e.g., [17, p. 441]), the characteristic function of a stable law with characteristic exponent $\alpha \in (0, 2)$ admits a canonical representation

$$\phi_\alpha(u) = \begin{cases} \exp\left\{i\mu u - b|u|^\alpha\left(1 - i\beta \operatorname{sgn} u \cdot \tan\dfrac{\pi\alpha}{2}\right)\right\} & (\alpha \neq 1), \\ \exp\left\{i\mu u - b|u|\left(1 + i\beta \operatorname{sgn} u \cdot \dfrac{2}{\pi}\log|u|\right)\right\} & (\alpha = 1), \end{cases} \tag{6.18}$$

where μ is a real constant, $b > 0$ and $-1 \le \beta \le 1$.

1) Suppose that $0 < \alpha < 1$. It is easy to verify that, due to (6.7) and (6.15), the characteristic function (6.6) can be rewritten in the form

$$\phi_\alpha(u) = \exp\left\{\alpha \int_0^\infty \frac{e^{iux} - 1}{x^{\alpha+1}}\, dx\right\}. \tag{6.19}$$

The integral in (6.19) can be evaluated (see [12, p. 43–44]):

$$\int_0^\infty \frac{e^{iux} - 1}{x^{\alpha+1}}\, dx = -\frac{\Gamma(1-\alpha)}{\alpha}\, |u|^\alpha\, e^{-(i\pi\alpha/2)\operatorname{sgn} u},$$

and (6.18) follows with $\mu = 0,\ b = \Gamma(1-\alpha)\cos(\pi\alpha/2) > 0,\ \beta = 1$.

2) Let now $1 < \alpha < 2$. Using (6.16), we can rewrite (6.6) in the form

$$\phi_\alpha(u) = \exp\left\{\alpha \int_0^\infty (e^{iux} - 1 - iux)\, \frac{dx}{x^{\alpha+1}}\right\}. \tag{6.20}$$

The integral in (6.20) is given by (see [12, p. 44–45])

$$\int_0^\infty (e^{iux} - 1 - iux)\, \frac{dx}{x^{\alpha+1}} = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\, |u|^\alpha\, e^{(i\pi\alpha/2)\operatorname{sgn} u},$$

which yields $\mu = 0,\ b = -\Gamma(2-\alpha)/(\alpha-1)\cdot\cos(\pi\alpha/2) > 0,\ \beta = 1$.

3) If $\alpha = 1$, by the substitution $y = |u|x$ in (6.7) we get

$$\phi_1(u) = \exp\left\{-|u| \int_0^\infty \frac{1 - \cos y}{y^2}\, dy - iu \int_0^\infty \left(\sin y - \frac{u^2 y}{u^2 + y^2}\right) \frac{dy}{y^2}\right\}. \tag{6.21}$$

It is well known (see [9, #3.782(2)]) that

$$\int_0^\infty \frac{1 - \cos y}{y^2}\, dy = \frac{\pi}{2}. \tag{6.22}$$

To evaluate the second integral in (6.21), let us represent it in the form

$$\int_0^\infty \left(\frac{\sin y}{y} - \frac{1}{1+y}\right) \frac{dy}{y} + \int_0^\infty \left(\frac{1}{1+y} - \frac{u^2}{u^2+y^2}\right) \frac{dy}{y}. \tag{6.23}$$

[5] See [10] for a nice review of the history of the canonical form of stable distributions.

It is known that (see [9, #3.781(1)])

$$\int_0^\infty \left(\frac{\sin y}{y} - \frac{1}{1+y} \right) \frac{dy}{y} = 1 - \gamma, \tag{6.24}$$

where γ is the Euler constant. Furthermore, note that

$$\int_0^\infty \left(\frac{1}{1+y} - \frac{u^2}{u^2+y^2} \right) \frac{dy}{y} = \frac{1}{2} \log \frac{u^2+y^2}{(1+y)^2} \bigg|_0^\infty = -\log |u|. \tag{6.25}$$

Returning to (6.23), from (6.24) and (6.25) we get

$$\int_0^\infty \left(\sin y - \frac{u^2 y}{u^2+y^2} \right) \frac{dy}{y^2} = 1 - \gamma - \log |u|. \tag{6.26}$$

Therefore, substituting expressions (6.22) and (6.26) into (6.21), we obtain the required canonical form (6.18) with $\mu = 1-\gamma,\ b = \pi/2,\ \beta = 1.$ □

7. Limit theorems at the critical points

7.1. Proof of Theorem 2.4

The statement of Theorem 2.4 follows from Theorem 2.3 (for $\alpha = 1$). Indeed, according to (2.13) and (5.31), we have $A(t) = NB_1(t) = Ne^{\pm t\eta_1 - h(\eta_1^\pm)}\, b_1(t)$. Furthermore, (2.12), (2.9), (5.27) and (5.32) imply

$$A^*(t) := \frac{A(t)}{B(t)} \sim e^{\lambda H_0(t) - h(\eta_1^\pm)}\, b_1(t) = b_1(t) \to \infty \qquad (t \to \infty). \tag{7.1}$$

Therefore, dividing (2.15) by $A^*(t) \to \infty$ we obtain $S_N(t)/A(t) = 1 + o_p(1)$, which is in agreement with (2.17).

7.2. Proof of Theorem 2.5

Denote

$$Y_i \equiv Y_i(t) := \frac{e^{tX_i}}{(NB_2(t))^{1/2}}, \tag{7.2}$$

where $B_2(t)$ is defined in (5.31). According to the classical CLT for independent summands (see [14, Theorem 18, p. 95]), it suffices to check that for any $\tau > 0$ the following three conditions are satisfied as $t \to \infty$:

$$N\,\mathsf{P}\{Y(t) > \tau\} \to 0, \tag{7.3}$$

$$N\Big(\mathsf{E}\Big[Y(t)^2 \mathbf{1}_{\{Y(t)\le\tau\}}\Big] - \big(\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)\le\tau\}}\big]\big)^2\Big) \to 1, \tag{7.4}$$

$$N\,\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)>\tau\}}\big] \to 0. \tag{7.5}$$

Firstly, note that condition (7.3) is guaranteed by (5.43). Next, let us show that

$$\frac{\big(\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big]\big)^2}{\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le\tau\}}\big]}=\frac{\big(\mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\tilde\eta_{2,\tau}\}}\big]\big)^2}{\mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X\le\pm\tilde\eta_{2,\tau}\}}\big]}\to 0\qquad(t\to\infty).\tag{7.6}$$

Indeed, taking into account inequality (5.41) and representation (5.32), the ratio in (7.6) is estimated from above by

$$\frac{\big(\mathsf{E}\big[e^{tX}\big]\big)^2}{\mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\big]}=\frac{e^{\pm 2H(t)}}{B_2(t)}=\frac{e^{\pm 2H(t)+h(\eta_1^{\pm})\mp 2t\eta_1}}{b_2(t)}.\tag{7.7}$$

Using the Basic Identity (5.27) and the limit (5.26), we have

$$\frac{\pm 2H(t)+h(\eta_1^{\pm})\mp 2t\eta_1(t)}{H_0(t)}\to\pm 2\pm(\varrho'-1)\,2^{\varrho'}\mp\varrho' 2^{\varrho'}=\pm(2-2^{\varrho'})<0,$$

and hence the numerator on the right of (7.7) tends to zero. Moreover, $b_2(t)\to\infty$ (see (5.32)), and therefore (7.6) is validated. Hence, condition (7.4) amounts to

$$N\,\mathsf{E}\Big[Y^2\mathbf{1}_{\{Y\le\tau\}}\Big]\to 1.\tag{7.8}$$

Noting that, according to (7.2), (5.45) and (2.18),

$$N\,\mathsf{E}\Big[Y^2\mathbf{1}_{\{Y\le y_2\}}\Big]=\frac{1}{B_2}\,\mathsf{E}\Big[e^{2tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\Big]\equiv 1,$$

we can rewrite (7.8) in the form $N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{y_2<Y\le\tau\}}\big]\to 0$. The latter is true by Lemma 5.17, and (7.4) follows.

Finally, condition (7.5) is fulfilled by Lemma 5.16 (with $p=1<2=\alpha$).

Appendix. Proof of Lemma 5.13

A.1. Proof of part (i)

1) We start by showing that for a suitable $\theta\in(0,1)$

$$\lim_{t\to\infty}e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})}\,\mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X\le\pm\theta^{\pm}\eta_\tau\}}\Big]=0.\tag{A.1}$$

Since $\mathsf{E}\big[e^{ptX}\mathbf{1}_{\{X\le\pm\theta^{\pm}\eta_\tau\}}\big]\le e^{\pm pt\theta^{\pm}\eta_\tau}$, it suffices to check that

$$e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})\pm pt\theta^{\pm}\eta_\tau}=e^{\pm(\theta^{\pm}-1)pt\eta_\tau+h(\eta_\tau^{\pm})}\to 0\qquad(t\to\infty).\tag{A.2}$$

Using the limit (5.29) of Lemma 5.10, we have

$$\frac{\pm(\theta^{\pm}-1)\,pt\eta_\tau+h(\eta_\tau^{\pm})}{t\eta_\tau}\to\pm(\theta^{\pm}-1)\,p+\frac{\alpha}{\varrho}\qquad(t\to\infty).\tag{A.3}$$

Since $t\eta_\tau(t) \sim \mu(t)H_0(t) \to +\infty$ as $t \to \infty$, the limit (A.2) will follow if there exists $\theta \in (0,1)$ such that the right-hand side of (A.3) is negative. The latter is guaranteed by the fact that $0 < (1 \mp \alpha/p\varrho)^{\pm} < 1$, which can be easily verified using that $p > \alpha > 0$ and $\varrho > 1$ (case B) or $\varrho > 0$ (case A).

2) Similarly to (5.33), integration by parts yields

$$\mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{\pm\theta^{\pm}\eta_\tau < X \le \pm\eta_\tau\}}\Big] = -\, e^{\pm pt\eta_\tau - h(\eta_\tau^{\pm})} + e^{\pm pt\theta^{\pm}\eta_\tau - h(\theta^{\pm}\eta_\tau^{\pm})} + pt\int_{\pm\theta^{\pm}\eta_\tau}^{\pm\eta_\tau} e^{ptx - h(\pm x^{\pm})}\,dx. \tag{A.4}$$

Using that $h(\cdot) \ge 0$, we have

$$e^{\pm pt\theta^{\pm}\eta_\tau - h(\theta^{\pm}\eta_\tau^{\pm})} \le e^{\pm pt\theta^{\pm}\eta_\tau} = o(1)\, e^{\pm pt\eta_\tau - h(\eta_\tau^{\pm})} \qquad (t \to \infty), \tag{A.5}$$

as shown above (see (A.2)).

3) Let us set $\tilde{\eta}_\tau(t) := \eta_\tau(t) \mp g(t)$, where $g(t) := t^{-1+\varrho'/2}$. Using that $\eta_\tau \in R_{\varrho'-1}$, we get $\tilde{\eta}_\tau/\eta_\tau \to 1$ $(t \to \infty)$ and so for all t large enough, $\pm\theta^{\pm}\eta_\tau \le \pm\tilde{\eta}_\tau \le \pm\eta_\tau$.

Let us now show that for any $x \in [\pm\theta^{\pm}\eta_\tau, \pm\tilde{\eta}_\tau]$ and all t large enough,

$$ptx - h(\pm x^{\pm}) \le \pm pt\tilde{\eta}_\tau - h(\tilde{\eta}_\tau^{\pm}). \tag{A.6}$$

Setting $\kappa_\tau(t) := \pm x^{\pm}/\tilde{\eta}_\tau^{\pm}$, we have

$$1 \ge \kappa_\tau(t) \ge \theta\left(\frac{\eta_\tau}{\tilde{\eta}_\tau}\right)^{\pm} \to \theta \qquad (t \to \infty),$$

so by Lemma 5.4 we can write

$$h(\pm x^{\pm}) - h(\tilde{\eta}_\tau^{\pm}) = h(\tilde{\eta}_\tau^{\pm})(\kappa_\tau^{\varrho} - 1)(1 + o(1)) \qquad (t \to \infty), \tag{A.7}$$

uniformly in $x \in [\pm\theta^{\pm}\eta_\tau, \pm\tilde{\eta}_\tau]$. Furthermore, inequality (3.8) yields

$$\kappa_\tau^{\varrho} - 1 = \left(\frac{\pm x}{\tilde{\eta}_\tau}\right)^{\pm\varrho} - 1 \ge (\pm\varrho)\left(\frac{\pm x}{\tilde{\eta}_\tau} - 1\right) = \frac{\varrho}{\tilde{\eta}_\tau}\,(x \mp \tilde{\eta}_\tau). \tag{A.8}$$

Combining (A.7) and (A.8) and using Lemma 5.10, we obtain that for all t large enough, uniformly in x,

$$\begin{aligned} h(\pm x^{\pm}) - h(\tilde{\eta}_\tau^{\pm}) &\ge \frac{h(\tilde{\eta}_\tau^{\pm})}{\tilde{\eta}_\tau}\,\varrho\,(x \mp \tilde{\eta}_\tau)(1 + o(1)) \\ &= \alpha t\,(x \mp \tilde{\eta}_\tau)(1 + o(1)) \\ &\ge pt\,(x \mp \tilde{\eta}_\tau), \end{aligned} \tag{A.9}$$

since $x \mp \tilde{\eta}_\tau \le 0$ and $\alpha < p$. Hence, inequality (A.6) follows.

4) We now want to prove that, as $t \to \infty$,

$$I(t) := pt\, e^{\mp pt\eta_\tau + h(\eta_\tau^{\pm})} \int_{\pm\theta^{\pm}\eta_\tau}^{\pm\tilde{\eta}_\tau} e^{ptx - h(\pm x^{\pm})}\,dx \to 0. \tag{A.10}$$

Applying the estimate (A.6) we get

$$I(t) \le pt\, e^{-ptg(t)+h(\eta_\tau^\pm)-h(\tilde{\eta}_\tau^\pm)}\big[\pm(1-\theta^\pm)\eta_\tau - g(t)\big]. \tag{A.11}$$

Recalling that $g(t) \ge 0$ and $0 < \theta < 1$, it is easy to check that $\pm(1-\theta^\pm)\eta_\tau - g(t) \le \eta_\tau(1-\theta)/\theta$. Therefore, from (A.11) it follows

$$I(t) \le \frac{p\,(1-\theta)}{\theta}\, t\,\eta_\tau\, e^{-ptg(t)+h(\eta_\tau^\pm)-h(\tilde{\eta}_\tau^\pm)}. \tag{A.12}$$

It remains to observe that the pre-exponential factor in (A.12) grows only polynomially, since $t\,\eta_\tau(t) \sim \mathrm{const}\cdot H_0(t) \in R_{\varrho'}$, while by Lemma 5.11, $-ptg(t) + h(\eta_\tau^\pm) - h(\tilde{\eta}_\tau^\pm) \sim -(p-\alpha)tg(t)$, where $p-\alpha > 0$ and $tg(t) = t^{\varrho'/2}$. Hence the right-hand side of (A.12) is exponentially small as $t \to \infty$, and (A.10) follows.

5) Let us check that

$$J(t) := pt\, e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)} \int_{\pm\tilde{\eta}_\tau}^{\pm\eta_\tau} e^{ptx - h(\pm x^\pm)}\, dx \to \frac{p}{p-\alpha} \qquad (t \to \infty). \tag{A.13}$$

By the substitution $\pm x = \eta_\tau(t) \mp yg(t) =: \tilde{\eta}_{\tau,y}(t)$, the left-hand side of (A.13) is rewritten in the form

$$J(t) = ptg(t) \int_0^1 e^{-ptg(t)y + h(\eta_\tau^\pm) - h(\tilde{\eta}_{\tau,y}^\pm)}\, dy. \tag{A.14}$$

Note that by Lemma 5.11, $h(\eta_\tau^\pm) - h(\tilde{\eta}_{\tau,y}^\pm) = \alpha tg(t)y(1+o(1))$ as $t \to \infty$, uniformly in $y \in [0,1]$. Therefore, given any ε such that $0 < \varepsilon < p-\alpha$, for all large enough t and all $y \in [0,1]$ we have

$$(\alpha-\varepsilon)tg(t)y \le h(\eta_\tau^\pm) - h(\tilde{\eta}_{\tau,y}^\pm) \le (\alpha+\varepsilon)tg(t)y.$$

Substituting these estimates into (A.14) and evaluating the integral, we obtain

$$J(t) \le ptg(t) \int_0^1 e^{-(p-\alpha-\varepsilon)tg(t)y}\, dy = \frac{p(1-e^{-(p-\alpha-\varepsilon)tg(t)})}{p-\alpha-\varepsilon}$$

and similarly

$$J(t) \ge ptg(t) \int_0^1 e^{-(p-\alpha+\varepsilon)tg(t)y}\, dy = \frac{p(1-e^{-(p-\alpha+\varepsilon)tg(t)})}{p-\alpha+\varepsilon}.$$

Using that $p-\alpha\pm\varepsilon > 0$ and $tg(t) \to \infty$, in the limit as $t \to \infty$ we get

$$\frac{p}{p-\alpha+\varepsilon} \le \liminf_{t\to\infty} J(t) \le \limsup_{t\to\infty} J(t) \le \frac{p}{p-\alpha-\varepsilon}.$$

Letting $\varepsilon \downarrow 0$, we obtain $\lim_{t\to\infty} J(t) = p/(p-\alpha)$, as required.

6) Finally, formulas (A.1), (A.4), (A.5), (A.10) and (A.13) yield

$$\lim_{t\to\infty} e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\, \mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X \le \pm\eta_\tau\}}\Big] = -1 + \frac{p}{p-\alpha} = \frac{\alpha}{p-\alpha}.$$

A.2. Proof of part (ii)

The proof follows similar steps as above.

1′) Let us start by showing that if $p < \alpha$ then for any $\theta > 1$

$$\lim_{t\to\infty} e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\,\mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X>\pm\theta^\pm\eta_\tau\}}\Big] = 0. \qquad (A.15)$$

Note that Lemma 4.2 (with $k = p$, $m = \alpha$) yields

$$\mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X>\pm\theta^\pm\eta_\tau\}}\Big] \le \mathsf{E}[e^{\alpha tX}]\cdot e^{\mp\theta^\pm(\alpha-p)t\eta_\tau} = e^{\pm H(\alpha t)\mp\theta^\pm(\alpha-p)t\eta_\tau}.$$

Hence, it suffices to check that

$$e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\cdot e^{\pm H(\alpha t)\mp\theta^\pm(\alpha-p)t\eta_\tau} = o(1) \qquad (t\to\infty). \qquad (A.16)$$

To this end, recall that $H \sim H_0 \in R_{\varrho'}$ and use (5.26), (5.29) and (3.3) to obtain

$$\begin{aligned}\lim_{t\to\infty}&\frac{\pm H(\alpha t)\mp\big(p+\theta^\pm(\alpha-p)\big)t\eta_\tau + h(\eta_\tau^\pm)}{H_0(t)}\\ &= \pm\alpha^{\varrho'}\mp(p+\theta^\pm(\alpha-p))\varrho'\alpha^{\varrho'-1} + \frac{\alpha}{\varrho}\varrho'\alpha^{\varrho'-1}\\ &= \pm(1-\theta^\pm)(\alpha-p)\varrho'\alpha^{\varrho'-1} < 0,\end{aligned}$$

since $\theta > 1$ and $\alpha > p$. Hence, the limit (A.16) follows.

2′) Similarly to (A.4), integration by parts yields

$$\begin{aligned}\mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{\pm\eta_\tau<X\le\pm\theta^\pm\eta_\tau\}}\Big] =& - e^{\pm pt\theta^\pm\eta_\tau - h(\theta\eta_\tau^\pm)} + e^{\pm pt\eta_\tau - h(\eta_\tau^\pm)}\\ &+ pt\int_{\pm\eta_\tau}^{\pm\theta^\pm\eta_\tau} e^{ptx-h(\pm x^\pm)}\,dx.\end{aligned}$$

Let us check here that

$$e^{\pm(\theta^\pm-1)pt\eta_\tau - [h(\theta\eta_\tau^\pm)-h(\eta_\tau^\pm)]} = o(1) \qquad (t\to\infty). \qquad (A.17)$$

Recalling that $h \in R_\varrho$ and using the limit (5.29), we obtain

$$h(\theta\eta_\tau^\pm) - h(\eta_\tau^\pm) \sim (\theta^\varrho - 1)h(\eta_\tau^\pm) \sim \frac{(\theta^\varrho-1)\alpha}{\varrho}\,t\eta_\tau.$$

Hence,

$$\frac{\pm(\theta^\pm-1)pt\eta_\tau - [h(\theta\eta_\tau^\pm)-h(\eta_\tau^\pm)]}{t\eta_\tau} \to \pm p(\theta^\pm-1) - \frac{(\theta^\varrho-1)\alpha}{\varrho}. \qquad (A.18)$$

Inequality (3.8) gives $\theta^\varrho - 1 = (\theta^\pm)^{\pm\varrho} - 1 \ge \pm\varrho(\theta^\pm - 1)$, so the right-hand side of (A.18) is estimated from above by $\pm p(\theta^\pm-1)\mp\alpha(\theta^\pm-1) = \pm(\theta^\pm-1)(p-\alpha) < 0$, because $\theta > 1$ and $p < \alpha$. Hence, the limit (A.17) follows.

3′) Let us set $\tilde{\eta}_\tau(t) := \eta_\tau(t) \pm g(t)$, where the function g is as in step 3 above, and check that for $x \in [\pm\tilde{\eta}_\tau, \pm\theta^\pm\eta_\tau]$ and all sufficiently large t

$$ptx - h(\pm x^\pm) \le \pm pt\tilde{\eta}_\tau - h(\tilde{\eta}_\tau^\pm).$$

To this end, similarly to (A.9) we show that

$$h(\pm x^\pm) - h(\tilde{\eta}_\tau^\pm) \ge \alpha t\,(x \mp \tilde{\eta}_\tau)(1 + o(1)) \ge pt\,(x \mp \tilde{\eta}_\tau),$$

using that $x \mp \tilde{\eta}_\tau \ge 0$ and $\alpha > p$.

4′) The goal here is to prove that, as $t \to \infty$,

$$I(t) := pt\,e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)} \int_{\pm\tilde{\eta}_\tau}^{\pm\theta^\pm\eta_\tau} e^{ptx - h(\pm x^\pm)}\,dx \to 0.$$

Using the estimate from step 3′, we obtain

$$\begin{aligned} I(t) &\le pt\,e^{ptg(t) + h(\eta_\tau^\pm) - h(\tilde{\eta}_\tau^\pm)} \left(\pm\theta^\pm\eta_\tau \mp \tilde{\eta}_\tau\right) \\ &\le p(\theta - 1)t\eta_\tau\,e^{ptg(t) + h(\eta_\tau^\pm) - h(\tilde{\eta}_\tau^\pm)}. \end{aligned}$$

We can now apply the same argument as in step 4 above, using that

$$ptg(t) + h(\eta_\tau^\pm) - h(\tilde{\eta}_\tau^\pm) \sim -(\alpha - p)tg(t) \qquad (t \to \infty).$$

5′) Similarly as in step 5 above [cf. (A.13)], one proves that

$$\lim_{t\to\infty} pt\,e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)} \int_{\pm\eta_\tau}^{\pm\tilde{\eta}_\tau} e^{ptx - h(\pm x^\pm)}\,dx = \frac{p}{\alpha - p}.$$

In so doing, the suitable substitution in the integral is of the form $\pm x = \eta_\tau(t) \pm yg(t)$, and an auxiliary ε involved in the estimation is taken to satisfy $0 < \varepsilon < \alpha - p$.

6′) Combining the limit formulas obtained in steps 1′–5′ we obtain

$$\lim_{t\to\infty} e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\,\mathsf{E}\Big[e^{ptX}\mathbf{1}_{\{X > \pm\eta_\tau\}}\Big] = 1 + \frac{p}{\alpha - p} = \frac{\alpha}{\alpha - p}.$$

Acknowledgements. The authors wish to thank Anton Bovier for useful discussions of the REM and Yijun Hu for drawing our attention to possible actuarial applications. The paper was completed while the second author was visiting the Isaac Newton Institute for Mathematical Sciences, Cambridge.

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