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Strengthening classical results on convergence rates in strong limit theorems

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Abstract. Let $X_1, X_2, \ldots$ be i.i.d. random variables, and set $S_n = X_1 + \ldots + X_n$. Several authors proved convergence of series of the type $f(\varepsilon) = \sum_n c_n P(|S_n| > \varepsilon a_n), \varepsilon > \alpha$, under necessary and sufficient conditions. We show that under the same conditions, in fact $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty, \delta > \alpha$, i.e. the finiteness of $\sum_n c_n P(|S_n| > \varepsilon a_n), \varepsilon > \alpha$, is equivalent to the convergence of the double sum $\sum_k \sum_n c_n P(|S_n| > k a_n)$. Two exceptional series required deriving necessary and sufficient conditions for $E[\sup_n |S_n| (\log n)^\eta / n] < \infty, 0 \le \eta \le 1$.

1. Introduction

Throughout this paper, $X, X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with $P(X \neq 0) > 0$, and $S_n = X_1 + \ldots + X_n, n \ge 1$. We consider series of the type

$$f(\varepsilon) = \sum_n c_n P(|S_n| > \varepsilon a_n), \varepsilon > 0, \tag{1}$$

where $c_n > 0$ and $\sum_n c_n = \infty$, and a_n is either $n^{1/p}, 0 < p < 2, \sqrt{n \log n}$ or $\sqrt{n \log\log n}$. Under appropriate necessary and sufficient conditions of the form $E[\varphi(|X|)] < \infty$ and $EX = 0$, several authors (Hsu and Robbins (1947), Erdős (1949,1950), Spitzer (1956), Baum and Katz (1965), Davis (1968a,1968b), Lai (1974), Gut (1980)) proved that $f(\varepsilon) < \infty$ for $\varepsilon >$ some α. The purpose of this paper is to strengthen these classical results by showing that, except for two remarkable cases,

$$E[\varphi(|X|)] < \infty \text{ and } EX = 0 \Longleftrightarrow \int_\delta^\infty f(\varepsilon)d\varepsilon < \infty, \delta > \alpha. \tag{2}$$

Thus the convergence of the series in (1) is in fact equivalent to the convergence of the double series $\sum_k \sum_n c_n P(|S_n| > k a_n)$. As expected, the exceptional situations

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pertain to the cases $c_n = 1/n, a_n = n$ (Spitzer (1956)), and $c_n = (\log n)/n, a_n = n$ (Baum and Katz (1965)). The range of tools used is rather broad, including Fuk-Nagaev type inequalities, stopping times, non-uniform estimates in the CLT, symmetrization techniques. Since the implication $\Longleftarrow$ in (2) is trivial, we shall prove only $\Longrightarrow$. Notice also that it suffices to prove (2) for $\delta >$ some $\alpha_0 \geq \alpha$, as f is nonincreasing.

We collect the auxiliary lemmas in Section 2, after which Sections 4–6 are essentially devoted to the proof of (2) for large, moderate and small a_n, respectively. To cover the exceptional cases alluded to above, an interlude on the expected supremum of $|S_n| (\log n)^\eta / n, 0 \leq \eta \leq 1$, is done in Section 3. In the sequel, $\log^+ x = \log(e \vee x), x \geq 0$, and Φ stands for the standard normal distribution function. Also C shall denote positive constants, independent of ε, possibly varying from place to place. For a random variable g, we will write $E[g : A] = \int_A g dP$ whenever the integral exists.

2. Auxiliary lemmas

Part (i) of the next lemma is to be found in Spătaru (1999), while (ii) follows by specializing an inequality in Nagaev and Pinelis (1977, p. 250). Both rest on an inequality by Fuk and Nagaev (1971).

Lemma 1. *Let $Y_1, \ldots, Y_n$ be i.i.d random variables with $EY_1 = 0$, set $T_n = Y_1 + \cdots + Y_n$, and let $x, a > 0$.*

(i) *If $E|Y_1|^q < \infty$ for some $1 \leq q \leq 2$, then*

$$P(|T_n| > x) \leq nP(|Y_1| > x/a) + C\left(\frac{nE|Y_1|^q}{x^q}\right)^a.$$

(ii) *If $E|Y_1|^q < \infty$ for some $q > 2$, then*

$$P(|T_n| > x) \leq nP(|Y_1| > x/a) + C\left(\frac{nE|Y_1|^q}{x^q}\right)^{aq/(q+2)} + 2e^{-Cx^2/nEY_1^2}.$$

The following result is due to Choi and Sung (1987, p.100).

Lemma 2. *Let $\{Y_n, n \geq 1\}$ be a sequence of independent random variables with $EY_n = 0, n \geq 1$, and let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers with $\lim_{n\to\infty} b_n = \infty$ such that $\sum_{n\geq 1} EY_n^2/b_n^2 < \infty$. Then*

$$E\left[\sup_{n\geq 1} \frac{|Y_1 + \ldots + Y_n|}{b_n}\right] < \infty.$$

Lemma 3. *Let $p, r, c > 0$ and assume that $E|X|^r < \infty$. Then*

$$\sum_{n\geq 1} n^{r/p-1} P(|X| > c\varepsilon n^{1/p}) \leq C\varepsilon^{-r} E|X|^r, \ \ \varepsilon > 0.$$

Proof. We have

$$\sum_{n\ge 1} n^{r/p-1} P(|X| > c\varepsilon n^{1/p}) = \sum_{n\ge 1} n^{r/p-1} \sum_{k\ge n} P(c\varepsilon k^{1/p} < |X| \le c\varepsilon (k+1)^{1/p})$$
$$= \sum_{k\ge 1} \left(\sum_{n=1}^{k} n^{r/p-1} \right) P(c\varepsilon k^{1/p} < |X| \le c\varepsilon (k+1)^{1/p})$$
$$\le C \sum_{k\ge 1} k^{r/p} P(c\varepsilon k^{1/p} < |X| \le c\varepsilon (k+1)^{1/p})$$
$$= C c^{-r}\varepsilon^{-r} \sum_{k\ge 1} c^r \varepsilon^r k^{r/p} P(c^r\varepsilon^r k^{r/p} < |X|^r \le c^r\varepsilon^r (k+1)^{r/p})$$
$$\le C\varepsilon^{-r} E|X|^r, \quad \varepsilon > 0.$$

□

Lemma 4. *Let $p, c > 0$ and assume that $E[|X|^p \log^+ |X|] < \infty$. Then there is $\varepsilon_0 > 0$ such that*

$$\sum_{n\ge 2} \log n \, P(|X| > c\varepsilon n^{1/p}) \le C\varepsilon^{-p} E[|X|^p \log^+ |X|], \quad \varepsilon > \varepsilon_0.$$

Proof. Taking $\varepsilon_0 = e^{1\vee 1/p}/c$, it is easily checked that, for $k \ge 2$, $\log k \le p \log^+(c\varepsilon k^{1/p})$, $\varepsilon > \varepsilon_0$. Then, arguing as above, we find

$$\sum_{n\ge 2} \log n \, P(|X| > c\varepsilon n^{1/p}) \le \sum_{k\ge 2} k \log k \, P(c\varepsilon k^{1/p} < |X| \le c\varepsilon (k+1)^{1/p})$$
$$\le C\varepsilon^{-p} \sum_{k\ge 2} c^p\varepsilon^p k \log^+(c\varepsilon k^{1/p}) P(c^p\varepsilon^p k \log^+(c\varepsilon k^{1/p}) < |X|^p \log^+ |X| \le c^p\varepsilon^p (k+1) \log^+(c\varepsilon (k+1)^{1/p}))$$
$$\le C\varepsilon^{-p} E[|X|^p \log^+ |X|], \quad \varepsilon > \varepsilon_0.$$

□

Lemma 5. *Let $p, c > 0$ and assume that $E[|X|^{2p} (\log^+ |X|)^{-p}] < \infty$. Then there is $\varepsilon_0 > e$ such that*

$$\sum_{n\ge 2} n^{p-1} P(|X| > c\varepsilon \sqrt{n \log n}) \le C\varepsilon^{-2p} (\log \varepsilon)^p E[|X|^{2p} (\log^+ |X|)^{-p}], \quad \varepsilon > \varepsilon_0.$$

Proof. Set $h(x) = x^{2p} (\log^+ x)^{-p}$, $x > 0$. As above, we see that

$$\sum_{n\ge 2} n^{p-1} P(|X| > c\varepsilon \sqrt{n \log n})$$
$$\le \sum_{k\ge 2} \frac{k^p}{h(c\varepsilon\sqrt{k \log k})} h(c\varepsilon\sqrt{k \log k}) P(h(c\varepsilon\sqrt{k \log k}) < h(|X| \le h(c\varepsilon \sqrt{(k+1)\log(k+1)})), \quad \varepsilon > 0. \tag{3}$$

Since, for any $k \ge 2$,

$$\frac{k^p}{h(c\varepsilon\sqrt{k\log k})} = c^{-2p}\varepsilon^{-2p}\left(\frac{\log^+(c\varepsilon\sqrt{k\log k})}{\log k}\right)^p \leq C\varepsilon^{-2p}(\log\varepsilon)^p$$

whenever $\varepsilon >$ some $\varepsilon_0 > e$, the result follows from (3). □

The proof of the next lemma is similar to that of Lemma 3.

Lemma 6. *If* $EX^2 < \infty$, *then*

$$\sum_{n\geq e^e} P(|X| > \varepsilon\sqrt{n\log\log n}) \leq \varepsilon^{-2}EX^2, \quad \varepsilon > 0.$$

The last lemma is useful in connection with exponential identities.

Lemma 7. *Let* $a_n, b_n, c_n, d_n \geq 0$, $n \geq 1$, *be such that* $\sum_{n\geq 1} a_n < \infty$, $\sum_{n\geq 1} b_n \leq 1$, $\sum_{n\geq 1} c_n < \infty$ *and* $\{d_n\}$ *is nonincreasing, and assume that*

$$\left(\sum_{n\geq 1} a_n t^n\right)\left(1 - \sum_{n\geq 1} b_n t^n\right) = \sum_{n\geq 1} c_n t^n, \ \ 0 < t < 1. \tag{4}$$

Then

$$\left(\sum_{n\geq 1} a_n d_n t^n\right)\left(1 - \sum_{n\geq 1} b_n t^n\right) \leq \sum_{n\geq 1} c_n d_n t^n, \ \ 0 < t < 1. \tag{5}$$

Proof. Equating the coefficients of t^n in (4), we get

$$a_1 = c_1, a_n - \sum_{k=1}^{n-1} a_k b_{n-k} = c_n, \ \ n > 1.$$

Hence, as $\{d_n\}$ is nonincreasing,

$$a_1 d_1 = c_1 d_1, \ a_n d_n - \sum_{k=1}^{n-1} a_k d_k b_{n-k} \leq c_n d_n, \ \ n > 1. \tag{6}$$

Clearly, (6) implies (5). □

3. On the expected supremum of $|S_n|\,(\log n)^\eta/n$

For the stopping times $T_\pm(\varepsilon)$ in Theorem 4, the finiteness of $E[|S_{T_\pm(\varepsilon)}|\,(\log T_\pm(\varepsilon))^\eta/T_\pm(\varepsilon)]$ is guaranteed by that of $E[\sup_n |S_n|\,(\log n)^\eta/n]$. Statements like Theorem 1 have long been of interest in ergodic theory, having also implications for stopping rules.

Theorem 1. *For* $0 \leq \eta \leq 1$, *the following are equivalent:*

(i) $E[|X|(\log^+|X|)^{\eta+1}] < \infty$;
(ii) $E[\sup_{n\geq 2} |S_n|\,(\log n)^\eta/n] < \infty$;
(iii) $E[\sup_{n\geq 2} |X_n|\,(\log n)^\eta/n] < \infty$.

Remark 1. For a sequence $\{c_n, n \geq 1\}$ of positive numbers, $E[\sup_{n\geq 1} c_n |S_n|] < \infty \iff E[\sup_{n\geq m} c_n |S_n|] < \infty$, $m \geq 2$. This will be tacitly used several times in the sequel.

Remark 2. If $\eta = 0$, Marcinkiewicz and Zygmund (1937) proved that (i)$\Longrightarrow$(ii), and Burkholder (1962) showed that (ii)$\Longrightarrow$(iii)$\Longrightarrow$(i).

Proof. Let $\psi(x) = x/(\log x)^\eta$, $x \geq 2$, and let $\varphi(x)$, $x \geq e$, denote the inverse function of ψ. Then

$$\varphi(x) \sim x(\log x)^\eta \text{ as } x \to \infty. \tag{7}$$

For $n \geq 2$, put $Y_n = X_n I\{|X_n| \leq \psi(n)\} - E[X_n I\{|X_n| \leq \psi(n)\}]$ and $Z_n = X_n I\{|X_n| > \psi(n)\}$. Assume that (i) holds. Then, on account of Fubini's theorem and (7), we have

$$\begin{aligned}\sum_{n\geq 3} EY_n^2/\psi(n)^2 &\leq \sum_{n\geq 3} E[X^2 I\{|X| \leq \psi(n)\}]/\psi(n)^2 \\ &\leq e^2 \sum_{n\geq 3} \frac{(\log n)^{2\eta}}{n^2} + \sum_{n\geq 3} E[X^2 I\{e \leq |X| \leq \psi(n)\}] \frac{(\log n)^{2\eta}}{n^2} \\ &\leq C + E\left[X^2 I\{e \leq |X|\} \sum_{n\geq \varphi(|X|)} \frac{(\log n)^{2\eta}}{n^2} \right] \\ &\leq C + CE\left[X^2 I\{e \leq |X|\} \frac{(\log \varphi(|X|))^{2\eta}}{\varphi(|X|)} \right] \\ &\leq C + CE\left[X^2 I\{e \leq |X|\} \frac{(\log |X| + \eta \log\log^+ |X|)^{2\eta}}{|X| (\log |X|)^\eta} \right] \\ &\leq C + CE[|X|(\log^+ |X|)^\eta] < \infty.\end{aligned}$$

This in conjunction with Lemma 2 entails that

$$E\left[\sup_{n\geq 2} \frac{|Y_2 + \ldots + Y_n|}{\psi(n)} \right] < \infty. \tag{8}$$

Now, as $\{\psi(n), n \geq 3\}$ is increasing, and by (7), we get

$$\begin{aligned} E\left[\sup_{n\geq 4} \frac{|Z_2 + ... + Z_n|}{\psi(n)} \right] &\leq E\left[\sup_{n\geq 4} \frac{|Z_2| + \ldots + |Z_n|}{\psi(n)} \right] \leq E\left[\sum_{n\geq 2} \frac{|Z_n|}{\psi(n)} \right] \\ &= \sum_{n\geq 2} E[|X| I\{|X| > \psi(n)\}] \frac{(\log n)^\eta}{n} \\ &= C + E\left[|X| I\{e \leq |X|\} \sum_{3\leq n<\varphi(|X|)} \frac{(\log n)^\eta}{n} \right] \\ &\leq C + E[|X| I\{e \leq |X|\} (\log \varphi(|X|))^{\eta+1}] \\ &\leq C + CE[|X|(\log^+ |X|)^{\eta+1}] < \infty. \end{aligned}$$

Hence, in fact

$$E\left[\sup_{n\geq 2}\frac{|Z_2+\ldots+Z_n|}{\psi(n)}\right]<\infty. \tag{9}$$

Finally, since $EX=0$, we have

$$\begin{aligned}|E[X_nI\{|X_n|\leq\psi(n)\}]|&\leq E[|X|I\{|X|>\psi(n)\}]\\&\leq E[|X|(\log^+|X|)^\eta]/(\log^+\psi(n))^\eta,\quad n\geq 2,\end{aligned}$$

and so, as $\sum_{k=2}^n(\log k)^{-\eta}\leq Cn(\log n)^{-\eta}$,

$$\begin{aligned}\frac{1}{\psi(n)}\sum_{k=2}^n|E[X_kI\{|X_k|\leq\psi(k)\}]|&\leq\frac{C}{\psi(n)}\sum_{k=2}^n\frac{1}{(\log^+\psi(k))^\eta}\\&\leq C\frac{(\log n)^\eta}{n}\sum_{k=2}^n\frac{1}{(\log k)^\eta}\leq C,\ n\geq 2.\end{aligned} \tag{10}$$

Since $X_k=Y_k+Z_k+E[X_kI\{|X_k|\leq\psi(k)\}]$, $k\geq 2$, (8)–(10) show that (i)$\Longrightarrow$(ii). The implication (ii)$\Longrightarrow$(iii) follows at once from $X_n=S_n-S_{n-1}$, $n\geq 3$. Finally, suppose that (iii) holds. Then, by the result of Burkholder (1962), we have

$$E[|X|\log^+|X|)]<\infty. \tag{11}$$

Now choose $M>0$ so that $P(|X|\leq M)>0$. From (11) and (7) we infer that $\sum_{i\geq 2}P(|X|>\psi(i)M)<\infty$, and so $\prod_{i\geq 2}P(|X|\leq\psi(i)M)>0$. Consequently, we may write

$$\begin{aligned}E[\sup_{n\geq 2}|X_n|/\psi(n)]&\geq\int_M^\infty P(\sup_{n\geq 2}|X_n|/\psi(n)>u)du\\&=\int_M^\infty\left(\sum_{n\geq 2}P(|X|>\psi(n)u)\prod_{i=2}^{n-1}P(|X|\leq\psi(i)u)\right)du\\&\geq\left(\prod_{i\geq 2}P(|X|\leq\psi(i)M)\right)\int_M^\infty\left(\sum_{n\geq 2}P(|X|>\psi(n)u)\right)du\\&\geq C\int_{\{|X|>\psi(3)M\}}\left(\int_M^\infty\left(\sum_{n\geq 3}I\{n<\varphi(\frac{|X|}{u})\}\right)du\right)dP\\&\geq C\int_{\{|X|>\psi(3)M\}}\left(\int_M^{|X|/\psi(3)}(\varphi(\frac{|X|}{u})-3)du\right)dP.\end{aligned} \tag{12}$$

By virtue of (7), it is not hard to check that

$$I\{|X| > \psi(3)M\} \int_M^{|X|/\psi(3)} (\varphi(\frac{|X|}{u})-3)du \geq CI\{|X| > \psi(3)M\}|X|(\log^+|X|)^{\eta+1}. \tag{13}$$

Combining (12) with (13) shows that (iii)$\Longrightarrow$(i). □

4. Strengthening the Hsu-Robbins-Erdős/Spitzer/Baum-Katz and Baum-Katz theorems

4.1. The following result was proved in Baum and Katz (1965)

Theorem A. *Let* $0 < p < 2$ *and* $r \geq 1 \vee p$*, and set* $f(\varepsilon) = \sum_{n\geq 1} n^{r/p-2} P(|S_n| > \varepsilon n^{1/p})$, $\varepsilon > 0$. *Then* $f(\varepsilon) < \infty$, $\varepsilon > 0$, $\Longleftrightarrow E|X|^r < \infty$ *and* $EX = 0$.

The result had been discovered earlier by Hsu and Robbins (1947) and Erdős (1949,1950) in case $p = 1$ and $r = 2$, and by Spitzer (1956) in case $p = r = 1$. It can be strengthened as follows.

Theorem 2. *Let* p, r *and* f *be as in Theorem A and assume* $r > 1$. *Then* $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty$, $\delta > 0$, $\Longleftrightarrow E|X|^r < \infty$ *and* $EX = 0$.

Proof. As stated above, we have to prove $\Longleftarrow$. Without loss of generality suppose $E|X|^r = 1$. We distinguish three cases.

Case $p < r \leq 2$. Applying Lemma 1.i with $q = r, x = \varepsilon n^{1/p}$ and $a = 2$, and then Lemma 3, we have

$$\begin{aligned} f(\varepsilon) &\leq \sum_{n\geq 1} n^{r/p-1} P(|X| > \varepsilon n^{1/p}/2) + C \sum_{n\geq 1} n^{r/p-2} n^2 \varepsilon^{-2r} n^{-2r/p} \\ &\leq C\varepsilon^{-r} + C\varepsilon^{-2r} \sum_{n\geq 1} n^{-r/p}, \quad \varepsilon > 0, \end{aligned}$$

whence $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty, \delta > 0$.

Case $r > 2$. Lemma 1.ii with $q = r, x = \varepsilon n^{1/p}$ and $a = r + 2$ yields

$$\begin{aligned} f(\varepsilon) &\leq \sum_{n\geq 1} n^{r/p-1} P(|X| > \varepsilon n^{1/p}/(r+2)) + C\varepsilon^{-2r} \sum_{n\geq 1} n^{-r/p} \\ &\quad + 2 \sum_{n\geq 1} n^{r/p-2} e^{-C\varepsilon^2 n^{r/p-1}}, \quad \varepsilon > 0. \end{aligned} \tag{14}$$

Now choose $s > \frac{r-p}{2-p}$. Then $s > 1$ and $\frac{r}{p} - 2 - (\frac{2}{p} - 1)s < -1$. Hence, on applying also Lemma 1.i, (14) shows that

$$\begin{aligned} f(\varepsilon) &\leq C\varepsilon^{-r} + C\varepsilon^{-2r} + C \sum_{n\geq 1} n^{r/p-2} \varepsilon^{-2s} n^{-(2/p-1)s} \\ &\leq C\varepsilon^{-r} + C\varepsilon^{-2r} + C\varepsilon^{-2s}, \quad \varepsilon > 0, \end{aligned}$$

and so $\int_{\delta}^{\infty} f(\varepsilon)d\varepsilon < \infty, \delta > 0$.

Case $p = r$. For $n \geq 1$, put $Y_{n,k} = X_k I\{|X_k| \leq \varepsilon n^{1/p}\}, 1 \leq k \leq n$, and $U_n = Y_{n,1} + \ldots + Y_{n,n}$. Since $EX = 0$,

$$\left|EY_{n,1}\right| \leq E[|X|I\{|X| > \varepsilon n^{1/p}\}] \leq (\varepsilon n^{1/p})^{1-p}, \quad n \geq 1.$$

Thus

$$\left|EY_{n,1}\right| \leq \varepsilon n^{1/p}/8, \quad \varepsilon > 0,\ n \geq 8\varepsilon^{-p}, \tag{15}$$

and

$$|EU_n| \leq \varepsilon n^{1/p}/2, \quad \varepsilon \geq 2^{-p}, n \geq 1.$$

Therefore, we have

$$\begin{aligned} P(|S_n| > \varepsilon n^{1/p}) &\leq nP(|X| > \varepsilon n^{1/p}) + P(|U_n| > \varepsilon n^{1/p}) \\ &\leq nP(|X| > \varepsilon n^{1/p}) \\ &\quad + P(|U_n - EU_n| > \varepsilon n^{1/p}/2), \quad \varepsilon \geq 2^{-p},\ n \geq 1. \end{aligned} \tag{16}$$

Further, by Lemma 1.i with $Y_1 = Y_{n,1} - EY_{n,1}$, $q = a = 2$ and $x = \varepsilon n^{1/p}/2$, and on account of (15), we obtain

$$\begin{aligned} P(|U_n - EU_n| > \varepsilon n^{1/p}/2) &\leq nP(\left|Y_{n,1} - EY_{n,1}\right| > \varepsilon n^{1/p}/4) \\ &\quad + Cn^2(EY_{n,1}^2)^2 \varepsilon^{-4} n^{-4/p} \\ &\leq nP(|X| > \varepsilon n^{1/p}/8) \\ &\quad + C\varepsilon^{-4} n^{2-4/p}(EY_{n,1}^2)^2,\ \varepsilon > 0,\ n \geq 8\varepsilon^{-p}. \end{aligned} \tag{17}$$

Combining (16) with (17), we get

$$\begin{aligned} P(|S_n| > \varepsilon n^{1/p}) &\leq 2nP(|X| > \varepsilon n^{1/p}/8) \\ &\quad + C\varepsilon^{-4} n^{2-4/p}(EY_{n,1}^2)^2, \quad \varepsilon \geq 2^{-p}, n \geq 8\varepsilon^{-p}. \end{aligned} \tag{18}$$

This, in connection with Lemma 3, shows that

$$\begin{aligned} f(\varepsilon) &\leq 8\varepsilon^{-p} + 2P(|X| > \varepsilon n^{1/p}/8) + C\varepsilon^{-4} \sum_{n \geq 1} n^{1-4/p}(EY_{n,1}^2)^2 \\ &\leq C\varepsilon^{-p} + C\varepsilon^{-4} \sum_{n \geq 1} n^{1-4/p}(E[X^2 I\{|X| \leq \varepsilon n^{1/p}\}])^2, \quad \varepsilon \geq 2^{-p}. \end{aligned} \tag{19}$$

Now let Y be a random variable which is independent of X and which has the same distribution as X. Then, on account of Fubini's theorem, we may write

$$\begin{aligned}
&\sum_{n\geq 1} n^{1-4/p}(E[X^2 I\{|X|\leq \varepsilon n^{1/p}\}])^2\\
&\quad = \sum_{n\geq 1} n^{1-4/p} E[X^2Y^2 I\{|X|\leq \varepsilon n^{1/p}\} I\{|Y|\leq \varepsilon n^{1/p}\}]\\
&\quad = E\left[X^2Y^2 I\{XY\neq 0\} \sum_{n\geq (|X|/\varepsilon)^p \vee (|Y|/\varepsilon)^p} n^{1-4/p}\right]\\
&\quad \leq CE[X^2Y^2 I\{XY\neq 0\}((|X|/\varepsilon)^p \vee (|Y|/\varepsilon)^p)^{2-4/p}]\\
&\quad \leq CE[X^2 I\{X\neq 0\}(|X|/\varepsilon)^{p-2} Y^2 I\{Y\neq 0\}(|Y|/\varepsilon)^{p-2}]\\
&\quad = C\varepsilon^{-2p+4}(E|X|^p)^2, \quad \varepsilon \geq 2^{-p}, \qquad (20)
\end{aligned}$$

where the last inequality comes from $a \vee b \geq \sqrt{ab}, a, b \geq 0$. From (19) and (20) we infer that $f(\varepsilon) \leq C\varepsilon^{-p} + C\varepsilon^{-2p}$, $\varepsilon \geq 2^{-p}$, which leads to $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty$, $\delta > 2^{-p}$. □

In some instances, even $\int_0^\infty f(\varepsilon)d\varepsilon$ is finite and computable.

Example 1. Let p, r and f be as in Theorem A, and assume that the distribution of X is stable with characteristic exponent γ, where either $r < \gamma < 2$ or $\gamma = 2$. Then

$$\int_0^\infty f(\varepsilon)d\varepsilon = \begin{cases} E|X|\sum_{n\geq 1} n^{r/p-2-1/p+1/\gamma} & \text{if } \frac{r}{p} - \frac{1}{p} + \frac{1}{\gamma} < 1 \\ \infty & \text{otherwise} \end{cases}.$$

Actually, since $S_n/n^{1/\gamma}$ has the same distribution as X,

$$\begin{aligned}
\int_0^\infty f(\varepsilon)d\varepsilon &= \sum_{n\geq 1} n^{r/p-2}\int_0^\infty P(|S_n| > \varepsilon n^{1/p})d\varepsilon\\
&= \sum_{n\geq 1} n^{r/p-2}\int_0^\infty P(|X| > \varepsilon n^{1/p-1/\gamma})d\varepsilon\\
&= E|X|\sum_{n\geq 1} n^{r/p-2-1/p+1/\gamma}.
\end{aligned}$$

4.2. The next result was also proved by Baum and Katz (1965)

Theorem B. *Let* $1 \leq p < 2$, *and put* $f(\varepsilon) = \sum_{n\geq 2}\frac{\log n}{n} P(|S_n| > \varepsilon n^{1/p})$, $\varepsilon > 0$. *Then* $f(\varepsilon) < \infty, \varepsilon > 0, \Longleftrightarrow E[|X|^p \log^+ |X|] < \infty$ *and* $EX = 0$.

We obtained the following strengthening.

Theorem 3. *Suppose* $1 < p < 2$, *and let* f *be as in Theorem B. Then* $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty$, $\delta > 0$, $\Longleftrightarrow E[|X|^p \log^+ |X|] < \infty$ *and* $EX = 0$.

Proof. We may and do assume $E|X|^p = 1$. For $n \geq 1$, let $Y_{n,k},\ 1 \leq k \leq n$, be as in the proof of Theorem 2, and set $\varepsilon_0 = 8e^{1\vee 1/p}$. Then (18) still holds, and so

$$f(\varepsilon) \leq 8\varepsilon^{-p} + 2\sum_{n\geq 2} \log n P(|X| > \varepsilon n^{1/p}/8)$$
$$+C\varepsilon^{-4}\sum_{n\geq 2}\frac{\log n}{n^{4/p-1}}(EY_{n,1}^2)^2,\quad \varepsilon \geq 2^{-p}.$$

On applying Lemma 4, this leads to

$$f(\varepsilon) \leq C\varepsilon^{-p}$$
$$+C\varepsilon^{-4}\sum_{n\geq 2}\frac{\log n}{n^{4/p-1}}E[X^2Y^2I\{|X| \leq \varepsilon n^{1/p}\}I\{|Y| \leq \varepsilon n^{1/p}\}],\quad \varepsilon > \varepsilon_0,$$

where Y is an independent copy of X. Hence, by Fubini's theorem, as $\sum_{n\geq m} n^{-t}\log n \leq Cm^{-t+1}\log m,\ t > 1$, we have

$$\begin{aligned}
f(\varepsilon) &\leq C\varepsilon^{-p} + C\varepsilon^{-4}E\left[X^2Y^2I\{XY \neq 0\}\sum_{n\geq(|X|/\varepsilon)^p\vee(|Y|/\varepsilon)^p}\frac{\log n}{n^{4/p-1}}\right]\\
&\leq C\varepsilon^{-p} + C\varepsilon^{-4}E\left[X^2Y^2I\{XY \neq 0\}\frac{\log((|X|/\varepsilon)^p \vee (|Y|/\varepsilon)^p)}{((|X|/\varepsilon)^p \vee (|Y|/\varepsilon)^p)^{4/p-2}}\right]\\
&\leq C\varepsilon^{-p} + C\varepsilon^{-4}\left(E\left[X^2Y^2\frac{\log^+(|X|/\varepsilon)^p}{(|X|/\varepsilon)^{2-p}(|Y|/\varepsilon)^{2-p}}\right]\right.\\
&\quad\left.+E\left[X^2Y^2\frac{\log^+(|Y|/\varepsilon)^p}{(|X|/\varepsilon)^{2-p}(|Y|/\varepsilon)^{2-p}}\right]\right)\\
&\leq C\varepsilon^{-p} + C\varepsilon^{-2p}(E[|X|^p\log^+|X|]E|Y|^p\\
&\quad+E[|Y|^p\log^+|Y|]E|X|^p),\quad \varepsilon > \varepsilon_0,
\end{aligned}\tag{21}$$

where the last but one inequality follows from $\log(a \vee b) \leq \log^+ a + \log^+ b$ and $a \vee b \geq \sqrt{ab},\ a, b > 0$. (21) shows that $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty, \delta > \varepsilon_0$. □

Example 2. Let p and f be as in Theorem B, and assume that the distribution of X is stable with characteristic exponent $\gamma \geq p$. Then

$$\int_0^\infty f(\varepsilon)d\varepsilon = \begin{cases} E|X|\sum_{n\geq 2}(\log n)n^{-1-1/p+1/\gamma} & \text{if } \gamma > p\\ \infty & \text{if } \gamma = p\end{cases}.$$

4.3 The remaining case $p = r = 1$, corresponding to the Spitzer theorem, is not covered by Theorem 2. Neither is the case $p = 1$ in Theorem 3. As the next result shows, the assumption $E|X| < \infty$ and $EX = 0$, respectively $E[|X| \log^+ |X|] < \infty$ and $EX = 0$, is no longer sufficient for the integrability of f in these exceptional cases. Actually, it turns out the necessary and sufficient condition is to be augmented to $E[|X| \log^+ |X|] < \infty$ and $EX = 0$, respectively $E[|X|(\log^+ |X|)^2] < \infty$ and $EX = 0$.

Theorem 4. *For $\varepsilon > 0$, define the stopping times $T_\pm(\varepsilon) = \inf\{n : 1 \le n \le \infty, \pm S_n > \varepsilon n\}$. Then, for $0 \le \eta \le 1$, the following are equivalent:*

(i) $E[|X|(\log^+ |X|)^{\eta+1}] < \infty$ *and* $EX = 0$;
(ii) $E[\sup_{n\ge1} |S_n|(\log n)^\eta/n] < \infty$ *and* $EX = 0$;
(iii) $E[S_{T_+(\varepsilon)}(\log T_+(\varepsilon))^\eta/T_+(\varepsilon) : T_+(\varepsilon) < \infty] < \infty$ *and* $E[-S_{T_-(\varepsilon)} (\log T_-(\varepsilon))^\eta/T_-(\varepsilon) : T_-(\varepsilon) < \infty] < \infty$ *for any* $\varepsilon > 0$, *and* $EX = 0$;
(iv) $\int_\delta^\infty \left(\sum_{n\ge1} \frac{(\log n)^\eta}{n} P(|S_n| > \varepsilon n) \right) d\varepsilon < \infty$ *for any* $\delta > 0$.

Remark 3. For $1 \le p < 2$ and $0 \le \eta \le 1$, one can convince oneself that $\sum_{n\ge1} \frac{(\log n)^\eta}{n} P(|S_n| > \varepsilon n^{1/p}) < \infty, \varepsilon > 0, \Longleftrightarrow E[|X|^p(\log^+ |X|)^\eta] < \infty$ and $EX = 0$.

Proof. The equivalence of (i) and (ii) is established in Theorem 1. Clearly, (ii)$\Longrightarrow$(iii). Assume that (iii) holds. For $\delta > 0$, set $X_n^* = X_n - \delta$, $S_n^* = X_1^* + \ldots + X_n^*$, $n \ge 1$, and put $T = \inf\{n : 1 \le n \le \infty, S_n^* > 0\} = T_+(\delta)$. Then

$$E[S_T^*(\log T)^\eta/T : T < \infty] < \infty. \tag{22}$$

Since $EX_1^* = -\delta < 0$, we have (see, e.g. Loéve (1977), pp. 396, 399)

$$\exp\left(-\sum_{n\ge1} \frac{1}{n} P(S_n^* > 0) \right) = P(T = \infty) > 0, \tag{23}$$

and so

$$\sum_{n\ge1} \frac{1}{n} P(S_n^* > 0) < \infty. \tag{24}$$

Now, for $0 < t < 1$ and $u > 0$, consider the exponential identity (see, e.g. Loéve (1977), p. 395)

$$\exp\left(-\sum_{n\ge1} \frac{t^n}{n} E[e^{-uS_n^*} : S_n^* > 0] \right) = 1 - E[t^T e^{-uS_T^*}],$$

that is

$$\sum_{n\ge1} \frac{t^n}{n} E[e^{-uS_n^*} : S_n^* > 0] = -\log(1 - E[t^T e^{-uS_T^*} : T < \infty]). \tag{25}$$

For $u \geq u_0 > 0$, in view of (24), we have

$$\sum_{n\geq 1} \left| \frac{t^n}{n} \frac{d}{du} (E[e^{-uS_n^*} : S_n^* > 0]) \right| \leq \frac{1}{eu_0} \sum_{n\geq 1} \frac{1}{n} P(S_n^* > 0) < \infty.$$

Thus we may differentiate term by term the series in (25) to get

$$\sum_{n\geq 1} \frac{t^n}{n} E[S_n^* e^{-uS_n^*} : S_n^* > 0] = \frac{E[t^T S_T^* e^{-uS_T^*} : T < \infty]}{1 - E[t^T e^{-uS_T^*} : T < \infty]}, \quad u > 0. \tag{26}$$

Next, on account of the monotone convergence theorem, by letting $u \to 0$ in (26), we obtain

$$\sum_{n\geq 1} \frac{t^n}{n} E[S_n^* : S_n^* > 0] = \frac{E[t^T S_T^* : T < \infty]}{1 - E[t^T : T < \infty]}.$$

Hence, on applying Lemma 7 with $a_n = E[S_n^* : S_n^* > 0]/n$, $b_n = P(T = n)$, $c_n = E[S_n^* : T = n]$, $n \geq 1$, and $d_1 = d_2 = d_3$, $d_n = (\log n)^\eta / n$, $n \geq 3$, we see that

$$\sum_{n\geq 1} \frac{t^n d_n}{n} E[S_n^* : S_n^* > 0] \leq \frac{E[t^T d_T S_T^* : T < \infty]}{1 - E[t^T : T < \infty]}. \tag{27}$$

Letting $t \to 1$ in (27), and taking into account (22) and (23), yields

$$\sum_{n\geq 1} \frac{d_n}{n} E[S_n^* : S_n^* > 0] \leq \frac{E[d_T S_T^* : T < \infty]}{P(T = \infty)} < \infty. \tag{28}$$

For $n \geq 1$, we have

$$\int_\delta^\infty P(S_n > \varepsilon n) d\varepsilon = E[\frac{S_n}{n} - \delta : \frac{S_n}{n} - \delta > 0] = \frac{1}{n} E[S_n^* : S_n^* > 0].$$

Therefore, (28) entails that

$$\int_\delta^\infty \left(\sum_{n\geq 1} d_n P(S_n > \varepsilon n) \right) d\varepsilon < \infty. \tag{29}$$

(29) in conjunction with the inequality derived from it on replacing X_n by $-X_n$ show that (iv) is satisfied. Finally, assume that (iv) holds. Then, by Remark 3, $E|X| < \infty$ and $EX = 0$, and so

$$\lim_{n\to\infty} n P(|X| > n) = 0. \tag{30}$$

Suppose first that X is symmetric. On account of Proposition 2.3 and Lemma 2.6 in Ledoux and Talagrand (1991), and by (30),

$$\begin{aligned} 2P(|S_n| > \varepsilon n) &\geq P(\max_{1\leq i\leq n} |X_i| > \varepsilon n) \\ &\geq \frac{nP(|X| > \varepsilon n)}{1 + nP(|X| > \varepsilon n)} \\ &\geq \frac{nP(|X| > \varepsilon n)}{1 + nP(|X| > n)} \\ &\geq CnP(|X| > \varepsilon n), \quad \varepsilon, n \geq 1. \end{aligned}$$

Therefore, we may write

$$\begin{aligned} \infty > \int_1^\infty \left(\sum_{n\geq 1} \frac{(\log n)^\eta}{n} P(|S_n| > \varepsilon n) \right) d\varepsilon &\geq C \int_1^\infty \left(\sum_{n\geq 1} (\log n)^\eta P(|X| > \varepsilon n) \right) d\varepsilon \\ = C \sum_{n\geq 1} \frac{(\log n)^\eta}{n} \int_n^\infty P(|X| > \varepsilon) d\varepsilon &\geq C \sum_{n\geq 1} \frac{(\log n)^\eta}{n} \sum_{k\geq n} P(|X| > k+1) \\ = C \sum_{k\geq 1} P(|X| > k+1) \sum_{n=1}^{k} \frac{(\log n)^\eta}{n} &\geq C \sum_{k\geq 1} (\log k)^{\eta+1} P(|X| > k+1), \quad (31) \end{aligned}$$

since $\sum_{n=1}^{k} \frac{(\log n)^\eta}{n} \sim \frac{(\log k)^{\eta+1}}{\eta+1}$ as $k \to \infty$. (31) shows that (i) is satisfied under the symmetry assumption. Considering now the general case, let α, α_1, $\alpha_2, \ldots$ be a sequence of i.i.d. Rademacher variables, independent of X, X_1, $X_2, \ldots$, with $P(\alpha = 1) = P(\alpha = -1) = 1/2$. Put $X' = \alpha X$, $X'_n = \alpha_n X_n$, $S'_n = X'_1 + \ldots + X'_n$, $n \geq 1$. It follows from Theorem 1 in Pruss (1997) that there is an absolute constant $C > 0$ such that, for all $t > 0$ and $n \geq 1$,

$$P(\left|S'_n\right| > t) \leq CP(|S_n| > t/C). \quad (32)$$

On account of (iv) and (32), it then follows that (iv) holds with S'_n in place of S_n. Hence, by the first part of the proof, $E[|X|(\log^+ |X|)^{\eta+1}] = E[\left|X'\right| (\log^+ \left|X'\right|)^{\eta+1}] < \infty$. Thus (iv)$\Longrightarrow$ (i). □

Remark 4. For $\eta = 0$, the following alternate proof of the implication (iv)$\Longrightarrow$(i) appears instructive.

In view of (23), as $1 - e^{-x} \leq x, \ x \geq 0$, we have

$$\begin{aligned} P(T_\pm(\varepsilon) < \infty) &= 1 - \exp\left(- \sum_{n\geq 1} \frac{1}{n} P(\pm S_n > \varepsilon n) \right) \\ &\leq \sum_{n\geq 1} \frac{1}{n} P(\pm S_n > \varepsilon n), \quad \varepsilon > 0. \end{aligned}$$

Consequently, for $\delta > 0$,

$$\begin{aligned}
E[\sup_{n\geq 1}|S_n|/n] &\leq \delta + \int_\delta^\infty P(\sup_{n\geq 1}|S_n|/n > \varepsilon)d\varepsilon \\
&= \delta + \int_\delta^\infty P(T_+(\varepsilon) \wedge T_-(\varepsilon) < \infty)d\varepsilon \\
&\leq \delta + \int_\delta^\infty (P(T_+(\varepsilon) < \infty) + P(T_-(\varepsilon) < \infty))d\varepsilon \\
&\leq \delta + \int_\delta^\infty \left(\sum_{n\geq 1} \frac{1}{n} P(|S_n| > \varepsilon n)\right) d\varepsilon < \infty.
\end{aligned}$$

5. Strengthening a theorem involving moderate deviations

The following result is due to Lai (1974).

Theorem C. *Let $p > 1$, and set $f(\varepsilon) = \sum_{n\geq 2} n^{p-2} P(|S_n| > \varepsilon\sqrt{n \log n}),\ \varepsilon > 0$. If $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty,\ EX^2 = \sigma^2$ and $EX = 0$, then $f(\varepsilon) < \infty$ for any $\varepsilon > \sigma\sqrt{2p-2}$. Conversely, if $f(\varepsilon) < \infty$ for some ε, then $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$ and $EX = 0$.*

We obtained the next strengthening.

Theorem 5. *Let p and f be as in Theorem C. If $E[|X|^{2p}(\log^+|X|)^{-p}] < \infty$, $EX^2 = \sigma^2$ and $EX = 0$, then*

$$\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty,\ \delta > \sigma\sqrt{2p-2}. \tag{33}$$

Proof. Assume $EX^2 = 1$ and let ε_0 be as in Lemma 5. Applying Lemma 1.ii with $q = p+1,\ x = \varepsilon\sqrt{n \log n}$ and $a = 2(p+3)/(p+1)$ yields

$$\begin{aligned}
P(|S_n| > \varepsilon\sqrt{n\log n}) &\leq nP(|X| > \varepsilon\sqrt{n\log n}(p+1)/2(p+3)) \\
&\quad + C\varepsilon^{-2(p+1)} n^{1-p}(\log n)^{-p-1} + 2n^{-C\varepsilon^2},\quad \varepsilon > 0,\ n \geq 2.
\end{aligned}$$

Therefore, in view of Lemma 5,

$$\begin{aligned}
f(\varepsilon) &\leq \sum_{n\geq 2} n^{p-1} P(|X| > \varepsilon\sqrt{n\log n}/(p+3)) + C\varepsilon^{-2(p+1)} \sum_{n\geq 2} n^{-1}(\log n)^{-p-1} \\
&\quad + 2\sum_{n\geq 2} n^{p-2-C\varepsilon^2} \\
&\leq C\varepsilon^{-2p}(\log\varepsilon)^p + C\varepsilon^{-2(p+1)} + C(C\varepsilon^2 - p + 1)^{-1},\quad \varepsilon > \varepsilon_0 \vee C\sqrt{p-1}.
\end{aligned}$$

This shows that (33) holds. □

Remark 5. Davis (1968b) proved that $f(\varepsilon) = \sum_{n\geq 2} \frac{\log n}{n} P(|S_n| > \varepsilon\sqrt{n\log n}) < \infty, \varepsilon > 0, \Longleftrightarrow EX^2 < \infty$ and $EX = 0$. In Gafurov and Siraždinov (1979, p. 283) the estimate $f(\varepsilon) \leq C\varepsilon^{-6} + C\varepsilon^{-2}$, $\varepsilon > 0$, is obtained. Thus in this case also $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty,\ \delta > 0, \Longleftrightarrow EX^2 < \infty$ and $EX = 0$.

6. Strengthening a result concerning the LIL

The first part of the next result follows from Theorem 4 in Davis (1968a), and the second part is due to Gut (1980).

Theorem D. *Let* $f(\varepsilon) = \sum_{n\geq e^e} \frac{1}{n} P(|S_n| > \varepsilon\sqrt{n\log\log n})$, $\varepsilon > 0$. *If* $EX^2 = \sigma^2 < \infty$ *and* $EX = 0$, *then* $f(\varepsilon) < \infty$ *for any* $\varepsilon > \sigma\sqrt{2}$. *Conversely, if* $f(\varepsilon) < \infty$ *for some* ε, *then* $EX^2 < \infty$ *and* $EX = 0$.

The following strengthening holds.

Theorem 6. *Let* f *be as in Theorem D. If* $EX^2 = \sigma^2 < \infty$ *and* $EX = 0$, *then* $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty, \delta > \sigma\sqrt{2}$.

Proof. Assume $EX^2 = 1$, define $\psi(x) = \sqrt{x\log\log x}$, $x \geq e$, and let $\varphi(x)$, $x \geq 0$, denote the inverse function of ψ. Notice that

$$x^2 = \varphi(x)\log\log\varphi(x), \quad x \geq 0. \tag{34}$$

For $\varepsilon > 0$ and $n \geq e^e$, set $Y_{n,k} = X_k I\{|X_k| \leq \varepsilon\psi(n)\}$, $1 \leq k \leq n$, $U_n = Y_{n,1} + \ldots + Y_{n,n}$, $\sigma_n(\varepsilon)^2 = EY_{n,1}^2 - (EY_{n,1})^2$ and $\rho_n(\varepsilon) = \sigma_n(\varepsilon)^{-3}E\left|Y_{n,1} - EY_{n,1}\right|^3$. As $EX = 0$ and $EX^2 = 1$, we see that

$$\left|EY_{n,1}\right| \leq \frac{1}{\varepsilon\psi(n)}, \quad \varepsilon > 0,\ n \geq e^e, \tag{35}$$

and so $|EU_n| \leq \varepsilon\psi(n)/2$, $\varepsilon \leq \sqrt{2}$, $n \geq e^e$. Consequently, we have

$$\begin{aligned} P(|S_n| > \varepsilon\psi(n)) &\leq P(S_n \neq U_n) + P(|U_n| > \varepsilon\psi(n)) \\ &\leq P(S_n \neq U_n) + P(|U_n - EU_n| > \varepsilon\psi(n)/2) \\ &\leq P(S_n \neq U_n) \\ &\quad + \left|P(|U_n - EU_n| > \varepsilon\psi(n)/2) - 2\Phi\left(-\frac{\varepsilon\psi(n)}{2\sigma_n(\varepsilon)\sqrt{n}}\right)\right| \\ &\quad + 2\Phi\left(-\frac{\varepsilon\psi(n)}{2\sigma_n(\varepsilon)\sqrt{n}}\right), \quad \varepsilon \geq \sqrt{2}, n \geq e^e. \end{aligned} \tag{36}$$

Now, on account of Lemma 6,

$$\sum_{n\geq e^e} \frac{1}{n} P(S_n \neq U_n) \leq \sum_{n\geq e^e} P(|X| > \varepsilon\psi(n)) \leq \varepsilon^{-2}, \quad \varepsilon > 0. \tag{37}$$

Also, in view of Nagaev's inequality (see, e.g. Petrov (1975), p. 125) and (36),

$$\begin{aligned}
&\left|P(|U_n - EU_n| > \varepsilon\psi(n)/2) - 2\Phi\left(-\frac{\varepsilon\psi(n)}{2\sigma_n(\varepsilon)\sqrt{n}}\right)\right| \\
&\quad \le C\frac{\rho_n(\varepsilon)}{\sqrt{n}}\left(\frac{2\sigma_n(\varepsilon)\sqrt{n}}{\varepsilon\psi(n)}\right)^3 \\
&\quad = Cn\frac{E\left|Y_{n,1} - EY_{n,1}\right|^3}{\varepsilon^3\psi(n)^3} \\
&\quad \le Cn\frac{E\left|Y_{n,1}\right|^3}{\varepsilon^3\psi(n)^3} + Cn\frac{\left|EY_{n,1}\right|^3}{\varepsilon^3\psi(n)^3} \\
&\quad \le Cn\frac{E\left|Y_{n,1}\right|^3}{\varepsilon^3\psi(n)^3} + \frac{C}{\varepsilon^6}\frac{1}{n^2}, \quad \varepsilon \ge \sqrt{2},\ n \ge e^e.
\end{aligned}$$

Therefore, by Fubini's theorem and (34), as

$$\sum_{n\ge m}\psi(n)^{-3} \le\ Cm^{-1/2}(\log\log m)^{-3/2}, m \ge e^e,$$

we get

$$\begin{aligned}
&\sum_{n\ge e^e}\frac{1}{n}\left|P(|U_n - EU_n| > \varepsilon\psi(n)/2) - 2\Phi\left(-\frac{\varepsilon\psi(n)}{2\sigma_n(\varepsilon)\sqrt{n}}\right)\right| \\
&\quad \le C\sum_{n\ge e^e}\frac{E[|X|^3 I\{|X| \le \varepsilon\psi(n)\}]}{\varepsilon^3\psi(n)^3} + \frac{C}{\varepsilon^6} \\
&\quad \le CE\left[\left(\frac{|X|}{\varepsilon}\right)^3\sum_{n\ge e^e\vee\varphi(|X|/\varepsilon)}\frac{1}{\psi(n)^3}\right] + C\varepsilon^{-6} \\
&\quad \le CE\left[\left(\frac{|X|}{\varepsilon}\right)^3\frac{1}{(e^e\vee\varphi(|X|/\varepsilon))^{1/2}(\log\log(e^e\vee\varphi(|X|/\varepsilon)))^{3/2}}\right] + C\varepsilon^{-6} \\
&\quad = CE\left[\frac{X^2}{\varepsilon^2}\frac{(\varphi(|X|/\varepsilon)\log\log\varphi(|X|/\varepsilon))^{1/2}}{(e^e\vee\varphi(|X|/\varepsilon))^{1/2}(\log\log(e^e\vee\varphi(|X|/\varepsilon)))^{3/2}}\right] + C\varepsilon^{-6} \\
&\quad \le C\varepsilon^{-2} + C\varepsilon^{-6}, \quad \varepsilon \ge \sqrt{2}.
\end{aligned} \tag{38}$$

As for the last term in (36), since $\sigma_n(\varepsilon) \le 1$ and

$$\Phi(-x) \sim \frac{1}{\sqrt{2\pi}}\cdot\frac{e^{-x^2/2}}{x} \text{ as } x \to \infty,$$

we obtain

$$\sum_{n\ge e^e} \frac{1}{n}\Phi\left(-\frac{\varepsilon\psi(n)}{2\sigma_n(\varepsilon)\sqrt{n}}\right) \le C\sum_{n\ge e^e}\frac{2}{n}\frac{\sigma_n(\varepsilon)\sqrt{n}}{\varepsilon\psi(n)}\exp\left(-\frac{\varepsilon^2\psi(n)^2}{8\sigma_n(\varepsilon)^2 n}\right)$$
$$\le \frac{C}{\varepsilon}\sum_{n\ge e^e}\frac{1}{n}\exp\left(-\frac{\varepsilon^2\log\log n}{8}\right)$$
$$= \frac{C}{\varepsilon}\sum_{n\ge e^e}\frac{1}{n(\log n)^{\varepsilon^2/8}}$$
$$\le \frac{C}{\varepsilon(\varepsilon^2-8)},\quad \varepsilon > 2\sqrt{2}. \tag{39}$$

Finally, (36)–(39) show that $\int_\delta^\infty f(\varepsilon)d\varepsilon < \infty$, $\delta > 2\sqrt{2}$. □

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