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Alice Guionnet · Mylène Maïda

Character expansion method for the first order asymptotics of a matrix integral

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Abstract. The estimation of various matrix integrals as the size of the matrices go to infinity is motivated by theoretical physics, geometry and free probability questions. On a rigorous ground, only integrals of one matrix or of several matrices with simple quadratic interaction (called AB interaction) could be evaluated so far (see e.g. [19], [17] or [9]). In this article, we follow an idea widely developed in the physics literature, which is based on character expansion, to study more complex interaction. In this context, we derive a large deviation principle for the empirical measure of Young tableaux. We then use it to study a matrix model defined in the spirit of the 'dually weighted graph model' introduced in [13], but with a cutoff function such that the matrix integral and its character expansion converge. We prove that the free energy of this model converges as the size of the matrices goes to infinity and study the critical points of the limit.

1. Introduction

The evaluation of matrix integrals was first motivated by theoretical physics and geometry since they can be related, via Feynman diagrams expansion (see [28] for a nice introduction), to the enumeration of maps. Thanks to this relation, matrix integrals can also be used to describe some models appearing in statistical mechanics, such as the Ising model or the q-Potts model, on random graphs (instead of the usual two-dimensional lattice). Using similar ideas, string theory models can be described via matrix integrals around criticality (see the course [7] for various applications to physics). Another motivation is the study of non-commutative entropies introduced by D. Voiculescu [23] in the context of free probability. Let us roughly say that the understanding of the asymptotic behaviour of all possible matrix integrals would be equivalent to the understanding of the so-called microstates entropy.

So, what is a matrix integral ? If we let, for $n \in \mathbb{N}$, $\mathbb{C}\langle X_1, \cdots, X_n\rangle$ be the set of polynomial functions of n non-commutative variables and if we choose, for some

A. Guionnet: Ecole Normale Supérieure de Lyon, Unité de Mathématiques pures et appliquées, UMR 5669, 46 Allée d'Italie, 69364 Lyon Cedex 07, France.
e-mail: aguionne@umpa.ens-lyon.fr

M. Maïda: *Current address:* Laboratoire Modal-X, Université Paris X-Nanterre, 200 av. de la république, 92001 Nanterre Cedex, France. e-mail: mmaida@umpa.ens-lyon.fr

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$m, p \in \mathbb{N}$, $P \in \mathbb{C}\langle X_1, \cdots, X_{n+p}\rangle^{\otimes m}$ and $\phi := (\phi_i)_{1\le i\le n+p} \in \mathcal{C}^o(\mathbb{R})^{n+p}$, then a matrix integral can be defined by

$$Z_N(P, \phi) = \int e^{N^2(N^{-1}\mathrm{tr})^{\otimes m}(P(\phi_1(A_1),\cdots,\phi_{n+p}(A_{n+p}))}dA_1\cdots dA_n,$$

where dA denotes the Lebesgue measure on the chosen state space of the matrices, included into $\mathcal{M}_N(\mathbb{C})$, the space of square matrices of dimension N with complex entries. In the following, the matrices will take their values in the set $\mathcal{H}_N(\mathbb{C})$ of Hermitian matrices of dimension N. The first order asymptotics of $Z_N(P, \phi)$ can easily be studied in the case where $n = 1$, when P is chosen going to infinity fast enough to insure existence of $Z_N(P, \phi)$, since then the joint distribution of the eigenvalues of the matrix A is known and described by the Coulomb gas law (see [1] for instance). In this setting, all the correction terms have been recently studied rigorously by N. Ercolani and K. McLaughlin in [6]. To this end, they use Riemann-Hilbert techniques together with a good understanding of the asymptotic behaviour of the spectral measure of the matrix with distribution given by the corresponding Gibbs measure

$$d\mu_N^{P,\phi}(A_1,\ldots,A_n) = \frac{e^{N^2(N^{-1}\mathrm{tr})^{\otimes m}(P(\phi_1(A_1),\cdots,\phi_{n+p}(A_{n+p})))}}{Z_N(P,\phi)}dA_1\cdots dA_n.$$

There are much less complete results in the case where $n \geq 2$. On a rigorous ground, let us however mention the work of M. Mehta and al. (see e.g. [19] and [17]) who considered symmetric models with AB interaction including the so-called Ising model or matrices coupled in chain model, i.e $m = 1$, $p = 0$ and

$$P(A_1, \cdots, A_n) = \sum_{i=1}^{n} P(A_i) + \sum_{i=1}^{n-1} A_i A_{i+1}.$$

By orthogonal polynomial techniques, they could find the asymptotic behaviour of the associated free energy when integration holds over Hermitian matrices. By using completely different techniques based on large deviations, similar asymptotics could be derived in [10] and [9] for AB interaction models where the symmetry between the matrices can be broken (i.e. we can choose $P(A_1, \cdots, A_n) = \sum_{i=1}^{n} P_i(A_i)+\sum_{i=1}^{n-1} A_i A_{i+1}$, possibly with different P_i's) and integration can also hold over the orthogonal ensemble. These techniques have moreover the advantage to allow the description of the asymptotic behaviour of the spectral measures of the matrices $(A_1, \cdots, A_n)$ with distribution μ_N^P, key step to try to obtain the full expansion of $Z_N(P)$.

On a less rigorous ground, a few other models have been studied. The main idea to study most of them is based on character expansion, a technique which was introduced by A. Migdal in [20] and by C. Itzykson and J.-B. Zuber in their famous article on planar approximation [12], and then widely developed in the 90's by various physicists (see for example [5], [15] for the so-called $ABAB$ model or refer to [13] for a review). This technique allows to express the involved matrix integrals in terms basically of a sum over characters which are simpler to deal with because the interaction is reduced to spherical integrals, whose asymptotics are described in [10]. However, this sum is in general an infinite signed series (which actually might diverge), point which is not addressed for instance in [13]. A formal expansion was also obtained by B. Collins in [3] in a very general setting. He could obtain a formula for the free energy of matrix integrals as formal series and study the convergence of each terms of these series. However, he could not prove that the series in fact converge.

In the present article, we show how the idea of character expansion can be used to estimate rigorously the specific matrix integral in which, A_N and B_N being two $N \times N$ given Hermitian matrices, the partition function is

$$
\begin{aligned}
Z_N(\Phi) &\equiv \int dM e^{-\frac{N}{2}\operatorname{tr} M^2 - \operatorname{tr}\otimes\operatorname{tr}\log(I\otimes I - B_N\otimes\Phi(M)A_N)}, \\
&= \int dM e^{-\frac{N}{2}\operatorname{tr} M^2 + \sum_{k\geq 1} k^{-1}\operatorname{tr}(B_N^k)\operatorname{tr}((\Phi(M)A_N)^k)} \qquad (1)
\end{aligned}
$$

with the following notations :

- dM is the Lebesgue measure over the set $\mathcal{H}_N(\mathbb{C})$ of Hermitian matrices of size N,
- tr is the usual trace on $\mathcal{M}_N(\mathbb{C})$ and I is the identity in $\mathcal{M}_N(\mathbb{C})$,
- Φ is a continuous function from $\mathbb{R}$ into $\mathbb{R}$. $\Phi(M)$ is then uniquely defined by $\Phi(M) = U\operatorname{diag}(\Phi(\lambda_1), \cdots, \Phi(\lambda_N))U^*$ when $M = U\operatorname{diag}(\lambda_1, \cdots, \lambda_N)U^*$ for some $U \in \mathcal{U}_N(\mathbb{C})$.

This model was studied in the case where $\Phi(x) = x$ in [14] where it was called the “dually weighted graphs model", because $N^{-2}\log Z_N(x)$ is a generating function for planar maps (that is oriented connected graphs drawn on the sphere modulo equivalent classes) having arbitrary coordination dependent weights for both vertices and faces in the large N limit. Note that in fact, in the case where $\Phi(x) = x$, the expansion is diverging (see [14], (2.7)). In this work, we shall restrict ourselves to functions Φ satisfying appropriate boundness conditions to insure that the partition function $Z_N(\Phi)$ and its character expansion are well defined. We discuss in section 6 the relation between our result, [14] and the enumeration of maps. Our main results can be sketched as follows

Theorem 1. *1. Under Hypotheses 2 and 3,*

$$F_N(\Phi) = \frac{1}{N^2}\log Z_N(\Phi)$$

converges as N goes to infinity and a formula is derived (see Theorem 5 for details).

2. Under the hypotheses of Proposition 1, we can give a weak characterization of the limit points of the spectral measure of M under the Gibbs measure associated to $Z_N(\Phi)$ (see Proposition 1)

The main advantage of this model is that its character expansion is not signed (i.e is a sum of non-negative terms), allowing standard Laplace method techniques.

But let us explain what we mean by "character expansion", i.e. expansion in terms of Schur polynomials. For that, we recall the following notions (see for example section 4.4. of the book [22] for more details):

Definition-Notation 1.

- *a Young shape λ is a finite sequence of non negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ written in non-increasing order. One should think of it as a diagram whose ith line is made of λ_i empty boxes. We denote by $|\lambda| = \sum_i \lambda_i$ the total number of boxes of the shape λ.*
  In the sequel, when we have a shape $\lambda = (\lambda_1, \lambda_2, \ldots)$ and an integer N greater than the number of lines of λ having a strictly positive length, we will define a sequence ℓ associated to λ and N, which is an N-uple of integers $\ell_i = \lambda_i + N - i$. In particular we have that $\ell_1 > \ell_2 > \ldots > \ell_N \geqslant 0$ and $\ell_i - \ell_{i+1} \geq 1$.
- *for some fixed $N \in \mathbb{N}$, a Young tableau will be any filling of the Young shape above with integers from 1 to N which is non-decreasing on each line and (strictly) increasing on each column. For each such filling, we define the content of a Young tableau as the N-uple $(\mu_1, \ldots, \mu_N)$ where μ_i is the number of i's written in the tableau.*
  Notice that, for $N \in \mathbb{N}$, a Young shape can be filled with integers from 1 to N if and only if $\lambda_i = 0$ for $i > N$.
- *for a Young shape λ and an integer N, the Schur polynomial s_λ is an element of $\mathbb{C}\langle x_1, \ldots, x_N\rangle$ defined by*

$$s_\lambda(x_1, \ldots, x_N) = \sum_T x_1^{\mu_1} \ldots x_N^{\mu_N}, \tag{2}$$

  where the sum is taken over all Young tableaux T of fixed shape λ and $(\mu_1, \ldots, \mu_N)$ is the content of T. Note that s_λ is positive whenever the x_i's are and, although it is not obvious from this definition (cf for example [22] for a proof), s_λ is a symmetric function of the x_i's.

If A is a matrix in $\mathcal{M}_N(\mathbb{C})$, then define $s_\lambda(A) \equiv s_\lambda(A_1, \ldots, A_N)$, where the A_i's are the eigenvalues of A.

Now the point is that we shall see in Theorem 3, whose derivation is the object of section 2, that we can write $Z_N(\Phi)$ as

$$Z_N(\Phi) = c_N \sum_\lambda s_\lambda(A_N) s_\lambda(B_N) Z_N(\Phi, \lambda)$$

where the sum runs over Young tableaux $\lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_N)$ and $Z_N(\Phi, \lambda)$ is a positive function of the shape λ which depends 'almost continuously' on the empirical measure

$$\hat{\mu}_\lambda^N := \frac{1}{N} \sum_{i=1}^N \delta_{\frac{\lambda_i + N - i}{N}} \in \mathcal{P}(\mathbb{R}^+)$$

where $\mathcal{P}(\mathbb{R}^+)$ denotes the set of probability measures on $\mathbb{R}^+$. Therefore, to study the asymptotic behaviour of $Z_N(\Phi)$ we are lead to estimate the deviations of more general measures Π_N which shall depend on a sequence $(F, c, (A_N, B_N)_{N\geq 0}, a, b)$ satisfying

Hypothesis 1. *1. F is a bounded continuous function from $\mathcal{P}(\mathbb{R}^+)$ equipped with its usual weak topology into $\mathbb{R}$.*

2. *$c : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function such that* $\liminf_{x\to+\infty} x^{-1}c(x) > 0$.
3. *$(A_N, B_N)_{N\geq 0}$ is a sequence of matrices with eigenvalues taking their values in $[\epsilon, 1]$ for some $\epsilon > 0$ and such that the spectral measures of A_N and B_N converge towards μ_A and μ_B respectively.*
4. *a, b are two non-negative real numbers.*

We then consider the non-negative measure Π^N on $\mathcal{P}(\mathbb{R}^+)$ given, for any measurable subset $M \in \mathcal{P}(\mathbb{R}^+)$, by

$$\Pi^N(M) = \sum_\lambda 1_{\hat{\mu}_\lambda^N \in M} s_\lambda(A_N)^a s_\lambda(B_N)^b e^{N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x)} \tag{3}$$

where the sum runs over Young tableaux λ. Note that when $F = 0$, $a = b = 1$, $A_N = B_N$ and $c(x) = \alpha x$ we get the Schur measure (cf. [21], section 4) evaluated at bounded Hermitian matrices $A_N = B_N$. We shall obtain large deviation bounds for $(\Pi^N)_{N\in\mathbb{N}}$ with rate function described as follows.

Definition-Notation 2.

- *Let $\mathcal{L}$ be the subset of $\mathcal{P}(\mathbb{R}^+)$ given by*

$$\mathcal{L} := \left\{ \nu \in \mathcal{P}(\mathbb{R}^+) : d\nu(x) \ll dx, \quad \frac{d\nu(x)}{dx} \leq 1 \right\} \tag{4}$$

- *Let, for $\mu \in \mathcal{P}(\mathbb{R})$, Σ be the non-commutative entropy*

$$\Sigma(\mu) = \iint \log|x-y| d\mu(x) d\mu(y).$$

- $S(\mu) = \iint \log(s(x,y))\, d\mu(x) d\mu(y)$, *with*

$$s(x,y) = \int_0^1 s, (\alpha x + (1-\alpha)y)^{-1} d\alpha \text{ if } x \neq y, \qquad s(x,x) = x^{-1}. \tag{5}$$

- *For $\mu \in \mathcal{P}(\mathbb{R})$ and any measurable function $f : \mathbb{R} \to \mathbb{R}$, we denote by $f_\# \mu$ the probability measure such that, for any bounded measurable function g on $\mathbb{R}$, $f_\#\mu(g) = \int g(f(x)) d\mu(x)$.*
- *We then define $H : \mathcal{P}(\mathbb{R}^+) \to \mathbb{R}$ infinite-valued on $\mathcal{L}^c$ and otherwise given by*

$$H(\nu) = \int c(x) d\nu(x) - \frac{a+b}{2}\Sigma(\nu) - F(\nu)$$
$$-aI(\log_\# \mu_A, \nu) - bI(\log_\# \mu_B, \nu) - \frac{a}{2}S(\mu_A) - \frac{b}{2}S(\mu_B),$$

where I is the limit of spherical integrals in a sense that will be properly settled in Lemma 1.

One of our main results is the following :

Theorem 2. *Let $(F, c, (A_N, B_N)_{N\geq 0}, a, b)$ satisfying Hypothesis 1. $(\Pi^N)_{N\geq 0}$ satisfies large deviation bounds with rate function H defined in 2. More precisely,*

1. *H has compactly supported level sets, i.e $\{\nu \in \mathcal{P}(\mathbb{R}^+) : H(\nu) \leq M\}$ is compact for all $M < \infty$.*
2. *For any closed set $F \in \mathcal{P}(\mathbb{R}^+)$*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \Pi^N(F) \leq -\inf\{H(\nu), \nu \in F\}$$

3. *For any open set $O \in \mathcal{P}(\mathbb{R}^+)$*

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \Pi^N(O) \geq -\inf\{H(\nu), \nu \in O\}$$

In particular,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \Pi^N(\mathcal{P}(\mathbb{R}^+)) = -\inf\{H(\nu)\}$$

and the infimum is achieved.

Theorem 5 would be a direct consequence of Theorem 2 according to (8) (with $a = b = 1$ and $\log Z_N(\Phi, \lambda) = N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x))$ if $Z_N(\Phi, \lambda)$ was indeed a continuous function of $\hat{\mu}_\lambda^N$ and decayed sufficiently fast as the size of the tableau goes to infinity. Although it is not exactly the case, most of the technicalities are already contained in the proof of Theorem 2, which, as we shall see in section 6, is of independent interest. The proof of Theorem 2 relies on techniques developed in [1] in a continuous setting, the relation of Schur functions with spherical integrals (see section 2) and on [10] where the asymptotics of such integrals were obtained. However, the proof remains rather technical for various reasons, the most severe being that we need to define the spherical integrals in a broader set than what was studied in [10]. In section 3, we prove Theorem 2 in details. We precise the strategy used to show Theorem 2 at the beginning of section 3, just after the precise statement of the theorem. We outline how to adapt the proofs to obtain Theorem 5 in section 4. Section 5 is devoted to the study of the minimizers of the rate function associated with the asymptotics of $Z_N(\Phi)$. They are reminiscent of [14] since they are described in terms of an additional measure describing the optimal shape of the Young tableau. They involve also, following [9] and [16], the solutions of an Euler equation for isentropic flow with negative pressure $p(\rho) = -\frac{\pi^2}{3}\rho^3$.
Finally, we comment our result, give other applications of our techniques, and their relations with the problem of the enumeration of maps in section 6.

2. Formulation of the matrix model as a sum over characters

Before going into the details of the large deviation principles we have announced in the introduction, we devote this section to show the character expansion for $Z_N(\Phi)$

(see Theorem 3). This will be useful in section 4 and can also be seen as a justification for the definition of Π^N we introduced above and therefore as a motivation to prove such a result like Theorem 2.

Since we shall later also be interested by the Gibbs measure associated with such a model we more generally define, after (1), if X is a measurable subset of $\mathcal{P}(\mathbb{R})$

$$Z_N(\Phi)(X) \equiv \int_{\hat{\mu}_M^N \in X} dM e^{-\frac{N}{2} tr M^2 - tr \otimes tr \log(I \otimes I - B_N \otimes \Phi(M) A_N)}, \tag{6}$$

where, for an Hermitian matrix $M \in \mathcal{H}_N(\mathbb{C})$ with eigenvalues $(M_1, \cdots, M_N) \in \mathbb{R}^N$, we shall denote $\hat{\mu}_M^N$ the spectral measure of M given by

$$\hat{\mu}_M^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{M_i}.$$

$\hat{\mu}_M^N$ is an element of the space $\mathcal{P}(\mathbb{R})$ of probability measures on the real line. We endow $\mathcal{P}(\mathbb{R})$ with its usual weak topology (i.e $\mu_n \in \mathcal{P}(\mathbb{R})$ converges towards μ iff $\mu_n(f) = \int f d\mu_n$ converges to $\mu(f)$ for all f in the space $\mathcal{C}_b(\mathbb{R})$ of bounded continuous functions).
We shall assume that

Hypothesis 2.

1. *If* $\|.\|_N$ *denotes the operator norm in* $\mathcal{M}_N(\mathbb{C})$*,* $\sup_{N \in \mathbb{N}} \|A_N\|_N$ *and* $\sup_{N \in \mathbb{N}} \|B_N\|_N$ *are finite and* Φ *is bounded. Without loss of generality, we will assume hereafter that*
$$\sup_{N \in \mathbb{N}} \|A_N\|_N \leq 1, \quad \sup_{N \in \mathbb{N}} \|B_N\|_N \leq 1$$
which amounts to multiply Φ *by* $\sup_{N \in \mathbb{N}} \|A_N\|_N . \sup_{N \in \mathbb{N}} \|B_N\|_N$.
2. *For all* $N \in \mathbb{N}$*,* A_N *and* B_N *are non-negative and* Φ *takes its value in* $\mathbb{R}^+$.
3. *If we define* $\rho_\Phi := -\log ||\Phi||_\infty$*, we assume that*
$$e^{-\rho_\Phi} := ||\Phi||_\infty < 1. \tag{7}$$

Note that this assumption insures that for each N, $I \otimes I - B_N \otimes \Phi(M) A_N$ has positive eigenvalues, so that its logarithm is well defined and $\operatorname{tr} \otimes \operatorname{tr} \log(I \otimes I - B_N \otimes \Phi(M) A_N)$ is bounded so that the partition function itself is well defined.

The goal of this section is to express the partition function $Z_N(\Phi)(X)$ in terms of spherical integrals, where a spherical integral I_N over the unitary group is given, for two real diagonal matrices D_N, E_N, by

$$I_N(D_N, E_N) := \int \exp\{N \operatorname{tr}(U D_N U^* E_N)\} dm_N(U),$$

where m_N denotes the probability Haar measure on the unitary group $\mathcal{U}_N$. In the sequel, we will denote Δ the VanderMonde determinant given, for any diagonal matrix $A_N = \operatorname{diag}(a_1, \cdots, a_N)$, by $\Delta(A_N) = \Delta(a) = \prod_{i<j} |a_i - a_j|$.
The main result of this section is

Theorem 3. *When Hypothesis 2 is satisfied, we have that*

$$Z_N(\Phi)(X) = c_N \sum_\lambda s_\lambda(A_N) s_\lambda(B_N) Z_N(\Phi,\lambda)(X) \tag{8}$$

where :

- $\mathcal{U}_N$ *is the unitary group of dimension* N*,*
- *the sum holds over all Young shapes with at most* N *lines,*
- s_λ *is the Schur polynomial corresponding to a Young shape* λ*,*
-

$$Z_N(\Phi,\lambda)(X) = \int_{\hat{\mu}_M^N \in X} I_N\left(\log\Phi(M), \frac{\ell}{N}\right) \frac{\Delta(\log\Phi(M))}{\Delta(\Phi(M))} \times \Delta(M)^2 e^{-\frac{N}{2}\sum_{i=1}^N M_i^2} \prod_{i=1}^N dM_i,$$

where ℓ *is the sequence associated to* λ *and* N*,*

- c_N *is a constant which only depends on* N*, equal to* $\frac{(2\pi N)^{\frac{N(N+1)}{2}}}{N!}$.

Denoting $|\lambda| = \sum_i \lambda_i$*, we can rewrite (8) into*

$$Z_N(\Phi)(X) = c_N \sum_\lambda s_\lambda(A_N) s_\lambda(B_N) Z_N(\Psi,\lambda)(X)\, e^{-\rho_\Phi |\lambda|} \tag{9}$$

where $\Psi = (||\Phi||_\infty)^{-1}\Phi$.

Proof. 1. *Expansion along Young tableaux*

By definition, if $(B_{N,i})_{1\leqslant i\leqslant N}$ and $((\Phi(M)A_N)_i)_{1\leqslant i\leqslant N}$ are respectively the eigenvalues of B_N and $\Phi(M)A_N$, we can rewrite :

$$e^{-tr\otimes tr\log(I\otimes I - B_N\otimes\Phi(M)A_N)} = \prod_{i,j=1}^N \frac{1}{1 - B_{N,i}(\Phi(M)A_N)_j}, \tag{10}$$

where condition (7) ensures the existence of the right hand side.

The Cauchy formula (for a reference and a proof, see for example formula 4.8.4 in the book of Sagan [22]) gives us that

$$\prod_{i,j=1}^N \frac{1}{1 - B_{N,i}(\Phi(M)A_N)_j} = \sum_\lambda s_\lambda(B_N) s_\lambda(\Phi(M)A_N), \tag{11}$$

where λ is the shape of a Young tableau and s_λ is the Schur polynomial corresponding to this shape.

Note that $s_\lambda(B_N) \geq 0$ since $B_N \geq 0$ as well as $s_\lambda(\Phi(M)A_N) = s_\lambda(A_N^{\frac{1}{2}}\Phi(M)A_N^{\frac{1}{2}}) \geq 0$. Hence, we can use Fubini's theorem to write the above series converges absolutely and our partition function

$$Z_N(\Phi)(X) = \sum_\lambda s_\lambda(B_N) \int_{\hat{\mu}_M^N \in X} e^{-\frac{N}{2}\mathrm{tr} M^2} s_\lambda(\Phi(M)A_N) dM. \tag{12}$$

2. *Formulating $Z_N(\Phi)(X)$ in terms of Schur polynomials*
It is useful to recall now the result of Weyl which establishes that s_λ coincides with the character of the unitary group associated to the shape λ (this is contained in theorem 7.5.B of [24]). Since s_λ's are characters of $\mathcal{U}_N$, we can apply the following property of orthogonality. If V and W are two unitary matrices of size N, this property reads, for any shape λ,

$$\int s_\lambda(UVU^*W)dm_N(U) = \frac{1}{d_\lambda} s_\lambda(V)s_\lambda(W), \tag{13}$$

where dm_N is the probability Haar measure on the unitary group $\mathcal{U}_N$ and $d_\lambda = s_\lambda(1, 1, \cdots, 1)$. Its explicit form is

$$d_\lambda = \frac{\Delta(\ell)}{\prod_{i=1}^{N-1} i!}, \tag{14}$$

with $\ell = \mathrm{diag}(\ell_1, \ldots, \ell_N)$ where we recall that $\ell_i = \lambda_i + N - i$.
A proof of formula (13) can be easily deduced from proposition II.4.2 of [2] (see also exercise 3 p.84 therein) whereas the explicit expression of d_λ given in (14) appears in [24].

As a consequence, with the notations introduced above,

$$\int s_\lambda(U\Phi(M)U^*A_N)dm_N(U) = \frac{1}{d_\lambda} s_\lambda(\Phi(M))s_\lambda(A_N). \tag{15}$$

Combining equations (12) and (15), since dM is invariant under the action of the unitary group, we can rewrite our partition function

$$Z_N(\Phi)(X) = c'_N \sum_\lambda \frac{1}{d_\lambda} s_\lambda(A_N)s_\lambda(B_N) \int_{\hat{\mu}_M^N \in X} s_\lambda(\Phi(M)) e^{-\frac{N}{2}\mathrm{tr}M^2} \Delta(M)^2 \prod_{i=1}^{N} dM_i, \tag{16}$$

where $\prod_{i=1}^N dM_i$ is the product Lebesgue measure on $\mathbb{R}^N$ and c'_N some normalizing constant, only depending on N.
3. *Relation between Schur polynomials and spherical integrals*
We can now recall the following determinantal formula for s_λ, that can be found for example in corollary 4.6.2 of [22]:

$$s_\lambda(\mathbf{x}) = \frac{\det(\mathrm{x}_\mathrm{i}^{\ell_\mathrm{j}})_{\mathrm{i,j}}}{\Delta(\mathbf{x})}, \tag{17}$$

where Δ is the VanderMonde determinant, $\mathbf{x} = (x_i)_{1 \leqslant i \leqslant N}$ and ℓ is the tableau associated to λ (that is to say $\ell_j = \lambda_j + N - j$ for $1 \leq j \leq N$).

We then use a formula due to Harish-Chandra (see [18]): if C_N and D_N are two $N \times N$ matrices whose eigenvalues $C_N(i)$ and $D_N(j)$ are distinct, we have that

$$I_N(C_N, D_N) = \frac{\det(\exp N C_N(i) D_N(j))_{i,j}}{\Delta(C_N)\Delta(D_N)}. \tag{18}$$

This last equation together with the determinantal formula (17) allows us to rewrite for any $M \in \mathcal{H}_N(\mathbb{C})$ with non-negative distinct eigenvalues:

$$s_\lambda(M) = I_N\left(\log M, \frac{\ell}{N}\right) \Delta\left(\frac{\ell}{N}\right) \frac{\Delta(\log M)}{\Delta(M)}, \tag{19}$$

Note that under the measure $e^{-\frac{N}{2}\mathrm{tr} M^2} dM$, the eigenvalues of the matrix M are almost surely distinct, and therefore so are the eigenvalues of the two matrices $\Phi(M)$ and $\log \Phi(M)$ by Hypothesis 2.3. Note however that (19) extends readily to any non-negative matrix by extending by continuity the definition

$$\frac{\Delta(\log M)}{\Delta(M)} = e^{\sum_{i<j} s(\lambda_i, \lambda_j)},$$

with s as defined in (5).

From (19), we conclude that there exists a constant c_N depending only on N such that,

$$Z_N(\Phi)(X) = c_N \sum_\lambda s_\lambda(A_N) s_\lambda(B_N) \times \int_{\hat{\mu}_M^N \in X} I_N\left(\log \Phi(M), \frac{\ell}{N}\right)$$
$$\times \frac{\Delta(\log \Phi(M))}{\Delta(\Phi(M))} \Delta(M)^2 e^{-\frac{N}{2}\sum_{i=1}^N M_i^2} \prod_{i=1}^N dM_i,$$

which completes the proof of Theorem 3 except from formula (9) which is easily obtained by dividing Φ by its norm before beginning the expansion.

It is also easy to deduce from equations (14), (16) and (19) above and from Selberg formula (see for example (25) in [1]) that we have indeed $c_N = \dfrac{(2\pi N)^{\frac{N(N+1)}{2}}}{N!}$. □

3. Large deviations estimates for the empirical distribution of Young tableaux following the distribution Π^N

The object of this section is to prove Theorem 2.

Throughout this section, we fix $(F, c, (A_N, B_N)_{N\geq 0}, a, b)$ satisfying Hypothesis 1.

From the definition (3) and following (19), we get that Π^N is the positive measure given, for any measurable subset M of $\mathcal{P}(\mathbb{R}^+)$, by :

$$\Pi^N(M) = e^{\frac{a}{2}N^2 S_N(\hat{\mu}_A^N) + \frac{b}{2}N^2 S_N(\hat{\mu}_B^N)} \sum_{\lambda : \hat{\mu}_\lambda^N \in M} \Delta\left(\frac{\ell}{N}\right)^{a+b} I_N\left(\log A_N, \frac{\ell}{N}\right)^a$$
$$\times I_N\left(\log B_N, \frac{\ell}{N}\right)^b e^{N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x)}$$

where

$$e^{\frac{N^2}{2} S_N(\hat{\mu}_A^N)} := \frac{\Delta(\log(A_N))}{\Delta(A_N)}.$$

Let us denote, for any measurable subset M of $\mathcal{P}(\mathbb{R}^+)$, $\tilde{\Pi}^N$ the non-negative measure

$$\tilde{\Pi}^N(M) = \sum_\lambda 1_{\hat{\mu}_\lambda^N \in M} \Delta\left(\frac{\ell}{N}\right)^{a+b} I_N\left(\log A_N, \frac{\ell}{N}\right)^a$$
$$\times I_N\left(\log B_N, \frac{\ell}{N}\right)^b e^{N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x)}.$$

We shall prove in this section a large deviation principle for $(\tilde{\Pi}^N)_{N\in\mathbb{N}}$ with rate function $\tilde{H}$ which, using the notations of Definition 2, is infinite on $\mathcal{L}^c$ and otherwise given by

$$\tilde{H}(\nu) = \int c(x)d\nu(x) - \frac{a+b}{2}\Sigma(\nu) - F(\nu) - aI(\log_\# \mu_A, \nu) - bI(\log_\# \mu_B, \nu), \tag{20}$$

I being in fact the limit of $N^{-2} \log I_N$ whose existence and description is discussed in Lemma 1.

Theorem 4. $(\tilde{\Pi}^N)_{N\geq 0}$ *satisfies large deviation bounds with rate function* $\tilde{H}$. *More precisely,*

1. $\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\}$ *is compact for all* $M < \infty$.
2. *For any closed set* $F \in \mathcal{P}(\mathbb{R}^+)$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \tilde{\Pi}^N(F) \leq -\inf\{\tilde{H}(\nu), \nu \in F\}$$

3. *For any open set* $O \in \mathcal{P}(\mathbb{R}^+)$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \tilde{\Pi}^N(O) \geq -\inf\{\tilde{H}(\nu), \nu \in O\}$$

Theorem 2 is easily deduced from Theorem 4 since, with s defined in (5),

$$S_N(\hat{\mu}_A^N) = \frac{2}{N^2} \sum_{i<j} s(A_i, A_j). \tag{21}$$

Hence, since s is a bounded continuous function on $[\epsilon, 1]^2$, we deduce (see Lemma 7.3.12 in [4]) that, as $\hat{\mu}_A^N$ converges to μ_A,

$$\lim_{N\to\infty} S_N(\hat{\mu}_A^N) = S(\mu_A)$$

and similarly for B_N.

The proof of Theorem 4 is heuristically simple since it amounts to perform a Laplace method and notice that the uniform measure on Young shape will not produce any entropy on the scale N^2 (see Lemma 4). On a rigorous ground, it becomes a bit technical, for mainly the two following reasons :

- The distribution of $\hat{\mu}_\lambda^N$ is discrete so that the arguments developed in [1] to obtain large deviation principles in similar scales and potentials have to be adapted. In particular, the discrete nature of the Young tableaux implies that $\tilde{H}$ is infinite on $\mathcal{L}^c$ (with $\mathcal{L}$ as defined in Definition-Notation 2).
- More cumbersome is the fact that the natural space where the empirical measure of the Young tableaux lives is $\mathcal{P}_1(\mathbb{R}^+) := \{\nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) < \infty\}$. Hence, all the limiting spherical integrals appearing are of the type $I(\mu, \nu)$ with μ in the set $\mathcal{P}_\infty(\mathbb{R})$ of compactly supported probability measures but $\nu \in \mathcal{P}_1(\mathbb{R}^+)$. Such limits were not proved to exist in [10] (where $\nu(x^2) < \infty$ was assumed), the formula obtained in [10] is not valid, and continuity statements for I are lacking a priori.

The proof nevertheless follows the usual scheme :

1. In subsection 3.1 we study the rate function and prove that its level sets are compact.
2. In subsection 3.2 we show that the family of measures $(\tilde{\Pi}^N)_{N\in\mathbb{N}}$ is exponentially tight. More precisely, if we let $\mathcal{K}_L$ be the compact subset
$$\mathcal{K}_L = \left\{\nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \le L\right\}$$
we prove that
$$\limsup_{L\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \log \tilde{\Pi}^N(\mathcal{K}_L^c) = -\infty.$$
3. In subsection 3.3 we prove the upper bound for arbitrarily small balls, i.e if d is a metric on $\mathcal{P}(\mathbb{R})$ compatible with the weak topology such as the Dudley's metric d given by
$$d(\mu, \nu) = \sup \left| \int f d\mu - \int f d\nu \right|,$$
where the supremum is taken over all Lipschitz functions f with Lipschitz norm less than 1 (note that this distance is compatible with the weak topology), and if we set
$$B(\nu, \delta) = \{\mu \in \mathcal{P}(\mathbb{R}^+); d(\mu, \nu) < \delta\}$$
we show that for any $\nu \in \cup_{L\in\mathbb{N}} \mathcal{K}_L$,
$$\limsup_{\delta\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) \le -\tilde{H}(\nu).$$
4. In subsection 3.4 we prove the lower bound for arbitrarily small balls, i.e that for any $\nu \in \cup_{L\in\mathbb{N}} \mathcal{K}_L$,
$$\liminf_{\delta\to\infty} \liminf_{N\to\infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) \ge -\tilde{H}(\nu).$$

By Theorem 4.1.11 in [4], the above results prove Theorem 4.

3.1. Study of $\tilde{H}$

We begin this section by describing more precisely the function I as the limit of spherical integrals. Then, we show that $\tilde{H}$ has compact level sets.

3.1.1. Definition and properties of I

Let us remind that it was proved in Theorem 1.1 of [10] that

$$I(\mu_D, \mu_E) := \lim_{N\to\infty} \frac{1}{N^2} \log I_N(D_N, E_N) \tag{22}$$

exists for all sequences of diagonal matrices $(D_N, E_N)_{N\in\mathbb{N}}$ with spectral measures converging towards μ_D and μ_E respectively and such that $\sup_N ||D_N||_N$ and $\sup_N \hat{\mu}_E^N(x^2)$ are finite. A formula for I is given in [10] when either $\Sigma(\mu_E)$ or $\Sigma(\mu_D)$ are finite. If they are not, the limit still exists since spherical integrals are uniformly continuous (see Lemma 1.4 below) and the measures with finite Σ are dense, but its formula is far from being clear (see a discussion in [11]). However, let us remark that since the spherical integrals under consideration are always bounded, the rate function $\tilde{H}(\nu)$ is infinite unless ν has finite entropy Σ (see (33)) so that we can always use the formula given in [10] when $\int x^2 d\nu(x) < \infty$.

Since $\tilde{H}(\nu)$ is infinite if $\int x d\nu(x) = +\infty$ (see (32)) and μ_A and μ_B are supposed to be supported on $[\epsilon, 1]$, it is enough to extend the definition of $I(\mu, \nu)$ to compactly supported measures μ with support in $\mathbb{R}^-$ but $\nu \in \mathcal{P}_1(\mathbb{R}^+)$. We shall prove

Lemma 1. *Let $R \in \mathbb{R}^+$ and μ be a probability measure on $[-R, 0]$ and $\nu \in \mathcal{P}_1(\mathbb{R}^+)$. Then*

1. *Let $\phi_M(x) = x \wedge M$. The sequence $I(\mu, (\phi_M)_\#\nu)$ is well defined and decreases towards a limit*
$$I(\mu, \nu) := \lim_{M\to\infty} I(\mu, (\phi_M)_\#\nu).$$
Moreover, for any $M \geq 0$,
$$I(\mu, (\phi_M)_\#\nu) - R\nu(x - \phi_M(x)) \leq I(\mu, \nu) \leq I(\mu, (\phi_M)_\#\nu).$$
2. *Let $\mathcal{P}^R(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) : \mu([-R, R]^c) = 0\}$ and $\mathcal{P}_q(\mathbb{R}^+) = \{\mu \in \mathcal{P}(\mathbb{R}^+) : \mu(|x|^q) \leq R\}$. Then there exists a function $\kappa(\delta, R)$ such that for any $R < \infty$, $\kappa(\delta, R)$ goes to zero as δ goes to zero and for any $(\mu, \mu') \in \mathcal{P}^R(\mathbb{R})$ any $(\nu, \nu') \in \mathcal{P}_2(\mathbb{R})$, such that $d(\mu, \mu') + d(\nu, \nu') < \delta$,*
$$|I(\mu, \nu) - I(\mu', \nu')| \leq \kappa(\delta, R).$$
3. *For any $\mu \in \mathcal{P}(\mathbb{R}^-)$ and $\nu \in \mathcal{P}_1(\mathbb{R}^+)$,*
$$\mu(x)\nu(x) \leq I(\mu, \nu) \leq 0.$$

4. *For any sequence (D_N, E_N) of diagonal Hermitian matrices with $D_N \leq 0$ and $E_N \geq 0$, for any $M \in \mathbb{R}^+$,*

$$I_N(D_N, \phi_M(E_N))e^{-N||D_N||_N tr(E_N - \phi_M(E_N))} \leq I_N(D_N, E_N) \leq I_N(D_N, \phi_M(E_N)). \quad (23)$$

Moreover there exists a function $g : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$, depending on the limiting measures μ_E, μ_D only, such that $g(\delta, M)$ goes to zero as δ does for any $M \in \mathbb{R}^+$, and so that

$$\left| \frac{1}{N^2} \log \frac{I_N(\hat{D}_N, \phi_M(\hat{E}_N))}{I_N(D_N, \phi_M(E_N))} \right| \leq g(\delta, M). \quad (24)$$

for any $N \in \mathbb{N}$ and any diagonal matrices $(D_N, E_N, \hat{D}_N, \hat{E}_N)$ such that $E_N, \hat{E}_N$ are non-negative and

$$d(\hat{\mu}^N_{D_N}, \hat{\mu}^N_{\hat{D}_N}) + d(\hat{\mu}^N_{E_N}, \hat{\mu}^N_{\hat{E}_N}) < \delta, \qquad \hat{\mu}^N_{E_N}(x^2) + \hat{\mu}^N_{\hat{E}_N}(x^2) \leq M.$$

Proof. • We first prove the last point. If we denote $D_N = \text{diag}(d_1, \cdots, d_N)$ and $E_N = \text{diag}(e_1, \cdots, e_N)$,

$$\begin{aligned} I_N(D_N, E_N) &= \int e^{N\text{tr}(D_N U E_N U^*)} dm_N(U) \\ &= \int e^{N\sum_{i,j=1}^N d_i e_j |u_{ij}|^2} dm_N(U) \\ &\leq \int e^{N\sum_{i,j=1}^N d_i \phi_M(e_j)|u_{ij}|^2} dm_N(U) \end{aligned}$$

where we used that $d_i \leq 0$. The opposite inequality of (23) is also trivial since

$$\begin{aligned} I_N(D_N, E_N) &\geq e^{N||D_N||_N \sum_{j=1}^N (e_j - \phi_M(e_j))} \int e^{N\sum_{i,j=1}^N d_i \phi_M(e_j)|u_{ij}|^2} dm_N(U) \\ &= e^{-N||D_N||_N \text{tr}(E_N - \phi_M(E_N))} I_N(D_N, \phi_M(E_N)) \end{aligned}$$

The continuity statement (24) is a direct consequence of Lemma 5.1 in [10] since $\phi_M(E_N)$ is uniformly bounded by M and $d((\phi_M)_\#\mu, (\phi_M)_\#\mu') \leq d(\mu, \mu')$ for any $\mu, \mu' \in \mathcal{P}(\mathbb{R})$.

• We can now prove the first point. From (23), we deduce that for any $M \in \mathbb{R}^+$, any $E_N \geq 0$ with spectral measure converging towards μ_E and any sequence of bounded non-positive diagonal matrices D_N with spectral measure converging towards μ_D

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N^2} \log I_N(D_N, E_N) &\leq \limsup_{N\to\infty} \frac{1}{N^2} \log I_N(D_N, \phi_M(E_N)) \\ &= I(\mu_D, (\phi_M)_\#\mu_E), \end{aligned} \quad (25)$$

where the last equality comes from the observation that $(\phi_M(D_N), E_N)$ are uniformly bounded by hypothesis so that the convergence holds by Theorem 1.1 in [10]. With $\mu_E = (\phi_L)_\# \nu$ for some $L \geq M$ and E_N chosen so that $\hat{\mu}^N_{E_N}(|x| > L) = 0$, the left hand side of (25) converges towards $I(\mu_D, (\phi_L)_\# \nu)$ showing that $M \mapsto I(\mu_D, (\phi_M)_\# \mu_E)$ is non-increasing. Hence, it converges towards some limit (maybe infinite at this stage). Now, we choose a special sequence $(E_N)_{N \in \mathbb{N}}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{tr}(E_N - \phi_M(E_N)) = \mu_E(x - \phi_M(x)).$$

We can construct it as follows ; assume first that μ_E has no atoms and set

$$E_{1,N} = \inf \left\{ x \; / \; \mu_E((-\infty, x]) \geqslant \frac{1}{N+1} \right\}$$
$$E_{i+1,N} = \inf \left\{ x \geqslant E_{i,N} \; / \; \mu_E((E_{i,N}, x]) \geqslant \frac{1}{N+1} \right\}.$$

Then it is not hard to see that $\hat{\mu}^N_{E_N} = \frac{1}{N} \sum_{i=1}^N \delta_{E_{i,N}}$ converges towards μ_E. Moreover,

$$\begin{aligned}
\hat{\mu}^N_{E_N}(x - \phi_M(x)) &= \frac{1}{N} \sum_{E_{i,N} \geq M} (E_{i,N} - M) \\
&\leq \frac{N+1}{N} \sum_{E_{i,N} \geq M} (E_{i,N} - M) \mu_E([E_{i,N}, E_{i+1,N}]) \\
&\leq \frac{N+1}{N} \mu_E((x - M) 1_{x \geq M}).
\end{aligned}$$

If μ_E has atoms, we consider a finite collection of atoms $\{a_1, \cdots, a_K\}$ such that each of the remaining atoms has mass smaller than $(N+1)^{-1}$. Then, E_N has $\lfloor N \mu_E(\{a_i\}) \rfloor$ eigenvalues equal to a_i for $1 \leq i \leq K$. The remaining eigenvalues are chosen as above.

Inequality (23) yields with this choice

$$\frac{1}{N^2} \log I_N(D_N, E_N) \geq I_N(D_N, \phi_M(E_N)) e^{-N(N+1) \sup_N ||D_N||_N \mu_E((x-M) 1_{x \geq M})}$$

and therefore

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^2} \log I_N(D_N, E_N) \\
&\qquad \geq -\sup_N ||D_N||_N \mu_E((x - M) 1_{x \geq M}) + I(\mu_D, (\phi_M)_\# \mu_E) \qquad (26)
\end{aligned}$$

(25) and (26) shows that for such a sequence

$$
\begin{aligned}
-\sup_N ||D_N||_N \mu_E((x-M)1_{x\geq M}) + I(\mu_D, (\phi_M)_\# \mu_E) \\
&\leq \liminf_{N\to\infty} \frac{1}{N^2} \log I_N(D_N, E_N) \\
&\leq \limsup_{N\to\infty} \frac{1}{N^2} \log I_N(D_N, E_N) \\
&\leq I(\mu_D, \mu_E) \qquad (27)
\end{aligned}
$$

This completes the proof of the first point.

• The second point is a direct consequence of the fourth too. Indeed, let (μ, μ', ν, ν') be such that

$$d(\mu, \mu') + d(\nu, \nu') < \delta.$$

Then, we choose a sequence (D_N, E_N) (resp. $(\hat{D}_N, \hat{E}_N)$) of matrices with spectral measure converging towards (μ, ν) (resp. (μ', ν')) such that

$$\max\{d(\hat{\mu}^N_{D_N}, \mu), d(\hat{\mu}^N_{\hat{D}_N}, \mu'), d(\hat{\mu}^N_{E_N}, \nu), d(\hat{\mu}^N_{\hat{E}_N}, \nu')\} < \delta$$

which implies

$$d(\hat{\mu}^N_{D_N}, \hat{\mu}^N_{\hat{D}_N}) < 2\delta, \quad d(\hat{\mu}^N_{E_N}, \hat{\mu}^N_{\hat{E}_N}) < 2\delta$$

so that 4. implies, by taking the limit as N goes to infinity (here $M = R$), that

$$|I(\mu, \nu) - I(\mu', \nu')| \leq g(2\delta, R).$$

• In point 3., the upper bound on I is trivial and the lower bound comes from Jensen's inequality which yields

$$
\begin{aligned}
I_N(D_N, E_N) &= \int e^{N\sum_{i,j=1}^N e_i d_j |u_{ij}|^2} dm_N(U) \\
&\geq e^{N\sum_{i,j=1}^N e_i d_j \int |u_{ij}|^2 dm_N(U)} \\
&= e^{\sum_{i,j=1}^N e_i d_j} = e^{N^2 \hat{\mu}^N_{E_N}(x)\hat{\mu}^N_{D_N}(x)}
\end{aligned}
$$

The result is then obtained by letting N go to infinity. □

3.1.2. $\tilde{H}$ has compact level sets

In this section, we prove Theorem 4.1 by proving first that $\tilde{H}$ is lower semi-continuous and then that its level sets are compact.

• $\tilde{H}$ is lower semi-continuous, i.e $\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\}$ is closed for any $M \in \mathbb{R}^+$. We recall that $\mathcal{L}$ is the set of probability measures which are absolutely continuous with respect to Lebesgue measure and with density bounded by one. If we introduce the function $\hat{H}$ given by

$$\hat{H}(\nu) := \int c(x)d\nu(x) - \frac{a+b}{2}\Sigma(\nu) - F(\nu) - aI(\log_{\#}\mu_A, \nu) - bI(\log_{\#}\mu_B, \nu),$$

for all $\nu \in \mathcal{P}(\mathbb{R}^+)$ ($\hat{H}$ coincides with $\tilde{H}$ on $\mathcal{L}$ but is not necessarily infinite outside $\mathcal{L}$), then we have that $\{\tilde{H} \le M\} = \mathcal{L} \cap \{\hat{H} \le M\}$. We first check that $\mathcal{L}$ is closed and then show that $\hat{H}$ is lower semi-continuous, these two points proving that $\{\tilde{H} \le M\}$ is closed.
To show that $\mathcal{L}$ is closed, take a sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{L}$ converging weakly to a measure ν. For any c and d, the function $\mathbf{1}_{[c,d]}$ is upper semi-continuous so that

$$|d-c| \ge \limsup_{n\to\infty} \nu_n([c,d]) \ge \nu([c,d]).$$

so that ν is in $\mathcal{L}$.

We now show that $\hat{H}$ is a supremum of continuous functions which we define as follows: we let, with $\phi_M(x) = x \wedge M$ for $M \ge 0$ as in Lemma 1, and for $\nu \in \mathcal{P}(\mathbb{R}^+)$,

$$\begin{aligned}\hat{H}^M(\nu) := -aI(\log_{\#}\mu_A, (\phi_M)_{\#}\nu) - bI(\log_{\#}\mu_B, (\phi_M)_{\#}\nu)\\ + \iint g(x,y) \wedge M d\nu(x)d\nu(y) - F(\nu)\end{aligned}$$

with

$$g(x,y) = \left(\frac{a+b}{2}\right)\log|x-y|^{-1} + \frac{1}{2}c(x) + \frac{1}{2}c(y) \tag{28}$$

We claim that for any finite M, $\hat{H}^M$ is continuous on $\mathcal{P}(\mathbb{R}^+)$. Indeed, by Lemma 1.2, for $C = A$ or B, $\nu \in \mathcal{P}(\mathbb{R}^+) \mapsto I(\log_{\#}\mu_C, (\phi_M)_{\#}\nu) \in \mathbb{R}$ is continuous since $\log_{\#}\mu_C$ is compactly supported by Hypothesis 2.1. Moreover, one checks that g is bounded below and continuous except when on the diagonal $\{x = y\}$ where it goes to infinity. Consequently, $g \wedge M$ is a bounded continuous function on $\mathbb{R}^2$. Thus $\mu \to \iint g(x,y) \wedge M d\mu(x)d\mu(x)$ is bounded continuous.

This last argument finishes to prove that $\hat{H}^M$ is a continuous function on $\mathcal{P}(\mathbb{R}^+)$. To deduce that $\hat{H}$ is lower semi-continuous, it is therefore enough to prove that

$$\hat{H}(\nu) = \sup_{M\ge 0}\{\hat{H}^M(\nu)\}. \tag{29}$$

But this is straightforward since monotone convergence theorem asserts that for any f bounded below

$$\lim_{M\uparrow\infty} \iint f(x,y) \wedge M d\mu(x)d\mu(y) = \iint f(x,y)d\mu(x)d\mu(y)$$

and by Lemma 1.1, $I(\mu, (\phi_M)_{\#}\nu)$ decreases towards its limit $I(\mu, \nu)$.

• As a consequence of the last point, for any $M \geq 0$, $\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\}$ is closed. We now check that it is compact by showing that it is contained in a compact set. In fact, by Lemma 1.3,

$$\tilde{H}(\nu) \geq \iint g(x, y) d\nu(x) d\nu(y) - \sup_{\nu \in \mathcal{P}(\mathbb{R}^+)} F(\nu) \tag{30}$$

and one checks that, by Hypothesis 1.2, there exists a finite constant C and $\rho > 0$ such that for any $(x, y) \in (\mathbb{R}^+)^2$

$$g(x, y) \geq \frac{\rho}{2} x + \frac{\rho}{2} y + C \tag{31}$$

yielding with (30) that for any $M \in \mathbb{R}^+$, if $C' = C - \sup_{\nu \in \mathcal{P}(\mathbb{R}^+)} F(\nu)$,

$$\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\} \subset \left\{\nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \leq \frac{2}{\rho}(M - C')\right\} := \mathcal{K}_{M,\rho}. \tag{32}$$

Since $\mathcal{K}_{M,\rho}$ is a compact subset of $\mathcal{P}(\mathbb{R}^+)$, the proof is complete.
Note that since $\iint g(x, y) d\nu(x) d\nu(y) = \int c(x) d\nu(x) - \Sigma(\nu)$ and c is bounded below, we also see from (30) that there exists a finite constant L such that for all $M \geq 0$

$$\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\} \subset \{\nu \in \mathcal{P}(\mathbb{R}^+) : \Sigma(\nu) \geq -M + L\}. \tag{33}$$

3.2. $\tilde{\Pi}^N$ *is exponentially tight*

The goal of this section is to prove that

Lemma 2. $\tilde{\Pi}^N$ *is exponentially tight, and more precisely if we set*

$$\mathcal{K}_L := \left\{\nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \leq L\right\},$$

then

$$\limsup_{L \to \infty} \limsup_{N \to \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(\mathcal{K}_L^c) = -\infty.$$

Proof. Since the spherical integrals under consideration are uniformly bounded above by one and F is uniformly bounded by a constant $||F||_\infty$,

$$\tilde{\Pi}^N(X) \leq e^{N^2 ||F||_\infty} \sum_{\lambda : \hat{\mu}_\lambda^N \in X} e^{-N^2 \int_{x \neq y} g(x,y) d\hat{\mu}_\lambda^N(x) d\hat{\mu}_\lambda^N(y)},$$

Choosing $X = \mathcal{K}_L^c$, we get by (31) that

$$\tilde{\Pi}^N(\mathcal{K}_L^c) \leq e^{N^2 ||F||_\infty + N^2 C} \sum_\lambda 1_{\hat{\mu}_\lambda^N \in \mathcal{K}_L^c} e^{-N^2 \rho \int x d\hat{\mu}_\lambda^N(x)} \Delta\left(\frac{\ell}{N}\right)^{a+b} \tag{34}$$

It remains to consider the sums over Young shapes. Let us recall that

$$\hat{\mu}_\lambda^N(x) = \frac{1}{N^2}\sum_{i=1}^{N} \ell_i = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\lambda_i}{N} - \frac{i}{N}\right) + 1 \le N^{-2}|\lambda|_N + 1$$

where $|\lambda|_N = \sum_{i\le N}\lambda_i$. Therefore, for any $L \ge 0$,

$$\begin{aligned}\sum_{\lambda:\hat{\mu}_\lambda^N(x)\ge L} e^{-\rho|\lambda|}\Delta\left(\frac{\ell}{N}\right)^{a+b} &\le \sum_{\lambda:|\lambda|_N\ge N^2(L-1)} e^{-\rho|\lambda|}\Delta\left(\frac{\ell}{N}\right)^{a+b} \\ &\le e^{-\frac{1}{2}\rho N^2(L-1)} \sum_{\lambda:|\lambda|_N\ge N^2(L-1)} e^{-\frac{1}{2}\rho|\lambda|}\Delta\left(\frac{\ell}{N}\right)^{a+b}.\end{aligned}$$

For any j,

$$\prod_{j<i}\left|\frac{\ell_i}{N} - \frac{\ell_j}{N}\right| \le \left(\frac{\ell_j}{N}\right)^{N-j},$$

therefore, for any shape,

$$\Delta\left(\frac{\ell}{N}\right)^{a+b} e^{-\frac{1}{2}\rho|\lambda|} \le e^{(a+b)\sum_j (N-j)\log\frac{\ell_j}{N} - \frac{1}{4}N\rho\frac{\ell_j}{N}} \le e^{N^2 C''},$$

where $C'' = \sup_x\left\{(a+b)\log x - \frac{1}{4}\rho x\right\} - \frac{1}{8}$.
Now the number of Young shapes λ such that $|\lambda|_N = m$ is bounded by C_m^N so that we conclude

$$\sum_{\lambda:|\lambda|_N\ge N^2(L-1)} e^{-\rho|\lambda|}\Delta\left(\frac{\ell}{N}\right)^{a+b} \tag{35}$$

$$\begin{aligned}&\le e^{N^2C} e^{-\frac{1}{2}\rho N^2(L-1)}\frac{1}{N!}\sum_{m\ge N^2(L-1)} m(m-1)\cdots(m-N+1)e^{-\frac{1}{4}\rho m}, \\ &\le e^{N^2C} e^{-\frac{1}{2}\rho N^2(L-1)}\frac{1}{N!}\sum_{m\ge N^2(L-1)} e^{N\log m}e^{-\rho m}\end{aligned}$$

$$\le e^{-\frac{1}{2}(\rho-\delta)N^2(L-1)} \tag{36}$$

where in the last line δ is any positive number and the inequality holds as soon as N and L are big enough. Equations (34) and (36) give Lemma 2. □

3.3. $(\tilde{\Pi}^N)_{N\geq 0}$ *satisfies a weak large deviation upper bound*

In this section, we shall prove the following

Lemma 3. *$\tilde{\Pi}^N$ satisfies a weak large deviation upper bound in the scale N^2 with rate function $\tilde{H}$ i.e for any $\nu \in \mathcal{P}(\mathbb{R}^+)$,*

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\tilde{\Pi}^N(B(\nu,\delta)) \leq -\tilde{H}(\nu). \tag{37}$$

Proof. • We first prove that for any $\epsilon > 0$, if ν is such that there exists two positive real numbers α and β ($\alpha < \beta$) such that $\nu([\alpha,\beta]) \geq (1+\epsilon)(\beta-\alpha)$, then,

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\tilde{\Pi}^N(B(\nu,\delta)) = -\infty \tag{38}$$

which implies that for all $\nu \in \mathcal{L}^c$,

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\tilde{\Pi}^N(B(\nu,\delta)) = -\infty. \tag{39}$$

The main remark is that, for any shape λ, as the ℓ_i are (strictly) decreasing we have that, for any $c < d$,

$$\begin{aligned}\hat{\mu}_\lambda^N([c,d]) &= \frac{1}{N}\sharp\left\{i : \frac{\ell_i}{N}\in[c,d]\right\} \leq \frac{1}{N}(\lfloor N(d-c)\rfloor + 1)\\ &\leq \left(1+\frac{\epsilon}{2}\right)(d-c),\end{aligned} \tag{40}$$

where the last inequality holds for N large enough.
Let be $\eta > 0$ and consider the function $f : \mathbb{R}\to\mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & \text{if } x < \alpha-\eta \text{ or } x > \beta+\eta,\\ 1, & \text{if } \alpha < x < \beta,\\ \eta^{-1}(x-\alpha+\eta), & \text{if } \alpha-\eta \leq x \leq \alpha,\\ \eta^{-1}(-x+\beta+\eta), & \text{if } \beta \leq x \leq \beta+\eta.\end{cases}$$

Note that the Lipschitz norm of f is bounded by $\eta^{-1}\vee 1$. We have, for any shape λ,

$$\begin{aligned}\int f d\nu - \int f d\hat{\mu}_\lambda^N = &\int_{\alpha-\eta}^{\alpha} f(d\nu - d\hat{\mu}_\lambda^N)\\ &+\int_{\beta}^{\beta+\eta} f(d\nu - d\hat{\mu}_\lambda^N) + \int_{\alpha}^{\beta} f(d\nu - d\hat{\mu}_\lambda^N).\end{aligned}$$

Using (40), we get that, for any shape λ and N large enough,

$$\int_{\alpha-\eta}^{\alpha} f(d\nu - d\hat{\mu}_\lambda^N) \geq -\int_{\alpha-\eta}^{\alpha} f d\hat{\mu}_\lambda^N = -\frac{1}{\eta}\int_{\alpha-\eta}^{\alpha}\hat{\mu}_\lambda^N([\alpha-\eta,x])dx \geq -\left(1+\frac{\epsilon}{2}\right)\frac{\eta}{2},$$

(and the same thing for β) and that

$$\int_\alpha^\beta f d\nu - \int_\alpha^\beta f d\hat{\mu}_\lambda^N = \nu([\alpha,\beta]) - \hat{\mu}_\lambda^N([\alpha,\beta]) \geq \frac{\epsilon}{2}(\beta-\alpha),$$

so that, if we choose $\eta = \frac{\epsilon}{4}(\beta-\alpha)$, we get that

$$\int f d\nu - \int f d\hat{\mu}_\lambda^N \geq \left(1-\frac{\epsilon}{2}\right)\frac{\epsilon}{4}(\beta-\alpha).$$

Thus, since $\int f d\nu - \int f d\hat{\mu}_\lambda^N \leq \eta^{-1} d(\hat{\mu}_\lambda^N, \nu)$, we conclude that, if we take $\delta < (1-\frac{\epsilon}{2})\left[\frac{\epsilon}{4}(\beta-\alpha)\right]^2$, the set $B(\nu,\delta) = \{\lambda : d(\hat{\mu}_\lambda^N, \nu) < \delta\}$ is empty for N large enough, which gives (38).

• We now take $\nu \in \mathcal{L}$. By lemma 1.4, for any $M \in \mathbb{R}^+$,

$$\tilde{\Pi}^N(B(\nu,\delta)) \leq \sum_{\lambda : d(\hat{\mu}_\lambda^N,\nu)<\delta} I\left(A_N, \phi_M\left(\frac{\ell}{N}\right)\right)^a I\left(B_N, \phi_M\left(\frac{\ell}{N}\right)\right)^b$$
$$\Delta\left(\frac{\ell}{N}\right)^{a+b} e^{-N^2\int c(x)d\hat{\mu}_\lambda^N(x) + N^2 F(\hat{\mu}_\lambda^N)}$$

Observe that with g defined in (28), since $|\lambda| = \sum \lambda_j = \sum \ell_j - \sum(N-j) = \sum \ell_j - 2^{-1}N(N-1)$,

$$\Delta\left(\frac{\ell}{N}\right)^{a+b} e^{-N^2\int c(x)d\hat{\mu}_\lambda^N(x)} = e^{-N^2\int_{y'\neq y} g(y',y)d\hat{\mu}_\lambda^N(y)d\hat{\mu}_\lambda^N(y') - N\int c(x)d\hat{\mu}_\lambda^N(x)},$$

we obtain

$$\begin{aligned}\tilde{\Pi}^N(B(\nu,\delta)) \leq & \sum_{\lambda : d(\hat{\mu}_\lambda^N,\nu)<\delta} I\left(A_N, \phi_M\left(\frac{\ell}{N}\right)\right)^a I\left(B_N, \phi_M\left(\frac{\ell}{N}\right)\right)^b \\ & \times e^{-N^2\int_{y'\neq y} g(y',y)\wedge M d\hat{\mu}_\lambda^N(y)d\hat{\mu}_\lambda^N(y') + N^2F(\hat{\mu}_\lambda^N) - N\int c(x)d\hat{\mu}_\lambda^N(x)} \\ \leq & \, e^{NM} \sum_{\lambda : d(\hat{\mu}_\lambda^N,\nu)<\delta} I\left(A_N, \phi_M\left(\frac{\ell}{N}\right)\right)^a I\left(B_N, \phi_M\left(\frac{\ell}{N}\right)\right)^b \\ & \times e^{-N^2\int g(y',y)\wedge M d\hat{\mu}_\lambda^N(y)d\hat{\mu}_\lambda^N(y') + N^2F(\hat{\mu}_\lambda^N) - N\int c(x)d\hat{\mu}_\lambda^N(x)} \end{aligned} \tag{41}$$

Now, following section 3.1.2, we know that all the functions appearing above are continuous for any finite M so that for each such M we find a $\kappa(\delta, M)$ going to zero as δ goes to zero so that

$$\tilde{\Pi}^N(B(\nu,\delta)) \leq e^{-N^2(\hat{H}^M(\nu)+\kappa(\delta,M))} e^{N(M+C)} \sum_{\lambda : d(\hat{\mu}_\lambda^N,\nu)<\delta} e^{-N\rho\int y d\hat{\mu}_\lambda^N(y)} \tag{42}$$

where we used again (31). By Lemma 4 stated below, the last entropy term will not contribute in the scale N^2.

By (42), and lemma 4 we conclude that, for all $M \geq 0$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu,\delta)) \leq -\hat{H}^M(\nu) + \kappa(\delta, M).$$

Letting δ going to zero and then M going to infinity (since we saw in section 3.1.2 that $\hat{H}^M$ converges towards $\hat{H}$) gives (37) in the case where $\nu \in \mathcal{L}$. This together with (39) finishes the proof of Lemma 3. □

As announced above, we have now to prove :

Lemma 4.

$$\frac{1}{N^2} \log \sharp\{\lambda / d(\hat{\mu}^N_\lambda, \nu) < \delta\} \to_{N\to\infty} 0,$$

Proof. We first show a lower bound for the number of tableaux λ whose empirical measure is such that, for a given $\epsilon > 0$ and a given $\nu \in \mathcal{P}(\mathbb{R}^+)$, $d(\frac{1}{N}\sum_{j=1}^N \delta_{\frac{\ell_j}{N}}, \nu)$ $< \epsilon$.
As this number is an integer, we just need to show that this set is non-empty. This is true thanks to two facts : first the set $\{\nu\}$ is tight so that we choose a convex compact K such that $\nu(K) \geqslant 1 - \frac{\epsilon}{3}$ and then the set $\mathcal{P}(K)$ of all probability measures on K endowed with the weak topology is a compact in the locally convex space of measures with mass less than 1, so that the Krein-Milman theorem tells us that $\mathcal{P}(K)$ is the closure of the convex envelope of its extremal points, which are the Dirac measures. We have the approximation announced above : for $\epsilon > 0$, there exists an integer $N(\epsilon)$ and some real number that we order $a_{1,N(\epsilon)} > a_{2,N(\epsilon)} > \dots$ such that $d(\frac{1}{N}\sum_{j=1}^N \delta_{a_{j,N}}, \nu) < \frac{\epsilon}{2}$. Then for each j between 1 and N, we choose for ℓ_j the integer for which $\frac{\ell_j}{N}$ is the closest from $a_{j,N}$. This gives us that, for N large enough

$$\sharp\left\{\lambda / d\left(\frac{1}{N}\sum_{j=1}^N \delta_{\frac{\ell_j}{N}}, \nu\right) < \epsilon\right\} \geqslant 1.$$

For the upper bound, we first find a compactly supported measure ν' (with support $K = [0, M]$) such that $d(\nu, \nu') < \frac{\epsilon}{2}$. This gives us that

$$\{\lambda / d(\hat{\mu}^N_\lambda, \nu) < \epsilon\} \subset \left\{\lambda / d(\hat{\mu}^N_\lambda, \nu') < 3\frac{\epsilon}{2}\right\}.$$

Let us consider the function f_2 given by

$$f_2(x) = \begin{cases} 0, & \text{if } x \leqslant M \\ x - M, & \text{if } M \leqslant x \leqslant M + 2N\epsilon \\ 2N\epsilon & \text{if } x \geq M + 2N\epsilon. \end{cases}$$

f_2 is a bounded Lipschitz function whose Lipschitz norm is bounded by 1 and such that $\int f_2 d\nu' = 0$. But, if there exists an ℓ_j greater or equal $2N^2\epsilon + NM$ then $\frac{1}{N}\sum_{i=1}^N f_2\left(\frac{\ell_i}{N}\right) \geqslant 2\epsilon \geqslant 3\frac{\epsilon}{2}$, so that we have the inclusion

$$\{\lambda / d(\hat{\mu}^N_\lambda, \nu) < \epsilon\} \subset \{\lambda / \forall j,\ \ell_j \leqslant 2N^2\epsilon + NM\}$$

and we get the upper bound as we know that

$$\sharp\{\lambda/\forall j,\ \ell_j \leqslant 2N^2\epsilon + NM\} \leqslant (2N^2\epsilon + NM)^N.$$

Upper and lower bound together give the result announced in Lemma 4. □

3.4. $(\tilde{\Pi}^N)_{N\geq 0}$ *satisfies a large deviation lower bound*

In this part we show that

Lemma 5. $\tilde{\Pi}^N$ *satisfies a large deviation lower bound, i.e for any* $\nu \in \mathcal{P}(\mathbb{R}^+)$*,*

$$\liminf_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{N^2}\log \tilde{\Pi}^N(B(\nu,\delta)) \geq -\tilde{H}(\nu).$$

Proof. To prove this lower bound, we follow [1] and consider discrete approximations of the probability measures $\nu \in \{\tilde{H} < \infty\}$ as follows. First note that $\tilde{H} < \infty$ implies that for any $\alpha < \beta$, $\nu([\alpha,\beta]) \leq (\beta-\alpha)$.
Recall from (33) and (32) that $\tilde{H}(\nu) \leq M$ implies that for some universal constant C and $\rho > 0$,

$$\rho\int x d\nu(x) \leq M + C \text{ and } \ \Sigma(\nu) > -M - C. \tag{43}$$

The last condition in particular implies that ν have no atoms. We now construct the following approximations.
Recall that $\phi_L(x) = x \wedge L$ and set $\nu^L = (\phi_L)_\# \nu$. By Chebychev inequality,

$$d(\nu,\nu^L) \leq \int_{x>L} d\nu \leq \rho^{-1}L^{-1}(M+C),$$

and if ν is in $\mathcal{L}$, so is ν^L.
We then consider

$$a_{N,N} = \inf\left\{x\ /\ \nu^L([0,x]) \geqslant \frac{1}{N}\right\}$$
$$a_{i-1,N} = \begin{cases} \inf\left\{x \geqslant a_{i,N}\ /\ \nu^L((a_{i,N},x]) \geqslant \frac{1}{N}\right\}, & \text{if } a_{i,N} < L \\ L + \frac{1}{N}, & \text{otherwise.}\end{cases}$$

It is easy to check that since ν has no atoms, for $N \geq N(\eta)$,

$$d\left(\nu, \frac{1}{N}\sum_{i=1}^N \delta_{a^{i,N}}\right) < \eta + \rho^{-1}L^{-1}(M+C). \tag{44}$$

Now, for N, L large enough so that the right hand sides of (44) is smaller that $2^{-1}\delta$,

$$\bigcap_{i=1}^N\left\{\left|\frac{\ell_i}{N} - a_{i,N}\right| < \frac{\delta}{2}\right\} \subset \left\{d\left(\hat{\mu}^N_\lambda, \frac{1}{N}\sum_{i=1}^N \delta_{a^{i,N}}\right) < \frac{\delta}{2}\right\} \subset \left\{d\left(\hat{\mu}^N_\lambda,\nu\right) < \delta\right\}$$

Therefore

$$\begin{aligned}\tilde{\Pi}^N(B(\nu,\delta) &\geq \tilde{\Pi}^N\left(\bigcap_{i=1}^N\left\{\left|\frac{\ell_i}{N}-a_{i,N}\right|<\frac{\delta}{2}\right\}\right)\\ &\geq \tilde{\Pi}^N\left(\bigcap_{i=1}^N\left\{\left|\frac{\ell_i}{N}-a_{i,N}\right|<\epsilon\right\}\right)\end{aligned}$$

for any $\epsilon\in(0,\frac{\delta}{2}]$. We now show that for any fixed L,

$$\liminf_{\epsilon\downarrow 0}\liminf_{N\to\infty}\frac{1}{N^2}\log\tilde{\Pi}^N\left(\bigcap_{i=1}^N\left\{\left|\frac{\ell_i}{N}-a_{i,N}\right|<\epsilon\right\}\right)\geq -\tilde{H}(\nu^L). \tag{45}$$

Observe first that $\frac{1}{N}\sum_{i=1}^N\delta_{a^{i,N}}$ is supported in $[-L-1,L+1]$ so that all the spherical integrals are well defined and uniformly continuous by Lemma 1. Therefore, we find a $\kappa(\epsilon)$, going to zero with ϵ such that for N sufficiently large,

$$\begin{aligned}\tilde{\Pi}^N\left(\bigcap_{i=1}^N\left\{\left|\frac{\ell_i}{N}-a_{i,N}\right|<\epsilon\right\}\right) &\geq e^{N^2(aI(\log_\#\mu_A,\nu^L)+bI(\log_\#\mu_B,\nu^L)+F(\nu)-\kappa(\epsilon))}\\ &\quad\times\sum_{\left|\frac{\ell_i}{N}-a_{i,N}\right|<\epsilon}\Delta\left(\frac{\ell}{N}\right)^{a+b}e^{-N^2\int c(x)d\hat{\mu}_\lambda^N(x)}\end{aligned} \tag{46}$$

Notice that

$$\begin{aligned}&\sum_{\left|\frac{\ell_i}{N}-a_{i,N}\right|<\epsilon}\Delta\left(\frac{\ell}{N}\right)^{a+b}e^{-N^2\int c(x)d\hat{\mu}_\lambda^N(x)}\\ &\qquad=\sum_{|\frac{\ell_i}{N}-a_{i,N}|<\epsilon}e^{N^2\left(\frac{a+b}{2}\iint_{x\neq y}\log|x-y|d\hat{\mu}_\lambda^N(x)d\hat{\mu}_\lambda^N(y)-\int c(x)d\hat{\mu}_\lambda^N(x)\right)}\\ &\qquad\geqslant e^{-N\sum_{j=1}^N\sup_{|x-a_{j,N}|\leq\frac{\delta}{2}}c(x)+\frac{a+b}{2}N^2\iint_{x\neq y}\log|x-y|d\hat{\mu}_\lambda^N(x)d\hat{\mu}_\lambda^N(y)}.\end{aligned}$$

where λ is a Young shape defined by $\ell_i:=\lfloor Na_{i,N}\rfloor$.
Note that such a tableau exists, since according to the definition of the $a_{i,N}$'s, we have that

$$\frac{1}{N}\leq\nu^L([a_{i+1,N},a_{i,N}])\leq a_{i,N}-a_{i+1,N},$$

so that

$$N(a_{i,N}-a_{i+1,N})\geq 1,$$

which insures that $\ell_i-\ell_{i+1}\geq 1$ and so $\lambda_i\geq\lambda_{i+1}$ for all $i\in\mathbb{N}$. Note that $|\frac{\ell_i}{N}-a_{i,N}|<\frac{1}{N}$ is smaller than ϵ for N large enough.
Furthermore, we also get the estimate

$$a_{i+1,N}\leq\frac{\ell_i}{N}\leq a_{i,N}.$$

Therefore, for i, j such that $i < j - 1$, we have the lower bound

$$\left|\frac{\ell_i}{N} - \frac{\ell_j}{N}\right| \geq \left|a_{i,N} - a_{j-1,N}\right|,$$

so that

$$\sum_{|\frac{\ell_i}{N} - a_{i,N}|<\epsilon} \Delta\Big(\frac{\ell}{N}\Big)^{a+b} e^{-N^2\int c(x)d\hat{\mu}_\lambda^N(x)} \geqslant \exp\Big(N^2\Big(-\frac{1}{N}\sum_{j=1}^{N}(c(a_{j,N}) + C(L,\delta))$$

$$+ \frac{a+b}{2}\frac{1}{N^2}\sum_{i+1<j}\log|a_{i,N} - a_{j,N}| + \frac{a+b}{4N^2}\sum_{i=1}^{N-1}\log\Big|\frac{\ell_{i+1}}{N} - \frac{\ell_i}{N}\Big|\Big)\Big)$$

where $C(L,\delta)$ is going to zero as δ goes to infinity for any given L. With our choice of the $a_{j,N}$'s, we have that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} c(a_{j,N}) = \int x d\nu^L(x),$$

and

$$\frac{1}{N^2}\sum_{i<j}\log|a_{i,N} - a_{j+1,N}| + \frac{1}{2N^2}\sum_{i=1}^{N-1}\log|a_{i,N} - a_{i+1,N}|$$

$$= \sum_{1\leqslant i\leqslant j\leqslant N-1}\log|a_{i,N} - a_{j+1,N}|\nu^L\otimes\nu^L(a_{i,N}\leqslant x\leqslant a_{i+1,N};\ a_{j,N}\leqslant y\leqslant a_{j+1,N})$$

$$\geqslant \int_{a_{1,N}\leqslant x<y\leqslant a_{N,N}}\log|x-y|d\nu^L(x)d\nu^L(y) \tag{47}$$

Let's turn our attention to the last term : for any choice of the ℓ_i's, as the ℓ_i are distinct integers, the difference of a pair of them is at least 1, so that we have

$$\prod_{i=1}^{N-1}\left|\frac{\ell_{i+1}}{N} - \frac{\ell_i}{N}\right| \geqslant \left(\frac{1}{N}\right)^{N-1},$$

which gives

$$\liminf_{N\to\infty}\frac{1}{N^2}\log\sum_{i=1}^{N-1}\log\left|\frac{\ell_{i+1}}{N} - \frac{\ell_i}{N}\right| = 0.$$

Putting everything together, we can conclude,

$$\liminf_{\epsilon\downarrow 0}\liminf_{N\to\infty}\frac{1}{N^2}\log\sum_{|\frac{\ell_i}{N} - a_{i,N}|<\epsilon}\Delta\left(\frac{\ell}{N}\right)^{a+b} e^{-N^2\int c(x)d\hat{\mu}_\lambda^N(x)}$$

$$\geq -\frac{a+b}{2}\Sigma(\nu^L) - \int c(x)d\nu^L(x)$$

(47) and (46) prove (45). To finish the proof , we take the supremum over L to obtain the lower bound thanks to Lemma 1.2 and monotone convergence theorem. $\square$

4. Laplace method for $Z_N(\Phi)(X)$

Let μ_ϕ^N be the measure on $\mathcal{P}(\mathbb{R})$ given, for any measurable set X of $\mathcal{P}(\mathbb{R})$, by

$$\mu_\phi^N(X) = \frac{Z_N(\Phi)(X)}{Z_N(\Phi)}.$$

The goal of this section is to prove a large deviation theorem for μ_ϕ^N.

We first need some definitions.

Definition 1. *With $\mathcal{L}$ and Σ as defined in Definition 2 and ρ_Φ given by (7), we let*

$$\mathcal{G}_\Phi(\nu) = \begin{cases} -I(\log_\# \mu_A, \nu) - I(\log_\# \mu_B, \nu) - \Sigma(\nu) + \rho_\Phi . \int x d\nu(x), & \text{if } \nu \in \mathcal{L}, \\ +\infty & \text{otherwise,} \end{cases}$$

and if $\Psi = ||\Phi||_\infty^{-1}\Phi$,

$$J_\Phi(\nu, \mu) := \begin{cases} -I(\log \Psi_\#\mu, \nu) - \frac{1}{2}S(\Psi_\#\mu) - \Sigma(\mu) + \frac{1}{2}\int x^2 d\mu(x), & \text{if } \nu \in \mathcal{L}, \\ +\infty & \text{otherwise,} \end{cases}$$

with I as defined in Lemma 1. The rate function governing our large deviation principle is then given, for $\mu \in \mathcal{P}(\mathbb{R})$, by

$$I_\Phi(\mu) := \inf_{\nu \in \mathcal{P}(\mathbb{R}^+)} (\mathcal{G}_\Phi(\nu) + J_\Phi(\nu, \mu)) - \inf_{\mu' \in \mathcal{P}(\mathbb{R})} \inf_{\nu' \in \mathcal{P}(\mathbb{R}^+)} \left(\mathcal{G}_\Phi(\nu') + J_\Phi(\nu', \mu')\right).$$

To prove the large deviation principle, we shall make the following additional hypothesis

Hypothesis 3.
The cut-off function Φ is bounded below :

$$\exists \epsilon > 0 \ s.t. \ \forall x \in \mathbb{R}, \ \ \Phi(x) \geq \epsilon. \tag{48}$$

The two sequences of matrices $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ and their spectral measures μ_{A_N} and μ_{B_N} are such that
• there exists an $\alpha > 0$ so that for all N, A_N and B_N are bounded below by αI. Hence, with $\mathcal{K}$ the compact set $[\alpha, 1]$, supp $\hat{\mu}_{A_N} \subset \mathcal{K}$ and supp $\hat{\mu}_{B_N} \subset \mathcal{K}$.
• $\hat{\mu}_{A_N}$ and $\hat{\mu}_{B_N}$ converge weakly respectively to μ_A and μ_B.

We shall then prove that

Theorem 5. *Under Hypotheses 2 and 3,*

1. *I_Φ is a good rate function on $\mathcal{P}(\mathbb{R})$, i.e. I_Φ is non-negative and for any $M \in \mathbb{R}^+$, $\{\nu \in \mathcal{P}(\mathbb{R}) : I_\Phi(\nu) \leq M\}$ is compact.*

2. $(\mu_\Phi^N)_{N\in\mathbb{N}}$ *satisfies a large deviation principle in the scale* N^2 *with good rate function* I_Φ, *i.e*
   - *For any closed subset* F *of* $\mathcal{P}(\mathbb{R})$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu_\Phi^N(F) \le -\inf_F I_\Phi,$$

   - *For any open subset* O *of* $\mathcal{P}(\mathbb{R})$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mu_\Phi^N(O) \ge -\inf_O I_\Phi.$$

3. *Under Hypothesis 3,* $S(\hat{\mu}_{A_N}^N)$ *(resp.* $S(\hat{\mu}_{B_N}^N)$*) converges towards* $S(\mu_A)$ *(resp.* $S(\mu_B)$*), and*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \frac{Z_N(\Phi)}{Z_N(0)} = -\inf_{\mu\in\mathcal{P}(\mathbb{R})} \inf_{\nu\in\mathcal{P}(\mathbb{R}^+)} (\mathcal{G}_\Phi(\nu) + J_\Phi(\nu,\mu)) + \frac{1}{2}S(\mu_A) + \frac{1}{2}S(\mu_B) + \frac{1}{2}\rho_\Phi.$$

The proof of this theorem is deduced from a large deviation principle obtained for the law of the couple $(\hat{\mu}_\lambda^N, \hat{\mu}_M^N)$ given by the Gibbs measure defined, for $X = (X_1, X_2) \subset \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$, by

$$\Pi_\Phi^N(X) = \frac{1}{Z_N(\Phi)} \sum_{\lambda:\hat{\mu}_\lambda^N\in X_1} s_\lambda(A_N)s_\lambda(B_N)Z_N(\Psi,\lambda)(X_2)e^{-\rho_\Phi|\lambda|} \tag{49}$$

that we can formulate as follows :

Theorem 6.

1. *For* $(\nu,\mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$, *we set*

$$\mathcal{I}_\Phi(\nu,\mu) := \begin{cases} +\infty \text{ if } \nu \notin \mathcal{L} \text{ or } \int x^2 d\mu(x) = +\infty, \\ J_\Phi(\nu,\mu) + \mathcal{G}_\Phi(\nu) \\ -\inf_{(\nu',\mu')\in\mathcal{P}(\mathbb{R}^+)\times\mathcal{P}(\mathbb{R})}\{J_\Phi(\nu',\mu') + \mathcal{G}_\Phi(\nu')\} \text{ otherwise.} \end{cases}$$

   Then $\mathcal{I}_\Phi$ *is a good rate function.*
2. $(\Pi_\Phi^N)_{N\in\mathbb{N}}$ *satisfies a full large deviation principle in the scale* N^2 *with rate function* $\mathcal{I}_\Phi$.

Theorem 5.1 and .2 are direct consequences of Theorem 6 and the contraction principle since the application $(\nu,\mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R}) \to \nu \in \mathcal{P}(\mathbb{R})$ is clearly continuous.

Proof of Theorem 6: This proof follows rather closely that of Theorem 4. Let us briefly outline it.

1. To prove that $\mathcal{I}_\phi$ is a good rate function, we note first that it is clearly non-negative. To show that its level sets are compact, we proceed exactly as in section 3.1 ; $\mathcal{G}_\Phi$ has compact level sets by direct application of Theorem 4.1 whereas for J_Φ we can proceed similarly once we notice that $\mu \to S(\Psi_\# \mu)$ is continuous since Ψ is bounded below by a positive constant and
$$S(\Psi_\#\mu) = \int\int \log\left(\int_0^1 (a\Psi(x) + (1-a)\Psi(y))^{-1} da\right) d\mu(x)d\mu(y)$$
and introducing the function
$$j(x,y) = \log|x-y|^{-1} + \frac{1}{4}x^2 + \frac{1}{4}y^2,$$
we can treat it as g (that was introduced in (28)) to show that $\mu \mapsto \int\int j(x,y)d\mu(x)d\mu(y)$ is lower semicontinuous on $\mathcal{P}(\mathbb{R})$ and infinite unless $\int x^2 d\mu(x)$ and $\Sigma(\mu)$ are finite.
Thus, we see that $\mathcal{I}_\Phi(\nu,\mu)$ is infinite unless
$$\nu \in \mathcal{L},\ \int x d\nu(x) < \infty,\ \Sigma(\nu) > -\infty,\ \int x^2 d\mu(x) < \infty,\ \Sigma(\mu) > -\infty.$$
This in particular shows that we can apply the formula for the spherical integral obtained in [10] when $\int x^2 d\nu(x) < \infty$ and hence the formula for $\mathcal{I}_\Phi$ is explicit.
2. To prove that Π_Φ^N is exponentially tight, we consider the compact set
$$K_L := \left\{\nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \le L\right\} \times \left\{\mu \in \mathcal{P}(\mathbb{R}) : \int x^2 d\mu(x) \le L\right\}.$$
It is not hard to bound below Z_Φ^N by some estimate of order $e^{-N^2 C}$ (for instance by proving the lower bound estimate as below). Then, using the fact that $S(\Psi_\#\mu)$ as well as the spherical integrals are bounded uniformly, we find a finite constant C' such that
$$\Pi_\Phi^N(K_L^c) \le e^{C'N^2}\left(\tilde{\Pi}^N(\mathcal{K}_L^c) + \int_{\sum x_i^2 \ge NL} \Delta(x)^2 e^{-\frac{N}{2}\sum_{i=1}^N x_i^2} \prod_{i=1}^N dx_i\right).$$
Following [1] (or the arguments of section 3.2) we easily see that for sufficiently large L
$$\limsup_{N\to\infty} \frac{1}{N^2} \log \int_{\sum x_i^2 \ge NL} \Delta(x)^2 e^{-\frac{N}{2}\sum_{i=1}^N x_i^2} \prod_{i=1}^N dx_i \le -\frac{1}{4}L$$
so that we can conclude again by section 3.2.
3. To prove the weak large deviation upper bound, we proceed as in section 3.3 by considering the functions g (with $c(x) = \rho_\Phi x$ and $a = b = 1$) and j. We then impose a cutoff on both functions and on the spherical integrals as in (41) to obtain a large deviation upper bound estimate, and then proceed again by optimizing over the cutoff.

4. For the large deviation lower bound, we restrict the sum and the integral also to configurations contained in small neighborhoods of well chosen values $(a_{i,N})_{1\le i\le N}$ and $(x_{i,N})_{1\le i\le N}$ and show convergence. This strategy works as well in the continuous setting as can be seen in [1].

□

5. Study of the minimizers of $\mathcal{I}_\Phi$

In this section, we wish to give some weak description of the minimizers of $\mathcal{I}_\Phi$. We have not been able to prove uniqueness of such minimizers. In [9], uniqueness of the minimizers of the rate function was deduced from convexity arguments which were actually lacking for instance for the q-Potts model. In fact, the spherical integrals are expressed as the sum of a convex complicated function and the entropies Σ which are concave. Hence, if the full rate function does not contain some term to kill these Σ terms, the convexity of the full rate function becomes unclear. The same phenomenon appears here and despite our efforts we could not overcome this difficulty. It is unclear here whether the minimizer should be unique or not. We here meet the additional difficulty that the formula obtained in [10] for the limit of the spherical integral concerned the case where both probability measures had finite covariance, which is not the case here (one of the argument has only a first moment which is finite, even if the other one is compactly supported). In this section, we show that the minimizers of $\mathcal{I}_\Phi$ are compactly supported. We then characterize the minimizers.

Proposition 1. *Assume that* $\Sigma(\log_\# \mu_A) > -\infty,\ \ \Sigma(\log_\# \mu_B) > -\infty$. *Then*

1. *There exists a real number* $M \ge 0$ *such that any minimizer* $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$ *of* $\mathcal{I}_\Phi$ *satisfies supp*$(\nu) \subset [0, M]$.
2. *If we additionally assume that there exists* $A < B$ *in* $\mathbb{R}$ *such that for L large enough* Φ *satisfies*

$$\max_{|x|\ge L} \Phi(x) \le \inf_{x\in[A,B]} \Phi(x) \tag{50}$$

   then there exists a real number M such that for any minimizer $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$ *of* $\mathcal{I}_\Phi$, μ *satisfies supp*$(\mu) \subset [-M, M]$.
3. $\mathcal{I}_\Phi$ *achieves its minimal value (which is zero). Let* $(\bar\nu, \bar\mu)$ *be a minimizer. Then, there exists 3 flows* $(\rho^i, u^i)_{1\le i\le 3}$ *such that*
   • $\mu^i_t(dx) = \rho^i_t(x)dx$ *is a probability measure for all* $t \in (0, 1)$. $t \in [0, 1] \to \mu^i_t \in \mathcal{P}(\mathbb{R})$ *is continuous.*

$$\lim_{t\to 0} \mu^1_t = \log_\# \mu_A, \quad \lim_{t\to 0} \mu^2_t = \log_\# \mu_B, \quad \lim_{t\to 0} \mu^3_t = \log \Psi_\# \mu,$$

$$\lim_{t\to 1} \mu^i_t = \nu, \quad 1 \le i \le 3.$$

• *For $i \in \{1, 2, 3\}$, (ρ^i, u^i) satisfies the Euler equation for isentropic flow described by the equations, for $t \in (0, 1)$,*

$$\partial_t \rho_t^i(x) = -\partial_x(\rho_t^i(x)u_t^i(x)) \tag{51}$$

$$\partial_t(\rho_t^i(x)u_t^i(x)) = -\partial_x\left(\rho_t^i(x)u_t^i(x)^2 - \frac{\pi^2}{3}\rho_t^i(x)^3\right) \tag{52}$$

in the sense of distributions that for all $f \in \mathcal{C}_c^{\infty,\infty}(\mathbb{R} \times [0, 1])$,

$$\int_0^1 \int \partial_t f(t, x) d\mu_t^i(x) dt + \int_0^1 \int \partial_x f(t, x) u_t^i(x) d\mu_t^i(x) dt = 0$$

and, for any $f \in \mathcal{C}_c^{\infty,\infty}(\Omega)$ with $\Omega_i := \{(x, t) \in \mathbb{R} \times [0, 1] : \rho_t^i(x) > 0\}$,

$$\int_0^1 \int \left(2u_t^i(x)\partial_t f(x, t) + \left(u_t^i(x)^2 - \pi^2\rho_t^i(x)^2\right)\partial_x f(x, t)\right) dxdt = 0, \tag{53}$$

where $\mathcal{C}_c^{\infty,\infty}(\mathcal{A})$ is the space of functions which are infinitely differentiable on both variables on the open set $\mathcal{A}$ and compactly supported.
(ρ^i, u^i) are smooth in the interior of Ω_i, which guarantees that (51) and (52) hold everywhere in the interior of Ω_i. Moreover, Ω_i is bounded in $\mathbb{R} \times [0, 1]$.
• *Let $\bar{\rho}$ be the density of $\bar{\nu}$ and $\bar{\Omega} = \{x : \bar{\rho}(x) > 0\}$ Then, for any continuously differentiable test function ϕ which is supported in the interior of $\bar{\Omega}$,*

$$\int \left(\rho_\Phi x - \frac{1}{2}x^2 + \int \log|x - y| d\bar{\nu}(y)\right)\partial_x\phi(x) dx = \sum_{i=1}^{3} \int \phi(x)u_1^i(x) dx.$$

• *For any $\phi \in \mathcal{C}^1(Im(\log\Psi)^c \cap supp(\bar{\mu}))$,*

$$\int \partial_x\phi(x)\left(\frac{1}{2}x^2 - 2\int \log|x - y| d\bar{\mu}(y)\right) dx = 0$$

To simplify, we shall assume that $\log\Psi$ is one to one from $\mathbb{R}$ into its image $Im(\log\Psi)$. Then, in a very weak sense of distribution, for any $\phi \in \mathcal{C}^1(Im(\log\Psi) \cap supp(\bar{\mu}))$

$$\int \partial_x\phi\big(-\frac{1}{2}x^2 + \frac{1}{2}(\log\Psi)^{-1}(x)^2 - 2\int \log|(\log\Psi)^{-1}(x) - y| d\bar{\mu}(y)$$
$$+ \int \log\left|e^x - \Psi(y)\right| d\bar{\mu}(y)\big)dx = -\int \phi(x)u_0^3(x) dx.$$

If $\bar{\mu}$ has a density with respect to Lebesgue measure, we obtain the usual sense of distribution in the interior of $Im(\log\Psi) \cap supp(\bar{\mu})$.

Note that the additional assumption $\Sigma(\log_\# \mu_A) > -\infty$ and $\Sigma(\log_\# \mu_B) > -\infty$ allows us to use an alternative formula for I obtained in [9].

Proof. • We first prove that for any minimizer $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$ of $\mathcal{I}_\Phi$, ν is compactly supported. In [9], such a result was obtained by going back to the matrix model. We shall here provide a new and more elegant proof based on the idea that if a minimizer would put mass far away from the origin, we can construct another measure with smaller entropy $\mathcal{I}_\Phi$ by transporting back this mass in a neighborhood of the origin.

To do so, the only property of the spherical integral we shall use is the following : Let ν and ν^* in $\mathcal{P}(\mathbb{R}^+)$ be such that there exists a coupling $\pi \in \mathcal{P}(\mathbb{R}^+ \times \mathbb{R}^+)$ of (ν, ν^*) such that $\pi(x \in .) = \nu(x \in .)$, $\pi(y \in .) = \nu^*(y \in .)$, and

$$\pi(x \leq y) = 1. \tag{54}$$

Then, for any $\mu \in \mathcal{P}(\mathbb{R}^-)$ which is compactly supported,

$$I(\nu^*, \mu) \leq I(\nu, \mu). \tag{55}$$

This is a direct consequence of the definition of the spherical integral ; indeed, by the above, we can construct discrete approximations $(\ell_i, 1 \leq i \leq N)$ and $(\ell_i^*, 1 \leq i \leq N)$ such that $N^{-1} \sum_{i=1}^N \delta_{\frac{\ell_i}{N}}$ (resp. $N^{-1} \sum_{i=1}^N \delta_{\frac{\ell_i^*}{N}}$) converges towards ν (resp. ν^*) and $\ell_i \leq \ell_i^*$. Therefore, if $N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$ approximates μ with $\lambda_i \leq 0$, it is clear that

$$I_N\left(\frac{\ell_i}{N}, \lambda_i\right) \geq I_N\left(\frac{\ell_i^*}{N}, \lambda_i\right)$$

yielding (55) at the limit $N \to \infty$.

Let now (ν^*, μ^*) be a minimizer and ν satisfying (54) belonging to $\mathcal{L}$. By definition,

$$\mathcal{I}_\Phi(\nu, \mu^*) \geq \mathcal{I}_\Phi(\nu^*, \mu^*),$$

and therefore by (55), since $\log_\# \mu_A$, $\log_\# \mu_B$ and $\log \Psi_\# \mu$ are supported in $\mathbb{R}^-$,

$$-\Sigma(\nu) + \rho_\Phi \int x d\nu(x) \geq -\Sigma(\nu^*) + \rho_\Phi \int x d\nu^*(x). \tag{56}$$

We shall use this inequality for a well chosen ν which is a modification of ν^*. We construct it as follows : recall that $\nu^* \in \mathcal{L}$ implies that $\nu^*(dx) = \rho^*(x)dx$ with $\rho^* \leq 1$. We assume that $\nu^*([0, M]) < 1$ and are going to show a contradiction for M large enough. Observe that $A := \int_0^3 1_{\{x:\rho^*(x) \leq \frac{1}{2}\}} dx \geq 1$ since $\int_0^\infty \rho^*(x)dx = 1$. Set for $M \geq 3$,

$$\nu = \nu_M = 1_{[0,M]}\nu^* + \frac{\alpha_M}{A} 1_{\{\rho^* \leq \frac{1}{2}, x \in [0,3]\}} dx,$$

with $\alpha_M = \nu^*([M, \infty))$.

We have on one side that

$$
\begin{aligned}
-\Sigma(\nu^*) &= -\Sigma(1_{[0,M]}\nu^*) + 2\int_{\substack{x<M\\ y>M}} \log|x-y|^{-1} d\nu^*(x)d\nu^*(y) \\
&\quad + \int_{\substack{x>M\\ y>M}} \log|x-y|^{-1} d\nu^*(x)d\nu^*(y) \\
&\geq -\Sigma(1_{[0,M]}\nu^*) + 2\int_{\substack{x<M, y>M\\ |x-y|>1}} \log|x-y|^{-1} d\nu^*(x)d\nu^*(y) \\
&\quad + \int_{\substack{x>M, y>M\\ |x-y|>1}} \log|x-y|^{-1} d\nu^*(x)d\nu^*(y)
\end{aligned}
$$

Using that for all $a \in (0, 1]$ there exists a finite constant C_a such that for all $x \geq 0$,

$$
\log(1+x) \leq C_a x^a
$$

we deduce

$$
\begin{aligned}
-\Sigma(\nu^*) &\geq -\Sigma(1_{[0,M]}\nu^*) - 2C_a \int_{\substack{x<M, y>M\\ |x-y|>1}} (|x-y|-1)^a d\nu^*(x)d\nu^*(y) \\
&\quad - C_a \int_{\substack{x>M, y>M\\ |x-y|>1}} (|x-y|-1)^a d\nu^*(x)d\nu^*(y) \\
&\geq -\Sigma(1_{[0,M]}\nu^*) - (2+\alpha_M)C_a \int_{y>M} y^a d\nu^*(y) \\
&\geq -\Sigma(1_{[0,M]}\nu^*) - (2+\alpha_M)C_a M^{a-1} \int_{y>M} y d\nu^*(y)
\end{aligned} \tag{57}
$$

where we used in the last line Chebyshev inequality.
On the other side,

$$
\begin{aligned}
-\Sigma(\nu_M) &= -\Sigma(1_{[0,M]}\nu^*) + 2\frac{\alpha_M}{A}\int_{x<M}\int_0^3 1_{\rho^*(y)\leq \frac{1}{2}} \log|x-y|^{-1} dy d\nu^*(x) \\
&\quad + \left(\frac{\alpha_M}{A}\right)^2 \int_0^3 1_{\rho^*(x)\leq\frac{1}{2}} \int_0^3 1_{\rho^*(y)\leq\frac{1}{2}} \log|x-y|^{-1} dy dx \\
&\leq -\Sigma(1_{[0,M]}\nu^*) \\
&\quad + 2\frac{\alpha_M}{A}\int_{x<M}\int_0^3 1_{\rho^*(y)\leq\frac{1}{2}} 1_{|x-y|\leq 1} \log|x-y|^{-1} dy \rho^*(x) dx \\
&\quad + \left(\frac{\alpha_M}{A}\right)^2 \int_0^3 1_{\rho^*(x)\leq\frac{1}{2}} \int_0^3 1_{\rho^*(y)\leq\frac{1}{2}} 1_{|x-y|\leq 1} \log|x-y|^{-1} dy dx \\
&\leq -\Sigma(1_{[0,M]}\nu^*) \\
&\quad + \left(2\frac{\alpha_M}{A} + \left(\frac{\alpha_M}{A}\right)^2\right) \int_{x<4}\int_0^3 1_{|x-y|\leq 1} \log|x-y|^{-1} dy dx \\
&\leq -\Sigma(1_{[0,M]}\nu^*) + 4\left(2\frac{\alpha_M}{A} + \left(\frac{\alpha_M}{A}\right)^2\right)
\end{aligned} \tag{58}
$$

Observe now that ν_M is in $\mathcal{L}$ for M large enough so that $A^{-1}\alpha_M \le 2^{-1}$. Furthermore, ν_M satisfies (54) since we have been transporting large values of the l_i's to smaller one. Hence, we can apply (56) and together with (57), (58), it gives that

$$\rho_\Phi \left(\int_{x>M} x d\nu^*(x) - \frac{\alpha_M}{A} \int_0^3 x 1_{\rho^* < \frac{1}{2}} dx \right) \\ \le (2+\alpha_M) C_a M^{a-1} \int_{y>M} y d\nu^*(y) + 4\left(2\frac{\alpha_M}{A} + \left(\frac{\alpha_M}{A}\right)^2 \right),$$

showing that for any $a \in (0, 1)$, for M large enough,

$$(\rho_\Phi - (2+\alpha_M) C_a M^{a-1}) \int_{x>M} x d\nu^*(x) \le \frac{15}{A}\alpha_M \le \frac{15}{AM} \int_{x>M} x d\nu^*(x). \tag{59}$$

Hence, $\int_{x>M} x d\nu^*(x)$ has to be null for M large enough so that $\rho_\Phi - (2 + \alpha_M) C_a M^{a-1} - \frac{15}{AM} > 0$.

• We now pass to the proof of the second point of the proposition. Let, with $\beta_M = \mu^*([-M, M]^c)$, for $B > A$,

$$\mu_M(dx) = 1_{[-M,M]} \mu^*(dx) + \frac{\beta_M}{B-A} 1_{[A,B]} dx$$

Because of our assumption, we see that if M is large enough and $[A, B]$ chosen so that

$$\inf_{[A,B]} \Phi \ge \sup_{[-M,M]^c} \Phi$$

for any $\nu \in \mathcal{P}(\mathbb{R}^+)$,

$$I(\log \Psi_\# \mu_M, \nu) \ge I(\log \Psi_\# \mu^*, \nu).$$

Hence, when (μ^*, ν^*) minimize $\mathcal{I}_\Phi$, we obtain

$$- \Sigma(\mu^*) + \frac{1}{2} \int x^2 d\mu^*(x) - \frac{1}{2} S(\Psi_\# \mu^*) \\ \le -\Sigma(\mu_M) + \frac{1}{2} \int x^2 d\mu_M(x) - \frac{1}{2} S(\Psi_\# \mu_M) \tag{60}$$

Arguing as above, we find that, for any $a \in (0, 2)$, there exists a finite constant C_a such that

$$\Sigma(\mu^*) - \Sigma(\mu_M) \le C_a M^{a-2} \int x^2 d\mu^*(x) \tag{61}$$

$$-S(\Psi_\# \mu_M) + S(\Psi_\# \mu^*) \le C\beta_M \tag{62}$$

where we observed in the last line that Ψ was bounded uniformly above and below. Hence, we get

$$\left(\frac{1}{2} - C_a M^{a-2}\right)\int_{x\geq M} x^2 d\mu^*(x) \leq C'\beta_M \leq C'M^{-2}\int_{x\geq M} x^2 d\mu^*(x) \qquad (63)$$

where $C' = C + B^2$. This is again a contradiction for sufficiently large M.

• We finally study the characterization of the minimizers. In [9], the characterization was done by going back to the matrix model description. We shall here tackle this problem by a direct study of the rate function. Note that by point 1., any minimizers $(\bar{\nu}, \bar{\mu})$ is such that $\bar{\nu}$ is compactly supported. Moreover $\log \Psi_{\#}\bar{\mu}$, $\log_{\#} \mu_A$ and $\log_{\#} \mu_B$ are also compactly supported by our hypotheses so that we can apply Property 2.2 in [9] which says that if μ, ν are two probability measures with finite covariance and such that $\Sigma(\mu) > -\infty$, $\Sigma(\nu) > -\infty$,

$$I(\mu,\nu) = -\frac{1}{2}\inf_{(\rho,u)\in C(\mu,\nu)}\{S(\rho,u)\} - \frac{1}{2}\left(\Sigma(\mu) + \Sigma(\nu) - \mu(x^2) - \nu(x^2)\right) + c \qquad (64)$$

where

$$S(\rho,u) := \int_0^1 \int u_t(x)^2 \rho_t(x) dx dt + \frac{\pi^2}{3}\int_0^1 \int \rho_t(x)^3 dx dt,$$

$$C(\mu,\nu) = \Big\{\rho_. \in L^1(dxdt), \int \rho_t(x)dx = 1 \ \forall t \in [0,1], \lim_{t\to 0} \rho_t(x)dx = \mu, \\ \lim_{t\to 1} \rho_t(x)dx = \nu, \partial_t \rho_t(x) + \partial_x(\rho_t(x)u_t(x)) = 0\Big\},$$

where the last equality is to be understood in the sense of distributions. It was shown in [9] that the infimum defining I is achieved at a unique $(u^*, \rho^*) \in C(\mu,\nu)$ which is described by an isentropic Euler equation with negative pressure $p(\rho) = -\frac{\pi^2}{3}\rho^3$. In (64), c is some universal constant. Since $\mathcal{I}_\Phi(\mu,\nu) < \infty$ implies that $\Sigma(\mu) > -\infty$, $\Sigma(\nu) > -\infty$ and $\mu(x^2) < \infty$, for any $\nu \in \mathcal{P}(\mathbb{R}^+)$ such that $\nu(x^2) < \infty$, we can apply (64) to obtain

$$\begin{aligned}\mathcal{I}_\Phi(\mu,\nu) &= \inf_{((\rho^i,u^i)\in C(\mu^i,\nu))_{1\leq i\leq 3}} \Big\{\frac{1}{2}\sum_{i=1}^{3} S(\rho^i,u^i) + \frac{1}{2}\Sigma(\nu) - \Sigma(\mu) + \frac{1}{2}\Sigma(\Psi_{\#}\mu) \\ &\qquad + \frac{1}{2}\mu(-\log\Psi(x)^2 + x^2) - \frac{3}{2}\nu(x^2) + \rho_\Phi \nu(x) + K(\mu_A,\mu_B)\Big\} \\ &:= \inf_{((\rho^i,u^i)\in C(\mu^i,\nu))_{1\leq i\leq 3}} \Xi\left((\rho^i,u^i)_{1\leq i\leq 3},\nu,\mu\right), \qquad (65)\end{aligned}$$

where $\mu^1 = \log_\# \mu_A$, $\mu^2 = \log_\# \mu_B$, $\mu^3 = \log \Psi_\# \bar{\mu}$ and $K(\mu_A, \mu_B)$ is a constant depending only on μ_A and μ_B .

We now consider a minimizer $((\bar{\rho}^i, \bar{u}^i)_{1\le i\le 3}, \bar{\mu}, \bar{\nu})$ of Ξ in $\Omega := \{\nu \in \mathcal{L}, \mu \in \mathcal{P}(\mathbb{R}), (\bar{\rho}^i, \bar{u}^i)_{1\le i\le 3} \in C(\log_\# \mu_A, \nu) \times C(\log_\# \mu_B, \nu) \times C(\log \Psi_\# \mu, \nu)\}$. We can follow the ideas of [9], Property 2.9, to perturbe this minimizer with respect to the source to see that $(\bar{u}^i, \bar{\rho}^i)_{1\le i\le 3}$ satisfies for any test functions ϕ such that

$$L(\phi) = \sum_{i=1}^{3} \int_0^1 \int \left(\frac{|\partial_t \phi^i(t,x)|^2}{\bar{\rho}^i(t,x)} \right) dxdt + \sum_{i=1}^{3} \sup_{t\in(0,1)} \left\| \frac{\partial_x \phi^i(t,x)}{\bar{\rho}^i(t,x)} \right\|_\infty < \infty,$$

$$\begin{aligned}
&\int \left(\rho_\Phi x - \frac{3}{2}x^2 \right) \partial_x \phi(1,x)dx - \frac{1}{2}\int x^2 \partial_x \phi^3(0,x)dx + \frac{1}{2}\int x^2 \partial_x \psi(x)dx \\
&+ \iint \log|x-y| d\bar{\nu}(y) \partial_x \phi(1,x)dx - 2\iint \log|x-y| d\bar{\mu}(y)\partial_x \psi(x)dx \\
&+ \iint \log\left|e^x - e^y\right| d\log \Psi_\# \bar{\mu}(y) \partial_x \phi^3(0,x)dx \qquad (66) \\
&+ \frac{1}{2}\sum_{i=1}^{3} \int \int_0^1 [-2\partial_t \phi^i(t,x)\bar{u}^i(t,x) - (\bar{u}^i(t,x))^2 \partial_x \phi^i(t,x) \qquad (67) \\
&+ \pi^2 (\bar{\rho}^i(t,x))^2 \partial_x \phi^i(t,x)]dxdt = 0.
\end{aligned}$$

Applying this result with $\phi^i(0,x) = \phi^i(1,x) = 0$ shows that $(\bar{u}^i, \bar{\rho}^i)_{1\le i\le 3}$ satisfies the Euler equation for isentropic flow described in the proposition.

We now turn to the boundary conditions expressed in the last two points of Proposition 1. We obtain a stronger result than in [9] where similar equations were obtained only under a differential form (i.e equations were obtained for the Hilbert transform rather than for its primitive). To characterize these equations, we will try to regularize the densities $\rho^i_\epsilon(t,.)$. We remark that by Property 2.8 in [9], since $\bar{\nu}$ and $\bar{\mu}$ are compactly supported under our hypothesis, we can find sequences of potentials $(h^{\epsilon,i}, \epsilon > 0, 1 \le i \le 3)$ in $\mathcal{C}_b^{1,1}(\mathbb{R} \times [0,1])$ such that if we set

$$\rho^i_\epsilon(t,x) := \pi^{-1}(\max\{\partial_t h^{\epsilon,i}(t,x) + 4^{-1}(\partial_x h^{\epsilon,i}(t,x))^2, 0\})^{\frac{1}{2}}$$

then for any $\epsilon > 0$,

$$\begin{aligned}
&\int \left(\bar{u}^i(t,x) - \partial_x \frac{h^{\epsilon,i}(t,x)}{2} \right)^2 \bar{\rho}^i(t,x)dxdt \\
&\quad + \frac{\pi^2}{3}\int_0^1 \int \left(\bar{\rho}^i(t,x) - \rho^i_\epsilon(t,x) \right)^2 \left(\bar{\rho}^i(t,x) + \rho^i_\epsilon(t,x) \right) dxdt \\
&\quad + \pi^2 \int_0^1 \int |\partial_t h^{\epsilon,i}(t,x) + 4^{-1}(\partial_x h^{\epsilon,i}(t,x))^2 - \pi^2 \rho^i_\epsilon(t,x)^2 | \bar{\rho}^i(t,x)dxdt \le \epsilon.
\end{aligned}$$

From this result, we deduce that

$$\sup_{1\le i\le 3}\Big|\int_0^1\int[-2\partial_t\phi^i(t,x)\bar u^i(t,x)-(\bar u^i(t,x))^2\partial_x\phi^i(t,x) +\pi^2(\bar\rho^i(t,x))^2\partial_x\phi^i(t,x)]dxdt$$
$$-\int_0^1\int[-\partial_t\phi^i(t,x)\partial_x h^{\epsilon,i}(t,x)-\frac14(\partial_x h^{\epsilon,i}(t,x))^2\partial_x\phi^i(t,x) + \pi^2(\rho_\epsilon^i(t,x))^2\partial_x\phi^i(t,x)]dxdt\Big|\le C_{L(\phi)}\sqrt{\epsilon}$$

with $C(L(\phi))<\infty$ when $L(\phi)<\infty$. Moreover, since $h^{i,\epsilon}\in\mathcal{C}^{1,1}(\mathbb{R}\times[0,1])$, we can integrate by part so that

$$\Big|\int_0^1\int[-\partial_t\phi^i(t,x)\partial_x h^{\epsilon,i}(t,x)-4^{-1}(\partial_x h^{\epsilon,i}(t,x))^2\partial_x\phi^i(t,x) + \pi^2(\rho_\epsilon^i(t,x))^2\partial_x\phi^i(t,x)]dxdt-2\left[\int h^{\epsilon,i}\partial_x\phi^i dx\right]_0^1\Big|\le C'(L(\phi))\sqrt{\epsilon}$$

We now can define in the sense of distribution

$$\int \Pi_t^i\partial_x\phi^i dx=-\int u_t^i\phi^i dx$$

and by letting ϵ going to zero we get that

$$\int[-2\partial_t\phi^i(t,x)\bar u^i(t,x)-(\bar u^i(t,x))^2\partial_x\phi^i(t,x)+\pi^2(\bar\rho^i(t,x))^2\partial_x\phi^i(t,x)]dxdt = 2\left[\int \Pi_t^i\partial_x\phi^i dx\right]_0^1.$$

Thus, we have proved that we can rewrite (66) under the form

$$\begin{aligned}&\int\left(\rho_\Phi x-\frac32 x^2\right)\partial_x\phi(1,x)dx-\frac12\int x^2\partial_x\phi^3(0,x)dx+\frac12\int x^2\partial_x\psi(x)dx\\ &+\iint\log|x-y|d\bar\nu(y)\partial_x\phi(1,x)dx-2\iint\log|x-y|d\bar\mu(y)\partial_x\psi(x)dx\\ &+\iint\log|\Psi(x)-\Psi(y)|d\bar\mu(y)\partial_x\psi(x)dx\\ &+\sum_{i=1}^3\left(\int\Pi_1^i\partial_x\phi(1,x)dx-\int\Pi_0^i\partial_x\phi^i(0,x)dx\right)=0\end{aligned}\tag{68}$$

From that we can deduce the boundary conditions we are seeking for.
As the equality (68) holds for any function $\partial_x\phi(1,x)$ such that $L(\phi)$ is finite, we find that

$$A(x,\bar\nu)=\rho_\Phi x-\frac32 x^2+\int\log|x-y|d\bar\nu(y)+\sum_{i=1}^3\Pi_1^i(x)\tag{69}$$

is constant in the sense of distribution.

Furthermore, it is not hard to deduce from the representation of ρ_t^i as a free Brownian motion given in [9] that for t close enough to one $\{x : \rho_t^i(x) \geq \epsilon\} \subset \{x : \bar{\rho}(x) \geq 2\epsilon\}$ with $\bar{\rho}$ the density of $\bar{\nu}$ with respect to Lebesgue measure. Therefore, for any $\mathcal{C}_b^1$ function ϕ with compact support in the interior of $\{x : \bar{\rho}(x) > 0\}$,

$$\int \partial_x \phi(x) A(x, \bar{\nu}) dx = 0.$$

Now only the last point of our proposition is left to establish.
The statement of the result is more obscure when dealing with $\bar{\mu}$ since we do not a priori know if $\bar{\mu}$ has a density with respect to Lebesgue measure. What we get from (68) is that :

For any $\psi \in \mathcal{C}_b^1(\mathrm{Im}(\log \Psi)^c \cap \mathrm{supp}(\bar{\mu}))$

$$\int \partial_x \psi(x) \left(\frac{1}{2} x^2 - 2 \int \log|x-y| d\bar{\mu}(y) \right) dx = 0$$

i.e $\frac{1}{2}x^2 - 2\int \log|x-y|d\bar{\mu}(y)$ is constant outside of the image $\mathrm{Im}(\log \Psi)$ of $\log \Psi$.

Inside $\mathrm{Im}(\log \Psi)$, if we assume that $\log \Psi$ is one to one from $\mathbb{R}$ onto its image, we have that

$$\begin{aligned} B(x, \bar{\mu}) = -\frac{1}{2}x^2 + \frac{1}{2}(\log \Psi)^{-1}(x)^2 - 2 \int \log|(\log \Psi)^{-1}(x) - y| d\bar{\mu}(y) \\ + \int \log \left| e^x - \Psi(y) \right| d\bar{\mu}(y) - \Pi_0^3(x) \end{aligned}$$

is constant in the weak sense of distribution that is its integral with respect to $\partial_x \phi^3(x, 0)$ vanishes. If $\bar{\mu}$ has a density with respect to Lebesgue measure, we find that $B(x, \bar{\mu})$ is constant in the sense of distribution inside $\{x : \frac{d\bar{\mu}}{dx} \neq 0\}$ as above, but it is not clear that a $\phi^3 \neq 0$ indeed exists in general ! □

6. Conclusion and remarks

In this paper, we studied the asymptotics of the model given by the partition function (1). In the course of doing so, we adapted the techniques of [1] to study large deviations of the profiles of Young tableaux with a density given by a Vandermonde determinant and Schur polynomial functions (see Theorem 2). We believe that these techniques might be useful to study other problems since these kinds of distributions appear in different contexts due to their combinatorial nature. For instance, following Migdal-Witten formula [26, 25], the partition function of two-dimensional Yang Mills theory on a cylinder with gauge group $U(N)$ is given by the central heat kernel defined, at time $t = TN^{-1}$, by

$$\mathcal{Z}_N \left(U_1, U_2; \frac{T}{N} \right) = \sum_{\lambda} s_\lambda(U_1) s_\lambda(U_2) e^{-\frac{T}{2N} C_2(\lambda)}$$

where $U_1, U_2 \in U(N)$, the sum runs over Young tableaux λ and

$$C_2(\lambda) = \sum_{i=1}^{N} \lambda_i(\lambda_i + 1 - 2i + N) = \sum_{i=1}^{N} \ell_i^2 - (N-1)\sum_{i=1}^{N} \ell_i + \sum_{i=1}^{N}(N-i)(i-1)$$

with $\ell_i = \lambda_i + N - i$ (see for example [8]).

S. Zelditch [27] asked us if we could study the asymptotics of $\mathcal{Z}_N(U_1, U_2; TN^{-1})$ when U_1, U_2 are not unitary but real diagonal matrices with converging spectral distributions. Our techniques apply readily to this context and we find

Theorem 7. *Let A_N, B_N be two sequences of uniformly bounded matrices bounded below by ϵI for some $\epsilon > 0$ with spectral measures converging towards μ_A, μ_B. Then for any time $T > 0$*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathcal{Z}_N\left(A_N, B_N; \frac{T}{N}\right) = Z(\mu_A, \mu_B, T)$$

with

$$\begin{aligned} Z(\mu_A, \mu_B; T) = \sup_{\nu\in\mathcal{L}} \Big\{ & I(\log_\# \mu_A, \nu) + I(\log_\# \mu_B, \nu) + \Sigma(\nu) \\ & -\frac{T}{2}\int x^2 d\nu(x) + \frac{T}{2}\int x d\nu(x)\Big\} + \frac{1}{2}S(\mu_A) + \frac{1}{2}S(\mu_B) - \frac{T}{12} \end{aligned}$$

This theorem is a direct consequence of Theorem 2 with $a = b = 1$ and $c(x) = x^2 - x$.

In addition to giving a rigorous basis to the study of such natural asymptotics, we gave a firm ground to begin the study of other matrix models where additional problems due for instance to signed series might appear. This step seems necessary since the proofs are already rather involved. Furthermore, we developed new arguments to study the critical points of our model based on transport of mass.

One of the weakness of our result is apparently the cut-off function Φ, since the matrix integral (1) is then hard to relate with the enumeration of maps as in [14]. Let us comment heuristically this point. Observe first that the matrix integral (1) with $\Phi(x) = x$ considered in [14] is always infinite. Indeed, for instance in the case $A = 1$, we are integrating

$$Z_N(Id) = \int_{x_i\in\mathbb{R}} \Delta(x)^2 \prod_{i,j=1}^{N} \frac{1}{1 - b_i x_j} e^{-N\sum x_j^2} \prod dx_j$$

which is clearly infinite for all $N \in \mathbb{N}^*$. Hence, everything should be understood formally. The same problem a priori also arises when one considers random triangulations generated by the one matrix integrals

$$\tilde{Z}_N(\lambda) = \int e^{\lambda N \mathrm{tr}(M^3) - \frac{N}{2}\mathrm{tr}(M^2)} dM$$

which is clearly infinite when $\lambda \neq 0$ is real. One way to bypass this problem is for instance to consider

$$\tilde{Z}_N(\lambda, \eta) = \int e^{-\eta N \operatorname{tr}(M^4) + \lambda N \operatorname{tr}(M^3) - \frac{N}{2}\operatorname{tr}(M^2)} dM$$

which is well defined for $\eta > 0$. Recall that planar maps are enumerated by

$$C(n) = \lim_{N\to\infty} \partial_\lambda^n \frac{1}{N^2} \log \tilde{Z}_N(\lambda)|_{\lambda=0} = \lim_{N\to\infty} \partial_\lambda^n \frac{1}{N^2} \log \tilde{Z}_N(\lambda, \eta)|_{\lambda=0,\eta=0}.$$

In the physics literature, these quantities are implicitely supposed to be given by

$$\tilde{C}(n) = \partial_\lambda^n \lim_{N\to\infty} \frac{1}{N^2} \log \tilde{Z}_N(\lambda, \eta)|_{\lambda=0,\eta=0}.$$

This is fine for rather general one matrix models according to Theorem 1.1 of [6], but this point is open in general.

Similarly, one could try to regularize the dually weighted graphs model by considering $Z_N(\Phi_{\epsilon,R})$ with

$$\Phi_{\epsilon,R}(x) = \frac{x}{1+\epsilon x^2} + R$$

with $\epsilon > 0$ and $R \geq \sqrt{2\epsilon}^{-1}$. For $||A||$ and $||B||$ small enough (which we can always assume since again only derivatives at the origin should be of interest), we obtain by our result a limit for $N^{-2} \log Z_N(\Phi_{\epsilon,R})$. Assuming that the limit can be extended analytically to R, ϵ small, we should be able to enumerate, modulo the above ansatz of interchanging derivation and limit, the enumeration of dually weighted graphs.

There is still a long way toward the rigorous understanding of the use of matrix integrals for the enumeration of maps in physics but we hope that this paper provides some useful steps in this direction.

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