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Fabrice Blache

Backward stochastic differential equations on manifolds

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Abstract. The problem of finding a martingale on a manifold with a fixed random terminal value can be solved by considering BSDEs with a generator with quadratic growth. We study here a generalization of these equations and we give uniqueness and existence results in two different frameworks, using differential geometry tools. Applications to PDEs are given, including a certain class of Dirichlet problems on manifolds.

1. Introduction

1.1. Martingales and BSDEs on manifolds

Unless otherwise stated, we shall work on a fixed finite time interval $[0; T]$; moreover, $(W_t)_{0\le t\le T}$ will always denote a Brownian Motion (BM for short) in $\mathbb{R}^{d_W}$, for a positive integer d_W. Moreover, Einstein's summation convention will be used for repeated indices in lower and upper position.

It is well-known that there is a deep interplay between on the one hand the probability theory of real martingales and Brownian motion and on the other hand the theory of PDEs and harmonic functions $h : M_1 \to \mathbb{R}$, defined on a manifold M_1. For instance, the Feynman-Kac formula gives a probabilistic interpretation for the solution of a PDE; besides, the Dirichlet problem for such harmonic functions h can be solved by considering real martingales with fixed terminal value.

It is natural to ask whether these links can be generalized to the nonlinear context of manifolds, i.e. if we replace the vector space $\mathbb{R}$ in the definition of h by a manifold M; this is what we now examine. Suppose that M is a manifold endowed with a connection Γ; then one can define the notion of Γ-martingale on M, which generalizes real local martingales (for an overview of the basic definitions and properties, see [10], [5] or [14]).

In $\mathbb{R}^n$, the problem of finding a martingale $(X_t)_{0\le t\le T}$ with terminal value $X_T = U$ consists of solving the Backward Stochastic Differential Equation (BSDE for short)

$$(E)\begin{cases}\overline{X}_{t+dt} = \overline{X}_t + \overline{Z}_t dW_t\\ \overline{X}_T = U,\end{cases}$$

F. Blache: Laboratoire de Mathématiques, Université Blaise Pascal, 63177 Aubière Cedex, France. e-mail: `Fabrice.Blache@math.univ-bpclermont.fr`

where $(\overline{Z}_t)_{0\leq t\leq T}$ is a $\mathbb{R}^{n\times dw}$-valued progressively measurable process. With the connection Γ on M, one can define an exponential mapping exp and the equation under infinitesimal form (E) becomes, for martingales on M,

$$(M)_0 \begin{cases} X_{t+dt} = \exp_{X_t}(Z_t dW_t) \\ X_T = U \end{cases} \tag{1.1}$$

where $Z_t \in \mathcal{L}(\mathbb{R}^{dw}, T_{X_t}M)$ is now a linear map. As in the linear context, studying martingales on M (or equivalently solving BSDE $(M)_0$) allows to solve in a probabilistic way some nonlinear PDEs and to study harmonic mappings. Let us recall the definition and some properties of these mappings.

A harmonic map $H : M_1 \to M$ between Riemannian manifolds M_1 and M is a smooth map which is a local extremal of the energy functional

$$\int \|\mathrm{grad} H\|^2 dvol$$

where $dvol$ is the Riemannian volume element on M_1.

A different but equivalent point of view about these mappings is the one of a system of elliptic PDEs (see [8]); let us make precise it. Consider a second-order differential operator $\mathcal{L}$ without term of order 0, defined on M_1. For $h : M_1 \to M$, one can define by means of $\mathcal{L}$ the tension field of h; it is a vector field along h, i.e.

$$\mathcal{L}_M(h) : M_1 \to TM, \quad \mathcal{L}_M(h)(x) \in T_{h(x)}M.$$

Then the equation $\mathcal{L}_M(h) = 0$ characterizes $\mathcal{L}$-harmonic maps (see [9] and [8], and probabilistic interpretations in the introductions of [28] and [29]). In coordinates (x^i) on M and (y^α) on M_1, this equation can be written as the following system of elliptic PDEs

$$\forall i, \quad \Delta_{M_1}\phi^i + \Gamma^i_{jk}(\phi) g^{\alpha\beta}(x) D_\alpha \phi^j D_\beta \phi^k = 0$$

with Δ_{M_1} denoting the Laplace-Beltrami operator and $(g^{\alpha\beta})$ the inverse metric tensor on M_1. Note that we have used the summation convention.

With the theory of martingales on manifolds, one can solve such a system of nonlinear elliptic PDEs; for further details, the reader is referred to [15] and [17].

Now we come to the aim of this article, by enlarging the class of processes studied. Let $(B^y_t)_{0\leq t\leq T}$ denote the $\mathbb{R}^d$-valued diffusion which is the unique strong solution of the following SDE :

$$\begin{cases} dB^y_t &= b(B^y_t)dt + \sigma(B^y_t)dW_t \\ B^y_0 &= y, \end{cases} \tag{1.2}$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times dw}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are C^3 bounded functions with bounded partial derivatives of order 1,2 and 3.

On $\mathbb{R}^n$, equation (E) is a very simple BSDE. A more general form of BSDE on $[0; T]$ is

$$(E+D)\begin{cases}\overline{X}_{t+dt} = \overline{X}_t + \overline{Z}_t dW_t + f(B_t^y, \overline{X}_t, \overline{Z}_t)dt \\ \overline{X}_T = U,\end{cases}$$

studied for instance in [24] and [25]; such an equation can be used to solve systems of quasilinear PDEs (see for example [25]).

If we try to extend equation $(E+D)$ to manifolds, by combining with equation $(M)_0$ we get the following equation (under infinitesimal form)

$$(M+D)_0\begin{cases}X_{t+dt} = \exp_{X_t}(Z_t dW_t + f(B_t^y, X_t, Z_t)dt) \\ X_T = U.\end{cases}$$

The aim of this work is to study existence and uniqueness of a solution to the generalized equation $(M+D)_0$.

As for martingales, which are linked to harmonic mappings h (i.e. to the above equation $\mathcal{L}_M(h) = 0$), this BSDE is related with a differential operator generalizing the tension field $\mathcal{L}_M$; moreover, in some cases, the class of mappings which solve the new PDE can be described in terms of the local extrema of another variational problem. This will be discussed in Section 5. It is known that when $M_1 \subset \mathbb{R}^3$ and $M = S^2$, harmonic mappings can be used to model the state of equilibrium of liquid crystals (see the introduction of [13] for a brief discussion); then mappings associated to the new variational problem could be used to model the equilibrium state of a liquid crystal in an exterior field equal to the drift term f in equation $(M+D)_0$.

1.2. Setting of the problem

In the whole paper, we will always suppose that a global system of coordinates is given on M. Then in these coordinates, we get from equation $(M)_0$ the following BSDE (see for instance [10] or the introduction of [6])

$$(M)\begin{cases}dX_t = Z_t dW_t - \frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k|[Z_t]^j)dt \\ X_T = U;\end{cases}$$

in this equation, we have used the following notations, which will be valid throughout the sequel : $(\cdot|\cdot)$ is the usual inner product in an Euclidean space, the summation convention is used, and $[A]^i$ denotes the i^{th} row of any matrix A; finally,

$$\Gamma_{jk}(x) = \begin{pmatrix}\Gamma^1_{jk}(x) \\ \vdots \\ \Gamma^n_{jk}(x)\end{pmatrix} \tag{1.3}$$

is a vector in $\mathbb{R}^n$, whose components are the Christoffel symbols of the connection. Remark also that here Z_t is a matrix in $\mathbb{R}^{n\times d_W}$.

In this case, the classical approach of [24] to solve BSDEs with Lipschitz coefficients fails since there is a quadratic term in Z_t in the drift (the reader is referred to [23] or [21] for an introduction to the theory of BSDEs in Euclidean spaces). However, uniqueness and existence results have been obtained using differential geometry tools, in particular by Arnaudon ([1]), Darling ([6]), Emery([10]), Kendall ([15]), Picard ([27] and [28]) or Thalmaier ([32] and [31]); note also the results of Estrade and Pontier ([11]) concerning some classes of Lie groups. Independently of geometric tools, a lot of works have tried to weaken the Lipschitz assumption : in the one-dimensional case, they include [18] (in dimension one, her results are more general than the ones of this paper because she deals with generators with quadratic growth), [20] or [12]; in higher dimensions, we refer the reader for instance to [4], [30] (who studies a Ricatti-type BSDE) and [22]. To the best of our knowledge, there is no paper that would include our results in dimensions greater than one.

Now in our global chart, the equation $(M+D)_0$ becomes

$$(M+D)\begin{cases} dX_t = Z_t dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k|[Z_t]^j) + f(B_t^y, X_t, Z_t)\right)dt \\ X_T = U \end{cases}$$

(the same notation will be used to denote the TM-valued function f and its image in local coordinates). The process X will take its values in a compact set, and a solution of equation $(M+D)_0$ will be a pair of processes (X, Z) in $M \times (\mathbb{R}^{dw} \otimes TM)$ such that X is continuous and $\mathbb{E}\left(\int_0^T \|Z_t\|_r^2 dt\right) < \infty$ ($\|\cdot\|_r^2$ is a Riemannian norm; see below). If we consider a global system of coordinates on an open set O of $\mathbb{R}^n$, it corresponds to processes (X, Z) in $O \times \mathbb{R}^{ndw}$, such that X is in a compact set and $\mathbb{E}\left(\int_0^T \|Z_t\|^2 dt\right) < \infty$, solving equation $(M+D)$.

Two different cases will be considered here : firstly when the drift f does not depend on z, and secondly the case of a general f in nonpositive curvatures. In the two cases, a Riemannian structure is fixed on M; in the former case, the connection may be independent of this Riemannian structure, while in the latter, only the Levi-Civita connection associated will be used. Note that the case for a general f with $K > 0$ involves more technical calculations; it will appear elsewhere.

We first give in Section 2 mild generalizations of well-known results, concerning the geometry of the manifold and a characterization of the solutions in the z-independent case by means of convex functions. In Section 3, we study the uniqueness problem. It is solved by generalizing to our context two methods : on the one hand, Emery's idea, used in [10] and [15]; on the other hand, the work of Picard ([28]). We obtain Theorems 3.3.2 and 3.4.6.

Section 4 is devoted to proving the existence of a solution of equation $(M+D)$. The main arguments are to exhibit a solution for "simple" terminal values (based on a strong bound on the process (Z_t) in Subsection 4.3) and to solve the equation for any terminal value using approximation procedures (Subsections 4.1 and 4.6). We need for the proof an additional (and necessary in fact) condition on the drift f : it is supposed to point outward on the boundary of the set on which we work. We give in Subsection 1.4 the main result (Theorem 1.4.1) which sums up the results obtained. In Section 5, we extend the results to random time intervals $[0; \tau]$, where

τ is successively a bounded stopping time (Theorem 5.3.1) and a stopping time verifying an exponential integrability condition (Theorem 5.3.2); then to conclude this paper, we give some generations and applications to the theory of PDEs, as well as the variational problem related to equation $(M + D)_0$.

The uniqueness part, as well as the applications to PDEs, are mainly adaptations of procedures already used; on the contrary, the approach for the existence seems to be novel.

1.3. Notations and hypothesis

In all the article, we suppose that a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\leq t\leq T})$ is given on which $(W_t)_t$ denotes a d_W-dimensional BM. Moreover, we always deal with a complete Riemannian manifold M of dimension n, endowed with a linear symmetric (i.e. torsion-free) connection whose Christoffel symbols Γ^i_{jk} are smooth; the connection does not depend *a priori* on the Riemannian structure.

On M, δ denotes the Riemannian distance; $|u|_r$ is the Riemannian norm for a tangent vector u and $|u'|$ the Euclidean norm for a vector u' in $\mathbb{R}^n$. If h is a smooth real function defined on M and u_1, u_2 are tangent vectors at x, the differential of h is denoted by $Dh(x) < u >$ or $h'(x) < u >$; the Hessian Hess $h(x)$ is a bilinear form the value of which is denoted by Hess $h(x) < u_1, u_2 >$.

For $\beta \in \mathbb{N}^*$, we say that a function is C^β on a closed set F if it is C^β on an open set containing F. For a matrix z with n rows and k columns, ${}^t z$ denotes its transpose,

$$\|z\| = \sqrt{\mathrm{Tr}(z\,{}^t z)} = \sqrt{\sum_{i=1}^{k} |[{}^t z]^i|^2}$$

(Tr is the trace of a square matrix) and $\|z\|_r = \sqrt{\sum_{i=1}^k |[{}^t z]^i|_r^2}$ where the columns of z are considered as tangent vectors. The notation $\Psi(x, x') \approx \delta(x, x')^\nu$ means that there is a constant $c > 0$ such that

$$\forall x, x', \ \frac{1}{c}\,\delta(x, x')^\nu \leq \Psi(x, x') \leq c\,\delta(x, x')^\nu.$$

Finally, recall that a real function χ defined on M is said to be convex if for any M-valued geodesic γ, $\chi \circ \gamma$ is convex in the usual sense (if χ is smooth, this is equivalent to require that Hess χ be nonnegative).

Before the general framework, let us give some additional notations which are specific to the Levi-Civita connection. In this case, we always assume that the injectivity radius R of M is positive and that its sectional curvatures are bounded above; we let K be the smallest nonnegative number dominating all the sectional curvatures. For the distance function δ, if $\tilde{x} = (x, x')$ is a point and $u, \overline{u}$ are tangent vectors at x', we consider the partial derivatives (when they are defined)

$$\delta_2'(\tilde{x}) < u >= \delta'(\tilde{x}) < (0, u) >$$

and

$$\mathrm{Hess}_{22}\,\delta(\tilde{x}) < u, \overline{u} >= \mathrm{Hess}\,\delta(\tilde{x}) < (0, u), (0, \overline{u}) > .$$

Now let us recall from [15] the definition of a regular geodesic ball. A closed geodesic ball $\mathcal{B}$ of radius ρ and center p is said to be regular if

(i) $\rho\sqrt{K} < \frac{\pi}{2}$
(ii) the cut locus of p does not meet $\mathcal{B}$.

For an introductory course in Riemannian geometry, the reader is referred to [3] and for further facts about curvature, to [19].

Throughout this article, we consider an open set $\omega \neq \emptyset$ relatively compact in an open subset O of M, such that there is a unique O-valued geodesic between any two points of O; O is also supposed to be relatively compact in a local chart, so that it provides a system of coordinates (global on $\overline{\omega}$); it will be as well considered as a subset of $\mathbb{R}^n$. We suppose that there exists a nonnegative, smooth and convex function Ψ on the product $\overline{\omega} \times \overline{\omega}$ (i.e. convex on an open set containing this set) which vanishes only on the diagonal $\Delta = \{(x, x)/x \in \overline{\omega}\}$ ($\overline{\omega}$ is said to have Γ-convex geometry); besides, we suppose that $\Psi \approx \delta^p$ for a $p \geq 2$ (note that since Ψ is smooth, p is an even integer). In fact, we take for $\overline{\omega}$ a sublevel set of a smooth convex function χ defined on $O : \{\chi \leq c\}$ (note that this hypothesis and the existence of Ψ guarantee the existence and uniqueness of a $\overline{\omega}$-valued geodesic between any two points of $\overline{\omega}$).

Emery has shown (see Lemma (4.59) of [10]) that, in the case of a general connection Γ, any point of M possesses a neighbourhood with Γ-convex geometry; when the Levi-Civita connection is used, this is true for a regular geodesic ball (see [16]).

Finally we always assume two hypothesis on f :

$$\begin{aligned}
&\exists L > 0,\ \forall b, b' \in \mathbb{R}^d, \forall (x, z) \in O \times \mathcal{L}(\mathbb{R}^{dw}, T_x M),\\
&\qquad \forall (x', z') \in O \times \mathcal{L}(\mathbb{R}^{dw}, T_{x'} M),\\
&\left| \overset{x'}{\underset{x}{\|}} f(b, x, z) - f(b', x', z') \right|_r \leq L\Big((|b - b'| + \delta(x, x'))(1 + \|z\|_r + \|z'\|_r) \\
&\qquad\qquad\qquad + \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r \Big)
\end{aligned} \tag{1.4}$$

and

$$\exists L_2 > 0, \exists x_0 \in O, \forall b \in \mathbb{R}^d, |f(b, x_0, 0)|_r \leq L_2. \tag{1.5}$$

The first one is a "geometrical" Lipschitz condition on f. This special form is needed to get an expression which is invariant under changes of coordinates. We will see that later, in (2.5). Remark that, in the z-independent case, it just means that f is Lipschitz with respect to the first two variables; otherwise,

$$\overset{x'}{\underset{x}{\|}} z$$

denotes the Riemannian parallel transport along the unique geodesic between x and x'. The second one means that f is bounded with respect to the first argument.

Remark that these conditions also imply the boundedness of f if it does not depend on the variable z.

Note to end this part that the same letter C will often stand for different constant numbers.

1.4. The main result

Before achieving calculations, we give the main theorem of the article. Let us first introduce a technical but natural hypothesis, which we will make explicit in Subsection 4.6 :

$$(H) \quad f \text{ is pointing outward on the boundary of } \overline{\omega}.$$

Then we can state :

Theorem 1.4.1. *We consider the BSDE* $(M + D)$ *with terminal random variable* $U \in \overline{\omega} = \{\chi \leq c\}$*, where* $\overline{\omega}$ *satisfies the above conditions. If* f *verifies conditions* (1.4)*,* (1.5) *and* (H)*, and if* χ *is strictly convex (i.e.* Hess χ *is positive definite), then*

(i) If f does not depend on z, the BSDE has a unique solution $(X_t, Z_t)_{0\leq t\leq T}$ such that X remains in $\overline{\omega}$.
(ii) If M is a Cartan-Hadamard manifold and the Levi-Civita connection is used, then the BSDE has yet a unique solution $(X_t, Z_t)_{0\leq t\leq T}$ with X in $\overline{\omega}$.

In particular, if the Levi-Civita connection is used, a “good” example of domain on which existence and uniqueness hold is a regular geodesic ball.

In Section 5, we will extend this theorem to random time intervals $[0; \tau]$ (instead of $[0; T]$), for stopping times τ which are bounded, or verify the exponential integrability condition :

$$\exists \rho > 0 : \mathbb{E}(e^{\rho\tau}) < \infty. \tag{1.6}$$

In the former case, Theorem 1.4.1 goes the same, while in the latter the constants L and L_2 in (1.4) and (1.5) are furthermore required to be small with respect to the constant ρ in (1.6).

2. Preliminary results

We first recall elementary results about Itô’s formula and parallel transport. Then we give some geometrical estimates for the distance function on $M \times M$ and characterize solutions of the equation $(M + D)$ using convex functions, but only when the drift f does not depend on z. As underlined in the introduction, these results are just mild generalizations of well-known results of [10] and [28].

In this section, the covariant derivative of a vector field z_t along a curve γ_t will be denoted $\nabla_{\dot{\gamma}_t} z_t$.

2.1. Itô's formula on manifolds

Consider two solutions (X^1, Z^1) and (X^2, Z^2) of equation $(M + D)$ with terminal values U^1 and U^2, such that X^1 and X^2 remain in O. Let

$$\tilde{X} = (X^1, X^2) \text{ and } \tilde{Z} = \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix};$$

then Itô's formula with the function Ψ is written

$$\begin{aligned}
&\Psi(\tilde{X}_t) - \Psi(\tilde{X}_0) \\
&= \int_0^t D\Psi(\tilde{X}_s)\left(\tilde{Z}_s dW_s\right) \\
&\quad + \int_0^t D\Psi(\tilde{X}_s)\begin{pmatrix} f(B_s^y, X_s^1, Z_s^1) - \frac{1}{2}\Gamma_{jk}(X_s^1)([Z_s^1]^k|[Z_s^1]^j) \\ f(B_s^y, X_s^2, Z_s^2) - \frac{1}{2}\Gamma_{jk}(X_s^2)([Z_s^2]^k|[Z_s^2]^j) \end{pmatrix} ds \\
&\quad + \frac{1}{2}\int_0^t \mathrm{Tr}\left({}^t\tilde{Z}_s D^2\Psi(\tilde{X}_s)\tilde{Z}_s\right) ds \\
&= \int_0^t D\Psi(\tilde{X}_s)\left(\tilde{Z}_s dW_s\right) \\
&\quad + \frac{1}{2}\int_0^t \left(\sum_{i=1}^{d_W} {}^t[{}^t\tilde{Z}_s]^i \text{Hess } \Psi(\tilde{X}_s)[{}^t\tilde{Z}_s]^i\right) ds \\
&\quad + \int_0^t D\Psi(\tilde{X}_s)\begin{pmatrix} f(B_s^y, X_s^1, Z_s^1) \\ f(B_s^y, X_s^2, Z_s^2) \end{pmatrix} ds
\end{aligned} \tag{2.1}$$

(remember notation (1.3) and that $[{}^t\tilde{Z}_s]^i$ denotes the i^{th} column of the matrix $\tilde{Z}$; it is a vector in $\mathbb{R}^{2n}$). Moreover, for a smooth function h on O and a solution (X, Z) of $(M + D)$, we get a similar formula, replacing $\tilde{X}$ by X and $\tilde{Z}$ by Z.

2.2. Two inequalities

We first give an equivalence result between the Euclidean and Riemannian norms; it follows easily from the relative compactness of O, considered as a subset of $\mathbb{R}^n$ (in particular, this means that we can identify each tangent space with $\mathbb{R}^n$).

Lemma 2.2.1. *There is a $c > 0$ such that for any $(x, z) \in O \times T_xM = O \times \mathbb{R}^n$,*

$$\frac{1}{c}|z| \le |z|_r \le c|z|.$$

Lemma 2.2.1 will often be useful in the sequel.

Proposition 2.2.2. *The Levi-Civita connection is used. There is a $C > 0$ such that for every $(x, x') \in O \times O$ and $(z, z') \in T_xM \times T_{x'}M$, we have*

$$\left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r \le C\left(|z - z'| + \delta(x, x')(|z| + |z'|)\right) \tag{2.2}$$

and

$$|z - z'| \le C \left(\left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r + \delta(x, x')(|z|_r + |z'|_r) \right). \tag{2.3}$$

Remark 1. In fact, by Lemma 2.2.1, we can use any of the two norms (except for $|z - z'|$ which is necessarily the Euclidean norm).

Proof. It is sufficient to prove

$$\forall (x, z), (x', z') \in O \times \mathbb{R}^n, \left| \overset{x'}{\underset{x}{\|}} z - z \right| \le C\delta(x, x')|z|. \tag{2.4}$$

Indeed, this implies

$$\left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r \le C_1 \left(|z - z'| + \left| \overset{x'}{\underset{x}{\|}} z - z \right| \right) \le C(|z - z'| + \delta(x, x')|z|)$$

and

$$|z - z'| \le \left| \overset{x'}{\underset{x}{\|}} z - z \right| + \left| \overset{x'}{\underset{x}{\|}} z - z' \right| \le C \left(\delta(x, x')|z| + \left| \overset{x'}{\underset{x}{\|}} z - z' \right| \right).$$

So let us prove (2.4) : let γ be the geodesic such that $\gamma(0) = x$ and $\gamma(1) = x'$, and $z(t)$ the parallel transport of z along γ :

$$\forall t \in [0; 1], \quad z(t) = \overset{\gamma_t}{\underset{x}{\|}} z.$$

In local coordinates, the equation $\nabla_{\dot{\gamma}_t} z(t) = 0$ gives for every k

$$\dot{z}^k(t) + \Gamma^k_{jl}(\gamma_t)\dot{\gamma}^j_t z^l(t) = 0.$$

Moreover, $|\dot{\gamma}_t|_r = \delta(x, x')$ so $|\dot{\gamma}_t| \le \tilde{C}\delta(x, x')$ and

$$\begin{aligned}
\left| \overset{x'}{\underset{x}{\|}} z - z \right|^2 &= |z(1) - z(0)|^2 \\
&= \sum_k |z^k(1) - z^k(0)|^2 \\
&= \sum_k \left| \int_0^1 \dot{z}^k(t)dt \right|^2 \\
&= \sum_k \left| \int_0^1 \Gamma^k_{jl}(\gamma_t)\dot{\gamma}^j_t z^l(t)dt \right|^2 \\
&\le C_1 \delta(x, x')^2 \sum_l \int_0^1 |z^l(t)|^2 dt \\
&\le C\delta(x, x')^2 |z|^2.
\end{aligned}$$

The last inequality comes from the equivalence on O of the Riemannian and Euclidean norms, and the fact that $|z(t)|_r$ is constant by definition of $z(t)$. The proof is completed. □

As a consequence, on the relatively compact set $O \subset \mathbb{R}^n$, (1.4) becomes

$$\begin{aligned} &\exists L' > 0,\ \forall b, b' \in \mathbb{R}^d,\ \forall (x,z),(x',z') \in O \times \mathbb{R}^{ndw}, \\ &|f(b,x,z) - f(b',x',z')| \leq L'((|b-b'| + |x-x'|) \\ &\times (1 + \|z\| + \|z'\|) + \|z - z'\|). \end{aligned} \tag{2.5}$$

Remark 2. In particular, (2.5) is verified for a drift f that is Lipschitz in (b,x,z) in O (the Lipschitz property of f is not necessarily preserved by a change of coordinates, but (2.5) is).

Remark 3. If f does not depend on z, it just means that f is Lipschitz in (b,x).

2.3. Estimates of the derivatives of the distance

This paragraph is based on Section 1 of [28]. The connection used is Levi-Civita's one.

The geodesic distance $(x,x') \mapsto \delta(x,x')$ is defined on $M \times M$ and is smooth except on the cut locus and the diagonal $\{x = x'\}$. We want to estimate its first and second derivatives when $M \times M$ is endowed with the product Riemannian metric. If $\tilde{x} = (x,x')$ is a point which is not in the cut locus or the diagonal, there exists a unique minimizing geodesic $\gamma(t)$, $0 \leq t \leq 1$, from x to x'. If u_t is a vector of $T_{\gamma(t)}M$, we can decompose u_t as $v_t + w_t$, where v_t is the orthogonal projection of u_t on $\dot{\gamma}(t)$; the vectors v_t and w_t are respectively called the tangential and orthogonal components of u_t. If $u = (u_0, u_1)$ is a vector of $T_{\tilde{x}}(M \times M)$, (v_0, v_1) and (w_0, w_1) are also called its tangential and orthogonal components.

Lemma 2.3.1. *Let $\tilde{x}$ be a point of $M \times M$ which is not in the cut locus or the diagonal. Let u be a vector of $T_{\tilde{x}}(M \times M)$ and let v and w be its tangential and orthogonal components. Then*

$$|\delta'(\tilde{x}) < u > | = \left| \overset{x'}{\underset{x}{\|}} v_0 - v_1 \right|_r ; \tag{2.6}$$

if moreover $K = 0$ (i.e. the sectional curvatures are nonpositive), then

$$\text{Hess}\, \delta(\tilde{x}) < u, u > \geq \frac{1}{\delta(\tilde{x})} \left| \overset{x'}{\underset{x}{\|}} w_0 - w_1 \right|_r^2 . \tag{2.7}$$

Proof. Let $J_v(t)$ (resp. $J_w(t)$) be the tangential (resp. normal) Jacobi field along $\gamma(t)$ satisfying $J_v(0) = v_0$ and $J_v(1) = v_1$ (resp. $J_w(0) = w_0$ and $J_w(1) = w_1$). From (1.1.5) and (1.1.7) of [28], we have

$$\delta'(\tilde{x}) < u > = \frac{(\dot{\gamma}(t) | \nabla_{\dot{\gamma}(t)} J_v(t))}{|\dot{\gamma}(t)|_r} \tag{2.8}$$

and

$$\text{Hess}\,\delta(\tilde{x}) < u, u > \geq \frac{1}{\delta(\tilde{x})} \int_0^1 |\nabla_{\dot{\gamma}(t)} J_w(t)|_r^2 dt - K\delta(\tilde{x}) \int_0^1 |J_w(t)|_r^2 dt. \quad (2.9)$$

It is easy to see with the Jacobi equation that $J_v(t) = (At + B)\dot{\gamma}(t)$. Then $\nabla_{\dot{\gamma}(t)} J_v(t) = A\dot{\gamma}(t)$ and the limit conditions $J_v(0) = v_0 = \alpha\dot{\gamma}(0)$ and $J_v(1) = v_1 = \beta\dot{\gamma}(1)$ imply $A = \beta - \alpha$. Hence (2.8) gives

$$|\delta'(\tilde{x}) < u > | = |A| \cdot |\dot{\gamma}(t)|_r = |\beta - \alpha|\delta(\tilde{x}) = \left| \overset{x'}{\underset{x}{\|}} v_0 - v_1 \right|_r .$$

This is (2.6).

Moreover, let us write $J_w(t) = \sum_i u_i(t)E_i(t)$ where $\{E_i(t)\}_i$ is a parallel orthonormal frame along γ. Its covariant derivative is $\nabla_{\dot{\gamma}(t)} J_w(t) = \sum_i \dot{u}_i(t)E_i(t)$ and $\overset{x'}{\underset{x}{\|}} J_w(0) = \sum_i u_i(0)E_i(1)$. Then

$$\begin{aligned} \int_0^1 |\nabla_{\dot{\gamma}(t)} J_w(t)|_r^2 dt &= \int_0^1 \sum_i |\dot{u}_i(t)|^2 dt \\ &\geq \sum_i \left(\int_0^1 \dot{u}_i(t) dt \right)^2 = \left| \overset{x'}{\underset{x}{\|}} J_w(0) - J_w(1) \right|_r^2 . \end{aligned}$$

Now this inequality together with (2.9) and the nonpositivity of the sectional curvatures give (2.7). □

Then we have the following estimate :

Proposition 2.3.2. *If $K = 0$ and $\tilde{x}$ is not in the cut locus, then*

$$\text{Hess}\,(\frac{1}{2}\delta^2)(\tilde{x}) < u, u > \geq \left| \overset{x'}{\underset{x}{\|}} u_0 - u_1 \right|_r^2 . \quad (2.10)$$

Proof. If $\tilde{x} = (x, x')$ is not on the diagonal, we recall the classical formula

$$\text{Hess}\,(\frac{1}{2}\delta^2)(\tilde{x}) < u, u > = \delta(\tilde{x}) \cdot \text{Hess}\,\delta(\tilde{x}) < u, u > + (\delta'(\tilde{x}) < u >)^2.$$

This formula and estimates (2.6) and (2.7) imply the proposition for $x \neq x'$ since the two terms $\overset{x'}{\underset{x}{\|}} v_0 - v_1$ and $\overset{x'}{\underset{x}{\|}} w_0 - w_1$ are orthogonal for the Riemannian norm. The case $x = x'$ follows by continuity since δ^2 is smooth on a neighbourhood of the diagonal. □

2.4. A characterization of the solutions of equation $(M + D)$ when the drift f does not depend on z

We give here a generalization of a well-known result (see (4.41)(ii) in [10]) which roughly says that a continuous M-valued process (Y_t) is a Γ-martingale if and only if its image under convex functions is a real local submartingale. In this paragraph, the filtration used is the natural one of $(W_t)_t$.

Proposition 2.4.1. *Suppose that the drift f does not depend on z. Then every point p of M has an open neighbourhood O_p, included in a local chart, with the following property :*

A pair of processes (X, Z) (with X continuous, adapted and O_p-valued) is a solution of $(M+D)$ iff for every convex function $\xi : O_p \to \mathbb{R}$, $\xi(X_t) - \int_0^t D\xi(X_s) \cdot f(B_s^y, X_s)ds$ is a local submartingale.

The proof given here is just an adaptation of Emery's one; first we recall Lemma (4.40) of [10].

Lemma 2.4.2. *On M, let ξ be a smooth function. Every point of M has an open neighbourhood O_p depending on ξ with the following property : For every $\varepsilon > 0$ and $a \in O_p$, there is a convex function $h_\varepsilon^a : O_p \to \mathbb{R}$ such that $(a, x) \mapsto h_\varepsilon^a(x)$ is smooth on $O_p \times O_p$, $h_\varepsilon^a(a) = 0$, $Dh_\varepsilon^a(a) = D\xi(a)$ and* Hess *$h_\varepsilon^a(a) = \varepsilon g(a)$, where g represents the metric.*

Now we complete the proof of Proposition (2.4.1) :

The "only if" part is just a consequence of Itô's formula (similar to (2.1)) applied to $\xi(X_t)$: as Hess ξ is nonnegative by convexity, $\xi(X_t) - \int_0^t D\xi(X_s) \cdot f(B_s^y, X_s)ds$ is indeed a local submartingale.

For the "if" part, notice first that around p there is a system (x^i) of local coordinates that are convex functions (if (y^i) are any local coordinates with $y^i(p) = 0$, then take $x^i = y^i + c\sum_j (y^j)^2$ for $c > 0$ large enough). Choose O_p relatively compact in the domain of such a local chart and in an open set on which Lemma (2.4.2) holds for $\xi = x^i$ and $\xi = -x^i$ (denote by S this set of $2n$ functions).

For a continuous adapted O_p-valued process X, suppose that $h \circ X - \int Dh(X) \cdot f(B^y, X)dt$ is a local submartingale for every convex h on O_p. Taking first for h the global (on O_p) coordinates (x^i) shows that each $x^i \circ X$ is a real semimartingale, so X is a semimartingale. For every fixed $\xi \in S$, it is sufficient to prove that $\xi \circ X - \frac{1}{2}\int (\text{Hess}\,\xi)_{ij} d < X^i, X^j > - \int D_i\xi(X) \cdot f^i(B^y, X)dt$ is a local submartingale; for then replacing ξ by $-\xi$ shows that (remember that $D_{ij}x^k = 0$ and $D_i x^k = 1$ if $i = k$ and 0 otherwise)

$$x^k \circ X + \frac{1}{2}\int \Gamma_{ij}^k(X)d < X^i, X^j > - \int f^k(B^y, X)dt$$

is a local martingale for each k. But the theorem of representation of local martingales in Brownian filtrations allows to write this local martingale explicitly as $\int Z_t dW_t$; thus $d < X^i, X^j >_t = ([Z_t]^i | [Z_t]^j)dt$ and (X, Z) solves equation $(M + D)$.

By the choice of O_p, given any $\varepsilon > 0$ we are provided with functions h_ε^a associated to ξ as in Lemma (2.4.2). Call σ the p^{th} dyadic subdivision of the time axis ($\sigma = \{t_k = \frac{k}{2^p} \ : \ k, p \in \mathbb{N}\}$) and let for $t \in [t_k; t_{k+1}[$, $\rho(t) = t_k$ and

$$S_t^\sigma = \sum_{l<k} \left(h_\varepsilon^{X_{t_l}}(X_{t_{l+1}}) - \int_{t_l}^{t_{l+1}} Dh_\varepsilon^{X_{t_l}}(X_u) \cdot f(B_u^y, X_u) du \right) \\ + h_\varepsilon^{X_{t_k}}(X_t) - \int_{t_k}^{t} Dh_\varepsilon^{X_{t_k}}(X_u) \cdot f(B_u^y, X_u) du.$$

As each h_ε^a is convex, $h_\varepsilon^{X_{t_k}}(X_t) - \int_{t_k}^t Dh_\varepsilon^{X_{t_k}}(X_u) \cdot f(B_u^y, X_u) du$ is a submartingale in the interval $[t_k; t_{k+1}]$, and S^σ is a continuous submartingale. Using the coordinates (x^i), write

$$dS_t^\sigma = D_i h_\varepsilon^{X_{t_k}}(X_t) dX_t^i + \frac{1}{2} D_{ij} h_\varepsilon^{X_{t_k}}(X_t) d < X^i, X^j >_t \\ - D_i h_\varepsilon^{X_{t_k}}(X_t) f^i(B_t^y, X_t) dt.$$

So, if X^i is decomposed into $N^i + A^i$ (i.e. local martingale + bounded variation part),

$$dS_t^\sigma - \left(D_i h_\varepsilon^{X_{t_k}}(X_t) dA_t^i + \frac{1}{2} D_{ij} h_\varepsilon^{X_{t_k}}(X_t) d < X^i, X^j >_t \\ - D_i h_\varepsilon^{X_{t_k}}(X_t) f^i(B_t^y, X_t) dt \right)$$

is a local martingale; hence the process

$$B_t^\sigma = \int_0^t D_i h_\varepsilon^{X_{\rho(s)}}(X_s) dA_s^i + \frac{1}{2} \int_0^t D_{ij} h_\varepsilon^{X_{\rho(s)}}(X_s) d < X^i, X^j >_s \\ - \int_0^t D_i h_\varepsilon^{X_{\rho(s)}}(X_s) \cdot f^i(B_s^y, X_s) ds$$

is increasing. The estimates (using relative compactness of O_p)

$$|D_i h_\varepsilon^u(v) - D_i h_\varepsilon^v(v)| \le C|u - v| \\ |D_{ij} h_\varepsilon^u(v) - D_{ij} h_\varepsilon^v(v)| \le C|u - v|$$

and the convergence of $X_{\rho(t)}$ to X_t when p goes to infinity (i.e. when σ becomes finer), yield a dominated convergence of $D_i h_\varepsilon^{X_{\rho(s)}}(X_s)$ to $D_i h_\varepsilon^{X_s}(X_s) = D_i \xi(X_s)$ (since $dh_\varepsilon^a(a) = d\xi(a)$) and of $D_{ij} h_\varepsilon^{X_{\rho(s)}}(X_s)$ to $\Gamma_{ij}^k(X_s) D_k \xi(X_s) + \varepsilon g_{ij}(X_s)$ (since Hess $h_\varepsilon^a(a) = \varepsilon g(a)$). Hence B^σ has a limit, equal to

$$\int D_i \xi(X) dA^i + \frac{1}{2} \int \Gamma_{ij}^k(X) D_k \xi(X) d < X^i, X^j > \\ + \frac{1}{2} \varepsilon \int g_{ij}(X) d < X^i, X^j > - \int D_i \xi(X) \cdot f^i(B^y, X) dt,$$

that is an increasing process too. Letting now ε tend to zero,

$$J = \int D_i\xi(X)dA^i + \frac{1}{2}\int \Gamma_{ij}^k(X)D_k\xi(X)d < X^i, X^j >$$
$$- \int D_i\xi(X) \cdot f^i(B^y, X)dt$$

is also increasing, and (remember that $D_{ij}\xi = 0$)

$$\xi \circ X - \xi \circ X_0 - \frac{1}{2}\int (\text{Hess } \xi)_{ij} d < X^i, X^j >$$
$$- \int D_i\xi(X) \cdot f^i(B^y, X)dt = \int D_i\xi(X)dN^i + J$$

is a local submartingale, as was to be proved. □

3. The uniqueness property

In the first paragraph, we set the problem and exhibit the sum (3.2), whose non-negativity suffices to have uniqueness. Then, we give a useful estimate and derive the result in the two cases considered in this paper. The calculus is rather longer if the drift f depends on z, for we have to prove exponential integrability.

3.1. The general method

Consider two solutions $(X_t, Z_t)_t$ and $(X'_t, Z'_t)_t$ of $(M + D)$ such that X and X' remain in $\overline{\omega}$ and $X_T = Y_T = U$ (we will often write "$\overline{\omega}$-valued solutions of $(M + D)$"). Let

$$\tilde{X}_s = (X_s, X'_s) \quad \text{and } \tilde{Z}_s = \begin{pmatrix} Z_s \\ Z'_s \end{pmatrix}.$$

In the martingale case ($f = 0$), Itô's formula (2.1) and the convexity of Ψ ensure that the process $(\Psi(\tilde{X}_t))_t$ is a submartingale. The other properties of Ψ (see the introduction) then imply that $\tilde{X}$ remains in the diagonal Δ, therefore the uniqueness required. This is Emery's method (see Corollary (4.61) in [10]).

For our purpose (i.e. f does not vanish identically), we want to keep the submartingale property and therefore control the integral involving f in (2.1). The idea is to study, rather than $(\Psi(\tilde{X}_t))_t$, the new process $(\exp(A_t)\Psi(\tilde{X}_t))_t$ where

$$A_t = \lambda t + \mu \int_0^t (\|Z_s\|_r + \|Z'_s\|_r)ds,$$

for appropriate nonnegative constants λ and μ. Apply Itô's formula to obtain

$$\begin{aligned}
e^{A_t}\Psi(\tilde{X}_t) - \Psi(\tilde{X}_0) &= \int_0^t e^{A_s} d(\Psi(\tilde{X}_s)) \\
&\quad + \int_0^t e^{A_s}(\lambda + \mu(\|Z_s\|_r + \|Z'_s\|_r))\Psi(\tilde{X}_s)ds \\
&= \int_0^t e^{A_s} D\Psi(\tilde{X}_s)\left(\tilde{Z}_s dW_s\right) \\
&\quad + \frac{1}{2}\int_0^t e^{A_s}\left(\sum_{i=1}^{d_W} {}^t[{}^t\tilde{Z}_s]^i \text{Hess } \Psi(\tilde{X}_s)[{}^t\tilde{Z}_s]^i\right) ds \\
&\quad + \int_0^t e^{A_s} D\Psi(\tilde{X}_s)\begin{pmatrix} f(B_s^y, X_s, Z_s) \\ f(B_s^y, X'_s, Z'_s)\end{pmatrix} ds \\
&\quad + \int_0^t e^{A_s}\Psi(\tilde{X}_s)(\lambda + \mu(\|Z_s\|_r + \|Z'_s\|_r))ds. \qquad (3.1)
\end{aligned}$$

It is clear that the submartingale property will be preserved if we show the nonnegativity of the sum

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{d_W} {}^t[{}^t\tilde{Z}_t]^i \text{Hess } \Psi(\tilde{X}_t)[{}^t\tilde{Z}_t]^i + D\Psi(\tilde{X}_t)\begin{pmatrix} f(B_t^y, X_t, Z_t) \\ f(B_t^y, X'_t, Z'_t)\end{pmatrix} \\
&\qquad + (\lambda + \mu(\|Z_t\|_r + \|Z'_t\|_r))\Psi(\tilde{X}_t). \qquad (3.2)
\end{aligned}$$

The remainder of the uniqueness part is mainly devoted to proving this result.

3.2. An upper bound

The first step towards the nonnegativity of the sum (3.2) is to give a bound on the term involving f. This is the purpose of Lemma 3.2.1 below.

Let (x, x') be a point in $\overline{\omega} \times \overline{\omega}$ and $b \in \mathbb{R}^d$. For notational convenience, we keep the same notation $\overline{\omega} \times \overline{\omega}$ for the image of this compact set in local coordinates considered below, and we write f for $f(b, x, z)$ and f' for $f(b, x', z')$ (note that the b is the same in f and f'). Take a local chart (ϕ, ϕ) in which (x, x') has coordinates $(\hat{x}, \hat{x}')$; if $(\partial_1, \ldots, \partial_{2n})$ denotes the natural dual basis of these coordinates, then

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \sum_{i=1}^{n}\left(f^i \partial_i + f'^i \partial_{i+n}\right). \qquad (3.3)$$

Now $v = (v_1, \ldots, v_{2n}) = (\hat{x} - \hat{x}', \hat{x}')$ are new coordinates in which the diagonal Δ is represented by the equation $\{v_1 = \cdots = v_n = 0\}$; moreover, if $(\varphi_1, \ldots, \varphi_{2n})$ is the natural dual basis of the v-coordinates, then $\varphi_i = \partial_i$ and $\varphi_{i+n} = \partial_i + \partial_{i+n}$ for $i = 1, \ldots, n$; thus (3.3) becomes

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \sum_{i=1}^{n}\left((f^i - f'^i)\varphi_i + f'^i \varphi_{i+n}\right)$$

and

$$D\Psi \cdot \begin{pmatrix} f \\ f' \end{pmatrix} = \sum_{i=1}^{n} \left(\frac{\partial \Psi}{\partial v_i}(f^i - f'^i) + \frac{\partial \Psi}{\partial v_{i+n}} f'^i \right). \tag{3.4}$$

Finally, we call $p(V)$ the projection of a vector V onto Δ : if $V = (v_1, \ldots, v_{2n})$, then $p(V) = (0, \ldots, 0, v_{n+1}, \ldots, v_{2n})$.

Lemma 3.2.1. *Suppose that $\Psi(x, x') \approx \delta(x, x')^\nu$ on $\overline{\omega} \times \overline{\omega}$ where ν is an even positive integer (since Ψ is smooth). Then there is $C > 0$ such that, for all b in $\mathbb{R}^d$, x, x' in $\overline{\omega}$ and z, z' in $\mathbb{R}^{n \times d_W}$*

$$\left| D\Psi \cdot \begin{pmatrix} f \\ f' \end{pmatrix} \right| \leq C\delta(x, x')^{\nu-1} \left(\delta(x, x')|f'| + |f - f'| \right). \tag{3.5}$$

Proof. First remark that on $\overline{\omega} \times \overline{\omega}$, we have

$$|V - p(V)| \approx \delta(x, x'). \tag{3.6}$$

Indeed, let $(x, x') \in M \times M$; in v-coordinates, it is represented by $V = (\hat{x} - \hat{x}', \hat{x}')$. Then $|V - p(V)| = |(\hat{x} - \hat{x}', 0)|$, the Euclidean distance between $\hat{x}$ and $\hat{x}'$; as x and x' belong to the compact set $\overline{\omega}$, the Euclidean and Riemannian distances are equivalent and (3.6) follows.

Write D_i for $\frac{\partial}{\partial v_i}$. Since $\Psi(x, x') \approx \delta(x, x')^\nu$, a Taylor expansion gives

$$\Psi(V) = \frac{1}{\nu!} \sum_{1 \leq i_1, \ldots, i_\nu \leq n} D_{i_1 \ldots i_\nu} \Psi(p(V))(V - p(V))_{i_1} \ldots \\ \times (V - p(V))_{i_\nu} + O(|V - p(V)|^{\nu+1}).$$

Using another Taylor expansion with $D_i \Psi$ gives

$$|D_i \Psi(V)| \leq C|V - p(V)|^{\nu-1} \text{ if } i \leq n$$

$$|D_i \Psi(V)| \leq C|V - p(V)|^{\nu} \text{ if } i > n \text{ (i.e. in the direction of the diagonal)}$$

with a uniform C on the compact set $\overline{\omega} \times \overline{\omega}$. As a consequence of (3.4), we obtain

$$\left| D\Psi \cdot \begin{pmatrix} f \\ f' \end{pmatrix} \right| \leq C|V - p(V)|^{\nu-1} \left(|V - p(V)| \cdot |f'| + |f - f'| \right).$$

Using once again (3.6) completes the proof. □

We now give two functions which verify the hypothesis of Lemma (3.2.1). In these two examples, the Levi-Civita connection is used.

Example 1. Consider the mapping $\Psi(x, x') = \delta^2(x, x')$; it is smooth, not convex in general, but this is true if the sectional curvatures are bounded above by 0.

Example 2. Take for $\overline{\omega}$ a regular geodesic ball $\mathcal{B}$ centered at $o \in M$, with the sectional curvatures bounded from above by a constant $K > 0$; then we can find a nonnegative convex function Ψ on $\mathcal{B} \times \mathcal{B}$ which vanishes only on the diagonal (see [16]) :

$$\Psi(x, x') = \left(\frac{1 - \cos(\sqrt{K}\delta(x, x'))}{\cos(\sqrt{K}\delta(x, o))\cos(\sqrt{K}\delta(x', o)) - h^2} \right)^p,$$

where $h > 0$ is small and $p \geq 2$ is an integer large enough (so that Ψ is smooth).

3.3. The case f independent of z

In all this paragraph, f does not depend on z, i.e. $f(b, x, z) = f(b, x)$.

Proposition 3.3.1. *If the drift f doesn't depend on z, then the process $(e^{\lambda t}\Psi(\tilde{X}_t))_t$ is a submartingale for λ positive large enough; this implies that two $\overline{\omega}$-valued continuous semimartingales (X_t) and (X'_t) verifying the same equation $(M + D)$, with the same terminal value, are indistinguishable.*

Proof. For $x, x' \in \overline{\omega}$, $|f(b, x) - f(b, x')| \leq L'\delta(x, x')$; thus using (3.5) and the boundedness of f

$$\left| D\Psi(x, x') \cdot \begin{pmatrix} f(b, x) \\ f(b, x') \end{pmatrix} \right| \leq C\delta(x, x')^{\nu} \leq \tilde{C}\Psi(x, x').$$

Then for $\lambda \geq \tilde{C}$, $\mu = 0$, the sum (3.2) is nonnegative (note that the convexity of Ψ gives the nonnegativity of the term involving Hess Ψ). Moreover, the local martingale in equation (3.1) is in fact a martingale since $D\Psi$ is bounded on $\overline{\omega} \times \overline{\omega}$, so the process $(e^{\lambda t}\Psi(\tilde{X}_t))_t$ is indeed a submartingale. As it is nonnegative and has terminal value 0, it vanishes identically; so $\Psi(\tilde{X}_t) = 0$ for all t. Finally, the definition of Ψ leads to $X_t = X'_t$ for all t and the proof is completed since we consider continuous processes. □

Remark. Of course, $X = X'$ implies that for any t, $Z_t = Z'_t$ a.s.

As an immediate corollary, we are now able to give the uniqueness property.

Theorem 3.3.2. *Suppose that $\overline{\omega}$ and Ψ verify the properties of the introduction (see paragraph 1.3) and moreover that the drift f depends only on (b, x) and verifies* (1.4) *and* (1.5). *Then, for a given terminal value U in $\overline{\omega}$, there is at most one $\overline{\omega}$-valued solution to the equation* $(M + D)$.

Example. Suppose that the Levi-Civita connection is used. Then Theorem 3.3.2 implies uniqueness on any compact set of a Cartan-Hadamard manifold, and on any regular geodesic ball (with $K > 0$); indeed it suffices to consider respectively the functions δ^2 and Ψ defined after Lemma 3.2.1.

3.4. The general case in nonpositive curvatures

In this subsection, the drift f depends also on z, $\Psi = \frac{1}{2}\delta^2$ and the connection used is Levi-Civita's one; moreover M is supposed to be a Cartan-Hadamard manifold (i.e. simply connected with nonpositive sectional curvatures); remark then that any closed geodesic ball is regular. By achieving explicit calculations we are going to derive the uniqueness property for any compact set.

The problem is to show that the process $(\exp(A_t)\Psi(\tilde{X}_t))_t$ is a submartingale. But to define such a process, we need to consider solutions in some class which we now define.

Definition 3.4.1. *If α is a positive constant, let $(\mathcal{E}_\alpha)$ be the set of $\overline{\omega}$-valued solutions of $(M + D)$ satisfying*

$$\mathbb{E}\exp\left(\alpha\int_0^T \|Z_s\|_r^2 ds\right) < \infty. \tag{3.7}$$

Actually, we now verify that for α small, $(\mathcal{E}_\alpha)$ contains any solution of equation $(M + D)$. The first step is the following lemma, which generalizes Proposition 2.1.2 of [28].

Lemma 3.4.2. *Suppose that we are given a positive constant α and a C^2 function ϕ on $\overline{\omega}$ satisfying $C_{min} \leq \phi(x) \leq C_{max}$ for some positive C_{min} and C_{max}. Suppose moreover that* Hess $\phi + 2\alpha\phi \leq 0$ *on $\overline{\omega}$; this means that*

$$\text{Hess}\ \phi(x) < u, u > +2\alpha\phi(x)|u|_r^2 \leq 0. \tag{3.8}$$

Then, for every $\varepsilon > 0$, any $\overline{\omega}$-valued solution of $(M + D)$ belongs to $(\mathcal{E}_{\alpha-\varepsilon})$.

Proof. Define

$$S_t = \phi(X_t)\exp\left(\alpha\int_0^t \|Z_s\|_r^2 ds - \frac{1}{C_{min}}\int_0^t |D\phi(X_s)\cdot f(B_s^y, X_s, Z_s)|ds\right).$$

Denote by $e(t)$ the exponential term above. It follows from Itô's formula that

$$\begin{aligned} dS_t = e(t)\Bigg(& D\phi(X_t)(Z_t dW_t) \\ & + \left(\frac{1}{2}\sum_{i=1}^{d_W} {}^t[{}^tZ_t]^i \text{Hess}\ \phi(X_t)[{}^tZ_t]^i + \alpha\phi(X_t)\|Z_t\|_r^2\right) dt \\ & + \left(-\frac{\phi(X_t)}{C_{min}}|D\phi(X_t)\cdot f(B_t^y, X_t, Z_t)| + D\phi(X_t)\cdot f(B_t^y, X_t, Z_t)\right) dt\Bigg). \end{aligned}$$

Thus condition (3.8) ensures that S_t is a local supermartingale; since it is nonnegative, we get $\mathbb{E}S_T \leq \mathbb{E}S_0$. By using the lower and upper bounds on ϕ, we deduce that

$$\mathbb{E}\exp\left(\alpha\int_0^T \|Z_s\|_r^2 ds - \frac{1}{C_{min}}\int_0^T |D\phi(X_s)\cdot f(B_s^y, X_s, Z_s)|ds\right) \leq \frac{C_{max}}{C_{min}}. \tag{3.9}$$

By conditions (1.5) and (2.5), there is a $\tilde{C} > 0$ such that

$$\begin{aligned} \frac{1}{C_{min}}|D\phi(X_s)\cdot f(B_s^y, X_s, Z_s)| &\leq \tilde{C}(1 + \|Z_s\|_r) \\ &\leq \varepsilon\|Z_s\|_r^2 + C_\varepsilon; \end{aligned}$$

in particular, we have (3.7) for $\alpha - \varepsilon$. □

Lemma 3.4.3. *Let $\overline{B(o,\rho)}$ be a (regular) geodesic ball of center o and radius ρ. Then for $\alpha > 0$ small enough, every $\overline{B(o,\rho)}$-valued solution of $(M + D)$ belongs to the set $(\mathcal{E}_\alpha)$ defined previously.*

Proof. Let $\phi(x) = \cos\left(\frac{\pi}{3\rho}\delta(o,x)\right)$ for x in the geodesic ball $\overline{B(o,\rho)}$. We want to prove that this function ϕ verifies the hypothesis of Lemma (3.4.2). Firstly, ϕ is obviously a smooth function and there is a $c > 0$ such that

$$\forall x \in \overline{B(o,\rho)},\ c \le \phi(x) \le 1.$$

Moreover, we have

$$\begin{aligned}\text{Hess}\,\phi(x) < u,u > &= -\cos\left(\frac{\pi}{3\rho}\delta(o,x)\right)\left(\frac{\pi}{3\rho}\right)^2 (\delta_2'(o,x) < u >)^2\\ &\quad -\sin\left(\frac{\pi}{3\rho}\delta(o,x)\right)\left(\frac{\pi}{3\rho}\right)\text{Hess}_{22}\,\delta(o,x) < u,u > .\end{aligned}$$

If we get back to the notations of Subsection 2.3, we have, as a consequence of (2.6) and (2.7), the estimates (for $x \neq o$) :

$$|\delta_2'(o,x) < u > | = |v|_r$$

and

$$\text{Hess}_{22}\,\delta(o,x) < u,u > \ge \frac{|w|_r^2}{\delta(o,x)}.$$

Then

$$\begin{aligned}\text{Hess}\,\phi(x) < u,u > &\le -\phi(x)\left(\frac{\pi}{3\rho}\right)^2 |v|_r^2 - \frac{\sin\left(\frac{\pi}{3\rho}\delta(o,x)\right)}{\frac{\pi}{3\rho}\delta(o,x)}\left(\frac{\pi}{3\rho}\right)^2 |w|_r^2\\ &\le -\left(\frac{\pi}{3\rho}\right)^2 \phi(x)|u|_r^2.\end{aligned}$$

So there is a $\alpha > 0$ (depending on the radius ρ) such that (3.8) holds. It suffices to apply Lemma 3.4.2 to conclude. □

Obviously, this is also true for any compact set $\overline{\omega}$. We can easily derive, from (3.9) and the preceding lemma, the following result, which will be useful in later calculations.

Corollary 3.4.4. *There is an $\alpha > 0$ and a finite positive constant C_u (both depending only on $\overline{\omega}$ and the constants L and L_2 in (1.4) and (1.5)) such that for any $\overline{\omega}$-valued solution (X, Z) of equation $(M + D)$,*

$$\mathbb{E}\exp\left(\alpha\int_0^T \|Z_s\|_r^2 ds\right) \le C_u.$$

We can now state the uniqueness result (Theorem 3.4.6).

Lemma 3.4.5. *For two solutions* (X, Z) *and* (X', Z') *of* $(M+D)$ *verifying* $\mathbb{E}(A_T) < \infty$ *for every* $\mu > 0$*, the expression* (3.2) *is nonnegative for* λ *and* μ *large enough.*

Proof. Using (2.10) we have for $\tilde{z} = \begin{pmatrix} z \\ z' \end{pmatrix}$

$$\sum_{i=1}^{d_W} {}^t[{}^t\tilde{z}]^i \text{Hess } \Psi(\tilde{x})[{}^t\tilde{z}]^i \geq \sum_{i=1}^{d_W} \left| \overset{x'}{\underset{x}{\|}} [{}^t z]^i - [{}^t z']^i \right|_r^2 = \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r^2. \tag{3.10}$$

Moreover, we also have using (1.4) and (1.5), together with (3.5) (or with (2.2) and (2.6))

$$\begin{aligned} &\left| D\Psi(\tilde{x}) \begin{pmatrix} f(b, x, z) \\ f(b, x', z') \end{pmatrix} \right| \\ &\quad \leq C_1 \delta(x, x') \left(\delta(x, x')(1 + \|z\|_r + \|z'\|_r) + \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r \right) \\ &\quad \leq C\delta^2(x, x')(1 + \|z\|_r + \|z'\|_r) + \frac{1}{4} \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r^2. \end{aligned} \tag{3.11}$$

Then (3.2) is greater than the following sum

$$-C\delta^2(X_t, X'_t)(1 + \|Z_t\|_r + \|Z'_t\|_r) + \frac{1}{2}\delta^2(X_t, X'_t)(\lambda + \mu(\|Z_t\|_r + \|Z'_t\|_r)).$$

Taking λ and μ greater than $2C$ makes obviously this sum (and the expression (3.2)) nonnegative. □

Theorem 3.4.6. *Suppose that* M *is a Cartan-Hadamard manifold and that the drift* f *verifies condition* (1.4) *and* (1.5)*. Then, for a given terminal value* U *in the compact* $\overline{\omega}$*, there is at most one* $\overline{\omega}$*-valued solution to the equation* $(M + D)$ *(i.e. for two solutions* (X, Z) *and* (X', Z')*, the (continuous) processes* X *and* X' *are indistinguishable).*

Proof. Every compact $\overline{\omega}$ is included in a closed geodesic ball. Hence any $\overline{\omega}$-valued solution of $(M + D)$ is in $(\mathcal{E}_\alpha)$ for a $\alpha > 0$ from Lemma 3.4.3; this easily gives the integrability of $\exp(A_T)$ for every $\mu > 0$. Consequently, it suffices to apply Lemma 3.4.5 and conclude as in the proof of Proposition 3.3.1. □

Remark. Note that the only hypothesis required for uniqueness in this case is the compactness of $\overline{\omega}$.

4. Existence results

In this section we are given an $\overline{\omega}$-valued random variable U and we want to construct a pair of processes (X, Z), satisfying equation $(M + D)$, with X in $\overline{\omega}$ and terminal value U. We limit ourselves to the case of a Wiener probability space and we recall that on $\overline{\omega}$, if (X, Z) and (X', Z') are two solutions of the equation $(M + D)$ and $\tilde{X} = (X, X')$, then for $\lambda > 0$ and $\mu > 0$ large enough the following processes

$$\left(e^{\lambda t}\Psi(\tilde{X}_t)\right)_{t\in[0;T]} \quad \text{if } f = f(B^y, X)$$

and, when M is Cartan-Hadamard,

$$\left(e^{\lambda t+\mu\int_0^t(\|Z_s\|_r+\|Z'_s\|_r)ds}\delta^2(\tilde{X}_t)\right)_{t\in[0;T]} \quad \text{if } f = f(B^y, X, Z)$$

are nonnegative submartingales.

The strategy of the proof can be described as follows :

1. Simplify the problem by considering only terminal values which can be expressed as functions of the diffusion B^y at time T, i.e. $U = F(B^y_T)$ (Subsection 4.1). This step needs to pass through the limit in equation $(M+D)$; when f is independent of z, it is a corollary of a well-known result, but in the other case, more technical calculations, using the uniqueness part, are involved.
2. Solve a Pardoux-Peng BSDE with parameter to construct a pair of processes in $\mathbb{R}^n \times \mathbb{R}^{nd_w}$ which is close to being a solution of $(M + D)$ with $X_T = U$ (Subsection 4.2).
3. Show that under an additional condition on f the solution of the preceding BSDE is a solution of the BSDE $(M + D)$ on a small time interval (Subsections 4.3 and 4.4). Note that the main argument in the existence proof is certainly Proposition 4.3.2, where we give an a.s. upper bound on the process (Z_t).
4. Use the convex function Ψ to show that we have a solution of $(M + D)$ on the whole time interval $[0; T]$ (Subsection 4.5).
   In fact, for technical reasons we suppose in the two last steps that f is sufficiently regular; then the proof of the existence is completed with the last subsection :
5. Solve BSDE $(M + D)$ for general f using classical approximation methods (Subsection 4.6).
   Note that we usually work within local coordinates in $\mathbb{R}^n$, i.e. we consider that $\overline{\omega} \subset O \subset \mathbb{R}^n$.

4.1. Reduction of the problem

Let $C^\infty_c(\mathbb{R}^d, \overline{\omega})$ denote $\overline{\omega}$-valued functions on $\mathbb{R}^d$ which are constant off a compact set. In this paragraph, it is shown that it suffices to check the existence result for $U = F(B^y_T)$ with $F \in C^\infty_c(\mathbb{R}^d, \overline{\omega})$.

The space of all functionals $\{G(W_{t_1}, W_{t_2}, \ldots, W_{t_q}), 0 < t_1 < \ldots < t_q \le T;$ $G \in C^\infty_c(\mathbb{R}^{qd_W}, \overline{\omega})\}$ is dense in $L^2(\mathcal{F}_T; \overline{\omega})$ (remark that $L^2(\mathcal{F}_T; \overline{\omega}) = L^\infty(\mathcal{F}_T; \overline{\omega})$ since $\overline{\omega}$ is compact) endowed with the distance

$$\delta^{(1)}(U_1, U_2) = \sqrt{\mathbb{E}\left(\delta^2(U_1, U_2)\right)}.$$

The same is true by replacing functions of the Brownian Motion at discrete times by functions of the diffusion B^y at discrete times. Indeed, if we add the d_W components $W_t^1, \dots, W_t^{dw}$ to the d components of the diffusion $B_t^y = (B_t^{1,y}, \dots, B_t^{d,y})$, it is easy to conclude that the diffusion obtained in this way generate the same filtration as $(W_t)_{0\le t\le T}$.

Let $U^l \in L^2(\mathcal{F}_T; \overline{\omega})$ and (X^l, Z^l) the $\overline{\omega}$-valued solution of $(M + D)$ with $X_T^l = U^l$. We are going to show that if $U^l \to U$ in $L^2(\mathcal{F}_T; \overline{\omega})$, then (X^l) tends, for the distance

$$\delta^{(2)}\left((X_t^l), (X_t)\right) = \sqrt{\mathbb{E}\left(\sup_{t\in[0,T]} \delta^2(X_t^l, X_t)\right)},$$

to a process X ending at U and that there is a process Z such that (X, Z) solves the BSDE $(M + D)$.

Definition 4.1.1. *Let $\mathcal{T}$ be the set of all terminal values $U \in \overline{\omega}$ of processes X solutions of $(M + D)$ (i.e. such that there is a process Z with (X, Z) solution of $(M + D)$) and $\mathcal{S}$ be the set of all these $\overline{\omega}$-valued processes. According to the uniqueness part, to every U in $\mathcal{T}$ corresponds a unique process $(X_t)_t$ in $\mathcal{S}$ such that $X_T = U$. Hence we can define a mapping c with $c(U)$ being this process :*

$$\begin{aligned} c : \mathcal{T} &\to \mathcal{S} \\ U &\mapsto (X_t)_t. \end{aligned}$$

We endow the space $\mathcal{T}$ with $\delta^{(1)}$ and $\mathcal{S}$ with $\delta^{(2)}$, the distances just defined.

It is obvious that c is one-to-one and onto, and that c^{-1} is uniformly continuous for the above distances. Next we want to prove the uniform continuity of c. This is the aim of the following lemma, whose proof is a (slightly) modified version of the one of theorem (5.5) from [15].

Lemma 4.1.2. *c is uniformly continuous for the distances $\delta^{(1)}$ and $\delta^{(2)}$.*

Proof. Suppose that we are given two terminal values U_1 and U_2 corresponding to two solutions (X_t, Z_t) and (X_t', Z_t'). Then by Hölder's inequality,

$$\delta^{(2)}((X_t), (X_t')) \le \mathbb{E}\left(\sup_t \delta^p(X_t, X_t')\right)^{\frac{1}{p}}, \tag{4.1}$$

and since $\Psi \approx \delta^p$

$$\begin{aligned} \mathbb{E}\left(\sup_t \delta^p(X_t, X_t')\right) &\le C\mathbb{E}\left(\sup_t \left(e^{\lambda t+\mu\int_0^t(\|Z_s\|_r+\|Z_s'\|_r)ds}\Psi(X_t, X_t')\right)^2\right)^{\frac{1}{2}} \\ &\le C\mathbb{E}\left(e^{2\lambda T+2\mu\int_0^T(\|Z_s\|_r+\|Z_s'\|_r)ds}\Psi^2(X_T, X_T')\right)^{\frac{1}{2}} \\ &\le C\mathbb{E}\left(e^{4\mu\int_0^T(\|Z_s\|_r+\|Z_s'\|_r)ds}\right)^{\frac{1}{4}}\mathbb{E}\left(\delta^{4p}(U_1, U_2)\right)^{\frac{1}{4}} \\ &\le C\mathbb{E}\left(e^{C_\eta T+\eta\int_0^T(\|Z_s\|_r^2+\|Z_s'\|_r^2)ds}\right)^{\frac{1}{4}}\mathbb{E}\left(\delta^2(U_1, U_2)\right)^{\frac{1}{4}}. \end{aligned} \tag{4.2}$$

The constant C above is allowed to vary from one inequality to another, but it depends only on T, $\overline{\omega}$ and Ψ (but not on the processes X and X'). The second inequality is Doob's L^2 one applied to the submartingale $(\exp(A_t)\Psi(X_t, X'_t))_t$; the third one is Cauchy-Schwarz's one and the last one uses the classical inequality $x \leq \eta x^2 + C_\eta$ and the boundedness of δ which implies $\delta^{4p} \leq \tilde{C}\delta^2$.

Inequalities (4.1) and (4.2) together give :

$$\delta^{(2)}((X_t), (X'_t)) \leq C\mathbb{E}\left(e^{C_\eta T + \eta \int_0^T (\|Z_s\|_r^2 + \|Z'_s\|_r^2) ds}\right)^{\frac{1}{4p}} \delta^{(1)}(U_1, U_2)^{\frac{1}{2p}}. \tag{4.3}$$

Now we specialize to each case : if f does not depend on z, take $\mu = 0$; then $\left(e^{\lambda t}\Psi(X_t, X'_t)\right)_t$ is indeed a submartingale, and with $\eta = C_\eta = 0$, the inequality (4.3) gives the uniform continuity required.

In the other case, $(\exp(A_t)\Psi(X_t, X'_t))_t$ is yet a submartingale with $\Psi = \frac{1}{2}\delta^2$; moreover, taking in (4.3) $\eta = \alpha$ (with α as in Corollary 3.4.4) leads to the conclusion again. □

Using the completeness of the space of all $\overline{\omega}$-valued processes endowed with the distance $\delta^{(2)}$, we get the following result as an easy consequence of the preceding lemma.

Proposition 4.1.3. *Let $(U^l)_l$ be a sequence in $\mathcal{T}$ converging to $U \in L^2(\mathcal{F}_T; \overline{\omega})$ for the distance $\delta^{(1)}$ and $X^l = c(U^l)$. Then there is a (continuous) process $(X_t)_{t\in[0;T]}$ such that $\delta^{(2)}(X^l, X) \xrightarrow{l} 0$. In particular, we have $X_T = U$ a.s.*

The result of this part is completed by proving the next proposition.

Proposition 4.1.4. *There is a process Z such that the pair (X, Z), with X defined in Proposition 4.1.3, is a solution of $(M + D)$ with $X_T = U$, limit of the sequence $(U^l)_l$.*

Proof. First, it is clear from the definition of $\delta^{(2)}$ that X is $\overline{\omega}$-valued. We give then two proofs, one for the z-independent case and one for the general case in nonpositive curvatures.

In the z-independent case, we use a proof very similar to the one of Theorem (4.43) of [10] based on the characterization of Proposition 2.4.1 : first we localize the processes (in order to work in the O_p defined in Proposition 2.4.1) and then it suffices to notice that the submartingale property passes through the limit in l.

In the other case, it is not possible to apply the preceding proof, as the drift depends also on z. In fact, we are going to find explicitly a process Z such that (X, Z) is a solution of $(M + D)$. We recall that for each l, we have in $\mathbb{R}^n$ (remember that $[Z_t^l]^j$ is the j^{th} row of the matrix Z_t^l)

$$\begin{cases} dX_t^l = Z_t^l dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t^l)([Z_t^l]^k \mid [Z_t^l]^j) + f(B_t^y, X_t^l, Z_t^l)\right) dt \\ X_T^l = U^l. \end{cases}$$

First step. Find a process Z, limit in the L^2 sense of the processes Z^l.

Let

$$\tilde{X}^{l,m} = (X^l, X^m) \quad \text{and} \quad \tilde{Z}^{l,m} = \begin{pmatrix} Z^l \\ Z^m \end{pmatrix};$$

apply Itô's formula (2.1) to $\Psi(\tilde{X}^{l,m})$; then use (3.10) to bound below the Hessian term and (3.11) to bound above the term involving f. Taking the expectation gives

$$\frac{1}{4}\mathbb{E}\int_0^T \left\| \overset{X_s^l}{\underset{X_s^m}{\|}} Z_s^m - Z_s^l \right\|_r^2 ds \le C\mathbb{E}\left(\sup_s \delta^2(X_s^l, X_s^m)\right) \\ +C\mathbb{E}\int_0^T \delta^2(X_s^l, X_s^m)(\|Z_s^l\|_r + \|Z_s^m\|_r)ds.$$

We know that the first expectation on the right tends to zero as l and m tend to $+\infty$. Moreover, if I_1 denotes the last integral, then using a α as in Corollary 3.4.4 and the fact that δ^4 is bounded above by $C\delta^2$ on the compact set $\overline{\omega} \times \overline{\omega}$, we get

$$\begin{aligned} I_1 &\le \sqrt{2}\mathbb{E}\left(\sup_s \delta^4(X_s^l, X_s^m)\right)^{\frac{1}{2}} \mathbb{E}\left(\int_0^T (\|Z_s^l\|_r^2 + \|Z_s^m\|_r^2)ds\right)^{\frac{1}{2}} \\ &\le \sqrt{\frac{2}{\alpha}}\mathbb{E}\left(\sup_s \delta^4(X_s^l, X_s^m)\right)^{\frac{1}{2}} \mathbb{E}\left(e^{\alpha\int_0^T (\|Z_s^l\|_r^2 + \|Z_s^m\|_r^2)ds}\right)^{\frac{1}{2}} \\ &\le \frac{C}{\sqrt{\alpha}}\mathbb{E}\left(\sup_s \delta^2(X_s^l, X_s^m)\right)^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

As C does not depend on l, m,

$$\mathbb{E}\int_0^T \left\| \overset{X_s^l}{\underset{X_s^m}{\|}} Z_s^m - Z_s^l \right\|_r^2 ds \overset{l,m\to\infty}{\longrightarrow} 0. \tag{4.5}$$

Now we use inequality (2.3) to have a bound on the Euclidean norm :

$$\mathbb{E}\int_0^T \|Z_s^m - Z_s^l\|^2 ds \le C\Bigg(\mathbb{E}\int_0^T \left\| \overset{X_s^l}{\underset{X_s^m}{\|}} Z_s^m - Z_s^l \right\|_r^2 ds \\ +\mathbb{E}\int_0^T \delta^2(X_s^l, X_s^m)(\|Z_s^l\|_r^2 + \|Z_s^m\|_r^2)ds\Bigg).$$

The first term on the right tends to 0 by (4.5), and an argument similar to (4.4) would show that it also holds for the second term. Hence

$$\mathbb{E}\int_0^T \|Z_s^m - Z_s^l\|^2 ds \overset{l,m\to\infty}{\longrightarrow} 0.$$

Now by completeness of the space $L^2([0;T] \times \Omega)$, we have the required result:

$$\exists (Z_t)_t \in L^2([0;T] \times \Omega) : (Z_t^l) \overset{L^2}{\longrightarrow} (Z_t).$$

Second step. (X, Z) is indeed a solution of equation $(M + D)$ with terminal value U.

In view of this, let us show that the following expectation tends to zero as l tends to ∞ :

$$\mathbb{E}\bigg|U - \int_t^T Z_s dW_s - \int_t^T \left(-\frac{1}{2}\Gamma_{jk}(X_s)([Z_s]^k \mid [Z_s]^j) + f(B_s^y, X_s, Z_s)\right) ds$$
$$-U^l + \int_t^T Z_s^l dW_s$$
$$+ \int_t^T \left(-\frac{1}{2}\Gamma_{jk}(X_s^l)(\left[Z_s^l\right]^k \mid \left[Z_s^l\right]^j) + f(B_s^y, X_s^l, Z_s^l)\right) ds\bigg|.$$

Obviously, this expectation is bounded above by

$$\mathbb{E}\left(|U - U^l|^2\right)^{\frac{1}{2}} + \mathbb{E}\left(\int_0^T \|Z_s^l - Z_s\|^2 ds\right)^{\frac{1}{2}}$$
$$+\mathbb{E}\left(\int_0^T |\Gamma_{jk}(X_s) - \Gamma_{jk}(X_s^l)| \cdot |([Z_s]^k \mid [Z_s]^j)| ds\right)$$
$$+\mathbb{E}\left(\int_0^T |\Gamma_{jk}(X_s^l)| \left|([Z_s]^k \mid [Z_s]^j) - (\left[Z_s^l\right]^k \mid \left[Z_s^l\right]^j)\right| ds\right)$$
$$+\mathbb{E}\left(\int_0^T |f(B_s^y, X_s, Z_s) - f(B_s^y, X_s^l, Z_s^l)| ds\right).$$

We know that the first two expectations tend to zero; the third term tends to zero by dominated convergence (at least for a subsequence of (X^l), but it doesn't matter since (X^l) is Cauchy). Let E_1 denote the next expectation; then we can write

$$E_1 \leq C\mathbb{E}\left(\int_0^T \|Z_s^l - Z_s\|(\|Z_s^l\| + \|Z_s\|) ds\right)$$
$$\leq \sqrt{2}C\mathbb{E}\left(\int_0^T \|Z_s^l - Z_s\|^2 ds\right)^{\frac{1}{2}} \mathbb{E}\left(\int_0^T (\|Z_s^l\|^2 + \|Z_s\|^2) ds\right)^{\frac{1}{2}}.$$

The first integral tends to zero and the second is bounded because Z^l converges in L^2; hence E_1 tends to zero.

Finally, let E_2 denote the last integral and use inequality (2.5) to obtain

$$E_2 \leq \mathbb{E}\left(\int_0^T \left(\delta(X_s^l, X_s)(1 + \|Z_s^l\| + \|Z_s\|) + \|Z_s^l - Z_s\|\right) ds\right);$$

using an argument similar to (4.4), this quantity tends also to zero as l tends to ∞.

Hence the limit in L^1 of X_t^l is

$$U - \int_t^T Z_s dW_s - \int_t^T \left(-\frac{1}{2}\Gamma_{jk}(X_s)([Z_s]^k \mid [Z_s]^j) + f(B_s^y, X_s, Z_s)\right) ds;$$

We know that it is also X_t, so by continuity :

$$a.s.,\ \forall t,\ X_t = U - \int_t^T Z_s dW_s \\ - \int_t^T \left(-\frac{1}{2}\Gamma_{jk}(X_s)([Z_s]^k \mid [Z_s]^j) + f(B_s^y, X_s, Z_s) \right) ds.$$

That finishes the proof of the proposition for a general f in nonpositive curvatures. $\square$

As a consequence of Propositions 4.1.3 and 4.1.4, it suffices to work with a dense subset of $L^2(\mathcal{F}_T; \overline{\omega})$, i.e. to consider terminal values U that are written $U = G(B_{t_1}^y, B_{t_2}^y, \ldots, B_{t_q}^y)$ where $G \in C_c^\infty(\mathbb{R}^{qd}, \overline{\omega})$ (in fact, we can obviously take $t_q = T$).

A further step of simplification is possible (see [15]): conditioning by $\mathcal{F}_{t_{q-1}}$ and working over the time interval $[t_{q-1}; T]$, $B_{t_1}^y, \ldots, B_{t_{q-1}}^y$ can be treated as constant. Then, if we know a solution (X_t, Z_t) to $(M + D)$ on $[t_{q-1}; T]$ with $X_T = G(B_T^y)$, the problem is to reach $X_{t_{q-1}}$. But this variable is in fact a measurable function of $B_{t_1}^y, B_{t_2}^y, \ldots, B_{t_{q-1}}^y$. A density argument enables us to suppose this function smooth and constant off a compact set. Thus by induction the problem is solved if we can find a solution to $(M + D)$ with terminal value $F(B_T^y)$ with $F \in C_c^\infty(\mathbb{R}^d, \overline{\omega})$.

So to prove the existence result for any $\overline{\omega}$-valued terminal variable U, it suffices to solve equation $(M + D)$ with a terminal value U that can be written $F(B_T^y)$ where $F \in C_c^\infty(\mathbb{R}^d, \overline{\omega})$. This is the aim of the next paragraphs.

4.2. Approximation by BSDEs with Lipschitz coefficients

We deal with a BSDE whose coefficients $\Gamma_{jk}(x)$ (we recall that it is a vector in $\mathbb{R}^n$; see the introduction) and $f(b, x)$ (or $f(b, x, z)$) are defined only for x in the open subset O of $\mathbb{R}^n$, and with a quadratic term in Z_t; but we would like to apply the existence result of [24] to BSDEs with Lipschitz coefficients. Our purpose in this part is to define such a BSDE, derived from the initial one and defined on all $\mathbb{R}^n$.

Firstly, we extend the definition of the BSDE $(M + D)$ to the whole space $\mathbb{R}^n$: let ϕ be a smooth function on $\mathbb{R}^n$ with compact support in O and such that $\phi = 1$ on $\overline{\omega}$. We will explicit ϕ a little more later in Subsection 4.4. Then we extend the drift f and Christoffel symbols to the whole space $\mathbb{R}^n$ by letting , for $b \in \mathbb{R}^d$, $\tilde{f}(b, x) = \phi(x) f(b, x)$ (or $\tilde{f}(b, x, z) = \phi(x) f(b, x, z)$) and $\tilde{\Gamma}(x) = \phi(x)\Gamma(x)$.

The new BSDE defined on all $\mathbb{R}^n$ (we keep X and Z for the notations) is :

$$(\widetilde{M + D}) \begin{cases} dX_t = Z_t dW_t + \left(-\frac{1}{2}\tilde{\Gamma}_{jk}(X_t)([Z_t]^k \mid [Z_t]^j) + \tilde{f}(B_t^y, X_t, Z_t) \right) dt \\ X_T = U. \end{cases}$$

Remark. We write $\tilde{f}(B_t^y, X_t, Z_t)$ but when it is not pointed out, it should also be interpreted as $\tilde{f}(B_t^y, X_t)$ as well.

Now let $\varepsilon \in]0;1[$ and $s_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth nondecreasing function such that $s_\varepsilon(t) = 0$ iff $t \in [0; \frac{1}{\varepsilon}]$, and s_ε is linear for t large enough. Then, for $z \in \mathbb{R}^{ndw}$, we define $h_\varepsilon(z) = s_\varepsilon(\|z\|)$ and $\overline{z} = \frac{z}{1+h_\varepsilon(z)}$. We have the following

Lemma 4.2.1. *The functions h_ε and $\overline{z}$ defined above satisfy the next assertions.*

(i) If $\|z\| \leq \frac{1}{\varepsilon}$, then $h_\varepsilon(z) = 0$ and $\overline{z} = z$;
(ii) $z \mapsto \overline{z}$ is smooth, bounded and Lipschitz on $\mathbb{R}^{ndw}$;
(iii) $(x,z) \mapsto g(x,z) := \tilde{\Gamma}_{jk}(x)([\overline{z}]^k|[\overline{z}]^j)$ is smooth and Lipschitz on $\mathbb{R}^n \times \mathbb{R}^{ndw}$.
(iv) $(b,x,z) \mapsto \tilde{f}(b,x,\overline{z})$ is bounded and Lipschitz on $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{ndw}$.

As a consequence, the function

$$\begin{aligned}\gamma : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{ndw} &\to \mathbb{R}^n \\ (b,x,z) &\mapsto \frac{1}{2}\tilde{\Gamma}_{jk}(x)([\overline{z}]^k|[\overline{z}]^j) - \tilde{f}(b,x,\overline{z})\end{aligned}$$

is bounded and Lipschitz.

Proof. (i) It is obvious on the definition.

(ii) The smoothness and boundedness of $\overline{z}$ are clear. Now we just prove that $\|\overline{z} - \overline{z'}\| \leq C\|z - z'\|$, considering the 3 cases below.

- $\|z\| \leq \frac{1}{\varepsilon}$ *and* $\|z'\| \leq \frac{1}{\varepsilon}$
  Then $\|\overline{z} - \overline{z'}\| = \|z - z'\|$.
- $\|z\| \geq \frac{1}{\varepsilon}$ *and* $\|z'\| \geq \frac{1}{\varepsilon}$
  Then, as $\overline{z}$ is bounded and h_ε Lipschitz, we can write

$$\begin{aligned}\left\|\overline{z} - \overline{z'}\right\| &\leq \left\|\frac{z - z'}{1 + h_\varepsilon(z)}\right\| + \|z'\| \left|\frac{1}{1 + h_\varepsilon(z)} - \frac{1}{1 + h_\varepsilon(z')}\right| \\ &\leq \|z - z'\| + \|\overline{z'}\| \left|\frac{h_\varepsilon(z') - h_\varepsilon(z)}{1 + h_\varepsilon(z)}\right| \\ &\leq C\|z - z'\|.\end{aligned}$$

- $\|z\| > \frac{1}{\varepsilon}$ *and* $\|z'\| \leq \frac{1}{\varepsilon}$
  Let $z'' \in [z; z']$ with $\|z''\| = \frac{1}{\varepsilon}$; then, using the first two cases, we have

$$\left\|\overline{z} - \overline{z'}\right\| \leq \left\|\overline{z} - \overline{z''}\right\| + \left\|\overline{z'} - \overline{z''}\right\| \leq C(\|z - z''\| + \|z' - z''\|) = C\|z - z'\|.$$

(iii) We have

$$\begin{aligned}|g(x,z) - g(x',z')| \leq\ & |\tilde{\Gamma}_{jk}(x) - \tilde{\Gamma}_{jk}(x')| \cdot |([\overline{z}]^k|[\overline{z}]^j)| \\ &+ |\tilde{\Gamma}_{jk}(x')| \cdot |([\overline{z}]^k|[\overline{z}]^j) - ([\overline{z'}]^k|[\overline{z'}]^j)|.\end{aligned}$$

There is no problem with the first term on the right because $\tilde{\Gamma}_{jk}$ is smooth on $\mathbb{R}^n$ with compact support and $\overline{z}$ is bounded. Using (ii), it is easy to deduce that the second term is bounded above by $C\|z - z'\|$ and that finishes the proof.

(iv) Recall that for $x, x' \in O$,

$$|f(b, x, \overline{z}) - f(b', x', \overline{z'})| \\ \leq L'\Big((|b - b'| + |x - x'|)(1 + \|\overline{z}\| + \|\overline{z'}\|) + \|\overline{z} - \overline{z'}\|\Big).$$

If $\overline{O_s}$ denotes the compact support of ϕ, this inequality and (1.5) imply that $f(b, x, \overline{z})$ is bounded, uniformly in $x \in \overline{O_s}$, $b \in \mathbb{R}^d$ and $z \in \mathbb{R}^{ndw}$. Thus $\tilde{f}$ is bounded on $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{ndw}$ because for x outside $\overline{O_s}$, $\tilde{f} = 0$. Besides, we consider the 3 following cases to study the Lipschitz property :

- $x, x' \notin \overline{O_s}$
  Then $\tilde{f}(b, x, \overline{z}) - \tilde{f}(b', x', \overline{z'}) = 0$ and it is obvious.
- $x \in \overline{O_s}, x' \notin \overline{O_s}$
  Then, as ϕ is Lipschitz,

$$\begin{aligned}|\tilde{f}(b, x, \overline{z}) - \tilde{f}(b', x', \overline{z'})| &= |\phi(x) f(b, x, \overline{z})| \\ &= |\phi(x) - \phi(x')| \cdot |f(b, x, \overline{z})| \\ &\leq \alpha|x - x'|.\end{aligned}$$

- $x, x' \in \overline{O_s}$
  Then

$$\begin{aligned}|\tilde{f}(b, x, \overline{z}) - \tilde{f}(b', x', \overline{z'})| &= |\phi(x) f(b, x, \overline{z}) - \phi(x') f(b', x', \overline{z'})| \\ &\leq |\phi(x) - \phi(x')| \cdot |f(b, x, \overline{z})| \\ &\quad + |\phi(x')| \cdot |f(b, x, \overline{z}) - f(b', x', \overline{z'})| \\ &\leq \alpha|x - x'| + \beta_1\|\overline{z} - \overline{z'}\| + \gamma|b - b'| \\ &\leq \alpha|x - x'| + \beta_2\|z - z'\| + \gamma|b - b'|\end{aligned}$$

since $z \mapsto \overline{z}$ is Lipschitz by (ii). The proof of (iv) is completed. □

Now we can introduce a new BSDE

$$(\widetilde{M + D})_\varepsilon \begin{cases} dX_t^\varepsilon = Z_t^\varepsilon dW_t - \gamma(B_t^y, X_t^\varepsilon, Z_t^\varepsilon)dt \\ X_T^\varepsilon = U. \end{cases}$$

The interest of this new equation lies in the following result.

Proposition 4.2.2. *The equation $(\widetilde{M + D})_\varepsilon$ has Lipschitz coefficients; it has a unique solution $(X_t^\varepsilon, Z_t^\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^{ndw}$ such that*

$$\mathbb{E}\left(\sup_{t\in[0;T]} |X_t^\varepsilon|^2\right) < \infty \ \textit{and} \ \mathbb{E}\left(\int_0^T \|Z_t^\varepsilon\|^2 dt\right) < \infty.$$

Proof. The first part is a consequence of Lemma 4.2.1; then existence and uniqueness are classical results of [24]. □

In Subsections 4.3, 4.4 and 4.5 below, f is also supposed to be a C^3 function such that $(b, x, z) \mapsto \tilde{f}(b, x, \overline{z})$ and γ are C^3 functions with all their partial derivatives of order 1, 2 and 3 bounded.

4.3. Existence of a solution of $(\widetilde{M+D})$ on a small time interval

In this paragraph we consider a terminal value $U = F(B_T^y)$ ($F \in C_c^\infty(\mathbb{R}^d, \overline{\omega})$) which lies in $\overline{\omega}$ and we show that for an ε small enough, the solution $(X_t^\varepsilon, Z_t^\varepsilon)$ of $(\widetilde{M+D})_\varepsilon$ is also a solution of the BSDE $(\widetilde{M+D})$ on a small time interval $[T_1^\varepsilon; T]$. This result relies on Proposition 4.3.2, which gives the very strong condition that Z is bounded a.s.

Firstly, let us give a classical link between BSDEs and PDEs.

Proposition 4.3.1. *Let $\varepsilon \in]0; 1[$; we use the notations introduced in paragraph 4.2 (recall in particular that the function γ depends on ε).*
Consider for $u_\varepsilon = (u_\varepsilon^1, \dots, u_\varepsilon^n) : [0; T] \times \mathbb{R}^d \to \mathbb{R}^n$ the following system of quasilinear parabolic partial differential equations

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t}(t, x) &= \mathcal{L}u_\varepsilon(t, x) + \gamma(x, u_\varepsilon(t, x), (\nabla_x u_\varepsilon \sigma)(t, x)) \\ u_\varepsilon(0, x) &= F(x) \end{cases} \tag{4.6}$$

where $\nabla_x u_\varepsilon$ is the $n \times d$ matrix whose rows are $(\nabla_x u_\varepsilon^i)_{i=1,\dots,n}$, the partial derivatives of the components of u_ε with respect to space; moreover,

$$\mathcal{L}u_\varepsilon = (Lu_\varepsilon^1, \dots, Lu_\varepsilon^n) \tag{4.7}$$

is a vector in $\mathbb{R}^n$, and

$$L = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma^t \sigma)_{i,j}(t, x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x)\frac{\partial}{\partial x_i} \tag{4.8}$$

is the infinitesimal generator of the diffusion (1.2).

Then

(i) This equation has a unique solution u_ε in $C^{1,2}([0; T] \times \mathbb{R}^d, \mathbb{R}^n)$ (i.e. u_ε has continuous first derivative with respect to time and continuous second derivatives with respect to space);
(ii) A.s., $X_t^\varepsilon = u_\varepsilon(T - t, B_t^y)$ $\forall t$;
(iii) $\forall t$, a.s., $Z_t^\varepsilon = \nabla_x u_\varepsilon(T - t, B_t^y)\sigma(B_t^y)$.

Proof. The assertion (i) is a classical result; for a probabilistic proof, see Theorem 3.2 of [25], and for a proof in dimension 1 (i.e. $n = 1$) with less regular functions γ, see [2].

Then (ii) and (iii) follow easily : both Itô's formula and equation (4.6) lead to

$$\begin{cases} du_\varepsilon(T - t, B_t^y) &= -\gamma(B_t^y, u_\varepsilon, (\nabla_x u_\varepsilon \sigma)(B_t^y))dt + (\nabla_x u_\varepsilon \sigma)dW_t \\ u_\varepsilon(0, B_T^y) &= F(B_T^y) = U. \end{cases}$$

Thus

$$\left(u_\varepsilon(T - t, B_t^y), \nabla_x u_\varepsilon(T - t, B_t^y)\sigma(B_t^y)\right)_t$$

is a solution of BSDE $(\widetilde{M+D})_\varepsilon$. But $(X^\varepsilon, Z^\varepsilon)$ is also a solution and this BSDE has Lipschitz coefficients, so it has a unique solution. Then (ii) and (iii) follow. □

Remark. Since $\nabla_x u_\varepsilon$ is continuous, the trajectories $t \mapsto Z_t^\varepsilon$ can be taken continuous.

Now we give the main result of this paragraph.

Proposition 4.3.2. *There is an* $\varepsilon \in]0;1[$ *and a* $T_1^\varepsilon \in [0;T[$ *(depending on* ε*) such that a.s. for* $t \in [T_1^\varepsilon; T]$*, we have* $\|Z_t^\varepsilon\| \le \frac{1}{\varepsilon}$*. This means, with the notations of paragraph 4.2, that* $h_\varepsilon(Z_t^\varepsilon) = 0$ *so* $\overline{Z}_t^\varepsilon = Z_t^\varepsilon$ *and* $(X_t^\varepsilon, Z_t^\varepsilon)$ *is a solution of BSDE* $(\widetilde{M+D})$ *on the time interval* $[T_1^\varepsilon; T]$.

Proof. We only deal with a drift f depending both on x and z, the z-independent case being similar (and easier).

We let α_ε be the global Lipschitz constant of the function γ (independent of b):

$$\begin{aligned}&\forall (x,z),(x',z') \in \mathbb{R}^n \times \mathbb{R}^{ndw},\\ &|\gamma(b,x,z) - \gamma(b,x',z')| \le \alpha_\varepsilon \left(|x-x'| + \|z-z'\|\right).\end{aligned} \tag{4.9}$$

(in general, α_ε will tend to ∞ as ε decreases to zero).

We have to show that $\|Z_t^\varepsilon\| = \left\|\nabla_x u_\varepsilon(T-t, B_t^y)\sigma(B_t^y)\right\| \le \frac{1}{\varepsilon}$ a.s., so it suffices to check that u_ε is $\frac{1}{\varepsilon\|\sigma\|_\infty}$-Lipschitz with respect to the space variable ($\|\sigma\|_\infty$ denoting the supremum of $\|\sigma(b)\|$ for $b \in \mathbb{R}^d$). This work will be achieved in several steps.

First step. We consider two solutions (Y,Z) and $(\hat{Y},\hat{Z})$ of $(\widetilde{M+D})_\varepsilon$ corresponding to two terminal values Y_T and $\hat{Y}_T$. We set $\delta Y_t = Y_t - \hat{Y}_t$ and $\delta Z_t = Z_t - \hat{Z}_t$. Then

$$\delta Y_t + \int_t^T \delta Z_s dW_s = \delta Y_T + \int_t^T \left(\gamma(B_s^y, Y_s, Z_s) - \gamma(B_s^y, \hat{Y}_s, \hat{Z}_s)\right) ds.$$

Independence gives :

$$\begin{aligned}&\mathbb{E}|\delta Y_t|^2 + \mathbb{E}\left(\int_t^T \|\delta Z_s\|^2 ds\right)\\ &= \mathbb{E}\left(\left|\delta Y_T + \int_t^T (\gamma(B_s^y, Y_s, Z_s) - \gamma(B_s^y, \hat{Y}_s, \hat{Z}_s))ds\right|^2\right).\end{aligned}$$

Using Hölder's inequality, we obtain :

$$\begin{aligned}\mathbb{E}|\delta Y_t|^2 + \mathbb{E}\left(\int_t^T \|\delta Z_s\|^2 ds\right) &\le 2\mathbb{E}|\delta Y_T|^2\\ &+2(T-t)\mathbb{E}\Bigg(\int_t^T |\gamma(B_s^y, Y_s, Z_s)\\ &-\gamma(B_s^y, \hat{Y}_s, \hat{Z}_s)|^2 ds\Bigg).\end{aligned}$$

Hence using (4.9) we have

$$\mathbb{E}|\delta Y_t|^2 + \mathbb{E}\left(\int_t^T \|\delta Z_s\|^2 ds\right) \le 2\mathbb{E}|\delta Y_T|^2$$
$$+4\alpha_\varepsilon^2(T-t)\Bigg(\mathbb{E}\left(\int_t^T |\delta Y_s|^2 ds\right)$$
$$+\mathbb{E}\left(\int_t^T \|\delta Z_s\|^2 ds\right)\Bigg).$$

Second step. We want to remove the terms $\mathbb{E}\left(\int_t^T \|\delta Z_s\|^2 ds\right)$, to get an equation with Y only; this leads to make the following assumption (which will be true in the end of the proof) : $t \in [T_1^\varepsilon; T]$ with $T_1^\varepsilon \in [0; T[$ such that $4\alpha_\varepsilon^2(T - T_1^\varepsilon) \le 1$.

Then, if $t \in [T_1^\varepsilon; T]$, we get

$$\mathbb{E}|\delta Y_t|^2 \le 2\mathbb{E}|\delta Y_T|^2 + \mathbb{E}\left(\int_t^T |\delta Y_s|^2 ds\right).$$

And Gronwall's lemma gives

$$\mathbb{E}|\delta Y_t|^2 \le 2e^{T-t}\mathbb{E}|\delta Y_T|^2. \tag{4.10}$$

Third step. Let us choose $Y_T = F(B_T^{t,x})$ and $\hat{Y}_T = F(B_T^{t,\hat{x}})$ (where $(B_\cdot^{t,x})$ denotes the diffusion starting at x at time t); then

$$\delta Y_t = u_\varepsilon(T-t, x) - u_\varepsilon(T-t, \hat{x}) \tag{4.11}$$

and

$$\mathbb{E}|\delta Y_T|^2 \le L_F^2 \mathbb{E}\left(|B_T^{t,x} - B_T^{t,\hat{x}}|^2\right) \le L_F^2 C_{\sigma,b}|x - \hat{x}|^2, \tag{4.12}$$

with L_F the Lipschitz constant of F, and the last inequality is a well-known result of L^2-continuity with respect to initial conditions. Hence from (4.10)

$$|u_\varepsilon(T-t,x) - u_\varepsilon(T-t,\hat{x})| \le L_F\sqrt{2C_{\sigma,b}e^{T-t}}|x-\hat{x}|$$

and

$$\|\nabla_x u_\varepsilon(T-t,x)\| \le L_F\sqrt{2C_{\sigma,b}e^T}.$$

(In fact, the right hand side is given up to a positive multiplicative constant, due to the different norms used; it doesn't matter in the sequel).

Conclusion. Let $\varepsilon \in]0; 1[$ such that $\frac{1}{\varepsilon} \ge \|\sigma\|_\infty L_F\sqrt{2C_{\sigma,b}e^T}$. Then the preceding proof shows that if $T_1^\varepsilon \in [0; T[$ is such that $4\alpha_\varepsilon^2(T - T_1^\varepsilon) \le 1$, then for $t \in [T_1^\varepsilon; T]$ we have

$$a.s.\ \ \|Z_t^\varepsilon\| = \left\|\nabla_x u_\varepsilon(T-t, B_t^y)\sigma(B_t^y)\right\| \le \frac{1}{\varepsilon}. \tag{4.13}$$

In fact, using the continuity in t of Z_t^ε, this inequality holds for any t, a.s.; that finishes the proof of the proposition. □

4.4. A solution of $(M + D)$ on a small time interval

The framework is the same as the one introduced in Subsections 4.2 and 4.3. The aim of this section is to prove that the above solution $(X_t, Z_t)_t$ (we omit in this paragraph the superscript ε for notational convenience) of $(\widetilde{M + D})$ is, under an additional condition on the drift f, a solution of $(M + D)$ (in fact we prove that (X_t) remains in $\overline{\omega}$, the compact where the terminal value U lies). We recall that $\overline{\omega} = \{\chi \leq c\}$ is the sublevel set of a smooth convex (for the connection Γ) function χ, defined on an open set O relatively compact in $\mathbb{R}^n$, and moreover that $\overline{\omega}$ is relatively compact in O. We make the following hypothesis

(H_s) f is pointing strictly outward on the boundary $\partial\overline{\omega}$ of $\overline{\omega}$.

It means that

$$\forall(b, x, z) \ : \ x \in \partial\overline{\omega}, \quad \inf_{b,x,z} (D\chi(x)|f(b, x, z))_r \geq \zeta > 0, \tag{4.14}$$

where $(\cdot|\cdot)_r$ denotes the Riemannian metric tensor (if $f^\perp$ is the component of f orthogonal to $\partial\overline{\omega} = \{\chi = c\}$, it is equivalent to require that $\inf_{b,x,z} (D\chi(x)|f^\perp(b, x, z))_r$ be bounded below by a positive constant, since χ is constant (equal to c) on $\partial\overline{\omega}$).

Remark. This condition arises naturally in the deterministic version of equation $(M + D)$

$$\begin{cases} dx_t = f(x_t)dt \\ x_T = u \end{cases}$$

where u is deterministic and so $Z_t = 0$ for any t (in fact, as we shall see in Subsection 4.6, the natural condition is to require that the infimum in (4.14) be only nonnegative).

Proposition 4.4.1. *Suppose that χ is strictly convex on O (this means that* Hess χ *is positive definite). Then, under the assumption (H_s), the process $(X_t)_{t\in[T_1^\varepsilon;T]}$ remains in $\overline{\omega}$, i.e. (X_t, Z_t) is a solution of $(M + D)$ on the time interval $[T_1^\varepsilon; T]$.*

Proof. Once again, the goal is to construct a nonnegative submartingale, null at time T, which vanishes if and only if the process X is in $\overline{\omega}$. The proof will be split into three steps.

First step. Framework of the proof.

Suppose that $c > 0$ and χ reaches its infimum at $p \in \omega$ with $\chi(p) = 0$. Then consider the following mapping, defined on a normal open neighbourhood of $\overline{\omega}$ centered at p (so it is in particular a neighbourhood of 0 in $\mathbb{R}^n$)

$$y \mapsto \frac{\sqrt{c_2}y}{\sqrt{c_2\|y\|^2 + c - \chi(y)}};$$

for $c_2 > 0$ small enough, it is a diffeomorphism from an open set O_1 (relatively compact in O and containing $\overline{\omega}$) onto an open neighbourhood N of $\overline{B(0, 1)}$, such that $\overline{\omega}$ is sent onto $\overline{B(0,1)}$ (in fact, it is sufficient to take $c_2 \leq \frac{1}{2}\lambda_\chi$, where λ_χ denotes the (positive) infimum on O_1 of the eingenvalues of Hess χ). Using this diffeomorphism, we can work in a local chart O (take $O := N$) such that $\overline{\omega} = \overline{B(0,1)}$ and $\chi(0) = 0$.

Second step. Construction of a "nearly convex" function H.

Choose $\rho > 1$ such that $\overline{B(0,\rho)} \subset O$; then we take for the mapping ϕ (see the beginning of Subsection 4.2) a smooth function $\mathbb{R}^n \to [0;1]$ equal to 1 on $B(0,1)$ and to 0 outside $B(0,\rho)$, which has moreover spherical symmetry. This gives $\tilde{\Gamma} = \phi\Gamma$ and $\tilde{f} = \phi f$. Then define on $\mathbb{R}^n$

$$k(x) = \phi(x)\chi(x) + (1 - \phi(x))\alpha(x)$$

where $\alpha(x) = a|x|$ and $a > 0$ is chosen so that $k(x) \leq c$ if and only if $x \in \overline{B(0,1)}$ (take for instance $a > \sup_{B(0,\rho)} \chi$); k is clearly a nonnegative smooth mapping. We would like to have a mapping which vanishes on $\overline{\omega}$, so we let $H = h \circ k$ where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth convex (so nondecreasing) function, vanishing on the interval $[0; c]$ (only) and growing linearly at infinity.

The mapping H so defined is convex for the connection $\tilde{\Gamma}$ on $B(0,1)$ and outside $B(0,\rho)$. Indeed, on $B(0,1)$, $H = 0$ and outside $B(0,\rho)$, $H = h \circ \alpha$ which is convex for the flat connection ($= \tilde{\Gamma}$).

Third step. We work on the time interval $[T_1^\varepsilon; T]$. The aim of this step is to show that, for λ large enough, the process $(e^{\lambda t} H(X_t))_t$ is a real submartingale. Reasoning as in the uniqueness part, we apply Itô's formula :

$$\begin{aligned}
e^{\lambda t} H(X_t) - e^{\lambda T_1^\varepsilon} H(X_{T_1^\varepsilon}) = &\int_{T_1^\varepsilon}^t e^{\lambda s} DH(X_s)(Z_s dW_s)\\
&+ \frac{1}{2}\int_{T_1^\varepsilon}^t e^{\lambda s}\left(\sum_{i=1}^{d_W} {}^t[{}^tZ_s]^i \widetilde{\mathrm{Hess}}\, H(X_s)[{}^tZ_s]^i\right) ds\\
&+ \int_{T_1^\varepsilon}^t e^{\lambda s} DH(X_s)\cdot \tilde{f}(B_s^y, X_s, Z_s) ds\\
&+ \int_{T_1^\varepsilon}^t \lambda e^{\lambda s} H(X_s) ds.
\end{aligned}$$

The stochastic integral is a martingale because DH is bounded; it remains to prove that the bounded variation term is an increasing process, i.e. show the nonnegativity of the sum

$$\frac{1}{2}\sum_{i=1}^{d_W} {}^t[{}^tz]^i \widetilde{\mathrm{Hess}}\, H(x)[{}^tz]^i + DH(x)\tilde{f}(b,x,z) + \lambda H(x).$$

But $\widetilde{\mathrm{Hess}}\, H(x) \geq h'(k(x))\mathrm{Hess}\, k(x)$ since k is nondecreasing (see for instance (4.36) p 42 in [10]); then it suffices to prove the nonnegativity of

$$h'(k(x))\left(\frac{1}{2}\sum_{i=1}^{d_W} {}^t[{}^tz]^i \widetilde{\mathrm{Hess}}\, k(x)[{}^tz]^i + Dk(x)\tilde{f}(b,x,z)\right) + \lambda H(x). \tag{4.15}$$

Remark that $h'(k(x)) \geq 0$ and that, because of the boundedness of the process Z_t (according to (4.13)), it is sufficient to consider z such that $\|z\| \leq \frac{1}{\varepsilon}$.

Let A_1, A_2 and A_3 denote the three terms in this order in the sum (4.15).

- if $x \in \overline{B(0,1)}$, A_1 and A_2 vanish because $H = 0$, so the sum is nonnegative;
- if $x \in {}^cB(0,\rho)$, $A_1 \geq 0$ because $k = \alpha$ is convex for the flat connection and $A_2 = 0$ because $\tilde{f} = 0$; then the sum is nonnegative.
- if $x \in B(0,\rho) \setminus \overline{B(0,1)}$, we consider two situations : firstly when x belongs to a neighbourhood of the sphere $S(0,1)$ (i.e. when $|x| \in]1; 1+\eta[$ where $\eta > 0$ is to be determined). Using continuity, $\widetilde{\text{Hess}}\, k(x)$ is nonnegative (in the sense of matrices) on $]1; 1+\eta_1[$ (with $\eta_1 > 0$ sufficiently small) because if $|x| = 1$, $\widetilde{\text{Hess}}\, k(x) = \widetilde{\text{Hess}}\, \chi(x)$ and χ is strictly convex on $B(0,1)$ for the connection $\tilde{\Gamma}$. This gives $A_1 \geq 0$.

For A_2, using hypothesis (H_s), we get that for $x \in \partial\overline{\omega}$ and $\|z\| \leq \frac{1}{\varepsilon}$,

$$Dk(x) \cdot \tilde{f}(b,x,z) = D\chi(x) \cdot \tilde{f}(b,x,z) = D\chi(x) \cdot f(b,x,z) \geq \zeta > 0.$$

So by uniform continuity and hypothesis (1.4) and (1.5), there is an $\eta_2 > 0$ such that

$$\forall b,\ \forall y : 1 < |y| < 1+\eta_2,\ \forall z : \|z\| \leq \frac{1}{\varepsilon},\ Dk(y) \cdot \tilde{f}(b,y,z) \geq 0.$$

In particular, $A_2 \geq 0$ for $1 < |x| < 1+\eta_2$. Let $\eta = \min(\eta_1, \eta_2, \rho - 1)$. Then $A_1 + A_2 \geq 0$ if $1 < |x| < 1+\eta \leq \rho$, and the sum (4.15) is nonnegative.

On the other hand, if $|x| \in]1+\eta; \rho[$ (i.e. if $|x|$ is "far" from 1), then $A_1 + A_2$ is bounded above since H is smooth, and $\tilde{f}(b,x,z)$ is bounded since z is bounded. But we have constructed H so that $H(x) \geq \theta > 0$ if $|x| > 1+\eta$. Then we can choose $\lambda > 0$ such that $A_1 + A_2 + A_3$ is nonnegative.

Conclusion. The end of the proof now goes on by classical arguments : the process $(e^{\lambda t} H(X_t))_{t \in [T_1^\varepsilon; T]}$ is a nonnegative submartingale for the λ chosen above, null at time T. Then $\forall t \in [T_1^\varepsilon; T],\ H(X_t) = 0$; and the mapping H has precisely been chosen so that this implies $X_t \in \overline{\omega}$. □

Remark. In our local coordinates, let $D_r\chi(x)$ denote the radial derivative of χ at x and f^r the radial component of f. Then, since $\chi = c$ on the sphere $S(0,1) = \partial\overline{\omega}$, we have

$$\forall x \in \partial\overline{\omega},\ D\chi(x) \cdot f(b,x,z) = D_r\chi(x) \cdot f^r(b,x,z).$$

But if Hess χ is positive definite, $D_r\chi(x)$ is a positive real number and in our coordinates, (H_s) is equivalent to require that $f^r(b,x,z)$ be bounded below by a positive constant, independent of $x \in \partial\overline{\omega}$ and of b, z.

Now we come back to the end of the proof of the existence.

4.5. The solution on the whole interval $[0; T]$

According to Subsections 4.3 and 4.4, $(X_t^\varepsilon, Z_t^\varepsilon)$ is a solution of BSDE $(M + D)$ on the time interval $[T_1^\varepsilon; T]$; moreover, ε and T_1^ε must verify (see the conclusion in the proof of Proposition 4.3.2)

$$\begin{cases} \frac{1}{\varepsilon} & \geq \|\sigma\|_\infty L_F \sqrt{2C_{\sigma,b}e^T} \\ 4\alpha_\varepsilon^2(T - T_1^\varepsilon) & \leq 1 \end{cases}$$

where α_ε tends *a priori* to ∞ when ε goes to zero. The aim of this subsection is to show that it is a solution on the time interval $[0; T]$.

In general, if we take back the proof of Proposition 4.3.2, with now T_1^ε and $u_\varepsilon(T - T_1^\varepsilon, B^y_{T_1^\varepsilon})$ as terminal time and variable, we get a solution of $(M + D)$ on a time interval $[T_2^\varepsilon; T_1^\varepsilon]$; but it is easy to show that ε and T_2^ε must verify now

$$\begin{cases} \frac{1}{\varepsilon} & \geq (\|\sigma\|_\infty L_F \sqrt{2C_{\sigma,b}e^T})\sqrt{2e^T} \\ 4\alpha_\varepsilon^2(T_1^\varepsilon - T_2^\varepsilon) & \leq 1; \end{cases}$$

as a consequence, the length of the interval $[T_2^\varepsilon; T_1^\varepsilon]$ may be less than the one of $[T_1^\varepsilon; T]$ and repeating this method inductively could lead to a solution on an interval $]T_0; T[$ with $T_0 > 0$ only (this means that the solution explodes).

In fact, the existence of Ψ prevents the solution from exploding and allows to build inductively a solution of $(M + D)$ on $[T_1^\varepsilon; T]$, $[T - 2(T - T_1^\varepsilon); T]$, $[T - 3(T - T_1^\varepsilon); T]$, ... and so to get at the end a solution on $[0; T]$.

Proposition 4.5.1. *Suppose that assumption* (H_s) *holds and that* χ *is strictly convex on* O. *If* $\varepsilon > 0$ *is small enough, then* $(X_t^\varepsilon, Z_t^\varepsilon)$*, the solution of BSDE* $(\widetilde{M + D})_\varepsilon$*, is also a solution of BSDE* $(M + D)$ *on the whole time interval* $[0; T]$.

Proof. Consider T_1^ε and $u_\varepsilon(T - T_1^\varepsilon, B^y_{T_1^\varepsilon})$ as terminal time and variable; applying the two preceding sections, we get a solution of $(M + D)$ (with $X_t^\varepsilon \in \overline{\omega}$) on a time interval $[T_2^\varepsilon; T_1^\varepsilon]$ where ε and T_2^ε verify

$$\begin{cases} \frac{1}{\varepsilon} & \geq \|\sigma\|_\infty L_{u_\varepsilon} \\ 4\alpha_\varepsilon^2(T_1^\varepsilon - T_2^\varepsilon) & \leq 1 \end{cases}$$

(L_{u_ε} is the Lipschitz constant of u_ε for the space variable, uniformly in t on $[T_2^\varepsilon; T]$).

As in Subsection 4.3, for $t \in [T_2^\varepsilon; T[$ we let $(Y_s, Z_s)_{s\in[t;T]}$ and $(\hat{Y}_s, \hat{Z}_s)_{s\in[t;T]}$ be two solutions of BSDE $(\widetilde{M + D})_\varepsilon$ such that $Y_T = F(B_T^{t,x})$ and $\hat{Y}_T = F(B_T^{t,\hat{x}})$ (by which we denote diffusions starting at x or $\hat{x}$ at time t). According to Subsection 4.4, these two processes remain in $\overline{\omega}$, so we can make use of the function Ψ. Then the same inequalities as in (4.2) give

$$\begin{aligned} \mathbb{E}\left(\Psi(Y_t, \hat{Y}_t)\right) &\leq C_1 \mathbb{E}\left(e^{\eta \int_0^T (\|Z_u\|_r^2 + \|\hat{Z}_u\|_r^2)du}\right)^{\frac{1}{4}} \mathbb{E}\left(\delta^{4p}(Y_T, \hat{Y}_T)\right)^{\frac{1}{4}} \\ &\leq C\mathbb{E}\left(\delta^{4p}(Y_T, \hat{Y}_T)\right)^{\frac{1}{4}} \end{aligned}$$

(the second inequality is obtained by letting $\eta = 0$ in the z-independent case; in the other case, by letting $\eta = \alpha$ as in Corollary 3.4.4 and $p = 2$).

The equivalence of the Riemannian and Euclidean distances on $\overline{\omega} \times \overline{\omega}$ and $\Psi \approx \delta^p$, together with (4.11) and (4.12) give

$$|u_\varepsilon(T-t,x) - u_\varepsilon(T-t,\hat{x})| \leq C_0 L_F |x - \hat{x}|,$$

where C_0 depends only on the diffusion $(B_t^y)_t$, T, Ψ, $\overline{\omega}$ and the drift f (then $L_{u_\varepsilon} = L_F C_0$).

Consequently, if we take ε such that

$$\frac{1}{\varepsilon} \geq \|\sigma\|_\infty L_F C_0, \tag{4.16}$$

T_1^ε s.t. $4\alpha_\varepsilon^2(T - T_1^\varepsilon) \leq 1$ and T_2^ε s.t. $4\alpha_\varepsilon^2(T_1^\varepsilon - T_2^\varepsilon) \leq 1$, we get a solution on an interval $[T_2^\varepsilon - T_1^\varepsilon]$ with the same length as $[T_1^\varepsilon; T]$. Repeating the same method inductively with the same ε at each step, we obtain that $(X_t^\varepsilon, Z_t^\varepsilon)$ is a solution of $(M + D)$ on $[T_1^\varepsilon; T]$, $[T - 2(T - T_1^\varepsilon); T]$, $[T - 3(T - T_1^\varepsilon); T]$, ... and so on the whole interval $[0; T]$. $\square$

Remark. As $Z_t^\varepsilon = \nabla_x u_\varepsilon(T-t, B_t^y)\sigma(B_t^y)$, it is a straightforward consequence of the proof that a.s., for all t, $\|Z_t^\varepsilon\|$ is bounded above by $\frac{1}{\varepsilon}$.

If we sum up the results obtained, we have the

Proposition 4.5.2. *We consider BSDE $(M + D)$ with a terminal value U in $\overline{\omega} = \{\chi \leq c\}$. Suppose that f is a C^3 function which verifies conditions* (1.4), (1.5) *and* (H_s). *If moreover χ is strictly convex (i.e.* Hess χ *is positive definite), then*

(i) If f does not depend on z, the BSDE has a (unique) solution such that X remains in $\overline{\omega}$.
(ii) If M is a Cartan-Hadamard manifold and the Levi-Civita connection is used, then the BSDE has a (unique) solution such that X remains in $\overline{\omega}$ too.

The last paragraph is devoted to generalize Proposition 4.5.2 to drifts f which are less regular and satisfy a weaker hypothesis than (H_s).

4.6. The solution for general f

Let f be a function verifying (1.4), (1.5) and the hypothesis

$$(H) \quad f \text{ is pointing outward on the boundary of } \overline{\omega}$$

introduced in Subsection 1.4. This means that

$$\forall (b,x,z) \ : \ x \in \partial\overline{\omega}, \quad (D\chi(x)|f(b,x,z))_r \geq 0;$$

equivalently, if $f^\perp$ is the component of f orthogonal to $\partial\overline{\omega} = \{\chi = c\}$, we may require $(D\chi(x)|f^\perp(b,x,z))_r$ to be nonnegative. This condition is obviously weaker than (H_s).

Firstly, we will derive from equation $(M + D)$ new BSDEs, each one having a unique $\overline{\omega}$-valued solution. Then we will show that these solutions converge to the solution of $(M + D)$. The function χ will supposed to be strictly convex and calculus will be achieved for a drift f depending both on x and z (the case when f does not depend on z being simpler).

Let us consider the local chart O defined in the first step of the proof of Proposition 4.4.1. Remember that $\overline{\omega} = \overline{B(0,1)}$ in these local coordinates and remark that hypothesis (H) means that the radial component $f^r(b,x,z)$ of $f(b,x,z)$ is nonnegative for $x \in \partial\overline{\omega}$ (see the remark at the end of Subsection 4.4). Extend the mapping f to $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{ndw}$ by putting $f(b,x,z) = 0$ if $x \notin O$ and define (on $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{ndw}$) the convolution product for $l \in \mathbb{N}^*$ $f_l = f * \rho_l$ where $\rho_l(b,x,z) = l\rho(l\|(b,x,z)\|)$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a bump function (i.e. a smooth function with $\rho'(0) = 0$, $\rho = 0$ outside $[0;1]$ and $\int_{\mathbb{R}_+} \rho(u)du = 1$). Besides, let us define a function g_l by

$$g_l(b,x,z) = f_l(b,x,z) + \frac{A}{l}x, \tag{4.17}$$

where A is a positive constant which will be chosen below; we introduce on O the following BSDE for $F \in C_c^\infty(\mathbb{R}^d)$ and $U \in \overline{\omega}$

$$(M+D)_l \begin{cases} dX_t = Z_t dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k|[Z_t]^j) + g_l(B_t^y, X_t, Z_t)\right)dt \\ X_T = F(B_T^y) = U. \end{cases}$$

Lemma 4.6.1. *Let O_1 be an open set such that $\overline{\omega} \subset O_1 \subset \overline{O_1} \subset O$. Then for $l \geq l_0$ the functions g_l are smooth on $\mathbb{R}^d \times O_1 \times \mathbb{R}^{ndw}$ and verify a Lipschitz condition like* (2.5) *with the same Lipschitz constant L''. Moreover, they also verify a boundedness condition like* (1.5).

Proof. For $x \in O_1$, we write

$$f_l(b,x,z) = \int_{\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{ndW}} f((b,x,z) - (\beta,y,w))\rho_l(\beta,y,w)d(\beta,y,w).$$

Thus, as soon as $dist(O_1, O) > 1/l_0 \geq 1/l$, $\rho_l(\beta,y,w) = 0$ if $|y| \geq dist(O_1, O)$; so the integrand vanishes if $x - y \notin O$ and we can use the Lipschitz property (2.5) of f on O. Then the properties of convolution give the result for f_l. As it also holds obviously for the functions $x \mapsto A/l\, x$, we have the result for g_l. The second assertion is an easy consequence of conditions (1.4) and (1.5) for the drift f. □

In the sequel, we will consider the sequence $(g_l)_l$ only for $l \geq l_0$.

Proposition 4.6.2. *For every l, the BSDE $(M+D)_l$ has a (unique) $\overline{\omega}$-valued solution (X^l, Z^l); moreover, there is an $\varepsilon > 0$, independent of l, such that a.s., $\|Z_t^l\| \leq \frac{1}{\varepsilon}$ for any t.*

Proof. Using Lemma 4.6.1, we apply Subsections 4.2 and 4.3 to $(M+D)_l$. We get an ε_l and a $T_1^{\varepsilon_l}$ for every l; but the Lipschitz constant of γ is independent of l

(since the g_l have the same one), so the proof of Proposition 4.3.2 shows that we can choose $\varepsilon_l = \varepsilon$ and $T_1^{\varepsilon_l} = T_1^{\varepsilon}$, independently of l.

In order to apply Subsection 4.4, we need to prove that g_l verifies condition (H_s); in fact, we have seen (see in particular the remark at the end of Subsection 4.4) that it suffices to show that

$$\forall (b,x,z)\ :\ x \in \partial\overline{\omega},\ \|z\| \leq \frac{1}{\varepsilon},\ \inf_{b,x,z} g_l^r(b,x,z) \geq \zeta > 0, \tag{4.18}$$

It is easy to see with the properties of convolution that, for a constant $\hat{C}$ depending only on ε and f,

$$\forall l,\ \forall b,\ \forall x \in \partial\overline{\omega},\ \forall z : \|z\| \leq \frac{1}{\varepsilon},\ |f_l(b,x,z) - f(b,x,z)| \leq \frac{\hat{C}}{l}.$$

This and the nonnegativity of $f^r(b,x,z)$ for $x \in \partial\overline{\omega}$ give

$$\forall l,\ \forall b,\ \forall x \in \partial\overline{\omega},\ \forall z : \|z\| \leq \frac{1}{\varepsilon},\ f_l^r(b,x,z) \geq -\frac{\hat{C}}{l}.$$

Now if we take in (4.17) $A = \hat{C} + 1$, obviously (4.18) holds with $\zeta = 1/l$ and Subsection 4.4 can be applied.

Then the results of Subsection 4.5 hold (in particular Proposition 4.5.2) with the same ε for every l (this comes from (4.16), remarking that the constant C_0 in this inequality is independent of l). The proof is completed. □

Now we prove that, as expected, the limit of these solutions when $l \to \infty$ solves equation $(M + D)$.

Proposition 4.6.3. *For $n \in \mathbb{N}$ and $s \in [0;T]$, we note for simplicity $V_s^n := (B_s^y, X_s^n, Z_s^n)$, When l tends to infinity, the above solution (X^l, Z^l) converges (for the usual L^2 norms for X and Z) to a pair (X, Z) which solves equation $(M + D)$.*

Proof. For l and m in $\mathbb{N}$, we note $\delta X_t = X_t^l - X_t^m$, $\delta Z_t = Z_t^l - Z_t^m$ and

$$A_{l,m} = \mathbb{E}\int_0^T |\delta X_s| \left(|f_m(V_s^l) - f_l(V_s^l)| + A\left|\frac{1}{l}X_s^l - \frac{1}{m}X_s^m\right|\right) ds.$$

Applying Itô's formula (in $\mathbb{R}^n$) to $|\delta X_t|^2$ between t and T, we get (note that a dot stands for the inner product in $\mathbb{R}^n$)

$$\begin{aligned}
-|\delta X_t|^2 = & \int_t^T 2\delta X_s \cdot (\delta Z_s dW_s) + \int_t^T 2\delta X_s \cdot \left(g_l(V_s^l) - g_m(V_s^m)\right) ds \\
& -\frac{1}{2}\int_t^T 2\delta X_s \cdot \left(\Gamma_{jk}(X^l)([Z_s^l]^k | [Z_s^l]^j) - \Gamma_{jk}(X^m)([Z_s^m]^k | [Z_s^m]^j)\right) ds \\
& + \int_t^T \|\delta Z_s\|^2 ds.
\end{aligned} \tag{4.19}$$

Using the uniform boundedness of the (Z_t^l) (proved in the preceding lemma), we can bound above the integral involving the Christoffel symbols by $C\int_t^T |\delta X_s|^2 ds$; besides, for the second term on the right, we write

$$\left|\int_t^T \delta X_s \cdot (g_l(V_s^l) - g_m(V_s^m))ds\right| \le A_{l,m} + \int_t^T |\delta X_s| \cdot |f_m(V_s^l) - f_m(V_s^m)| ds$$
$$\le C_1 \left(A_{l,m} + \int_t^T |\delta X_s|(|\delta X_s| + |\delta Z_s|)ds\right)$$
$$\le C \left(A_{l,m} + \int_t^T |\delta X_s|^2 ds\right) + \frac{1}{2}\int_t^T \|\delta Z_s\|^2 ds$$

where C is independent of l and m. Then, we obtain by taking the expectation in (4.19)

$$\mathbb{E}|\delta X_t|^2 + \frac{1}{2}\mathbb{E}\int_t^T \|\delta Z_s\|^2 ds \le C\left(\int_t^T \mathbb{E}|\delta X_s|^2 ds + A_{l,m}\right). \qquad (4.20)$$

Gronwall's lemma gives $\mathbb{E}|\delta X_t|^2 \le CA_{l,m}$, where C is again independent of l and m. Moreover, using (1.4), f_l converges uniformly to f on $\mathbb{R}^d \times \overline{\omega} \times B(0,r)$ for any $r > 0$ (with $B(0,r) = \{z \in \mathbb{R}^{nd_W} : \|z\| < r\}$) and X^l, X^m are bounded, so $A_{l,m}$ tends to zero when l, m tend to infinity; therefore $(X^l)_l$ converges to a process X in $L^2(\Omega \times [0;T])$.

Using (4.20) again, we get

$$\mathbb{E}\int_0^T \|\delta Z_s\|^2 ds \le CA_{l,m};$$

hence the sequence of processes (Z^l) has also a limit in $L^2(\Omega \times [0;T])$; let Z denote this limit process.

The pair (X, Z) solves BSDE $(M + D)$ and X is $\overline{\omega}$-valued; the proof is just an adaptation of the second step in the proof of Proposition 4.1.4. This remark completes the proof. □

Remark. As a consequence, a.s., $\|Z_t\| \le \frac{1}{\varepsilon}$ for any t.

According to Subsection 4.1, this result can be extended to every $\overline{\omega}$-valued and $\mathcal{F}_T$-measurable terminal variable U. Then Theorem 1.4.1 of existence and uniqueness of a solution follows.

Note that uniqueness and existence hold in particular on any regular geodesic ball (or geodesic ball if the sectional curvatures are nonpositive).

5. Applications and related PDEs

5.1. The martingale case

The drift $f = 0$ verifies hypothesis (H). Hence in this case the results of this paper apply to the martingale case. As already underlined, any regular geodesic ball verifies the condition of Theorem 1.4.1; so we recover the well-known results of existence and uniqueness of a martingale with prescribed terminal value in such domains (see [15]). These results hold in nonpositive curvatures, they will be achieved in positive curvatures elsewhere.

5.2. The one-dimensional case

The nonpositive curvature case gives the existence and uniqueness of a solution to the one-dimensional BSDE

$$(E)_1 \begin{cases} dX_t = Z_t dW_t - \Gamma(X_t) Z_t^2 + f(B_t^y, X_t, Z_t) dt \\ X_T = U \end{cases}$$

for a bounded terminal condition U, a drift f satisfying (1.5), (2.5) and any smooth function Γ defined on $\mathbb{R}$.

Note that a change of coordinates (in fact a reparametrization of the one-dimensional manifold by arclength) reduces equation $(E)_1$ to

$$\begin{cases} dX_t = Z_t dW_t + f(B_t^y, X_t, Z_t) dt \\ X_T = U; \end{cases}$$

moreover, it is a very particular case of the results of Kobylansky in [18].

One can ask whether such results can be extended to higher dimensions. In fact, the original problem is geometric and to deal with general BSDEs, we would start with smooth functions (Γ_{ij}^k) and should give conditions in order to interpret these functions as the Christoffel symbols of a given Levi-Civita connection. This problem is out of the scope of this paper.

5.3. Case of a random terminal time

In this paragraph, we will only sketch the proofs.

We are interested in the following equation

$$(M+D)_\tau \begin{cases} dX_t = Z_t dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k | [Z_t]^j) + f(B_t^y, X_t, Z_t)\right) dt \\ X_\tau = U^\tau \end{cases}$$

where τ is a stopping time with respect to the filtration used and U^τ is a $\overline{\omega}$-valued, $\mathcal{F}_\tau$-measurable random variable. It is the counterpart of equation $(M+D)$ on the random interval $[0; \tau]$.

Let us first consider the case of a bounded stopping time τ, i.e. $\tau \le T$ where T is a deterministic constant. We have the following result :

Theorem 5.3.1. *We consider BSDE $(M+D)_\tau$ with $\overline{\omega} = \{\chi \le c\}$ and $\tau \le T$ a.s. If f verifies conditions* (1.4), (1.5) *and* (H), *and if χ is strictly convex (i.e.* Hess χ *is positive definite), then this BSDE has a unique solution (X, Z), with $X \in \overline{\omega}$, in the same two cases as in Theorem 1.4.1.*

Proof. First remark that the uniqueness part goes the same as in the deterministic case; for the existence part, it can be completed in several steps :

First step. We work in local coordinates introduced in Subsection 4.4. Let $c_1 > c$ and put $\overline{\omega}_1 = \{\chi \leq c_1\}$; suppose that c_1 is such that $\overline{\omega} \subset \overline{\omega}_1 \subset O$ and that χ is yet strictly convex on $\overline{\omega}_1$. Now let ϕ be a cut-off function with $\phi = 1$ on $\overline{\omega}$ and $\phi = 0$ outside $\overline{\omega}_1$. For any nonzero integer l, we define a new drift by $f_l(b, x, z) := \phi(x)(f(b, x, z) + (1/l)x)$; note that f_l verifies hypothesis (H) with respect to $\overline{\omega}_1$ and (H_s) with respect to $\overline{\omega}$.

Then solve path by path on $[\tau; T]$ the following differential equation

$$\begin{cases} dX_t^l = f_l(B_t^y, X_t^l, 0)dt \\ X_\tau^l = U^\tau \end{cases} \tag{5.1}$$

and set $U^l = U^\tau + \int_\tau^T f_l(B_t^y, X_t^l, 0)dt$. Since f_l vanishes outside $\overline{\omega}_1$, U^l is in $\overline{\omega}_1$; besides, U^l is $\mathcal{F}_T$-measurable.

Second step. Considering the random variable U^l and the drift f_l introduced in the first step, we solve on $[0; T]$ equation $(M + D)$ with drift f_l and terminal value $X_T^l = U^l$. The hypothesis of Theorem 1.4.1 are satisfied considering the set $\overline{\omega}_1$ instead of $\overline{\omega}$. So this BSDE has a solution (X^l, Z^l) with $X^l \in \overline{\omega}_1$.

We condition by $\mathcal{F}_\tau$ and consider the above solution (X^l, Z^l) on the random time interval $[\tau; T]$. It is a solution on this interval of BSDE $(M + D)$ with drift f_l and terminal value U^l. The uniqueness property for such equations implies that X^l is the solution of equation (5.1), i.e.

$$\forall t \in [\tau; T], \; X_t^l = U^l - \int_t^T f_l(B_s^y, X_s^l, 0)ds$$

and $Z_t^l = 0$ for $\tau \leq t \leq T$. In particular,

$$X_\tau^l = U^l - \int_\tau^T f_l(B_s^y, X_s^l, 0)ds = U^\tau.$$

We now show that actually, $(X_t^l)_{0 \leq t \leq \tau}$ remains in $\overline{\omega}$, and not only in $\overline{\omega}_1$. For this purpose, we want to construct as in Subsection 4.4 a submartingale which is written as $(e^{\lambda t} H(X_t^l))$, where now $H = h \circ \chi$, since X^l is $\overline{\omega}_1$-valued. Recall from (4.15) that the keypoint is to show for $x \in \overline{\omega}_1$ the nonnegativity of

$$h'(\chi(x)) \left(\frac{1}{2} \sum_{i=1}^{d_W} {}^t[{}^t z]^i \text{Hess}\, \chi(x)[{}^t z]^i + D\chi(x) f_l(b, x, z) \right) + \lambda H(x).$$

In fact, as χ is strictly convex on the compact domain $\overline{\omega}_1$, we have Hess $\chi \geq \alpha Id$ (in the sense of matrices) for $\alpha > 0$ and it turns out that it suffices to prove the nonnegativity of

$$h'(\chi(x)) \left(\frac{1}{2} \alpha \|z\|^2 + D\chi(x) f_l(b, x, z) \right) + \lambda H(x). \tag{5.2}$$

From (1.4) and (1.5) we deduce

$$|D\chi(x) f_l(b, x, z)| \leq C(1 + \|z\|) \leq \frac{1}{2}\alpha\|z\|^2,$$

the last inequality holding for $\|z\|$ large enough, say $\|z\| \geq A$. Obviously in this case, (5.2) is nonnegative.

Now suppose that $\|z\| \leq A$. If $x \in \overline{B(0,1)} = \overline{\omega}$ then $h' = 0$ and the required result holds. Otherwise we write for $x \in \overline{\omega}_1 \setminus \overline{\omega}$ and $x_0 \in \partial\overline{\omega}$ (i.e. in our local coordinates $|x| \geq 1$ and $|x_0| = 1$) :

$$|D\chi(x) f_l(b, x, z) - D\chi(x_0) f_l(b, x_0, z)| \leq C|x - y|$$

for a constant C. But the hypothesis (H_s) for f_l writes

$$D\chi(x_0) f_l(b, x_0, z) \geq \zeta > 0.$$

Then we distinguish two cases (x near 1 or "far" from 1) and get the nonnegativity of (5.2) in both situations. This can be done by using similar arguments to those displayed at the end of the Third Step in the proof of Proposition 4.4.1; in particular, a λ large enough is needed.

Third step. The first two steps give the existence of processes (X^l, Z^l) (with $X^l \in \overline{\omega}$) solving equation $(M + D)_\tau$ associated to the drift f^l and terminal value U^τ. But f^l converges to f uniformly on $\overline{\omega}$ so, passing through the limit as in Subsection 4.6, we get a pair of processes $(X_t, Z_t)_{0 \leq t \leq \tau}$, with $X \in \overline{\omega}$ and solving the initial equation $(M + D)_\tau$ with drift f and terminal value U^τ. This completes the proof.

□

We consider again a stopping time τ and the corresponding equation $(M + D)_\tau$; now, we only suppose that τ is finite a.s. and verifies the exponential integrability condition (1.6). Examples of such stopping times are exit times of uniformly elliptic diffusions from bounded domains in Euclidean spaces.

In this case, we need to add restrictions on the drift f; indeed, the main thrust in the proof of uniqueness and existence is the construction of a submartingale on the product manifold $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$ with $\mu = 0$ if f does not depend on z. To extend this approach to a random (non necessarily bounded) interval, we have to keep the integrability of S_τ. An accurate examination of the method to obtain the submartingale (in particular inequalities (3.5) and (3.9)) shows that this integrability holds for "small" drifts; more precisely there is a constant h with $0 < h < \rho$ such that, under the following condition on the constants in (1.4) and (1.5)

$$L < h, \quad L_2 < h, \tag{5.3}$$

the integrability required holds, so $(S_t)_{0 \leq t \leq \tau}$ is a true submartingale.

Remarks. 1- Such a condition guarantees in particular that we have

$$\mathbb{E} \int_0^\tau |f(B_s^y, X_s, Z_s)| ds < \infty.$$

2- This condition is rather natural; actually, it is very similar to conditions yet introduced for BSDEs with Lipschitz coefficients and random terminal time : see condition (24) and Propositions 3.2 and 3.3 in [7], or (2.6) and the condition before in [26].

3- *A priori*, the process $(Z_t)_t$ verifies the integrability condition

$$\mathbb{E}\left(\int_0^\tau \|Z_s\|^2 ds\right) < \infty;$$

in fact, it results from the existence part that in any case (i.e. f depending or not on z), $(Z_t)_{0\le t\le\tau}$ belongs to $(\mathcal{E}_\alpha)$ (see Definition 3.4.1) for α small enough, which is a stronger property. In particular, we get

$$\forall\theta < \rho,\ \ \mathbb{E}\int_0^\tau e^{\theta s}\|Z_s\|^2 ds < \infty;$$

this condition is usual for BSDEs with random terminal time (see again Propositions 3.2 and 3.3 in [7], or Theorem 2.2 in [26]).

Once we have constructed the submartingale as on a deterministic interval, uniqueness is straightforward. Let us indicate how existence can be deduced.

We are given a $\overline{\omega}$-valued and $\mathcal{F}_\tau$-measurable variable U^τ. As in the proof of Theorem 5.3.1, we consider again local coordinates introduced in Subsection 4.4, $\overline{\omega}_1$ such that $\overline{\omega} \subset \overline{\omega}_1 \subset O$ and a cut-off function ϕ with $\phi = 1$ on $\overline{\omega}$ and $\phi = 0$ outside $\overline{\omega}_1$. We put $f_1(b, x, z) = \phi(x) f(b, x, z)$.

The first step here is to solve on $[0;\tau]$ a BSDE whose terminal value is near U^τ : on $[0;\tau\wedge n]$, using Theorem 5.3.1, we solve equation $(M + D)$ with drift f_1 and terminal value at time $\tau\wedge n$, $\mathbb{E}[U^\tau|\mathcal{F}_n]$; let $(X_t^n, Z_t^n)_{0\le t\le\tau\wedge n}$ denote the solution; on $[\tau\wedge n;\tau]$, we put $Z_t = 0$ and solve

$$\begin{cases} dX_t^n = f_1(B_t^y, X_t^n, 0)dt \\ X_{\tau\wedge n}^n = \mathbb{E}[U^\tau|\mathcal{F}_n]. \end{cases}$$

Then it is easily seen, since $\mathbb{E}[U^\tau|\mathcal{F}_n]$ is $\mathcal{F}_{\tau\wedge n}$-measurable, that $(X_t^n, Z_t^n)_{0\le t\le\tau}$ is a solution to BSDE $(M + D)_\tau$ with terminal value $U^{\tau,n}$, where

$$U^{\tau,n} = X_{\tau\wedge n}^n + \int_{\tau\wedge n}^\tau f_1(B_s^y, X_s^n, 0)ds.$$

The second step is to show that when n tends to infinity, we get the solution of BSDE $(M + D)_\tau$ with terminal value U^τ.

We have that $U^{\tau,n}$ tends to U^τ in $L^2(\Omega)$; indeed,

$$\mathbb{E}|U^\tau - U^{\tau,n}|^2 = \mathbb{E}\left(1_{n\le\tau}\left|U^\tau - \mathbb{E}[U^\tau|\mathcal{F}_n] - \int_{\tau\wedge n}^\tau f_1(B_s^y, X_s^n, 0)ds\right|^2\right)$$

and the last expectation tends to zero as n tends to infinity; this is a consequence of dominated convergence, using the exponential integrability condition of τ.

Then we apply the results of Subsection 4.1 to the random interval $[0;\tau]$; indeed, an accurate examination shows that these results rely essentially on the exponential integrability condition (according to the uniqueness part)

$$\mathbb{E}\left(e^{\alpha\int_0^\tau(\|Z_s^n\|^2+\|Z_s^m\|^2)ds}\right)\leq C<\infty$$

for a constant C independent of m, n. At the end, we get the existence of a pair of processes $(X_t, Z_t)_{0\leq t\leq\tau}$ solution of BSDE $(M+D)_\tau$ with drift f_1 and terminal value U^τ. But, since U^τ is $\overline{\omega}$-valued and $\overline{\omega}=\overline{B(0,1)}$, for each n the process $(X_t^n)_{0\leq t\leq\tau\wedge n}$ remains in $\overline{\omega}$ by Theorem 5.3.1; thus the whole process $(X_t)_{0\leq t\leq\tau}$ remains in $\overline{\omega}$ and this completes the existence part.

As a consequence, we can state the following result

Theorem 5.3.2. *We consider BSDE $(M+D)_\tau$ with τ a stopping time verifying the integrability condition* (1.6)*; the function χ used to define the domain $\overline{\omega}$ is supposed as usual to be strictly convex. Then if f verifies conditions* (1.4), (1.5), (H) *and moreover is "small" (i.e. verifies condition* (5.3) *above), this BSDE has a unique solution (X, Z), in the same cases as in Theorem 1.4.1.*

5.4. Application to nonlinear elliptic PDEs

In this paragraph, we make precise the Dirichlet problem that we briefly discussed in the introduction.

Suppose (N, g) is a Riemannian manifold, and B^x a Brownian motion on (N, g) (started at x at time 0). Alternatively, think of B^x as the diffusion process on $\mathbb{R}^d$, defined by (1.2); in this case,

$$\forall i, j=1,\dots,d,\quad \sum_{l=1}^{d}\sigma_{il}\sigma_{jl}=g^{ij},$$

the inverse metric tensor, and

$$\forall i=1,\dots,d,\quad b^i+\sum_{k,l=1}^{d_W}g^{kl}\Gamma_{kl}^i=0.$$

Let $\overline{M}_1$ be a compact submanifold of N, with boundary ∂M_1 and interior M_1. For $x\in\overline{M}_1$, we call ζ the first time B^x hits the boundary; we assume that ζ verifies an integrability condition like (1.6). Given a regular mapping

$$\overline{\phi}:\partial M_1\to\overline{\omega}\subset M,$$

we wish to find a mapping $\phi:\overline{M}_1\to\overline{\omega}$ which solves the following Dirichlet problem

$$(D)\begin{cases}\mathcal{L}_M\phi(x)-f(x,\phi(x),\nabla\phi(x)\sigma(x))=0, & x\in M_1\\ \phi(x)=\overline{\phi}(x), & x\in\partial M_1 n\sigma.\end{cases}$$

where $\mathcal{L}_M\phi$ is the tension field of the mapping ϕ (see [9], or for a probabilistic point of view the introduction of [28]).

We recall from the introduction that, in coordinates (x^i) on M and (y^α) on M_1, the equation $\mathcal{L}_M\phi = 0$ characterizes harmonic mappings, and is written

$$\forall i,\quad \Delta_{M_1}\phi^i + g^{\alpha\beta}\Gamma^i_{jk}(\phi)D_\alpha\phi^j D_\beta\phi^k = 0.$$

Using the same Wiener process W with which we constructed B^x, we can solve according to Theorem 5.3.2 the BSDE $(M+D)_\zeta$ with terminal value $\overline{\phi}(B^x_\zeta)$. Let $(X^x_t, Z^x_t)_{0\le t\le\zeta}$ be the unique solution and put $\phi(x) := X^x_0$. Then under sufficient regularity on ϕ, it is not difficult to verify that ϕ is a solution to the Dirichlet problem (D). Note that when $f \equiv 0$ (i.e. in the martingale case), Kendall ([17]) has proved regularity results on ϕ using almost only probability theory, so that ϕ is a strong solution of the equation $\mathcal{L}_M\phi \equiv 0$ (i.e. a harmonic mapping).

When $f(b,x,z) = f(b,x)$ and is written as $f(b,x) = D_2G(b,x)$ (the differential of G with respect to the second variable), the elliptic nonlinear PDE in the Dirichlet problem (D) is associated with a variational problem; more precisely, solutions of this equation are critical points of the functional

$$\mathcal{F}(u) = \frac{1}{2}\int_{M_1} \|\mathrm{grad} u(b)\|^2 dvol(b) + \int_{M_1} G(b,u(b))dvol(b)$$

and the elliptic PDE in equation (D) is the Euler-Lagrange equation associated.

5.5. *Application to nonlinear parabolic PDEs*

We conclude this part by studying the time-dependent equation associated with the stationary equation described in the Dirichlet problem (D) above. More precisely, we are interested in the following equation, for mappings $u : [0;T]\times N \to \overline{\omega} \subset M$:

$$\begin{cases} \frac{\partial u}{\partial t} &= \mathcal{L}_M u - f(x,u,\nabla u\sigma) \\ u_{|t=0} &= F \end{cases}$$

where F is sufficiently regular and has range $\overline{\omega}$. In local coordinates, this equation becomes

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\mathcal{L}u(t,x) + \frac{1}{2}\Gamma_{jk}(u(t,x))([(\nabla_x u\sigma)(t,x)]^k|[(\nabla_x u\sigma)(t,x)]^j) \\ &\qquad -f(x,u(t,x),(\nabla_x u\sigma)(t,x)) \\ u(0,x) &= F(x) \end{cases}$$

in the case of $\mathcal{L}$ being the Laplace-Beltrami operator on N. This is equation (4.6). As a by-product of Section 4, we have the existence and uniqueness of a regular solution to this system of quasilinear parabolic PDEs; it is based on the boundedness of $\nabla_x u$, proved in Subsection 4.3.

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