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Some new examples of Markov processes which enjoy the time-inversion property

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Abstract. In this paper we give a sufficient condition on the semi group densities of an homogeneous Markov process taking values in $\mathbb{R}^n$ which ensures that it enjoys the time-inversion property. Our condition covers all previously known examples of Markov processes satisfying this property. As new examples we present a class of Markov processes with jumps, the Dunkl processes and their radial parts.

1. Introduction

Let $\{(X_t, t \geq 0); (P_x)_{x\in\mathbb{R}^n}\}$ be a Markov process taking values in $\mathbb{R}^n$. The aim of this paper is to give some sufficient condition on the Markov semi-group of X :

$$P_t(x, dy) = p_t(x, y)dy$$

(we assume the existence of the densities $p_t(x, y)$) which will ensure that for all $x \in \mathbb{R}^n$ the process :

$$(tX_{1/t}, t > 0), \text{under} P_x, \tag{1}$$

is an homogeneous Markov process. We shall then say that X enjoys the time-inversion property.

To our knowledge, the only known examples of Markov processes which enjoy the time-inversion property are Brownian motions with drift (in $\mathbb{R}^n$), and Bessel processes with drifts (see Watanabe [22], [23], [24], Pitman-Yor [17]). In particular, there were, until now, no Markov processes with jumps known with this property; the present paper provides such examples.

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In Section 2, we give some sufficient condition on the densities p_t which ensure the time-inversion property. In particular, we assume that our Markov process of reference (X_t), which takes values in $\mathbb{R}^n$, or $\mathbb{R}_+$, enjoys the Brownian scaling property, together with some technical condition on its semigroup densities (see (4) and (6) below). Then any Markov process $(\widetilde{X}_t)$, which is a h-process of (X_t), will also enjoy the time-inversion property. Moreover, we give an explicit description of the extended infinitesimal generator of the time-inverted processes, using the "opérateur carré du champ".

In Section 3, we present, as new examples, a class of càdlàg Markov processes associated with a finite reflection subgroup of $O(\mathbb{R}^n)$, the Dunkl processes, and their "W-radial" parts (with Brownian motions in a Weyl chamber as particular cases), which are shown to satisfy our criterion, hence they enjoy the time inversion property.

2. A sufficient condition on the semi-group of a Markov process in $\mathbb{R}^n$, which implies the time-inversion property

2.1. Preliminary remarks

We keep the notation and hypothesis presented in the Introduction. Fix $x \in \mathbb{R}^n$; it is easily shown that, under P_x, the process $(tX(1/t), t > 0)$ is Markov, inhomogeneous, with densities $q_{s,t}^{(x)}(z, y)(s < t; z, y \in \mathbb{R}^n)$ for its transition mechanism such that :

$$E_x[f(tX_{1/t})|sX_{1/s} = z] = \int dy \ f(y)q_{s,t}^{(x)}(z, y) \tag{2}$$

where

$$q_{s,t}^{(x)}(a, b) = \frac{1}{t^n} \frac{p_{\frac{1}{t}}(x, \frac{b}{t})p_{\frac{1}{s}-\frac{1}{t}}(\frac{b}{t}, \frac{a}{s})}{p_{\frac{1}{s}}(x, \frac{a}{s})}. \tag{3}$$

We note the following easy result:

Lemma 2.1. *: Assume that another Markov family is given by :*

$$P_x^{\varphi}|_{\mathcal{F}_t} = \frac{\varphi(X_t)}{\varphi(x)} exp(-\lambda t).P_x|_{\mathcal{F}_t} \quad ,$$

for some function φ, and some constant $\lambda > 0$.
Then, formula (2) holds also with E_x^{φ} instead of E_x, with $q^{(x)}$ unchanged.

Proof. It suffices to use (with obvious notation) :

$$p_t^{\varphi}(x, y) = \frac{\varphi(y)}{\varphi(x)} e^{-\lambda t} p_t(x, y).$$

We then see that the right hand side of (3), where we replace p by p^{φ} is unchanged. □

Remark. A more general argument for the proof of Lemma 2.1 is that under the hypothesis of the Lemma, the P^{φ}-and P-bridges are the same, which implies the result.

2.2. The main result

From now on, in our discussion, we shall assume that our Markov process $(X_t, t \geq 0)$ is semi-stable with index $(\frac{1}{2})$, in the sense of Lamperti [16], that is :

$$\{(X_{ct}, t \geq 0);\ P_x\} \overset{(d)}{=} \{(\sqrt{c}X_t, t \geq 0);\ P_{x/\sqrt{c}}\}. \tag{4}$$

This property has the following consequence for the family $(p_t(x, y))$ of the densities (assumed to exist) of the semi-group of the process :

$$p_t(x, y) = \frac{1}{t^{n/2}} p(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}), \tag{5}$$

where we have written simply p for p_1.

Here is our main general result.

Proposition 2.2. *1) Assume that p is of the form :*

$$p(a, b) = \Phi(a, b)\theta(b)\exp[-\{\rho(a) + \rho(b)\}], \tag{6}$$

where :

i. $\rho(\lambda a) = \lambda^2 \rho(a)$;
ii. $\theta(\lambda b) = \lambda^\alpha \theta(b)$;
iii. $\Phi(\lambda a, b) = \Phi(a, \lambda b)$, *for all* $\lambda > 0, a, b \in \mathbb{R}^n$, *and some* $\alpha \in \mathbb{R}$.

Then, the density $q^{(x)}$ *in (3) depends only on* $(t - s)$, *and* $(X_t, t \geq 0)$ *under* P_x *enjoys the time-inversion property.*

2) Denote by $\{Q_a^{(x)}\}_{a\in\mathbb{R}^n}$ *the family of the laws induced by the densities* $q^{(x)}$ *under the above conditions ; assume furthermore that :*

iv. $\Phi(a, b) = \Phi(b, a)$ *Then, there is the following h-transform relationship:*

$$Q_a^{(x)}|_{\mathcal{F}_t} = \frac{\Phi(x, X_t)}{\Phi(x, a)} \exp(-t\rho(x)).P_a|_{\mathcal{F}_t}. \tag{7}$$

Before we give the proof of the Proposition, we make two remarks:
a) The following consequence is immediate from the Lemma:

Corollary 2.3. *If* $\{Q_a\}_{a\in\mathbb{R}^n}$ *is any Markovian family of laws which is in h-transform relationship with* $\{P_a\}_{a\in\mathbb{R}^n}$ *satisfying the hypothesis of the Proposition, then* $\{Q_a\}$ *enjoys the time-inversion property; this applies in particular to the family* $\{Q_a^{(x)}\}_{a\in\mathbb{R}^n}$.

For example, a pair $(X, \widetilde{X})$ of processes, as indicated in the introduction, with X satisfying (4) and (6) and $\widetilde{X}$ a h-process of X, is obtained if we take for X the Bessel(3) process, which is the euclidean norm of the 3-dimensional Brownian motion $(\beta_t)_{t>0}$ and $\widetilde{X}$ the radial hyperbolic Bessel(3) process, with parameter $x > 0$, which takes values in $\mathbb{R}_+$, and has infinitesimal generator:

$$\frac{1}{2}\frac{d^2}{dy^2} + x\coth(xy)\frac{d}{dy}. \tag{8}$$

This process may be realized as the euclidean norm of $(\beta_t + at, t \geq 0)$ with $a \in \mathbb{R}^3$, and $x = |a|$ (see [18]).
b) It is of some interest to give an expression of the extended generator $\mathcal{L}^{(x)}$ of $\{Q_a^{(x)}\}_{a\in\mathbb{R}^n}$ in terms of $\mathcal{L}$, the extended generator of $\{P_a\}_{a\in\mathbb{R}^n}$ (more details are given in subsection 2.4). Together with the hypothesis of Proposition 2.2, we assume that the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ is an algebra, so that, for $f, g \in \mathcal{D}(\mathcal{L})$,

$$\Gamma(f, g) = \mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f \tag{9}$$

is well defined (see Kunita [15], and Dellacherie-Maisonneuve-Meyer [6] who call Γ the "opérateur carré du champ", following Roth [21] [14]).
Now it follows from (7), (9), and Girsanov's theorem, that if $f \in \mathcal{D}(\mathcal{L})$, then f also belongs to $\mathcal{D}(\mathcal{L}^{(x)})$, and:

$$\mathcal{L}^{(x)}(f)(y) = \mathcal{L}(f)(y) + \frac{1}{\Phi(x,y)}\Gamma(\Phi(x,.), f)(y). \tag{10}$$

In our discussion of examples in section 3, we shall give a more explicit form of $\mathcal{L}$ and Γ, hence of $\mathcal{L}^{(x)}$.
We now give a

Proof of the Proposition. It will be convenient to decompose the function p, as given in the Proposition as :

$$p(a, b) = \prod_{i=1}^{3} p^{(i)}(a, b),$$

where

$$p^{(1)}(a, b) = \Phi(a, b); \quad p^{(2)}(a, b) = \theta(b); \quad p^{(3)}(a, b) = \exp[-(\rho(a) + \rho(b))].$$

From formulae (3) and (5), we get that $q_{s,t}^{(x)}$ has the following form :

$$q_{s,t}^{(x)}(a, b) = \frac{1}{(t-s)^{n/2}}\left\{\frac{p(\sqrt{t}x, \frac{b}{\sqrt{t}})p(\frac{b\sqrt{s}}{\sqrt{t-s}\sqrt{t}}, \frac{a\sqrt{t}}{\sqrt{s}\sqrt{t-s}})}{p(x\sqrt{s}, \frac{a}{\sqrt{s}})}\right\}.$$

We denote the quantity appearing between $\{\dots\}$ as R_p, and we want to show that, under the hypothesis of the Proposition, R_p depends only on $(t-s)$, instead of (s, t). Since $R_p = \prod_{i=1}^3 R_p^{(i)}$, it suffices to verify this only dependency in $(t-s)$ for $R_p^{(i)}$, $i = 1, 2, 3$, separately. We obtain easily the following:

$$R_p^{(1)} = \frac{\Phi(x, b)\Phi(\frac{b}{t-s}, a)}{\Phi(x, a)}$$

$$R_p^{(2)} = \theta(\frac{b}{\sqrt{t-s}})$$

$$R_p^{(3)} = \exp\left(-\{(t-s)\rho(x) + \frac{1}{t-s}(\rho(a) + \rho(b))\}\right).$$

Thus, the desired property is obtained, and we get :

$$q_{s,t}^{(x)}(a,b) = q_{t-s}^{(x)}(a,b),$$

with

$$q_t^{(x)}(a,b) = \frac{1}{t^{n/2}} \frac{\Phi(x,b)}{\Phi(x,a)} \Phi\left(\frac{b}{t}, a\right) \theta\left(\frac{b}{\sqrt{t}}\right) \exp\left\{-\left(t\rho(x) + \frac{\rho(a)+\rho(b)}{t}\right)\right\}. \tag{11}$$

Furthermore, under the symmetry assumption (iv), it follows from the equality (11) that :

$$q_t^{(x)}(a,b) = \frac{\Phi(x,b)}{\Phi(x,a)} \exp(-t\rho(x)) p_t(a,b), \tag{12}$$

from which the h-transform relationship (7) follows. □

We now deduce from the h-transform relation (7) a self-reproducing property of $\{\Phi(a,b); a,b \in \mathbb{R}^n\}$.

Corollary 2.4. *Under the hypothesis of the Proposition, the following identity of self-reproduction holds:*

$$\Phi(x,a) = \int dy \Phi(x,y)\Phi(a,y)\theta(y) \exp[-(\rho(a) + \rho(x) + \rho(y))]. \tag{13}$$

Proof. We start from the identity :

$$E_a[\frac{\Phi(x,X_t)}{\Phi(x,a)} \exp(-t\rho(x))] = 1 \tag{14}$$

which implies :

$$\Phi(x,a) = \int dy \Phi(x,y) p_t(a,y) \exp(-t\rho(x)).$$

It then remains to use the special form of $p_t(a,y)$ as given in (5) and (6), to obtain the following identity :

$$\Phi\left(x\sqrt{t}, \frac{a}{\sqrt{t}}\right) = \int dy \Phi(x, y\sqrt{t}) \Phi\left(\frac{a}{\sqrt{t}}, y\right) \theta(y) \times \exp\left(-\left\{\rho(\frac{a}{\sqrt{t}}) + \rho(y)) + \rho(\sqrt{t}x)\right\}\right).$$

Then, we simply take $t = 1$. □

Remarks. Corollary 2 gives us an opportunity to single out a few important properties of the semigroup $P_t(a,db)$ of our Markov process :

i. for convenience, we write Φ_x for $\Phi(x,\cdot)$; then :

$$P_t(\Phi_x)(a) = \exp(t\rho(x))\Phi_x(a).$$

This is simply a rewriting of the identity (14) above.

ii. The semi-group (P_t) is symmetric with respect to the measure $\theta(y)dy$; again, this follows directly from the explicit formulas (5) and (6) which give $p_t(a,b)$.

2.3. The time inversion hypothesis

In Pitman-Yor [17], the following property is called the time-inversion hypothesis:

$$i(Q_x^{(y)}) = Q_y^{(x)} \tag{15}$$

where $i(P)$ denotes the image of the probability P by :

$$X \to \{tX((1/t)-)\}$$

(ie : if X is right continuous, so is $X((1/t)-)$).

To prove (15), it suffices to show the identity of the one-dimensional marginals of both sides of (15), which easily follows from, e.g.,(11).

We now recall the following

Theorem 1 (= Theorem 5.8 in [17]). . *Let* $\{Q_x^{(y)}\}$ *be a family of Markov processes which enjoy the time-inversion hypothesis (15).*
Let $t > 0$ *; then the process :*

(a) $\{X(u), 0 < u < t;\ Q_x^{(y)} / X(t) = z\}$
(b) $\{uX((\frac{1}{u})_- - \frac{1}{t}), 0 < u < t;\ Q_{z/t}^{(x)}\}$
(c) $\{\frac{t-u}{t} X(\frac{tu}{t-u}), 0 < u < t;\ Q_x^{(z/t)}\}$

are identical in law, ie : the $Q^{(y)}$ *bridges may be explicitly realized as* (b) *or* (c).

2.4. On extended generators and "carré du champ" operator

This subsection is intended to give an explicit description of the extended infinitesimal generator considered in (10).

i) Following Kunita [15], let us call *extended generator* $\mathcal{L}$ of a Markov process $(X_t)_{t\geq 0}$, with nice state space E, the operator $\mathcal{L}$ defined on $\mathcal{D}(\mathcal{L})$, the space of functions f on E such that there exists g, another function on E, with:

$$M_t^f = f(X_t) - f(X_0) - \int_0^t g(X_s)ds \tag{16}$$

is a local martingale for every P_x, $x \in E$. From the general theory of semimartingales, the function g featured in (16) is unique, in the sense that two functions g and $\widetilde{g}$ which fit in (16) will satisfy

$$g(X_s) = \widetilde{g}(X_s), \quad ds P_x \quad \text{a.s., for every} \quad x \in E.$$

We denote $g = \mathcal{L}(f)$.
Assuming that $\mathcal{D}(\mathcal{L})$ is an algebra, the "opérateur carré du champ" defined in (9), satisfies the important identity:

$$d < M^f, M^g >_t = \Gamma(f, g)(X_t)dt.$$

ii) We now particularize the above discussion to a class of Markov processes (X_t) taking values in $\mathbb{R}^n$, such that $C_c^2(\mathbb{R}^n) \subset \mathcal{D}(\mathcal{L})$, and for $f \in C_c^2(\mathbb{R}^n)$:

$$\mathcal{L}f(y) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 f(y)}{\partial y_i \partial y_j} + \sum_{i=1}^{n} b_i(y)\frac{\partial f}{\partial y_i}(y) + \int n(y,dz)(f(z)-f(y)), \tag{17}$$

where $\{n(y,dz)\}_{y\in\mathbb{R}^n}$ is a family of positive, finite measures (this is quite restrictive, but sufficient for our purposes in this paper), such that for each y, $n(y,dz)$ does not charge $\{y\}$. When $\mathcal{L}$ is given by (17), the "opérateur carré du champ" is given, for $u, v \in C_c^2(\mathbb{R}^n)$, by:

$$\Gamma(u,v)(y) = <\nabla u(y), a(y)\nabla v(y)> + \int n(y,dz)(u(z)-u(y))(v(z)-v(y)). \tag{18}$$

iii) Since the semi-stable hypothesis for (X_t), our Markov process of reference, plays such an important rôle in our discussion, we now study the consequences of the semi-stable hypothesis on the form (17) of $\mathcal{L}$.
First it is easily shown that for $f \in \mathcal{D}(\mathcal{L})$, and $c > 0$ (4) implies:

$$c\mathcal{L}f(y) = \mathcal{L}(f(\sqrt{c}(.)))\left(\frac{y}{\sqrt{c}}\right). \tag{19}$$

We now note that, when $\mathcal{L}$ is given by (17), the right-hand side of (19) takes the form:

$$\frac{c}{2}\sum_{i,j=1}^{n} a_{ij}\left(\frac{y}{\sqrt{c}}\right)\frac{\partial^2 f}{\partial y_i \partial y_j}(y) + \sum_{i=1}^{n} b_i\left(\frac{y}{\sqrt{c}}\right)\sqrt{c}\frac{\partial f}{\partial y_i}(y) + \int n\left(\frac{y}{\sqrt{c}}, dz\right)(f(\sqrt{c}z)-f(y)).$$

Thus in order that (19) be satisfied, the following identities must hold, for any $c > 0$, $y \in \mathbb{R}^n$ and every function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$, bounded, with compact support:

$$a(y) = a\left(\frac{y}{\sqrt{c}}\right); \quad b(y) = \frac{1}{\sqrt{c}}b\left(\frac{y}{\sqrt{c}}\right)$$
$$\int n(y,dz)\varphi(z) = \frac{1}{c}\int n\left(\frac{y}{\sqrt{c}}, dz\right)\varphi(\sqrt{c}z).$$

Taking $\lambda = \frac{1}{\sqrt{c}}$ as a new parameter, the preceding identities may be written as:

$$\begin{cases} a(\lambda y) = a(y); \quad \lambda b(\lambda y) = b(y) \\ \int n(y,dz)\varphi(z) = \lambda^2 \int n(\lambda y, dz)\varphi(\frac{z}{\lambda}) \end{cases} \tag{20}$$

As a consequence, the fields a, b, and n are determined from their restrictions to the unit sphere: a_1, b_1, n_1 and we get

$$\begin{cases} a(y) & = a_1(\frac{y}{|y|}) \\ b(y) & = \frac{1}{|y|} b_1(\frac{y}{|y|}) \\ \int n(y, dz)\varphi(z) & = \frac{1}{|y|^2} \int n(\frac{y}{|y|}, dz)\varphi(\frac{z}{|y|}) \end{cases} \tag{21}$$

3. Old and new examples

The examples we know of Markov processes with the semi-stable scaling property (in the sense of Lamperti [16]) which satisfy our hypothesis are the following :

(i) Bessel processes (see e.g., Watanabe [22], [23] Pitman-Yor, [17]), and, more generally Bilateral Bessel processes (Watanabe [24])

(ii) Wishart processes ([2]) : in fact, these processes are matrix valued extensions of the squared Bessel processes, hence, they satisfy :

$$(X_{ct}^{(x)}, t \geq 0) \stackrel{(d)}{=} (cX_t^{(x/c)}, t \geq 0),$$

but, apart from this, our conditions are satisfied, mutatis mutandis.

(iii) The Dunkl Markov processes (see, e.g.,[11], [12], [20]) and their "radial" parts. We now give some details about these 3 classes of processes in the 3 following subsections.

3.1. Bessel and Bilateral Bessel processes

Bessel processes (BES (ν), $\nu > -1$), are continuous Markov processes which take values in $\mathbb{R}_+$, and have infinitesimal generators :

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}.$$

For $-1 < \nu < 0$, they are instantaneously reflecting at 0. The densities of the semi-group of BES (ν) are given by :

$$p_t^\nu(x, y) = \frac{1}{t}\left(\frac{y}{x}\right)^\nu y \ \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right), x, y > 0,$$

where I_ν is the modified Bessel function of the first kind with index ν [17]. Thus, our hypothesis in Proposition 2.2 is satisfied with

$$\Phi(x, y) = \frac{1}{(xy)^\nu} I_\nu(xy), \quad \theta(y) = y^{\nu+1}, \quad \rho(x) = \frac{1}{2}|x|^2.$$

The extended infinitesimal generator of $\left\{Q_y^{(x)}\right\}_{y\in\mathbb{R}_+}$ in this case is given from formula (10) by:

$$\begin{aligned}\mathcal{L}^{(x)} &= \frac{1}{2}\frac{d^2}{dy^2} + \frac{2\nu+1}{2y}\frac{d}{dy} + \left(\frac{\partial}{\partial y}(\log(\frac{I_\nu(xy)}{(xy)^\nu}))\right).\frac{d}{dy} \\ &= \frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2y}\frac{d}{dy} + \left(\frac{xI'_\nu(xy)}{I_\nu(xy)}\right).\frac{d}{dy} \\ &= \frac{1}{2}\frac{d^2}{dy^2} + \left(\frac{2\nu+1}{2y} + x\left(\frac{I_{\nu+1}}{I_\nu}\right)(xy)\right).\frac{d}{dy}\end{aligned}$$

Note that in the particular case $\nu = \frac{1}{2}$ we obtain for $\mathcal{L}^{(x)}$ the operator given by formula (8).

3.2. Wishart processes

For $m = 1, 2, \ldots$, the Wishart process WIS (δ, m) is a Markov process with continuous paths taking values in the space of positive symmetric mxm matrices S_m^+ ; WIS (δ, m) solves the Stochastic Differential Equation :

$$dX_t = \sqrt{X_t}dB_t + dB'_t\sqrt{X_t} + \delta I_m dt,\ X_0 = x,$$

where $(B_t, t \geq 0)$ is an mxm Brownian matrix, B'_t is the transpose of B_t,I_m is the identity matrix in $\mathbb{R}^m$.

To our knowledge, the first papers on Wishart processes were written by M.F. Bru [2], and further results have been obtained in [3], [4], [7]; in particular, the transition probability density $q_\delta(t, x, dy)$ with respect to the Lebesgue measure $dy = \Pi_{i\leq j}(dy_{ij})$ is given by:

$$\begin{aligned}q_\delta(t, x, y) = &\frac{1}{(2t)^{\delta m/2}\Gamma_m(\delta/2)}\ \exp\left(-\frac{1}{2t}Tr(xy)\right) \\ &\times \left(det(y)\right)^{(\delta-m-1)/2}{}_0F_1\left(\frac{\delta}{2}; \frac{xy}{4t^2}\right),\end{aligned}$$

$(x, y \in S_m^+)$ where Γ_m is the multivariate gamma function, and ${}_0F_1$ a hypergeometric function whose second argument is a generic mxm matrix.

3.3. Dunkl processes and their radial parts

The Dunkl Markov processes in $\mathbb{R}^n$ are càdlàg Markov processes with infinitesimal generators $\frac{1}{2}L_k = \frac{1}{2}\sum_{i=1}^{n} T_i^2$, where for $1 \leq i \leq n$, T_i is the differential-difference operator defined for $u \in C^1(\mathbb{R}^n)$ by

$$T_i u(x) = \frac{\partial u(x)}{\partial x_i} + \sum_{\alpha\in R_+} k(\alpha)\alpha_i\frac{u(x) - u(\sigma_\alpha x)}{<\alpha, x>}, \quad (x \in \mathbb{R}^n)$$

where $< \cdot, \cdot >$ is the usual scalar product, R is a root system in $\mathbb{R}^n$, R_+ a positive subsystem, k a non negative multiplicity function defined on R and invariant by the finite reflection group W associated with R and σ_α is the reflection with respect to the hyperplane H_α orthogonal to α ([8], [9]).

For $u \in C^2(\mathbb{R}^n)$, we can write

$$L_k u(x) = \Delta u(x) + 2\sum_{\alpha \in R_+} k(\alpha)\left[\frac{<\nabla u(x), \alpha>}{<x,\alpha>} - \frac{u(x) - u(\sigma_\alpha x)}{<x,\alpha>^2}\right],$$

where Δ is the usual Laplacian and ∇u denotes the gradient of u. Then the "opérateur carré du champ" Γ_k associated with $\mathcal{L}_k \equiv \frac{1}{2}L_k$, is given, for $u, v \in C^2(\mathbb{R}^n)$, by:

$$\Gamma_k(u,v)(x) = <\nabla u(x), \nabla v(x)> + \sum_{\alpha \in R_+} \frac{(u(\sigma_\alpha x) - u(x))(v(\sigma_\alpha x) - v(x))}{<x,\alpha>^2}, \tag{22}$$

as follows from the general formula (18).

The semi-group densities of the process $(X_t^{(k)})_{t>0}$ with generator $\frac{1}{2}L_k$ are given by the formulae obtained by M. Rösler ([19], (see also [12]) :

$$p_t^{(k)}(x,y) = \frac{1}{c_k t^{\gamma+n/2}} \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) D_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)\omega_k(y), \tag{23}$$

where $D_k(u,v) > 0$ is the Dunkl kernel, $\omega_k(y) = \Pi_{\alpha \in R_+} | < \alpha, y > |^{2k(\alpha)}$ the weight function which is homogeneous of degree $2\gamma = 2\sum_{\alpha \in R_+} k(\alpha)$ and $c_k = \int_{\mathbb{R}^n} e^{-|x|^2/2}\omega_k(x)dx$.

Moreover the Dunkl kernel satisfies

$$\forall \lambda \in \mathbb{R}, \forall x, y \in \mathbb{R}^n, D_k(x, \lambda y) = D_k(\lambda x, y).$$

Thus, our hypothesis in Proposition 1 is satisfied with

$$\Phi \equiv D_k, \quad \theta \equiv \omega_k, \quad \rho(a) \equiv \frac{1}{2}|a|^2. \tag{24}$$

We must emphasize that, unlike Bessel processes, Dunkl processes have jumps (see [11] for a study of the behaviour of the trajectories in the case $n = 1$).

We also remark that, for $x \in \mathbb{R}^n$, the semigroup densities of the process $\widetilde{X}_t^{(k),x} = tX_{\frac{1}{t}}^{(k)}$ under P_x are given, as a consequence of (7), by:

$$q_t^{(x)}(a,b) = \exp(-\frac{1}{2}|x|^2 t)\frac{D_k(x,b)}{D_k(x,a)} p_t^{(k)}(a,b), \tag{25}$$

and they induce a family of laws $(Q_a^{(x)})_{a \in \mathbb{R}^n}$ on $\mathbb{R}^n$. The process $\left\{(\widetilde{X}_t^{(k),x})_{t>0}, (Q_a^{(x)})_{a\in\mathbb{R}^n}\right\}$ will be called the Dunkl process with drift x. It can be easily deduced from the semigroup densities (25) that the infinitesimal generator of this process is of the form

$$\mathcal{L}^{(k),x} : f \longmapsto \frac{1}{2}\frac{1}{D_k(x,.)}L_k(D_k(x,.)f) - \frac{|x|^2}{2}f,$$

($f \in C^2(\mathbb{R}^n)$), which from formulas (10) and (22), is given by:

$$\mathcal{L}^{(k),x}(f)(y) = \frac{1}{2}L_k(f)(y) + \frac{1}{D_k(x,y)}\Big\{< \nabla_y(D_k(x,y)), \nabla f(y) > + \sum_{\alpha\in R_+} \frac{k(\alpha)}{< y,\alpha >^2}(D_k(x,\sigma_\alpha y) - D_k(x,y))(u(\sigma_\alpha y) - u(y))\Big\}.$$

Note that if $k(\alpha) \equiv 0$, this process is the usual Brownian motion with drift x.

Let us now consider a fixed Weyl chamber C of the root-system R (i.e. C:= a connected component of $\mathbb{R}^n \setminus \bigcup_{\alpha\in R} H_\alpha$). The operator $\frac{1}{2}L_k^W$, where

$$L_k^W u(x) = \Delta u(x) + 2\sum_{\alpha\in R_+} k(\alpha)\frac{< \nabla u(x), \alpha >}{< \alpha, x >}, \tag{26}$$

($u \in C_0^2(\overline{C})$, $< \nabla u(x), \alpha >= 0$ for $x \in H_\alpha, \alpha \in R_+$), is the infinitesimal generator of a diffusion process $(X_t^W)_{t>0}$ in $\overline{C}$ (see [20]), which we call the W-radial part of the Dunkl process $(X_t^{(k)})_{t>0}$. Indeed, the space $\mathbb{R}^n/W$ of W-orbits in $\mathbb{R}^n$ can be identified to $\overline{C}$ and we have $X_t^W = \pi(X_t^{(k)})$ if $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/W$ denotes the canonical projection. For example, in the one dimensional case, $\overline{C} = \mathbb{R}_+$ and $\pi(X_t^{(k)}) = |X_t^{(k)}|$ is a Bessel process [11]. We deduce immediately from (23) that the semi-group densities of $(X_t^W)_{t>0}$ are of the form

$$p_t^W(x,y) = \frac{1}{c_k t^{\gamma+n/2}} \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) D_k^W\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)\omega_k(y)dy,\ (x,y\in C),$$

where

$$D_k^W(u,v) = \sum_{w\in W} D_k(u, wv).$$

Clearly the hypothesis of Proposition 1 is also satisfied using (24) with D_k replaced by D_k^W.

Example. It is interesting to notice that the Brownian motion process in a Weyl chamber as studied by Biane, Bougerol and O'Connell in [1], is an example of a W-radial Dunkl process. Indeed, its infinitesimal generator is of the form $\frac{1}{2}L_k^W$ with the particular multiplicity function $k(\alpha) \equiv 1$. To prove this assertion, consider the function

$$h(x) = \prod_{\alpha\in R_+} < \alpha, x >,$$

which is Δ-harmonic (see e.g. [10], Theorem 4.2.6) and strictly positive in the Weyl chamber C (if suitably chosen). Then formula (26) with $k \equiv 1$, can be written

$$L_k^W u(x) = \Delta u(x) + \frac{2}{h(x)} < \nabla u(x), \nabla h(x) > .$$

This shows immediately ([18], p.357) that the corresponding process $(X_t^W)_{t>0}$ is the h-transform of the usual Brownian motion in C, killed when it reaches the walls of C i.e. the Brownian motion in the Weyl chamber C of [1].

4. Conclusion and summary

In this paper, the criterion which is presented in Proposition 2.2 for a Markov process to enjoy the time-inversion property may look a little strange in several respects: indeed, we ask for our Markov process of reference (X_t) to be a semi-stable Markov process of index $(\frac{1}{2})$, in short: a $SSMP(\frac{1}{2})$, in the sense of Lamperti [16], and that its semigroup density be of a special form. This second condition (or some variant of it) is needed, since not every $SSMP(\frac{1}{2})$ satisfies the time inversion property as we show with the two following examples:

i) For every $\alpha \in]0, 1[$, if we denote by $(T_t^{(\alpha)}, t \geq 0)$, the subordinator of index α, then the process $((T_t^{(\alpha)})^{\alpha/2}, t \geq 0)$ is a $SSMP(\frac{1}{2})$ and it does not satisfy the time inversion property. Indeed if we denote by $p_t^{(\alpha)}(.,.)$ its transition densities and if we consider $t = s + 1$ in formula (3) and let $s \to +\infty$, then for fixed $x > 0$, we would have

$$p^{(\alpha)}_{\frac{1}{s+1}}(x, \frac{b}{s+1}) = 0,$$

for s big enough and this would imply $q^{(x)}_{s,s+1}(a, b) = 0$ for all $a, b \in \mathbb{R}_+$ which is impossible. Another proof based on the harness property satisfied by subordinators (see Exercise 6.19 in [5]), is obtained by considering the process $\hat{T}_t \equiv t(T^{(\alpha)}_{1/t})^{\alpha/2}$ which satisfies the following identity :

$$E[(\hat{T}_t)^{2/\alpha} | \hat{T}_s] = (1 + \frac{t-s}{s})^{(2/\alpha)-1} (\hat{T}_s)^{2/\alpha}. \tag{27}$$

Then we clearly see that the conditional expectation on the left hand side of (27) given $\hat{T}_s = x$, depends both on s and $t - s$.

ii) With the same ideas used in the first example, we can also show that if $(C_t, t \geq 0)$ is a symmetric standard Cauchy process on the real line, then the process $(\sqrt{|C_t|}, t \geq 0)$ is also a $SSMP(\frac{1}{2})$ which does not satisfy the time inversion property.

Finally, we are led to ask the following question :

Does a $SSMP(\frac{1}{2})$ which satisfies the time inversion property, necessarily have transition densities of the form given in Proposition 2.2?

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