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A study of a class of stochastic differential equations with non-Lipschitzian coefficients

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Abstract. We study a class of stochastic differential equations with non-Lipschitz coefficients. A unique strong solution is obtained and the non confluence of the solutions of stochastic differential equations is proved. The dependence with respect to the initial values is investigated. To obtain a continuous version of solutions, the modulus of continuity of coefficients is assumed to be less than $|x-y|\log\frac{1}{|x-y|}$. Finally a large deviation principle of Freidlin-Wentzell type is also established in the paper.

1. Introduction and statement of results

Let $\sigma : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^m$ and $b : \mathbf{R}^d \to \mathbf{R}^d$ be respectively matrix valued and $\mathbf{R}^d$ valued continuous functions. It is well-known that the following Itô stochastic differential equation

$$dX_t = \sigma(X_t)\,dW_t + b(X_t)\,dt, \quad X_0 = x_o \in \mathbf{R}^d \tag{1.1}$$

has a weak solution up to a lifetime ζ (see [SV], [IW, p.155-163]), where $t \to W_t$ is a $\mathbf{R}^m$-valued standard Brownian motion. It is also known that under the assumption of linear growth of coefficients σ and b, the lifetime ζ is infinite almost surely. The well-known result of Yamada and Watanabe says that if the stochastic differential equation (1.1) has the pathwise uniqueness, then it admits a unique strong solution (see [IW, p.149], [RY, p.341]). So the study of pathwise uniqueness is of great interest. It is a classical result that under the Lipschitz conditions, the pathwise uniqueness holds and the solution of stochastic differential equation (1.1) can be constructed using Picard iteration; moreover the solution depends continuousely on the initial data (see [Ku]). The main tool to these studies is the Gronwall lemma. When the coefficients σ and b do not satisfy the Lipschitz conditions, the use of Gronwall lemma is not possible. Therefore, there are few results of pathwise uniqueness of solutions of stochastic differential equations beyond the Lipschitz

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(or local Lipschitz) conditions in the literature except in the one dimensional case (see [IW, p.168], [RY, Ch. IX-3]). We refer also to [MX] for related discussions. In the case of ordinary differential equations, the Gronwall lemma was generalized to establish uniqueness results (see e.g. [La]). However the method is not applicable to stochastic differential equations. For stochastic differential equations with less regular coefficients, a new concept of solutions has been introduced recently and many interesting phenomena have been obtained in [LJR1,2]. In this work, we shall study a class of stochastic differential equations for which the Lipschitzian conditions are relaxed mainly by a logarithmic factor. For the pathwise uniqueness, we impose a condition on the modules of the continuity of the coefficients only in an arbitrarily small neigbourhood of the diagonal. This seems the first time in the literature to notice this phenomena. Our method of proving the uniqueness and the non-explosion is dimension-free and does not require that the control function of the modules of the continuity of the coefficients is concave in contrast to the existing literature .

Now let's describe the main results in the paper. For a matrix σ, we denote by $||\sigma||$ the Hilbert-Schmidt norm: $||\sigma||^2 = \sum_{ij} \sigma_{ij}^2$; for a vector $x \in \mathbf{R}^d$, $|x|$ the Euclidean norm.

Theorem A. *Let ρ be a strictly positive, C^1-function defined on a neighborhood $[K, +\infty[$ of $+\infty$, satisfying (i) $\lim_{s\to+\infty} \rho(s) = +\infty$, (ii) $\lim_{s\to+\infty} \frac{s\rho'(s)}{\rho(s)} = 0$ and (iii) $\int_K^{+\infty} \frac{ds}{s\rho(s)+1} = +\infty$. Assume that for $|x| \geq K$,*

$$\begin{cases} ||\sigma(x)||^2 \leq C\,(|x|^2\,\rho(|x|^2) + 1), \\ \ |b(x)| \ \leq \ C\,(|x|\,\rho(|x|^2) + 1). \end{cases} \tag{H1}$$

Then the stochastic differential equation (1.1) has no explosion: $P(\zeta = +\infty) = 1$. ∎

This result will be proved in section 4. Moreover if we denote by $X_t(x_o)$ a solution of (1.1) with the initial value x_o, then the condition $(H1)$ implies that $\lim_{|x_o|\to+\infty} |X_t(x_o)| = +\infty$ in probability.

Functions $\rho(s) = \log s$, $\rho(s) = \log s \cdot \log\log s, \cdots$ are typical examples satisfying the above conditions (i)-(iii). When the coefficients σ and b grow at most as $|x|$, the hypothesis $(H1)$ holds obviously with $\rho(s) = \log s$.
Our second result deals with pathwise uniqueness.

Theorem B. *Let r be a strictly positive, C^1-function defined on a neighborhood $]0, c_o]$ of 0, satisfying (i) $\lim_{s\to 0} r(s) = +\infty$, (ii) $\lim_{s\to 0} \frac{sr'(s)}{r(s)} = 0$ and (iii) $\int_0^a \frac{ds}{sr(s)} = +\infty$ for any $a > 0$. Assume that for $|x - y| \leq c_o$,*

$$\begin{cases} ||\sigma(x) - \sigma(y)||^2 \leq C\,|x-y|^2\, r(|x-y|^2), \\ \ |b(x) - b(y)| \ \leq \ C\,|x-y|\, r(|x-y|^2). \end{cases} \tag{H2}$$

Then the pathwise uniqueness holds for stochastic differential equation (1.1). ∎

The proof of this result will be given in section 5. The crucial idea is to construct a family of positive increasing functions $(\Phi_\delta)_{\delta>0}$ on $\mathbf{R}_+$ so that the Gronwall lemma can be applied to the composition of Φ_δ with appropriate processes. Functions $r(s) = \log 1/s$, $r(s) = \log 1/s \cdot \log\log 1/s, \cdots$ are typical examples satisfying the conditions (i)-(iii) in theorem **B**.

It is known that for ordinary differential equations, pathwise uniqueness is equivalent to the property of non confluence by reversing the time. For stochastic differential equations, reversing the time is delicate. However, using this idea, we shall construct another function Φ_δ so that we shall obtain the following result

Theorem C. *Under the same hypothesis as in theorem* **B** *and assume that the solution does not explode at a finite time. Then for $x_o \neq y_o$, almost surely $X_t(x_o) \neq X_t(y_o)$ for all $t > 0$.* ∎

Such kind of non confluence property was studied by M. Emery in an early work [Em] for general stochastic differential equations under Lipschitz conditions, and by T. Yamada and Y. Ogura for non-Lipschitz case in [YO]. However the mixing condition imposed in [YO] for coefficients σ and b seems difficult to check and not natural.

Our third result concerns the dependence with respect to initial data.

Theorem D. *Assume that the hypothesis $(H2)$ holds with $r(s) = \log 1/s$ and the stochastic differential equation (1.1) has no-explosion. Then there exists a version $\tilde{X}_t(x_o)$ of $X_t(x_o)$ such that $(t, x_o) \to \tilde{X}_t(x_o)$ is continuous over $[0, +\infty[\times\mathbf{R}^d$ almost surely.* ∎

We shall prove this result in section 6. The last result is devoted to Freidlin-Wentzell type large deviation principle.

Theorem E. *Suppose that the coefficients σ and b satisfy the hypothesis $(H1)$ with $\rho(s) = \log s$ and the hypothesis $(H2)$ with $r(s) = \log 1/s$. For $\varepsilon > 0$, consider the solution $(X_t^\varepsilon)_{t\geq 0}$ of the stochastic differential equation*

$$dX_t^\varepsilon = \sqrt{\varepsilon}\,\sigma(X_t^\varepsilon)\,dW_t + b(X_t^\varepsilon)\,dt, \quad X_0^\varepsilon = x_o. \tag{1.2}$$

Let μ_ε be the law of $\omega \to X_\cdot^\varepsilon(\omega)$ on the space $C_{x_o}([0,1], \mathbf{R}^d)$ of continuous functions starting from $x_o \in \mathbf{R}^d$. Then $\{\mu_\varepsilon,\ \varepsilon > 0\}$ satisfies a large deviation principle. ∎

The good rate function and the proof of theorem **E** will be given in section 7. The method of estimating moments used in the literature ([DS],[DZ], [S]) does not work here because of the non-Lipschitz feature of coefficients. We again appeal to a family of positive functions $(\Phi_{\rho,\lambda})$.

The organization of the paper is as follows. In section 2, we shall discuss the case of ordinary differential equations. It will be useful for the study on skeletons of stochastic differential equations in section 7. In section 3, we shall prepare some preliminary lemmas in order to avoid repetitions of same kind of computations in the sequel. Section 4,5,6, and 7 will be devoted to the proof of main results. A preliminary version of theorem **A** and theorem **B** has been given in [FZ1]. A study on stochastic flows of homeomorphisms and critical Sobolev flows on the sphere S^d will be discussed in a separate paper.

2. Ordinary differential equations

Let $b : \mathbf{R}^d \to \mathbf{R}^d$ be a continuous function. It is essentially due to Ascoli-Arzela theorem that the following differential equation:

$$\frac{dX_t}{dt} = b(X_t), \qquad X(0) = x_0 \tag{2.1}$$

admits a solution up to a lifetime ζ. The following result weakens the linear growth condition for non explosion.

Theorem 2.1. *Let $\rho : [0, +\infty[\to [1, +\infty[$ be a continuous function such that $\int_0^{+\infty} \frac{ds}{s\rho(s)+1} = +\infty$. Assume that*

$$|b(x)| \le C\,(|x|\rho(|x|^2) + 1) \tag{2.2}$$

where $C > 0$ is a constant. Then the lifetime of any solution of (2.1) is infinite: $\zeta = +\infty$.

Proof. Define for $\xi \ge 0$,

$$\psi(\xi) = \int_0^{\xi} \frac{ds}{s\rho(s)+1} \qquad \text{and } \Phi(\xi) = e^{\psi(\xi)}.$$

We have

$$\Phi'(\xi) = \frac{\Phi(\xi)}{\xi\rho(\xi)+1}. \tag{2.3}$$

Let $\xi_t = |X_t|^2$. Taking the derivative of $\Phi(\xi_t)$ with respect to the time t, we get

$$\frac{d}{dt}\Phi(\xi_t) = 2\Phi'(\xi_t)\big\langle X_t, b(X_t)\big\rangle,$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbf{R}^d$. By growth condition (2.2), we have

$$\left|\frac{d}{dt}\Phi(\xi_t)\right| \le 2C\Phi'(\xi_t)|X_t|\,\big(|X_t|\rho(\xi_t) + 1\big).$$

It is easy to see that

$$\sup_{\xi\ge 0} \frac{\xi\rho(\xi) + \sqrt{\xi}}{\xi\rho(\xi)+1} \le 2.$$

Therefore, according to (2.3),

$$\left|\frac{d}{dt}\Phi(\xi_t)\right| \le 4C\Phi(\xi_t). \tag{2.4}$$

It follows that for $t < \zeta$,

$$\Phi(\xi_t) \le \Phi(|x_0|^2) + 4C\int_0^t \Phi(\xi_s)\,ds.$$

By Gronwall lemma, we have

$$\Phi(\xi_t) \leq \Phi(|x_0|^2)\, e^{4C\,t} \quad \text{for all } t < \zeta. \tag{2.5}$$

If $\zeta < +\infty$, letting $t \uparrow \zeta$ in (2.5), we get $\Phi(\xi_\zeta) \leq \Phi(|x_0|^2)\, e^{4C\zeta}$. Since $\xi_\zeta = +\infty$, by condition on ρ, $\Phi(+\infty) = +\infty$. The left hand side of (2.5) is infinite, while the right hand side is finite; which is impossible. So $\zeta = +\infty$. □

Remark 2.2. By considering the inequality $\frac{d}{dt}\Phi(\xi_t) \geq -4C\,\Phi(\xi_t)$ in (2.4), we have

$$\Phi(\xi_t) \geq \Phi(|x_o|^2) - 4C \int_0^t \Phi(\xi_s)\, ds,$$

which yields

$$\Phi(\xi_t) \geq \Phi(|x_o|^2) e^{-4Ct}.$$

If we denote by $X_t(x_o)$ the solution of the differential equation (2.1) with initial value x_o, then we get $\lim_{|x_o|\to+\infty} \Phi(|X_t(x_o)|^2) = +\infty$, which implies that

$$\lim_{|x_o|\to+\infty} |X_t(x_o)| = +\infty. \;\blacksquare$$

For simplicity, we shall assume that solutions of (2.1) do not explode at a finite time.

Theorem 2.3. *Let* $r\ :]0, c_o] \to [1, +\infty[$ *be a continuous function defined on a neighborhood* $]0, c_o]$ *of* 0 *such that* $\int_0^a \frac{ds}{sr(s)} = +\infty$ *for any* $a > 0$. *Assume that for* $|x-y| \leq c_o$,

$$|b(x) - b(y)| \leq C\,|x-y|\, r(|x-y|^2) \tag{2.6}$$

where $C > 0$ *is a constant. Then the differential equation* (2.1) *has a unique solution.*

Proof. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two solutions of the equation (2.1). Set $\eta_t = X_t - Y_t$ and $\xi_t = |\eta_t|^2$. Let $\delta > 0$ be a parameter, consider functions

$$\psi_\delta(\xi) = \int_0^\xi \frac{ds}{sr(s) + \delta} \quad \text{and} \quad \Phi_\delta(\xi) = e^{\psi_\delta(\xi)}. \tag{2.7}$$

We have

$$\Phi'_\delta(\xi) = \frac{\Phi_\delta(\xi)}{\xi\, r(\xi) + \delta}.$$

Define the stopping time

$$\tau = \inf\{t > 0,\ \xi_t \geq c_0^2\}.$$

By assumption (2.6), we have for $t < \tau$

$$\left|\langle \eta_t, b(X_t) - b(Y_t)\rangle\right| \leq C\, \xi_t\, r(\xi_t).$$

Therefore

$$\left|\frac{d}{dt}\Phi_\delta(\xi_t)\right| \le 2C\;\Phi_\delta(\xi_t). \tag{2.8}$$

It follows that, for $t < \tau$,

$$\Phi_\delta(\xi_t) \le 1 + 2C\int_0^t \Phi_\delta(\xi_s)\,ds,$$

which implies that $\Phi_\delta(\xi_t) \le e^{2C\,t}$ for $t < \tau$. Letting $\delta \downarrow 0$, we get that $e^{\psi_0(\xi_t)} \le e^{2Ct}$. Now by assumption on r, we obtain that $\xi_t = 0$ for all $t < \tau$. If $\tau < +\infty$, letting $t \uparrow \tau$, we get

$$c_o^2 = \xi_\tau = 0,$$

which is absurd. Therefore $\xi_t = 0$ for all $t \ge 0$. In other words, $X_t = Y_t$ for $t \ge 0$. □

In what follows, we shall study the dependence of the solutions on the initial data.

Theorem 2.4. *Assume that the coefficient b satisfies the condition* (2.6) *and that the solution of (2.1) has no explosion. Then* $x_o \to X_t(x_o)$ *is continuous, uniformly with respect to t in any compact subset.*

Proof. Let $\varepsilon \in]0, \sqrt{c_o}]$, where c_0 is given in definition of r. Let $(x_o, y_o) \in \mathbf{R}^d \times \mathbf{R}^d$ such that $|x_o - y_o| < \varepsilon$. Set $\eta_t = X_t(x_o) - X_t(y_o)$ and $\xi_t = |\eta_t|^2$. Define

$$\tau(x_o, y_o) = \inf\{t > 0,\ \xi_t \ge \varepsilon^2\}.$$

Let Φ_δ be the function defined in (2.7), as in proof of Theorem 2.3, we have for $t < \tau(x_o, y_o)$,

$$\Phi_\delta(\xi_t) \le \Phi_\delta(\xi_o)e^{2C\,t}. \tag{2.9}$$

Taking $\delta = |x_o - y_o|$ in (2.9), we get

$$\Phi_\delta(\xi_t) \le e^{\delta}\,e^{2C\,t}, \quad \text{for } t < \tau(x_o, y_o). \tag{2.10}$$

Fix the point x_o. If $\liminf_{y_o \to x_o} \tau(x_o, y_o) = \tau$ is finite, we can choose a sequence $y_n \to x_o$ such that $\lim_{n\to+\infty} \tau(x_o, y_n) = \tau$. Set $\delta_n = |x_o - y_n|$. Applying (2.10) for (x_o, y_n) and letting $t \uparrow \tau(x_o, y_n)$, we get

$$\Phi_{\delta_n}(\varepsilon) = \Phi_{\delta_n}\left(\xi_{\tau(x_o,y_n)}\right) \le e^{\delta_n}\,e^{2C\,\tau(x_o,y_n)}.$$

Letting $n \to +\infty$ in the above inequality, we see that

$$+\infty = \Phi_o(\varepsilon) \le e^{2C\,\tau}$$

which is absurd. Therefore

$$\lim_{y_o \to x_o} \tau(x_o, y_o) = +\infty,$$

which means that for any $t > 0$, there exists $\delta > 0$ such that for $|y_o - x_o| < \delta$, $\tau(x_o, y_o) > t$. In other words,

$$\sup_{0\le s\le t} |X_s(x_o) - X_s(y_o)| \le \varepsilon.$$

□

Remark 2.5. This argument does not work, when we deal with stochastic differential equations, as it is not possible to choose a common sequence y_n. ∎

Proposition 2.6. *Under the same hypothesis as in theorem 2.4, for $x_o \neq y_o$, we have $X_t(x_o) \neq X_t(y_o)$ for all $t \ge 0$.*

Proof. Let $\eta_t = X_t(x_o) - X_t(y_o)$ and $\xi_t = |\eta_t|^2$. Without loss of generality, assume that $c_0^2 > 2\xi_0$. Let

$$\tau = \inf\left\{t > 0,\ \xi_t \ge \frac{3}{4}c_o^2\right\}.$$

By starting from τ again , it is enough to prove that for any $c_0^2 > 2\xi_0$, we have $\xi_t > 0$ for $t < \tau$. Consider

$$\psi_\delta(\xi) = \int_0^\xi \frac{ds}{sr(s) + \delta} \quad \text{and} \quad \Phi_\delta(\xi) = e^{\psi_\delta(\xi)}.$$

We have for $t < \tau$,

$$\left|\frac{d\Phi_\delta(\xi_t)}{dt}\right| \le 2C\Phi_\delta(\xi_t).$$

It follows that $\Phi_\delta(\xi_t) \ge \Phi_\delta(\xi_o) - 2C\int_0^t \Phi_\delta(\xi_s)ds$ or

$$\Phi_\delta(\xi_t) \ge \Phi_\delta(\xi_o)e^{-2Ct} \quad \text{for} \quad t < \tau.$$

For $\delta > 0$ small enough, $\Phi_\delta(\xi_o)e^{-2Ct} > 1$. It follows that $\Phi_\delta(\xi_t) > 1$ or $\xi_t > 0$. □

Now by the standard arguments, we obtain

Theorem 2.7. *Suppose that the coefficient b satisfies the hypothesis (2.2) and (2.6). Then for any $t > 0$, $x_o \to X_t(x_o)$ defines a flow of homeomorphisms of* $\mathbf{R}^d$.

Before ending this section, we would like to give an example of function b satisfying the condition (2.6), but having not the local Lipschitz continuity property.

Example 2.8. Define the function V on $\mathbf{R}$ by the series

$$V(x) = \sum_{k=1}^{+\infty} \frac{|\sin kx|}{k^2}. \tag{2.11}$$

Obviously the function V is continuous on $\mathbf{R}$. We claim that

$$|V(x) - V(y)| \le C\,|x - y| \log\{1/|x - y|\} \quad \text{for } |x - y| < 2/e \tag{2.12}$$

where $C > 0$ is a constant. To prove (2.12), we start with the following preliminary lemma.

Lemma 2.9. *For* $0 < \theta < 1/e$, *we have*

$$V(\theta) = \sum_{k=1}^{\infty} \frac{|\sin k\theta|}{k^2} \le C_1\, \theta\, \log \frac{1}{\theta} \tag{2.13}$$

Proof. Consider the function $\phi(s) = \dfrac{\sin s\theta}{s^2}$. We compute the derivative $\phi'(s)$:

$$\phi'(s) = \frac{s^2\theta \cos s\theta - 2s \sin s\theta}{s^4}.$$

Then $|\phi'(s)| \le \dfrac{3\theta}{s^2}$. Consider the integral $W(\theta) = \displaystyle\int_1^{+\infty} \frac{|\sin s\theta|}{s^2}\, ds$. We have

$$\begin{aligned} |V(\theta) - W(\theta)| &\le \sum_{k=1}^{+\infty} \int_k^{k+1} |\phi(s) - \phi(k)|\, ds \\ &\le 3\theta \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2\theta}{2}. \end{aligned} \tag{2.14}$$

Now for $0 < \theta < 1$,

$$W(\theta) = \theta \int_\theta^{+\infty} \frac{|\sin t|}{t^2}\, dt \le \theta \int_\theta^1 \frac{\sin t}{t}\frac{dt}{t} + \theta \int_1^{+\infty} \frac{ds}{s^2},$$

which is dominated by

$$\theta\Big(\log\frac{1}{\theta} + 1\Big).$$

Therefore, according to (2.14)

$$V(\theta) \le \theta\Big(\log\frac{1}{\theta} + 1 + \frac{\pi^2}{2}\Big),$$

which is less than $(\dfrac{\pi^2}{2} + 2)\, \theta \log \dfrac{1}{\theta}$ for $0 < \theta < 1/e$. □

Proof of (2.12). We have

$$|V(x) - V(y)| \le \sum_{k=1}^{+\infty} \frac{|\sin kx - \sin ky|}{k^2} \le 2 \sum_{k=1}^{+\infty} \frac{|\sin k\frac{x-y}{2}|}{k^2}.$$

Applying (2.13), we get (2.12). □

Now we define for $X = (x_1, x_2) \in \mathbf{R}^2$,

$$f(X) = V(x_1) + V(x_2). \tag{2.15}$$

Using the concavity of the function $s \to s \log(1/s)$, we see that for $|X - Y| < 1/e$,

$$|f(X) - f(Y)| \le C\, |X - Y| \log\{1/|X - Y|\}. \ ∎$$

Remark 2.10. By further analysis on the function V, it is known (see e.g. [Ma]) that $V(\theta) \sim \theta \log(1/\theta)$ as $\theta \to 0$. So the function f on $\mathbf{R}^2$ constructed in (2.15) is not locally Lipschitzian. Furthermore functions satisfying the condition (2.6) are mostly defined by a series or a generalized integral. ∎

3. Preparing lemmas

Let $(\Omega, \mathcal{F}, P)$ be a probability space, endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(W_t)_{t\geq 0}$ be a $\mathcal{F}_t$-Brownian motion taking values in $\mathbf{R}^m$. Consider the following Itô process in $\mathbf{R}^d$:

$$\eta_t = \eta_0 + \int_0^t e_s dW_s + \int_0^t f_s ds, \quad \eta_0 \in \mathbf{R}^d \tag{3.1}$$

where $(e_t(\omega))_{t\geq 0}$ is a matrices-valued adapted process such that $\int_0^T ||e_s||^2\, ds < +\infty$ for any $T > 0$ and $(f_t(\omega))_{t\geq 0}$ is a $\mathbf{R}^d$-valued adapted process such that $\int_0^T |f_s|\, ds < +\infty$ for any $T > 0$.

Lemma 3.1. *Let $\xi_t = |\eta_t|^2$. Then*

$$d\xi_t = 2\langle e_t^* \eta_t, dW_t\rangle + 2\langle \eta_t, f_t\rangle\, dt + ||e_t||^2\, dt \tag{3.2}$$

where e_t^* denotes the transpose matrix of e_t. The stochastic contraction $d\xi_t \cdot d\xi_t$ is given by

$$d\xi_t \cdot d\xi_t = 4|e_t^*\eta_t|^2\, dt \tag{3.3}$$

Proof. It follows directly from Itô formula. □

Lemma 3.2. *Let ρ be a continuous function on $[0, +\infty[$ such that $\rho \geq 1$. Let Φ be a strictly positive, C^2-function on $[0, +\infty[$ satisfying the conditions*

$$|\Phi'(\xi)| \leq \frac{C_1\Phi(\xi)}{\xi\rho(\xi)+1}, \quad \Phi''(\xi) \leq \frac{C_2\Phi(\xi)\rho(\xi)}{(\xi\rho(\xi)+1)^2} \tag{3.4}$$

where C_1, C_2 are two positive constant. Keeping the notations in lemma 3.1, assume that almost surely and for all $t \geq 0$,

$$\begin{cases} ||e_t||^2 \leq \ C_3\,(\xi_t\,\rho(\xi_t)+1), \\ \ |f_t| \ \ \leq C_4\,(\xi_t^{1/2}\,\rho(\xi_t)+1) \end{cases} \tag{3.5}$$

where C_3, C_4 are two positive constant. Define the stopping time $\tau_R = \inf\{t > 0,\ \xi_t \geq R\}$. Let

$$K = (C_1 + 2C_2)C_3 + 4C_1C_4. \tag{3.6}$$

Then the following estimate holds

$$\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau_R})\Big) \leq \Phi(|\eta_0|^2)\, e^{Kt}, \quad \textit{for any } t \geq 0,\ R > 0.$$

Proof. Using Itô formula and according to (3.2) and (3.3), we have

$$\begin{aligned}\Phi(\xi_{t\wedge\tau_R}) &= \Phi(\xi_0) + 2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\langle e_s^*\eta_s, dW_s\rangle \\ &\quad +2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\langle\eta_s, f_s\rangle\, ds + \int_0^{t\wedge\tau_R} \Phi'(\xi_s)||e_s||^2\, ds \\ &\quad +2\int_0^{t\wedge\tau_R} \Phi''(\xi_s)\ |e_s^*\eta_s|^2\, ds \\ &= \Phi(\xi_0) + I_1(t) + I_2(t) + I_3(t) + I_4(t)\end{aligned}$$

respectively. By conditions (3.5), we see that $I_1(t)$ is a martingale; therefore $\mathbb{E}(I_1(t)) = 0$. Using (3.4) and (3.5),

$$\begin{aligned}|\Phi'(\xi_s)\langle\eta_s, f_s\rangle| &\le \frac{C_1C_4\Phi(\xi_s)}{\xi_s\rho(\xi_s)+1}\cdot|\eta_s|\ (|\eta_s|\rho(\xi_s)+1) \\ &= C_1C_4\Phi(\xi_s)\frac{\xi_s\rho(\xi_s)+\xi_s^{1/2}}{\xi_s\rho(\xi_s)+1}.\end{aligned}$$

Since $\rho \ge 1$ and

$$\sup_{\xi\ge 0}\frac{\xi\rho(\xi)+\xi^{1/2}}{\xi\rho(\xi)+1} \le 2,$$

we get the estimate

$$\mathbb{E}(I_2(t)) \le 4C_1C_4\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\, ds.$$

In the same way, $\mathbb{E}(I_3(t)) \le C_1C_3\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\, ds$. Now

$$\Phi''(\xi_s) \le \frac{C_2\Phi(\xi_s)\rho(\xi_s)}{(\xi_s\rho(\xi_s)+1)^2} \le \frac{C_2\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)}$$

and

$$\Phi''(\xi_s)|e_s^*\eta_s|^2 \le \frac{C_2C_3\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)}\cdot\xi_s(\xi_s\rho(\xi_s)+1) = C_2C_3\Phi(\xi_s),$$

so that $E(I_4(t)) \le 2C_2C_3\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\, ds$. Let K be the constant defined in (3.6), we obtain the inequality

$$\mathbb{E}(\Phi(\xi_{t\wedge\tau_R})) \le \Phi(\xi_0) + K\int_0^t \mathbb{E}(\Phi(\xi_{s\wedge\tau_R}))\, ds.$$

By Gronwall lemma, we get that for all $t \ge 0$ and $R > 0$,

$$\mathbb{E}(\Phi(\xi_{t\wedge\tau_R})) \le \Phi(\xi_0)e^{Kt}.$$

□

Lemma 3.3. *Let r be a continuous function defined on a neighborhood of* 0*, say* $]0, c_o]$*, such that* $r \geq 1$*. Let* Φ *be a strictly positive,* C^2*-function defined on* $[0, c_o]$*. Suppose that there exists* $\delta > 0$ *such that*

$$|\Phi'(\xi)| \leq \frac{C_1\Phi(\xi)}{\xi r(\xi) + \delta}, \quad \Phi''(\xi) \leq \frac{C_2\Phi(\xi)r(\xi)}{(\xi r(\xi) + \delta)^2}. \tag{3.7}$$

Keep the notations in lemma 3.1 and suppose that $|\eta_0|^2 < c_o$*. Define the stopping time*

$$\tau = \inf\{t > 0,\ \xi_t \geq c_o\}.$$

Assume that for $t < \tau$*,*

$$||e_t||^2 \leq C_3\,(\xi_t r(\xi_t) + \delta), \quad |\langle \eta_t, f_t\rangle| \leq C_4\,(\xi r(\xi_t) + \delta). \tag{3.8}$$

Let

$$K = (C_1 + 2C_2)C_3 + 4C_1C_4. \tag{3.9}$$

Then

$$I\!E\Big(\Phi(\xi_{t\wedge\tau})\Big) \leq \Phi(|\eta_0|^2)\,e^{Kt}, \quad \text{for any } t \geq 0.$$

Proof. Using Itô formula and according to (3.2) and (3.3), we have

$$\begin{aligned}\Phi(\xi_{t\wedge\tau}) &= \Phi(\xi_0) + 2\int_0^{t\wedge\tau} \Phi'(\xi_s)\big\langle e_s^*\eta_s, dW_s\big\rangle \\ &\quad +2\int_0^{t\wedge\tau} \Phi'(\xi_s)\big\langle \eta_s, f_s\big\rangle\,ds + \int_0^{t\wedge\tau} \Phi'(\xi_s)||e_s||^2\,ds \\ &\quad +2\int_0^{t\wedge\tau_R} \Phi''(\xi_s)|e_s^*\eta_s|^2\,ds \\ &= \Phi(\xi_0) + I_1(t) + I_2(t) + I_3(t) + I_4(t)\end{aligned}$$

respectively. By assumption (3.8), for any $s \leq \tau$,

$$|e_s^*\eta_s|^2 \leq ||e_s||^2|\eta_s|^2 \leq C_3\,(\xi_s r(\xi_s) + \delta)\xi_s.$$

According to (3.7),

$$|\Phi'(\xi_s)e_s^*\eta_s|^2 \leq C_1^2C_3\Phi(\xi_s)^2\frac{\xi_s\,(\xi_s r(\xi_s) + \delta)}{(\xi_s r(\xi_s) + \delta)^2} \leq C_1^2C_3 \sup_{0\leq\xi\leq c_o} \Phi(\xi)^2 < +\infty.$$

Therefore $I_1(t)$ is a martingale and $I\!E(I_1(t)) = 0$. On the other hand, by assumption (3.7) and (3.8),

$$|\Phi'(\xi_s)\big\langle\eta_s, f_s\big\rangle| \leq C_1C_4\Phi(\xi_s) \text{ and } |\Phi'(\xi_s)|\,||e_s||^2 \leq C_1C_3\Phi(\xi_s)$$

and

$$\Phi''(\xi_s)\,|e_s^*\eta_s|^2 \leq C_2C_3\Phi(\xi_s).$$

Let K be the constant defined in (3.9). We have

$$I\!E\Big(\Phi(\xi_{t\wedge\tau})\Big) \leq \Phi(|\eta_0|^2) + K\int_0^t I\!E\Big(\Phi(\xi_{s\wedge\tau})\Big)\,ds.$$

It follows that $I\!E\Big(\Phi(\xi_{t\wedge\tau})\Big) \leq \Phi(|\eta_0|^2)\,e^{K\,t}$ for all $t > 0$. □

Lemma 3.4. *Keeping the same notations, assume that the coefficients e and f are bounded, namely*

$$||e_t(w)|| \leq A, \quad |f_t(w)| \leq B \quad \textit{for all } (t, \omega).$$

Then for any $T > 0$ *and* $R > \sqrt{d}BT$, *we have*

$$P\Big(\sup_{0 \leq s \leq T} |\xi_s| \geq R\Big) \leq 2de^{-(R-\sqrt{d}BT)^2/2dA^2T}. \tag{3.10}$$

Proof. It is a classical result and can be deduced from exponential martingale method. For a proof, see for example [S, p.81]. □

4. Criterion of non-explosion for stochastic differential equations

Let $\sigma : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^m$ and $b : \mathbf{R}^d \to \mathbf{R}^d$ be continuous functions. Let (X_t, W_t) be a solution of the Itô stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x_o \in \mathbf{R}^d \tag{4.1}$$

with the lifetime ζ. The main purpose of this section is to prove theorem **A**.

Proof of theorem A. Extend the function ρ to a strictly positive C^1 function defined on the whole half line $[0, \infty[$. It is easy to see that there exists a constant $C > 0$ such that (H1) holds for any $x \in \mathbf{R}^d$. Define the functions

$$\psi(\xi) = \int_0^\xi \frac{ds}{s\rho(s) + 1} \quad \text{and} \quad \Phi(\xi) = e^{\psi(\xi)}, \quad \xi \geq 0.$$

We have

$$\Phi'(\xi) = \frac{\Phi(\xi)}{\xi\,\rho(\xi) + 1} \quad \text{and} \quad \Phi''(\xi) = \frac{\Phi(\xi)\left(1 - \rho(\xi) - \xi\rho'(\xi)\right)}{(\xi\rho(\xi) + 1)^2}.$$

By conditions (i) and (ii) on the function ρ, we see that $|1-\rho(\xi)-\xi\rho'(\xi)| \leq C_1\rho(\xi)$ for a large constant $C_1 > 0$, so that

$$\Phi''(\xi) \leq C_1 \frac{\Phi(\xi)\rho(\xi)}{(\xi\rho(\xi) + 1)^2} \quad \text{for all } \xi \geq 0.$$

The conditions in (3.4) are satisfied. Now according to notations in lemma 3.2, let $\eta_t = X_t$ and $\xi_t = |\eta_t|^2$. Then we have $e_t = \sigma(X_t)$ and $f_t = b(X_t)$. By hypothesis (H1),

$$||e_t||^2 \leq C\,(\xi_t\rho(\xi_t) + 1), \quad |f_t| \leq (\xi_t^{1/2}\rho(\xi_t) + 1).$$

This means that conditions in (3.5) are satisfied. Now define

$$\tau_R = \inf\left\{t > 0,\ \xi_t \geq R\right\}, \quad R > 0.$$

It is clear that τ_R tends to the lifetime ζ as $R \to +\infty$. Using lemma 3.2, there exists a constant $C_2 > 0$ such that

$$E\Big(\Phi(\xi_{t\wedge\tau_R})\Big) \leq \Phi(\xi_o)e^{C_2 t}.$$

Letting $R \to +\infty$ in above inequality, by Fatou lemma, we get

$$\mathbb{E}\Big(\Phi(\xi_{t\wedge\zeta})\Big) \le \Phi(\xi_o)e^{C_2 t}. \tag{4.2}$$

Now if $P(\zeta < +\infty) > 0$, then for a large $T > 0$, $P(\zeta \le T) > 0$. Taking $t = T$ in (4.2), we get

$$\mathbb{E}\Big(\mathbf{1}_{(\zeta\le T)}\Phi(\xi_\zeta)\Big) \le \Phi(\xi_o)e^{C_2 T}. \tag{4.3}$$

Since $\Phi(\xi_\zeta) = \Phi(+\infty) = +\infty$ on a positive measure subset $\{\zeta \le T\}$, the left hand side of (4.3) is infinite, while the right hand side is finite: which is impossible. Therefore $P(\zeta = +\infty) = 1$. □

Let $X_t(x_o)$ be a solution of stochastic differential equation (4.1) with the initial value x_o. Comparing to remark 2.2, we have the following result

Theorem 4.1. *Under the same hypothesis as in theorem* **A***, we have*

$$\lim_{|x_o|\to+\infty} |X_t(x_o)| = +\infty \quad \text{in probability.} \tag{4.4}$$

Proof. Let ψ be the function defined in the proof of theorem A:
$\psi(\xi) = \displaystyle\int_0^\xi \frac{ds}{s\rho(s)+1}$. Define the function Φ by

$$\Phi(\xi) = e^{-\psi(\xi)}.$$

In this case, Φ is a decreasing function, but $|\Phi'(\xi)| = \dfrac{\Phi(\xi)}{\xi\,\rho(\xi)+1}$ and for a large constant $C_1 > 0$,

$$\Phi''(\xi) = \frac{\Phi(\xi)\left(1+\rho(\xi)+\xi\rho'(\xi)\right)}{(\xi\rho(\xi)+1)^2} \le C_1\,\frac{\Phi(\xi)\rho(\xi)}{(\xi\rho(\xi)+1)^2}.$$

The conditions in (3.4) are satisfied. Let R, M be two positive constant such that $M > |x_o| > R$. Define

$$\hat\tau_R = \inf\Big\{t > 0,\ |X_t(x_o)| \le R\Big\} \quad \text{and} \quad \tau_M = \inf\Big\{t > 0,\ |X_t(x_o)| \ge M\Big\}.$$

By theorem **A**, we see that $\tau_M \uparrow +\infty$ as $M \uparrow +\infty$. Let $\eta_t = X_{t\wedge\hat\tau_R}$, which is a Itô process. According to notations in lemma 3.1, we have

$$e_s(\omega) = \mathbf{1}_{\{\hat\tau_R\ge s\}}\sigma(X_s), \quad f_s(\omega) = \mathbf{1}_{\{\hat\tau_R\ge s\}}b(X_s).$$

By hypothesis $(H1)$, we have

$$||e_s||^2 \le C(\xi_s\rho(\xi_s)+1), \quad |f_s| \le C(\xi_s^{1/2}\rho(\xi_s)+1).$$

Now using lemma 3.2, we get

$$\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau_M})\Big) \le \Phi(|x_0|^2)\,e^{Ct}.$$

Letting $M \to +\infty$ in above inequality, we get

$$\mathbb{E}\Big(\Phi(|X_{t\wedge\hat{\tau}_R}(x_o)|^2)\Big) \le \Phi(|x_o|^2)\, e^{C\,t}.$$

This gives that

$$P\big(\hat{\tau}_R \le t\big)\Phi(R^2) \le \mathbb{E}\Big(\Phi(|X_{t\wedge\hat{\tau}_R}(x_o)|^2)\Big) \le \Phi(|x_0|^2)\, e^{Ct}.$$

Therefore,

$$P\big(\inf_{0\le s\le t} |X_s(x_o)| \le R\big) \le e^{Ct}\ \exp\Big\{-\int_{R^2}^{|x_o|^2} \frac{ds}{s\rho(s)+1}\Big\},$$

which tends to 0 when $|x_o| \to +\infty$. □

5. Pathwise uniqueness and non contact property

The main purpose of this section is to prove theorem **B** and **C**. After establishing these results, we shall prove that the solutions $X_t(x_o)$ will define a Feller semigroup.

Proof of theorem B. Without loss of generality, we can assume the lifetime ζ of stochastic differential equation (4.1) is infinite; otherwise we have the pathwise uniqueness up to the lifetime. Let X_t and Y_t be two solutions of (4.1) having the same initial data. Consider $\eta_t = X_t - Y_t$ and $\xi_t = |\eta_t|^2$. According to notations in lemma 3.1,

$$e_t = \sigma(X_t) - \sigma(Y_t), \quad f_t = b(X_t) - b(Y_t).$$

Let $\tau = \inf\{t > 0,\ \xi_t \ge c_o^2\}$. By hypothesis $(H2)$, for $t \le \tau$,

$$||e_t||^2 \le C\,\xi_t r(\xi_t) \quad \text{and} \quad |\langle \eta_t, f_t\rangle| \le C\,\xi_t r(\xi_t).$$

According to the condition (i) on the function r, we can assume that $r(\xi) \ge 1$ for all $\xi \in]0, c_o]$. Let $\delta > 0$, we define

$$\psi_\delta(\xi) = \int_0^\xi \frac{ds}{sr(s)+\delta} \quad \text{and} \quad \Phi_\delta(\xi) = e^{\psi_\delta(\xi)}.$$

By condition (iii) on r, we see that $\Phi_0(\xi) = +\infty$ for any $\xi > 0$. We have

$$\Phi_\delta'(\xi) = \frac{\Phi_\delta(\xi)}{\xi\, r(\xi)+\delta}, \quad \Phi_\delta''(\xi) = \Phi_\delta(\xi)\frac{1 - r(\xi) - \xi\, r'(\xi)}{(\xi\, r(\xi)+\delta)^2}.$$

By conditions (i) and (ii) on the function r, there exists a large constant $C_1 > 0$ such that

$$|1 - r(\xi) - \xi\, r'(\xi)| \le C_1\, r(\xi)$$

so that

$$\Phi_\delta''(\xi) \le C_1 \frac{\Phi_\delta(\xi) r(\xi)}{(\xi\, r(\xi)+\delta)^2}.$$

The conditions in (3.7) are satisfied. Now using lemma 3.3, there exists a constant $C_2 > 0$ such that for any $t > 0$,

$$\mathbb{E}\Big(\Phi_\delta(\xi_{t\wedge\tau})\Big) \le e^{C_2 t}.$$

Letting $\delta \downarrow 0$ in the above inequality, we get

$$\mathbb{E}\Big(e^{\psi_0(\xi_{t\wedge\tau})}\Big) \le e^{C_2 t}$$

which implies that for t given,

$$\xi_{t\wedge\tau} = 0 \quad \text{almost surely.} \tag{5.1}$$

If $P(\tau < +\infty) > 0$, then for some large $T > 0$, $P(\tau \le T) > 0$. By (5.1), almost surely for all $t \in \mathbf{Q} \cap [0, T]$, $\xi_{t\wedge\tau} = 0$. It follows that on $\{\tau \le T\}$, $\xi_\tau = 0$ which is absurd by the definition of τ. Therefore $\tau = +\infty$ almost surely. So for any given t , $\xi_t = 0$ almost surely. Now by the continuity of samples, the two solutions are indistinguishable. □

Remark 5.1. In the case of $d = m = 1$, stronger results on pathwise uniqueness have been established. Namely σ was allowed to be Hölder of exponent $\ge 1/2$ (see [RY, Ch. IX-3], [IW, p.168]).

Proof of theorem C. Without loss of generality, we may assume that $|x_o - y_o| < c_o/2$. Let $0 < \varepsilon < |x_o - y_o|$ and define

$$\hat\tau_\varepsilon = \inf\{t > 0,\ |X_t(x_o) - X_t(y_o)| \le \varepsilon\}, \quad \hat\tau = \inf\{t > 0,\ X_t(x_o) = X_t(y_o)\}.$$

It is clear that $\hat\tau_\varepsilon \uparrow \hat\tau$ as $\varepsilon \downarrow 0$. Let

$$\tau = \inf\{t > 0,\ |X_t(x_o) - X_t(y_o)| \ge \frac{3}{4}c_o\}.$$

Consider $\eta_t = X_{t\wedge\hat\tau_\varepsilon}(x_o) - X_{t\wedge\hat\tau_\varepsilon}(y_o)$ and $\xi_t = |\eta_t|^2$. Then using notations in lemma 3.1, we have expressions

$$e_t = \mathbf{1}_{\{\hat\tau_\varepsilon \ge t\}}\big(\sigma(X_t(x_o)) - \sigma(X_t(y_o))\big), \quad f_t = \mathbf{1}_{\{\hat\tau_\varepsilon \ge t\}}\big(b(X_t(x_o)) - b(X_t(y_o))\big).$$

By hypothesis of (H2), for $t < \tau$,

$$||e_t||^2 \le C\,\xi_t\, r(\xi_t) \quad \text{and} \quad |\langle \eta_t, f_t\rangle| \le C\,\xi_t^{1/2}\, r(\xi_t).$$

Now define the functions

$$\psi_\delta(\xi) = \int_\xi^{c_o} \frac{ds}{sr(s) + \delta} \quad \text{and} \quad \Phi_\delta = e^{\psi_\delta(\xi)} \quad \text{for } \xi \le c_o.$$

We have $|\Phi_\delta'(\xi)| = \dfrac{\Phi_\delta(\xi)}{\xi\, r(\xi) + \delta}$ and for some large constant $C_1 > 0$,

$$\Phi_\delta''(\xi) = \Phi_\delta(\xi)\frac{1 + r(\xi) + \xi\, r'(\xi)}{(\xi\, r(\xi) + \delta)^2} \le C_1\, \frac{\Phi_\delta(\xi) r(\xi)}{(\xi\, r(\xi) + \delta)^2}.$$

So we can apply lemma 3.3 to get the inequality

$$\mathbb{E}\Big(\Phi_\delta(\xi_{t\wedge\tau})\Big) \le \Phi_\delta(\xi_o)e^{C_2 t} \quad \text{for some } \ C_2 > 0 \quad \text{and for all } \ t > 0.$$

Letting $\delta \downarrow 0$, by Fatou lemma, we get

$$\mathbb{E}\Big(\Phi_0(\xi_{t\wedge\tau})\Big) \le \Phi_0(\xi_o)e^{C_2 t}.$$

Replacing ξ_t by its expression, we have

$$\mathbb{E}\Big(\Phi_0(|X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(x_o) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(y_o)|^2)\Big) \le \Phi_0(\xi_o)e^{C_2 t}.$$

On subset $\{\hat{\tau}_\varepsilon < t\wedge\tau\}$, $|X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(x_o) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(y_o)| = \varepsilon$. From the above inequality, we obtain

$$P(\hat{\tau}_\varepsilon < t\wedge\tau)\Phi_0(\varepsilon^2) \le \Phi_0(\xi_o)e^{C_2 t},$$

or

$$P(\hat{\tau}_\varepsilon < t\wedge\tau) \le \exp\Big\{-\int_{\varepsilon^2}^{\xi_o} \frac{ds}{sr(s)}\Big\} \to 0 \quad \text{as} \ \ \varepsilon \downarrow 0.$$

Therefore $P(\hat{\tau}_\varepsilon < t\wedge\tau) = 0$ for all t. Letting $t \to \infty$ we get $P(\hat{\tau} < \tau) = 0$. Therefore, $\xi_.$ is positive almost surely on the interval $[0, \tau]$. Now define $T_0 := 0$,

$$T_1 := \tau, \qquad T_2 = \inf\big\{t > 0,\ |X_t(x_o) - X_t(y_o)| \le \frac{c_o}{2}\big\}$$

and generally

$$T_{2n} = \inf\big\{t > T_{2n-1},\ |X_t(x_o) - X_t(y_o)| \le \frac{c_o}{2}\big\},$$
$$T_{2n+1} = \inf\big\{t > T_{2n},\ |X_t(x_o) - X_t(y_o)| \ge \frac{3c_o}{4}\big\}$$

Clearly $T_n \to \infty$ almost surely as $n \to \infty$. By definition, $\xi_.$ is positive on the interval $[T_{2n-1}, T_{2n}]$. By pathwise uniqueness of solutions, X enjoys the strong Markovian property. Starting again from T_{2n} and applying the same arguments as in the first part of the proof, one can show that $\xi_.$ is positive almost surely also on the interval $[T_{2n}, T_{2n+1}]$. This completes the proof. □

Theorem 5.2. *Under the same hypothesis as in theorem C, for any $\varepsilon > 0$, we have*

$$\lim_{y_o\to x_o} P(\sup_{0\le s\le t} |X_s(y_o) - X_s(x_o)| > \varepsilon) = 0. \tag{5.2}$$

Proof. Let x_0, y_0 be such that $|y_o - x_o| < \varepsilon < c_0$, where c_0 is the parameter in definition of function r. Let $\xi_t = |X_t(y_o) - X_t(x_o)|^2$. Denote explictly

$$\tau(x_o, y_o) = \inf\{t > 0,\ \xi_t > \varepsilon^2\}.$$

Let Φ_δ be the function defined in the proof of theorem **B**. By lemma 3.3, for some large constant $C > 0$,

$$I\!E\Big(\Phi_\delta(\xi_{t\wedge\tau(x_o,y_o)})\Big) \le \Phi_\delta(\xi_o)\, e^{Ct} \quad \text{for all } \ t > 0 \quad \text{and} \quad \delta > 0.$$

Taking $\delta = |x_o - y_o|$ in the above inequality, we have $I\!E\Big(\Phi_\delta(\xi_{t\wedge\tau(x_o,y_o)})\Big) \le e^\delta\, e^{Ct}$. Hence

$$P(\tau(x_o, y_o) < t)\Phi_\delta(\varepsilon^2) \le E\Big(\Phi_\delta(\xi_{t\wedge\tau(x_o,y_o)})\Big) \le e^\delta\, e^{Ct}.$$

It follows

$$P(\sup_{0\le s\le t} |X_s(y_o) - X_s(x_o)| > \varepsilon) = P(\tau(x_o, y_o) < t) \le e^{-\psi_\delta(\varepsilon^2)}\, e^\delta\, e^{Ct} \to 0$$

as $\delta = |y_o - x_o| \to 0$. We obtain (5.2). □

Corollary 5.3. *The diffusion process* $(X_t(x))$ *given by solutions of the stochastic differential equation* (4.1) *is Feller, i.e., the associated semigroup* $(T_t, t \ge 0)$ *maps* $C_b(R^d)$ *into* $C_b(R^d)$.

Proof. It is a direct consequence of Theorem 5.2 and the definition

$$T_t f(x) = I\!E[f(X_t(x))], \qquad f \in C_b(R^d).$$

□

6. Continuous dependence of initial data

Having pathwise uniqueness under the hypothesis $(H2)$ in theorem **B**, the stochastic differential equation (4.1) has a unique solution $X_t(x_o)$. In this section, we are mainly interested in the continuous modification of $X_t(x_o)$. To this end, we need to assume that for $|x - y| \le c_o$, where c_0 is a small enough constant,

$$\begin{aligned} ||\sigma(x) - \sigma(y)||^2 &\le C\,|x - y|^2 \log\frac{1}{|x - y|}, \\ |b(x) - b(y)| &\le C\,|x - y| \log\frac{1}{|x - y|}. \end{aligned} \tag{6.1}$$

In what follows, we shall first construct the strong solution of stochastic differential equation (4.1) via Euler approximation.

Theorem 6.1. *Assume that the coefficients* σ *and* b *satisfy the condition* (6.1) *and are bounded:*

$$||\sigma(x)|| \le A, \quad |b(x)| \le B \quad \text{for all } \ x \in \mathbf{R}^d.$$

For $n \ge 1$, *define* $(X_n(t))_{n\ge 1}$ *by* $X_n(0) = x$ *and*

$$X_n(t) = X_n(k2^{-n}) + \sigma(X_n(k2^{-n}))(W_t - W_{k2^{-n}}) + b(X_n(k2^{-n}))(t - k2^{-n})$$

for $k2^{-n} \le t \le (k+1)2^{-n}$. *Then for any* $T > 0$, *almost surely,* $X_n(t)$ *converges uniformly in* $t \in [0, T]$, *to the solution* X_t *of stochastic differential equation (4.1).*

Proof. Define $\phi_n(t) = k2^{-n}$ for $t \in [k2^{-n}, (k+1)2^{-n}[$, $k \geq 0$. Then $X_n(t)$ can be expressed by

$$X_n(t) = x_o + \int_0^t \sigma\big(X_n(\phi_n(s))\big)dW_s + \int_0^t b\big(X_n(\phi_n(s))\big)ds. \tag{6.2}$$

Let $1 < a < \sqrt{2}$. Introduce the stopping time

$$\tau_n = \inf\big\{t > 0,\ |X_n(t) - X_n(\phi_n(t))| \geq a^{-n}\big\}.$$

For $t \in [k2^{-n}, (k+1)2^{-n}[$, by expression (6.2), we have

$$\begin{aligned} X_n(t) - X_n(\phi_n(t)) = &\int_0^{t-k2^{-n}} \sigma\big(X_n(\phi_n(k2^{-n}+s))\big)d\tilde{W}_s \\ &+ \int_0^{t-k2^{-n}} b\big(X_n(\phi_n(k2^{-n}+s))\big)ds \end{aligned}$$

where $\tilde{W}_s = W_{k2^{-n}+s} - W_{k2^{-n}}$. Using lemma 3.4,

$$\begin{aligned} &P\Big(\sup_{k2^{-n}\leq t<(k+1)2^{-n}} |X_n(t) - X_n(\phi_n(t))| \geq a^{-n}\Big) \\ &\leq 2d\exp\big\{-(a^{-n} - \sqrt{d}B2^{-n})^2/2dA^2 2^{-n}\big\} \\ &= 2d\exp\big\{-(\frac{2}{a^2})^n(1 - \sqrt{d}B(\frac{2}{a})^{-n})^2/2dA^2\big\}. \end{aligned}$$

Let $c = 2/a^2$, which is strictly bigger than 1. Therefore for large n,

$$P\Big(\sup_{k2^{-n}\leq t<(k+1)2^{-n}} |X_n(t) - X_n(\phi_n(t))| \geq a^{-n}\Big) \leq 2de^{-c^n/4dA^2}$$

and for integer $T > 0$,

$$P(\tau_n \leq T) \leq 2d2^nT\exp\big\{-c^n/4dA^2\big\}.$$

Hence for sufficiently large n,

$$P(\tau_n \leq T) \leq e^{-c^n/8dA^2}. \tag{6.3}$$

Now define $\eta_n(t) = X_{n+1}(t) - X_n(t)$ and $\xi_n(t) = |\eta_n(t)|^2$. Introduce the notations

$$\begin{aligned} e_s &= \sigma\big(X_{n+1}(\phi_{n+1}(s))\big) - \sigma\big(X_n(\phi_n(s))\big), \\ f_s &= b\big(X_{n+1}(\phi_{n+1}(s))\big) - b\big(X_n(\phi_n(s))\big). \end{aligned}$$

By lemma 3.1, we have

$$d\xi_n(t) = 2\langle e_t^*\eta_n(t), dW_t\rangle + 2\langle\eta_n(t), f_t\rangle dt + ||e_t||^2dt \tag{6.4}$$

and the stochastic contraction $d\xi_n(t)\cdot d\xi_n(t)$ is given by

$$d\xi_n(t)\cdot d\xi_n(t) = 4|e_t^*\eta_n(t)|^2\,dt. \tag{6.5}$$

Define the stopping time

$$\zeta_n = \inf\Big\{t > 0,\ \xi_n(t) \geq \frac{1}{n^{2\beta}}\Big\} \tag{6.6}$$

where $\beta > 1$ is a parameter. Then for $s \leq \tau_{n+1}$ and n large enough, we can use (6.1) to obtain

$$\begin{aligned} &||\sigma\big(X_{n+1}(\phi_{n+1}(s))\big) - \sigma\big(X_{n+1}(s)\big)||^2 \\ &\quad \leq C\,\big|X_{n+1}(\phi_{n+1}(s)) - X_{n+1}(s)\big|^2 \log\Big\{1/|X_{n+1}(\phi_{n+1}(s)) - X_{n+1}(s)|\Big\} \\ &\quad \leq C\, a^{-2(n+1)} \log a^{n+1} \leq C\, a^{-2n} \log a^n \end{aligned}$$

where we used the fact that $s \to s \log 1/s$ is increasing over $[0, 1/e]$. In the same way,

$$||\sigma\big(X_n(\phi_n(s))\big) - \sigma\big(X_n(s)\big)||^2 \leq C\, a^{-2n} \log a^n,$$

and for $s \leq \tau_n \wedge \tau_{n+1} \wedge \zeta_n$,

$$\begin{aligned} ||e_s||^2 \leq 2\Big\{&||\sigma\big(X_{n+1}(\phi_{n+1}(s))\big) - \sigma\big(X_{n+1}(s)\big)||^2 \\ &+||\sigma\big(X_{n+1}(s)\big) - \sigma\big(X_n(s)\big)||^2 \\ &+||\sigma\big(X_n(\phi_n(s))\big) - \sigma\big(X_n(s)\big)||^2\Big\} \\ \leq 2C\,&\{\xi_n(s) \log\{1/\xi_n(s)\} + 2a^{-2n} \log a^n\}. \end{aligned}$$

On the other hand, for $s \leq \tau_n \wedge \tau_{n+1} \wedge \zeta_n$,

$$\begin{aligned} |\langle \eta_n(t), f_t\rangle| \leq C\,|\eta_n(t)|\Big\{&|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)| \\ &\times \log\{1/|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)|\} \\ &+|X_n(\phi_n(t)) - X_n(t)|\, \log\{1/|X_n(\phi_n(t)) - X_n(t)|\} \\ &+|\eta_n(t)|\, \log\{1/|\eta_n(t)|\Big\} \\ \leq C\,\Big\{&\xi_n(t) \log\{1/\xi_n(t)\} + \frac{2}{n^\beta} a^{-n} \log a^n\Big\}. \end{aligned}$$

Choose the parameter ρ_n by

$$\rho_n = \frac{2}{n^\beta} a^{-n} \log a^n. \tag{6.7}$$

Then the conditions (3.8) in lemma 3.3 are satisfied with $C_3 = 2C$, $C_4 = C$ and δ replaced by ρ_n.

Now consider the function $\psi_n(\xi) = \displaystyle\int_0^\xi \frac{ds}{s \log(1/s) + \rho_n}$ and $\Phi_n(\xi) = e^{4\psi_n(\xi)}$. We have

$$\Phi_n'(\xi) = \frac{4\Phi_n(\xi)}{\xi \log(1/\xi) + \rho_n} \quad \text{and} \quad \Phi_n''(\xi) = \frac{4\Phi_n(\xi)(5 + \log\xi)}{(\xi \log 1/\xi + \rho_n)^2} \leq 0$$

for $\xi \leq c_o$ small enough. The conditions in (3.7) are satisfied with $C_1 = 4$, $C_2 = 0$. Let

$$\tilde{\tau}_n = \tau_n \wedge \tau_{n+1} \wedge \zeta_n.$$

For large n, $\xi_n(t \wedge \tilde{\tau}_n) \leq c_o$. Let $K = 24C$. Then by lemma 3.3, we have the following estimate

$$\mathbb{E}\Big(\Phi_n(\xi_n(t \wedge \tilde{\tau}_n))\Big) \leq e^{Kt} \quad \text{for all} \quad t,$$

from which we get

$$\mathbb{E}\Big(\mathbf{1}_{\{\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T\}} \Phi_n(\xi_n(T \wedge \tilde{\tau}_n))\Big) \leq e^{KT},$$

or

$$P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) \cdot \Phi_n(\frac{1}{n^{2\beta}}) \leq e^{KT},$$

Therefore

$$P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) \leq e^{KT} \exp\Big\{-4\int_0^{n^{-2\beta}} \frac{ds}{s \log 1/s + \rho_n}\Big\}. \tag{6.8}$$

Since $0 < \rho_n < n^{-2\beta} \log n^{2\beta}$, there exists $c_n \in]0, n^{-2\beta}[$ such that

$$c_n \log \frac{1}{c_n} = \rho_n = \frac{2}{n^\beta} a^{-n} \log a^n < a^{-n} \log a^n. \tag{6.9}$$

The function $s \to s \log \frac{1}{s}$ being increasing over $[0, 1/e]$, from (6.9), we see that

$$0 < c_n < a^{-n}. \tag{6.10}$$

Now

$$\int_0^{n^{-2\beta}} \frac{ds}{s \log \frac{1}{s} + \rho_n} \geq \int_{c_n}^{n^{-2\beta}} \frac{ds}{2s \log \frac{1}{s}} = -\frac{1}{2} \log\Big(\frac{\log n^{-2\beta}}{\log c_n}\Big).$$

According to (6.8),

$$P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) \leq e^{KT} \Big(\frac{\log n^{-2\beta}}{\log c_n}\Big)^2 \leq e^{KT} \Big(\frac{2\beta \log n}{n \log a}\Big)^2,$$

where the last inequality was deduced by (6.10). Therefore for some constant $C_1 > 0$ and n big enough,

$$P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) \leq C_2 \Big(\frac{\log n}{n}\Big)^2. \tag{6.11}$$

Now combining (6.3) and (6.11), we get

$$P(\zeta_n \leq T) \leq \frac{1}{n^\gamma} \quad \text{for some} \quad \gamma > 1,$$

or

$$P\Big(\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|^2 \ge \frac{1}{n^{2\beta}}\Big) \le \frac{1}{n^\gamma}.$$

By Borel-Cantelli lemma, almost surely

$$\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)| \le \frac{1}{n^\beta} \quad \text{for large } n.$$

It follows that the series

$$X_t := \sum_{n\ge 1}\big(X_{n+1}(t)-X_n(t)\big) + X_1(t)$$

converges uniformly in $t \in [0, T]$. It is easy to check that X_t is the solution of stochastic differential equation (4.1). □

Now we turn to establish theorem **D**. Let's begin with the following lemma.

Lemma 6.2. *Let r be a strictly positive continuous function defined on $]0, c_o]$, where $0 < c_o < 1$. Assume that the coefficients σ and b are compactly supported, say,*

$$\sigma(x) = 0 \quad \text{and} \quad b(x) = 0 \quad \text{for} \quad |x| \ge R, \tag{6.12}$$

and satisfy the hypothesis $(H2)$ in theorem **B**. *Let $p \ge 1$. If $s \to r(s)$ is decreasing on $]0, c_o]$, then there exists a constant $C_p > 0$ such that for all $|x| \le R+1$, $|y| \le R+1$,*

$$\begin{cases} ||\sigma(x)-\sigma(y)||^2 \le C_p\,|x-y|^2\, r\Big(\frac{|x-y|^{2p}}{M^p}\Big), \\ \quad |b(x)-b(y)| \;\le\; C_p\,|x-y|\, r\Big(\frac{|x-y|^{2p}}{M^p}\Big) \end{cases} \tag{6.13}$$

where $M = \dfrac{4(R+1)^2}{c_o}$.

Proof. Because of the similarity, we only prove the conclusion for b. If $|x-y| \le c_o$, by hypothesis $(H2)$,

$$|b(x)-b(y)| \le C\,|x-y| r(|x-y|^2) \le C\,|x-y|\, r\Big((\frac{|x-y|^2}{M})^p\Big), \tag{6.14}$$

as r is supposed to be decreasing. Remark that

$$\inf_{c_o\le\xi\le 2(R+1)} \xi\, r\Big(\Big[\frac{\xi^2}{M}\Big]^p\Big) \ge c_o\, r(c_o^p) > 0$$

and $\sup_{x,y}|b(x)-b(y)| \le 2||b||_\infty$, where $||b||_\infty$ denotes the uniform norm of b over $\mathbf{R}^d$. Therefore there exists a constant $C_p > 0$ such that

$$|b(x)-b(y)| \le C_p\,|x-y|\, r\Big((\frac{|x-y|^2}{M})^p\Big) \quad \text{for } |x-y| \ge c_o. \tag{6.15}$$

Combining (6.14) and (6.15), we get the result. □

Lemma 6.3. *Let σ and b be continuous functions satisfying the support condition (6.12). If the stochastic differential equation (4.1) has the pathwise uniqueness, then for any $|x_o| \leq R+1$, $|X_t(x_o)| \leq R+1$ almost surely for all $t > 0$.*

Proof. Define the stopping time

$$\tau = \inf\{t \geq 0;\ |X_t(x_o)| \geq R+1\}$$

Set $Y_t = X_{t\wedge\tau}(x)$. Then

$$Y_t = x_o + \int_0^{t\wedge\tau} \sigma(Y_s)dW_s + \int_0^{t\wedge\tau} b(Y_s)ds.$$

We have

$$\begin{aligned} I\!E\Big(\int_0^t ||\sigma(Y_s)||^2\big(\mathbf{1}_{(s<\tau)} - 1\big)^2\,ds\Big) &= I\!E\Big(\int_\tau^t ||\sigma(Y_s)||^2\,ds\Big)\\ &= I\!E\Big(\int_\tau^t ||\sigma(X_\tau)||^2\,ds\Big)\end{aligned}$$

X_τ being on the sphere of radius $R+1$, $\sigma(X_\tau) = 0$ by hypothesis (6.12), the last term in the above equality is equal to zero. Therefore $t \geq 0$,

$$\int_0^{t\wedge\tau} \sigma(Y_s)dW_s = \int_0^t \sigma(Y_s)dW_s \quad \text{and} \int_0^{t\wedge\tau} b(Y_s)ds = \int_0^t b(Y_s)ds,$$

almost surely. We see that $\{Y_t,\ t \geq 0\}$ satisfies the same stochastic differential equation as $\{X_t,\ t \geq 0\}$. By pathwise uniqueness, we conclude that $Y_t = X_t$ almost surely for all $t \geq 0$, which proves the lemma. □

Lemma 6.4. *Assume the same hypothesis as in lemma 6.2 and furthermore r satisfies the condition (i)-(iii) in theorem* **B** *and $\xi \to \xi r(\xi)$ is concave over $]0, c_o]$. Let $p \geq 2$ be an integer. For $|x_o| \leq R+1$ and $|y_o| \leq R+1$, set*

$$\eta_t = X_t(x_o) - X_t(y_o),\ \xi_t = |\eta_t|^2 \quad \text{and} \quad z_t = (\frac{\xi_t}{M})^p$$

where M is the constant defined in Lemma 6.2. Put $\varphi(t) = I\!E(z_t)$. Then for some constant C_p,

$$\varphi'(t) \leq C_p\,\varphi(t)\,r(\varphi(t)). \tag{6.16}$$

Proof. By hypothesis imposed on the function r, we can apply lemma 6.2 and 6.3 so that z_t is a bounded process. Now we shall proceed as in [F]. Let $e_t = \sigma(X_t(x_o)) - \sigma(X_t(y_o))$ and $f_t = b(X_t(x_o)) - b(X_t(y_o))$. By Itô formula and applying (3.2) and (3.3),

$$\begin{aligned} dz_t = \frac{1}{M^p}\Big(&2p\xi^{p-1}\langle e_t^*\eta_t, dW_t\rangle + 2p\xi^{p-1}\langle \eta_t, f_t\rangle dt\\ &+p\xi_t^{p-1}\,||e_t||^2\,dt + 2p(p-1)\xi_t^{p-2}|e_t^*\eta_t|^2\,dt\Big).\end{aligned}$$

By lemma 6.3, $X_t(x_o)$ and $X_t(y_o)$ are bounded by $R+1$. Now using (6.13), we have

$$\frac{\xi_s^{p-1}}{M^p}\big|\langle\eta_s, f_s\rangle\big| \le C_p\,\frac{\xi_s^p}{M^p}r\Big(\frac{\xi_s^p}{M^p}\Big) = C_p z_s r(z_s) \tag{6.17}$$

and

$$\xi_s^{p-1}||e_s||^2 \le C_p z_s r(z_s), \qquad \frac{\xi_s^{p-2}}{M^p}\big|e_s^*\eta_s\big|^2 \le C_p z_s r(z_s). \tag{6.18}$$

By concavity of $\xi \to \xi r(\xi)$ over $]0, c_o]$, we see that $\sup\limits_{0<\xi\le c_o}\big(\xi r(\xi)\big)$ is finite. Therefore the first term in the expression of dz_t is a martingale and $\varphi(t) = I\!E(z_t)$ is a derivable function with respect to t and

$$\varphi'(t) = \frac{1}{M^p}\Big(2pI\!E(\xi_t^{p-1}\langle\eta_t, f_t\rangle) + pI\!E(\xi_t^{p-1}\,||e_t||^2) \\ +2p(p-1)I\!E(\xi_t^{p-2}|e_t^*\eta_t|^2)\Big)$$

which is less, according to (6.17) and (6.18), than

$$(2p^2+p)C_p\,I\!E(z_t r(z_t)) \le (2p^2+p)C_p\,\varphi(t)r(\varphi(t)).$$

So we get the result. □

Remark 6.5. Consider $r(s) = \log\frac{1}{s}\cdot\log\log\frac{1}{s}$ for $s\ \in]0, 1/2e]$. Clearly $s \to r(s)$ is decreasing and $s \to sr(s)$ is concave over $[0, 1/2e]$. Solving (6.16), we get

$$\varphi(t) \le \exp\Big(-\big[\log\frac{1}{\varphi(0)}\big]^{e^{-C_p t}}\Big).$$

In order to apply the Kolmogorov's modification theorem, we have to find $\alpha > 0$ such that

$$\exp\Big(-\big[\log\frac{1}{\varphi(0)}\big]^{e^{-C_p t}}\Big) \le \varphi(0)^\alpha,$$

or

$$\big[\log\frac{1}{\varphi(0)}\big]^{e^{-C_p t}} \ge \alpha\log\frac{1}{\varphi(0)}$$

which is impossible when $|x_o - y_o|$ is small for any $t>0$. ∎

Proof of theorem D. We split the proof into two steps.
Step 1. Assume that σ and b are compactly supported, say,

$$\sigma(x) = 0 \quad \text{and} \quad b(x) = 0 \quad \text{for } |x| \ge R.$$

Let φ be defined as in Lemma 6.4. Solving (6.16) with $r(s) = \log\dfrac{1}{s}$, we get $\varphi(t) \le (\varphi(0))^{e^{-C_p t}}$ or explicitly

$$I\!E\Big(|X_t(x_o) - X_t(y_o)|^{2p}\Big) \le C_p\,|x_o - y_o|^{2pe^{-C_p t}}.$$

On the other hand, it is easy to see that

$$\mathbb{E}\Big(|X_t(x_o) - X_s(x_o)|^{2p}\Big) \le C_p|t-s|^p.$$

Therefore,

$$\mathbb{E}\Big(|X_t(x_o) - X_s(y_o)|^{2p}\Big) \le C_p\Big[|t-s|^p + |x_o - y_o|^{2pe^{-C_p t}}\Big]. \tag{6.19}$$

Fix $p > d+1$. Choose a constant $T_o > 0$ small enough such that $2pe^{-C_pT_0} > d+1$. It follows from (6.19) and Kolmogorov's modification theorem that there exists a version of $X_t(x_o, w)$, denoted by $\tilde{X}_t(x_o, w)$, such that $(t, x_o) \to \tilde{X}_t(x_o, w)$ is continuous over $[0, T_o] \times \{|x_o| \le R+1\}$ almost surely. But

$$X_t(x_o, w) = x_o \quad \text{if} \quad |x_o| > R.$$

We conclude that $(t, x_o) \to \tilde{X}_t(x_o, w)$ can be extended continuously to $[0, T_o]\times\mathbf{R}^d$. Let $(\theta_{T_o}w)(t) = w(t+T_o) - w(T_o)$. Define for $0 < t \le T_o$,

$$\tilde{X}_{T_o+t}(x_o, w) = \tilde{X}_t\big(\tilde{X}_{T_o}(x_o, w), \theta_{T_o}w\big).$$

Then $\tilde{X}_{T_o+\cdot}(x_o, w)$ satisfies the stochastic differential equation (4.1) driven by the Brownian motion $\theta_{T_o}w$ with the initial condition $\tilde{X}_{T_o}(x_o, w)$. By pathwise uniqueness, we see that $\tilde{X}_{T_o+t}(x_o, w) = X_{T_o+t}(x_o, w)$ almost surely for all $t \in [0, T_o]$. This means that $\tilde{X}_t(x_o, w)$ is a continuous version of $X_t(x_o, w)$ over $|0, 2T_o] \times \mathbf{R}^d$. Continuing in this way, we get a continuous version on the whole space $[0, +\infty[\times\mathbf{R}^d$.

Step 2: General case. We shall proceed as in [Pr] for locally Lipschitzian coefficients.

For $R > 0$, let $f_R(x)$ denote a smooth function with compact support satisfying

$$f_R(x) = 1 \quad \text{for } |x| \le R \quad \text{and} \quad f_R(x) = 0 \quad \text{for } |x| > R+1.$$

Define

$$\sigma_R(x) = \sigma(x)f_R(x) \quad \text{and} \quad b_R(x) = b(x)f_R(x).$$

Let $X_t^R(x, w)$ be the unique solution of the stochastic differential equation (4.1) with σ and b replaced by σ_R and b_R. Let $\tilde{X}_t^R(x, w)$ denote a continuous version of $X_t^R(x, w)$. Such a version exists according to step 1. For $K > 0$, set

$$\tau_K^R(x) = \inf\{t > 0;\ |\tilde{X}_t^R(x, w)| \ge K\},$$

$$\tau_K(x) = \inf\{t > 0;\ |X_t(x, w)| \ge K\}.$$

By pathwise uniqueness, for $|x| \le R$,

$$X_t(x, w) = \tilde{X}_t^N(x, w) \quad \text{for any} \quad N > R+1 \quad \text{and} \quad t < \tau_{R+1}^N,$$

or

$$\tau_{R+1}(x) = \tau_{R+1}^N(x) \quad \text{for any} \quad N > R+1.$$

For $|x| \leq R$, we define

$$\tilde{X}_t(x, w) = \tilde{X}_t^{R+2}(x, w) \quad \text{on} \quad [0, \tau_{R+1}^{R+2}(x)[.$$

Then $\tilde{X}_\cdot(x, w)$ is a version of $X_\cdot(x, w)$. Let us prove that $\tilde{X}_t(x, w)$ is continuous in (t, x) for almost all w. Fix x_0 with $|x_0| \leq R$. Since the lifetime of the solution is infinite, there exists $R > 0$ such that $\tau_{R+1}^{R+2}(x_0) > t + \varepsilon$ for a small ε. This implies that $\sup_{0 \leq s \leq t+\varepsilon} |\tilde{X}_s^{R+2}(x_o, w)| < R+1$. By the continuity, we can find a neighborhood $B_\delta(x_o)$ of x_o such that $\sup_{0 \leq s \leq t+\varepsilon} |\tilde{X}_s^{R+2}(x, w)| < R + 1$ or $\tau_{R+1}^{R+2}(x) > t + \varepsilon$ for all $x \in B_\delta(x_o)$. Hence, $\tilde{X}_s(x, w) = \tilde{X}_s^{R+2}(x, w)$ for all $x \in B_\delta(x_o)$ and $s \leq t + \varepsilon$, which implies that $\tilde{X}_s(x_o, w)$ is continuous at the point (t, x_o). □

7. Large deviations principle

The theory of large deviations for diffusion processes under Lipschitzian coefficients is well established (see [A], [S]). Some new developments in infinite dimensional situations are discussed in [FZ2,3], [Z1,2]. The main task of this section is again to handle the non-Lipschitzian feature. In what follows, we shall restrict ourselves on the intervall [0, 1]. Let $n \geq 1$, $\varepsilon > 0$, consider the following Euler approximation of stochastic differential equation (1.2):

$$X_n^\varepsilon(t) = x_o + \sqrt{\varepsilon} \int_0^t \sigma\big(X_n^\varepsilon(\phi_n(s))\big)\, dW_s + \int_0^t b\big(X_n^\varepsilon(\phi_n(s))\big)\, ds$$

where ϕ_n is the step function defined in the proof of theorem 6.1; namely $\phi_n(t) = k2^{-n}$ if $t \in [k2^{-n}, (k+1)2^{-n}[,\ k \geq 0$.

Lemma 7.1. *Under the same hypothesis as in theorem 6.1, for any fixed $\delta_o > 0$,*

$$\lim_{n \to +\infty} \limsup_{\varepsilon \to 0} \varepsilon \log P(\sup_{0 \leq t \leq 1} |X_t^\varepsilon - X_n^\varepsilon(t)| \geq \delta_o) = -\infty. \tag{7.1}$$

Proof. Let $\delta > 0$ be a small parameter, define the stopping time

$$\tau_n^\varepsilon = \inf\big\{t > 0,\ |X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t))| \geq \delta\big\}.$$

Replacing A^2 by εA^2, $c^n = 2^n/a^{2n}$ by $2^n \delta^2$ and T by 1 in the estimate (6.3), we get

$$P(\tau_n^\varepsilon \leq 1) \leq \exp\{-2^n \delta^2 / 8d\varepsilon A^2\}. \tag{7.2}$$

Now let $\eta_n^\varepsilon(t) = X_t^\varepsilon - X_n^\varepsilon(t)$ and $\xi_n^\varepsilon(t) = |\eta_n^\varepsilon(t)|^2$. According to notations in lemma 3.1, set

$$e_t = \sqrt{\varepsilon}\big(\sigma(X_t^\varepsilon) - \sigma(X_n^\varepsilon(\phi_n(t)))\big) \quad \text{and} \quad f_t = b(X_t^\varepsilon) - b(X_n^\varepsilon(\phi_n(t))).$$

Introduce the stopping time

$$\zeta_n^\varepsilon = \inf\{\, t > 0,\ \xi_n^\varepsilon(t) \geq \delta_o^2 \,\}.$$

We shall take $\delta_o \vee \delta < c_o < \frac{1}{e}$ where c_o is given in the condition (6.1). Then for $t < \tau_n^\varepsilon \wedge \zeta_n^\varepsilon$,

$$||e_t||^2 \leq 2\varepsilon \left\{||\sigma(X_t^\varepsilon) - \sigma(X_n^\varepsilon(t))||^2 + ||\sigma(X_n^\varepsilon(t)) - \sigma(X_n^\varepsilon(\phi_n(t)))||^2\right\}.$$

Using condition (6.1), the first term in the brace is dominated by $\frac{C}{2}\xi_n^\varepsilon(t)\log\{1/\xi_n^\varepsilon(t)\}$; while the second term is dominated by

$$\begin{aligned}
&\frac{C}{2}|X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t))|^2 \log\{1/|X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t))|^2\} \\
&\quad \leq \frac{C}{2}\delta^2 \log\{1/\delta^2\} \leq C\delta \log\{1/\delta\}.
\end{aligned}$$

Therefore for $t < \tau_n^\varepsilon \wedge \zeta_n^\varepsilon$,

$$||e_t||^2 \leq 2C\varepsilon\big(\xi_n^\varepsilon(t)\log\frac{1}{\xi_n^\varepsilon(t)} + \delta\log\frac{1}{\delta}\big). \tag{7.3}$$

In the same way for $t < \tau_n^\varepsilon \wedge \zeta_n^\varepsilon$, δ_o being chosen less than $1/e$, we have

$$\begin{aligned}
|\langle\eta_n^\varepsilon(t), f_t\rangle| &\leq \frac{C}{2}\big(\xi_n^\varepsilon(t)\log\frac{1}{\xi_n^\varepsilon(t)} + \delta_o\delta\log\frac{1}{\delta^2}\big) \\
&\leq C\left(\xi_n^\varepsilon(t)\log\frac{1}{\xi_n^\varepsilon(t)} + \delta\log\frac{1}{\delta}\right).
\end{aligned}$$

Therefore the conditions in (3.8) are satisfied with $C_3 = 2C\varepsilon, C_4 = C$ and δ replaced by

$$\rho = \delta\log\frac{1}{\delta}. \tag{7.4}$$

Now let $\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s\log(1/s)+\rho}$ for $0 < \xi \leq c_o^2 < c_o$, and

$$\Phi_{\rho,\lambda}(\xi) = e^{\lambda\psi_\rho(\xi)} \quad \text{for } \lambda > 0.$$

We have $\Phi'_{\rho,\lambda}(\xi) = \dfrac{\lambda\Phi_{\rho,\lambda}(\xi)}{\xi\log\frac{1}{\xi} + \rho}$, and

$$\begin{aligned}
\Phi''_{\rho,\lambda}(\xi) &= \lambda^2\frac{\Phi_{\rho,\lambda}(\xi)}{(\xi\log\frac{1}{\xi}+\rho)^2} + \lambda\Phi_{\rho,\lambda}(\xi)\frac{1+\log\xi}{(\xi\log\frac{1}{\xi}+\rho)^2} \\
&\leq \lambda^2\frac{\Phi_{\rho,\lambda}(\xi)}{(\xi\log\frac{1}{\xi}+\rho)^2}
\end{aligned}$$

The conditions in (3.7) are satisfied with $C_1 = \lambda, C_2 = \lambda^2$. Let $K_\lambda = 2C\lambda + 2C\lambda\varepsilon + 4C\lambda^2\varepsilon$. By lemma 3.3, we have

$$I\!E\Big(\Phi_{\rho,\lambda}(\xi_n^\varepsilon(t \wedge \tau_n^\varepsilon \wedge \zeta_n^\varepsilon))\Big) \leq e^{K_\lambda t}, \quad \text{for all } t > 0. \tag{7.5}$$

Let $t = 1$. From (7.5), we get easily the following estimate

$$P\Big(\tau_n^\varepsilon \geq 1, \zeta_n^\varepsilon \leq 1\Big) \leq e^{K_\lambda} \cdot e^{-\lambda \psi_\delta(\delta_o^2)}. \tag{7.6}$$

Now let $\lambda = \dfrac{1}{\varepsilon}$, then $K_\lambda = 2C + 6C\dfrac{1}{\varepsilon}$. Using (7.6), we have

$$\varepsilon \log P\big(\tau_n^\varepsilon \geq 1, \zeta_n^\varepsilon \leq 1\big) \leq (2C\varepsilon + 6C) - \psi_\rho(\delta_o^2).$$

Therefore for any $\rho = \delta \log \frac{1}{\delta}$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log P\big(\tau_n^\varepsilon \geq 1, \zeta_n^\varepsilon \leq 1\big) \leq 6C - \psi_\rho(\delta_o^2). \tag{7.7}$$

Now

$$\begin{aligned} P\Big(\sup_{0 \leq t \leq 1} |X_t^\varepsilon - X_n^\varepsilon(t)| \geq \delta_o\Big) &= P(\zeta_n^\varepsilon \leq 1) \\ &\leq P\big(\tau_n^\varepsilon \geq 1, \zeta_n^\varepsilon \leq 1\big) + P(\tau_n^\varepsilon \leq 1). \end{aligned}$$

By (7.2), for any $\delta > 0$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log P(\tau_n^\varepsilon \leq 1) \leq -2^n \delta / 8dA^2. \tag{7.8}$$

Using the inequality

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \varepsilon \log\big\{P(\zeta_n^\varepsilon \leq 1,\ \tau_n^\varepsilon \geq 1) + P(\tau_n^\varepsilon \leq 1)\big\} \\ &\quad \leq \limsup_{\varepsilon \to 0} \varepsilon \log P(\zeta_n^\varepsilon \leq 1,\ \tau_n^\varepsilon \geq 1) \vee \limsup_{\varepsilon \to 0} \varepsilon \log P(\tau_n^\varepsilon \leq 1) \end{aligned}$$

and according to (7.7) and (7.8), we get

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \varepsilon \log P\Big(\sup_{0 \leq t \leq 1} |X_t^\varepsilon - X_n^\varepsilon(t)| \geq \delta_o\Big) \\ &\quad \leq \big(6C - \psi_\rho(\delta_o^2)\big) \vee \big(-2^n \delta / 8dA^2\big). \end{aligned} \tag{7.9}$$

Letting first $n \to +\infty$ in (7.9), we get

$$\limsup_{\varepsilon \to 0} \varepsilon \log P\Big(\sup_{0 \leq t \leq 1} |X_t^\varepsilon - X_n^\varepsilon(t)| \geq \delta_o\Big) \leq 6C - \psi_\rho(\delta_o^2).$$

Hence as $\delta \to 0$, the right hand side in the above inequality tends to $-\infty$. The proof of (7.1) is complete. □

Let $m \geq 1$ be an integer. Fix $x \in \mathbf{R}^m$, denote by $C_x([0, 1], \mathbf{R}^m)$ the space of continuous functions from $[0, 1]$ into $\mathbf{R}^m$ with initial value x. Let $g \in C_0([0, 1], \mathbf{R}^m)$, define

$$e(g) = \begin{cases} \int_0^1 |\dot{g}(t)|^2\, dt & \text{if } g \text{ is absolutely continuous,} \\ +\infty & \text{otherwise}\ . \end{cases} \tag{7.10}$$

For an absolutely continuous function h in $C_0([0,1],\mathbf{R}^m)$, we consider the following ordinary differential equation on $\mathbf{R}^d$,

$$dX_h(t) = \Big(\sigma(X_h(t))\dot h(t) + b(X_h(t))\Big)\,dt, \quad X_h(0) = x_o \in \mathbf{R}^d. \tag{7.11}$$

The uniqueness and non-explosion of (7.11) under the hypothesis as in lemma 7.1 could be obtained in a similar way as in section 2. In this case, we have the following result.

Lemma 7.2. *Let $h \in C_0([0,1],\mathbf{R}^m)$ such that $e(h) < +\infty$ and $X_h^n(t)$ be the Euler approximation of the same scale as in lemma 7.1 for differential equation (7.11), then for any $\alpha > 0$,*

$$\lim_{n\to+\infty} \sup_{\{h; e(h)\le\alpha\}} \Big(\sup_{0\le t\le 1} |X_h^n(t) - X_h(t)|\Big) = 0. \tag{7.12}$$

Proof. Let h such that $e(h) \le \alpha$. For any $n \ge 1$, $X_h^n(t)$ satisfies the equation

$$X_h^n(t) = x_o + \int_0^t \Big[\sigma\big(X_h^n(\phi_n(s))\big)\dot h(s) + b\big(X_h^n(\phi_n(s))\big)\Big]\,ds. \tag{7.13}$$

Let $t \in [k2^{-n}, (k+1)2^{-n}]$, we have

$$X_h^n(t) - X_h^n(k2^{-n}) = \sigma\big(X_h^n(k2^{-n})\big)(h(t) - h(k2^{-n})) + b\big(X_h^n(k2^{-n})\big)(t - k2^{-n}).$$

Assume that $||\sigma(x)|| \le A,\ |b(x)| \le B$ for all $x \in \mathbf{R}^d$. Using the above expression and the fact that

$$|h(t) - h(k2^{-n})| \le 2^{-n/2}\sqrt{e(h)},$$

we get the estimate for the difference $X_h^n(t) - X_h^n(\phi_n(t))$, namely

$$|X_h^n(t) - X_h^n(\phi_n(t))| \le (A\sqrt{e(h)} + B)2^{-n/2}, \quad \text{for any } t > 0. \tag{7.14}$$

Therefore for sufficient large n, the quantity $|X_h^n(t) - X_h^n(\phi_n(t))|$ is bounded by c_o, uniformly over the subset $\{h;\ e(h) \le \alpha\}$. Let $0 < \delta < c_o \le \frac{1}{e}$. Define

$$\tau_n(h) = \inf\big\{t > 0,\ |X_h(t) - X_h^n(t)| \ge \delta\big\}.$$

As before, set $\eta_t = X_h^n(t) - X_h(t)$ and $\xi_t = |\eta_t|^2$. According to (7.13), we have

$$\begin{aligned}\frac{d\xi_t}{dt} &= 2\big\langle \eta_t, \big(\sigma(X_h^n(\phi_n(t))) - \sigma(X_h(t))\big)\dot h(t)\big\rangle \\ &\quad + 2\big\langle \eta_t, b(X_h^n(\phi_n(t))) - b(X_h(t))\big\rangle. \end{aligned}\tag{7.15}$$

For large n and $t < \tau_n(h)$ and using the hypothesis (6.1) and (7.14), we get

$$\begin{aligned}&||\sigma(X_h^n(\phi_n(t))) - \sigma(X_h(t))|| \\ &\quad \le ||\sigma(X_h^n(\phi_n(t))) - \sigma(X_h^n(t))|| + ||\sigma(X_h^n(t)) - \sigma(X_h(t))|| \\ &\quad \le C\left\{a_n(h)\sqrt{\log\frac{1}{a_n(h)}} + |\eta_t|\sqrt{\log\frac{1}{|\eta_t|}}\right\}\end{aligned}$$

where we set

$$a_n(h) = (A\sqrt{e(h)} + B)2^{-n/2}. \tag{7.16}$$

It is clear that $\sup_{e(h)\leq\alpha} a_n(h) \leq (A\sqrt{\alpha} + B)2^{-n/2}$. Therefore there exists $n_o \geq 1$ (independent of h satisfying $e(h) \leq \alpha$) such that for $n \geq n_o$ and for $t < \tau_n(h)$,

$$||\sigma(X_h^n(\phi_n(t))) - \sigma(X_h(t))|| \leq C\left\{a_n(h)\log\frac{1}{a_n(h)} + |\eta_t|\log\frac{1}{|\eta_t|}\right\}.$$

In the same way,

$$||b(X_h^n(\phi_n(t))) - b(X_h(t))|| \leq C\left\{a_n(h)\log\frac{1}{a_n(h)} + |\eta_t|\log\frac{1}{|\eta_t|}\right\}.$$

According to (7.15), we get, for $t < \tau_n(h)$,

$$\begin{aligned}
\left|\frac{d\xi_t}{dt}\right| &\leq 2C\left\{\delta a_n(h)\log\frac{1}{a_n(h)} + \frac{1}{2}\xi_t\log\frac{1}{\xi_t}\right\}(|\dot{h}(t)|+1)\\
&\leq 2C\left\{a_n(h)\log\frac{1}{a_n(h)} + \xi_t\log\frac{1}{\xi_t}\right\}(|\dot{h}(t)|+1)\\
&\leq 2C\left\{\beta_n\log\frac{1}{\beta_n} + \xi_t\log\frac{1}{\xi_t}\right\}(|\dot{h}(t)|+1)
\end{aligned} \tag{7.17}$$

where

$$\beta_n = (A\sqrt{\alpha} + B)2^{-n/2}. \tag{7.18}$$

Let $\rho_n = \beta_n\log\frac{1}{\beta_n}$. Define $\psi_n(\xi) = \int_0^\xi \frac{ds}{s\log\frac{1}{s}+\rho_n}$ and $\Phi_n(\xi) = e^{\psi_n(\xi)}$. By (7.17), we have for $t < \tau_n(h)$,

$$\left|\frac{d}{dt}\Phi_n(\xi_t)\right| \leq 2C\,\frac{\Phi_n(\xi)}{\xi_t\log\frac{1}{\xi_t}+\rho_n}\left\{\beta_n\log\frac{1}{\beta_n} + \xi_t\log\frac{1}{\xi_t}\right\}(|\dot{h}(t)|+1)$$

which is less than $2C\Phi_n(\xi_t)(|\dot{h}(t)|+1)$. It follows that

$$\Phi_n(\xi_{t\wedge\tau_n(h)}) \leq 1 + 2C\int_0^t \Phi_n(\xi_{s\wedge\tau_n(h)})(|\dot{h}(s)|+1)\,ds.$$

By Gronwall lemma, for any $t > 0$ and $n \geq n_o$, $e(h) \leq \alpha$,

$$\Phi_n(\xi_{t\wedge\tau_n(h)}) \leq \exp\left\{2C\int_0^t(|\dot{h}(s)|+1)\,ds\right\} \leq e^{2C(\sqrt{\alpha}+1)}. \tag{7.19}$$

The function $\xi \to \Phi_n(\xi)$ being increasing, by taking the suprenium over $\{h;\ e(h) \leq \alpha\}$ in (7.19), we get

$$\Phi_n(\sup_{e(h)\leq\alpha}\xi_{1\wedge\tau_n(h)}) \leq e^{2C(\sqrt{\alpha}+1)}. \tag{7.20}$$

To complete the proof, it is sufficient to prove that for any $\delta > 0$, there exists an integer N such that if $n \geq N$, then $\tau_n(h) > 1$ for all $h \in \{g;\ e(g) \leq \alpha\}$. Otherwise, there exists $\delta_o > 0$, a subsequence $\{n_k;\ k \geq 1\}$ of positive integers and $h_{n_k} \in \{g;\ e(g) \leq \alpha\}$ such that

$$\tau_{n_k}(h_{n_k}) \leq 1.$$

By (7.20),

$$\Phi_{n_k}(\delta_o^2) = \Phi_{n_k}(\xi_{1\wedge\tau_{n_k}(h_{n_k})}) \leq e^{2C(\sqrt{\alpha}+1)}. \tag{7.21}$$

As $k \to +\infty, n_k \to +\infty$, $\rho_{n_k} = \beta_{n_k} \log 1/\beta_{n_k}$ tends to 0. Therefore letting $k \to +\infty$ in (7.21), the left hand side tends to $+\infty$: it is impossible. This completes the proof of (7.12). □

Theorem 7.3. *Let σ and b be continuous functions on $\mathbf{R}^d$, taking values respectively in $\mathbf{R}^d \otimes \mathbf{R}^m$ and $\mathbf{R}^d$. Suppose that they are bounded and satisfy the hypothesis (6.1). Let $\varepsilon > 0$, consider the stochastic differential equation*

$$dX_t^\varepsilon = \sqrt{\varepsilon}\sigma(X_t^\varepsilon)\,dW_t + b(X_t^\varepsilon)\,dt, \quad X_0^\varepsilon = x_o. \tag{7.22}$$

Denote by μ_ε the law of $w \to X_\cdot^\varepsilon(w)$ on the space $C_{x_o}([0,1], \mathbf{R}^d)$. Then $\{\mu_\varepsilon,\ \varepsilon > 0\}$ satisfies a large deviation principle with the following good rate function defined by $I(f) = \inf\{\frac{1}{2}e(g);\ X_g = f\}$ for $f \in C_{x_o}([0,1], \mathbf{R}^d)$; namely

(i) for any closed subset $C \subset C_{x_o}([0,1], \mathbf{R}^d)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \leq -\inf_{f\in C} I(f),$$

(ii) for any open subset $G \subset C_{x_o}([0,1], \mathbf{R}^d)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \geq -\inf_{f\in G} I(f).$$

Proof. Let $n \geq 1$. Define the map $F_n : C_0([0,1], \mathbf{R}^m) \to C_{x_o}([0,1], \mathbf{R}^d)$ by $F_n(w)(0) = x_o$ and

$$\begin{aligned} F_n(w)(t) = {} & F_n(w)(k2^{-n}) + \sigma\big(F_n(w)(k2^{-n})\big)(w(t) - w(k2^{-n})) \\ & + b\big(F_n(w)(k2^{-n})\big)(t - k2^{-n}). \end{aligned}$$

It is clear that F_n is a continuous map from $C_0([0,1], \mathbf{R}^m)$ into $C_{x_o}([0,1], \mathbf{R}^d)$. Notice that

$$X_n^\varepsilon(t) = F_n(\sqrt{\varepsilon}w)(t).$$

By Schilder large deviations principle for $\{\sqrt{\varepsilon}w;\ \varepsilon > 0\}$ and the continuity of F_n, the large deviations principle holds for X_n^ε. Now according to lemmas 7.1 and 7.2 and theorem 4.2.23 in [DZ], we obtain the result. □

In what follows, we shall drop the hypothesis of boundedness on the coefficients σ and b.

Proposition 7.4. *Assume that σ and b satisfy the following growth condition*

$$\begin{cases} ||\sigma(x)||^2 \leq C\,(|x|^2 \log|x| + 1), \\ \ |b(x)| \ \ \leq \ C\,(|x|\log|x| + 1). \end{cases} \tag{7.24}$$

Let $(X_t^\varepsilon)_{t\geq 0}$ be a solution of stochastic differential equation (7.22). Then

$$\lim_{R\to+\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{0\leq t\leq 1} |X_t^\varepsilon| \geq R\Big) = -\infty. \tag{7.25}$$

Proof. Let $\rho \geq 1$ be a C^1-function defined on $[0,+\infty[$ such that $\rho(\xi) = \log\xi$ for $\xi \geq e^2$. According to (7.24), there exists a constant $C_1 > 0$ such that

$$\begin{cases} ||\sigma(x)||^2 \leq C_1\,(|x|^2\rho(|x|^2) + 1), \\ \ |b(x)| \ \ \leq \ C_1\,(|x|\rho(|x|^2) + 1). \end{cases} \tag{7.26}$$

Now define $\psi(\xi) = \int_0^\xi \frac{ds}{s\rho(s)+1}$ and $\Phi(\xi) = e^{\lambda\psi(\xi)}$ for $\lambda > 0$. We have $\Phi'(\xi) = \dfrac{\lambda\Phi(\xi)}{\xi\rho(\xi)+1}$ and

$$\begin{aligned} \Phi''(\xi) &= \frac{\lambda^2\Phi(\xi)}{(\xi\rho(\xi)+1)^2} - \frac{\lambda\Phi(\xi)(\rho(\xi)+\xi\rho'(\xi))}{(\xi\rho(\xi)+1)^2} \\ &\leq \frac{\lambda^2\Phi(\xi)}{(\xi\rho(\xi)+1)^2} - \frac{\lambda\Phi(\xi)\xi\rho'(\xi)}{(\xi\rho(\xi)+1)^2}. \end{aligned}$$

By choice of the function ρ, we see that $\sup_{\xi\geq 0}(|\rho'(\xi)|\,\xi) < +\infty$. Therefore there exists a constant $C_2 > 0$ such that

$$\Phi''(\xi) \leq \frac{(\lambda^2 + \lambda C_2)\Phi(\xi)}{(\xi\rho(\xi)+1)^2}. \tag{7.27}$$

Let $\eta_t^\varepsilon = X_t^\varepsilon$ and $\xi_t^\varepsilon = |\eta_t^\varepsilon|^2$. According to lemma 3.1, we set

$$e_t = \sqrt{\varepsilon}\sigma(X_t^\varepsilon), \quad f_t = b(X_t^\varepsilon).$$

By (7.26), we have the estimates

$$||e_t||^2 \leq C_1\varepsilon\,(\xi_t^\varepsilon\rho(\xi_t^\varepsilon) + 1), \quad |f_t| \leq C_1\,(\sqrt{\xi_t^\varepsilon}\rho(\xi_t^\varepsilon) + 1).$$

Define $\tau_R^\varepsilon = \inf\{t > 0,\ \xi_t^\varepsilon \geq R^2\}$ and set

$$K = (\lambda + 2\lambda^2 + 2\lambda C_2)\varepsilon\, C_1 + 4\lambda C_1. \tag{7.28}$$

Applying lemma 3.2, we have

$$I\!E\Big(\Phi(\xi_{t\wedge\tau_R^\varepsilon}^\varepsilon)\Big) \leq e^{Kt}\,\Phi(|x_o|^2). \tag{7.29}$$

Let $t = 1$ and $\lambda = \dfrac{1}{\varepsilon}$ in (7.29), we get

$$P(\tau_R^\varepsilon \leq 1)\cdot\Phi(R^2) \leq e^K\,\Phi(|x_o|^2)$$

or more precisely

$$P(\tau_R^\varepsilon \le 1) \le e^{-\frac{1}{\varepsilon}\psi(R^2)} \cdot e^{(2C+1)C_1+6C_1/\varepsilon} \cdot e^{\psi(|x_o|^2)/\varepsilon}$$

which implies that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(\tau_R^\varepsilon \le 1) \le -\psi(R^2) + 6C_1 + \psi(|x_o|^2).$$

Letting $R \to +\infty$ in the above inequality, we get (7.25). □

Remark 7.5. Let $\varepsilon = 1$ and $t = 1$ in (7.29), similarly we obtain

$$P(\sup_{0\le t\le 1} |X_t| \ge R) \le C_2\, e^{\lambda\psi(|x_o|^2)} \cdot e^{-\lambda\psi(R^2)} \quad \text{for some constant } \; C_2 > 0.$$

Since $\psi(R^2) = \int_0^{R^2} \frac{ds}{s\rho(s)+1} \ge \frac{1}{2}\int_{e^2}^{R^2} \frac{ds}{s\log s} = \frac{1}{2}\log\log R$, we have

$$P(\sup_{0\le t\le 1} |X_t| \ge R) \le C_2\, e^{\lambda\psi(|x_o|^2)} \cdot (\log R)^{-\lambda/2} \quad \text{for any } \; \lambda > 0. \;\blacksquare \tag{7.30}$$

Now for $R > 0$, define $m_R = \sup\{|b(x)|, ||\sigma(x)||; |x| \le R\}$ and $b_i^R = (-m_R - 1) \vee b_i \wedge (m_R + 1)$, $\sigma_{i,j}^R = (-m_R - 1) \vee \sigma_{i,j} \wedge (m_R + 1)$, $1 \le i \le d, 0 \le j \le m$. Put $b_R = (b_1^R, b_2^R, ..., b_d^R)$ and $\sigma_R = (\sigma_{i,j}^R)_{1\le i\le d, 1\le j\le m}$. Then for $|x| \le R$,

$$b_R(x) = b(x), \quad \sigma_R(x) = \sigma(x).$$

and b_R, σ_R satisfy (6.1) and (7.24) with the same constants.

Let $X_R^\varepsilon(\cdot)$ be the solution to

$$dX_R^\varepsilon(t) = \sqrt{\varepsilon}\sigma_R(X_R^\varepsilon(s))dW_s + b_R(X_R^\varepsilon(s))ds, \quad X_R^\varepsilon(0) = x_o. \tag{7.31}$$

For h with $e(h) < \infty$, let $X_R^h(t)$ be the solution to

$$dX_R^h(t) = \Big[\sigma_R(X_R^h(t))\, \dot h(t) + b_R(X_R^h(t))\Big] dt, \quad X_R^h = x_o. \tag{7.32}$$

For $f \in C_{x0}\big([0, 1], R^d\big)$, define

$$I_R(f) = \inf\left\{\frac{1}{2}e(g);\; X_R^h = f\right\}, \quad \text{and} \quad I(f) = \inf\left\{\frac{1}{2}e(g);\; X_h = f\right\}$$

where X_h is the solution of differential equation (7.11). If $\sup_{0\le t\le 1} |X_h(t)| \le R$, then X_h solves the differential equation (7.32) up to the time $t = 1$. By uniqueness of solutions, we see that $X_h(t) = X_R^h(t)$ for $0 \le t \le 1$. Therefore for $f \in C_{x_o}([0, 1], \mathbf{R}^d)$ satisfying $\sup_{0\le t\le 1} |f(t)| \le R$, $I(f) = I_R(f)$.

Lemma 7.6. *Let $\alpha > 0$. Then under the condition (7.24),*

$$\sup_{\{h; e(h)\le\alpha\}} \sup_{0\le t\le 1} |X_h(t)| < +\infty. \tag{7.33}$$

Proof. Let $\xi_h(t) = |X_h(t)|^2$. Then for some constant $C > 0$,

$$\left|\frac{d\xi_h(t)}{dt}\right| \le C\,\big(\xi_h(t)\rho(\xi_h(t)) + 1\big)(|\dot{h}(t)| + 1)$$

where ρ is the function defined on $[0, +\infty[$, considered in the proof of proposition 7.4. Introduce again the function $\psi(\xi) = \int_0^\xi \frac{ds}{s\rho(s) + 1}$ and $\Phi(\xi) = e^{\psi(\xi)}$. Then

$$\left|\frac{d\Phi(\xi_h(t))}{dt}\right| \le C\,(|\dot{h}(t)| + 1)\cdot\Phi(\xi_h(t))$$

from which we get

$$\Phi(\xi_h(t)) \le \Phi(|x_o|^2) + C\int_0^t (|\dot{h}(s)| + 1)\cdot\Phi(\xi_h(s))\,ds.$$

Again by Gronwall lemma, we obtain

$$\Phi(\xi_h(t)) \le \Phi(|x_o|^2)\cdot e^{C(\sqrt{\alpha}+1)} \quad \text{for } e(h) \le \alpha. \tag{7.34}$$

Φ being increasing, taking the suprenium over $\{h;\ e(h) \le \alpha\}$ and $t \in [0, 1]$ in (7.34), we get

$$\Phi\big(\sup_{e(h)\le\alpha}\sup_{0\le t\le 1}\xi_h(t)\big) \le \Phi(|x_o|^2)\,e^{C(\sqrt{\alpha}+1)}.$$

It follows that $\sup_{e(h)\le\alpha}\sup_{0\le t\le 1}\xi_h(t) < +\infty$. □

Lemma 7.7. *The rate function I is a good rate function, namely for any $\beta > 0$, the level set $Q_\beta = \{f;\ I(f) \le \beta\}$ is compact.*

Proof. Let (f_n) be a sequence in $Q_\beta : I(f_n) \le \beta$. Then there exist $h_n \in C_o([0, 1], \mathbf{R}^m)$ such that $\frac{1}{2}e(h_n) \le \beta + \frac{1}{n}$ and $X_{h_n} = f_n$. Let $\alpha = \beta + 1$. By (7.33), for some $R > 0$, we have

$$X_{h_n} = X_R^{h_n} = f_n.$$

It is known that there exists a $h \in C_o([0, 1], \mathbf{R}^m)$ such that $\frac{1}{2}e(h) \le \beta$ and $X_R^{h_n}$ converges to X_R^h uniformly over $[0, 1]$, up to a subsequence. Therefore $f = X_h \in Q_\beta$. □

Now we are ready to prove the main result **E**.

Proof of theorem E. Recall that the process $X_R^\varepsilon(\cdot)$ is defined by (7.31). Let μ_ε^R denote the law of $X_R^\varepsilon(\cdot)$ on $C_{x_0}([0, 1], R^d)$. By theorem 7.3, $\{\mu_\varepsilon^R, \varepsilon > 0\}$ satisfies a large deviation principle with good rate function $I_R(\cdot)$. Note that μ_ε^R and μ_ε coincide on the ball $\{f; \sup_{0\le t\le 1}|f(t)| \le R\}$. For $R > 0$ and a closed subset $C \subset C_{x_0}([0, 1], R^d)$, set $C_R = C \cap \{f; \sup_{0\le t\le 1}|f(t)| \le R\}$. Then,

$$\begin{aligned}\mu_\varepsilon(C) &\le \mu_\varepsilon(C_R) + P(\sup_{0\le t\le 1}|X_t^\varepsilon| > R)\\ &= \mu_\varepsilon^R(C_R) + P(\sup_{0\le t\le 1}|X_t^\varepsilon| > R).\end{aligned}$$

By large deviation principle for $\{\mu_\varepsilon^R, \varepsilon > 0\}$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon^R(C_R) \le -\inf_{f\in C_R}\{I_R(f)\} \le -\inf_{f\in C} I(f).$$

Hence

$$\limsup_{\varepsilon\to 0}\ \varepsilon \log \mu_\varepsilon(C) \le \big(-\inf_{f\in C} I(f)\big) \vee \big(\limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\le t\le 1} |X^\varepsilon(t)| > R)\big).$$

Applying (7.25) and letting $R \to \infty$, we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \le -\inf_{f\in C} I(f),$$

which is the upper bound.

Let G be an open subset of $C_{x_0}([0, 1], R^d)$. Fix any $\phi_0 \in G$. Choosing $\delta > 0$ such that $B(\phi_0, \delta) = \{f; \sup_{0\le t\le 1} |f(t) - \phi_0(t)| \le \delta\} \subset G$. Let $R = \sup_{0\le t\le 1} |\phi_0(t)| + \delta$. Since

$$B(\phi_0, \delta) \subset \{f;\ \sup_{0\le t\le 1} |f(t)| \le R\},$$

we have

$$\begin{aligned} -I(\phi_0) = -I_R(\phi_0) &\le \liminf_{\varepsilon\to 0}\ \varepsilon \log \mu_\varepsilon^R\Big(B(\phi_0, \delta)\Big) \\ &= \liminf_{\varepsilon\to 0}\ \varepsilon \log \mu_\varepsilon\Big(B(\phi_0, \delta)\Big) \\ &\le \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G). \end{aligned}$$

Since ϕ_0 is arbitrary, it follows that

$$-\inf_{f\in G} I(f) \le \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G),$$

which is the lower bound. □

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