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On singularity of energy measures on self-similar sets

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Abstract. We provide general criteria for energy measures of regular Dirichlet forms on self-similar sets to be singular to Bernoulli type measures. In particular, every energy measure is proved to be singular to the Hausdorff measure for canonical Dirichlet forms on 2-dimensional Sierpinski carpets.

1. Introduction

During the development of the analysis on self-similar sets, or fractals, various anomalous properties have been observed. For example, a typical behavior of transition densities $p(t, x, y)$ of diffusion processes on good fractal sets can be described as

$$c_{1.1}t^{-d_s/2}\exp\left(-c_{1.2}(d(x,y)^{d_w}/t)^{-1/(d_w-1)}\right) \le p(t,x,y) \le c_{1.3}t^{-d_s/2}\exp\left(-c_{1.4}(d(x,y)^{d_w}/t)^{-1/(d_w-1)}\right), \qquad (1.1)$$

where the spectral dimension d_s is different from the fractal dimension, and the walk dimension d_w is greater than 2. This is in contrast to the case of symmetric diffusion processes on $\mathbb{R}^d$ associated with uniformly elliptic operators of divergence form. The estimates of the transition densities of these processes are similar to (1.1) but with $d_s = d$ and $d_w = 2$. Moreover, the domains of Dirichlet forms associated with canonical symmetric diffusions on a broad class of fractals are represented by Lipschitz spaces with a differential order of $d_w/2$ ([14, 18, 26]), while that of the Brownian motion on $\mathbb{R}^d$ is given by a first order Sobolev space.

In this paper, we demonstrate another anomalous property concerning the energy measures associated with regular Dirichlet forms on self-similar sets. In the case of the Brownian motion on $\mathbb{R}^d$, the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on

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$L^2(\mathbb{R}^d, dx)$ is given by

$$\mathcal{E}(f, g) = \frac{1}{2}\int_{\mathbb{R}^d} (\nabla f, \nabla g)\, dx, \quad f, g \in \mathcal{F} = H^1(\mathbb{R}^d),$$

and the energy measure $\mu_{\langle f \rangle}$, where $f \in \mathcal{F}$, is equal to $|\nabla f|^2\, dx$. This, in particular, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. It has been suspected that energy measures are singular to the Hausdorff measure for generic fractals due to the lack of differential structures. This was first proved by Kusuoka [19] for a class of fractals including Sierpinski gaskets. Later, Ben-Bassat, Strichartz, and Teplyaev [9] proved the singularity for generic p.c.f. self-similar sets. In this article, we treat further generalized self-similar sets including infinitely ramified ones, and provide criteria for energy measures of self-similar Dirichlet forms to be singular to Bernoulli type measures. Broadly speaking, main theorems imply that singularity to the canonical Bernoulli measure is verified if the elliptic Harnack inequality holds and the walk dimension is greater than 2. In particular, every energy measure is proved to be singular to the Hausdorff measure for standard Dirichlet forms on 2-dimensional Sierpinski carpets. This partially answers the questions posed in [2].

The organization of this paper is as follows. In Section 2, we set up a framework and state the main claims. In Section 3, we prove a series of lemmas. In Section 4, the main theorems are proved. Certain examples are presented in the last section.

Throughout this article, $c_{i,j}$ denotes a positive constant appearing in Section i for the first time.

2. Framework

First, we introduce a self-similar structure. Let K be a compact metrizable topological space, S a finite set with $\#S \geq 2$, and $\psi_i : K \to K$ a continuous injective map for $i \in S$. Set $\Sigma = S^{\mathbb{N}}$. For $i \in S$, define a shift operator $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\cdots) = i\omega_1\omega_2\cdots$. Let us suppose that there exists a continuous surjective map $\pi : \Sigma \to K$ such that $\psi_i \circ \pi = \pi \circ \sigma_i$ for each $i \in S$. We call $(K, S, \{\psi_i\}_{i\in S})$ a self-similar structure, following Kigami [15].

Define $W_0 = \{\emptyset\}$, $W_m = S^m$ for $m \in \mathbb{N}$, and $W_* = \bigcup_{m\in\mathbb{Z}_+} W_m$. When $w \in W_m$, we write $|w| = m$ and call m the length of w. For $w = w_1w_2\cdots w_m \in W_m \subset W_*$, we define $\psi_w = \psi_{w_1}\circ\psi_{w_2}\circ\cdots\circ\psi_{w_m}$, $\sigma_w = \sigma_{w_1}\circ\sigma_{w_2}\circ\cdots\circ\sigma_{w_m}$, $K_w = \psi_w(K)$, and $\Sigma_w = \sigma_w(\Sigma)$. Here, we use the convention that $\psi_\emptyset$ and $\sigma_\emptyset$ represent the identity maps. When $w = w_1w_2\cdots w_m \in W_m$ and $w' = w'_1w'_2\cdots w'_{m'} \in W_{m'}$, ww' denotes $w_1w_2\cdots w_m w'_1 w'_2\cdots w'_{m'} \in W_{m+m'}$.

Define a subset $\mathcal{A}$ of $\mathbb{R}^S$ by

$$\mathcal{A} = \left\{\theta = \{\theta_i\}_{i\in S} \in \mathbb{R}^S \;\middle|\; \theta_i > 0 \text{ for every } i \in S \text{ and } \sum_{i\in S}\theta_i = 1\right\}.$$

Given $\theta \in \mathcal{A}$, let λ_θ denote the Bernoulli measure on Σ with weight θ. Specifically, λ_θ is a unique Borel probability measure such that $\lambda_\theta(\Sigma_w) = \theta_{w_1}\theta_{w_2}\cdots\theta_{w_m}$

for every $w = w_1 w_2 \cdots w_m \in W_m \subset W_*$. We will use the notation $\theta_w = \theta_{w_1}\theta_{w_2}\cdots\theta_{w_m}$ for $\theta \in \mathbb{R}^S$ and $w \in W_m$, and $\theta_\emptyset = 1$ for the remainder of the paper. Define a Borel measure μ_θ on K by $\mu_\theta = \pi_*\lambda_\theta$, that is, $\mu_\theta(A) = \lambda_\theta(\pi^{-1}(A))$. It is called a self-similar measure on K with weight θ.

We impose the following assumption.

(A1) For every $x \in K$, $\pi^{-1}(x)$ is a finite set.

Let $K^b = \{x \in K \mid \#(\pi^{-1}(x)) > 1\}$. According to Theorem 1.4.5 and Lemma 1.4.7 in [17], every $\theta \in \mathcal{A}$ satisfies $\mu_\theta(K^b) = 0$ and $\mu_\theta(K_w) = \theta_w$ for all $w \in W_*$.

We fix $p \in \mathcal{A}$ and abbreviate λ_p and μ_p as λ and μ, respectively. Assume that we are given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mu) = L^2(K, \mu)$. Let $\{T_t\}_{t>0}$ denote the associated Markovian semigroup on $L^2(\mu)$. For any $\alpha \in [1, \infty)$, $\{T_t\}_{t>0}$ extends (or is restricted) to a strongly continuous contraction semigroup on $L^\alpha(\mu)$. Furthermore, the generator of $\{T_t\}_{t>0}$ on $L^\alpha(\mu)$ is denoted by $\mathcal{L}^{(\alpha)}$ with domain $\mathrm{Dom}(\mathcal{L}^{(\alpha)})$. Note that $\mathrm{Dom}(\mathcal{L}^{(\alpha_2)}) \subset \mathrm{Dom}(\mathcal{L}^{(\alpha_1)})$ and $\mathcal{L}^{(\alpha_1)}|_{\mathrm{Dom}(\mathcal{L}^{(\alpha_2)})} = \mathcal{L}^{(\alpha_2)}$ when $\alpha_1 \le \alpha_2$. We write $\mathcal{L}$ for $\mathcal{L}^{(2)}$, which is a nonpositive self-adjoint operator on $L^2(\mu)$. Set $\mathcal{F}_b = \mathcal{F} \cap L^\infty(\mu)$, $\mathcal{F}_+ = \{f \in \mathcal{F} \mid f \ge 0\ \mu\text{-a.e.}\}$, and $\mathcal{F}_{b,+} = \mathcal{F}_b \cap \mathcal{F}_+$. We equip $\mathcal{F}$ with norm $\|f\|_{\mathcal{F}} = (\mathcal{E}(f, f) + \|f\|^2_{L^2(\mu)})^{1/2}$. We further impose the following assumptions.

(A2) $1 \in \mathcal{F}$ and $\mathcal{E}(1, 1) = 0$.

(A3) (Self-similarity) $\psi_i^* f \in \mathcal{F}$ for every $f \in \mathcal{F}$ and $i \in S$, and there exists $s = \{s_i\}_{i \in S}$ with $s_i > 0$ for all $i \in S$ such that

$$\mathcal{E}(f, f) = \sum_{i \in S} s_i \mathcal{E}(\psi_i^* f, \psi_i^* f), \qquad f \in \mathcal{F}. \tag{2.1}$$

Here, $\psi_i^* f$ is a pullback of f by the map ψ_i.

(A4) (Spectral gap) There exists a constant $c_N > 0$ such that

$$\left\| f - \int_K f\, d\mu \right\|^2_{L^2(\mu)} \le c_N \mathcal{E}(f, f) \qquad \text{for all } f \in \mathcal{F}. \tag{2.2}$$

According to the polarization argument and by repeatedly using (2.1), any $f, g \in \mathcal{F}$ and $m \in \mathbb{N}$ satisfies

$$\mathcal{E}(f, g) = \sum_{w \in W_m} s_w \mathcal{E}(\psi_w^* f, \psi_w^* g). \tag{2.3}$$

Let $\tilde{f}$ denote a quasi-continuous Borel modification of $f \in \mathcal{F}$. For each $f \in \mathcal{F}$, let $\mu_{\langle f \rangle}$ denote the energy measure of f with respect to $(\mathcal{E}, \mathcal{F})$. When $f \in \mathcal{F}_b$, $\mu_{\langle f \rangle}$ is a unique smooth Borel measure on K satisfying

$$\int_K \tilde{g}\, d\mu_{\langle f \rangle} = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \qquad g \in \mathcal{F}_b.$$

The following inequalities are also useful (see e.g. [11, p. 111]). For $f_1, f_2 \in \mathcal{F}$, and a nonnegative Borel function g on K,

$$\left| \left(\int_K g\, d\mu_{\langle f_1 \rangle} \right)^{1/2} - \left(\int_K g\, d\mu_{\langle f_2 \rangle} \right)^{1/2} \right| \le \left(\int_K g\, d\mu_{\langle f_1 - f_2 \rangle} \right)^{1/2}; \tag{2.4}$$

in particular,

$$\left|\mu_{\langle f_1\rangle}(A)^{1/2} - \mu_{\langle f_2\rangle}(A)^{1/2}\right| \le \mu_{\langle f_1 - f_2\rangle}(A)^{1/2} \tag{2.5}$$

for any Borel subset A of K.

To state additional assumptions, we introduce a number of other notations. When $w \in W_*$ and $f \in L^2(\mu)$, we define $\Psi_w f \in L^2(\mu)$ by

$$\Psi_w f(x) = \begin{cases} f(\psi_w^{-1}(x)) & \text{if } x \in K_w \\ 0 & \text{otherwise} \end{cases}.$$

Since $\mu(K^b) = 0$, it should be noted that $\psi_{w'}^* \Psi_w f = 0$ μ-a.e. if w and w' are different elements in some W_m.

For a measurable function f on K, supp f denotes the smallest closed set F such that $f = 0$ μ-a.e. on $K \setminus F$. We fix a Borel subset K^∂ of K, which is regarded as a boundary of K. (In most cases, K^∂ denotes the image of the post-critical set by π; see Section 5.) Set

$$\mathcal{F}_0 = \{f \in \mathcal{F} \mid \operatorname{supp} f \cap K^\partial = \emptyset\}.$$

We impose the following assumptions.

(A5) $K \setminus K^\partial$ has a nonempty interior.
(A6) $\Psi_i f \in \mathcal{F}_0$ for any $f \in \mathcal{F}_0$ and $i \in S \subset W_*$.

By (A6), it is easy to prove that $\Psi_w f \in \mathcal{F}_0$ for any $f \in \mathcal{F}_0$ and $w \in W_*$. Denote the closure of $\mathcal{F}_0$ in $\mathcal{F}$ by $\mathcal{F}_D$.

Let

$$\mathcal{H} = \{h \in \mathcal{F} \mid \mathcal{E}(h,h) \le \mathcal{E}(h+f, h+f) \text{ for all } f \in \mathcal{F}_D\},$$
$$\mathcal{H}_+ = \{h \in \mathcal{H} \mid h \ge 0 \ \mu\text{-a.e.}\}.$$

We term the elements in $\mathcal{H}$ harmonic functions. As is seen later, $\mathcal{H}$ is a closed subspace of $\mathcal{F}$. The following is a key condition to the main theorem.

(C) There exists some $u \in W_*$ such that $\psi_u^* : \mathcal{H} \to \mathcal{F}$ is a compact operator.

Theorem 2.1. *Assume* (C). *Then, for every $q \in \mathcal{A}$, either of the following holds*:

(i) *There exists some $h \in \mathcal{H}$ such that $\mu_{\langle h\rangle} = \mu_q$.*
(ii) *For every $f \in \mathcal{F}$, $\mu_{\langle f\rangle}$ and μ_q are mutually singular.*

We will provide the sufficient conditions for (C). Consider the following conditions.

(EHI) (Elliptic Harnack inequality on a certain subset) There exist $v \in W_*$ and $c_{2.1} > 0$ such that for any $h \in \mathcal{H}_+$,

$$\mu\text{-}\operatorname*{esssup}_{x \in K_v} h(x) \le c_{2.1}\, \mu\text{-}\operatorname*{essinf}_{x \in K_v} h(x).$$

(D) There exist $v \in W_*$ and $c_{2.2} > 0$ such that

$$\mu\text{-}\operatorname*{esssup}_{x \in K_v} |h(x)| \le c_{2.2} \|h\|_{\mathcal{F}} \qquad \text{for every } h \in \mathcal{H}.$$

(R) $\mathcal{L}$ has compact resolvents. In other words, $\mathcal{F}$ is compactly imbedded in $L^2(\mu)$.

Theorem 2.2. (EHI) *implies* (D)*, and* (D)+(R) *implies* (C).

We will also provide criteria to guarantee that case (i) of Theorem 2.1 does not occur.

Theorem 2.3. *Suppose that every Borel subset A, B of K with positive μ-measure satisfies*

$$\varlimsup_{t \downarrow 0} t \log \int_K T_t 1_A \cdot 1_B \, d\mu \ge 0. \tag{2.6}$$

Then, if $f \in \mathcal{F}$ satisfies $\mu_{\langle f \rangle} \ll \mu (= \mu_p)$ and the Radon-Nikodym derivative $\frac{d\mu_{\langle f \rangle}}{d\mu}$ belongs to $L^\infty(\mu)$, then f will be a constant function. In particular, case (i) of Theorem 2.1 does not occur for $q = p$.

According to Lemma 3.12 below and Theorem 1.1 in [13], $\lim_{t\downarrow 0} t \log \int_K T_t 1_A \cdot 1_B \, d\mu$ always exists and is less than or equal to 0. Therefore, (2.6) is equivalent to the condition $\lim_{t\downarrow 0} t \log \int_K T_t 1_A \cdot 1_B \, d\mu = 0$.

The assumption in Theorem 2.3 holds if T_t has an integral kernel $p(t, x, y)$ satisfying

$$p(t, x, y) \ge c_{2.3} t^{-d_s/2} \exp\left(-c_{2.4} t^{-1/(d_w - 1)}\right), \quad t \in (0, 1] \text{ and } x, y \in K$$

for some positive constants $c_{2.3}$, $c_{2.4}$, and d_s, and some $d_w > 2$. This is because we have

$$\begin{aligned} \int_K T_t 1_A \cdot 1_B \, d\mu &= \iint_{A \times B} p(t, x, y) \, \mu(dx)\mu(dy) \\ &\ge \mu(A)\mu(B) c_{2.3} t^{-d_s/2} \exp\left(-c_{2.4} t^{-1/(d_w - 1)}\right), \end{aligned}$$

which implies (2.6) since $1/(d_w - 1) < 1$.

To state another criterion, we define distance-like functions as follows. Let $m \in \mathbb{N}$. For $x, x' \in K$, denote $x \overset{m}{\sim} x'$ if there exist $w, w' \in W_m$ such that $x \in K_w$, $x' \in K_{w'}$, and $K_w \cap K_{w'} \ne \emptyset$. Set

$$d_m(x, x') = \min\{j \in \mathbb{N} \mid x_i \overset{m}{\sim} x_{i+1},\ i = 0, 1, \dots, j-1,\ x_0 = x,\ x_j = x'\}. \tag{2.7}$$

We introduce the following condition.

(O) There exist some $z \in W_*$, $N \in \mathbb{N}$, and $C > 0$ such that for any $w \in W_*$ with $K_w \subset K_z$, there exists $A(w) \subset W_{|w|}$ with $\#A(w) \le N$ satisfying the condition that every $h \in \mathcal{H}$ has a continuous modification on K_z and that

$$\left(\mu\text{-}\operatorname*{esssup}_{x\in K_w} h(x) - \mu\text{-}\operatorname*{essinf}_{x\in K_w} h(x)\right)^2 \le C \sum_{\xi\in A(w)} \mathcal{E}(\psi_\xi^* h, \psi_\xi^* h).$$

Theorem 2.4. *Assume* (O). *Let* $q = \{q_i\}_{i\in S} \in \mathcal{A}$ *and* $r = \min_{i\in S} s_i/q_i$. *Suppose*

$$\varliminf_{m\to\infty} \frac{d_m(x, y)}{r^{m/2}} = 0$$

for each $x, y \in K$. *Then, if* $f \in \mathcal{H}$ *satisfies* $\mu_{\langle f\rangle} \ll \mu_q$ *and* $\frac{d\mu_{\langle f\rangle}}{d\mu_q}$ *belongs to* $L^\infty(\mu_q)$*, then* f *will be constant on* K_z*, where* z *is given in* (O). *In particular, case (i) of Theorem 2.1 does not occur for such* q.

As is seen in Section 5, with regard to typical examples such as nested fractals and Sierpinski carpets, the set of q satisfying the assumption of Theorem 2.4 is a neighborhood of p in $\mathcal{A}$, where p is given such that μ_p is the normalized Hausdorff measure of the self-similar set.

The following proposition is an easy application of singularity of energy measures.

Proposition 2.1. *Suppose* $f \in \mathrm{Dom}(\mathcal{L})$, $f^2 \in \mathrm{Dom}(\mathcal{L}^{(1)})$, *and* $\mu_{\langle f\rangle} \perp \mu$. *Then,* f *is a constant function. In particular, if* $\mu_{\langle f\rangle} \perp \mu$ *for any* $f \in \mathcal{F}$, *then no nonconstant function* f *can satisfy both* $f \in \mathrm{Dom}(\mathcal{L})$ *and* $f^2 \in \mathrm{Dom}(\mathcal{L}^{(1)})$.

3. Preliminary lemmas

Lemma 3.1. *For any* $w \in W_*$, ψ_w^* *is a bounded operator on* $\mathcal{F}$. *To be more precise,* $\|\psi_w^* f\|_{\mathcal{F}} \le (p_w^{-1/2} \vee s_w^{-1/2})\|f\|_{\mathcal{F}}$.

Proof. This is evident from the inequalities $\|\psi_w^* f\|^2_{L^2(\mu)} \le p_w^{-1}\|f\|^2_{L^2(\mu)}$ and $\mathcal{E}(f, f) \ge s_w \mathcal{E}(\psi_w^* f, \psi_w^* f)$ due to the self-similarities of μ and $\mathcal{E}$. □

Lemma 3.2. *There exists a constant* $c_{3.1}$ *such that* $\|f\|^2_{\mathcal{F}} \le c_{3.1}\mathcal{E}(f, f) + \left(\int_K f\,d\mu\right)^2$ *for all* $f \in \mathcal{F}$.

Proof. The claim follows with $c_{3.1} = c_N + 1$ from the identity $\|f\|^2_{L^2(\mu)} = \|f - \int_K f\,d\mu\|^2_{L^2(\mu)} + \left(\int_K f\,d\mu\right)^2$ and inequality (2.2). □

Lemma 3.3. $\Psi_w f \in \mathcal{F}_D$ *for any* $f \in \mathcal{F}_D$ *and* $w \in W_*$.

Proof. Take a sequence $\{f_n\}$ from $\mathcal{F}_0$ converging to f in $\mathcal{F}$. Then, $\Psi_w f_n \in \mathcal{F}_0$ by (A6), and we have

$$\mathcal{E}(\Psi_w f_n - \Psi_w f_k, \Psi_w f_n - \Psi_w f_k) = s_w \mathcal{E}(f_n - f_k, f_n - f_k)$$

and

$$\|\Psi_w f_n - \Psi_w f_k\|_{L^2(\mu)}^2 = p_w \|f_n - f_k\|_{L^2(\mu)}^2.$$

This implies that $\{\Psi_w f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}$. Since its limit should be $\Psi_w f$, we conclude that $\Psi_w f$ belongs to $\mathcal{F}_D$. □

Lemma 3.4. *If $f \in \mathcal{F}_D$ and $g \in \mathcal{F}_+$, then $f \wedge g \in \mathcal{F}_D$.*

Proof. When $f \in \mathcal{F}_0$, we have $f \wedge g \in \mathcal{F}_0$ since $\operatorname{supp}(f \wedge g) \subset \operatorname{supp} f$.

When $f \in \mathcal{F}_D$, take a sequence $\{f_n\}$ from $\mathcal{F}_0$ converging to f in $\mathcal{F}$ and μ-a.e. It is easy to see that $\{f_n \wedge g\} \subset \mathcal{F}_0$ is bounded in $\mathcal{F}$. Therefore, we can take a subsequence converging weakly to $f \wedge g$ in $\mathcal{F}$, which implies that $f \wedge g \in \mathcal{F}_D$. □

Lemma 3.5. *There exists a constant $c_D > 0$ such that*

$$\|f\|_{L^2(\mu)}^2 \le c_D \mathcal{E}(f, f) \quad \textit{for all } f \in \mathcal{F}_D.$$

In particular, $1 \notin \mathcal{F}_D$.

Proof. Take an arbitrary $i \in S = W_1$ and let $a = \mu(K \setminus K_i) > 0$. Let b be a positive number and $f \in \mathcal{F}_D$ with $\mathcal{E}(f, f) = 1$. From Chebyshev's inequality, (2.1), (2.2), and Lemma 3.3, we have

$$\begin{aligned}
\mu\left(\left\{\left|\Psi_i f - \int_K \Psi_i f\,d\mu\right| > b\right\}\right) &\le \frac{1}{b^2}\left\|\Psi_i f - \int_K \Psi_i f\,d\mu\right\|_{L^2(\mu)}^2 \\
&\le \frac{c_N}{b^2}\mathcal{E}(\Psi_i f, \Psi_i f) \\
&= \frac{c_N s_i}{b^2}\mathcal{E}(f, f) = \frac{c_N s_i}{b^2}. \qquad (3.1)
\end{aligned}$$

Set $b = (2c_N s_i/a)^{1/2}$. Then, the last term of (3.1) is less than a. Since $\Psi_i f = 0$ on $K \setminus K_i$, $\left|\int_K \Psi_i f\,d\mu\right|$ must be less than or equal to b. Therefore, we have

$$\left|\int_K f\,d\mu\right| = \frac{1}{p_i}\left|\int_K \Psi_i f\,d\mu\right| \le \frac{b}{p_i}$$

and

$$\begin{aligned}
\|f\|_{L^2(\mu)}^2 &= \left\|f - \int_K f\,d\mu\right\|_{L^2(\mu)}^2 + \left|\int_K f\,d\mu\right|^2 \\
&\le c_N \mathcal{E}(f, f) + \frac{b^2}{p_i^2} = c_N + \frac{b^2}{p_i^2}.
\end{aligned}$$

This concludes the assertion. □

Lemma 3.6. (i) *For $h \in \mathcal{F}$, $h \in \mathcal{H}$ if and only if $\mathcal{E}(f, h) = 0$ for every $f \in \mathcal{F}_D$.*
(ii) *$\mathcal{H}$ is a closed subspace of $\mathcal{F}$.*

Proof. (i) Suppose $h \in \mathcal{H}$. For any $f \in \mathcal{F}_D$ and $\varepsilon \neq 0$, we have $\mathcal{E}(h,h) \le \mathcal{E}(h+\varepsilon f, h+\varepsilon f)$, which implies that $2\varepsilon^{-1}\mathcal{E}(f,h)+\mathcal{E}(f,f) \ge 0$. If $\mathcal{E}(f,h) \neq 0$, we get a contradiction by letting $\varepsilon \downarrow 0$ or $\varepsilon \uparrow 0$.

Next, suppose that $\mathcal{E}(f,h) = 0$ for every $f \in \mathcal{F}_D$. Then, $\mathcal{E}(h+f,h+f) = \mathcal{E}(h,h)+\mathcal{E}(f,f) \ge \mathcal{E}(h,h)$ for every $f \in \mathcal{F}_D$, which implies that $h \in \mathcal{H}$.

Claim (ii) is a straightforward consequence of (i). □

Lemma 3.7. *For any $w \in W_*$, $\psi_w^*(\mathcal{H}) \subset \mathcal{H}$.*

Proof. Let $m = |w|$ and take $h \in \mathcal{H}$. For any $f \in \mathcal{F}_D$, $\Psi_w f \in \mathcal{F}_D$ by Lemma 3.3. Then,

$$0 = \mathcal{E}(h, \Psi_w f) = \sum_{\xi \in W_m} s_\xi \mathcal{E}(\psi_\xi^* h, \psi_\xi^* \Psi_w f) = s_w \mathcal{E}(\psi_w^* h, f).$$

Therefore, $\mathcal{E}(\psi_w^* h, f) = 0$. This implies that $\psi_w^* h \in \mathcal{H}$ by Lemma 3.6. □

Lemma 3.8. *For each $f \in \mathcal{F}$, there exists a unique $h \in \mathcal{H}$ such that $f - h \in \mathcal{F}_D$.*

Proof. Fix $f \in \mathcal{F}$ and let $\mathcal{F}_f := \{g \in \mathcal{F} \mid f - g \in \mathcal{F}_D\}$. It is sufficient to prove that there exists a unique element in $\mathcal{F}_f$ that attains the infimum of $\{\mathcal{E}(g,g) \mid g \in \mathcal{F}_f\}$. Take a sequence $\{g_n\} \subset \mathcal{F}_f$ such that $\mathcal{E}(g_n, g_n) \downarrow \inf\{\mathcal{E}(g,g) \mid g \in \mathcal{F}_f\}$ as $n \to \infty$. By Lemma 3.5,

$$\begin{aligned}\|g_n\|_{L^2(\mu)} &\le \|g_n - f\|_{L^2(\mu)} + \|f\|_{L^2(\mu)} \\ &\le \sqrt{c_D}\mathcal{E}(g_n - f, g_n - f)^{1/2} + \|f\|_{L^2(\mu)} \\ &\le \sqrt{c_D}\{\mathcal{E}(g_n, g_n)^{1/2} + \mathcal{E}(f,f)^{1/2}\} + \|f\|_{L^2(\mu)}.\end{aligned}$$

Therefore, $\{g_n\}$ is bounded in $L^2(\mu)$, and hence, bounded in $\mathcal{F}$. By taking a subsequence if necessary, g_n converges weakly to some g_∞ in $\mathcal{F}$. We also have $\mathcal{E}(g_\infty, g_\infty) \le \underline{\lim}_{n\to\infty} \mathcal{E}(g_n, g_n)$. Since $\mathcal{F}_f$ is weakly closed in $\mathcal{F}$, we conclude that $g_\infty \in \mathcal{F}_f$ and that g_∞ attains the infimum. If both g' and g'' in $\mathcal{F}_f$ attain the infimum of $\{\mathcal{E}(g,g) \mid g \in \mathcal{F}_f\}$, then

$$\mathcal{E}\left(\frac{g'-g''}{2}, \frac{g'-g''}{2}\right) = -\mathcal{E}\left(\frac{g'+g''}{2}, \frac{g'+g''}{2}\right) + \frac{1}{2}\mathcal{E}(g', g') + \frac{1}{2}\mathcal{E}(g'', g'') \le 0$$

since $(g'+g'')/2 \in \mathcal{F}_f$. Therefore, $g' - g''$ is a constant function. In view of Lemma 3.5, we conclude that $g' = g''$. □

Define a map $H : \mathcal{F} \to \mathcal{H} \subset \mathcal{F}$ by $Hf = h$, where h is given in the lemma above.

Lemma 3.9. (i) *H is a bounded linear operator on $\mathcal{F}$.*
(ii) *For $f \in \mathcal{F}$, μ-*essinf $f \le \mu$-essinf $Hf \le \mu$-esssup $Hf \le \mu$-esssup f.

Proof. (i) The linearity of the map H follows from Lemma 3.6. We have $\mathcal{E}(Hf, Hf) \le \mathcal{E}(f,f)$ by definition. We also have

$$\begin{aligned}\|Hf\|_{L^2(\mu)} &\le \|Hf - f\|_{L^2(\mu)} + \|f\|_{L^2(\mu)} \\ &\le \sqrt{c_D}\mathcal{E}(Hf - f, Hf - f)^{1/2} + \|f\|_{L^2(\mu)} \\ &\le 2\sqrt{c_D}\mathcal{E}(f,f)^{1/2} + \|f\|_{L^2(\mu)}.\end{aligned}$$

Therefore, $\|Hf\|_{\mathcal{F}}$ is dominated by a constant times $\|f\|_{\mathcal{F}}$, which implies the first assertion.
(ii) Suppose $f \le b$ μ-a.e. for $b \in \mathbb{R}$. Since $Hf - f \in \mathcal{F}_D$ and $b - f \in \mathcal{F}_+$, Lemma 3.4 assures that

$$Hf \wedge b - f = (Hf - f) \wedge (b - f) \in \mathcal{F}_D.$$

Since $\mathcal{E}(Hf \wedge b, Hf \wedge b) \le \mathcal{E}(Hf, Hf)$ and Hf is the unique element attaining the infimum of $\{\mathcal{E}(g, g) \mid g \in \mathcal{F}_f\}$, we conclude that $Hf \wedge b = Hf$ μ-a.e., that is, $Hf \le b$ μ-a.e. Considering $-f$ in place of f, we derive all the relations. □

Let

$$\mathcal{H}_{loc} = \left\{ f \in \mathcal{F} \;\middle|\; \begin{array}{l} \text{there exists some } m \in \mathbb{Z}_+ \text{ such that} \\ \psi_w^* f \in \mathcal{H} \text{ for every } w \in W_m \end{array} \right\}.$$

Lemma 3.10. *$\mathcal{H}_{loc}$ is dense in $\mathcal{F}$.*

Proof. For $f \in \mathcal{F}$ and $m \in \mathbb{Z}_+$, set

$$f_m(x) = H(\psi_w^* f)(\psi_w^{-1}(x)) \quad \text{for } x \in K_w,\ w \in W_m.$$

This is well-defined up to μ-equivalence because $\mu(K^b) = 0$. Then, f_m has another expression:

$$f_m = f + \sum_{w \in W_m} \Psi_w(H(\psi_w^* f) - \psi_w^* f),$$

which implies that $f_m \in \mathcal{F}$. Since $\psi_w^* f_m = H(\psi_w^* f) \in \mathcal{H}$ for any $w \in W_m$, f_m belongs to $\mathcal{H}_{loc}$. By (2.3), we also have

$$\begin{aligned} \mathcal{E}(f_m, f_m) &= \sum_{w \in W_m} s_w \mathcal{E}(H(\psi_w^* f), H(\psi_w^* f)) \\ &\le \sum_{w \in W_m} s_w \mathcal{E}(\psi_w^* f, \psi_w^* f) = \mathcal{E}(f, f). \end{aligned} \tag{3.2}$$

In order to prove the lemma, it is enough to show that any function $f \in \mathcal{F} \cap C(K)$ is approximated by functions in $\mathcal{H}_{loc}$ in the weak topology of $\mathcal{F}$. On account of Lemma 3.9 (ii), we may assume

$$\min_{x \in K_w} f(x) \le \inf_{x \in K_w} f_m(x) \le \sup_{x \in K_w} f_m(x) \le \max_{x \in K_w} f(x)$$

for any $w \in W_m$ by taking a suitable μ-modification if necessary. In particular, we see that $\{f_m\}_{m \in \mathbb{Z}_+}$ is bounded in $\mathcal{F}$ by combining the estimate (3.2). Let $\omega = \omega_1\omega_2 \cdots \in \Sigma$ and $y = \pi(\omega)$. For each $m \in \mathbb{Z}_+$,

$$|f(y) - f_m(y)| \le \max_{x \in K_{\omega_1\omega_2\cdots\omega_m}} f(x) - \min_{x \in K_{\omega_1\omega_2\cdots\omega_m}} f(x),$$

which converges to 0 as $m \to \infty$ because $\bigcap_{m \in \mathbb{Z}_+} K_{\omega_1\omega_2\cdots\omega_m} = \{y\}$ by [17, Proposition 1.3.3]. Therefore, f_m converges to f μ-a.e. and we conclude that f_m converges weakly to f in $\mathcal{F}$. □

Lemma 3.11. (i) *Let $w \in W_*$. For any exceptional set N of K, $\psi_w^{-1}(N)$ is also an exceptional set. In particular, for every $f \in \mathcal{F}$, $\psi_w^* \tilde{f}$ is a quasi-continuous modification of $\psi_w^* f$.*

(ii) *For $f \in \mathcal{F}$ and $m \in \mathbb{Z}_+$, we have*

$$\mu_{\langle f \rangle} = \sum_{w \in W_m} s_w (\psi_w)_* \mu_{\langle \psi_w^* f \rangle}, \tag{3.3}$$

that is, $\mu_{\langle f \rangle}(A) = \sum_{w \in W_m} s_w \mu_{\langle \psi_w^* f \rangle}(\psi_w^{-1}(A))$ for any Borel subset A of K.

Proof. (i) Let $\varepsilon > 0$. Take an open set $O \supset N$ and a function $e_O \in \mathcal{F}$ such that $e_O \geq 1$ μ-a.e. on O and $\|e_O\|_{\mathcal{F}} < \varepsilon$. Then, $\psi_w^* e_O \geq 1$ μ-a.e. on $\psi_w^{-1}(O) \supset \psi_w^{-1}(N)$. Moreover, by Lemma 3.1, $\|\psi_w^* e_O\|_{\mathcal{F}} \leq (p_w^{-1/2} \vee s_w^{-1/2})\varepsilon$. Therefore, $\psi_w^{-1}(N)$ is also exceptional.

(ii) Suppose $f \in \mathcal{F}_b$. Then for any $g \in \mathcal{F}_b$,

$$\begin{aligned}
\int_K \tilde{g}\, d\mu_{\langle f \rangle} &= 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g) \\
&= \sum_{w \in W_m} s_w \left\{ 2\mathcal{E}(\psi_w^* f, \psi_w^* f \psi_w^* g) - \mathcal{E}(\psi_w^* f^2, \psi_w^* g) \right\} \\
&= \sum_{w \in W_m} s_w \int_K \psi_w^* \tilde{g}\, d\mu_{\langle \psi_w^* f \rangle}.
\end{aligned}$$

From the uniqueness of the energy measure, we obtain (3.3). For general $f \in \mathcal{F}$, we simply take an approximate sequence in $\mathcal{F}_b$ and use (2.4). □

Lemma 3.12. *$(\mathcal{E}, \mathcal{F})$ is a local Dirichlet form.*

Proof. We note that for each $x \in K$, $\left\{ \bigcup_{w \in W_m : x \in K_w} K_w \,\middle|\, m \in \mathbb{Z}_+ \right\}$ gives a fundamental system of neighborhoods of x by Proposition 1.3.6 in [17]. Suppose that f, $g \in \mathcal{F}$ satisfies $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. Then, for sufficiently large $m \in \mathbb{Z}_+$, each $w \in W_m$ satisfies either $\operatorname{supp} f \cap K_w = \emptyset$ or $\operatorname{supp} g \cap K_w = \emptyset$. Then,

$$\mathcal{E}(f, g) = \sum_{w \in W_m} s_w \mathcal{E}(\psi_w^* f, \psi_w^* g) = 0.$$

This implies that $(\mathcal{E}, \mathcal{F})$ is local. □

4. Proof of Theorems

We set $\Sigma_A = \bigcup_{w \in A} \Sigma_w$ for $A \subset W_*$.

For $f \in \mathcal{F}$, we will construct a finite measure $\lambda_{\langle f \rangle}$ on Σ as follows. For each $m \in \mathbb{Z}_+$, define

$$\lambda_{\langle f \rangle}^{(m)}(A) = 2 \sum_{w \in A} s_w \mathcal{E}(\psi_w^* f, \psi_w^* f), \qquad A \subset W_m.$$

Then, $\lambda^{(m)}_{\langle f\rangle}$ is a measure on W_m. When $A \subset W_m$ and $A' = \{wi \in W_{m+1} \mid w \in A,\ i \in S\}$,

$$\begin{aligned}\lambda^{(m+1)}_{\langle f\rangle}(A') &= 2\sum_{w\in A}\sum_{i\in S} s_{wi}\mathcal{E}(\psi^*_{wi}f, \psi^*_{wi}f)\\ &= 2\sum_{w\in A} s_w \sum_{i\in S} s_i\mathcal{E}(\psi^*_i\psi^*_w f, \psi^*_i\psi^*_w f)\\ &= 2\sum_{w\in A} s_w\mathcal{E}(\psi^*_w f, \psi^*_w f) = \lambda^{(m)}_{\langle f\rangle}(A).\end{aligned}$$

Therefore, $\{\lambda^{(m)}_{\langle f\rangle}\}_{m\in\mathbb{Z}_+}$ has a consistency condition. We also note that $\lambda^{(m)}_{\langle f\rangle}(W_m) = 2\mathcal{E}(f, f) < \infty$. By the Kolmogorov extension theorem, there exists a unique Borel finite measure $\lambda_{\langle f\rangle}$ on Σ such that $\lambda_{\langle f\rangle}(\Sigma_w) = \lambda^{(|w|)}_{\langle f\rangle}(\{w\})$ for every $w \in W_*$.

Lemma 4.1. $\pi_*\lambda_{\langle f\rangle} = \mu_{\langle f\rangle}$.

Proof. We define a set function χ_m for $m \in \mathbb{Z}_+$ by

$$\chi_m(A) = \sum_{w\in W_m} s_w\mu_{\langle\psi^*_w f\rangle}(\pi(\sigma_w^{-1}(A))),$$

where A is a σ-compact set of Σ. χ_m does not necessarily satisfy the additive property but has monotonicity. Let B be a closed subset of K. From Lemma 3.11 (ii), $\mu_{\langle f\rangle}(B) = \sum_{w\in W_m} s_w\mu_{\langle\psi^*_w f\rangle}(\psi_w^{-1}(B))$. Since $\psi_w\circ\pi = \pi\circ\sigma_w$ and π is surjective, we have

$$\psi_w^{-1}(B) = \pi(\pi^{-1}(\psi_w^{-1}(B))) = \pi(\sigma_w^{-1}(\pi^{-1}(B))).$$

Therefore, we get

$$\mu_{\langle f\rangle}(B) = \chi_m(\pi^{-1}(B)). \tag{4.1}$$

When $C \subset W_m \subset W_*$, we have

$$\begin{aligned}\lambda_{\langle f\rangle}(\Sigma_C) &= \lambda^{(m)}_{\langle f\rangle}(C)\\ &= \sum_{w\in C} s_w\mu_{\langle\psi^*_w f\rangle}(K)\\ &= \sum_{w\in W_m} s_w\mu_{\langle\psi^*_w f\rangle}(\pi(\sigma_w^{-1}(\Sigma_C)))\\ &= \chi_m(\Sigma_C).\end{aligned} \tag{4.2}$$

Here, in the third equality, we used the identity

$$\pi(\sigma_w^{-1}(\Sigma_C)) = \begin{cases} K & \text{if } w \in C\\ \emptyset & \text{if } w \notin C\end{cases}.$$

Now, let D be a closed set of K. $\pi^{-1}(D)$ is also a closed set of Σ. For each $m \in \mathbb{Z}_+$, let $C_m = \{w \in W_m \mid \Sigma_w \cap \pi^{-1}(D) \neq \emptyset\}$. Then, $\{\Sigma_{C_m}\}_{m\in\mathbb{Z}_+}$ is a decreasing sequence and $\bigcap_{m\in\mathbb{Z}_+}\Sigma_{C_m} = \pi^{-1}(D)$. In fact, if we set a distance ρ on

Σ by $\rho(\omega, \eta) = \exp(-\inf\{j \mid \omega_j \neq \eta_j\})$ for $\omega = \{\omega_j\}_{j\in\mathbb{N}}$ and $\eta = \{\eta_j\}_{j\in\mathbb{N}} \in \Sigma$, then Σ_{C_m} is simply $\{\omega \in \Sigma \mid \rho(\omega, \pi^{-1}(D)) \le e^{-m-1}\}$.

By virtue of (4.1) and (4.2),

$$\mu_{\langle f\rangle}(D) = \chi_m(\pi^{-1}(D)) \le \chi_m(\Sigma_{C_m}) = \lambda_{\langle f\rangle}(\Sigma_{C_m}).$$

Letting $m \to \infty$, we get

$$\mu_{\langle f\rangle}(D) \le \lambda_{\langle f\rangle}(\pi^{-1}(D)) = \pi_*\lambda_{\langle f\rangle}(D).$$

Since both $\mu_{\langle f\rangle}$ and $\pi_*\lambda_{\langle f\rangle}$ are Borel measures on K, we have $\mu_{\langle f\rangle}(B) \le \pi_*\lambda_{\langle f\rangle}(B)$ for every Borel set B. Since the total masses are the same for $\mu_{\langle f\rangle}$ and $\pi_*\lambda_{\langle f\rangle}$, the reverse inequality also holds by considering $K \setminus B$ in place of B. This completes the proof. □

The following general criterion for the singularity of probability measures, which is a slight modification of Theorem VII.6.4 in [28], is a key to the proof of Theorem 2.1. For $x \in \mathbb{R}$, $x^\oplus$ is defined by $x^\oplus = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Theorem 4.1. *Let $(\Omega, \mathcal{B})$ be a measurable space on which a filtration $\{\mathcal{B}_n\}_{n\in\mathbb{Z}_+}$ is defined such that $\bigvee_{n\in\mathbb{Z}_+} \mathcal{B}_n = \mathcal{B}$. Let P and $\tilde{P}$ be two probability measures on $(\Omega, \mathcal{B})$. Assume that $\tilde{P}|_{\mathcal{B}_n} \ll P|_{\mathcal{B}_n}$ for each $n \in \mathbb{Z}_+$. Set $z_n = \frac{d(\tilde{P}|_{\mathcal{B}_n})}{d(P|_{\mathcal{B}_n})}$ for $n \in \mathbb{Z}_+$ and $\alpha_n = z_n z_{n-1}^\oplus$ for $n \in \mathbb{N}$. If*

$$\sum_{n=1}^{\infty}(1 - \mathbb{E}^P[\sqrt{\alpha_n} \mid \mathcal{B}_{n-1}]) = \infty \quad P\text{-a.e.}, \tag{4.3}$$

where $\mathbb{E}^P[\,\cdot \mid \mathcal{B}_{n-1}]$ denotes the conditional expectation for P given $\mathcal{B}_{n-1}$, then P and $\tilde{P}$ are mutually singular.

Remark 4.1. Note that by Jensen's inequality, $1 - \mathbb{E}^P[\sqrt{\alpha_n} \mid \mathcal{B}_{n-1}] \ge 0$ always holds. In Theorem VII.6.4 in [28], it is proved that the singularity of the two measures is equivalent to the same relation as (4.3) but with $\tilde{P}$-a.e. instead of P-a.e. A characterization for absolute continuity is also provided there.

For the proof of Theorem 4.1, we recall the following result.

Theorem 4.2 ([28, Theorem VII.6.1]). *With the same notation as in Theorem 4.1, $z_\infty = \lim_{n\to\infty} z_n$ exists $(P + \tilde{P})$-a.e. and*

$$\tilde{P}(A) = \int_A z_\infty \, dP + \tilde{P}(A \cap \{z_\infty = \infty\}), \qquad A \in \mathcal{B}.$$

Moreover, $\tilde{P}(\cdot \cap \{z_\infty = \infty\})$ and P are mutually singular.

Proof of Theorem 4.1. Combining the estimates (19), (20), (25), (23), (28), and (24) in the proof of Theorem VII.6.4 in [28], we have

$$\{z_\infty = \infty\} = \left\{\sum_{n=1}^{\infty} \mathbb{E}^P[(1-\sqrt{\alpha_n})^2 \mid \mathcal{B}_{n-1}] = \infty\right\}$$
$$= \left\{\sum_{n=1}^{\infty} (1-\mathbb{E}^P[\sqrt{\alpha_n} \mid \mathcal{B}_{n-1}]) = \infty\right\} \quad \tilde{P}\text{-a.e.}$$

By (4.3), there exists $B \in \mathcal{B}$ such that $P(B) = 1$ and $\sum_{n=1}^{\infty}(1-\mathbb{E}^P[\sqrt{\alpha_n} \mid \mathcal{B}_{n-1}]) = \infty$ on B. Then, $z_\infty = \infty$ $\tilde{P}$-a.e. on B. Applying Theorem 4.2 to $A = \Omega \setminus B$, we have $\tilde{P}(\Omega \setminus B) = \tilde{P}((\Omega \setminus B) \cap \{z_\infty = \infty\})$, that is, $z_\infty = \infty$ $\tilde{P}$-a.e. on $\Omega \setminus B$. Hence, $\tilde{P}(\{z_\infty = \infty\}) = 1$ and we obtain $\tilde{P} \perp P$ by the latter part of Theorem 4.2. □

Proof of Theorem 2.1. Consider a projective system $\{\mathbb{R}^{W_m}\ (m \in \mathbb{N}), \varphi_{m,n}\ (m, n \in \mathbb{N},\ m \le n)\}$, where $\varphi_{m,n}$ is a continuous map from $\mathbb{R}^{W_n}$ to $\mathbb{R}^{W_m}$ given by

$$\varphi_{m,n}(\{a_w\}_{w \in W_n}) = \{b_{w'}\}_{w' \in W_m}, \quad b_{w'} = \sum_{w'' \in W_{n-m}} a_{w'w''}.$$

Note that the consistency condition $\varphi_{l,m} \circ \varphi_{m,n} = \varphi_{l,n}$ holds for $l \le m \le n$. The projective limit $\varprojlim \mathbb{R}^{W_m}$ associated with $\{\mathbb{R}^{W_n}, \varphi_{m,n}\}$ becomes a Hausdorff space. We set $\Theta_w(f) = 2s_w\mathcal{E}(\psi_w^* f, \psi_w^* f)$ for $w \in W_*$ and $f \in \mathcal{F}$. For each $m \in \mathbb{N}$, we define a map $\Theta^{(m)} : \mathcal{F} \to \mathbb{R}^{W_m}$ by $\Theta^{(m)}(f) = \{\Theta_w(f)\}_{w \in W_m}$. When a sequence $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $\mathcal{F}$,

$$|\mathcal{E}(\psi_w^* f_n, \psi_w^* f_n)^{1/2} - \mathcal{E}(\psi_w^* f, \psi_w^* f)^{1/2}|^2 \le \mathcal{E}(\psi_w^* f_n - \psi_w^* f, \psi_w^* f_n - \psi_w^* f)$$
$$\le s_w^{-1}\mathcal{E}(f_n - f, f_n - f) \to 0$$

as $n \to \infty$ for any $w \in W_m$. Thus, $\Theta^{(m)}$ is a continuous map. Moreover, by the self-similarity (A3), $\Theta^{(m)} = \varphi_{m,n} \circ \Theta^{(n)}$ for $m \le n$. Therefore, there exists a unique continuous map $\Theta : \mathcal{F} \to \varprojlim \mathbb{R}^{W_m}$ such that $\Theta^{(n)} = \varphi_n \circ \Theta$ for every n, where φ_n is a canonical map from $\varprojlim \mathbb{R}^{W_m}$ to $\mathbb{R}^{W_n}$.

Set $q^{(m)} = \{q_w\}_{w \in W_m} \in \mathbb{R}^{W_m}$ for $m \in \mathbb{N}$ and let $\mathbf{q}$ denote the element of $\varprojlim \mathbb{R}^{W_m}$ represented by $\{q^{(m)}\}_{m \in \mathbb{N}}$.

First, assume that there exists some $h \in \mathcal{H}$ such that $\Theta(h) = \mathbf{q}$. For each $w \in W_*$, we have

$$\lambda_{\langle h \rangle}(\Sigma_w) = 2s_w\mathcal{E}(\psi_w^* h, \psi_w^* h) = \Theta_w(h) = q_w = \lambda_q(\Sigma_w).$$

This implies that $\lambda_{\langle h \rangle} = \lambda_q$. By Lemma 4.1, we get $\mu_{\langle h \rangle} = \pi^*\lambda_{\langle h \rangle} = \pi^*\lambda_q = \mu_q$. Therefore, case (i) of Theorem 2.1 holds.

Next, assume that $\Theta(h) \ne \mathbf{q}$ for every $h \in \mathcal{H}$. Let $N = |u|$ and $\delta = q_u/(4s_u) > 0$, where u is given in assumption (C). Define

$$\mathcal{K}' = \left\{h \in \mathcal{H} \,\middle|\, \int_K h\,d\mu = 0,\ \mathcal{E}(h,h) \le 1/2\right\}, \qquad \mathcal{K} = \psi_u^*(\mathcal{K}').$$

According to (2.2), $\mathcal{K}'$ is a bounded closed set of $\mathcal{F}$. Moreover, since $\mathcal{K}'$ is convex, it is closed for the weak topology of $\mathcal{F}$. Therefore, $\mathcal{K}'$ is weakly compact in $\mathcal{F}$. From Lemma 3.7 and assumption (C), $\mathcal{K}$ is a subset of $\mathcal{H}$ and compact in $\mathcal{F}$. The set $\mathcal{K}_\delta = \{f \in \mathcal{K} \mid \mathcal{E}(f, f) \ge \delta\}$ is also compact in $\mathcal{F}$. We set

$$\bar{\Theta}_w(f) := \Theta_w\left(f \Big/ \sqrt{2\mathcal{E}(f, f)}\right) = s_w \cdot \frac{\mathcal{E}(\psi_w^* f, \psi_w^* f)}{\mathcal{E}(f, f)}$$

for $f \in \mathcal{K}_\delta$ and $w \in W_*$. For each $m \in \mathbb{N}$, define $\bar{\Theta}^{(m)} : \mathcal{K}_\delta \to \mathbb{R}^{W_m}$ by $\bar{\Theta}^{(m)}(f) = \{\bar{\Theta}_w(f)\}_{w \in W_m}$. Further, define $\bar{\Theta} : \mathcal{K}_\delta \to \varprojlim \mathbb{R}^{W_m}$ by $\bar{\Theta}(f) = \Theta\left(f \big/ \sqrt{2\mathcal{E}(f, f)}\right)$. Then, $\mathbf{q} \notin \bar{\Theta}(\mathcal{K}_\delta)$ and $\bar{\Theta}$ is continuous when $\mathcal{K}_\delta$ is equipped with a relative topology of $\mathcal{F}$. By the definition of the topology of $\varprojlim \mathbb{R}^{W_m}$, there exist $M \in \mathbb{N}$ and $\gamma > 0$ such that $|\bar{\Theta}^{(M)}(f) - q^{(M)}| \ge \gamma$ for every $f \in \mathcal{K}_\delta$. Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{W_M}$. Set

$$\beta_1 = \sup\left\{ \sum_{w \in W_N} \sqrt{q_w a_w} \;\middle|\; \begin{array}{l} a = \{a_w\}_{w \in W_N} \in \mathbb{R}^{W_N},\ a_w \ge 0 \text{ for every } w, \\ \sum_{w \in W_N} a_w = 1, \text{ and } a_u \le q_u/2 \end{array} \right\},$$

$$\beta_2 = \sup\left\{ \sum_{w \in W_M} \sqrt{q_w b_w} \;\middle|\; \begin{array}{l} b = \{b_w\}_{w \in W_M} \in \mathbb{R}^{W_M},\ b_w \ge 0 \text{ for every } w, \\ \sum_{w \in W_M} b_w = 1, \text{ and } |b - q^{(M)}| \ge \gamma \end{array} \right\}.$$

Then, $\beta_1 < 1$ and $\beta_2 < 1$. Indeed, $\beta_1 < 1$ follows from the facts that $\sum_{w \in W_N} \sqrt{q_w a_w} \le \sum_{w \in W_N} (q_w + a_w)/2 = 1$ and the equality holds only when $a = q^{(N)}$, and a is taken over a compact set in $\mathbb{R}^{W_N}$ that does not contain $q^{(N)}$. The same rationale is applied to β_2. Let $\beta = \beta_1 \vee \beta_2 < 1$. We define a filtration $\{\mathcal{B}_n\}_{n \in \mathbb{Z}_+}$ on Σ by

$$\begin{aligned} \mathcal{B}_{2k} &= \sigma(\{\Sigma_w \mid w \in W_{(M+N)k}\}), \\ \mathcal{B}_{2k+1} &= \sigma(\{\Sigma_w \mid w \in W_{(M+N)k+N}\}) \qquad (k = 0, 1, 2, \dots). \end{aligned}$$

It is clear that $\bigvee_{n \in \mathbb{Z}_+} \mathcal{B}_n$ is identical with the Borel σ-field of Σ.

Let $h \in \mathcal{H}$ with $\mathcal{E}(h, h) = 1/2$. For each $n \in \mathbb{Z}_+$, $\lambda_{\langle h \rangle}|_{\mathcal{B}_n} \ll \lambda_q|_{\mathcal{B}_n}$ since only an empty set is a $\mathcal{B}_n$-measurable set with λ_q-null measure. Define $z_n = \frac{d(\lambda_{\langle h \rangle}|_{\mathcal{B}_n})}{d(\lambda_q|_{\mathcal{B}_n})}$ for $n \in \mathbb{Z}_+$ and $\alpha_n = z_n z_{n-1}^{\oplus}$ for $n \in \mathbb{N}$. We will prove that $\sum_{n=1}^{\infty} \left(1 - \mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-1}]\right) = \infty$ λ_q-a.e.

Taking $k \in \mathbb{Z}_+$, $w \in W_{(M+N)k}$, $w' \in W_N$, and $w'' \in W_M$, we have

$$z_{2k} = \frac{\lambda_{\langle h \rangle}(\Sigma_w)}{\lambda_q(\Sigma_w)} = \frac{2 s_w \mathcal{E}(\psi_w^* h, \psi_w^* h)}{q_w} \quad \text{on } \Sigma_w,$$

$$z_{2k+1} = \frac{\lambda_{\langle h \rangle}(\Sigma_{ww'})}{\lambda_q(\Sigma_{ww'})} = \frac{2 s_{ww'} \mathcal{E}(\psi_{ww'}^* h, \psi_{ww'}^* h)}{q_{ww'}} \quad \text{on } \Sigma_{ww'},$$

and

$$z_{2k+2} = \frac{\lambda_{\langle h\rangle}(\Sigma_{ww'w''})}{\lambda_q(\Sigma_{ww'w''})} = \frac{2s_{ww'w''}\mathcal{E}(\psi^*_{ww'w''}h, \psi^*_{ww'w''}h)}{q_{ww'w''}} \quad \text{on } \Sigma_{ww'w''}.$$

On $\Sigma_{ww'}$, we have

$$\alpha_{2k+1} = z_{2k+1}z_{2k}^{\oplus} = \frac{s_{w'}}{q_{w'}} \cdot \mathcal{E}(\psi^*_{w'}\psi^*_w h, \psi^*_{w'}\psi^*_w h)\mathcal{E}(\psi^*_w h, \psi^*_w h)^{\oplus}.$$

If $\mathcal{E}(\psi^*_w h, \psi^*_w h) = 0$, we get $\alpha_{2k+1} = 0$ on Σ_w, and therefore, $1 - \mathbb{E}^{\lambda_q}[\sqrt{\alpha_{2k+1}} \mid \mathcal{B}_{2k}] = 1$ on Σ_w.

Let us assume that $\mathcal{E}(\psi^*_w h, \psi^*_w h) \neq 0$. Set

$$g = \frac{\psi^*_w h - \int_K \psi^*_w h \, d\mu}{\sqrt{2\mathcal{E}(\psi^*_w h, \psi^*_w h)}}.$$

Then, $\int_K g\, d\mu = 0$, $\mathcal{E}(g, g) = 1/2$, and $\psi^*_u g \in \mathcal{K}$. If $\psi^*_u g \notin \mathcal{K}_\delta$, that is, $\mathcal{E}(\psi^*_u g, \psi^*_u g) < \delta$, then

$$\alpha_{2k+1} = \frac{2s_{w'}}{q_{w'}} \cdot \mathcal{E}(\psi^*_{w'}g, \psi^*_{w'}g) \qquad \text{on } \Sigma_{ww'}$$

and

$$\begin{aligned}
\mathbb{E}^{\lambda_q}[\sqrt{\alpha_{2k+1}} \mid \mathcal{B}_{2k}] &= \sum_{w' \in W_N} q_{w'}\sqrt{\frac{2s_{w'}}{q_{w'}} \cdot \mathcal{E}(\psi^*_{w'}g, \psi^*_{w'}g)} \\
&= \sum_{w' \in W_N} \sqrt{q_{w'}\Theta_{w'}(g)} \\
&\leq \beta_1 \leq \beta \quad \text{on } \Sigma_w
\end{aligned}$$

since $\Theta_u(g) < 2s_u\delta = q_u/2$ by the definition of δ and $\sum_{w' \in W_N} \Theta_{w'}(g) = 2\mathcal{E}(g, g) = 1$.

If $\psi^*_u g \in \mathcal{K}_\delta$, then we have, on $\Sigma_{wuw''}$,

$$\begin{aligned}
\alpha_{2k+2} &= z_{2k+2}z_{2k+1}^{\oplus} \\
&= \frac{s_{w''}}{q_{w''}} \cdot \frac{\mathcal{E}(\psi^*_{wuw''}h, \psi^*_{wuw''}h)}{\mathcal{E}(\psi^*_{wu}h, \psi^*_{wu}h)} \\
&= \frac{s_{w''}}{q_{w''}} \cdot \frac{\mathcal{E}(\psi^*_{w''}\psi^*_u g, \psi^*_{w''}\psi^*_u g)}{\mathcal{E}(\psi^*_u g, \psi^*_u g)};
\end{aligned}$$

therefore, on Σ_{wu},

$$\mathbb{E}^{\lambda_q}[\sqrt{\alpha_{2k+2}} \mid \mathcal{B}_{2k+1}] = \sum_{w'' \in W_M} q_{w''}\sqrt{\frac{\bar{\Theta}_{w''}(\psi^*_u g)}{q_{w''}}} \leq \beta_2 \leq \beta$$

since $|\bar{\Theta}^{(M)}(\psi^*_u g) - q^{(M)}| \geq \gamma$ and $\sum_{w'' \in W_M} \bar{\Theta}_{w''}(\psi^*_u g) = 1$.

Accordingly, in any case,

$$(1-\mathbb{E}^{\lambda_q}[\sqrt{\alpha_{2k+1}} \mid \mathcal{B}_{2k}]) + (1-\mathbb{E}^{\lambda_q}[\sqrt{\alpha_{2k+2}} \mid \mathcal{B}_{2k+1}]) \geq 1-\beta > 0 \quad \text{on } \Sigma_{wu}.$$

Set

$$\hat{\Sigma} = \{\omega \in \Sigma \mid \sigma^{(N+M)k}(\omega) \in \Sigma_u \text{ for an infinitely number of } k \in \mathbb{Z}_+\},$$

where $\sigma^m : \Sigma \to \Sigma$ is defined by $\sigma^m(\omega_1\omega_2\cdots) = \omega_{m+1}\omega_{m+2}\cdots$ for $m \in \mathbb{Z}_+$. Then, $\sum_{n=1}^{\infty}(1-\mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-1}](\omega)) = \infty$ if $\omega \in \hat{\Sigma}$. Since $\lambda_q(\hat{\Sigma}) = 1$ by the law of large numbers, we can apply Theorem 4.1 to conclude that $\lambda_q \perp \lambda_{\langle h\rangle}$. Take a σ-compact set A in Σ such that $\lambda_{\langle h\rangle}(A) = 1$ and $\lambda_q(A) = 0$. Recall the μ_q-null set $K^b = \{x \in K \mid \#(\pi^{-1}(x)) > 1\}$. Set $B = A \cup \pi^{-1}(K^b)$. Since $\pi^{-1}(\pi(B)) = B$, we have

$$\begin{aligned} \mu_q(\pi(B)) &= \lambda_q(\pi^{-1}(\pi(B))) = \lambda_q(B) = 0,\\ \mu_{\langle h\rangle}(\pi(B)) &= \lambda_{\langle h\rangle}(B) \geq \lambda_{\langle h\rangle}(A) = 1. \end{aligned}$$

Therefore, $\mu_q \perp \mu_{\langle h\rangle}$. Evidently, this relation is now true for all $h \in \mathcal{H}$.

When $h \in \mathcal{H}_{loc}$, we can also prove that $\mu_q \perp \mu_{\langle h\rangle}$ in view of expression (3.3). Take an arbitrary $f \in \mathcal{F}$. By Lemma 3.10, there exists a sequence $\{f_n\}$ in $\mathcal{H}_{loc}$ converging to f in $\mathcal{F}$. Take $A_n \subset K$ such that $\mu_q(A_n) = 0$ and $\mu_{\langle f_n\rangle}(K \setminus A_n) = 0$. Let $A = \bigcup_{n\in\mathbb{N}} A_n$. By (2.5), $\mu_{\langle f\rangle}(K \setminus A) = \lim_{n\to\infty} \mu_{\langle f_n\rangle}(K \setminus A) = 0$, while $\mu_q(A) = 0$. Hence, $\mu_q \perp \mu_{\langle f\rangle}$. This completes the proof. □

Remark 4.2. In the proof of Theorem 2.1, assumption (A1) is used only to assure that $\mu_p(K^b) = \mu_q(K^b) = 0$. Therefore, in view of [17, Lemma 1.4.7], (A1) can be replaced by a weaker condition, $\lambda_p(I_\infty) = \lambda_q(I_\infty) = 0$, where $I_\infty = \{\omega \in \Sigma \mid \#(\pi^{-1}(\pi(\omega))) = \infty\}$.

Proof of Theorem 2.2. (EHI) ⇒ (D): Let $h \in \mathcal{H}_+$. By (EHI),

$$\mu\text{-}\operatorname*{esssup}_{x\in K_v} h(x) \leq c_{2.1}\, \mu\text{-}\operatorname*{essinf}_{x\in K_v} h(x) \leq c_{2.1}\|\psi_v^* h\|_{L^2(\mu)}.$$

Next, suppose that $h \in \mathcal{H}$ and let $h_+ = h \vee 0$ and $h_- = (-h) \vee 0$. Since $h = Hh = Hh_+ - Hh_-$ and $Hh_\pm \in \mathcal{H}_+$ by Lemma 3.9,

$$\begin{aligned} \mu\text{-}\operatorname*{esssup}_{x\in K_v} |h(x)| &\leq \mu\text{-}\operatorname*{esssup}_{x\in K_v} Hh_+(x) + \mu\text{-}\operatorname*{esssup}_{x\in K_v} Hh_-(x)\\ &\leq c_{2.1}(\|\psi_v^*(Hh_+)\|_{L^2(\mu)} + \|\psi_v^*(Hh_-)\|_{L^2(\mu)})\\ &\leq c_{2.1} p_v^{-1/2}(\|Hh_+\|_{L^2(\mu)} + \|Hh_-\|_{L^2(\mu)})\\ &\leq c_{4.1}(\|h_+\|_{\mathcal{F}} + \|h_-\|_{\mathcal{F}}) \qquad \text{(by Lemma 3.9)}\\ &\leq 2c_{4.1}\|h\|_{\mathcal{F}}. \end{aligned}$$

(D)+(R) ⇒ (C): By assumption (A5) and the regularity of the Dirichlet form, we can take $\xi \in W_*$ and $g \in \mathcal{F}_0 \cap C(K)$ such that $0 \leq g \leq 1$ on K and $g = 1$ on K_ξ. We will show condition (C) with $u = v\xi$. It is sufficient to prove the following claim.

(∗) If a sequence $\{h_n\}$ in $\mathcal{H}$ converges weakly to 0 in $\mathcal{F}$, then there exists a subsequence $\{h_{n(k)}\}$ such that $\psi_u^* h_{n(k)}$ converges strongly to 0 in $\mathcal{F}$.

In order to deduce condition (C) from (∗), suppose that $\{f_m\}$ is a sequence in $\mathcal{H}$ that is bounded in $\mathcal{F}$. By the Banach-Alaoglu theorem, we can take a subsequence $\{f_{m(n)}\}$ converging weakly to some f in $\mathcal{F}$. Since $\mathcal{H}$ is weakly closed in $\mathcal{F}$, $f \in \mathcal{H}$. Applying (∗) to $h_n := f_{m(n)} - f$, we can take a sequence $\{n(k)\}$ diverging to ∞ such that $\psi_u^* f_{m(n(k))} \to \psi_u^* f$ in $\mathcal{F}$. This implies that condition (C) holds.

We now prove (∗). Since $\mathcal{F}$ is compactly imbedded in $L^2(\mu)$ by (R), $\{h_n\}$ converges to 0 in $L^2(\mu)$. Take a subsequence $\{h_{n(k)}\}$ converging to 0 μ-a.e. Define $f_k = \psi_v^* h_{n(k)}$. Then, $f_k \in \mathcal{F}_b \cap \mathcal{H}$ and $\sup_k \|f_k\|_{L^\infty(\mu)} < \infty$ by (D). Since $f_k g \in \mathcal{F}_0$, we have

$$0 = 2\mathcal{E}(f_k, f_k g) = \mathcal{E}(f_k^2, g) + \int_K g \, d\mu_{\langle f_k \rangle}.$$

Note that $\mathcal{E}(f_k^2, f_k^2) \le 4\|f_k\|_{L^\infty(\mu)}^2 \mathcal{E}(f_k, f_k)$, which is bounded in k. A suitable subsequence $\{f_{k'}\}$ can be taken so that $\{f_{k'}^2\}$ converges weakly in $\mathcal{F}$. Since $f_{k'} \to 0$ μ-a.e., $f_{k'}^2 \to 0$ weakly in $\mathcal{F}$. In particular, $\mathcal{E}(f_{k'}^2, g) \to 0$ as $k' \to \infty$. Applying Lemma 3.11 (ii) with $m = |\xi|$, we have

$$\begin{aligned}
\int_K g \, d\mu_{\langle f_{k'} \rangle} &= \sum_{w \in W_m} s_w \int_K \psi_w^* g \, d\mu_{\langle \psi_w^* f_{k'} \rangle} \\
&\ge s_\xi \int_K \psi_\xi^* g \, d\mu_{\langle \psi_\xi^* f_{k'} \rangle} \\
&= s_\xi \mu_{\langle \psi_\xi^* f_{k'} \rangle}(K) = 2 s_\xi \mathcal{E}(\psi_\xi^* f_{k'}, \psi_\xi^* f_{k'}).
\end{aligned}$$

Combining these estimates, we obtain $\overline{\lim}_{k' \to \infty} \mathcal{E}(\psi_\xi^* f_{k'}, \psi_\xi^* f_{k'}) \le 0$. Therefore, $\psi_\xi^* f_{k'} (= \psi_u^* h_{n(k')})$ converges to 0 in $\mathcal{F}$. This proves (∗). □

Proof of Theorem 2.3. Assume that there exists a nonconstant Borel function $f \in \mathcal{F}$ such that $\mu_{\langle f \rangle} \ll \mu$ and $\frac{d\mu_{\langle f \rangle}}{d\mu} \le c$ μ-a.e. for some $c > 0$. Take $a, b, R \in \mathbb{R}$ such that $a < b$, $R > 0$, and both $A = \{-R < f/\sqrt{c} < a\}$ and $B = \{b < f/\sqrt{c} < R\}$ have μ-positive measures. Then, by [13, Theorem 2.8] for example,

$$\int_K T_t 1_A \cdot 1_B \, d\mu \le \sqrt{\mu(A)\mu(B)} \exp\left(-\frac{(b-a)^2}{2t}\right), \quad t > 0.$$

Therefore, $\overline{\lim}_{t \downarrow 0} t \log \int_K T_t 1_A \cdot 1_B \, d\mu \le -(b-a)^2/2 < 0$, which is a contradiction. □

Proof of Theorem 2.4. Assume $\mu_{\langle h \rangle} \ll \mu_q$ and $\frac{d\mu_{\langle h \rangle}}{d\mu_q} \le c$ μ_q-a.e. for some $h \in \mathcal{H}$ and $c > 0$. Take an arbitrary $x, y \in K_z$. Set $x' = \psi_z^{-1}(x)$ and $y' = \psi_z^{-1}(y)$. Fix $m \in \mathbb{N}$ and let $k = d_m(x', y')$. Then, we can choose $x_0', x_1', \dots, x_{k+1}'$ from K such that $x_0' = x'$, $x_{k+1}' = y'$, and both x_i' and x_{i+1}' belong to some $K_{\xi_i'}$ with $\xi_i' \in W_m$ for each i. Setting $x_i = \psi_z(x_i')$ and $\xi_i = z\xi_i' \in W_{|z|+m}$, we obtain a sequence

$x_0, x_1, \dots, x_{k+1}$ in K_z such that $x_0 = x$, $x_{k+1} = y$, and both x_i and x_{i+1} belong to K_{ξ_i}.

Condition (O) assures that (after taking a continuous modification of h on K_z)

$$|h(x_i) - h(x_{i+1})|^2 \le C \sum_{w \in A(\xi_i)} \mathcal{E}(\psi_w^* h, \psi_w^* h).$$

By Lemma 3.11 (ii), we have

$$2 s_w \mathcal{E}(\psi_w^* h, \psi_w^* h) = s_w \mu_{\langle \psi_w^* h \rangle}(K) \le \mu_{\langle h \rangle}(K_w) \le c \mu_q(K_w) = c q_w$$

for any $w \in W_*$. Then,

$$|h(x_i) - h(x_{i+1})|^2 \le C \sum_{w \in A(\xi_i)} \frac{c q_w}{2 s_w} \le \frac{cCN}{2} \cdot r^{-(|z|+m)},$$

and therefore,

$$\begin{aligned} |h(x) - h(y)| &\le \sum_{i=0}^{d_m(x',y')} |h(x_i) - h(x_{i+1})| \\ &\le (d_m(x', y') + 1)(cCN/2)^{1/2} r^{-(|z|+m)/2}. \end{aligned}$$

Letting $m \to \infty$ and using the assumption in the theorem, we obtain $h(x) = h(y)$. This implies that h is constant on K_z. By virtue of Lemma 3.11 (ii), case (i) of Theorem 2.1 does not occur. □

Proof of Proposition 2.1. Let $\mathcal{D} = \{f \in \mathrm{Dom}(\mathcal{L}) \mid f \in L^\infty(\mu),\ \mathcal{L} f \in L^\infty(\mu)\}$. It is a dense subset of $\mathcal{F}$ since $(1 - \mathcal{L})^{-1}(L^\infty(\mu))$ is dense in $\mathcal{F}$ and is a subset of $\mathcal{D}$.

Suppose that $f \in \mathrm{Dom}(\mathcal{L})$ and $f^2 \in \mathrm{Dom}(\mathcal{L}^{(1)})$. Take $\{f_n\}_{n \in \mathbb{N}} \subset \mathrm{Dom}(\mathcal{L}) \cap L^\infty(\mu)$ such that $f_n \to f$ in $\mathrm{Dom}(\mathcal{L})$ with respect to the graph norm. Note that $f_n^2 \to f^2$ in $L^1(\mu)$.

For $g \in \mathcal{D}$, we have

$$\begin{aligned} \int_K \tilde{g}\, d\mu_{\langle f_n \rangle} &= 2\mathcal{E}(f_n, f_n g) - \mathcal{E}(f_n^2, g) \\ &= -2 \int_K (\mathcal{L} f_n) f_n g\, d\mu + \int_K f_n^2 \mathcal{L} g\, d\mu. \end{aligned}$$

The first term of the right-hand side converges to $-2 \int_K (\mathcal{L} f) f g\, d\mu$ when n tends to ∞, and the second term converges to $\int_K f^2 \mathcal{L} g\, d\mu = \int_K \mathcal{L}^{(1)}(f^2)\, g\, d\mu$. Therefore,

$$\int_K \tilde{g}\, d\mu_{\langle f \rangle} = -2 \int_K (\mathcal{L} f) f g\, d\mu + \int_K \mathcal{L}^{(1)}(f^2)\, g\, d\mu.$$

Since $\mathcal{D}$ is dense in $\mathcal{F}$, $d\mu_{\langle f \rangle} = \{-2(\mathcal{L} f) f + \mathcal{L}^{(1)}(f^2)\} d\mu$. In particular, $\mu_{\langle f \rangle} \ll \mu$, which implies that $\mu_{\langle f \rangle} = 0$ by combining $\mu_{\langle f \rangle} \perp \mu$. This concludes that f is a constant function. □

5. Examples

In the following examples, we set $\mathcal{P} = \bigcup_{m\in\mathbb{N}} \sigma^m \left(\pi^{-1}\left(\bigcup_{i,j\in S,\, i\neq j}(K_i \cap K_j)\right)\right)$ and $V_0 = \pi(\mathcal{P})$. The set $\mathcal{P}$ is called a post-critical set.

5.1. P. c. f. self-similar sets

Let us suppose that $\mathcal{P}$ is a finite set. In this case, $(K, S, \{\psi_i\}_{i\in S})$ is called post-critically finite (abbreviated to p. c. f.), which was introduced by Kigami [15]. By the proof of Lemma 4.2.3 in [17], the assumption (A1) is satisfied. Furthermore, assume that there exists a regular harmonic structure $(D, \mathbf{r})$ (see e.g. [17] for the detail). Then, it is known that we can construct a regular local Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with $(D, \mathbf{r})$ that satisfies (A2)–(A6) with $K^\partial = V_0$. Moreover, $\mathcal{F}$ is continuously imbedded in $C(K)$. Therefore, when we set $\hat{\mathcal{F}}_D = \{f \in \mathcal{F} \subset C(K) \mid f(x) = 0 \text{ for all } x \in K^\partial\}$, we can easily prove that $\mathcal{F}_D = \hat{\mathcal{F}}_D$ as follows. Since $\mathcal{F}_0 \subset \hat{\mathcal{F}}_D$ and $\hat{\mathcal{F}}_D$ is closed, we have $\mathcal{F}_D \subset \hat{\mathcal{F}}_D$. To prove the converse inclusion, let $f \in \hat{\mathcal{F}}_D$ and set $f_n = (f - 1/n)_+ - (f + 1/n)_-$. Then, $f_n \in \mathcal{F}_0$ and $f_n \to f$ in $\mathcal{F}$ as $n \to \infty$, which implies that $f \in \mathcal{F}_D$. Therefore, $\mathcal{H}$ is identical with the space of harmonic functions in [17]. The dimension of $\mathcal{H}$ is equal to $\#V_0 < \infty$, which implies that condition (C) with $u = \emptyset$ is satisfied. (In practice, (EHI), (D), and (R) are also true for $v = \emptyset$.) Thus, Theorem 2.1 can be applied. The result of Theorem 2.1 in the case that $q = p$ and s_i/p_i is independent of $i \in S$ is the same as that of Theorem 5.1 in [9]. The condition (O) is also assured by the following general lemma.

Lemma 5.1. *Suppose $\mathcal{F}$ is continuously imbedded in $C(K)$, that is, there exists $c_{5.1}$ such that $\|f\|_{L^\infty(\mu)} \le c_{5.1}\|f\|_{\mathcal{F}}$ for all $f \in \mathcal{F}$. Then,* (O) *holds with $z = \emptyset$, $N = 1$, and $A(w) = \{w\}$.*

Proof. Letting $f_+ = f \vee 0$ and $f_- = -(f \wedge 0)$ for $f \in \mathcal{F}$, we have

$$\begin{aligned} &|\,\mu\text{-esssup}\, f - \mu\text{-essinf}\, f|^2 \\ &\quad \le 4\|f\|_{L^\infty(\mu)}^2 \le 4c_{5.1}^2\|f\|_{\mathcal{F}}^2 \\ &\quad \le c_{5.2}\left\{\mathcal{E}(f,f) + \left(\int_K f\,d\mu\right)^2\right\}. \qquad \text{(by Lemma 3.2)} \end{aligned}$$

Taking $f - \int_K f\,d\mu$ in place of f, we obtain (O). □

As a special case, let $(K, S, \{\psi_i\}_{i\in S})$ be a nested fractal with length scaling factor L and mass scaling factor $M = \#S$. This object was introduced by Lindstrøm [23]. Here, we also refer to [1] for the details. We can construct a local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ satisfying (A2)–(A6) with $p_i = 1/M$ and $s_i = \rho$ for $i \in S$ for some resistance scaling factor ρ ([1, Theorem 6.23], originally [23]). By Proposition 6.30 in [1], $M\rho \ge L^2$. The shortest path scaling factor $\gamma \ge L(>1)$ can be defined (see Definition 5.42 in [1]); furthermore, in the same way as the proof of Corollary 5.41 in [1], it is proved that there exists a constant $c_{5.3}$ such that $d_m(x, y) \le c_{5.3}\gamma^m$ for every $x, y \in K$ and $m \in \mathbb{N}$. Let $d_w = \log(M\rho)/\log\gamma$ and assume that $d_w > 2$. Since $\gamma^m/(\min_{i\in S} s_i/q_i)^{m/2} = (\gamma^{2-d_w} M \max_{i\in S} q_i)^{m/2}$, the assumption of Theorem 2.4 is true if $q = \{q_i\}_{i\in S} \in \mathcal{A}$ satisfies $M \max_{i\in S} q_i < \gamma^{d_w-2}$. Combining Theorems 2.4 and 2.1, we obtain the following theorem.

Theorem 5.1. *If $q = \{q_i\}_{i\in S} \in \mathcal{A}$ satisfies $M \max_{i\in S} q_i < \gamma^{d_w-2}$, then $\mu_{\langle f\rangle} \perp \mu_q$ for every $f \in \mathcal{F}$.*

Since we can apply Theorem 5.1 for $q_i = 1/M(= p_i)$, $i \in S$, the set of q satisfying the assumption of the theorem above is an open neighborhood of p in $\mathcal{A}$.

We have the following heat kernel estimate (see e.g. [1, Theorem 8.18])

$$\begin{aligned} &c_{5.4} t^{-d_s/2} \exp\left(-c_{5.5}(d(x, y)^{d_w}/t)^{-1/(d_w-1)}\right) \le p(t, x, y) \\ &\le c_{5.6} t^{-d_s/2} \exp\left(-c_{5.7}(d(x, y)^{d_w}/t)^{-1/(d_w-1)}\right), \\ &t \in (0, 1],\ x, y \in K, \end{aligned} \tag{5.1}$$

where $d_s = 2\log M/\log(M\rho)$ and $d(\cdot,\cdot)$ is a suitable metric on K. Therefore, we may also apply Theorem 2.3 to deduce the singularity with respect to $\mu = \mu_p$.

For specific fractals, we can prove a stronger assertion. Let us consider the 2-dimensional Sierpinski gasket. Let $S = \{1, 2, 3\}$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and let $\{a_1, a_2, a_3\}$ be a set of vertices of an equilateral triangle in $\mathbb{C}$. Let T be a convex hull of $\{a_1, a_2, a_3\}$. Define $\psi_i(z) = (z - a_i)/2 + a_i$ for $i \in S$ and $K = \bigcap_{m\in\mathbb{N}} \bigcup_{w\in W_m} \psi_w(T)$. Then, $(K, S, \{\psi_i\}_{i\in S})$ is a self-similar structure and K is called the Sierpinski gasket. Let $p = (1/3, 1/3, 1/3) \in \mathcal{A}$, $\mu = \mu_p$, and $K^\partial = \{a_1, a_2, a_3\}$. The standard harmonic structure $(D, \mathbf{r})$ is given by $D = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ and $\mathbf{r} = (3/5, 3/5, 3/5)$. In other words, if we set

$$\mathcal{E}_0(f, f) = -{}^t R(f) D R(f), \quad R(f) = \begin{pmatrix} f(a_1) \\ f(a_2) \\ f(a_3) \end{pmatrix}$$

for a continuous function f on K, the canonical Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ is given by

$$\mathcal{E}(f, f) = \lim_{m\to\infty} \sum_{w\in W_m} \left(\frac{5}{3}\right)^m \mathcal{E}_0(\psi_w^* f, \psi_w^* f).$$

Here, the limit is an increasing limit, and $\mathcal{F}$ is the space of all f such that the limit above is finite. $(\mathcal{E}, \mathcal{F})$ has the self-similarity (A3) for $s = (5/3, 5/3, 5/3)$. For a harmonic function h, $\mathcal{E}(h, h) = \mathcal{E}_0(h, h)$ holds, and for each $j \in S$,

$$\begin{pmatrix} \psi_j^* h(a_1) \\ \psi_j^* h(a_2) \\ \psi_j^* h(a_3) \end{pmatrix} = A_j \begin{pmatrix} h(a_1) \\ h(a_2) \\ h(a_3) \end{pmatrix},$$

where

$$A_1 = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad A_2 = \frac{1}{5}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad A_3 = \frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix}.$$

See Examples 3.1.5 and 3.2.6 in [17] for further details. Using these data, we can calculate the value $\mathcal{E}(\psi_w^* h, \psi_w^* h)$ explicitly for any harmonic function h and $w \in W_*$. For any $q \in \mathcal{A}$, we can check that every harmonic function h satisfies $\Theta^{(2)}(h) \neq q^{(2)}$, particularly $\Theta(h) \neq \mathbf{q}$. Here, we used the same terminology as that in the proof of Theorem 2.1. Therefore, this concludes that $\mu_{\langle f \rangle} \perp \mu_q$ for every $f \in \mathcal{F}$ and every $q \in \mathcal{A}$.

5.2. *Sierpinski carpets*

As typical infinitely ramified self-similar sets, we consider Sierpinski carpets. Let $d \geq 2$, $l \geq 3$, and S be a finite set whose cardinality M is less than l^d. Assume that we are given a family $\{\psi_i\}_{i \in S}$ of contractive affine transformations on $\mathbb{R}^d$ of type $\psi_i(x) = a_i x + b_i$ for some $a_i \in \mathbb{R}_+$ and $b_i \in \mathbb{R}^d$ such that each ψ_i maps $F_0 = [0,1]^d$ onto $\prod_{j=1}^d [k_j/l, (k_j+1)/l]$ for some $k_j = 0, 1, \ldots, l-1$, and $\psi_i \neq \psi_{i'}$ if $i \neq i'$. Let $F_m = \bigcup_{w \in W_m} \psi_w(F_0)$ for $m \in \mathbb{N}$ and $K = \bigcap_{m \in \mathbb{N}} F_m$. Then, $(K, S, \{\psi_i\}_{i \in S})$ is a self-similar structure and K is called a Sierpinski carpet. We assume the following:

- (Symmetry) F_1 is preserved by all the isometries of the unit cube F_0.
- (Connectedness) $\mathrm{Int}(F_1)$ is connected and contains a path connecting the hyperplane $\{x_1 = 0\}$ and $\{x_1 = 1\}$.
- (Nondiagonality) Let B be a cube in F_0 with length $2/l$ and with vertices on $l^{-1}\mathbb{Z}$. Then, if $\mathrm{Int}(F_1 \cap B)$ is nonempty, it is connected.
- (Borders included) F_1 contains the line segment $\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leq x_1 \leq 1,\ x_2 = \cdots = x_d = 0\}$.

Above, $\mathrm{Int}(A)$ denotes the interior of A in $\mathbb{R}^d$. Barlow and Bass [3] constructed nondegenerate symmetric diffusions on K when $d = 2$ for the first time by taking a limit of the Brownian motions on Lipschitz domains of $\mathbb{R}^2$ converging to K. An analogous construction was devised in [6] for higher dimensions. On the other hand, Kusuoka and Zhou [21] provided symmetric diffusions on K when $d = 2$ using a limit of random walks on graphs. Such a method was generalized in [12] for higher dimensional spaces.

Here, we briefly review the method of construction by graph approximations for later convenience. We set $p = \{p_i\}_{i \in S}$ by $p_i = M^{-1}$ and let $\mu = \mu_p$, which is simply the normalized Hausdorff measure on K. For $m \in \mathbb{N}$, let $\mathcal{E}_m$ be a symmetric bilinear form in $C(W_m)$ defined by

$$\mathcal{E}_m(f, g) = \sum_{w, z \in W_m} q_{wz}^{(m)} (f(w) - f(z))(g(w) - g(z)), \qquad f, g \in C(W_m),$$

where $q_{wz}^{(m)} = 1$ if the Hausdorff dimension of $\psi_w(F_0) \cap \psi_z(F_0)$ is $d-1$, and $q_{wz}^{(m)} = 0$ otherwise. Let $B_m^{(1)} = \{w \in W_m \mid \psi_w(F_0) \cap (\{0\} \times [0,1]^{d-1}) \neq \emptyset\}$ and $B_m^{(2)} = \{w \in W_m \mid \psi_w(F_0) \cap (\{1\} \times [0,1]^{d-1}) \neq \emptyset\}$. Let

$$R_m = \min\{\mathcal{E}_m(f, f) \mid f \in C(W_m),\ f = 0 \text{ on } B_m^{(1)},\ f = 1 \text{ on } B_m^{(2)}\}^{-1}.$$

According to the result in [24], there exist some $c_{5.8}$, $c_{5.9}$, and ρ such that $c_{5.8}\rho^m \le R_m \le c_{5.9}\rho^m$ for every $m \in \mathbb{N}$. Set $T_m = R_m M^m$ for $m \in \mathbb{N}$ and $T = \rho M$. We define the operator $P_m : L^1(K, \mu) \to C(W_m)$ for $m \in \mathbb{N}$ by

$$P_m f(w) = \mu(K_w)^{-1} \int_{K_w} f(x)\,\mu(dx), \quad f \in L^1(K, \mu),\ w \in W_m.$$

Let $\mathcal{E}^{(m)}$, $m \in \mathbb{N}$, be a Dirichlet form on $L^2(K, \mu)$ defined by

$$\mathcal{E}^{(m)}(f, g) = R_m \mathcal{E}_m(P_m f, P_m g), \quad f, g \in L^2(K, \mu).$$

Let $(X(t);\ P_w^{(m)},\ w \in W_m)$ denote the Markov process on W_m associated with the Dirichlet form $(\mathcal{E}_m, L^2(W_m, M^{-m} \sum_{w\in W_m} \delta_w))$. Fix $x \in K$. Let $Q^{(m)}$ denote the law of the process $\{\psi_{X(T_m t)}(x)\}_{t\in\mathbb{Q}_+}$ on K with the initial law of $\{X(t)\}$ being $M^{-m} \sum_{w\in W_m} P_w^{(m)}$. Note that $Q^{(m)}$ is a probability measure on $K^{\mathbb{Q}_+}$. For any cluster point $\tilde{Q}$ of $\{Q^{(m)}\}_{m\in\mathbb{N}}$, we have the following theorem.

Theorem 5.2 ([21, 12])**.** *There exists a strongly continuous symmetric Markovian semigroup* $\{T_t\}_{t>0}$ *on* $L^2(K, \mu)$ *such that*

$$\begin{aligned}&\int_{K^{\mathbb{Q}_+}} f_0(\omega(t_0)) f_1(\omega(t_1)) \cdots f_n(\omega(t_n))\, \tilde{Q}(d\omega)\\ &= \int_K f_n \cdot T_{t_n - t_{n-1}}(f_{n-1}(T_{t_{n-1}-t_{n-2}}(f_{n-2}(\cdots(T_{t_1-t_0} f_0)\cdots))))\, d\mu\end{aligned}$$

for any $0 \le t_0 \le t_1 \le \cdots \le t_n \in \mathbb{Q}_+$ *and* $f_0, \ldots, f_n \in C(K)$*. Each* T_t *has an integral kernel* $p(t, x, y)$ *and the Aronson type estimate* (5.1) *holds with* $d_s = 2\log M/\log T$*,* $d_w = \log T/\log l$*, and the Euclidean distance* $d(\cdot,\cdot)$*. Moreover, the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *associated with* $\{T_t\}$ *satisfies the following*:

- $(\mathcal{E}, \mathcal{F})$ *is local.*
- $\mathcal{F} = \{f \in L^2(K, \mu) \mid \sup_m \mathcal{E}^{(m)}(f, f) < \infty\}$.

We will attempt to prove that every energy measure $\mu_{\langle f\rangle}$ of $(\mathcal{E}, \mathcal{F})$ is singular to μ. Since it is unknown whether $(\mathcal{E}, \mathcal{F})$ has the self-similarity (A3), we require additional arguments. For each $m \in \mathbb{Z}_+$, define

$$\bar{\mathcal{E}}_m(f, g) = \sum_{w\in W_m} \rho^m \mathcal{E}(\psi_w^* f, \psi_w^* g), \quad f, g \in \mathcal{F}.$$

Then, by the result presented in section 6 of [21], each $(\bar{\mathcal{E}}_m, \mathcal{F})$ is a Dirichlet form equivalent to $(\mathcal{E}, \mathcal{F})$ in the sense that there exist constants $c_{5.10}$ and $c_{5.11}$ independent of m satisfying

$$c_{5.10}\mathcal{E}(f, f) \le \bar{\mathcal{E}}_m(f, f) \le c_{5.11}\mathcal{E}(f, f), \qquad f \in \mathcal{F}.$$

Moreover, there exists a divergent sequence $\{m_k\}$ such that $m_k^{-1} \sum_{j=1}^{m_k} \bar{\mathcal{E}}_j(f, f)$ converges to some $\bar{\mathcal{E}}(f, f)$ for any $f \in \mathcal{F}$, and (after the polarization procedure)

$(\bar{\mathcal{E}}, \mathcal{F})$ becomes a regular Dirichlet form ([16]). In addition, (A1)–(A6) and (R) are true for this $\bar{\mathcal{E}}$ with $K^\partial = [0,1]^d \setminus (0,1)^d$ and $s_i = \rho$ for $i \in S$. Let $\bar{\mu}_{\langle f\rangle}$ denote the energy measure of $f \in \mathcal{F}$ associated with $(\bar{\mathcal{E}}, \mathcal{F})$. Since

$$c_{5.10}\mathcal{E}(f,f) \le \bar{\mathcal{E}}(f,f) \le c_{5.11}\mathcal{E}(f,f), \qquad f \in \mathcal{F},$$

Proposition 1.5.5(b) in [22] implies that $c_{5.10}\mu_{\langle f\rangle} \le \bar{\mu}_{\langle f\rangle} \le c_{5.11}\mu_{\langle f\rangle}$ for every $f \in \mathcal{F}$. (See also [25, p. 389] for simpler proof.) Therefore, it is sufficient to prove that $\bar{\mu}_{\langle f\rangle} \perp \mu_q$ in order to prove that $\mu_{\langle f\rangle} \perp \mu_q$. In order to apply Theorem 2.1 to $(\bar{\mathcal{E}}, \mathcal{F})$, it is necessary to check condition (C). The Harnack inequality is accepted for $(\mathcal{E}, \mathcal{F})$ for a general $d \ge 2$ ([6, 12]); unfortunately, the author is unable to determine whether $(\bar{\mathcal{E}}, \mathcal{F})$ satisfies (C) (see, however, Remark 5.1 below). At present, the case $d = 2$ will have to suffice. In this case, the following strong property holds: $\mathcal{F}$ is continuously imbedded in $C(K)$ ([21]). In particular, (D) holds and (O) is satisfied according to Lemma 5.1. The walk dimension d_w is greater than 2; see e.g. Remark 5.4 in [6]. Regarding the distance-like function defined in (2.7), we have the following estimate: there exists a constant $c_{5.12}$ such that

$$d_m(x,y) \le c_{5.12} l^m, \qquad m \in \mathbb{N},\ x, y \in K.$$

This is proved by the same idea as that used to prove Lemma 7.3 in [5]. Therefore, combining Theorems 2.1 and 2.4, we have the following theorem.

Theorem 5.3. *If $q = \{q_i\}_{i\in S} \in \mathcal{A}$ satisfies $\max_{i\in S} q_i < \rho/l^2$, then $\bar{\mu}_{\langle f\rangle} \perp \mu_q$ and $\mu_{\langle f\rangle} \perp \mu_q$ for every $f \in \mathcal{F}$.*

Note that $\rho/l^2 > 1/M$ since $d_w > 2$, and we can always take $q = p$.

For another class of symmetric diffusions on K due to Barlow and Bass [3, 6], it is also unknown whether the associated Dirichlet forms satisfy the self-similarity (A3). However, they have a transition estimate (5.1) with the same d_s and d_w ([5, 6]). Therefore, at least when $d = 2$, we can prove the same singularity as Theorem 5.3 for these Dirichlet forms by the following proposition.

Proposition 5.1. *Suppose that $\{T_t\}_{t>0}$ and $\{\hat{T}_t\}_{t>0}$ are symmetric and conservative Markovian semigroups on $L^2(\mu)$ having transition semigroups $p(t,x,y)$ and $\hat{p}(t,x,y)$, respectively, and that both have the Aronson-type estimate* (5.1) *with the same d_s and d_w and possibly other different constants. Let $(\mathcal{E}, \mathcal{F})$ and $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ be the Dirichlet forms associated with $\{T_t\}_{t>0}$ and $\{\hat{T}_t\}_{t>0}$, respectively. Then, $\mathcal{F} = \hat{\mathcal{F}}$ and there exist $c_{5.13}$ and $c_{5.14}$ such that*

$$c_{5.13}\mu_{\langle f\rangle} \le \hat{\mu}_{\langle f\rangle} \le c_{5.14}\mu_{\langle f\rangle}, \qquad f \in \mathcal{F},$$

where $\mu_{\langle f\rangle}$ (resp. $\hat{\mu}_{\langle f\rangle}$) is the energy measure of $f \in \mathcal{F}$ with respect to $(\mathcal{E}, \mathcal{F})$ (resp. $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$). In particular, $\mu_{\langle f\rangle}$ and $\hat{\mu}_{\langle f\rangle}$ are mutually absolutely continuous.

Proof. From the estimates of transition densities, there exist $c_{5.15}, \dots, c_{5.18}$ such that

$$p(t,x,y) \le c_{5.15}\hat{p}(c_{5.16}t,x,y), \quad \hat{p}(t,x,y) \le c_{5.17}p(c_{5.18}t,x,y), \quad x, y \in K,$$

when t, $c_{5.16}t$, and $c_{5.18}t$ belong to (0, 1]. Combining these estimates with the fact that $f \in \mathcal{F}$ if and only if

$$\infty > \lim_{t\to 0} t^{-1}(\|f\|^2_{L^2(\mu)} - \|T_{t/2}f\|^2_{L^2(\mu)})$$
$$= \lim_{t\to 0}(2t)^{-1}\int_{K\times K}(f(x)-f(y))^2 p(t,x,y)\,\mu(dx)\mu(dy),$$

and the same fact for $\hat{\mathcal{F}}$, we obtain $\mathcal{F} = \hat{\mathcal{F}}$.

For $f \in \mathcal{F}_b$ and $g \in \mathcal{F}_{b,+}$, let

$$I^t_f(g) = \frac{1}{t}\int_{K\times K} g(x)(f(x)-f(y))^2 p(t,x,y)\,\mu(dx)\mu(dy), \quad t > 0;$$

$\hat{I}^t_f(g)$ is similarly obtained. Then, $I^t_f(g) \le c_{5.15}c_{5.16}\hat{I}^{c_{5.16}t}_f(g)$ for small t. Letting $t \to 0$, we obtain

$$\int_K \tilde{g}\, d\mu_{\langle f\rangle} \le c_{5.15}c_{5.16}\int_K \tilde{g}\, d\hat{\mu}_{\langle f\rangle}.$$

Therefore, $\mu_{\langle f\rangle} \le c_{5.15}c_{5.16}\hat{\mu}_{\langle f\rangle}$. Similarly, we have the converse inequality. □

Remark 5.1. After submitting this paper, the author was informed that the elliptic Harnack inequality of the averaged Dirichlet form $(\bar{\mathcal{E}}, \mathcal{F})$ was proved for higher dimensional Sierpinski carpets in [8] by means of stability results for the parabolic Harnack inequality. Therefore, singularity of energy measures to the Hausdorff measure is now true for d-dimensional Sierpinski carpets with $d \ge 2$. For further details, see [8].

Remark 5.2. The notion of singularity of the energy measure is stable under a product in the following manner. Suppose that $(\mathcal{E}_i, \mathcal{F}_i)$ is a regular Dirichlet form on $L^2(X_i, \mu_i)$ for a measure space (X_i, μ_i) such that $\mu_i(X_i) = 1$, $1 \in \mathcal{F}_i$, and $\mathcal{E}_i(1, 1) = 0$, for $i = 1, \dots, n$. Let $\mathcal{L}_i$ be a nonpositive self-adjoint operator on $L^2(X_i, \mu_i)$ with domain $\mathrm{Dom}(\mathcal{L}_i)$ associated with $(\mathcal{E}_i, \mathcal{F}_i)$. Define $X = \prod_{i=1}^n X_i$ and $\mu = \bigotimes_{i=1}^n \mu_i$. Let $\bigotimes_{i=1}^n \mathrm{Dom}(\mathcal{L}_i)$ denote the set of all finite linear combinations of vectors $f_1 \otimes \cdots \otimes f_n$, where $f_i \in \mathrm{Dom}(\mathcal{L}_i)$. A linear operator $(\sum_{i=1}^n \mathcal{L}_i, \bigotimes_{i=1}^n \mathrm{Dom}(\mathcal{L}_i))$ on $\bigotimes_{i=1}^n L^2(X_i, \mu_i) \simeq L^2(X, \mu)$ is given by $(\sum_{i=1}^n \mathcal{L}_i)(f_1 \otimes \cdots \otimes f_n) := \sum_{i=1}^n f_1 \otimes \cdots \otimes f_{i-1} \otimes \mathcal{L}_i f_i \otimes f_{i+1} \otimes \cdots \otimes f_n$. It is known that this operator is essentially self-adjoint (see e.g. [27, p. 301, Corollary]). Let its closure be denoted by $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on (X, μ) associated with $\mathcal{L}$ and assume that it is regular. For $f = f_1 \otimes \cdots \otimes f_n \in \bigotimes_{i=1}^n \mathrm{Dom}(\mathcal{L}_i)$ and $g = g_1 \otimes \cdots \otimes g_n \in \bigotimes_{i=1}^n \mathrm{Dom}(\mathcal{L}_i)$, $\mathcal{E}(f, g) = \sum_{i=1}^n \mathcal{E}_i(f_i, g_i)\prod_{j\ne i}(f_j, g_j)_{L^2(\mu_j)}$. This expression implies that the energy measure $\mu_{\langle f\rangle}$ for such f is equal to

$$\sum_{i=1}^n f_1^2\mu_1 \otimes \cdots \otimes f_{i-1}^2\mu_{i-1} \otimes \mu_{i,\langle f_i\rangle} \otimes f_{i+1}^2\mu_{i+1} \otimes \cdots \otimes f_n^2\mu_n,$$

where $\mu_{i,\langle f_i\rangle}$ is an energy measure of f_i with respect to $(\mathcal{E}_i, \mathcal{F}_i)$. Let us assume that for some j, the energy measure $\mu_{j,\langle h\rangle}$ is singular to μ_j for all $h \in \mathcal{F}_j$. By the above expression, $\mu_{\langle f \rangle}$ is singular to μ for any $f \in \bigotimes_{i=1}^{n} \mathrm{Dom}(\mathcal{L}_i)$. Since $\mathrm{Dom}(\mathcal{L}_i)$ is dense in $\mathcal{F}$, by (2.5) and by a usual approximation argument we can conclude that $\mu_{\langle f \rangle} \perp \mu$ for all $f \in \mathcal{F}$.

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