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BSDEs with two reflecting barriers : the general result

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Abstract. In this paper we show the existence of a solution for the BSDE with two reflecting barriers when those latter are completely separated. Neither Mokobodzki's condition nor the regularity of the barriers are supposed. The main tool is the notion of local solution of reflected BSDEs. Applications related to Dynkin games and double obstacle variational inequality are given.

0. Introduction

A solution for a backward stochastic differential equation (BSDE in short) with two reflecting barriers $L := (L_t)_{t\le T}$ and $U := (U_t)_{t\le T}$ $(L \le U)$ and whose coefficient and terminal value are respectively f and ξ is a quadruple of adapted processes $(Y, Z, K^+, K^-) := (Y_t, Z_t, K_t^+, K_t^-)_{t\le T}$ with values in $R^{1+d+1+1}$ which mainly satisfies:

$$\begin{cases} K^{\pm} \text{ are continuous non-decreasing processes} \\ -dY_t = f(t, \omega, Y_t, Z_t) + dK_t^+ - dK_t^- - Z_t dB_t,\ t \le T \text{ and } Y_T = \xi \\ \forall t \le T,\ \ L_t \le Y_t \le U_t \text{ and } \displaystyle\int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (U_t - Y_t) dK_t^- = 0. \end{cases} \tag{1}$$

Here $B := (B_t)_{t\le T}$ is a Brownian motion and ξ is F_T-measurable where for any $t \le T$, $F_t = \sigma\{B_s, s \le t\}$; the adaptation is with respect to $(F_t)_{t\le T}$.

BSDEs with two reflecting barriers have been first introduced by J.Cvitanic & I.Karatzas [CK]. Their work generalizes the one of El-Karoui et al. [EKal] related to reflected BSDEs with just one barrier. Since then the interest to those equations grows steadily because they are an important tool in many mathematical fields especially in stochastic games ([HL],[H]) and mathematical finance ([MC], [H]).

However the known results which provide a solution for (1) are not very satisfactory and are of two types. Actually authors suppose either Mokobodzki's assumption

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([CK], [HL]), which means the existence of a difference of non-negative supermartingales between L and U, or the regularity of one of the barriers L and U ([CK],[HLM]), which roughly speaking has to be a semi-martingale.

Mokobodzki's condition is a bit troublesome since it is difficult to verify in practice. On the other hand, the regularity of one of the barriers is somewhat restrictive.

So the main objective of this work is to investigate under which conditions on the barriers, as weak as possible and easy to verify, the BSDE (1) has a solution. Indeed, we show that if the barriers are completely separated, *i.e.*, $L_t < U_t, \forall t \le T$, then a solution for (1) exits. In addition it is unique when the function f is Lipschitz with respect to (y, z). As a by-product we deduce that the value function of a Dynkin game is a semi-martingale when the barriers are completely separated. On the other hand, under the same assumption, we deduce also the existence of a semi-martingale which passes between L and U. This latter result has an important application in mathematical finance in connection with the problem of the existence of an optimal investment strategy in real options ([HJ]).

This paper is organized as follows:

In Section 1 we set up accurately the problem and we give some preliminary results related to BSDEs with one or two barriers (existence and comparison). In Section 2, we deal with the notion of local solution of the BSDE (1) which is a solution of that equation but just between two comparable stopping times. Some properties of local solutions are given, especially comparison, uniqueness and representation as a value function of a Dynkin game. Section 3 is devoted to the existence of some useful local solutions. Therefore step-by-step we construct a solution for the BSDE (1). This is the main result of this paper. The solution is unique when f is Lipschitz. In Section 4, we study some properties of the solution of (1) and we show, once again, the existence of a solution when f is just continuous with at most linear growth. Finally in Section 5, we assume that the randomness stems from a diffusion process and we study the connection of the solution of (1) with its related double obstacle variational inequality. A solution in viscosity sense for this latter is proved.

1. Setting of the problem. Preliminary results

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{t\le T}$ whose natural filtration is $(F_t^0 := \sigma\{B_s, s \le t\})_{t\le T}$. On the other hand let $(F_t)_{t\le T}$ be the completed filtration of $(F_t^0)_{t\le T}$ with the P-null sets of $\mathcal{F}$, hence $(F_t)_{t\le T}$ satisfies the usual conditions, *i.e.*, it is right continuous and complete. Now let :

- $\mathcal{P}$ be the σ-algebra on $[0, T] \times \Omega$ of F_t-progressively measurable sets
- for any stopping time $\tau \in [0, T]$, $\mathcal{T}_\tau$ denotes the set of all stopping times θ such that $\tau \le \theta \le T$
- $\mathcal{M}^{2,k}$ be the set of $\mathcal{P}$-measurable and R^k-valued processes $w = (w_t)_{t\le T}$ which belong to $L^2([0, T] \times \Omega, dP \otimes dt)$

- $\mathcal{M}$ the set of $\mathcal{P}$-measurable processes $Z = (Z_t)_{t\leq T}$ with values in R^d such that $\int_0^T |Z_s|^2 ds < \infty$, P-$a.s.$.
- $\mathcal{S}^2$ the set of $\mathcal{P}$-measurable and continuous processes $\bar{w} = (\bar{w}_t)_{t\leq T}$ such that $E[\sup_{t\leq T} |\bar{w}_t|^2] < \infty$
- $\mathcal{S}_{ci}$ (resp. $\mathcal{S}^2_{ci}$) the set of continuous $\mathcal{P}$-measurable non-decreasing processes $K := (K_t)_{t\leq T}$ such that $K_0 = 0$ (resp. and $E[(K_T)^2] < \infty$).

Now we are given four objects :

(i) a terminal value ξ which is a random variable F_T-measurable such that $E[\xi^2] < \infty$

(ii) two processes $U := (U_t)_{t\leq T}$ and $L := (L_t)_{t\leq T}$ which belong to $\mathcal{S}^2$ and satisfy $L_t \leq U_t, \forall t \leq T$, and $L_T \leq \xi \leq U_T$

(iii) a function $f : [0, T]\times\Omega\times R^{1+d} \longrightarrow R$ such that for any $(y, z) \in R^{1+d}$, the process $(f(t, \omega, y, z))_{t\leq T}$ is $\mathcal{P}$-measurable and $(f(t, \omega, 0, 0))_{t\leq T}$ belongs to $\mathcal{M}^{2,1}$. In addition, it is uniformly Lipschitz with respect to (y, z) uniformly in (t, ω), $i.e$:

[H1] : there exists a constant $C \geq 0$ such that

$$\begin{aligned}P - a.s.,\ \ & |f(t, y, z) - f(t, y', z')| \\ & \leq C(|y - y'| + |z - z'|),\ \text{for any } t, y, y', z \text{ and } z' \square\end{aligned}$$

A solution for the two reflecting barrier BSDE associated with (f, ξ, L, U) is a quadruple of $\mathcal{P}$-measurable processes $(Y, Z, K^+, K^-) := (Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ with values in $R^{1+d+1+1}$ such that :

$$\begin{cases} Y \in \mathcal{S}^2\,,\ Z \in \mathcal{M} \text{ and } K^+, K^- \in \mathcal{S}_{ci} \\ Y_t = \xi + \displaystyle\int_t^T f(s, Y_s, Z_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s,\ \ \forall t \leq T \\ L_t \leq Y_t \leq U_t,\ \ \forall t \leq T \text{ and } \displaystyle\int_0^T (Y_s - L_s)dK_s^+ = \int_0^T (U_s - Y_s)dK_s^- = 0. \end{cases} \tag{2}$$

In our setting, with respect to the ones of [CK] or [HLM], we do not require strong integrability assumptions on Z and $K^\pm$ since in many applications, especially in stochastic games or mathematical finance, we do not need such properties for those processes. However, as we will show it later, there exits a solution for (2) for which we can construct a sequence of stopping times $(\tau_n)_{n\geq 0}$ which is “almost independent” of (Y, Z, K^+, K^-) such that for any $n \geq 0$ the processes $(Z_t 1_{[t\leq\tau_n]})_{t\leq T}$ and $(K^\pm_{t\wedge\tau_n})_{t\leq T}$ belong respectively to $\mathcal{M}^{2,d}$ and $\mathcal{S}^2_{ci}$.

It is well known that the backward equation (2) has a unique solution under either one of the following conditions. The first is the regularity of one of the barriers which, roughly speaking, has to be a semi-martingale ([CK],[HLM]). The other one is Mokobodzki’s hypothesis which turns into the existence of a difference of non-negative super-martingales between L and U. Also, without those assumptions we do not have any result which provides a solution for (2). On the other hand, this equation may not have a solution for general processes L and U. Indeed, assume that P-a.s. and for any $s \in [0, T]$ we have $L_s = U_s$. Therefore if the process L is not a semi-martingale then obviously the BSDE (2) cannot have a solution.

So the main objective of this work is to study under which conditions, especially on L and U, as weak as possible and easier to verify in practice, the BSDE (2) has a solution. Mainly we show that if $L_t < U_t, \forall t \leq T$ then it has a solution. □

A. Preliminary results

First let us give the following result, established in [EKal], related to reflected BSDEs with just one upper barrier .

1.1. Theorem. *There exists a unique $\mathcal{P}$-measurable process $(Y, Z, K^-) = (Y_t, Z_t, K_t^-)_{t \leq T}$ with values in R^{1+d+1} solution of the single upper barrier reflected BSDE associated with (f, ξ, U), i.e., which satisfies*

$$\begin{cases} Y \in \mathcal{S}^2, Z \in \mathcal{M}^{2,d} \text{ and } K^- \in \mathcal{S}^2_{ci} \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - (K_T^- - K_t^-) - \int_t^T Z_s dB_s, \forall t \leq T \\ Y_t \leq U_t, \ \forall t \leq T \text{ and } \int_0^T (U_s - Y_s)dK_s^- = 0 \end{cases} \quad \square \tag{3}$$

1.2. Remark. The existence and uniqueness of a solution for equation (3) still valid if instead of a deterministic terminal time T we have a bounded stopping time τ.

B. Comparison of solutions of BSDEs with two reflecting barriers

Assume now that the reflected BSDE associated with (f, ξ, L, U) has a solution (Y, Z, K^+, K^-) (in the sense of (2)). On the other hand let $(f'(t, \omega, y, z), \xi', L', U')$ be another quadruple which satisfies :

- the process $(f'(t, 0, 0))_{t \leq T}$ belongs to $\mathcal{M}^{2,1}$ and for any $(y, z) \in R^{1+d}$, $(f'(t, y, z))_{t \leq T}$ is $\mathcal{P}$-measurable
- L' and U' belong to $\mathcal{S}^2$, $L'_t \leq U'_t, \forall t \leq T$; $\xi' \in L^2(\Omega, F_T, dP)$ and satisfies $L'_T \leq \xi' \leq U'_T$.

Suppose now that the two barrier reflected BSDE associated with (f', ξ', L', U') has a solution (Y', Z', K'^+, K'^-). The following result allows us to compare Y and Y', K^+ and K'^+, and finally K^- and K'^- since we can compare the quadruples (f, ξ, L, U) and (f', ξ', L', U'). Namely we have :

1.3. Theorem. *Assume that $\xi \leq \xi'$ and for any $t \leq T$, $L_t \leq L'_t$, $U_t \leq U'_t$, $f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$, then P-a.s. $Y \leq Y'$. In addition if :*

(i) $f(t, y, z) \leq f'(t, y, z)$ for any (t, y, z)
(ii) $\forall t \leq T$, $L_t = L'_t$, $U_t = U'_t$ and $L_t < U_t$
(iii) the function f' is uniformly Lipschitz with respect to (y, z) uniformly in (t, ω).

Then we have also, $\forall t \leq T$, $K_t^- \leq K_t'^-$ and $K_t^+ \geq K_t'^+$.

Proof. First let us point out that if $(X_t)_{t\le T}$ is an R-valued continuous semimartingale and if $X_t^+ := \max\{X_t, 0\}$ then

$$d(X_t^+)^2 = 2X_t^+ dX_t + 1_{[X_t>0]} d < X, X >_t, \ t \le T. \quad (4)$$

Actually using Tanaka's formula we obtain

$$dX_t^+ = 1_{[X_t>0]} dX_t + \frac{1}{2} dL_t, t \le T$$

where $(L_t)_{t\le T}$ is an increasing adapted process such that $\int_0^t X_s dL_s = 0, \forall t \le T$. Now it order to deduce the result it is enough to remark that $(X_t^+)^2 = X_t.X_t^+$ and to use Itô's formula.

Let us show that $Y \le Y'$. For $k \ge 0$, let τ_k be the stopping time defined as follows :

$$\tau_k = \inf\{t \ge 0, \int_0^t (|Z_s| + |Z'_s|)^2 ds \ge k\} \wedge T$$

(we always assume that $\inf\{\emptyset\} = \infty$).

Next relation (4) with $Y - Y'$ yields: $\forall t \le T$,

$$\begin{aligned}
&(Y_{t\wedge\tau_k} - Y'_{t\wedge\tau_k})^{+2} + \int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} |Z_s - Z'_s|^2 ds \\
&= 2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)) ds \\
&\quad +2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (dK_s - dK'_s) \\
&\quad -2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (Z_s - Z'_s) dB_s + (Y_{\tau_k} - Y'_{\tau_k})^{+2} \\
&\le 2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) ds \\
&\quad +2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (dK_s - dK'_s) \\
&\quad -2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (Z_s - Z'_s) dB_s + (Y_{\tau_k} - Y'_{\tau_k})^{+2}.
\end{aligned}$$

since $f(t, Y'_t, Z'_t) \le f'(t, Y'_t, Z'_t)$. But $\int_0^t 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (dK_s - dK'_s) = \int_0^t 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (-dK_s^- - dK_s'^+) \le 0$ since when $Y_t > Y'_t$ we have $Y_t > L_t$ and $U'_t > Y'_t$. Henceforth

$$\begin{aligned}
&(Y_{t\wedge\tau_k} - Y'_{t\wedge\tau_k})^{+2} + \int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} |Z_s - Z'_s|^2 ds \\
&\le 2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) ds \\
&\quad -2\int_{t\wedge\tau_k}^{\tau_k} 1_{[Y_s>Y'_s]} (Y_s - Y'_s)^+ (Z_s - Z'_s) dB_s + (Y_{\tau_k} - Y'_{\tau_k})^{+2}, \ \forall t \le T.
\end{aligned}$$

As f is Lipschitz then we can write $f(t,Y_t,Z_t)-f(t,Y'_t,Z'_t)=a_t(Y_t-Y'_t)+b_t(Z_t-Z'_t)$, $t\le T$, where $(a_t)_{t\le T}$ and $(b_t)_{t\le T}$ are bounded $\mathcal{P}$-measurable processes. Therefore for any $t\le T$,

$$\begin{aligned}&(Y_{t\wedge\tau_k}-Y'_{t\wedge\tau_k})^{+2}+\int_{t\wedge\tau_k}^{\tau_k}1_{[Y_s>Y'_s]}|Z_s-Z'_s|^2ds\\&\quad\le 2\int_{t\wedge\tau_k}^{\tau_k}1_{[Y_s>Y'_s]}(Y_s-Y'_s)^+\{a_s(Y_s-Y'_s)+b_s(Z_s-Z'_s)\}ds\\&\qquad-2\int_{t\wedge\tau_k}^{\tau_k}1_{[Y_s>Y'_s]}(Y_s-Y'_s)^+(Z_s-Z'_s)dB_s+(Y_{\tau_k}-Y'_{\tau_k})^{+2}.\end{aligned}$$

Now using the inequality $|x.y|\le\epsilon|x|^2+\epsilon^{-1}|y|^2$, $\forall\epsilon>0$ and $x,y\in R$ we obtain,

$$\begin{aligned}(Y_{t\wedge\tau_k}-Y'_{t\wedge\tau_k})^{+2}&\le(Y_{\tau_k}-Y'_{\tau_k})^{+2}+C\int_{t\wedge\tau_k}^{\tau_k}(Y_s-Y'_s)^{+2}ds\\&\quad-2\int_{t\wedge\tau_k}^{\tau_k}1_{[Y_s>Y'_s]}(Y_s-Y'_s)^+(Z_s-Z'_s)dB_s\end{aligned}$$

where C is a constant. But for any $t\le T$ we have,

$$\begin{aligned}&\int_{t\wedge\tau_k}^{\tau_k}(Y_s-Y'_s)^{+2}ds=1_{[t\le\tau_k]}\int_t^{\tau_k}(Y_s-Y'_s)^{+2}ds\\&\quad=1_{[t\le\tau_k]}\int_t^{\tau_k}(Y_{s\wedge\tau_k}-Y'_{s\wedge\tau_k})^{+2}ds\le\int_t^T(Y_{s\wedge\tau_k}-Y'_{s\wedge\tau_k})^{+2}ds.\end{aligned}$$

Therefore for any $t\le T$ we have

$$\begin{aligned}(Y_{t\wedge\tau_k}-Y'_{t\wedge\tau_k})^{+2}&\le(Y_{\tau_k}-Y'_{\tau_k})^{+2}+C\int_t^T(Y_{s\wedge\tau_k}-Y'_{s\wedge\tau_k})^{+2}ds\\&\quad-2\int_{t\wedge\tau_k}^{\tau_k}1_{[Y_s>Y'_s]}(Y_s-Y'_s)^+(Z_s-Z'_s)dB_s.\end{aligned}$$

But the process $(\int_0^{t\wedge\tau_k}1_{[Y_s>Y'_s]}(Y_s-Y'_s)^+(Z_s-Z'_s)dB_s)_{t\le T}$ is a martingale, then taking the expectation in both sides, and using Gronwall's inequality implies that $E[(Y_{t\wedge\tau_k}-Y'_{t\wedge\tau_k})^{+2}]\le CE[(Y_{\tau_k}-Y'_{\tau_k})^{+2}]$, $\forall t\le T$. Finally taking the limit in both sides as $k\to\infty$ we get $(Y_t-Y'_t)^{+2}=0$ and then $Y\le Y'$.

The proof of the second point is given in [BHM]. However for the reader's convenience we give it again.

Let us prove that $K^{'-}\ge K^-$. Let $\tau=\inf\{t\ge 0, K_t^->K_t^{'-}\}\wedge T$. We are going to show that $P[\tau<T]=0$ which implies that $K_t^-\le K_t^{'-}$, $\forall t<T$ and then $K^-\le K^{'-}$ by continuity.

Suppose that $P[\tau<T]>0$. Since K^- and $K^{'-}$ are continuous processes we have $K_\tau^-=K_\tau^{'-}$ on the set $\{\tau<T\}$. On the other hand we have $Y_\tau=Y'_\tau=U_\tau$ on the set $\{\tau<T\}$. Indeed, let $\omega\in\{\tau<T\}$; if $Y_{\tau(\omega)}(\omega)\ne U_{\tau(\omega)}(\omega)$, then there exists a real number $\eta(\omega)>0$ such that $\forall t\in]\tau(\omega)-\eta(\omega),\tau(\omega)+\eta(\omega)[$ we have

$Y_t(\omega) < U_t(\omega)$ which implies that $K^-_{\tau(\omega)}(\omega) = K^{'-}_{\tau(\omega)}(\omega) = K^-_t(\omega) \leq K^{'-}_t(\omega)$, $\forall t \in [\tau(\omega), \tau(\omega) + \eta(\omega)[$. But this contradicts the definition of $\tau(\omega)$, henceforth $Y_{\tau(\omega)}(\omega) = U_{\tau(\omega)}(\omega) = Y'_{\tau(\omega)}(\omega)$ since $Y \leq Y' \leq U$.

Now let $\delta = \inf\{t \geq \tau, Y_t = L_t\} \wedge T$. We have $\{\tau < T\} \subset \{\delta > \tau\}$. Indeed, if ω is such that $\tau(\omega) < T$ then $Y_{\tau(\omega)}(\omega) = U_{\tau(\omega)}(\omega)$. Now if $\delta(\omega) = \tau(\omega)$ then $Y_{\delta(\omega)}(\omega) = L_{\delta(\omega)}(\omega) = U_{\tau(\omega)}(\omega) = L_{\tau(\omega)}(\omega)$ which is absurd since $U_t > L_t, \forall t < T$. Hence $\{\tau < T\} \subset \{\delta > \tau\}$ and then $P[\delta > \tau] > 0$.

For $t \in [\tau, \delta]$ we have $K^+_t = K^+_\delta$ and $K^{'+}_t = K^{'+}_\delta$ since the processes K^+ (resp. $K^{'+}$) moves only when Y (resp. Y') reaches the obstacle L and $Y \leq Y'$. It follows that, $\forall t \in [\tau, \delta]$,

$$Y_t = Y_\delta + \int_t^\delta f(s, Y_s, Z_s)ds - (K^-_\delta - K^-_t) - \int_t^\delta Z_s dB_s \text{ and}$$

$$Y'_t = Y'_\delta + \int_t^\delta f'(s, Y'_s, Z'_s)ds - (K^{'-}_\delta - K^{'-}_t) - \int_t^\delta Z'_s dB_s.$$

Now let $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t)_{t\leq\delta}$ (resp. $(\bar{Y}'_t, \bar{Z}'_t, \bar{K}'_t)_{t\leq\delta}$) be the unique solution on $[0, \delta]$ of the BSDE whose coefficient is f (resp. f'), the terminal value Y_δ (resp. Y'_δ) and reflected in the upper obstacle U, *i.e.*,

$$\bar{Y}_t = Y_\delta + \int_t^\delta f(s, \bar{Y}_s, \bar{Z}_s)ds - (\bar{K}^-_\delta - \bar{K}^-_t) - \int_t^\delta \bar{Z}_s dB_s$$

$$\text{(resp. } \bar{Y}'_t = Y'_\delta + \int_t^\delta f'(s, \bar{Y}'_s, \bar{Z}'_s)ds - (\bar{K}^{'-}_\delta - \bar{K}^{'-}_t) - \int_t^\delta \bar{Z}'_s dB_s,\ \forall t \leq \delta).$$

The comparison theorem for one upper barrier reflected BSDEs (see e.g. [HLM], Prop.2.3) implies that $\bar{Y} \leq \bar{Y}'$ and $\bar{K}^-_t - \bar{K}^-_s \leq \bar{K}^{'-}_t - \bar{K}^{'-}_s, \forall s \leq t \leq \delta$. Since f and f' are Lipschitz in (y, z) then $\forall t \in [\tau, \delta]$ we have $\bar{Y}_t = Y_t$, $\bar{Y}'_t = Y'_t$, $\bar{Z}_t = Z_t$ and $\bar{Z}'_t = Z'_t$. It follows that $\bar{K}^-_\delta - \bar{K}^-_t = K^-_\delta - K^-_t$ and $\bar{K}'^-_\delta - \bar{K}'^-_t = K^{'-}_\delta - K^{'-}_t, \forall t \in [\tau, \delta]$. Henceforth we have $K^{'-}_t - K^{'-}_s \geq K^-_t - K^-_s$ for any $\tau(\omega) \leq s \leq t \leq \delta(\omega)$. As on $\{\tau < T\}$, $K^{'-}_\tau = K^-_\tau$ then $K^{'-}_t(\omega) \geq K^-_t(\omega), \forall t \in [\tau(\omega), \delta(\omega)]$. But this contradicts the definition of τ, hence $P[\tau < T] = 0$ and then $K^- \leq K^{'-}$. In the same way we show that P-*a.s.*, $K^+ \geq K^{'+}$. The proof is now complete □

BSDEs with one upper reflecting barrier is a particular case of (2) when we assume that $L \equiv -\infty$ and then $K^+ \equiv 0$. Therefore we can deduce from Theorem 1.3, a comparison result for the solutions of BSDEs with one upper barrier. Actually, let ξ' be a random variable of $L^2(\Omega, F_T, dP)$, $f'(t, \omega, y, z)$ be another function such that $(f'(t, \omega, 0, 0))_{t\leq T} \in \mathcal{M}^{2,1}$ and $(f'(t, \omega, y, z))_{t\leq T}$ is $\mathcal{P}$-measurable and finally U' another process of $\mathcal{S}^2$ such that $U'_T \geq \xi'$. Assume now that the one upper barrier reflected BSDE associated with (f', ξ', U') has a solution (Y', Z', K'). Then we have,

1.4. Corollary. *Assume that :*

(i) $\xi \leq \xi'$ *and* $U_t \leq U'_t, \forall t \leq T$

(ii) $\forall t \leq T,\ f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$.

Then P-a.s. $\forall t \leq T,\ Y_t \leq Y'_t$ □

2. Local solutions

Let τ and γ be two stopping times such that $\tau \leq \gamma$, P-$a.s.$. Let $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ be a $\mathcal{P}$-measurable process with values in $R^{1+d+1+1}$.

2.1. Definition. *The process $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ is called a local solution on $[\tau, \gamma]$ for the two barrier reflected BSDE associated with (f, ξ, L, U) if :*

$$\begin{cases} Y \in \mathcal{M}^{2,1},\ Z \in \mathcal{M}^{2,d},\ \ K^{\pm} \in \mathcal{S}^2_{ci} \\ Y_t = Y_\gamma + \displaystyle\int_t^\gamma f(s, Y_s, Z_s)ds + (K^+_\gamma - K^+_t) - (K^-_\gamma - K^-_t) \\ \qquad - \displaystyle\int_t^\gamma Z_s dB_s,\ \forall t \in [\tau, \gamma];\ Y_T = \xi \\ L_t \leq Y_t \leq U_t,\ \forall t \in [\tau, \gamma]\ \text{and}\ \displaystyle\int_\tau^\gamma (Y_s - L_s)dK^+_s = \int_\tau^\gamma (U_s - Y_s)dK^-_s = 0 \end{cases} \tag{5}$$

The notion of a local solution is in way a solution for (2) but just between two stopping times and $Y_T = \xi$ □

2.2. Connection with Dynkin games

Let us consider a process $g := (g_s)_{s\leq T}$ which belongs to $\mathcal{M}^{2,1}$ and τ a stopping time. The Dynkin game on $[\tau, T]$ associated with (g, ξ, L, U) is a zero-sum game on stopping times where the payoff after τ is given by:

$$\begin{aligned} \tilde{\Gamma}_\tau(\nu, \sigma) := E[&\int_\tau^{\nu\wedge\sigma} g_s ds + L_\sigma 1_{[\sigma\leq\nu<T]} \\ &+U_\nu 1_{[\nu<\sigma]} + \xi 1_{[\nu=\sigma=T]} | F_\tau],\ \ \forall \nu,\ \sigma \in \mathcal{T}_\tau. \end{aligned}$$

Dynkin games arise naturally when two agents c_1 and c_2, whose advantages are antagonistic, act on a system up to the time when one of them decides to stop its intervention (see e.g. [H] for more details on this subject).

The value function of the Dynkin game on $[\tau, T]$ is an $(F_t)_{t\leq T}$-adapted process $(\tilde{Y}_t)_{t\in[0,T]}$ such that P$-a.s.$,

$$\forall t \in [\tau, T],\ \tilde{Y}_t = \text{essinf}_{\nu\in\mathcal{T}_t}\text{esssup}_{\sigma\in\mathcal{T}_t}\tilde{\Gamma}_t(\nu, \sigma) = \text{esssup}_{\sigma\in\mathcal{T}_t}\text{essinf}_{\nu\in\mathcal{T}_t}\tilde{\Gamma}_t(\nu, \sigma).$$

In that case, the random variable $\tilde{Y}_\tau$ is just called the value of the game on $[\tau, T]$. On the other hand a pair of stopping times (ν_τ, σ_τ) which belongs to $\mathcal{T}_\tau \times \mathcal{T}_\tau$ and which satisfies

$$\tilde{\Gamma}_\tau(\nu_\tau, \sigma) \leq \tilde{\Gamma}_\tau(\nu_\tau, \sigma_\tau) \leq \tilde{\Gamma}_\tau(\nu, \sigma_\tau), \forall \nu, \sigma \in \mathcal{T}_\tau$$

is called a saddle-point for the Dynkin game on $[\tau, T]$.

Let $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ be a local solution on $[\tau, \gamma]$ for the BSDE with two reflecting barriers associated with (f, ξ, L, U). Let ν_τ, σ_τ be the stopping times defined as:

$$\nu_\tau := \inf\{s \geq \tau, Y_s = U_s\} \wedge T \text{ and } \sigma_\tau := \inf\{s \geq \tau, Y_s = L_s\} \wedge T.$$

On the other hand let $\Gamma_\tau(\nu, \sigma)$ be the payoff associated with the Dynkin game on $[\tau, T]$ associated with $((f(t, Y_t, Z_t))_{t\geq 0}, \xi, L, U)$. Then we have :

2.2.1. Proposition. *If P-a.s.* $\max\{\nu_\tau, \sigma_\tau\} \le \gamma$ *then :*

(i) $Y_\tau = \Gamma_\tau(\nu_\tau, \sigma_\tau)$

(ii) $\Gamma_\tau(\nu_\tau, \sigma) \le Y_\tau \le \Gamma_\tau(\nu, \sigma_\tau)$ *for any* $\nu, \sigma \in \mathcal{T}_\tau$.

Therefore Y_τ *is the value of the Dynkin game on* $[\tau, T]$ *and* (ν_τ, σ_τ) *is a saddle-point for the game.*

Proof. Since Y is continuous on $[\tau, \gamma]$ and P-$a.s.$ $\max\{\nu_\tau, \sigma_\tau\} \le \gamma$ then $Y_{\nu_\tau} = U_{\nu_\tau}$ on $[\nu_\tau < T]$ and $Y_{\sigma_\tau} = L_{\sigma_\tau}$ on $[\sigma_\tau < T]$. On the other hand we have :

$$\begin{aligned} Y_\tau = Y_{\nu_\tau\wedge\sigma_\tau} &+ \int_\tau^{\nu_\tau\wedge\sigma_\tau} f(s, Y_s, Z_s)ds + (K^+_{\nu_\tau\wedge\sigma_\tau} - K^+_\tau) \\ &-(K^-_{\nu_\tau\wedge\sigma_\tau} - K^-_\tau) - \int_\tau^{\nu_\tau\wedge\sigma_\tau} Z_s dB_s. \end{aligned} \tag{6}$$

But $\int_\tau^\gamma (Y_s - L_s)dK^+_s = \int_\tau^\gamma (U_s - Y_s)dK^-_s = 0$ therefore $K^+_{\nu_\tau\wedge\sigma_\tau} - K^+_\tau = 0$ and $K^-_{\nu_\tau\wedge\sigma_\tau} - K^-_\tau = 0$. In addition we have

$$\begin{aligned} Y_{\nu_\tau\wedge\sigma_\tau} &= Y_{\sigma_\tau} 1_{[\sigma_\tau \le \nu_\tau < T]} + Y_{\nu_\tau} 1_{[\nu_\tau < \sigma_\tau]} + \xi 1_{[\nu_\tau = \sigma_\tau = T]} \\ &= L_{\sigma_\tau} 1_{[\sigma_\tau \le \nu_\tau < T]} + U_{\nu_\tau} 1_{[\nu_\tau < \sigma_\tau]} + \xi 1_{[\nu_\tau = \sigma_\tau = T]} \end{aligned}$$

since P-$a.s.$, $Y_{\nu_\tau} = U_{\nu_\tau}$ on $[\nu_\tau < T]$ and $Y_{\sigma_\tau} = L_{\sigma_\tau}$ on $[\sigma_\tau < T]$. It follows that

$$\begin{aligned} Y_\tau = E[&\int_\tau^{\nu_\tau\wedge\sigma_\tau} f(s, Y_s, Z_s)ds + L_{\sigma_\tau} 1_{[\sigma_\tau \le \nu_\tau < T]} + U_{\nu_\tau} 1_{[\nu_\tau < \sigma_\tau]} \\ &+\xi 1_{[\nu_\tau = \sigma_\tau = T]} | F_\tau] = \Gamma_\tau(\nu_\tau, \sigma_\tau) \end{aligned}$$

after taking the conditional expectation in (6).

Now let ν be a stopping time of $\mathcal{T}_\tau$. Since $\nu \wedge \sigma_\tau \le \gamma$ then

$$\begin{aligned} Y_\tau = Y_{\nu\wedge\sigma_\tau} &+ \int_\tau^{\nu\wedge\sigma_\tau} f(s, Y_s, Z_s)ds + (K^+_{\nu\wedge\sigma_\tau} - K^+_\tau) \\ &-(K^-_{\nu\wedge\sigma_\tau} - K^-_\tau) - \int_\tau^{\nu\wedge\sigma_\tau} Z_s dB_s. \end{aligned}$$

But $K^+_{\nu\wedge\sigma_\tau} - K^+_\tau = 0$ and $K^-_{\nu\wedge\sigma_\tau} - K^-_\tau \ge 0$ therefore we have :

$$Y_\tau \le Y_{\nu\wedge\sigma_\tau} + \int_\tau^{\nu\wedge\sigma_\tau} f(s, Y_s, Z_s)ds - \int_\tau^{\nu\wedge\sigma_\tau} Z_s dB_s.$$

As

$$\begin{aligned} Y_{\nu\wedge\sigma_\tau} &= Y_{\sigma_\tau} 1_{[\sigma_\tau \le \nu < T]} + Y_\nu 1_{[\nu < \sigma_\tau]} + \xi 1_{[\nu = \sigma_\tau = T]} \\ &\le L_{\sigma_\tau} 1_{[\sigma_\tau \le \nu < T]} + U_\nu 1_{[\nu < \sigma_\tau]} + \xi 1_{[\nu = \sigma_\tau = T]}, \end{aligned}$$

then, after taking the conditional expectation, we obtain

$$\begin{aligned} Y_\tau \le E[&\int_\tau^{\nu\wedge\sigma_\tau} f(s, Y_s, Z_s)ds + L_{\sigma_\tau} 1_{[\sigma_\tau \le \nu < T]} \\ &+U_\nu 1_{[\nu < \sigma_\tau]} + \xi 1_{[\nu = \sigma_\tau = T]} | F_\tau] = \Gamma_\tau(\nu, \sigma_\tau). \end{aligned}$$

In the same way we can show that $Y_\tau \geq \Gamma_\tau(\nu_\tau, \sigma)$, $\forall \sigma \in \mathcal{T}_\tau$. It follows that $\Gamma_\tau(\nu_\tau, \sigma) \leq Y_\tau \leq \Gamma_\tau(\nu, \sigma_\tau)$ which implies

$$Y_\tau = \text{essinf}_{\nu \in \mathcal{T}_\tau} \text{esssup}_{\sigma \in \mathcal{T}_\tau} \Gamma_\tau(\nu, \sigma) = \text{esssup}_{\sigma \in \mathcal{T}_\tau} \text{essinf}_{\nu \in \mathcal{T}_\tau} \Gamma_\tau(\nu, \sigma). \quad (7)$$

Therefore Y_τ is the value of the Dynkin game on $[\tau, T]$ □

2.3. Comparison of local solutions and uniqueness

Let (f', ξ', L', U') be another quadruple where f' is a function from $[0, T] \times \Omega \times R^{1+d}$ into R which is $\mathcal{P} \otimes \mathcal{B}(R^{1+d})$-measurable and $(f'(t, \omega, 0, 0))_{t \leq T}$ belongs to $\mathcal{M}^{2,1}$, ξ' a random variable of $L^2(\Omega, F_T, dP; R)$ and finally L' and U' processes of $\mathcal{S}^2$ such that $L'_t \leq U'_t$ for any $t \leq T$ and $L'_T \leq \xi' \leq U'_T$, P-$a.s.$

Assume now that there exist a stopping time γ' such P-$a.s.$, $\tau \leq \gamma'$ and a quadruple $(Y'_t, Z'_t, K'^{+}_t, K'^{-}_t)_{t \leq T}$ of $\mathcal{P}$-measurable process with values in $R^{1+d+1+1}$ which is a local solution on $[\tau, \gamma']$ for the two barrier reflected BSDE associated with (f', ξ', L', U'). Let us set

$$\nu'_\tau := \inf\{s \geq \tau, Y'_s = U'_s\} \wedge T \ \text{ and } \ \sigma'_\tau := \inf\{s \geq \tau, Y'_s = L'_s\} \wedge T.$$

The following result gives conditions under which we can compare local solutions. Recall here that $(Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}$ is a local solution on $[\tau, \gamma]$ for the BSDE associated with (f, ξ, L, U).

2.3.1. Proposition. *Assume that :*

(i) *P-a.s. we have* $\sigma_\tau \wedge \nu'_\tau \leq \gamma \wedge \gamma'$, $Y_{\sigma_\tau} = L_{\sigma_\tau}$ *on* $[\sigma_\tau < T]$ *and* $Y'_{\nu'_\tau} = U'_{\nu'_\tau}$ *on* $[\nu'_\tau < T]$

(ii) *P-a.s.* $f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$ *for* $t \in [\tau, \gamma \wedge \gamma']$

(iii) $\xi \leq \xi'$, $L_s \leq L'_s$ *and* $U_s \leq U'_s$, $\forall s \leq T$.

Then we have $Y_\tau \leq Y'_\tau$, $P - a.s.$.

Proof. First we have :

$$\begin{aligned}
Y_{\sigma_\tau \wedge \nu'_\tau} &= Y_{\sigma_\tau} 1_{[\sigma_\tau \leq \nu'_\tau]} + Y_{\nu'_\tau} 1_{[\nu'_\tau < \sigma_\tau]} \\
&= Y_{\sigma_\tau} 1_{[\sigma_\tau \leq \nu'_\tau] \cap [\sigma_\tau < T]} + Y_{\sigma_\tau} 1_{[\sigma_\tau \leq \nu'_\tau] \cap [\sigma_\tau = T]} + Y_{\nu'_\tau} 1_{[\nu'_\tau < \sigma_\tau]} \\
&\leq L_{\sigma_\tau} 1_{[\sigma_\tau \leq \nu'_\tau] \cap [\sigma_\tau < T]} + \xi 1_{[\sigma_\tau \leq \nu'_\tau] \cap [\sigma_\tau = T]} + U'_{\nu'_\tau} 1_{[\nu'_\tau < \sigma_\tau]} \\
&\leq L'_{\sigma_\tau} 1_{[\sigma_\tau \leq \nu'_\tau] \cap [\sigma_\tau < T]} + \xi' 1_{[\sigma_\tau \leq \nu'_\tau] \cap [\sigma_\tau = T]} + Y'_{\nu'_\tau} 1_{[\nu'_\tau < \sigma_\tau]} \\
&\leq Y'_{\sigma_\tau \wedge \nu'_\tau}.
\end{aligned}$$

Therefore applying formula (4) with $((Y_t - Y'_t)^+)^2$ for $t \in [\tau, \sigma_\tau \wedge \nu'_\tau]$ we can argue as in Theorem 1.3, to obtain that $E[((Y_{(t \vee \tau) \wedge \sigma_\tau \wedge \nu'_\tau} - Y'_{(t \vee \tau) \wedge \sigma_\tau \wedge \nu'_\tau})^+)^2] = 0$ for any $t \leq T$ which implies $Y_\tau \leq Y'_\tau$ □

As a by-product we deduce the following result related to uniqueness of the local solution.

Let $\tau \leq T$ be a stopping time and γ, γ' two other stopping times such that P-a.s. $\gamma \geq \tau$ and $\gamma' \geq \tau$. Assume now that the BSDE associated with (f, ξ, L, U) has a local solution $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ (resp. $(Y'_t, Z'_t, K'^+_t, K'^-_t)_{t\leq T}$) on $[\tau, \gamma]$ (resp. $[\tau, \gamma']$). Therefore we have:

2.3.2. Proposition. *Uniqueness of the local solution.*

If $(\sigma_\tau \wedge \nu'_\tau) \vee (\sigma'_\tau \wedge \nu_\tau) \leq \gamma \wedge \gamma'$, $Y_{\sigma_\tau} = L_{\sigma_\tau}$ on $[\sigma_\tau < T]$, $Y'_{\sigma'_\tau} = L_{\sigma'_\tau}$ on $[\sigma'_\tau < T]$, $Y_{\nu_\tau} = U_{\nu_\tau}$ on $[\nu_\tau < T]$ and $Y'_{\nu'_\tau} = U_{\nu'_\tau}$ on $[\nu'_\tau < T]$ then P-a.s. $Y_\tau = Y'_\tau$.

Proof. It is just enough to remark that $\sigma_\tau \wedge \nu'_\tau \leq \gamma \wedge \gamma'$ and $\sigma'_\tau \wedge \nu_\tau \leq \gamma \wedge \gamma'$ therefore, in combination with the other properties and Prop.2.3.1, we have $Y_\tau \leq Y'_\tau$ and $Y'_\tau \leq Y_\tau$ then $Y_\tau = Y'_\tau$, P-*a.s.* □

3. Existence of useful local solutions and main result

In this section we are going to show that for any stopping time τ there exists another stopping time $\theta_\tau \geq \tau$ such that the reflected BSDE associated with (f, ξ, L, U) has a local solution on $[\tau, \theta_\tau]$. In addition we have $\theta_\tau \geq \sigma_\tau \vee \nu_\tau$. Therefore using a concatenation procedure we will prove that the reflected BSDE (2) has a global solution in the case when the barriers are completely separated, *i.e.*, $L < U$.

So for $n \geq 0$, let f_n be the function from $[0, T] \times \Omega \times R^{1+m}$ into R which with (t, ω, y, z) associates $f_n(t, \omega, y, z) = f(t, \omega, y, z) + n(L_t - y)^+$. The function f_n is uniformly Lipschitz with respect to (y, z) then, according to Theorem 1.1, there exists a process $(Y^n, Z^n, K^{-,n})$ solution of the reflected BSDE associated (f_n, ξ, U), *i.e.*,

$$\begin{cases} Y^n \in \mathcal{S}^2,\ Z^n \in \mathcal{M}^{2,d};\ \ K^{-,n} \in \mathcal{S}^2_{ci} \\ Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s)ds + n\int_t^T (L_s - Y^n_s)^+ ds - (K^{-,n}_T - K^{-,n}_t) - \int_t^T Z^n_s dB_s \\ Y^n_t \leq U_t,\ \forall t \leq T \text{ and } \int_0^T (U_s - Y^n_s)dK^{-,n}_s = 0. \end{cases} \tag{8}$$

Since $f_n \leq f_{n+1}$, then Corollary 1.4 implies that for any $n \geq 0$, $Y^n \leq Y^{n+1} \leq U$ and then the sequence of processes $(Y^n)_{n\geq 0}$ converges in $\mathcal{M}^{2,1}$ to a lower semi-continuous process $Y = (Y_t)_{t\leq T}$ which satisfies $E[\sup_{s\leq T} |Y_s|^2] < \infty$ and $Y_t \leq U_t$, $\forall t \leq T$.

First let us show that we have also $Y \geq L$.

3.1. Proposition. *P-a.s.*, $\forall t \leq T$ *we have* $Y_t \geq L_t$.

Proof. For any $n \geq 0$ and any stopping time $\tau \leq T$, the process Y^n satisfies (see e.g. [EKal])

$$\begin{aligned} Y^n_\tau = \operatorname{essinf}_{\nu\in\mathcal{T}_\tau} E\Big[\int_\tau^\nu f(s, Y^n_s, Z^n_s)ds \\ +n\int_\tau^\nu (L_s - Y^n_s)^+ds + U_\nu 1_{[\nu<T]} + \xi 1_{[\nu=T]}|F_\tau\Big]. \end{aligned}$$

Let $\nu^n_\tau = \inf\{s \geq \tau, Y^n_s = U_s\} \wedge T$, then

$$Y^n_\tau = E\Big[\int_\tau^{\nu^n_\tau} f(s, Y^n_s, Z^n_s)ds + n\int_\tau^{\nu^n_\tau} (L_s - Y^n_s)^+ds + U_{\nu^n_\tau} 1_{[\nu^n_\tau<T]} + \xi 1_{[\nu^n_\tau=T]}|F_\tau\Big].$$

Since $Y^n \leq Y^{n+1}$, the sequence of stopping times is decreasing and then converges to a stopping time $\nu_\tau := \lim_{n\to\infty} \nu^n_\tau$. Therefore we have

$$\begin{aligned} Y^n_\tau \geq E\Big[-\int_\tau^{\nu^n_\tau} |f(s, Y^n_s, Z^n_s)|ds + n\int_\tau^{\nu_\tau} (L_s - Y^n_s)^+ds \\ +U_{\nu^n_\tau} 1_{[\nu^n_\tau<T]} + \xi 1_{[\nu^n_\tau=T]}|F_\tau\Big]. \end{aligned} \tag{9}$$

Now since the process $K^{n,-}$ moves only when Y^n reaches the barrier U, then for any $s \in [\tau, \nu^n_\tau]$ we have (from (8)),

$$Y^n_s = Y^n_{\nu^n_\tau} + \int_s^{\nu^n_\tau} f(r, Y^n_r, Z^n_r)dr + n\int_s^{\nu^n_\tau} (L_r - Y^n_r)^+dr - \int_s^{\nu^n_\tau} Z^n_r dB_r.$$

Therefore basic calculations imply the existence of a constant C depending only on T and $\|f\|_{Lip}$, the Lipschitz constant of f, such that

$$\begin{aligned} E\Big[\int_\tau^{\nu^n_\tau} |f(s, Y^n_s, Z^n_s)|^2ds\Big] + E\Big[\int_\tau^{\nu^n_\tau} |Z^n_r|^2dr\Big] \\ \leq CE\Big[\sup_{t\leq T}\{|L_t|^2 + |U_t|^2\} + \int_0^T |f(s, 0, 0)|^2ds\Big], \end{aligned} \tag{10}$$

since $Y^n_{\nu^n_\tau}$ is uniformly bounded in $L^2(\Omega, F, P)$. As $Y^0 \leq Y^n \leq U$, f is Lipschitz and finally taking into account (10), we deduce from (9), in taking expectation, that

$$\int_\tau^{\nu_\tau} (L_r - Y_r)^+dr = 0, \mathrm{P} - a.s. \tag{11}$$

On the other hand for any $s \in [\tau, \nu_\tau]$ we have,

$$Y^n_s = Y^n_{\nu_\tau} + \int_s^{\nu_\tau} f(r, Y^n_r, Z^n_r)dr + n\int_s^{\nu_\tau} (L_r - Y^n_r)^+dr - \int_s^{\nu_\tau} Z^n_r dB_r$$

which implies

$$Y^n_s = Y^n_\tau - \int_\tau^s f(r, Y^n_r, Z^n_r)dr - n\int_\tau^s (L_r - Y^n_r)^+dr + \int_\tau^s Z^n_r dB_r, \ \forall s \in [\tau, \nu_\tau].$$

Since $Y^n \leq Y^{n+1}$ and taking into account (10), we can deduce from a result by S.Peng ([P], Theorem 2.1) that Y is right continuous on $[\tau, \nu_\tau]$. Henceforth from (11) we obtain $Y_\tau \geq L_\tau$ on the set $[\tau < \nu_\tau]$. Finally by (9) and (10) we obtain

$$Y_\tau = U_\tau \text{ on } [\tau = \nu_\tau < T]. \tag{12}$$

It implies that for any stopping time τ we have $Y_\tau \geq L_\tau$. Therefore using the optional section theorem ([DM], pp.220) we have P-$a.s.$, $Y \geq L$ □

Now for a stopping time $\tau \leq T$ and $n \geq 0$, let $\delta_\tau^n = \inf\{s \geq \tau, Y_s^n = U_s\} \wedge T$ (δ_τ^n is the same as ν_τ^n of the proof of Prop.3.1). The random variable δ_τ^n is a stopping time with respect to $(F_t)_{t\leq T}$, moreover since for any $n \geq 0$, $Y^n \leq Y^{n+1}$ we have $\delta_\tau^n \geq \delta_\tau^{n+1} \geq \tau$. Let us denote $\delta_\tau = \lim_{n\to\infty} \delta_\tau^n$; δ_τ is also a stopping time with respect to $(F_t)_{t\leq T}$ and $\delta_\tau \geq \tau$, P-$a.s.$.

First let us show that the reflected BSDE (2) has a local solution on $[\tau, \delta_\tau]$.

3.2. Proposition. *There exist two $\mathcal{P}$-measurable processes $(\bar{K}_s^+)_{s\leq T}$ and $(\bar{Z}_s)_{s\leq T}$ with values respectively in R^+ and R^d such that the process $(Y_s, \bar{Z}_s, \bar{K}_s^+, 0)_{s\leq T}$ is a local solution for (2) on $[\tau, \delta_\tau]$, i.e.,*

$$\begin{cases} \bar{Z} \in \mathcal{M}^{2,d}, \ \ \bar{K}^+ \in \mathcal{S}_{ci}^2 \\ Y_s = Y_{\delta_\tau} + \int_s^{\delta_\tau} f(u, Y_u, \bar{Z}_u)ds + (\bar{K}_{\delta_\tau}^+ - \bar{K}_s^+) - \int_s^{\delta_\tau} \bar{Z}_u dB_u, \ \forall s \in [\tau, \delta_\tau]; \ \ Y_T = \xi \\ U_s \geq Y_s \geq L_s, \ \forall s \in [\tau, \delta_\tau] \text{ and } \int_\tau^{\delta_\tau} (Y_s - L_s)d\bar{K}_s^+ = 0. \end{cases} \tag{13}$$

Proof. For any $n \geq 0$ and $s \in [\tau, \delta_\tau]$ we have

$$Y_s^n = Y_{\delta_\tau}^n + \int_s^{\delta_\tau} f(u, Y_u^n, Z_u^n)du + n\int_s^{\delta_\tau} (L_u - Y_u^n)^+ du - \int_s^{\delta_\tau} Z_u^n dB_u,$$

since the process $K^{-,n}$ moves only when Y^n reaches the barrier U and then $K_\tau^{-,n} = K_{\delta_\tau}^{-,n}$. On the other hand, for $n \geq 0$, let $(\bar{Y}_t^n, \bar{Z}_t^n)_{t\leq\delta_\tau}$ be the $\mathcal{P}$-measurable process with values in R^{1+d} such that:

$$\begin{cases} E[\sup_{s\leq\delta_\tau} |\bar{Y}_s^n|^2 + \int_0^{\delta_\tau} |\bar{Z}_s^n|^2 ds] < \infty \\ \bar{Y}_s^n = Y_{\delta_\tau}^n + \int_s^{\delta_\tau} f(u, \bar{Y}_u^n, \bar{Z}_u^n)du + n\int_s^{\delta_\tau} (L_u - \bar{Y}_u^n)^+ du - \int_s^{\delta_\tau} \bar{Z}_u^n dB_u, \ \forall s \leq \delta_\tau. \end{cases} \tag{14}$$

The process $(\bar{Y}_t^n, \bar{Z}_t^n)_{t\leq\delta_\tau}$ is the unique solution of the standard BSDE associated with the coefficient $f(t, y, z) + n(L_t - y)^+$, the terminal value $Y_{\delta_\tau}^n$ and bounded terminal time δ_τ.

We have $(Y_{\delta_\tau}^n)_{n\geq0} \nearrow Y_{\delta_\tau} \leq U_{\delta_\tau}$, hence from the Lebesgue dominated convergence theorem we get $E[|Y_{\delta_\tau}^n - Y_{\delta_\tau}|^2] \to 0$ as $n \to \infty$. Therefore the sequence of processes $((\bar{Y}_s^n, \bar{Z}_s^n, \int_0^s n(L_u - Y_u^n)^+ du)_{s\leq\delta_\tau})_{n\geq0}$ converges in $\mathcal{S}_{\delta_\tau}^2 \times \mathcal{M}_{\delta_\tau}^{2,d} \times \mathcal{S}_{\delta_\tau}^2$

($\mathcal{S}^2_{\delta_\tau}$ and $\mathcal{M}^{2,d}_{\delta_\tau}$ are the same as $\mathcal{S}^2$ and $\mathcal{M}^{2,d}$ except for that T is replaced by the stopping time δ_τ) to $(\underline{Y}_t, \underline{Z}_t, \underline{K}_t)_{t\le\delta_\tau}$ a $\mathcal{P}$-measurable process with values in R^{1+d+1} such that:

$$\begin{cases} E[\sup_{s\le\delta_\tau} |\underline{Y}_s|^2 + \int_0^{\delta_\tau} |\underline{Z}_s|^2 ds] < \infty\,;\ (\underline{K}_s)_{s\le\delta_\tau} \text{ is continuous non-decreasing and } \underline{K}_0=0 \\ \underline{Y}_s = Y_{\delta_\tau} + \int_s^{\delta_\tau} f(u, \underline{Y}_u, \underline{Z}_u)du + \underline{K}_{\delta_\tau} - \underline{K}_s - \int_s^{\delta_\tau} \underline{Z}_u dB_u,\ \forall s \le \delta_\tau \\ \underline{Y}_s \ge L_s,\ \forall s \le \delta_\tau \text{ and } \int_0^{\delta_\tau} (L_u - \underline{Y}_u) d\underline{K}_u = 0. \end{cases}$$

Since $Y_{\delta_\tau} \ge L_{\delta_\tau}$ (Prop.3.1), then the sketch of the proof of the existence and uniqueness of the process $(\underline{Y}_t, \underline{Z}_t, \underline{K}_t)_{t\le\delta_\tau}$ is similar to the one which has been done in El-Karoui et al.'s paper [EKal] in the part related to the penalization method, except for that there the terminal time is deterministic and it is a stopping time in our frame. However the difference is irrelevant since δ_τ is bounded.

Now uniqueness of the solution of (14) on $[\tau, \delta_\tau]$ implies that for any $s \in [\tau, \delta_\tau]$, $Y^n_s = \bar{Y}^n_s$ and $Z^n_s = \bar{Z}^n_s$. Therefore $Y_s = \underline{Y}_s$ for any $s \in [\tau, \delta_\tau]$ and

$$\begin{cases} E[\sup_{s\in[\tau,\delta_\tau]} |Y_s|^2 + \int_\tau^{\delta_\tau} |\underline{Z}_s|^2 ds] < \infty;\ (\underline{K}_s)_{s\le\delta_\tau} \text{ is continuous non-decreasing and } \underline{K}_0=0 \\ Y_s = Y_{\delta_\tau} + \int_s^{\delta_\tau} f(u, Y_u, \underline{Z}_u)du + \underline{K}_{\delta_\tau} - \underline{K}_s - \int_s^{\delta_\tau} \underline{Z}_u dB_u,\ \forall s \in [\tau, \delta_\tau] \\ U_s \ge Y_s \ge L_s,\ \forall s \in [\tau, \delta_\tau] \text{ and } \int_\tau^{\delta_\tau} (L_u - Y_u) d\underline{K}_u = 0. \end{cases}$$

For any $s \le T$, let us set $\bar{K}^+_s = (\underline{K}_{s\wedge\delta_\tau} - \underline{K}_\tau)1_{[s\ge\tau]}$ and $\bar{Z}_s = \underline{Z}_s 1_{[\tau\le s\le\delta_\tau]}$, we deduce that $(Y_s, \bar{Z}_s, \bar{K}^+_s, 0)_{s\le T}$ is a local solution for the equation (5) on $[\tau, \delta_\tau]$ □

Now, let $\theta^n_\tau = \inf\{s \ge \delta_\tau, Y^n_s \le L_s\} \wedge T$; θ^n_τ is a stopping time and P-$a.s.$, $\theta^n_\tau \le \theta^{n+1}_\tau$ since $Y^n \le Y^{n+1}$. Let us set $\theta_\tau := \lim_{n\to\infty} \theta^n_\tau$, then θ_τ is also a stopping time.

On the other hand, let $\tau^*_\tau = \inf\{s \ge \tau, Y_s = U_s\} \wedge T$. So for any $n \ge 0$, $\tau^*_\tau \le \delta^n_\tau$ and then $\tau^*_\tau \le \delta_\tau$. In other respects when $\delta_\tau(\omega) < T$, for n large enough, we have $U_{\delta^n_\tau}(\omega) = Y^n_{\delta^n_\tau}(\omega) \le Y_{\delta^n_\tau}(\omega) \le U_{\delta^n_\tau}(\omega)$, therefore we have $\lim_{n\to\infty} Y_{\delta^n_\tau}(\omega) = U_{\delta_\tau}(\omega)$ on $\{\delta_\tau < T\}$.

3.3. Proposition.
(i) There exists a $\mathcal{P}$-measurable process $(\tilde{Z}_t, \tilde{K}^-_t)_{t\le T}$ with values in R^{d+1} such that $(Y_t, \tilde{Z}_t, 0, \tilde{K}^-_t)_{t\le T}$ is a local solution for the reflected BSDE (5) on $[\delta_\tau, \theta_\tau]$.
(ii) P-a.s. we have $Y_{\delta_\tau} = U_{\delta_\tau}$ on $[\delta_\tau < T]$ and $Y_{\theta_\tau} = L_{\theta_\tau}$ on $[\theta_\tau < T]$.

Proof. Let us show (i). First let us recall that, from (8), for any $n \ge 0$ we have

$$\begin{aligned} Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s)ds + n\int_t^T (L_s - Y^n_s)^+ ds \\ -(K^{-,n}_T - K^{-,n}_t) - \int_t^T Z^n_s dB_s, t \le T. \end{aligned}$$

Then for any $s \in [\delta_\tau, \theta_\tau^n]$ we have,

$$Y_s^n = Y_{\theta_\tau^n}^n + \int_s^{\theta_\tau^n} f(s, Y_s^n, Z_s^n)ds - (K_{\theta_\tau^n}^{-,n} - K_s^{-,n}) - \int_s^{\theta_\tau^n} Z_s^n dB_s$$

since either $\theta_\tau^n = \delta_\tau$ or $\theta_\tau^n > \delta_\tau$ and then $Y_s^n \geq L_s$ for $s \in [\delta_\tau, \theta_\tau^n]$. Let us set

$$\tilde{Y}_s^n := Y_{s\wedge\theta_\tau^n}^n, \;\; \tilde{Z}_s^n := Z_s^n 1_{[\delta_\tau \leq s \leq \theta_\tau^n]} \text{ and } \tilde{K}_s^{-,n} := (K_{s\wedge\theta_\tau^n}^{-,n} - K_{\delta_\tau}^{-,n})1_{[s\geq\delta_\tau]}, \;\; s \leq T.$$

It follows that $(\tilde{Y}^n, \tilde{Z}^n, \tilde{K}^n)$ is a solution on $[\delta_\tau, \theta_\tau]$ for a single upper obstacle reflected BSDE and satisfies: $\forall s \in [\delta_\tau, \theta_\tau]$,

$$\begin{cases} \tilde{Y}_s^n = Y_{\theta_\tau^n}^n + \int_s^{\theta_\tau} 1_{[s\leq\theta_t^n]} f(s, \tilde{Y}_s^n, \tilde{Z}_s^n)ds - (\tilde{K}_{\theta_\tau}^{-,n} - \tilde{K}_s^{-,n}) - \int_s^T \tilde{Z}_s^n dB_s \\ \tilde{Y}_s^n \leq U_s \; ; \; \int_{\delta_\tau}^{\theta_\tau} (\tilde{Y}_s^n - U_s)d\tilde{K}_s^{-,n} = 0. \end{cases} \tag{15}$$

Standard calculations yield

$$E[\int_{\delta_\tau}^{\theta_\tau} |\tilde{Z}_s^n|^2 ds + \sup_{s\in[\delta_\tau,\theta_\tau]} |\tilde{K}_s^{-,n}|^2] \leq C(T, \|f\|_{Lip})$$
$$\times E[\sup_{t\leq T} (|L_t|^2 + |U_t|^2) + \int_0^T |f(s, 0, 0)|^2 ds]. \tag{16}$$

Now let us focus on the pointwise convergence of $(\tilde{Y}_s^n)_{n\geq0}$ for $s \in [\delta_\tau, \theta_\tau]$. If $s \in [\delta_\tau(\omega), \theta_\tau(\omega)[$, then for n large enough we have

$$\tilde{Y}_s^n = Y_s^n \to Y_s \text{ as } n \to \infty.$$

If $s = \theta_\tau(\omega)$, then once again for n large enough we have

$$\begin{aligned} \tilde{Y}_{\theta_\tau}^n &= Y_{\theta_\tau^n}^n \\ &= L_{\theta_\tau^n} 1_{[\delta_\tau<\theta_\tau<T]} + Y_{\delta_\tau}^n 1_{[\delta_\tau=\theta_\tau]} + L_{\theta_\tau^n} 1_{[\delta_\tau<\theta_\tau^n<\theta_\tau=T]} + \xi 1_{[\delta_\tau<\theta_\tau^n=\theta_\tau=T]} \\ &= L_{\theta_\tau^n} 1_{[\delta_\tau<\theta_\tau<T]} + Y_{\delta_\tau}^n 1_{[\delta_\tau=\theta_\tau]} + \xi 1_{[\delta_\tau<\theta_\tau=T]} + (L_{\theta_\tau^n} - \xi) 1_{[\delta_\tau<\theta_\tau^n<\theta_\tau=T]}. \end{aligned}$$

It implies that

$$\begin{aligned} \lim_{n\to\infty} \tilde{Y}_{\theta_\tau}^n = L_{\theta_\tau} 1_{[\delta_\tau<\theta_\tau<T]} + \xi 1_{[\delta_\tau<\theta_\tau=T]} \\ + Y_{\delta_\tau} 1_{[\delta_\tau=\theta_\tau]} + (L_T - \xi) 1_{\{\cap_{n\geq0}[\delta_\tau<\theta_\tau^n<\theta_\tau=T]\}}. \end{aligned}$$

But for any $n \geq 0$, $Y^n \leq Y^{n+1}$, $Y_T^n = \xi$ and $t \to Y_t^n$ is continuous then

$$(L_T - \xi) 1_{\{\cap_{n\geq0}[\delta_\tau<\theta_\tau^n<\theta_\tau=T]\}} = 0 \quad \text{a.s..}$$

Indeed if $\xi(\omega) > L_T(\omega)$ then we cannot have $\delta_\tau(\omega) < \theta_\tau^n(\omega) < \theta_\tau(\omega) = T$, at least for n large enough.

So, we have shown, P-$a.s.$, for any $s \in [\delta_\tau, \theta_\tau]$, the sequence $(\tilde{Y}_s^n)_{n\geq 0}$ converges to

$$\tilde{Y}_s := Y_s 1_{[s<\theta_\tau]} + 1_{[s=\theta_\tau]}(L_{\theta_\tau} 1_{[\delta_\tau<\theta_\tau<T]} + \xi 1_{[\delta_\tau<\theta_\tau=T]} + Y_{\delta_\tau} 1_{[\delta_\tau=\theta_\tau]}). \qquad (17)$$

Now let us use Itô's formula with $(\tilde{Y}^n - \tilde{Y}^m)^2$ and $s \in [\delta_\tau, \theta_\tau]$. It yields that

$$\begin{aligned}(\tilde{Y}_s^n - \tilde{Y}_s^m)^2 &+ \int_s^{\theta_\tau} (\tilde{Z}_u^n - \tilde{Z}_u^m)^2 du = (Y^n_{\theta_\tau^n} - Y^m_{\theta_\tau^m})^2 \\ &+2\int_s^{\theta_\tau} (\tilde{Y}_u^n - \tilde{Y}_u^m)\{1_{[u\leq\theta_\tau^n]} f(u, \tilde{Y}_u^n, \tilde{Z}_u^n) - 1_{[u\leq\theta_\tau^m]} f(u, \tilde{Y}_u^m, \tilde{Z}_u^m)\}du \\ &-2\int_s^{\theta_\tau} (\tilde{Y}_u^n - \tilde{Y}_u^m) d(\tilde{K}_u^{-,n} - \tilde{K}_u^{-,m}) - 2\int_s^{\theta_\tau} (\tilde{Y}_u^n - \tilde{Y}_u^m)(\tilde{Z}_u^n - \tilde{Z}_u^m) dB_u.\end{aligned}$$

But $E[(Y^n_{\theta_\tau^n} - Y^m_{\theta_\tau^m})^2] \to 0$ as $n, m \to \infty$ since $(Y^n_{\theta_\tau^n})_{n\geq 0}$ converges to $(L_{\theta_\tau} 1_{[\delta_\tau<\theta_\tau<T]} + \xi 1_{[\delta_\tau<\theta_\tau=T]} + Y_{\delta_\tau} 1_{[\delta_\tau=\theta_\tau]})$. On the other hand for any $n \leq m$ and $s \in [\delta_\tau, \theta_\tau]$ we have

$$\int_s^{\theta_\tau} (\tilde{Y}_u^n - \tilde{Y}_u^m) d(\tilde{K}_u^{-,n} - \tilde{K}_u^{-,m}) \geq - \sup_{s\in[\theta_\tau^n, \theta_\tau^m]} (L_s - Y^n_{\theta_\tau^n})^- \times K^{-,m}_{\theta_\tau^m}.$$

Therefore taking the supremum, using the Burkholder-Davis-Gundy inequality, Gronwall's one and finally the estimates (16) we obtain

$$\begin{aligned}E[\sup_{s\in[\delta_\tau,\theta_\tau]} |\tilde{Y}_s^n - \tilde{Y}_s^m|^2 &+ \int_{\delta_\tau}^{\theta_\tau} |\tilde{Z}_u^n - \tilde{Z}_u^m|^2 du \\ &+ \sup_{s\in[\delta_\tau,\theta_\tau]} |\tilde{K}_s^{-,n} - \tilde{K}_s^{-,m}|^2] \to 0 \text{ as } n, m \to \infty. \qquad (18)\end{aligned}$$

It implies that $\tilde{Y}$ is continuous on $[\delta_\tau, \theta_\tau]$.

Now let us show that $Y_s = \tilde{Y}_s \quad$ for all $s \in [\delta_\tau, \theta_\tau]$. From (17) we have

$$Y = \tilde{Y} \text{ on the sets } [\delta_\tau, \theta_\tau[,\ \ [\delta_\tau, \theta_\tau] \cap [\theta_\tau = T] \text{ and } [\delta_\tau = \theta_\tau]. \qquad (19)$$

Now let ω be such that $\delta_\tau(\omega) < \theta_\tau(\omega) < T$ and consider a sequence s_ω^n of $[\delta_\tau(\omega), \theta_\tau(\omega)[$ which converges to $\theta_\tau(\omega)$. Since $\tilde{Y}$ is continuous on $[\delta_\tau, \theta_\tau]$ we have $\lim_{n\to\infty} Y_{s_\omega^n}(\omega) = \lim_{n\to\infty} \tilde{Y}_{s_\omega^n}(\omega) = \tilde{Y}_{\theta_\tau}(\omega) = L_{\theta_\tau}(\omega)$. As Y is lower semi-continuous then $\lim_{n\to\infty} Y_{s_\omega^n}(\omega) \geq Y_{\theta_\tau}(\omega)$, $i.e.$, $L_{\theta_\tau}(\omega) \geq Y_{\theta_\tau}(\omega)$. But $Y \geq L$ then $Y_{\theta_\tau} = \tilde{Y}_{\theta_\tau}$ on $[\delta_\tau < \theta_\tau < T]$. Thereby, in combination with (19), we have

$$Y_s = \tilde{Y}_s \quad \text{for all } s \in [\delta_\tau, \theta_\tau].$$

Now let $\tilde{Z} := \mathcal{M}^{2,d} - \lim_{n\to\infty} \tilde{Z}^n$ and $\tilde{K}^- := \mathcal{S}^2_{ci} - \lim_{n\to\infty} \tilde{K}^{-,n}$. Since $\int_{\delta_\tau}^{\theta_\tau} (\tilde{Y}_s^n - U_s) dK_s^{-,n} = 0$ then through Helly's theorem ([KF], pp. 370) we have also

$$\int_{\delta_\tau}^{\theta_\tau} (Y_s - U_s) d\tilde{K}_s^- = 0. \qquad (20)$$

Therefore $(Y_s, \tilde{Z}_s, 0, \tilde{K}_s^-)_{s\le T}$ is a local solution for (5) in $[\delta_\tau, \theta_\tau]$. Indeed $Y_T = \xi$, $Y \ge L$ and for any $s \in [\delta_\tau, \theta_\tau]$ we have, in taking the limit in (15),

$$Y_s = Y_{\theta_\tau} + \int_s^{\theta_\tau} f(s, Y_s, \tilde{Z}_s)ds - (\tilde{K}_{\theta_\tau}^- - \tilde{K}_s^-) - \int_s^{\theta_\tau} \tilde{Z}_s dB_s \ ; \tag{21}$$

the integrability properties for Y, $\tilde{Z}$ and $\tilde{K}^-$ are obviously satisfied.

Now let us show (ii). As it is underlined previously we have

$$Y_s = \tilde{Y}_s \text{ on the set } [\delta_\tau, \theta_\tau]$$

which implies in particular that $Y_{\theta_\tau} = L_{\theta_\tau}$ on the set $[\delta_\tau < \theta_\tau < T]$ and since $\delta_\tau^n \searrow \delta_\tau$, Y is continuous on $[\delta_\tau, \theta_\tau]$ and $\lim_{n\to\infty} Y_{\delta_\tau^n} = U_{\delta_\tau}$ we have also $Y_{\delta_\tau} = U_{\delta_\tau}$ on the set $[\delta_\tau < T] \cap [\delta_\tau < \theta_\tau]$.

In order to finish the proof of (ii) it is enough to show that $Y_{\delta_\tau} = U_{\delta_\tau} = L_{\delta_\tau}$ on $[\delta_\tau = \theta_\tau < T]$. First observe that for all $n \ge 0$, $\theta_\tau^n = \delta_\tau$, hence $Y_{\delta_\tau} = \lim_n Y_{\theta_\tau^n}^n \le L_{\delta_\tau}$, on the set $[\delta_\tau = \theta_\tau < T]$. On the other hand, we have:

$$\begin{aligned}\forall n \ge 0,\ \ Y_{\delta_\tau}^n = Y_{\delta_\tau^n}^n + \int_{\delta_\tau}^{\delta_\tau^n} f(u, Y_u^n, Z_u^n)du + n\int_{\delta_\tau}^{\delta_\tau^n} (L_u - Y_u^n)^+ du \\ -(K_{\delta_\tau^n}^{-,n} - K_{\delta_\tau}^{-,n}) - \int_{\delta_\tau}^{\delta_\tau^n} Z_u^n dB_u.\end{aligned}$$

As $n(L_u - Y_u^n)^+ \ge 0$, $K_{\delta_\tau^n}^{-,n} = K_{\delta_\tau}^{-,n}$ and $\sup_{n\ge 0} E[\int_{\delta_\tau}^{\delta_\tau^n} |f(s, Y_s^n, Z_s^n)|^2 ds] < \infty$ (e.g. from (10)) then

$$E[1_{[\delta_\tau=\theta_\tau<T]} Y_{\delta_\tau}^n] \ge E[1_{[\delta_\tau=\theta_\tau<T]} Y_{\delta_\tau^n}^n] - C\sqrt{E[\delta_\tau^n - \delta_\tau]}, \tag{22}$$

for some constant C which does not depend on n. So, taking the limit in (22), we get $E[1_{[\delta_\tau=\theta_\tau<T]} L_{\delta_\tau}] \ge E[1_{[\delta_\tau=\theta_\tau<T]} U_{\delta_\tau}]$. But $L_{\theta_\tau} \le U_{\theta_\tau}$ then $L_{\theta_\tau} = U_{\theta_\tau}$ on the set $[\delta_\tau = \theta_\tau < T]$.

As a consequence of Propositions 3.2 and 3.3, the reflected BSDE (5) has a local solution on $[\tau, \theta_\tau]$.

3.4. Theorem. *The double obstacle reflected BSDE associated with (f, ξ, L, U) has a local solution $(Y_s, Z_s, K_s^+, K_s^-)_{s\le T}$ on $[\tau, \theta_\tau]$.*

Proof. Let $(Y_s, \bar{Z}_s, \bar{K}_s^+, 0)_{s\le T}$ (resp. $(Y_t, \tilde{Z}_t, 0, \tilde{K}_t^-)_{t\le T}$) be a local solution of (5) on $[\tau, \delta_\tau]$ (resp. $[\delta_\tau, \theta_\tau]$) which exists according to Prop.3.2 (resp. Prop.3.3).

Now for $s \le T$, let $Z_s = \bar{Z}_s 1_{[s\le\delta_\tau]} + \tilde{Z}_s 1_{[\delta_\tau\le s\le\theta_\tau]}$, $K_s^+ = \bar{K}_{s\wedge\delta_\tau}^+$ and $K_s^- = \tilde{K}_{s\wedge\theta_\tau}^- 1_{[s\ge\delta_\tau]}$. For any $s \in [\tau, \theta_\tau]$ we have,

$$Y_s = Y_{\theta_\tau} + \int_s^{\theta_\tau} f(s, Y_s, Z_s)ds + (K_{\theta_\tau}^+ - K_s^+) - (K_{\theta_\tau}^- - K_s^-) - \int_s^{\theta_\tau} Z_s dB_s. \tag{23}$$

Indeed, if $s \in [\delta_\tau, \theta_\tau]$ then $K^+_{\theta_\tau} - K^+_s = 0$ and (23) is satisfied from (21). On the other hand if $s \in [\tau, \delta_\tau]$, then from (5) we have,

$$Y_s = Y_{\delta_\tau} + \int_s^{\delta_\tau} f(u, Y_u, Z_u)ds + (K^+_{\delta_\tau} - K^+_s) - \int_s^{\delta_\tau} Z_u dB_u.$$

As

$$Y_{\delta_\tau} = Y_{\theta_\tau} + \int_{\delta_\tau}^{\theta_\tau} f(s, Y_s, \tilde{Z}_s)ds - \tilde{K}^-_{\theta_\tau} - \int_{\delta_\tau}^{\theta_\tau} \tilde{Z}_s dB_s$$

then (23) is also satisfied since $K^+_{\delta_\tau} = K^+_{\theta_\tau}$ and $\bar{K}^-_s = 0$.

Now for any $s \in [\tau, \theta_\tau]$, $L_s \le Y_s \le U_s$ and $\int_\tau^{\theta_\tau} (Y_s - L_s)dK^+_s = \int_\tau^{\delta_\tau} (Y_s - L_s)d\bar{K}^+_s = 0$ and $\int_\tau^{\theta_\tau} (U_s - Y_s)dK^-_s = \int_{\delta_\tau}^{\theta_\tau} (U_s - Y_s)d\tilde{K}^-_s = 0$.

Finally $Y_T = \xi$ and the integrability properties are satisfied. Henceforth the process $(Y_s, \bar{Z}_s, \bar{K}^+_s, \bar{K}^-_s)_{s\le T}$ is a local solution for (5) on $[\tau, \theta_\tau]$ □

3.5. Remark. The construction of Y does not depend on τ but the ones of Z, K^+ and K^- do □

As a consequence we have :

3.6. Proposition. *There exists a unique continuous $\mathcal{P}$-measurable process $(Y_t)_{t\le T}$ such that:*

(i) $\forall t \le T$, $L_t \le Y_t \le U_t$ and $Y_T = \xi$

(ii) for any stopping time τ, there exist another stopping time $\theta_\tau \ge \tau$ and a triple of $\mathcal{P}$-measurable processes $(Z^\tau_t, K^{+,\tau}_t, K^{-,\tau}_t)_{t\le T}$ such that on $[\tau, \theta_\tau]$ the process $(Y_t, Z^\tau_t, K^{+,\tau}_t, K^{-,\tau}_t)_{t\le T}$ is a local solution for the reflected BSDE associated (f, ξ, L, U)

(iii) If $\nu_\tau = \inf\{s \ge \tau, Y_s = U_s\} \wedge T$ and $\sigma_\tau = \inf\{s \ge \tau, Y_s = L_s\} \wedge T$ then $\nu_\tau \vee \sigma_\tau \le \theta_\tau$, $Y_{\sigma_\tau} = L_{\sigma_\tau}$ on $[\sigma_\tau < T]$ and $Y_{\nu_\tau} = L_{\nu_\tau}$ on $[\nu_\tau < T]$.

Proof. The existence of the process $(Y_t)_{t\le T}$, θ_τ and the triple $(Z^\tau_t, K^{+,\tau}_t, K^{-,\tau}_t)_{t\le T}$ such that $(i) - (iii)$ are satisfied stems from Thm. 3.4 and Prop. (3.3). It remains to show that $(Y_t)_{t\le T}$ is unique and continuous. But uniqueness is a direct consequence of Prop.2.3.2. Let us focus on the continuity.

In the construction of the process $(Y_t)_{t\le T}$ we have chosen an increasing scheme. Had we chosen a decreasing scheme, which is possible in making the penalization on the upper barrier U, we would have constructed in a symmetric way a process $\tilde{Y} = (\tilde{Y}_t)_{t\le T}$ which is $\mathcal{P}$-measurable and upper semi-continuous such that:

(i) $\forall t \le T$, $L_t \le \tilde{Y}_t \le U_t$ and $\tilde{Y}_T = \xi$

(ii) for any stopping time τ, there exist another stopping time $\tilde{\theta}_\tau \ge \tau$ such that on $[\tau, \tilde{\theta}_\tau]$ there exists a triple $(\tilde{Z}^\tau_t, \tilde{K}^{+,\tau}_t, \tilde{K}^{-,\tau}_t)_{t\le T}$ such that the process $(\tilde{Y}_t, \tilde{Z}^\tau_t, \tilde{K}^{+,\tau}_t, \tilde{K}^{-,\tau}_t)_{t\le T}$ is a local solution for the reflected BSDE associated with (f, ξ, L, U)

(iii) if $\tilde{\nu}_\tau = \inf\{s \ge \tau, \tilde{Y}_s = U_s\} \wedge T$ and $\tilde{\sigma}_\tau = \inf\{s \ge \tau, \tilde{Y}_s = L_s\} \wedge T$ then $\tilde{\nu}_\tau \vee \tilde{\sigma}_\tau \le \tilde{\theta}_\tau$, $\tilde{Y}_{\tilde{\sigma}_\tau} = L_{\tilde{\sigma}_\tau}$ on $[\tilde{\sigma}_\tau < T]$ and $\tilde{Y}_{\tilde{\nu}_\tau} = L_{\tilde{\nu}_\tau}$ on $[\tilde{\nu}_\tau < T]$.

Therefore using the uniqueness result of Prop. 2.3.2, we have $Y_\tau = \tilde{Y}_\tau$ which implies that Y is lower and upper semi-continuous in the same time and then is continuous □

We are now ready to give the main result of this section.

3.7. Theorem. *Assume that $\forall t \le T$, $L_t < U_t$. Then there exists a unique process $(Y_t, Z_t, K_t^+, K_t^-)_{t\le T}$ $\mathcal{P}$-measurable with values in $R^{1+d+1+1}$ solution of the reflected BSDE (2), i.e.,*

$$\begin{cases} Y \in \mathcal{S}^2,\ Z \in \mathcal{M} \text{ and } K^+, K^- \in \mathcal{S}_{ci} \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + (K_T^+ - K_t^-) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s,\ \forall t \le T \\ L_t \le Y_t \le U_t,\ \forall t \le T \text{ and } \int_0^T (Y_s - L_s)dK_s^+ = \int_0^T (U_s - Y_s)dK_s^- = 0. \end{cases} \tag{24}$$

Proof. Let $(Y_t)_{t\le T}$ be the continuous process defined in Prop.3.6. Therefore Y is continuous and satisfies $L_t \le Y_t \le U_t$, and $Y_T = \xi$.

Now let $\tau_0 = 0$ and for $n \ge 0$, $\tau_{2n+1} = \inf\{s \ge \tau_{2n}, Y_s = U_s\} \wedge T$ and $\tau_{2n+2} = \inf\{s \ge \tau_{2n+1}, Y_s = L_s\} \wedge T$. Henceforth, for any $n \ge 0$ there exists a triple $(Z_t^n, K_t^{+,n}, K_t^{n,-})_{t\le T}$ of $\mathcal{P}$-measurable processes with values in R^{d+1+1} such that the process $(Y_t, Z_t^n, K_t^{+,n}, K_t^{n,-})_{t\le T}$ is a local solution for the reflected BSDE associated with (f, ξ, L, U) on the set $[\tau_{2n}, \tau_{2n+2}]$.

On the other hand we have P-a.s., $\tau_n < \tau_{n+1}$ on the set $[\tau_{n+1} < T]$, $\forall n \ge 0$ since Y, L and U are continuous processes and $L < U$. In addition the sequence $(\tau_n)_{n\ge 0}$ is of *stationary type i.e.* P-$a.s.$ for $\omega \in \Omega$ there exists $n_0(\omega) \ge 0$ such that $\tau_{n_0}(\omega) = T$. Actually let us show that $P[\tau_n < T, \forall n \ge 0] = 0$.

Indeed let us set $A = \{\omega : \tau_n(\omega) < T, \forall n \ge 0\}$ and assume that $P(A) > 0$. Therefore for $\omega \in A$, we have for any $n \ge 0$, $Y_{\tau_{2n+1}} = U_{\tau_{2n+1}}$ and $Y_{\tau_{2n}} = L_{\tau_{2n}}$. Since the sequence of stopping times $(\tau_n)_{n\ge 0}$ is non-decreasing then it converges to another stopping time τ. It follows that $Y_\tau = L_\tau = U_\tau$ on A which is a contradiction since $L_t < U_t, \forall t \le T$. Thereby we deduce $P(A) = 0$ $i.e.$ P-$a.s.$ for $\omega \in \Omega$, there exists $n_0(\omega) \ge 0$ such that $\tau_{n_0}(\omega) = T$.

Now let us set for any $t \le T$,

$$\begin{aligned} K_t^+ &= K_{\tau_{2n}}^+ + (K_t^{+,n} - K_{\tau_{2n}}^{+,n}) \text{ if } t \in]\tau_{2n}, \tau_{2n+2}]\ (K_0^+ = 0) \\ K_t^- &= K_{\tau_{2n}}^- + (K_t^{-,n} - K_{\tau_{2n}}^{-,n}) \text{ if } t \in]\tau_{2n}, \tau_{2n+2}]\ (K_0^- = 0) \\ Z_t &= Z_t^0 1_{[0,\tau_2]}(t) + \sum_{n\ge 1} Z_t^n 1_{]\tau_{2n},\tau_{2n+2}]}(t). \end{aligned}$$

Therefore $K^\pm = (K_t^\pm)_{t\le T}$ are continuous non-decreasing processes and since the sequence $(\tau_n)_{n\ge 0}$ is P-$a.s.$ of stationary type and for any $n \ge 0$, $E[\int_0^{\tau_{2n}} |Z_s|^2 ds] < \infty$ then $\int_0^T |Z_s|^2 ds < \infty$, P-$a.s.$.

Now let $\omega \in \Omega$ and $t \le T$. Then there exits $n_1(\omega)$ such that $t \in]\tau_{2n_1}(\omega), \tau_{2n_1+2}(\omega)]$ and

$$Y_t(\omega) = Y_{\tau_{2n_1+2}}(\omega) + \int_t^{\tau_{2n_1+2}(\omega)} f(s, Y_s, Z_s)ds + (K^+_{\tau_{2n_1+2}}(\omega) - K^+_t)$$
$$-(K^-_{\tau_{2n_1+2}}(\omega) - K^-_t) - \int_t^{\tau_{2n_1+2}(\omega)} Z_s dBs.$$

But there exists $n_0(\omega)$ such that for $n \geq n_0$ we have $\tau_{2n} = T$, therefore step by step we obviously have

$$Y_{\tau_{2n_1+2}} = \xi + \int_{\tau_{2n_1+2}}^T f(s, Y_s, Z_s)ds + (K^+_T - K^+_{\tau_{2n_1+2}})$$
$$-(K^-_T - K^-_{\tau_{2n_1+2}}) - \int_{\tau_{2n_1+2}}^T Z_s dBs.$$

It follows that :

$$Y_t(\omega) = \xi + \int_t^T f(s, Y_s, Z_s)ds + (K^+_T - K^+_t) - (K^-_T - K^-_t) - \int_t^T Z_s dBs.$$

Finally the processes $K^{\pm}$ and Y satisfy $\int_0^T (Y_s - L_s)dK^+_s = \int_0^T (U_s - Y_s)dK^-_s = 0$ since on the intervals $[\tau_{2n}, \tau_{2n+2}]$ those properties are satisfied and the sequence $(\tau_n)_{n\geq 0}$ is P-$a.s.$ of stationary type. Henceforth the process (Y, Z, K^+, K^-) is a solution for the BSDE with two reflecting barriers associated with (f, ξ, L, U) in the sense of (2). Uniqueness is a direct consequence of the comparison theorem (1.3) □

Now let us consider a process $g := (g_s)_{s\leq T}$ which belongs to $\mathcal{M}^{2,1}$ and let $t \in [0, T]$. Let us recall that the Dynkin game on $[t, T]$ associated with (g, ξ, L, U) is a zero-sum game on stopping times where the payoff is given by:

$$\forall \nu, \sigma \in \mathcal{T}_t, \Gamma_t(\nu, \sigma)$$
$$:= E[\int_t^{\nu\wedge\sigma} g_s ds + L_\sigma 1_{[\sigma\leq\nu<T]} + U_\nu 1_{[\nu<\sigma]} + \xi 1_{[\nu=\sigma=T]}|F_t].$$

In the following result we give some feature, which is fairly not known, of the value function of the Dynkin game. The proof is a direct consequence of Thm.3.7 and Prop. 2.2.1, therefore we skip it.

3.8. Theorem. *Assume that for any $t \leq T$ we have $L_t < U_t$ and let $(Y_t, Z_t, K^+_t, K^-_t)_{t\leq T}$ be the solution of the BSDE with two reflecting barriers associated with (g, ξ, L, U), then $(Y_t)_{t\leq T}$, which is a continuous semimartingale, is the value function of the Dynkin game on $[t, T]$.* □

4. Further properties of the solution of the BSDE (24)

In Thm.3.7 we just know that $K^+_T + K^-_T + \int_0^T |Z_s|^2 ds < \infty$ P-$a.s.$ We are going to show the existence of a sequence of stopping times $(\gamma_n)_{\geq 0}$, P-$a.s.$ of stationary type and whose limit is T, which depends only on L, U, the process

$(f(t,\omega,0,0))_{t\le T}$ and the constant of linear growth of f, such that $E[(K^+_{\gamma_n})^2 + (K^-_{\gamma_n})^2 + \int_0^{\gamma_n} |Z_s|^2 ds] < \infty$ for any $n \ge 0$.

4.1. Proposition. *Assume that $L_t < U_t$, $\forall t \le T$, then there exists an increasing sequence of stopping times $(\gamma_n)_{\ge 0}$ such that :*

(i) P-a.s. the sequence is of stationary type and converges to T
(ii) the sequence depends only on L, U, the process $(f(t,\omega,0,0))_{t\le T}$ and C
(iii) $\forall\ n \ge 0$ we have

$$E[(K^+_{\gamma_n})^2 + (K^-_{\gamma_n})^2 + \int_0^{\gamma_n} |Z_s|^2 ds] < \infty.$$

Proof. As $L < U$ then there exists a quadruple of processes $(\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-)$ solution of the reflected BSDE associated with $(|f(t,\omega,0,0)|+C(|y|+|z|), U_T, L, U)$, *i.e.*,

$$\begin{cases} \tilde{Y} \in \mathcal{S}^2\,, \tilde{Z} \in \mathcal{M} \text{ and } \tilde{K}^+, \tilde{K}^- \in \mathcal{S}_{ci} \\ \tilde{Y}_t = U_T + \int_t^T \{|f(s,\omega,0,0)| + C(|\tilde{Y}_s| + |\tilde{Z}_s|)\}ds \\ \qquad +(\tilde{K}^+_T - \tilde{K}^+_t) - (\tilde{K}^-_T - \tilde{K}^-_t) - \int_t^T \tilde{Z}_s dB_s,\ \forall t \le T \\ L_t \le \tilde{Y}_t \le U_t,\ \forall t \le T \text{ and } \int_0^T (\tilde{Y}_s - L_s) d\tilde{K}^+_s = \int_0^T (U_s - \tilde{Y}_s) d\tilde{K}^-_s = 0. \end{cases}$$

On the other hand there exists a stationary sequence of stopping times $(\gamma_n)_{n\ge 0}$ such that $E[(\tilde{K}^+_{\gamma_n})^2 + (\tilde{K}^-_{\gamma_n})^2] < \infty$.

Now the comparison theorem (1.3) implies that P-a.s. for any $t \le T$, $K^-_t \le \tilde{K}^-_t$ and then $E[(K^-_{\gamma_n})^2] < \infty$. Henceforth standard calculations in (24) imply also that $E[(K^+_{\gamma_n})^2 + \int_0^{\gamma_n} |Z_s|^2 ds] < \infty$, whence the desired result □

5. Reflected BSDEs with continuous coefficient

Suppose now that the function f is no longer Lipschitz but just continuous. We are going to show, once again, that the reflected BSDE associated with (f, ξ, L, U) has a solution in the sense of Theorem.3.7.

So namely assume that the function $(t,\omega,y,z) \longmapsto f(t,\omega,y,z)$ is continuous with respect to (y,z) and is at most with linear growth, *i.e.*, there exists a constant C such that $|f(t,\omega,y,z)| \le C(1+|y|+|z|)$. Then we have the following result:

5.1. Theorem. *There exists a quadruple $(Y, Z, K^+, K^-) := (Y_t, Z_t, K^+_t, K^-_t)_{t\le T}$ solution of the double barrier reflected BSDE associated with (f, ξ, L, U).*

Proof. There exists a sequence of functions $(f_n)_{n\ge 0}$, obtained via inf-convolution, such that for any $n \ge 0$, $f_n \le f_{n+1}$, f_n is Lipschitz with respect to (y,z) and finally $|f_n(t,\omega,y,z)| \le C(1+|y|+|z|)$, $\forall (t,y,z) \in [0,T] \times R^{1+d}$ (see e.g. [HLM]).

Now for $n \geq 0$, let $(Y^n, Z^n, K^{n,+}, K^{n,-})$ be the solution of the reflected BSDE associated with (f_n, ξ, L, U) and $(\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)$ the solution of the reflected BSDE associated with $(C(1+|y|+|z|), \xi, L, U)$. Therefore using the comparison theorem (1.3) we have for any $n \geq 0$, $Y^n \leq Y^{n+1} \leq U$, $K^{n,-} \leq K^{-,n+1} \leq \bar{K}^-$ and $K^{n,+} \geq K^{+,n+1} \geq \bar{K}^+$. So for any $t \leq T$ let us set Y_t (resp. K_t^-; resp. K_t^+) the pointwise limit of the sequence $(Y_t^n)_{n\geq 0}$ (resp. $(K_t^{n,-})_{n\geq 0}$; resp. $(K_t^{n,+})_{n\geq 0}$).

Now let $(\gamma_k)_{k\geq 0}$ be the stationary sequence of stopping times such that $E[(\bar{K}^+_{\gamma_k})^2 + (\bar{K}^-_{\gamma_k})^2 + \int_0^{\gamma_k} |\bar{Z}_s|^2 ds] < \infty$ for any $k \geq 0$. Using standard calculations we obtain for any fixed $k \geq 0$,

$$E[\sup_{s\leq\gamma_k} |Y_s^m - Y_s^n|^2 + \int_0^{\gamma_k} |Z_s^m - Z_s^n|^2 ds] \longrightarrow 0 \text{ as } n, m \ \to 0.$$

It follows that the process $(Y_{t\wedge\gamma_k})_{t\leq T}$ is continuous for any $k \geq 0$. As $(\gamma_k)_{k\geq 0}$ is a stationary sequence then the process $(Y_t)_{t\leq T}$ is continuous. On the other hand the sequence $((Z_t^n 1_{[t\leq\gamma_k]})_{t\leq T})_{n\geq 0}$ converges in $\mathcal{M}^{2,d}$ to a process which we denote $(Z_t^k)_{t\leq T}$. In addition it satisfies, for any $p \geq 1$ and $k \geq 0$, $Z^{k+p}_{t\wedge\gamma_k} = Z^k_{t\wedge\gamma_k}$, $dt \otimes dP - a.s.$ Therefore for any $k \geq 0$ we have,

$$\begin{aligned} Y_{t\wedge\gamma_k} = Y_{\gamma_k} &+ \int_{t\wedge\gamma_k}^{\gamma_k} f(s, Y_s, Z_s^k)ds + (K^+_{\gamma_k} - K^+_{t\wedge\gamma_k}) \\ &-(K^-_{\gamma_k} - K^-_{t\wedge\gamma_k}) - \int_{t\wedge\gamma_k}^{\gamma_k} Z_s^k dB_s, \ \forall t \leq T \end{aligned} \tag{25}$$

since $E[\int_0^{\gamma_k} |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s^k)|ds] \to 0$ as $n \to 0$. Now (25) implies also that

$$Y_{t\wedge\gamma_k} = Y_0 - \int_0^{t\wedge\gamma_k} f(s, Y_s, Z_s^k)ds - K^+_{t\wedge\gamma_k} + K^-_{t\wedge\gamma_k} + \int_0^{t\wedge\gamma_k} Z_s^k dB_s, \ \forall t \leq T.$$

As K^- is lower semi-continuous and K^+ is upper semi-continuous then the processes $(K^+_{t\wedge\gamma_k})_{t\leq T}$ and $(K^-_{t\wedge\gamma_k})_{t\leq T}$ are continuous. Henceforth K^+ and K^- are continuous since $(\gamma_k)_{k\geq 0}$ is of stationary type. In addition, from Dini's theorem, the sequences $(K^{n,+})_{n\geq 0}$ and $(K^{n,-})_{n\geq 0}$ converge P-$a.s.$ uniformly to K^+ and K^- respectively.

Now for any $t \leq T$, let us set $Z_t = Z_t^0 1_{[0,\gamma_0]}(t) + \sum_{k\geq 0} Z_t^{k+1} 1_{]\gamma_k,\gamma_{k+1}]}(t)$. As $E[\int_0^{\gamma_k} |Z_s^k|^2 ds] < \infty$ for any $k \geq 0$ and the sequence $(\gamma_k)_{k\geq 0}$ is of stationary type then $\int_0^T |Z_s|^2 ds < \infty$, P-$a.s.$. On the other hand, with the definition of Z and (25) we have,

$$\begin{aligned} Y_{t\wedge\gamma_k} = Y_{\gamma_k} &+ \int_{t\wedge\gamma_k}^{\gamma_k} f(s, Y_s, Z_s)ds + (K^+_{\gamma_k} - K^+_{t\wedge\gamma_k}) \\ &-(K^-_{\gamma_k} - K^-_{t\wedge\gamma_k}) - \int_{t\wedge\gamma_k}^{\gamma_k} Z_s dB_s, \ \forall t \leq T. \end{aligned} \tag{26}$$

Now taking k great enough in (26) yields

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s, \ \forall t \leq T.$$

Finally it remains to show that $\int_0^T (Y_s - U_s)dK_s^- = \int_0^T (Y_s - L_s)dK_s^+ = 0$. But this is a direct consequence of the P-$a.s.$ uniform convergence of Y^n (resp. $K^{n,+}$; resp. $K^{n,-}$) to Y (resp. K^+; resp. K^-) and the facts that $\int_0^T (Y_s^n - U_s)dK_s^{n,-} = \int_0^T (Y_s^n - L_s)dK_s^{n,+} = 0$ (see e.g. the Helly's Theorem in [KF], pp.370). The proof is now complete □

6. Relation with double obstacle variational inequality

Let $b : [0, T] \times R^k \to R^k$ and $\sigma : [0, T] \times R^k \to R^{k\times d}$ be continuous mappings and Lipschitz with respect to the second variable, uniformly with respect to $t \in [0, T]$. For $(t, x) \in [0, T] \times R^k$, let $(X_s^{t,x})_{s\in[0,T]}$ be the unique R^k-valued process solution of the following standard SDE :

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r, \ t \leq s \leq T \\ X_s^{t,x} = x, \ \ s < t. \end{cases}$$

Now let us consider the functions $g : x \longmapsto g(x)$, $f : (r, x, y, z) \longmapsto f(r, x, y, z)$, $h : (r, x) \longmapsto h(r, x)$ and $h' : (r, x) \longmapsto h'(r, x)$, $(r, x, y, z) \in [0, T] \times R^{k+1+d}$. We assume that they are continuous and satisfy : for any $r \in [0, T]$, $x \in R^k$, $y, y' \in R$, $z, z' \in R^d$

$$\begin{cases} |g(x)| + |f(r, x, 0, 0)| + |h(r, x)| + |h'(r, x)| \leq C(1 + |x|^p), \\ |f(r, x, y, z) - f(r, x, y', z')| \leq C(|y - y'| + |z - z'|), \\ h(r, x) < h'(r, x) \text{ and } h(T, x) \leq g(x) \leq h'(T, x) \end{cases}$$

where C and p are some positive constants.

Now let $(Y_s^{t,x}, Z_s^{t,x}, K_s^{+,t,x}, K_s^{-,t,x})_{s\leq T}$ be the unique solution of the BSDE with two reflecting barriers associated with $(f(r, X_r^{t,x}, y, z), g(X_T^{t,x}), h(r, X_r^{t,x}), h'(r, X_r^{t,x}))$. On the other hand for $n \geq 0$, let $({}^nY_s^{t,x})_{s\leq T}$ (resp. $({}^n\bar{Y}_s^{t,x})_{s\leq T}$) be the first component of the unique solution of the BSDE with one reflecting lower (resp. upper) barrier associated with $(f(r, X_r^{t,x}, y, z) - n(h'(r, X_r^{t,x}) - y)^-, g(X_T^{t,x}), h(r, X_r^{t,x}))$ (resp. $(f(r, X_r^{t,x}, y, z) + n(h(r, X_r^{t,x}) - y)^+, g(X_T^{t,x}), h'(r, X_r^{t,x})))$ (${}^nY$ and ${}^n\bar{Y}$ exist through Theorem 1.1). It has been shown in [EKal] that, for any $n \geq 0$ there exist functions ${}^nu(t, x)$ and ${}^n\bar{u}(t, x)$, $(t, x) \in [0, T] \times R^k$, such that

$$\forall s \in [t, T], \ {}^nY_s^{t,x} = {}^nu(s, X_s^{t,x}) \text{ and } {}^n\bar{Y}_s^{t,x} = {}^n\bar{u}(s, X_s^{t,x}). \qquad (27)$$

In addition ${}^n u$ (resp. ${}^n \bar{u}$) is continuous and is a viscosity solution for the following obstacle problem :

$$\begin{cases} \min\{v(t,x) - h(t,x), -\frac{\partial v}{\partial t}(t,x) - L_t v(t,x) \\ \quad -f(t,x,v(t,x),\nabla v\sigma(t,x)) + n(h'(t,x) - v(t,x))^-\} = 0, \\ u(T,x) = g(x), \end{cases} \tag{28}$$

(resp.

$$\begin{cases} \max\{v(t,x) - h'(t,x), -\frac{\partial v}{\partial t}(t,x) - L_t v(t,x) \\ \quad -f(t,x,v(t,x),\nabla v\sigma(t,x)) - n(h(t,x) - v(t,x))^+\} = 0, \\ u(T,x) = g(x),) \end{cases} \tag{29}$$

where

$$L_t = \frac{1}{2}\sum_{i,j=1}^{k} (\sigma\sigma^*(t,x))_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{k} b_i(t,x)\frac{\partial}{\partial x_i}.$$

Now the comparison result (Corollary 1.4.) allows us to infer that $({}^n Y^{t,x})_{n\geq 0}$ (resp. $({}^n \bar{Y}^{t,x})_{n\geq 0}$) is a decreasing (resp. an increasing) sequence. Moreover they converge in $\mathcal{S}^2$ to $Y^{t,x}$. Therefore for any $(t,x) \in [0,T] \times R^k$, the sequence $({}^n u(t,x))_{n\geq 0}$ (resp. $({}^n \bar{u}(t,x))_{n\geq 0}$) converges decreasingly (resp. increasingly) to the same limit $u(t,x) := Y_t^{t,x}$ which satisfies $Y_s^{t,x} = u(s, X_s^{t,x})$ for any $s \in [t,T]$. Now since ${}^n u$ and ${}^n \bar{u}$ are continuous then u is, in the same time, lower and upper semicontinuous therefore it is continuous. It implies that the convergence of $({}^n u)_{n\geq 0}$ and $({}^n \bar{u})_{n\geq 0}$ to u are uniform on compact subsets of $[0,T] \times R^k$.

Consider now the following obstacle problem :

$$\begin{cases} \min\{v(t,x) - h(t,x), \max[-\frac{\partial v}{\partial t}(t,x) - L_t v(t,x) \\ \quad -f(t,x,v(t,x),\nabla v\sigma(t,x)), (v - h')(t,x)]\} = 0, \\ u(T,x) = g(x). \end{cases} \tag{30}$$

First we start by the definition of the viscosity solution for (30).

6.1. Definition. *Let v be a function which belongs to $\mathcal{C}([0,T] \times R^k)$. It is called a viscosity :*
(i) subsolution of (30) if $v(T,x) \leq g(x)$ and for any $\phi \in \mathcal{C}^{1,2}((0,T) \times R^k)$ and any local maximum point $(t,x) \in (0,T) \times R^k$ of $v - \phi$, we have

$$\min\{(v-h)(t,x), \max[-\frac{\partial \phi}{\partial t}(t,x) - L_t\phi(t,x) \\ -f(t,x,v(t,x),\nabla\phi\sigma(t,x)), (v-h')(t,x)]\} \leq 0$$

(ii) supersolution of (30) if $v(T,x) \geq g(x)$ and for any $\phi \in \mathcal{C}^{1,2}((0,T) \times R^k)$ and any local minimum point $(t,x) \in (0,T) \times R^k$ of $v - \phi$, we have

$$\min\{(v-h)(t,x), \max[-\frac{\partial \phi}{\partial t}(t,x) - L_t\phi(t,x) \\ -f(t,x,v(t,x),\nabla\phi\sigma(t,x)), (v-h')(t,x)]\} \geq 0$$

(iii) solution of (30) if it is both a viscosity subsolution and supersolution □

6.2. Theorem. *The function u defined above is a viscosity solution of (30) and for any* $(t, x) \in [0, T] \times R^k$ *we have,*

$$u(t,x) = inf_{\nu \in \mathcal{T}_t} sup_{\sigma \in \mathcal{T}_t} J_{t,x}(\nu, \sigma) = sup_{\sigma \in \mathcal{T}_t} inf_{\nu \in \mathcal{T}_t} J_{t,x}(\nu, \sigma) = J_{t,x}(\delta_{t,x}, \theta_{t,x}), \tag{31}$$

where for any stopping times ν *and* σ *in* $\mathcal{T}_t$,

$$J_{t,x}(\nu,\sigma) := \begin{cases} E[\int_t^{\nu\wedge\sigma} f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + h(\sigma, X_\sigma^{t,x})1_{[\sigma \le \nu < T]} \\ \quad +h'(\nu, X_\nu^{t,x})1_{[\nu<\sigma]} + g(X_T^{t,x})1_{[\nu=\sigma=T]}] \\ \quad if \;\; E[\int_t^{\nu\wedge\sigma} |f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})|ds] < \infty \\ +\infty \quad else\,; \end{cases}$$

here $\delta_{t,x} := \inf\{s \ge t, Y_s^{t,x} = h'(s, X_s^{t,x})\} \wedge T$ *and* $\theta_{t,x} := \inf\{s \ge t, Y_s^{t,x} = h(s, X_s^{t,x})\} \wedge T$.

Proof. Let us show that u is a viscosity subsolution of (30). Since $u(T, x) = g(x)$ and $h(t, x) \le u(t, x) \le h'(t, x)$, it is sufficient to prove that for any $\phi \in \mathcal{C}^{1,2}((0, T) \times R^k)$ and for any local maximum point $(t, x) \in (0, T) \times R^k$ of $u - \phi$ such that $u(t, x) > h(t, x)$, we have

$$-\frac{\partial \phi}{\partial t}(t,x) - L_t\phi(t,x) - f(t,x,u(t,x), \nabla\phi\sigma(t,x)) \le 0.$$

So let (t_n, x_n) be a sequence of local maximum points of ${}^n u - \phi$ such that (t_n, x_n) converges to (t, x) (the existence of such a sequence follows from the uniform convergence of ${}^n u$ to u (see e.g.[KM], pp.117)). Note that for n large enough we have ${}^n u(t_n, x_n) > h(t_n, x_n)$ then, using the fact that ${}^n u$ is a viscosity solution of (28) we have,

$$\begin{aligned} &-\frac{\partial \phi}{\partial t}(t_n,x_n) - L_{t_n}\phi(t_n,x_n) - f(t_n,x_n, {}^n u(t_n,x_n), \nabla\phi\sigma(t_n,x_n)) \\ &\le -n(h'(t_n,x_n) - {}^n u(t_n,x_n))^- \le 0. \end{aligned}$$

Now the continuity of the functions and the uniform convergence yields the desired result. In a similar way we can show that u is also a viscosity supersolution.

The second part of the theorem follows from Proposition 2.2.1 and the fact that for all $(t, x) \in [0, T] \times R^k$ we have

$$E[\int_t^{\delta_{t,x}\vee\theta_{t,x}} |Z_s^{t,x}|^2 ds] < \infty$$

as mentioned in Proposition 3.6 □

We now deal with the issue of uniqueness of the viscosity solution of (30). So assume furthermore that f satisfies the following assumption:

[H2] For each $R > 0$, there is a continuous function φ_R such that $\varphi_R(0) = 0$ and

$$|f(t,x,y,z) - f(t,x',y,z)| \leq \varphi_R((1+|z|)|x-x'|)$$

for all $t \in (0,T)$, $|x|, |x'|, |y| \leq R$ and $z \in R^d$.

6.3. Proposition. *Uniqueness of the viscosity solution of (30)*

If v (resp. u) is a viscosity supersolution (resp. subsolution) of (30) then for all $(t,x) \in [0,T] \times R^k$ we have $u(t,x) \leq v(t,x)$.

Proof. Note that $v \geq h$ and $u \leq h'$ then $v' := v \wedge h'$ (resp. $u' := u \vee h$) is a viscosity supersolution (resp. subsolution) of (30). Therefore it is enough to show that $u' \leq v'$.

In the same spirit as in [EKal], one can suppose that $y \to f(t,x,y,z)$ is strictly decreasing for all t, x, z and $|h(t,x)| + |h'(t,x)| \leq C(1+|x|^2)^{-1}$. Otherwise one can take, instead of u', v', g, ..., the following functions $\hat{u}$, $\hat{v}$,... defined by :

$$\begin{aligned}
\hat{u}(t,x) &:= u'(t,x)e^{\lambda t}\zeta(x),\ \ \hat{v}(t,x) := v'(t,x)e^{\lambda t}\zeta(x),\\
\hat{h}(t,x) &:= h(t,x)e^{\lambda t}\zeta(x), \hat{h}'(t,x) := h'(t,x)e^{\lambda t}\zeta(x),\\
\hat{g}(x) &:= g(x)e^{\lambda T}\zeta(x),\ \ \hat{L}. := L. + \langle \sigma\sigma^*\eta; \nabla.\rangle \text{ and}\\
\hat{f}(t,x,y,z) &:= e^{\lambda t}\zeta^{-1}(x) f(t,x,e^{-\lambda t}\zeta(x)y, e^{-\lambda t}\zeta(x)z\\
&\qquad + e^{-\lambda t}\nabla\zeta(x)\sigma(t,x)y) + [\frac{1}{2}trace(\sigma\sigma^*\kappa) + \langle b;\eta\rangle - \lambda]y
\end{aligned}$$

where $\zeta(x) := (1+|x|^2)^{p+2}$, $\eta(x) := \zeta(x)^{-1}\nabla\zeta(x)$, $\kappa(x) := \zeta(x)^{-1}\nabla^2\zeta(x)$ and λ is large enough such that the mapping $y \mapsto \hat{f}(t,x,y,z)$ is strictly decreasing for all t, x, z.

Using the same arguments as in [EKal], we have for any ε and $R > 0$

$$\sup_{t\in[0,T],|x|\leq R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+ \leq \sup_{t\in[0,T],|x|=R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+. \tag{32}$$

Indeed, assume that

$$\begin{aligned}
\delta :=& \sup_{t\in[0,T],|x|\leq R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+\\
>& \sup_{t\in[0,T],|x|=R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+ \geq 0.
\end{aligned}$$

Hence from ([EKal] pp.733) we have the existence of a sequence

$$(t_n, x_n, y_n, p_n, X_n, Y_n) \in (0,T) \times B_R^2 \times R \times (R^{d\times d})^2$$

such that:

(i) $n|x_n - y_n|^2 \to 0$ as $n \to \infty$

(ii) $u'(t_n,x_n) \geq v'(t_n,y_n) + \frac{\varepsilon}{t_n} + \delta$

(iii) $(p_n, n(x_n - y_n), X_n) \in \bar{\mathcal{P}}^{2,+}(u'(t_n,x_n))$

(iv) $(p_n, n(x_n - y_n), Y_n) \in \bar{\mathcal{P}}^{2,-}(v'(t_n, y_n) + \frac{\varepsilon}{t_n})$

(v)

$$\begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

where $B_R := \{x \in R^k : |x| \leq R\}$ and $\bar{\mathcal{P}}^{2,+}(u(t,x))$, $\bar{\mathcal{P}}^{2,-}(u(t,x))$ are defined in ([EKal], pp.728).

On the other hand, since h and h' are uniformly continuous on compact subsets we have from (ii), for n large enough, $u'(t_n, x_n) > h(t_n, x_n)$ and $v'(t_n, x_n) < h'(t_n, x_n)$. Hence since u' (resp. v') is a subsolution (resp. supersolution) and,

$$\begin{aligned} -p_n - \frac{1}{2} trace(\sigma\sigma^*(t_n, x_n)X_n) - \langle b; n(x_n - y_n)\rangle \\ -f(t_n, x_n, u'(t_n, x_n), n(x_n - y_n)) \leq 0, \\ -p_n - \frac{1}{2} trace(\sigma\sigma^*(t_n, y_n)Y_n) - \langle b; n(x_n - y_n)\rangle \\ -f(t_n, y_n, v'(t_n, y_n) + \frac{\varepsilon}{t_n}, n(x_n - y_n)) \geq \frac{\varepsilon}{t_n^2} \end{aligned}$$

then,

$$\begin{aligned} \frac{\varepsilon}{t_n^2} \leq \Lambda_n =: \frac{1}{2} trace(\sigma\sigma^*(t_n, x_n)X_n - \sigma\sigma^*(t_n, y_n)Y_n) \\ + f(t_n, x_n, u'(t_n, x_n), n(x_n - y_n)) - f(t_n, y_n, v'(t_n, y_n) + \frac{\varepsilon}{t_n}, n(x_n - y_n)). \end{aligned}$$

Now arguing as in ([EKal] pp. 734), we obtain that $\liminf_{n\to\infty} \Lambda_n \leq 0$ and then $\varepsilon \leq 0$ which is contradictory. Finally taking the limits in (32), first when $R \to \infty$ then $\epsilon \to 0$, we obtain $u' \leq v'$.

It implies that if $\tilde{u}$ is another solution for (30), then $u \leq \tilde{u}$ and $\tilde{u} \leq u$, therefore $u = \tilde{u}$ □

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