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Brownian survival among Poissonian traps with random shapes at critical intensity

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Abstract. In this paper we consider a standard Brownian motion in $\mathbb{R}^d$, starting at 0 and observed until time t. The Brownian motion takes place in the presence of a Poisson random field of traps, whose centers have intensity ν_t and whose shapes are drawn randomly and independently according to a probability distribution Π, on the set of closed subsets of $\mathbb{R}^d$, subject to appropriate conditions. The Brownian motion is killed as soon as it hits one of the traps. With the help of a large deviation technique developed in an earlier paper, we find the tail of the probability S_t that the Brownian motion survives up to time t when

$$\nu_t = \begin{cases} ct^{-2/d}, & d \geq 3, \\ ct^{-1}\log^2 t, & d = 2, \end{cases}$$

where $c \in (0, \infty)$ is a parameter. This choice of intensity corresponds to a critical scaling. We give a detailed analysis of the rate constant in the tail of S_t as a function of c, including its limiting behaviour as $c \to \infty$ or $c \downarrow 0$. For $d \geq 3$, we find that there are two regimes, depending on the choice of Π. In one of the regimes there is a collapse transition at a critical value $c^* \in (0, \infty)$, where the optimal survival strategy changes from being diffusive to being subdiffusive. At c^*, the slope of the rate constant is discontinuous. For $d = 2$, there is again a collapse transition, but the rate constant is independent of Π and its slope at $c = c^*$ is continuous.

1. Introduction and main results

1.1. Motivation

The model studied in this paper has two random ingredients:

1. Let $\beta = \{\beta(s) \colon s \geq 0\}$ be the standard Brownian motion in $\mathbb{R}^d$ – the Markov process with generator $\Delta/2$ – starting at 0. We write P, E to denote probability and expectation with respect to β.

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2. For $t \geq 0$, let

$$K_t = \bigcup_{x \in \omega_t} [x + A_x], \tag{1.1.1}$$

where ω_t is a Poisson point process with intensity

$$\nu_t = \begin{cases} ct^{-2/d}, & d \geq 3, \\ ct^{-1} \log^2 t, & d = 2, \end{cases} \tag{1.1.2}$$

$c \in (0, \infty)$ is a parameter and, given ω_t,

$$A_x, \quad x \in \omega_t, \tag{1.1.3}$$

are i.i.d. random sets drawn from $\mathcal{C} = \{A \subset \mathbb{R}^d\colon\ A \text{ closed}\}$ according to a probability distribution Π. We write $\mathbb{P}_t$, $\mathbb{E}_t$ to denote probability and expectation with respect to K_t.

Formally, $\mathcal{C}$ is endowed with the topology generated by the Hausdorff metric $\rho_H\colon \mathcal{C} \times \mathcal{C} \to [0, \infty]$ given by

$$\rho_H(A_1, A_2) = \inf\{\epsilon > 0\colon\ A_1 \subset A_2^\epsilon,\ A_2 \subset A_1^\epsilon\}, \tag{1.1.4}$$

where $A^\epsilon = \cup_{x \in A} B_\epsilon(x)$ is the ϵ-environment of A (with $B_\epsilon(x)$ the closed ball of radius ϵ centred at x). The probability distribution Π lives on the Borel sigma-algebra generated by ρ_H.

Throughout the paper, we assume that Π satisfies the following two conditions:

(C1) $\Pi(\mathcal{Q}) = 1$ with

$$\mathcal{Q} = \left\{A \subset \mathbb{R}^d\colon\ A \text{ compact},\ A = \mathrm{cl}(\mathrm{int}(A)),\ A \neq \emptyset\right\}, \tag{1.1.5}$$

where $\mathrm{cl}(A)$ denotes the closure of A and $\mathrm{int}(A)$ the interior of A.

(C2) $\lim_{M \to \infty} \delta_M = 0$ with

$$\delta_M = \int_{\mathcal{Q}} \frac{|(A + B_M) \cap B_M^c|}{|B_M|}\, \Pi(dA), \tag{1.1.6}$$

where $B_M = [-M/2, M/2]^d$, $B_M^c = \mathbb{R}^d \setminus B_M$ and $A + B_M = \cup_{x \in B_M} [x + A]$.

Condition (C1) is a regularity property for A, while condition (C2) allows us to control large A.

Let

$$\tau_{K_t} = \inf\{s \geq 0\colon\ \beta(s) \in K_t\} \tag{1.1.7}$$

and

$$S_t = (\mathbb{E}_t \times P)(\tau_{K_t} > t). \tag{1.1.8}$$

In other words, we view K_t as a collection of randomly located and randomly shaped traps, τ_{K_t} as the trapping time for the Brownian motion, and S_t as the probability of survival up to time t. The goal of the present paper is to identify the asymptotic behaviour of S_t for large t. As will become clear later on, the choice of intensity in (1.1.2) corresponds to a *critical scaling*. Our main results show that the tail of S_t has an interesting dependence on the parameter c, with *two regimes* for $d \geq 3$, depending on the choice of Π, and *one regime* for $d = 2$. The proof of these results relies on a large deviation technique developed in van den Berg, Bolthausen and den Hollander [2]. For each of the regimes we provide a detailed analysis of the rate constant controlling the tail behaviour of S_t, including its scaling as $c \to \infty$ or $c \downarrow 0$. We show that for $d \geq 3$, in one of the regimes, the rate constant exhibits a *collapse transition* in the optimal survival strategy at a critical value $c^* \in (0, \infty)$. We analyse the behaviour of the rate constant near c^* and show that a slope discontinuity occurs. For $d = 2$ there is a collapse transition too, but no slope discontinuity at $c = c^*$.

1.2. Representation in terms of Wiener sausages

The starting point of our analysis is a representation formula expressing S_t as an exponential functional of a family of Wiener sausages with varying shape. This formula is the analogue of the well-known formula for the fixed shape case.

The Wiener sausage with shape $A \in \mathcal{Q}$ is the random process defined by

$$W^A(t) = \bigcup_{0 \leq s \leq t} [\beta(s) + A], \qquad t \geq 0. \tag{1.2.1}$$

Proposition 1.2.1. *For any $d \geq 1$, $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$ and $t \geq 0$,*

$$S_t = E\left(\exp\left[-\nu_t \int_{\mathcal{Q}} \Pi(dA)\ |W^A(t)|\right]\right). \tag{1.2.2}$$

Proof. The trap field is a *marked* Poisson point process: the points $x \in \omega_t$ carry random labels A_x. Consider those points whose label is in dA, an infinitesimally small subset of $\mathcal{Q}$. These points form a Poisson point process with intensity $\nu_t \Pi(dA)$. The probability, under the law $\mathbb{P}_t$, that up to time t these traps avoid a given Brownian path β equals $\exp[-\nu_t \Pi(dA)\,|W^A(t)|]$. The probability that up to time t all the traps avoid the given β therefore equals $\exp[-\nu_t \int_{\mathcal{Q}} \Pi(dA)\,|W^A(t)|]$. Average over β to get the claim. □

Since $\int_{\mathcal{Q}} \Pi(dA)|A| \leq |B_M|(1 + \delta_M)$ for all $M > 0$, condition (C2) implies that $\int_{\mathcal{Q}} \Pi(dA)|A| < \infty$. The integral in the right-hand side of (1.2.2) is finite P-a.s. for all $t \geq 0$. Indeed, let $M(t) = \inf\{M > 0\colon\ \beta(s) \in B_M \text{ for all } 0 \leq s \leq t\}$. Then

$$|W^A(t)| \leq |B_{M(t)}| + |(A + B_{M(t)}) \cap B^c_{M(t)}|, \tag{1.2.3}$$

and hence

$$\int_{\mathcal{Q}} \Pi(dA)\,|W^A(t)| \leq |B_{M(t)}|(1 + \delta_{M(t)}), \tag{1.2.4}$$

with $M(t) < \infty$ P-a.s.

1.3. Survival theorems

This section contains our main results for the tail behaviour of S_t as $t \to \infty$.

For $d \geq 3$, let $\kappa(A)$ be the Newtonian capacity of A associated with the Green function of $(-\Delta/2)^{-1}$.

Theorem 1.3.1. *Let $d \geq 3$ and let Π satisfy* (C1) *and* (C2). *For every $c > 0$,*

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log S_t = -J_d^{\Pi}(c) \tag{1.3.1}$$

with

$$J_d^{\Pi}(c) = \inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2 + cF_d^{\Pi}(\phi^2)\colon\ \phi \in H^1(\mathbb{R}^d),\ \|\phi\|_2^2 = 1\right\}, \tag{1.3.2}$$

where

$$F_d^{\Pi}(\phi^2) = \int_{\mathbb{R}^d} dx \int_{\mathcal{Q}} \Pi(dA)\left(1 - e^{-\kappa(A)\phi^2(x)}\right). \tag{1.3.3}$$

Theorem 1.3.1 identifies the tail of S_t for $d \geq 3$ in terms of a variational problem involving Π. Since the dependence on Π enters only via the capacity of the random set A, we may rewrite (1.3.3) as

$$F_d^{\Pi}(\phi^2) = \int_{\mathbb{R}^d} dx \int_0^{\infty} \Theta(d\kappa)\left(1 - e^{-\kappa\phi^2(x)}\right) \tag{1.3.4}$$

with $\Theta = \Pi \circ \kappa^{-1}$ the probability distribution on $(0,\infty)$ induced from Π by κ. Therefore actually $F_d^{\Pi} = F_d^{\Theta}$ and $J_d^{\Pi} = J_d^{\Theta}$. Nevertheless, we prefer to keep Π in the notation. Note that $\kappa(A) \in (0,\infty)$ for all $A \in \mathcal{Q}$.

A similar result holds for $d = 2$, but without a role for Π.

Theorem 1.3.2. *Let $d = 2$ and let Π satisfy* (C1) *and* (C2). *For every $c > 0$,*

$$\lim_{t\to\infty} \frac{1}{\log t} \log S_t = -J_2(c) \tag{1.3.5}$$

with

$$J_2(c) = \inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2 + cF_2(\phi^2)\colon\ \phi \in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1\right\}, \tag{1.3.6}$$

where

$$F_2(\phi^2) = \int_{\mathbb{R}^2} dx \left(1 - e^{-2\pi\phi^2(x)}\right). \tag{1.3.7}$$

The scale of the large deviation in Theorem 1.3.2 is different from that in Theorem 1.3.1. This is due to the different choice of intensity in (1.1.2). However, the variational formula has the same structure. The difference is that $\kappa(A)$ is replaced by 2π, so that the dependence on Π drops out. This fact turns out to be related to the recurrence of planar Brownian motion.

1.4. Analysis of the variational problems

In this section we give a detailed analysis of $c \mapsto J_d^\Pi(c)$ in (1.3.2) and $c \mapsto J_2(c)$ in (1.3.6). We first note the following.

Proposition 1.4.1. *Let $d \geq 3$ and let Π satisfy* (C1) *and* (C2). *Then*

$$\langle \kappa \rangle = \int_{\mathcal{Q}} \Pi(dA)\kappa(A) < \infty. \tag{1.4.1}$$

The variational problem in (1.3.2) certainly makes sense also when $\langle \kappa \rangle = \infty$, but apparently this regime is not caught by our conditions (C1) and (C2).

Let $\langle \cdot \rangle$ denote expectation over Θ. For $d \geq 3$ there are two regimes:

(I) There exists $c^* \in (0, \infty)$ such that

$$\begin{aligned} J_d^\Pi(c) &= c\langle \kappa \rangle, \text{ for } 0 \leq c \leq c^*, \\ &< c\langle \kappa \rangle, \text{ for } c > c^*. \end{aligned} \tag{1.4.2}$$

(II) (1.4.2) with $c^* = 0$.

We consider two subclasses for Θ:

$$\begin{aligned} \mathcal{S}_I &= \Big\{\Theta\colon \text{ there exist } 0 < \kappa_0 < \infty \text{ and } 0 < K < \infty \text{ such that} \\ &\qquad \Theta(d\kappa) \leq K\kappa^{-1-\frac{d+2}{d}}d\kappa \text{ for all } \kappa > \kappa_0\Big\}, \\ \mathcal{S}_{II} &= \Big\{\Theta\colon \text{there exist } 0 < \kappa_1 < \infty \text{ and } L\colon\ (\kappa_1, \infty) \to (0, \infty) \\ &\qquad \text{non-decreasing with } \lim_{\kappa\to\infty} L(\kappa) = \infty \text{ such that} \\ &\qquad \Theta(d\kappa) \geq L(\kappa)\kappa^{-1-\frac{d+2}{d}}d\kappa \text{ for all } \kappa > \kappa_1\Big\}. \end{aligned} \tag{1.4.3}$$

Note that the separation between the classes $\mathcal{S}_I$ and $\mathcal{S}_{II}$ is thin, and is very close to where $\langle \kappa^{(d+2)/d} \rangle$ diverges.

Theorem 1.4.2. *Let $d \geq 3$.*
(i) For every Π, $c \mapsto J_d^\Pi(c)$ is continuous, strictly increasing and concave on $(0, \infty)$*, with* $J_d^\Pi(0) = 0$.
(ii) If $\Theta \in \mathcal{S}_I$, then J_d^Π falls in regime (I). *Moreover, if $\langle \kappa^\eta \rangle < \infty$ for some $\eta > \frac{d+2}{d}$, then*

$$[J_d^\Pi]'(c^*+) < \langle \kappa \rangle. \tag{1.4.4}$$

(iii) If $\Theta \in \mathcal{S}_{II}$, then J_d^Π falls in regime (II)*, and*

$$[J_d^\Pi]'(0+) = \langle \kappa \rangle. \tag{1.4.5}$$

(iv) The variational problem in (1.3.2) *has a minimiser with full support for*

$$\begin{aligned} &c > c^*, \text{ when } \Theta \in \mathcal{S}_I, \\ &c > 0, \quad \text{when } \Theta \in \mathcal{S}_{II}, \\ &c = c^*, \text{ when } \langle \kappa^\eta \rangle < \infty \text{ for some } \eta > \tfrac{d+2}{d}. \end{aligned} \tag{1.4.6}$$

Theorem 1.4.3. *Let $d \geq 3$.*
(i) For every Π,

$$J_d^\Pi(c) \leq c^{2/(d+2)} \frac{d+2}{2} \left(\frac{\lambda_d}{d}\right)^{d/(d+2)}, \qquad c \in (0,\infty), \tag{1.4.7}$$

and

$$\lim_{c\to\infty} c^{-2/(d+2)} J_d^\Pi(c) = \frac{d+2}{2} \left(\frac{\lambda_d}{d}\right)^{d/(d+2)}, \tag{1.4.8}$$

where λ_d is the principal Dirichlet eigenvalue of $-\Delta$ on the ball of unit volume.
(ii) For $\Theta \in \mathcal{S}_{II}$, let $\Theta(d\kappa) = \theta(\kappa)d\kappa$ with $\theta(\kappa) = K\kappa^{-1-\gamma}[1+o(1)]$ as $\kappa \to \infty$ and $1 < \gamma < \frac{d+2}{d}$, $0 < K < \infty$. Then

$$\lim_{c\downarrow 0} \{2K\Gamma(-\gamma)c\}^{-2/(2-d(\gamma-1))} \left[c\langle\kappa\rangle - J_d^\Pi(c)\right] = \tfrac{1}{2} M_d(\gamma), \tag{1.4.9}$$

where

$$M_d(\gamma) = -\inf\left\{\|\nabla\psi\|_2^2 - \int |\psi|^{2\gamma} : \ \psi \in H^1(\mathbb{R}^d),\ \|\psi\|_2^2 = 1\right\} \in (0,\infty). \tag{1.4.10}$$

The analogous results for $d = 2$ read as follows.

Theorem 1.4.4. *Let $d = 2$.*
(i) $c \mapsto J_2(c)$ is continuous, strictly increasing and concave on $(0,\infty)$, with $J_2(0) = 0$.
(ii) There exists a number $c^* \in (0,\infty)$, given by

$$c^* = \frac{1}{4\pi^2} \inf\left\{\frac{\|\nabla\phi\|_2^2}{\|\phi\|_4^4} : \ \phi \in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1\right\}, \tag{1.4.11}$$

such that

$$\begin{aligned} J_2(c) &= 2\pi c, \text{ for } 0 \leq c \leq c^*, \\ &< 2\pi c, \text{ for } c > c^*, \end{aligned} \tag{1.4.12}$$

and

$$[J_2]'(c^*+) = 2\pi. \tag{1.4.13}$$

(iii) Equations (1.4.7–1.4.8) hold with $d = 2$.
(iv) The variational problem in (1.3.6) has a minimiser if and only if $c > c^*$. This minimiser has full support.

The qualitative behaviour of $c \mapsto J_d^\Pi(c)$ found in Theorems 1.4.2, 1.4.3 and 1.4.4 is summarised as follows:

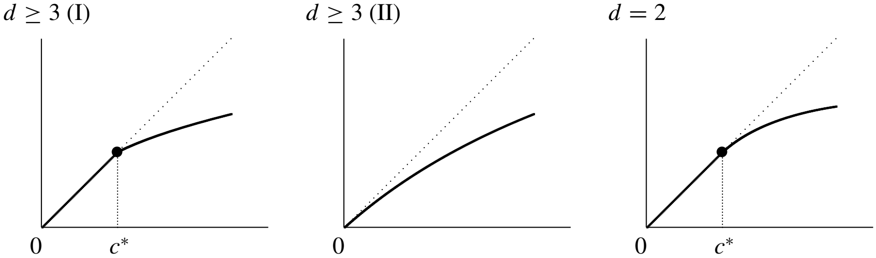

Fig. 1. Qualitative picture of $c \mapsto J_d^\Pi(c)$ for $d \geq 3$, regimes (I) and (II), and $d = 2$, respectively.

1.5. Discussion

The idea behind Theorem 1.3.1 is that for $d \geq 3$ the optimal strategy for the Brownian motion to survive the traps is to behave like a Brownian motion in a drift field $xt^{1/d} \mapsto (\nabla\phi/\phi)(x)$ for some smooth $\phi\colon \mathbb{R}^d \mapsto [0,\infty)$. The cost, under the law P, of adopting this drift during a time t is

$$\exp\left[-t \times t^{-2/d}\,\frac{1}{2}\int_{\mathbb{R}^d} dx\ |\nabla\phi(x)|^2\right]. \tag{1.5.1}$$

The effect of the drift is to push the Brownian motion towards the origin, so that it lives on space scale $t^{1/d}$, which is well below the diffusive scale. Conditioned on adopting the drift, the Brownian motion spends time $\phi^2(x)$ per unit volume in the neighbourhood of $xt^{1/d}$. It turns out that, for each A, the Wiener sausage with shape A associated with the Brownian motion covers a fraction $1 - \exp[-\kappa(A)\phi^2(x)]$ of that unit volume. The cost, under the law $\mathbb{P}_t$, of the traps avoiding the Brownian motion is

$$\exp\left[-ct^{-2/d} \times t\int_{\mathbb{R}^d} dx \int_{\mathcal{Q}} \Pi(dA)\ \left(1 - e^{-\kappa(A)\phi^2(x)}\right)\right] \tag{1.5.2}$$

(recall the proof of Proposition 1.2.1). Combining (1.5.1) and (1.5.2), we see that the best choice of the drift field is therefore given by a minimiser of the variational problem in (1.3.2), or by a minimising sequence.

Theorem 1.3.2 shows that for $d = 2$ the survival probability decays polynomially rather than exponentially fast. The optimal survival strategy is of the same type as for $d \geq 3$, but now the Brownian motion lives on space scale $\sqrt{t/\log t}$, which is only slightly below the diffusive scale. The limiting behaviour does not depend on Π. Apparently, the Brownian motion manages to stay far away from the traps.

Theorems 1.4.2 and 1.4.3 show that for $d \geq 3$ there are two regimes: [1]

[1] Even though we interpret our results in terms of an optimal survival strategy, we have no pathwise statements to offer. More work would be needed to prove that, conditional on survival, the Brownian motion and the trap field indeed behave as suggested. Thus, *all interpretations in this section remain to be proved.*

(I) There is a critical threshold ($c^* > 0$). For $c < c^*$, the Brownian motion prefers to ignore the survival strategies parametrised by ϕ and to move on space scale $\sqrt{t}$. In doing so, it behaves like a typical Brownian motion and sees the average trap capacity, i.e., also the trap field is typical. For $c > c^*$, on the other hand, the Brownian motion prefers to follow the survival strategy parametrised by a minimiser $\bar{\phi}$ and to move on space scale $t^{1/d}$. In doing so, it does a large deviation and sees less than the average trap capacity. Also the trap field does a large deviation, because it keeps traps out of the "spongy structure" that is formed by the Brownian motion. Since $\bar{\phi}$ has full support, the Brownian motion "sneaks around the traps and moves about" rather than "finds a large trap free hole and stays there". At $c = c^*$ there is a *collapse transition* from diffusive behavior to subdiffusive behavior. This collapse transition is *discontinuous* because a minimiser persists at the critical threshold, which leads to a slope discontinuity of J_d^Π at $c = c^*$.

(II) There is no critical threshold ($c^* = 0$). There is a minimiser $\bar{\phi}$ for all $c > 0$, meaning that the optimal survival strategy is always subdiffusive. As $c \downarrow 0$, this minimiser flattens out, the Brownian motion gradually covers more space and gradually sees the average trap capacity. The thinner the tail of Π (i.e., the closer Π to the boundary with regime (I)), the faster J_d^Π approaches the line with slope $\langle\kappa\rangle$.

Theorem 1.4.4 shows that for $d = 2$ the behaviour is similar to that for $d \geq 3$ in regime (I). There is again a collapse transition, associated with a crossover in the optimal strategy. This collapse transition is *continuous* because no minimiser persists as $c \downarrow c^*$.

The high intensity limit $c \to \infty$ corresponds to the minimiser contracting to a high and narrow peak. This corresponds to the optimal survival strategy looking more and more like "find a large trap free hole and stay there". This is the optimal survival strategy for all intensities that are larger than the one in (1.1.2), which is why the choice in (1.1.2) is critical.

Finally, the results in the present paper belong to a regime of critical scaling. A related reference is Merkl and Wüthrich [7], [8], [9]. Here, the principal eigenvalue of the Schrödinger operator $-\Delta + V_t$ on a box of size t with Dirichlet boundary conditions is considered, with V_t a potential consisting of a Poisson field of obstacles with a fixed shape but with a height that shrinks to zero in a critical manner with t. A critical threshold similar to the one in our regime (I) is found. Another related reference is van den Berg, Bolthausen and den Hollander [3], where the large deviation behaviour of the volume of the intersection of two Wiener sausages is identified. A critical threshold appears in the time horizon up to which the intersection volume is observed.

1.6. Examples

In this section we give examples of discrete Π for which regimes (I) and (II) hold.

Let $d \geq 3$, choose α, β, γ with $\alpha > 1$ and $0 < \beta < \gamma \wedge \frac{d}{2}$, and, for $n \in \mathbb{N}$, define

$$A_n = B_{n^\beta} + C_n \quad \text{with} \quad C_n = \{kn^\gamma : \ k \in \mathbb{Z}^d, \ \|k\|_\infty \leq n\}. \tag{1.6.1}$$

Then A_n consists of $(2n+1)^d$ cubes of volume $n^{\beta d}$ that are disjoint for all $n \geq 2$ because $\gamma > \beta$. Let Π be given by

$$\Pi(A_n) = \frac{1}{\zeta(\alpha)} n^{-\alpha}, \qquad n \in \mathbb{N}, \tag{1.6.2}$$

where ζ is the Riemann function. For this Π, condition (C1) trivially holds, while it is easily checked that condition (C2) holds if and only if $\alpha > d + \beta d + 1$. Moreover, for γ sufficiently large, $\kappa(A_n)$ is asymptotically subadditive, i.e.,

$$\kappa(A_n) = \kappa_d \, (2n+1)^d \, n^{\beta(d-2)} \, [1 + o(1)] \qquad \text{as } n \to \infty, \tag{1.6.3}$$

where κ_d is the capacity of the unit cube in $\mathbb{R}^d$. Combining (1.6.2–1.6.3), we see that

$$\begin{aligned} &\langle \kappa^{(d+2)/d} \rangle < \infty : \alpha \in (\beta \tfrac{d^2-4}{d} + d + 3, \infty), \\ &\langle \kappa^{(d+2)/d} \rangle = \infty : \alpha \in (d + \beta d + 1, \beta \tfrac{d^2-4}{d} + d + 3]. \end{aligned} \tag{1.6.4}$$

The latter interval is non-empty since $0 < \beta < \frac{d}{2}$.

1.7. Outline

Theorems 1.3.1 and 1.3.2 are proved in Section 2. The proof closely follows Sections 2, 3 and 4 in van den Berg, Bolthausen and den Hollander [2] (henceforth referred to as vdBBdH). We sketch the main line of the argument, so that the present paper can be read almost independently. Proposition 1.4.1 and Theorems 1.4.2, 1.4.3 and 1.4.4 are proved in Section 3 and rely on variational calculus and Sobolev inequalities.

2. Proof of Theorems 1.3.1 and 1.3.2

2.1. Scaling, compactifying and coarse-graining

Recalling (1.1.2), we have from Proposition 1.2.1 that

$$S_t = \begin{cases} E\left(\exp\left[-t^{(d-2)/d} \times cV^\Pi(t)\right]\right), & d \geq 3, \\ E\left(\exp\left[-\log t \times cV^\Pi(t)\right]\right), & d = 2, \end{cases} \tag{2.1.1}$$

where we define

$$V^\Pi(t) = \begin{cases} \frac{1}{t} \int_Q \Pi(dA) \, |W^A(t)|, & d \geq 3, \\ \frac{\log t}{t} \int_Q \Pi(dA) \, |W^A(t)|, & d = 2. \end{cases} \tag{2.1.2}$$

It follows from Spitzer [10] that, for every $A \in \mathcal{Q}$,

$$E|W^A(t)| = [1+o(1)] \times \begin{cases} \kappa(A)t, & d \geq 3, \\ 2\pi t/\log t, & d = 2, \end{cases} \qquad t \to \infty. \tag{2.1.3}$$

Hence

$$\lim_{t\to\infty} EV^{\Pi}(t) = \begin{cases} \langle \kappa \rangle, & d \geq 3, \\ 2\pi, & d = 2. \end{cases} \tag{2.1.4}$$

(Condition (C2) allows us to interchange limit and integral. For $d \geqslant 3$, by subadditivity, we have $E|W^A(t)|/t \leq E|W^A(1)|$ for $t \geq 1$, while, by (1.2.4) and (C2), this bound is integrable. Recall also Proposition 1.4.1.) Thus, in (2.1.1), the large deviations of $V^{\Pi}(t)$ driving Theorems 1.3.1 and 1.3.2 take place *on the scale of its mean*.

2.1.1. Scaling

By Brownian scaling,

$$\begin{aligned} \frac{1}{t}|W^A(t)| &\doteq |W^{At^{-1/d}}(t^{(d-2)/d})|, d \geq 3, \\ \frac{\log t}{t}|W^A(t)| &\doteq |W^{A\sqrt{\log t/t}}(\log t)|, \quad d = 2, \end{aligned} \tag{2.1.5}$$

where $\doteq$ denotes equality in distribution. Hence, abbreviating

$$\tau = \begin{cases} t^{(d-2)/d}, & d \geq 3, \\ \log t, & d = 2, \end{cases} \qquad T_\tau = \begin{cases} \tau^{2/(d-2)}, & d \geq 3, \\ e^\tau/\tau, & d = 2, \end{cases} \tag{2.1.6}$$

we find from (2.1.1) and (2.1.2) that

$$S_t = E\left(\exp\left[-c\tau V_\tau^{\Pi}(\tau)\right]\right) \tag{2.1.7}$$

with

$$V_\tau^{\Pi}(\tau) = \int_{\mathcal{Q}} \Pi(dA)\ |W^{A/\sqrt{T_\tau}}(\tau)|. \tag{2.1.8}$$

The right-hand side involves Wiener sausages at time τ with a shape that shrinks with $1/\sqrt{T_\tau}$. We are aiming for the large deviations of $V_\tau^{\Pi}(\tau)$.

2.1.2. Compactifying

We will obtain upper and lower bounds on S_t by *wrapping the scaled Brownian motion around a finite torus*, respectively, by *killing it at the boundary of this torus*. This compactification will be exploited in Sections 2.2 and 2.3, where we prove a large deviation principle (LDP) for $(V_\tau^{\Pi}(\tau))_{\tau>0}$ restricted to the torus and use it to compute asymptotics of exponential moments. This LDP will lead to a lower, respectively, upper bound on the variational characterisation of the rate functions J_d^{Π} and J_2 in Theorems 1.3.1 and 1.3.2. By letting the torus tend to $\mathbb{R}^d$ afterwards, we will obtain the variational characterisation as claimed. We will do the compactification for Π with *finite support* and remove this restriction afterwards.

2.1.3. Coarse-graining

The proof of the LDP for the Brownian motion on the torus consists of three steps, taken from vdBBdH:

- Step 1: For $\epsilon > 0$, we chop the Brownian motion into excursions of length ϵ, and define

$$\mathbb{X}_{\tau,\epsilon} = \{\beta_N(i\epsilon)\}_{1\le i\le \tau/\epsilon}, \tag{2.1.9}$$

  which is the collection of the endpoints of the excursions. The lower index N refers to the restriction to the torus of size N, and for notational convenience we assume that τ/ϵ is integer. Let $V^{\Pi}_{\tau,N}(\tau)$ be the analogue of (2.1.8) for the Brownian motion on the torus of size N. We approximate $V^{\Pi}_{\tau,N}(\tau)$ by $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$, where $\mathbb{E}_{\tau,\epsilon}$ denotes the *conditional expectation given* $\mathbb{X}_{\tau,\epsilon}$. We prove that the difference between $V^{\Pi}_{\tau,N}(\tau)$ and $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$ is negligible in the limit as $\tau \to \infty$ followed by $\epsilon \downarrow 0$. This is done by an application of a concentration inequality of Talagrand.
- Step 2: We represent $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$ as a functional of the *bivariate empirical measure*

$$L_{\tau,\epsilon} = \frac{\epsilon}{\tau}\sum_{i=1}^{\tau/\epsilon} \delta_{\left(\beta_N((i-1)\epsilon),\beta_N(i\epsilon)\right)}. \tag{2.1.10}$$

  According to Donsker and Varadhan, $(L_{\tau,\epsilon})_{\tau>0}$ satisfies an LDP. We need some further approximations to get the dependence of $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$ on $L_{\tau,\epsilon}$ in a suitable form. Based on just this LDP we get an LDP for $(\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)))_{\tau>0}$ via a contraction principle.
- Step 3: We take the limit $\epsilon \downarrow 0$. By Step 2, we already know that $V^{\Pi}_{\tau,N}(\tau)$ is well approximated by $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$. It therefore suffices to have an appropriate approximation for the variational formula in the LDP for $(\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)))_{\tau>0}$.

These three steps were used in vdBBdH to derive an LDP for the quantity in (2.1.8) when $\Pi = \delta_{B_a(0)}$, the point measure on the ball of radius $a \in (0,\infty)$ centred at 0. In the present context, the integral over Π represents an additional ingredient, and we have to see how this can be incorporated and carried along. The argument in vdBBdH is rather delicate, involving various estimates on Brownian motion and hitting times of shrinking balls. We need to check that these estimates can be handled when the balls are replaced by sets with a random shape. Therefore we provide a sketch of the main ingredients.

2.1.4. Outline

In Sections 2.2 and 2.3 we give the proof for $d \ge 3$ when Π has *finite support*, i.e.,

$$\Pi = \sum_{m=1}^{n} a_m \delta_{A_m}, \qquad \sum_{m=1}^{n} a_m = 1,\ a_m \ge 0,\ A_m \in \mathcal{Q},\ n \in \mathbb{N}. \tag{2.1.11}$$

In Section 2.4 we explain why the proof for arbitrary Π follows from that for finite Π via a sandwiching argument in combination with a truncation argument. Here conditions (C1) and (C2) in Section 1.1 will play a crucial role. In Section 2.5 we briefly indicate how to amend the proof for $d = 2$, taking (2.1.6), (2.1.7) and (2.1.8) into account.

2.2. Upper bound in $d \geq 3$

Write Λ_N to denote the torus of size N, i.e., $[-N/2, N/2]^d$ with periodic boundary conditions. Let $\beta_N(s)$, $s \geq 0$, be the Brownian motion wrapped around Λ_N. Let

$$W_N^{A/\sqrt{T_\tau}}(s), \qquad s \geq 0, \tag{2.2.1}$$

denote its Wiener sausage with shape A scaled down by $\sqrt{T_\tau}$, and let

$$V_{\tau,N}^{\Pi}(\tau) = \int_Q \Pi(dA)\ |W_N^{A/\sqrt{T_\tau}}(\tau)|. \tag{2.2.2}$$

The wrapping lowers the volume of the Wiener sausages, and so we have, recalling (2.1.7) and (2.1.8),

$$S_t \leq E\left(\exp\left[-c\tau V_{\tau,N}^{\Pi}(\tau)\right]\right). \tag{2.2.3}$$

The desired upper bound on S_t will therefore come out of the following LDP:

Theorem 2.2.1. *$(V_{\tau,N}^{\Pi}(\tau))_{\tau>0}$ satisfies the LDP on $(0,\infty)$ with rate τ and with rate function $I_{d,N}^{\Pi}$ given by*

$$I_{d,N}^{\Pi}(b) = \inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2 \colon\ \phi \in H^1(\Lambda_N),\ \|\phi\|_2^2 = 1,\ F_d^{\Pi}(\phi^2) = b\right\} \tag{2.2.4}$$

with F_d^{Π} given by (1.3.3).

Proof. We assume that Π has the form (2.1.11) and follow the three steps indicated in Section 2.1.3.
Step 1:

Proposition 2.2.2. *For any $\delta > 0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log P\left(|V_{\tau,N}^{\Pi}(\tau) - \mathbb{E}_{\tau,\epsilon}(V_{\tau,N}^{\Pi}(\tau))| \geq \delta\right) = -\infty. \tag{2.2.5}$$

Proof. For Π of the form (2.1.11), we decompose (2.2.2) as

$$V_{\tau,N}^{\Pi}(\tau) = \sum_{m=1}^{n} a_m |W_N^{A_m/\sqrt{T_\tau}}(\tau)|. \tag{2.2.6}$$

The proof of Proposition 4 in vdBBdH (p. 366), which consists of a series of estimates, can be copied to show that, for any $\delta > 0$ and $1 \leq m \leq n$,

$$\lim_{\epsilon\downarrow 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log P\left(\left|\,|W_N^{A_m/\sqrt{T_\tau}}(\tau)| - \mathbb{E}_{\tau,\epsilon}\left(|W_N^{A_m/\sqrt{T_\tau}}(\tau)|\right)\right| \geq \delta\right) = -\infty, \tag{2.2.7}$$

which yields the claim. The only property we need to check for (2.2.7) is the analogue of Equation (2.23) in vdBBdH (p. 367), which plays a pivotal role in the proof and which reads here

$$\sup_{T\geq 1} E\left(\exp\left[\frac{1}{T}|W^{A_m}(T)|\right]\right) < \infty. \tag{2.2.8}$$

Now, the left-hand side is bounded above by the same expression with A_m replaced by $B_R(0)$, where $R = \max_{1\leq m\leq n} R(A_m)$ with $R(A_m)$ the radius of the smallest ball containing A_m centred at 0. But for a ball with an arbitrary finite radius the bound in (2.2.8) is known to be true (see van den Berg and Bolthausen [1]). □

Step 2: Let $I_\epsilon^{(2)}\colon \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \mapsto [0,\infty]$ be the entropy function

$$I_\epsilon^{(2)}(\mu) = \begin{cases} h(\mu|\mu_1 \otimes \pi_\epsilon), & \text{if } \mu_1 = \mu_2, \\ \infty, & \text{otherwise}, \end{cases} \tag{2.2.9}$$

where $h(\cdot\,|\,\cdot)$ denotes relative entropy between measures, μ_1 and μ_2 are the two marginals of μ, and $\pi_\epsilon(x, dy) = p_\epsilon(y-x)dy$ is the Brownian transition kernel on Λ_N associated with an ϵ-excursion. Furthermore, let $\Phi_{1/\epsilon}^\Pi\colon \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \mapsto [0,\infty)$ be the function

$$\begin{aligned}\Phi_{1/\epsilon}^\Pi(\mu) = &\int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx \\ &\times \left(1 - \exp\left[-\frac{\kappa(A)}{\epsilon}\int_{\Lambda_N\times\Lambda_N} \varphi_\epsilon(y-x, z-x)\mu(dy,dz)\right]\right)\end{aligned} \tag{2.2.10}$$

with

$$\varphi_\epsilon(y,z) = \frac{\int_0^\epsilon ds\; p_s(-y)p_{\epsilon-s}(z)}{p_\epsilon(z-y)}. \tag{2.2.11}$$

Proposition 2.2.3. $(\mathbb{E}_{\tau,\epsilon}(V_{\tau,N}^\Pi(\tau)))_{\tau>0}$ *satisfies the LDP on* $(0,\infty)$ *with rate* τ *and with rate function*

$$b \mapsto \inf\left\{\frac{1}{\epsilon}I_\epsilon^{(2)}(\mu)\colon\ \mu \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N),\ \Phi_{1/\epsilon}^\Pi(\mu) = b\right\}. \tag{2.2.12}$$

Proof. The claim is the analogue of Proposition 5 in vdBBdH (p. 371). We indicate how the proof is adapted.

First, we fix $K > 0$ and cut out holes of radius $K/\sqrt{T_\tau}$ around the endpoints of the ϵ-excursions. To that end, we define

$$W_{i,N}^{A,K} = W_{i,N}^A \setminus \left[B_{K/\sqrt{T_\tau}}(\beta_N((i-1)\epsilon)) \cup B_{K/\sqrt{T_\tau}}(\beta_N(i\epsilon))\right] \tag{2.2.13}$$

with

$$W_{i,N}^A = \bigcup_{(i-1)\epsilon\leq s\leq i\epsilon} \left[\beta(s) + A/\sqrt{T_\tau}\right], \tag{2.2.14}$$

and we put

$$V_{\tau,N}^{\Pi,K}(\tau) = \int \Pi(dA) \left| \bigcup_{i=1}^{\tau/\epsilon} W_{i,N}^{A,K} \right|, \tag{2.2.15}$$

which is $V_{\tau,N}^{\Pi}(\tau)$ in (2.2.2) but with the holes cut out. Note that

$$0 \le V_{\tau,N}^{\Pi}(\tau) - V_{\tau,N}^{\Pi,K}(\tau) \le (\tau/\epsilon + 1)\, \omega_d (K/\sqrt{T_\tau})^d \le 2K^d \omega_d/\epsilon T_\tau \tag{2.2.16}$$

(recall (2.1.6); ω_d is the volume of the ball with unit radius). The right-hand side tends to zero as $\tau \to \infty$ for any $K < \infty$, so the cutting is harmless when we let $K \to \infty$ afterwards.

Next, we express $\mathbb{E}_{\tau,\epsilon}(V_{\tau,N}^{\Pi,K}(\tau))$ in terms of the empirical measure $L_{\tau,\epsilon}$ defined in (2.1.10):

$$\begin{aligned}
&\mathbb{E}_{\tau,\epsilon}\left(V_{\tau,N}^{\Pi,K}(\tau)\right)\\
&\quad = \int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx \left(1 - \mathbb{P}_{\tau,\epsilon}\left(x \notin \bigcup_{i=1}^{\tau/\epsilon} W_{i,N}^{A,K}\right)\right)\\
&\quad = \int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx \left(1 - \prod_{i=1}^{\tau/\epsilon} \left\{1 - \mathbb{P}_{\tau,\epsilon}\left(x \in W_{i,N}^{A,K}\right)\right\}\right)\\
&\quad = \int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx \Big(1 - \exp\Big[\frac{\tau}{\epsilon} \int_{\Lambda_N \times \Lambda_N} L_{\tau,\epsilon}(dy, dz)\\
&\qquad \times \log\left(1 - q_{\tau,\epsilon}^A(y - x, z - x)\, 1_{\{y-x, z-x \notin B_{K/\sqrt{T_\tau}}(0)\}}\right)\Big]\Big).
\end{aligned} \tag{2.2.17}$$

Here,

$$q_{\tau,\epsilon}^A(y, z) = P_{\epsilon,y,z}\left(\sigma_{A/\sqrt{T_\tau}} \le \epsilon\right) \tag{2.2.18}$$

with $\sigma_{A/\sqrt{T_\tau}}$ the first time the Brownian motion enters $A/\sqrt{T_\tau}$, and

$$P_{\epsilon,y,z}(\cdot) = P\big(\beta_N([0,\epsilon]) \in \cdot \mid \beta_N(0) = y, \beta_N(\epsilon) = z\big) \tag{2.2.19}$$

the probability law of the Brownian bridge of length ϵ from y to z.

The key property of the quantity in (2.2.18) needed in the proof is the following analogue of Lemma 2 in vdBBdH (p. 372):

$$\begin{aligned}
&\text{(a)} \lim_{K\to\infty} \limsup_{\tau\to\infty} \sup_{y,z \notin B_{K/\sqrt{T_\tau}}(0)} q_{\tau,\epsilon}^A(y, z) = 0 \text{ for all } A \in \mathcal{Q},\ \epsilon > 0,\\
&\text{(b)} \lim_{\tau\to\infty} \sup_{y,z \notin B_\rho(0)} |\tau q_{\tau,\epsilon}^A(y, z) - \kappa(A)\varphi_\epsilon(y, z)| = 0\\
&\qquad\qquad \text{for all } 0 < \rho < N/4 \text{ and } A \in \mathcal{Q},\ \epsilon > 0.
\end{aligned} \tag{2.2.20}$$

Property (2.2.20)(a) is immediate, since $q^A_{\tau,\epsilon}(y,z)$ is non-decreasing in A and for $A = B_R(0)$ the proof is in vdBBdH. For property (2.2.20)(b) the key ingredient is the analogue of Equation (2.64) in vdBBdH (p. 374), which reads

$$\lim_{b\downarrow 0} \frac{1}{\kappa(bA)} P_y\left(\sigma_{bA} \le t\right) = \int_0^t p_s(-y)ds \quad \text{for all } y \in \mathbb{R}^d,\ t \ge 0,\ A \in \mathcal{Q} \tag{2.2.21}$$

with $P_y(\cdot) = P_y(\beta([0,\infty)) \in \cdot \mid \beta(0) = y)$ (see Le Gall [5]). *It is through this relation that $\kappa(A)$ appears on the stage.*

Next, (2.2.20) allows us to linearise the logarithm in the last line of (2.2.17) and to replace it by $-\kappa(A)\varphi_\epsilon(y-x, z-x)/\tau$, which brings us to the expression in (2.2.10) with $\mu = L_{\tau\epsilon}$. To do this properly we need some continuity properties, which are the analogues of Lemmas 3 and 4 in vdBBdH (pp. 375–376) and which rely on (2.2.20)(b). Since Π has finite support, this part of the extension is again straightforward.

The combination of (2.2.16), (2.2.17) and (2.2.20) leads us to the conclusion that

$$\lim_{\tau\to\infty} \left\| \mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)) - \Phi^{\Pi}_{1/\epsilon}(L_{\tau,\epsilon}) \right\|_\infty = 0 \quad \text{for all } \epsilon > 0. \tag{2.2.22}$$

Finally, we note the following:

(1) $\mu \mapsto \Phi^{\Pi}_{1/\epsilon}(\mu)$ is continuous in the total variation norm.

(2) $(L_{\tau,\epsilon})_{\tau>0}$ satisfies the LDP on $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$ with rate τ and with rate function $\frac{1}{\epsilon} I^{(2)}_\epsilon$.

Therefore the claim in Proposition 2.2.3 now follows by using the contraction principle in combination with (2.2.22). □

<u>Step 3</u>: This step consists of two approximation lemmas.

• Let $\Psi^{\Pi}_{1/\epsilon} \colon \mathcal{M}_1^+(\Lambda_N) \mapsto [0,\infty)$ be the function

$$\Psi^{\Pi}_{1/\epsilon}(\nu) = \int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx \left(1 - \exp\left[-\frac{\kappa(A)}{\epsilon}\int_0^\epsilon ds \int_{\Lambda_N} p_s(x-y)\nu(dy)\right]\right). \tag{2.2.23}$$

Lemma 2.2.4. *For any $K > 0$,*

$$\lim_{\epsilon\downarrow 0} \sup_{\mu:\ \frac{1}{\epsilon} I^{(2)}_\epsilon(\mu) \le K} \left| \Phi^{\Pi}_{1/\epsilon}(\mu) - \Psi^{\Pi}_{1/\epsilon}(\mu_1) \right| = 0. \tag{2.2.24}$$

Proof. For Π of the form (2.1.11), we decompose (2.2.10) and (2.2.23) as

$$\Phi^{\Pi}_{1/\epsilon}(\mu) = \sum_{m=1}^n a_m \Phi^{\delta_{A_m}}_{1/\epsilon}(\mu), \qquad \Psi^{\Pi}_{1/\epsilon}(\nu) = \sum_{m=1}^n a_m \Psi^{\delta_{A_m}}_{1/\epsilon}(\nu). \tag{2.2.25}$$

The proof of Lemma 6 in vdBBdH (p. 379) can be copied to show that, for any $K > 0$ and $1 \le m \le n$,

$$\lim_{\epsilon\downarrow 0} \sup_{\mu:\ \frac{1}{\epsilon} I_\epsilon^{(2)}(\mu)\le K} \left|\Phi_{1/\epsilon}^{\delta_{A_m}}(\mu) - \Psi_{1/\epsilon}^{\delta_{A_m}}(\mu_1)\right| = 0, \tag{2.2.26}$$

which yields the claim. The only property needed for the proof of (2.2.26) is $\kappa(A_m) < \infty$. □

• Let $I\colon \mathcal{M}_1^+(\Lambda_N) \mapsto [0,\infty]$ be the standard large deviation rate function for the empirical distribution of the Brownian motion, given by

$$\begin{aligned} I(\nu) &= \tfrac{1}{2}\textstyle\int_{\Lambda_N} |\nabla\phi|^2(x)dx, && \text{if } \tfrac{d\nu}{dx} = \phi^2 \text{ with } \phi \in H^1(\Lambda_N), \\ &= \infty, && \text{otherwise.} \end{aligned} \tag{2.2.27}$$

Let $I_\epsilon\colon \mathcal{M}_1^+(\Lambda_N) \mapsto [0,\infty]$ be the projection of $I_\epsilon^{(2)}$ onto $\mathcal{M}_1^+(\Lambda_N)$, given by

$$I_\epsilon(\nu) = \inf\left\{I_\epsilon^{(2)}(\mu)\colon\ \mu_1 = \nu\right\}. \tag{2.2.28}$$

Then

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} I_\epsilon(\nu) = I(\nu) \qquad \text{for all } \nu \in \mathcal{M}_1^+(\Lambda_N) \tag{2.2.29}$$

(see Lemma 5 in vdBBdH (p. 378)).

Lemma 2.2.5. *For any $K > 0$,*

$$\lim_{\epsilon\downarrow 0} \sup_{\nu:\ \frac{1}{\epsilon} I_\epsilon(\nu)\le K} \left|\Psi_{1/\epsilon}^{\Pi}(\nu) - F_d^{\Pi}\left(\tfrac{d\nu}{dx}\right)\right| = 0 \tag{2.2.30}$$

with F_d^Π given by (1.3.3)*. (Note that if $I_\epsilon(\nu) < \infty$, then $d\nu \ll dx$ because $\nu\otimes\pi_\epsilon \ll dx \otimes dy$ by* (2.2.9) *and* (2.2.28)*).*

Proof. For Π of the form (2.1.11), we decompose (2.2.23) and (1.3.3) as

$$\Psi_{1/\epsilon}^{\Pi}(\nu) = \sum_{m=1}^{n} a_m \Psi_{1/\epsilon}^{\delta_{A_m}}(\nu), \qquad F_d^{\Pi}(\phi^2) = \sum_{m=1}^{n} a_m F_d^{\delta_{A_m}}(\phi^2). \tag{2.2.31}$$

The proof of Lemma 7 in vdBBdH (p. 380) can be copied to show that, for any $K > 0$ and $1 \le m \le n$,

$$\lim_{\epsilon\downarrow 0} \sup_{\nu:\ \frac{1}{\epsilon} I_\epsilon(\nu)\le K} \left|\Psi_{1/\epsilon}^{\delta_{A_m}}(\nu) - F_d^{\delta_{A_m}}\left(\tfrac{d\nu}{dx}\right)\right| = 0, \tag{2.2.32}$$

which yields the claim. The only property needed for the proof of (2.2.32) is $\kappa(A_m) < \infty$. □

Having completed Steps 1–3, the proof of Theorem 2.2.1 now follows easily. Indeed, for any $f\colon (0,\infty)\mapsto\mathbb{R}$ bounded and continuous we have

$$
\begin{aligned}
&\lim_{\tau\to\infty}\frac{1}{\tau}\log E\Big(\exp\big[\tau f(V^{\Pi}_{\tau,N}(\tau))\big]\Big)\\
&\quad=\lim_{\epsilon\downarrow 0}\lim_{\tau\to\infty}\frac{1}{\tau}\log E\left(\exp\left[\tau f(\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)))\right]\right)\\
&\quad=\lim_{\epsilon\downarrow 0}\sup_{\mu}\left\{f(\Phi^{\Pi}_{1/\epsilon}(\mu))-\frac{1}{\epsilon}I^{(2)}_{\epsilon}(\mu)\right\}\\
&\quad=\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\mu\colon\ \frac{1}{\epsilon}I^{(2)}_{\epsilon}(\mu)\le K}\left\{f(\Phi^{\Pi}_{1/\epsilon}(\mu))-\frac{1}{\epsilon}I^{(2)}_{\epsilon}(\mu)\right\}\\
&\quad=\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\mu\colon\ \frac{1}{\epsilon}I^{(2)}_{\epsilon}(\mu)\le K}\left\{f(\Psi^{\Pi}_{1/\epsilon}(\mu_1))-\frac{1}{\epsilon}I^{(2)}_{\epsilon}(\mu)\right\}\qquad(2.2.33)\\
&\quad=\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\nu\colon\ \frac{1}{\epsilon}I_{\epsilon}(\nu)\le K}\left\{f(\Psi^{\Pi}_{1/\epsilon}(\nu))-\frac{1}{\epsilon}I_{\epsilon}(\nu)\right\}\\
&\quad=\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\nu\colon\ \frac{1}{\epsilon}I_{\epsilon}(\nu)\le K}\left\{f\left(F^{\Pi}_{d}\left(\frac{d\nu}{dx}\right)\right)-\frac{1}{\epsilon}I_{\epsilon}(\nu)\right\}\\
&\quad=\sup_{\nu}\left\{f\left(F^{\Pi}_{d}\left(\frac{d\nu}{dx}\right)\right)-I(\nu)\right\}\\
&\quad=\sup_{\phi\in H^1(\Lambda_N)\colon\ \|\phi\|_2^2=1}\left\{f(F^{\Pi}_{d}(\phi^2))-\frac{1}{2}\|\nabla\phi\|_2^2\right\}.
\end{aligned}
$$

Here, the first equality uses Proposition 2.2.2, the second equality Proposition 2.2.3, the fourth equality Lemma 2.2.4, the fifth equality (2.2.29), the sixth equality Lemma 2.2.5, while the last equality comes from (2.2.27). The claim in Theorem 2.2.1 follows by applying to (2.2.33) the inverse of Varadhan's lemma due to Bryc [4]. □

It follows from (2.2.3) and Theorem 2.2.1 that

$$\limsup_{\tau\to\infty}\frac{1}{\tau}\log S_t\le -J^{\Pi}_{d,N}(c)\tag{2.2.34}$$

with

$$
\begin{aligned}
J^{\Pi}_{d,N}(c)&=\inf\left\{cb+I^{\Pi}_{d,N}(b)\colon\ b\in(0,\infty)\right\}\\
&=\inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2+cF^{\Pi}_{d}(\phi^2)\colon\ \phi\in H^1(\Lambda_N),\ \|\phi\|_2^2=1\right\}.
\end{aligned}\tag{2.2.35}
$$

This is the same as (1.3.2), but with $\mathbb{R}^d$ replaced by Λ_N. Thus, to complete the proof of the upper bound for Π with finite support it suffices to show that

$$\lim_{N\to\infty}J^{\Pi}_{d,N}(c)=J^{\Pi}_{d}(c).\tag{2.2.36}$$

The latter is a standard exercise, for which the reader is referred to vdBBdH Section 2.6. There a proof was given for $\Pi = \delta_{B_a(0)}$, which easily extends to Π with finite (or countable) support.

2.3. Lower bound in $d \geq 3$

We again assume that Π has the form (2.1.11). Let

$$R = \inf\{M > 0\colon\ A_m \subset B_M \text{ for } m = 1, \dots, n\}. \tag{2.3.1}$$

Fix N. Consider the event $C_{N,R}(\tau)$ that the Brownian motion does not hit $\partial\Lambda_{N-R/\sqrt{T_\tau}}$ until time τ. Then, recalling (2.1.7) and (2.1.8), we have

$$S_t \geq E\left(\exp\left[-c\tau V_\tau^\Pi(\tau)\right] 1_{C_{N,R}(\tau)}\right). \tag{2.3.2}$$

On the event $C_{N,R}(\tau)$, recalling (2.2.2), we have

$$V_\tau^\Pi(\tau) = V_{\tau,N}^\Pi(\tau). \tag{2.3.3}$$

Hence

$$\begin{aligned} \liminf_{\tau\to\infty} \frac{1}{\tau} \log S_t &\geq -\lambda_N + \lim_{\tau\to\infty} \frac{1}{\tau} \log E\left(\exp\left[-c\tau V_{\tau,N}^\Pi(\tau)\right] \mid C_{N,R}(\tau)\right) \\ &= -\lambda_N - J_{d,N,*}^\Pi(c), \end{aligned} \tag{2.3.4}$$

where

$$-\lambda_N = \lim_{\tau\to\infty} \frac{1}{\tau} \log P(C_{N,R}(\tau)) \tag{2.3.5}$$

is minus the principal Dirichlet eigenvalue of $-\Delta/2$ on Λ_N, and $J_{d,N,*}^\Pi(c)$ is given by (2.2.4), except that ϕ has the additional restriction $\mathrm{supp}(\phi) \cap \partial\Lambda_N = \emptyset$. Here, note that the dependence on R drops out with the limit $\tau \to \infty$, because $C_{N,R}$ keeps the Brownian motion a distance $R/\sqrt{T_\tau}$ away from $\partial\Lambda_N$, while $\lim_{\tau\to\infty} T_\tau = \infty$. Let $N \to \infty$ and use that $\lim_{N\to\infty} \lambda_N = 0$, to see that it suffices to show that

$$\lim_{N\to\infty} J_{d,N,*}^\Pi(c) = J_d^\Pi(c). \tag{2.3.6}$$

The latter is again a standard exercise, for which the reader is referred to vdBBdH Section 2.6.

2.4. Continuum limit of Π

In Sections 2.2 and 2.3 we proved Theorem 1.3.1 for Π with finite support. It remains to show that the result can be extended to arbitrary Π subject to the conditions (C1) and (C2) in Section 1.1. For this we need the notion of stochastic ordering by inclusion, namely, Π_1 is stochastically smaller than Π_2, written

$$\Pi^1 \preceq \Pi^2, \tag{2.4.1}$$

if there exists a coupling $\Pi^{1,2}$ of Π^1 and Π^2 such that $\Pi^{1,2}(A_1 \subset A_2) = 1$.

We begin by noting the following continuity property of the variational problem in (1.3.2).

Lemma 2.4.1. *Suppose that* (Π_n) *and* Π *satisfy* (C1) *and* (C2). *If*

$$\begin{array}{l}\text{(i)}\ \ \Pi_n \preceq \Pi \textit{ for all } n,\\ \text{(ii)}\ \int_{\mathcal{Q}} \Pi_n(dA)\,\kappa(A) \to \int_{\mathcal{Q}} \Pi(dA)\,\kappa(A) \textit{ as } n \to \infty,\end{array} \tag{2.4.2}$$

then $J_d^{\Pi_n}(c) \to J_d^{\Pi}(c)$ *as* $n \to \infty$ *for all* $c \in (0,\infty)$.

Proof. Note that

$$\{A_n, A \in \mathcal{Q},\ \ \rho_H(A_n, A) \to 0\} \quad \Longrightarrow \quad \{\kappa(A_n) \to \kappa(A)\}. \tag{2.4.3}$$

Consequently, under condition (C1),

$$(\Pi_n \Rightarrow \Pi) \quad \Longrightarrow \quad (\Theta_n \Rightarrow \Theta) \tag{2.4.4}$$

with $\Theta = \Pi \circ \kappa^{-1}$. Hence it suffices to prove continuity of the variational problem in $\Theta = \Pi \circ \kappa^{-1}$.

Since $A \mapsto \kappa(A)$ is non-decreasing in the partial order induced by inclusion, (2.4.2)(i) implies that $\Theta_n \preceq \Theta$ for all n. It therefore follows from (1.3.4) that $F_d^{\Pi_n} \le F_d^{\Pi}$ for all n. For a reverse estimate, write

$$\begin{aligned} 0 &\le F_d^{\Pi}(\phi^2) - F_d^{\Pi_n}(\phi^2)\\ &= \int_{\mathbb{R}^d} dx \int_0^\infty [\Theta(d\kappa) - \Theta_n(d\kappa)] \left(1 - e^{-\kappa\phi^2(x)}\right)\\ &\le \int_{\mathbb{R}^d} dx\, \phi^2(x) \int_0^\infty [\Theta(d\kappa) - \Theta_n(d\kappa)]\,\kappa\\ &= \int_0^\infty \Theta(d\kappa)\,\kappa - \int_0^\infty \Theta_n(d\kappa)\,\kappa. \end{aligned} \tag{2.4.5}$$

The right-hand side tends to zero as $n \to \infty$ by (2.4.2)(ii), where we recall from Section 1.6 that condition (C2) implies that both integrals are finite. □

We will now prove the extension of Theorem 1.3.1 via a sandwiching argument in combination with a truncation argument. Note that $\Pi^1 \preceq \Pi^2$ implies

$$S_t(\Pi^1) \ge S_t(\Pi^2), \qquad t \ge 0. \tag{2.4.6}$$

Upper bound: Any Π satisfying conditions (C1) and (C2) can be approximated from *below*, as in (2.4.2), by a sequence (Π_n) with *finite* support. By (2.4.6), we have $S_t(\Pi) \le S_t(\Pi_n)$, $t \ge 0$, for each n. Since Theorem 1.3.1 holds for Π_n, we therefore have

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \log S_t(\Pi) \le \limsup_{\tau\to\infty} \frac{1}{\tau} \log S_t(\Pi_n) = -J_d^{\Pi_n}(c). \tag{2.4.7}$$

Lemma 2.4.1 now gives us the desired upper bound.

Lower bound: If Π has unbounded support, then it *cannot* be approximated from *above*, as in (2.4.2), by a sequence (Π_n) with *finite* support. However, we can use the following truncation argument based on (C2). Fix N and R. Let

$$\mathcal{Q}_R = \{A \in \mathcal{Q}\colon\ A \subset B_R\}. \tag{2.4.8}$$

On the event $C_{N,R}(\tau)$ defined in Section 2.3, we have

$$|W^{A/\sqrt{T_\tau}}(\tau)| \leq \begin{cases} |W_N^{A/\sqrt{T_\tau}}(\tau)| + \left|\left(A/\sqrt{T_\tau} + B_N\right) \cap B_N^c\right|, & A \in \mathcal{Q}_R, \\ |B_N| + \left|\left(A/\sqrt{T_\tau} + B_N\right) \cap B_N^c\right|, & A \in \mathcal{Q} \setminus \mathcal{Q}_R. \end{cases} \tag{2.4.9}$$

Therefore, on the event $C_{N,R}(\tau)$, we get the bound

$$\begin{aligned} V_\tau^\Pi(\tau) &= \int_{\mathcal{Q}} \Pi(dA)\, |W^{A/\sqrt{T_\tau}}(\tau)| \\ &\leq V_{\tau,N,R}^\Pi(\tau) + |B_N|\, p_R + \int_{\mathcal{Q}} \Pi(dA)\ \left|\left(A/\sqrt{T_\tau} + B_N\right) \cap B_N^c\right| \end{aligned} \tag{2.4.10}$$

with

$$\begin{aligned} V_{\tau,N,R}^\Pi(\tau) &= \int_{\mathcal{Q}_R} \Pi(dA)\, |W_N^{A/\sqrt{T_\tau}}(\tau)|, \\ p_R &= \int_{\mathcal{Q}\setminus\mathcal{Q}_R} \Pi(dA). \end{aligned} \tag{2.4.11}$$

The third term in the right-hand side of (2.4.10) equals $|B_N|\delta_{N\sqrt{T_\tau}}$ (recall (1.1.6)) and tends to zero as $\tau \to \infty$ for fixed N by (C2). The key observation now is that Π *can* be approximated from *above*, as in (2.4.2), by a sequence (Π_n) with *countable* support such that $\Pi_n 1_{\mathcal{Q}_R}$ has *finite* support for each R. By (2.4.10–2.4.11), we have

$$\begin{aligned} S_t(\Pi) &\geq E\left(\exp\left[-c\tau V_\tau^\Pi(\tau)\right] 1_{C_{N,R}(\tau)}\right) \\ &\geq e^{-c|B_N|[\,p_R + \delta_{N\sqrt{T_\tau}}\,]}\ E\left(\exp\left[-c\tau V_{\tau,N,R}^{\Pi_n 1_{\mathcal{Q}_R}}(\tau)\right] 1_{C_{N,R}(\tau)}\right). \end{aligned} \tag{2.4.12}$$

We can now apply the argument in Section 2.3 to the expectation in the right-hand side of (2.4.12), for fixed N and R, and take the limit $\tau \to \infty$, to obtain

$$\begin{aligned} \liminf_{\tau\to\infty} \frac{1}{\tau} \log S_t(\Pi) &\geq -c|B_N|\, p_R - \lambda_N - J_{d,N,*}^{\Pi_n 1_{\mathcal{Q}_R}}(c) \\ &\geq -c|B_N|\, p_R - \lambda_N - J_{d,N,*}^{\Pi_n}(c). \end{aligned} \tag{2.4.13}$$

Note that $J_{d,N,*}^{\Pi_n}(c)$ depends on Π_n only via $\Theta_n = \Pi_n \circ \kappa^{-1}$. Therefore no harm was done in the last inequality of (2.4.13), which removes the truncation in (2.4.8) on the *finite* torus. We can now let $R \to \infty$ and use that $\lim_{R\to\infty} p_R = 0$, to get

$$\liminf_{\tau\to\infty} \frac{1}{\tau} \log S_t(\Pi) \geq -\lambda_N - J_{d,N,*}^{\Pi_n}(c). \tag{2.4.14}$$

Let $N \to \infty$, use that $\lim_{N\to\infty} \lambda_N = 0$ and recall (2.3.6), to get

$$\liminf_{\tau\to\infty} \frac{1}{\tau} \log S_t(\Pi) \geq -J_d^{\Pi_n}(c). \tag{2.4.15}$$

Finally, use Lemma 2.4.1 to arrive at the desired lower bound.

2.5. Extension to $d = 2$

The extension to $d = 2$ is minor and follows vdBBdH Section 4. The ingredients (2.2.8), (2.2.20) and (2.2.21) need to be properly modified, so as to adapt them to the different scaling (recall (2.1.6)). We refer to Equation (4.8), Lemma 8 and Equation (4.14) in vdBBdH (pp. 385–386). The rest of the argument is the same: the formulas in terms of τ and T_τ are unaltered. We leave the details to the reader.

3. Proof of Proposition 1.4.1 and of Theorems 1.4.2, 1.4.3 and 1.4.4

3.1. Proof of Proposition 1.4.1

Proof. By Lieb and Loss [6], p. 255, for $A \in \mathcal{Q}$ we have

$$\kappa(A) = \frac{1}{2} \inf \left\{ \|\nabla\phi\|_2^2 \colon\ \phi \in D^1(\mathbb{R}^d) \cap C^0(\mathbb{R}^d),\ \phi \geq 1 \text{ on } A \right\}. \tag{3.1.1}$$

For $A \in \mathcal{Q}$, define $\rho_A \colon \mathbb{R}^d \to [0,\infty)$ by $\rho_A(x) = \inf\{|y - x| \colon\ y \in A\}$ and, for $M > 0$, put

$$\phi_{A,M}(x) = \left(1 - \frac{2\rho_A(x)}{M}\right) \vee 0. \tag{3.1.2}$$

Then $\phi_{A,M} \in D^1(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$, $\phi_{A,M} \geq 1$ on A, and

$$|\nabla\phi_{A,M}(x)| = \begin{cases} \frac{2}{M}, & \text{when } 0 < \rho_A(x) < \frac{M}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.3}$$

Hence

$$\begin{aligned} \kappa(A) &\leq \frac{2}{M^2} \left| \left\{ x \in \mathbb{R}^d \colon\ 0 < \rho_A(x) < \tfrac{M}{2} \right\} \right| \\ &\leq \frac{2}{M^2} |A + B_M| \leq \frac{2}{M^2} \left(|B_M| + |(A + B_M) \cap B_M^c| \right). \end{aligned} \tag{3.1.4}$$

By (1.1.6),

$$\langle\kappa\rangle = \int_{\mathcal{Q}} \Pi(dA)\, \kappa(A) \leq \frac{2}{M^2} |B_M| (1 + \delta_M) = 2M^{d-2}(1 + \delta_M), \tag{3.1.5}$$

which is finite for M large enough by (C2). □

3.2. Proof of Theorem 1.4.2(i) and Theorem 1.4.4(i)

According to (1.3.2) and (1.3.6), $c \mapsto J_d^{\Pi}(c)$ and $c \mapsto J_2(c)$ are infima over functions that are linear. Consequently, both are concave, and therefore also continuous except possibly at the boundary point $c = 0$. It is obvious that $J_d^{\Pi}(0) = J_2(0) = 0$. From the general upper bound in Theorems 1.4.3(i) and 1.4.4(iii), it follows that $\lim_{c\downarrow 0} J_d^{\Pi}(c) = \lim_{c\downarrow 0} J_2(c) = 0$. Therefore continuity extends to the boundary. It is further obvious from (1.3.2) and (1.3.6) that $J_d^{\Pi}(c)$ and $J_2(c)$ are non-decreasing in c. By concavity, both are strictly increasing in c unless they are constant from some finite c onwards. But this is ruled out by the asymptotics for $c \to \infty$ in Theorems 1.4.3(i) and 1.4.4(iii).

3.3. Proof of Theorem 1.4.2(ii)

Lemma 3.3.1. *Let $d \geq 3$. Then $J_d^{\Pi}(c) \leq c\langle\kappa\rangle$ for all $c \geq 0$.*

Proof. Since $1 - e^{-x} \leq x$, $x \geq 0$, we have from (1.3.3) that $F_d^{\Pi}(\phi^2) \leq \langle\kappa\rangle\|\phi\|_2^2$. Hence the claim follows from (1.3.2), since $\inf\{\|\nabla\phi\|_2^2\colon\ \|\phi\|_2^2 = 1\} = 0$. □

The critical value c^* is the unique threshold such that $J_d^{\Pi}(c) < c\langle\kappa\rangle$ if and only if $c > c^*$. It follows from Theorem 1.4.3(i) that $c^* < \infty$. In Lemma 3.3.2 below we derive a lower bound on c^* in regime (I). To do so, we first rewrite (1.3.2) as

$$c\langle\kappa\rangle - J_d^{\Pi}(c) = -\inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2 - c\, G_d^{\Pi}(\phi^2)\colon\ \|\phi\|_2^2 = 1,\ \phi \text{ RSNI}\right\}, \quad (3.3.1)$$

where RSNI means radially symmetric and non-increasing (see Lemma 10 in vdBBdH (p. 387)), and

$$G_d^{\Pi}(\phi^2) = \int_{\mathbb{R}^d} dx \int_0^{\infty} \Theta(d\kappa)\ \left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \quad (3.3.2)$$

From (3.3.1) we see that

$$c^* = \inf\left\{\frac{\frac{1}{2}\|\nabla\phi\|_2^2}{G_d^{\Pi}(\phi^2)}\colon\ \|\phi\|_2^2 = 1, \phi \text{ RSNI}\right\}. \quad (3.3.3)$$

Lemma 3.3.2. *Let $d \geq 3$. If $\Theta \in \mathcal{S}_I$, then*

$$c^* \geq S_d\left(4\kappa_0^{d/(d-1)} + \frac{d^2 K}{d-2}\right)^{-1} \quad (3.3.4)$$

with S_d the Sobolev constant in (3.3.15) *below.*

Proof. We estimate the contribution to the double integral in (3.3.2) as follows.

First, let $A < \infty$. The contribution of the rectangle $(0, \kappa_0) \times \{x \in \mathbb{R}^d\colon\ \phi^2(x) < A\}$ is bounded from above by

$$\int_0^{\kappa_0} \Theta(d\kappa) \int_{\{\phi^2 < A\}} dx\, \kappa^2\phi^4(x) \leq \kappa_0^2 A^{2(d-2)/d} \int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x), \quad (3.3.5)$$

where we use that $e^{-x} - 1 + x \leq x^2$, $x \geq 0$. On the other hand, the contribution of the rectangle $(0, \kappa_0) \times \{x \in \mathbb{R}^d \colon \phi^2(x) \geq A\}$ is bounded from above by

$$\begin{aligned}\int_0^{\kappa_0} \Theta(d\kappa) \int_{\{\phi^2 \geq A\}} dx\, \kappa\phi^2(x) &\leq \kappa_0 \int_{\{\phi^2 \geq A\}} dx\, \phi^2(x) \left(\frac{\phi^2(x)}{A}\right)^{2/d} \\ &\leq \kappa_0 A^{-2/d} \int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x).\end{aligned} \tag{3.3.6}$$

We choose $A = \kappa_0^{-d/2(d-1)}$ to get from (3.3.5) and (3.3.6) that the contribution of $(0, \kappa_0)$ is bounded from above by

$$2\kappa_0^{d/(d-1)} \int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x). \tag{3.3.7}$$

Next, the contribution of the rectangle $[\kappa_0, \infty) \times \{x \in \mathbb{R}^d \colon \phi^2(x) < 1/\kappa_0\}$ is bounded from above by

$$\begin{aligned}&\int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{\kappa_0}^{1/\phi^2(x)} \Theta(d\kappa)\, \kappa^2\phi^4(x) \\ &\qquad\qquad + \int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{1/\phi^2(x)}^{\infty} \Theta(d\kappa)\, \kappa\phi^2(x) \\ &\leq K \int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{\kappa_0}^{1/\phi^2(x)} d\kappa\, \kappa^{-2/d}\phi^4(x) \\ &\qquad\qquad + K \int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{1/\phi^2(x)}^{\infty} d\kappa\, \kappa^{-(d+2)/d}\phi^2(x) \\ &\leq K \int_{\{\phi^2 < 1/\kappa_0\}} dx\, \phi^4(x) \int_0^{1/\phi^2(x)} d\kappa\, \kappa^{-2/d} \\ &\qquad\qquad + K \int_{\{\phi^2 < 1/\kappa_0\}} dx\, \phi^2(x) \int_{1/\phi^2(x)}^{\infty} d\kappa\, \kappa^{-(d+2)/d} \\ &= \frac{d^2 K}{2(d-2)} \int_{\{\phi^2 < 1/\kappa_0\}} dx\, \phi^{2(d+2)/d}(x),\end{aligned} \tag{3.3.8}$$

where we use the upper bound on $\Theta(d\kappa)$ that defines $\mathcal{S}_I$. On the other hand, the contribution of the rectangle $[\kappa_0, \infty) \times \{x \in \mathbb{R}^d \colon \phi^2(x) \geq 1/\kappa_0\}$ is bounded from above by

$$\begin{aligned}\int_{\{\phi^2 \geq 1/\kappa_0\}} dx \int_{\kappa_0}^{\infty} \Theta(d\kappa)\, \kappa\phi^2(x) &\leq K \int_{\{\phi^2 \geq 1/\kappa_0\}} dx\, \phi^2(x) \int_{\kappa_0}^{\infty} d\kappa\, \kappa^{-(d+2)/d} \\ &= \frac{dK}{2} \int_{\{\phi^2 \geq 1/\kappa_0\}} dx\, \phi^2(x)\, \kappa_0^{-2/d} \\ &\leq \frac{dK}{2} \int_{\{\phi^2 \geq 1/\kappa_0\}} dx\, \phi^{2(d+2)/d}(x).\end{aligned} \tag{3.3.9}$$

Combining (3.3.7), (3.3.8) and (3.3.9), we arrive at

$$G_d^\Pi(\phi^2) \le \left(2\kappa_0^{d/(d-1)} + \frac{d^2K}{2(d-2)}\right)\int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x). \tag{3.3.10}$$

Next, for any $0 < \alpha < 1$ and conjugate exponents $p, q > 1$, we estimate

$$\int \phi^{2(d+2)/d} \le \left(\int \phi^{[2(d+2)/d]\alpha p}\right)^{1/p} \left(\int \phi^{[2(d+2)/d](1-\alpha)q}\right)^{1/q}. \tag{3.3.11}$$

Choosing α, p, q such that

$$[2(d+2)/d]\alpha p = 2d/(d-2), \quad [2(d+2)/d](1-\alpha)q = 2, \tag{3.3.12}$$

i.e.,

$$p = d/(d-2), \quad q = d/2, \quad \alpha = d/(d+2), \tag{3.3.13}$$

and using that $\|\phi\|_2^2 = 1$, we obtain

$$\int \phi^{2(d+2)/d} \le \left(\int \phi^{2d/(d-2)}\right)^{(d-2)/d} = \|\phi\|^2_{2d/(d-2)}. \tag{3.3.14}$$

Together with the Sobolev inequality (see Lieb and Loss [6], p. 190)

$$\|\nabla\phi\|_2^2 \ge S_d\, \|\phi\|^2_{2d/(d-2)} \tag{3.3.15}$$

this gives

$$\int \phi^{2(d+2)/d} \le \frac{1}{S_d}\|\nabla\phi\|_2^2. \tag{3.3.16}$$

We obtain the claim in (3.3.3) by combining (3.3.10) and (3.3.16). □

Lemma 3.3.3. *Let $d \ge 3$. For $\Theta \in \mathcal{S}_I$, if $\langle\kappa^\eta\rangle < \infty$ for some $\eta > \frac{d+2}{d}$, then* $\lim_{c\downarrow c^*}[J_d^\Pi(c) - J_d^\Pi(c^*)]/(c-c^*) < \langle\kappa\rangle$.

Proof. Let ψ_{c^*} be any minimiser for (1.3.2) at $c = c^*$, the existence of which we prove in Lemma 3.5.2 below under the condition stated. Then

$$J_d^\Pi(c^*) = \tfrac{1}{2}\|\nabla\psi_{c^*}\|_2^2 + c^*\, F_d^\Pi(\psi_{c^*}^2). \tag{3.3.17}$$

But, for any $\delta > 0$, we have

$$J_d^\Pi(c^*+\delta) \le \tfrac{1}{2}\|\nabla\psi_{c^*}\|_2^2 + (c^*+\delta)F_d^\Pi(\psi_{c^*}^2). \tag{3.3.18}$$

Combining this with (3.3.17), we get

$$\frac{1}{\delta}\left[J_d^\Pi(c^*+\delta) - J_d^\Pi(c^*)\right] \le F_d^\Pi(\psi_{c^*}^2) < \frac{1}{c^*}J_d^\Pi(c^*) = \langle\kappa\rangle. \tag{3.3.19}$$

□

3.4. Proof of Theorem 1.4.2(iii)

Lemma 3.4.1. *Let $d \geq 3$. If $\Theta \in \mathcal{S}_{II}$, then $c^* = 0$.*

Proof. By (3.3.2) and the lower bound on $\Theta(d\kappa)$ that defines $\mathcal{S}_{II}$, we have

$$G_d^\Pi(\phi^2) \geq \int_{\mathbb{R}^d} dx \int_{\kappa_1}^\infty d\kappa\, L(\kappa)\kappa^{-1-(d+2)/d} \left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \quad (3.4.1)$$

Hence we get, for all $\phi \in H^1(\mathbb{R}^d)$ that are RSNI with $\|\phi\|_2^2 = 1$ and $\kappa_1\phi^2(0) \leq 1$,

$$\begin{aligned} G_d^\Pi(\phi^2) &\geq \int_{\mathbb{R}^d} dx \int_{1/\phi^2(x)}^\infty d\kappa\, L(\kappa)\kappa^{-1-(d+2)/d} \left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right) \\ &\geq L\left(\frac{1}{\phi^2(0)}\right) \int_{\mathbb{R}^d} dx \int_{1/\phi^2(x)}^\infty d\kappa\, \kappa^{-1-(d+2)/d} \left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right) \\ &\geq L\left(\frac{1}{\phi^2(0)}\right) \int_{\mathbb{R}^d} dx \int_{1/\phi^2(x)}^\infty d\kappa\, \kappa^{-1-(d+2)/d} \left(\frac{\kappa\phi^2(x)}{e}\right) \qquad (3.4.2) \\ &= \frac{d}{2e} L\left(\frac{1}{\phi^2(0)}\right) \int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x), \end{aligned}$$

where we have used that $e^{-x} - 1 + x \geq x/e$, $x \geq 1$. Inserting (3.4.2) into (3.3.3), we find

$$c^* \leq \frac{e}{d} \inf \left\{ \frac{1}{L(1/\phi^2(0))} \frac{\|\nabla\phi\|_2^2}{\int \phi^{2(d+2)/d}} \colon \|\phi\|_2^2 = 1,\ \phi \text{ RSNI},\ \kappa_1\phi^2(0) \leq 1 \right\}. \quad (3.4.3)$$

The choice

$$\phi(x) = \epsilon^{d/2} e^{-\pi\epsilon^2|x|^2/2}, \qquad \epsilon > 0, \quad (3.4.4)$$

yields that, for all $0 < \epsilon \leq \kappa_1^{-1/d}$,

$$c^* \leq \frac{e\pi}{2}[(d+2)/d]^{d/2} \frac{1}{L(\epsilon^{-d})}. \quad (3.4.5)$$

We obtain the claim by letting $\epsilon \downarrow 0$ and using that $\lim_{\kappa\to\infty} L(\kappa) = \infty$. □

Lemma 3.4.2. *Let $d \geq 3$. If $\Theta \in \mathcal{S}_{II}$, then* $\lim_{c\downarrow 0} \frac{1}{c} J_d^\Pi(c) = \langle\kappa\rangle$.

Proof. As shown in Lemma 3.5.1 below, for all $c > 0$ we have the existence of a minimiser for (1.3.2), say ψ_c. Hence

$$\frac{1}{c} J_d^\Pi(c) = \frac{1}{c}\|\nabla\psi_c\|_2^2 + \int_{\mathbb{R}^d} dx \int_0^\infty \Theta(d\kappa) \left(1 - e^{-\kappa\psi_c^2(x)}\right). \quad (3.4.6)$$

Let $\epsilon > 0$ and $R < \infty$. Then, since $e^{-x} - 1 + x \le \frac{1}{2}x^2$, $x \ge 0$, we have

$$\begin{aligned}\frac{1}{c} J_d^{\Pi}(c) &\ge \int_{\{\psi_c^2 \le \epsilon\}} dx \int_{\{\kappa \le R\}} \Theta(d\kappa) \left(1 - e^{-\kappa \psi_c^2(x)}\right) \\ &\ge \int_{\{\psi_c^2 \le \epsilon\}} dx \int_{\{\kappa \le R\}} \Theta(d\kappa) \left(\kappa \psi_c^2(x) - \frac{1}{2}\kappa^2 \psi_c^4(x)\right) \\ &\ge \int_{\{\psi_c^2 \le \epsilon\}} dx\, \psi_c^2(x) \int_{\{\kappa \le R\}} \Theta(d\kappa)\, \kappa - \frac{1}{2} R^2 \epsilon,\end{aligned} \tag{3.4.7}$$

where we use that $\|\psi_c\|_2^2 = 1$. We will show that, for any $\epsilon > 0$,

$$\lim_{c \downarrow 0} \int_{\{\psi_c^2 > \epsilon\}} dx\, \psi_c^2(x) = 0. \tag{3.4.8}$$

Combining this with (3.4.7) and again using that $\|\psi_c\|_2^2 = 1$, we obtain

$$\liminf_{c \downarrow 0} \frac{1}{c} J_d^{\Pi}(c) \ge \int_{\{\kappa \le R\}} \Theta(d\kappa)\, \kappa - \frac{1}{2} R^2 \epsilon. \tag{3.4.9}$$

By letting $\epsilon \downarrow 0$ and then letting $R \to \infty$, we arrive at

$$\liminf_{c \downarrow 0} \frac{1}{c} J_d^{\Pi}(c) \ge \langle \kappa \rangle. \tag{3.4.10}$$

This proves the claim, since we already know from Lemma 3.3.1 that $\frac{1}{c} J_d^{\Pi}(c) \le \langle \kappa \rangle$.

It remains to prove (3.4.8). We have

$$\begin{aligned}\int_{\{\psi_c^2 > \epsilon\}} dx\, \psi_c^2(x) &\le \int_{\{\psi_c^2 > \epsilon\}} dx\, \psi_c^2(x) \left(\frac{\psi_c^2(x)}{\epsilon}\right)^{2/(d-2)} \\ &\le \epsilon^{-2/(d-2)} \int_{\mathbb{R}^d} dx\, \psi_c^{2d/(d-2)}(x) \\ &\le \epsilon^{-2/(d-2)} S_d^{-d/(d-2)} \|\nabla \psi_c\|_2^{2d/(d-2)},\end{aligned} \tag{3.4.11}$$

where we use the Sobolev inequality (3.3.15). But $\lim_{c \downarrow 0} J_d^{\Pi}(c) = 0$ by Lemma 3.3.1, and therefore $\lim_{c \downarrow 0} \|\nabla \psi_c\|_2^2 = 0$. Consequently, (3.4.11) implies (3.4.8). □

3.5. Proof of Theorem 1.4.2(iv)

Lemma 3.5.1. *Let* $d \ge 3$. *In regimes* (I) *and* (II), (1.3.2) *has a minimiser with full support for all* $c > c^*$ *(with* $c^* = 0$ *in regime* (II)*).*

Proof. By the definition of c^*, we have $J_d^\Pi(c) < c\langle\kappa\rangle$ for $c > c^*$. For $c > 0$, define

$$\begin{aligned} K_d^\Pi(c) = \inf \Bigg\{ & \frac{1}{2}\|\nabla\psi\|_2^2 - c\int_0^\infty \Theta(d\kappa) \\ & \times \int_{\mathbb{R}^d} dx \, \left(e^{-\kappa\psi^2(x)} - 1 + \kappa\psi^2(x)\right) : \ \|\psi\|_2^2 = 1, \ \psi \text{ RSNI} \Bigg\}, \\ \widehat{K}_d^\Pi(c) = \inf \Bigg\{ & \frac{1}{2}\|\nabla\psi\|_2^2 - c\int_0^\infty \Theta(d\kappa) \\ & \times \int_{\mathbb{R}^d} dx \, \left(e^{-\kappa\psi^2(x)} - 1 + \kappa\psi^2(x)\right) : \ \|\psi\|_2^2 \le 1, \ \psi \text{ RSNI} \Bigg\}. \end{aligned} \tag{3.5.1}$$

Then, for $c > c^*$,

$$\widehat{K}_d^\Pi(c) \le K_d^\Pi(c) < 0. \tag{3.5.2}$$

Let (ψ_j) be a minimising sequence of the variational problem for $\widehat{K}_d^\Pi(c)$. Then, by a compactness argument along the lines of the proof of Lemma 13 in vdBBdH (pp. 390–391), we may extract a subsequence, also denoted by (ψ_j), such that $\psi_j \to \psi_c$ as $j \to \infty$ for some ψ_c almost everywhere and in $D^1(\mathbb{R}^d)$. It follows that ψ_c is RSNI and a minimiser for $\widehat{K}_d^\Pi(c)$. Moreover, $\|\psi_c\|_2^2 > 0$ (because $\|\psi_c\|_2^2 = 0$ would imply $\psi_c = 0$ almost everywhere, which in turn would imply $\widehat{K}_d^\Pi(c) = 0$, in contradiction with (3.5.2)). Suppose that $\|\psi_c\|_2^2 = 1-\rho$ with $0 \le \rho < 1$. Define

$$\phi(x) = \frac{1}{q}\psi_c(qx), \tag{3.5.3}$$

where we choose $q > 0$ such that $\|\phi\|_2^2 = 1$, i.e.,

$$q = (1-\rho)^{1/(d+2)}. \tag{3.5.4}$$

Then

$$\|\nabla\phi\|_2^2 = (1-\rho)^{-d/(d+2)}\|\nabla\psi_c\|_2^2 \tag{3.5.5}$$

and

$$\begin{aligned} & c\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right) \\ & \quad = c(1-\rho)^{-d/(d+2)}\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d} dx \\ & \qquad \times \left(e^{-\kappa(1-\rho)^{-2/(d+2)}\psi_c^2(x)} - 1 + \kappa(1-\rho)^{-2/(d+2)}\psi_c^2(x)\right) \\ & \quad \ge c(1-\rho)^{-d/(d+2)}\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d} dx\left(e^{-\kappa\psi_c^2(x)} - 1 + \kappa\psi_c^2(x)\right). \end{aligned} \tag{3.5.6}$$

Inserting (3.5.5) and (3.5.6) into the definition of $K_d^{\Pi}(c)$, and using the definition of $\widehat{K}_d^{\Pi}(c)$, we get

$$K_d^{\Pi}(c) \le (1-\rho)^{-d/(d+2)} \widehat{K}_d^{\Pi}(c). \tag{3.5.7}$$

By (3.5.2) and (3.5.7), we conclude that $\rho = 0$. Hence $\|\psi_c\|_2^2 = 1$, and ψ_c is a minimiser for $K_d^{\Pi}(c) = J_d^{\Pi}(c) - c\langle\kappa\rangle$. The fact that ψ_c has full support follows from the analogue of Lemma 11 in vdBBdH (p. 388): ψ_c satisfies the Euler-Lagrange equation associated with the variational problem, from which it follows that ψ_c is smooth and strictly decreasing in the radial component. □

Lemma 3.5.2. *Let $d \ge 3$. For $\Theta \in \mathcal{S}_I$, if $\langle\kappa^\eta\rangle < \infty$ for some $\eta > \frac{d+2}{d}$, then* (1.3.2) *has a minimiser for $c = c^*$.*

Proof. Define

$$\widehat{c} = \inf\left\{\frac{\frac{1}{2}\|\nabla\phi\|_2^2}{G_d^{\Pi}(\phi^2)} \colon\ 0 < \|\phi\|_2^2 \le 1,\ \phi \text{ RSNI}\right\}. \tag{3.5.8}$$

We begin by showing that $\widehat{c} = c^*$. Trivially, by comparing (3.3.3) and (3.5.8), we get $\widehat{c} \le c^*$. To prove the converse, let $(\widehat{\phi}_j)$ be a minimising sequence for (3.5.8). Put $0 < a_j = \|\widehat{\phi}_j\|_2^2 \le 1$, and

$$\phi_j(x) = a_j^{-1/(d+2)} \widehat{\phi}_j\left(a_j^{1/(d+2)} x\right). \tag{3.5.9}$$

Then $\|\phi_j\|_2^2 = 1$, and

$$c^* \le \frac{\frac{1}{2}\|\nabla\phi_j\|_2^2}{G_d^{\Pi}(\phi_j^2)} = \frac{\frac{1}{2}\|\nabla\widehat{\phi}_j\|_2^2}{G_d^{\Pi}(a_j^{-2/(d+2)}\widehat{\phi}_j^2)} \le \frac{\frac{1}{2}\|\nabla\widehat{\phi}_j\|_2^2}{G_d^{\Pi}(\widehat{\phi}_j^2)}. \tag{3.5.10}$$

But the right-hand side of (3.5.10) converges to $\widehat{c}$ as $j \to \infty$. Hence, $c^* \le \widehat{c}$.

By extracting a subsequence, also denoted by $(\widehat{\phi}_j)$, we may assume that $\widehat{\phi}_j \to \widehat{\phi}$ as $j \to \infty$ for some $\widehat{\phi}$ almost everywhere and weakly in $D^1(\mathbb{R}^d)$. It follows that $\widehat{\phi}$ is RSNI. Below we will show that $\|\widehat{\phi}\|_2^2 > 0$. If $\|\widehat{\phi}\|_2^2 = 1$, then $\widehat{\phi}$ is a minimiser of (3.3.3). If, on the other hand, $0 < \|\widehat{\phi}\|_2^2 = 1 - \rho < 1$, then define, as in (3.5.3),

$$\phi^*(x) = \frac{1}{q}\widehat{\phi}(qx), \tag{3.5.11}$$

where q is given by (3.5.4). Then $\|\phi^*\|_2^2 = 1$ and, as in (3.5.10),

$$c^* \le \frac{\frac{1}{2}\|\nabla\phi^*\|_2^2}{G_d^{\Pi}(\phi^{*2})} \le \frac{\frac{1}{2}\|\nabla\widehat{\phi}\|_2^2}{G_d^{\Pi}(\widehat{\phi}^2)} = \widehat{c} = c^*. \tag{3.5.12}$$

It follows that ϕ^* is a minimiser of (3.3.3). It then obviously also is a minimiser of (1.3.2) for $c = c^*$ (recall (3.3.1), (3.3.2) and (3.3.3)). The fact that ψ^* has full support again follows from the analogue of Lemma 11 in vdBBdH (p. 388).

It remains to prove that $\|\widehat{\phi}\|_2^2 > 0$. For this it suffices to show that there exist $\delta, \epsilon > 0$ such that, for any minimising sequence $(\widehat{\phi}_j)$ of (3.5.8),

$$|\{x \in \mathbb{R}^d\colon\ \widehat{\phi}_j^2(x) \geq \epsilon\}| \geq \delta \qquad \text{for all } j. \tag{3.5.13}$$

Indeed, (3.5.13) implies that $\|\widehat{\phi}_j\|_2^2 \geq \epsilon\delta$ for all j, and hence that

$$\|\widehat{\phi}\|_2^2 \geq \epsilon\delta. \tag{3.5.14}$$

To prove (3.5.13), we argue by contradiction. Suppose that there exists a minimising sequence $(\widehat{\phi}_j)$ of (3.5.8) with the property that, for all $\epsilon > 0$,

$$\lim_{j\to\infty} |\{x \in \mathbb{R}^d\colon\ \widehat{\phi}_j^2(x) \geq \epsilon\}| = 0. \tag{3.5.15}$$

Then, for all $\epsilon > 0$, there exists an $L_1(\epsilon) \in \mathbb{N}$ such that, for all $j \geq L_1(\epsilon)$,

$$|\{x \in \mathbb{R}^d\colon\ \widehat{\phi}_j^2(x) \geq \epsilon\}| < \epsilon^{d[\eta-(d+2)/d]/2}. \tag{3.5.16}$$

We already know that there exists an $L_2 \in \mathbb{N}$ such that, for all $j \geq L_2$,

$$\frac{\frac{1}{2}\|\nabla\widehat{\phi}_j\|_2^2}{G_d^\Pi(\widehat{\phi}_j^2)} \leq 2\widehat{c}. \tag{3.5.17}$$

To arrive at a contradiction, we will show that the left-hand side of (3.5.17) is at least $5\widehat{c}/2$ for $j \geq L_1(\epsilon_0) \vee L_2$ for some $\epsilon_0 > 0$.

By the Sobolev inequality (3.3.15), we have

$$\|\nabla\widehat{\phi}_j\|_2^2 \geq S_d\|\widehat{\phi}_j\|_{2d/(d-2)}^2. \tag{3.5.18}$$

Since $\langle\kappa^{\eta'}\rangle < \infty$ implies that $\langle\kappa^\eta\rangle < \infty$ for $\eta \leq \eta'$, we may assume that $\frac{d+2}{d} < \eta \leq 2$. To estimate the contribution of the strip $\{\widehat{\phi}_j^2 < \epsilon\}$ to the integral in $G_d^\Pi(\widehat{\phi}_j^2)$, we use that $e^{-x} + 1 - x \leq x^\eta$, $x \geq 0$, to obtain, via (3.3.14),

$$\begin{aligned}\int_0^\infty \Theta(d\kappa)\int_{\{\widehat{\phi}_j^2<\epsilon\}} dx\,(\kappa\widehat{\phi}_j^2(x))^\eta &= \langle\kappa^\eta\rangle\int_{\{\widehat{\phi}_j^2<\epsilon\}} dx\,\widehat{\phi}_j^{2\eta}(x)\\ &\leq \langle\kappa^\eta\rangle\epsilon^{\eta-(d+2)/d}\int_{\{\widehat{\phi}_j^2<\epsilon\}} dx\,\widehat{\phi}_j^{2(d+2)/d}(x)\\ &\leq \langle\kappa^\eta\rangle\epsilon^{\eta-(d+2)/d}\|\widehat{\phi}_j\|_{2d/(d-2)}^2.\end{aligned} \tag{3.5.19}$$

Furthermore, by Hölder's inequality and (3.5.16) we have, for $j \geq L_1(\epsilon)$,

$$\begin{aligned}\int_0^\infty \Theta(d\kappa)\int_{\{\widehat{\phi}_j^2\geq\epsilon\}} dx\,\kappa\widehat{\phi}_j^2(x) &= \langle\kappa\rangle\int_{\mathbb{R}^d} dx\,\widehat{\phi}_j^2(x)1_{\{\widehat{\phi}_j^2(x)\geq\epsilon\}}\\ &\leq \langle\kappa\rangle\left(\int_{\mathbb{R}^d} dx\,\widehat{\phi}_j^{2d/(d-2)}(x)\right)^{(d-2)/d}\\ &\qquad\times\left(\int_{\mathbb{R}^d} dx\,1_{\{\widehat{\phi}_j^2(x)\geq\epsilon\}}\right)^{2/d}\\ &\leq \langle\kappa\rangle\|\widehat{\phi}_j\|_{2d/(d-2)}^2\epsilon^{\eta-(d+2)/d}.\end{aligned} \tag{3.5.20}$$

Combining (3.5.18), (3.5.19) and (3.5.20) we have, for $j \geq L_1(\epsilon)$,

$$G_d^{\Pi}(\widehat{\phi}_j^2) \leq \big(\langle\kappa\rangle + \langle\kappa^\eta\rangle\big)\,\frac{1}{S_d}\epsilon^{\eta-(d+2)/d}\|\nabla\widehat{\phi}_j\|_2^2, \tag{3.5.21}$$

or

$$\frac{\frac{1}{2}\|\nabla\widehat{\phi}_j\|_2^2}{G_d^{\Pi}(\widehat{\phi}_j^2)} \geq \frac{1}{2}S_d\epsilon^{-[\eta-(d+2)/d]}\big(\langle\kappa\rangle + \langle\kappa^\eta\rangle\big)^{-1}. \tag{3.5.22}$$

Now choose $\epsilon = \epsilon_0$ with ϵ_0 the root of

$$\frac{1}{2}S_d\epsilon_0^{-[\eta-(d+2)/d]}\big(\langle\kappa\rangle + \langle\kappa^\eta\rangle\big)^{-1} = \frac{5}{2}\widehat{c}, \tag{3.5.23}$$

to get that (3.5.22) contradicts (3.5.17) for all $j \geq L_1(\epsilon_0) \vee L_2$. □

3.6. Proof of Theorem 1.4.3(i) and Theorem 1.4.4(iii)

We give the proof for $d \geq 3$. The proof for $d = 2$ is the same but uses (1.3.7) instead of (1.3.4).

From (1.3.4) we have

$$F_d^{\Pi}(\phi^2) \leq |\mathrm{supp}(\phi)|, \tag{3.6.1}$$

and so (1.3.2) gives

$$J_d^{\Pi}(c) \leq \inf\left\{\tfrac{1}{2}\,\|\nabla\phi\|_2^2 + c\,|\mathrm{supp}(\phi)|\colon\ \|\phi\|_2^2 = 1\right\}. \tag{3.6.2}$$

We get an upper bound on the infimum by restricting $\mathrm{supp}(\phi)$ to a ball B with volume $|B|$. Therefore

$$J_d^{\Pi}(c) \leq \inf\left\{\frac{1}{2}\,\frac{\|\nabla\phi\|_2^2}{\|\phi\|_2^2} + c\,|B|\colon\ \mathrm{supp}(\phi) \subset B\right\} = \tfrac{1}{2}\lambda_d(B) + c\,|B|, \tag{3.6.3}$$

with $\lambda_d(B)$ the principal Dirichlet eigenvalue of $-\Delta$ on B. By scaling B, we have

$$\lambda_d(B) = |B|^{-2/d}\lambda_d, \tag{3.6.4}$$

Substituting this into (3.6.3) and taking the infimum over $|B|$, we arrive at

$$J_d^{\Pi}(c) \leq \inf_{|B|}\left\{\frac{1}{2}\lambda_d|B|^{-2/d} + c|B|\right\} = \frac{d+2}{2}\left(\frac{\lambda_d}{d}\right)^{d/(d+2)} c^{2/(d+2)}. \tag{3.6.5}$$

This proves the upper bound in (1.4.8).

To prove the lower bound, we first scale ϕ to obtain

$$c^{-2/(d+2)}J_d^{\Pi}(c) = \inf\left\{\frac{1}{2}\,\|\nabla\phi\|_2^2 + \int_{\mathbb{R}^d}dx \right.$$
$$\left. \times \int_0^\infty \Theta(d\kappa)\left(1 - e^{-\kappa c^{d/(d+2)}\phi^2(x)}\right)\colon\ \|\phi\|_2^2 = 1,\ \phi\ \mathrm{RSNI}\right\}. \tag{3.6.6}$$

We know that this variational problem has a minimiser when $c > c^*$. Call this minimiser ψ. Pick $0 < \delta < 1/(2 \vee \lambda_d)$, and let

$$B_\delta = \left\{ x \in \mathbb{R}^d : \ \psi(x) \geq \delta \right\}. \tag{3.6.7}$$

Restricting the x-integration to B_δ, we get

$$\text{rhs (3.6.6)} \geq \frac{1}{2} \int_{B_\delta} dx \, |\nabla \psi(x)|^2 + |B_\delta| - \int_{B_\delta} dx \int_0^\infty \Theta(d\kappa) \, e^{-\kappa c^{d/(d+2)} \delta^2}. \tag{3.6.8}$$

By Lebesgue's dominated convergence theorem, for every $\epsilon > 0$ there exists a $C = C(\delta, \epsilon, \Theta)$ such that

$$\int_{B_\delta} dx \int_0^\infty \Theta(d\kappa) \, e^{-\kappa c^{d/(d+2)} \delta^2} \leq \epsilon \qquad \forall \, c \geq C. \tag{3.6.9}$$

Hence

$$\text{rhs (3.6.8)} \geq \tfrac{1}{2} \textstyle\int_{B_\delta} dx \, |\nabla \psi(x)|^2 + |B_\delta| - \epsilon. \tag{3.6.10}$$

Next, define ϕ by

$$\phi(x) = \begin{cases} \psi(x) - \delta, & x \in B_\delta, \\ 0, & x \in \mathbb{R}^d \setminus B_\delta. \end{cases} \tag{3.6.11}$$

Then ϕ is RSNI and satisfies the Dirichlet boundary condition on ∂B_δ. Since $\|\psi\|_2^2 = 1$, we have

$$\int_{B_\delta} \phi = \int_{B_\delta} \psi - \delta |B_\delta| \leq |B_\delta|^{1/2} - \delta |B_\delta|. \tag{3.6.12}$$

Hence

$$\begin{aligned} 1 = \int_{\mathbb{R}^d} \psi^2 &= \int_{B_\delta} (\phi + \delta)^2 = \delta^2 |B_\delta| + 2\delta \int_{B_\delta} \phi + \int_{B_\delta} \phi^2 \\ &\leq -\delta^2 |B_\delta| + 2\delta |B_\delta|^{1/2} + \int_{B_\delta} \phi^2 \leq 2\delta |B_\delta|^{1/2} + \|\phi\|_2^2. \end{aligned} \tag{3.6.13}$$

By (3.6.11) and the Rayleigh-Ritz variational characterisation of $\lambda_d(B_\delta)$, we have

$$\int_{B_\delta} |\nabla \psi|^2 = \int_{B_\delta} |\nabla \phi|^2 \geq \lambda_d(B_\delta) \, \|\phi\|_2^2. \tag{3.6.14}$$

Combining (3.6.6), (3.6.8), (3.6.10), (3.6.13) and (3.6.14), we obtain for $c \geq C$,

$$\begin{aligned} c^{-2/(d+2)} J_d^{\Pi}(c) &\geq \frac{1}{2}\lambda_d(B_\delta)\left(1 - 2\delta|B_\delta|^{1/2}\right) + |B_\delta| - \epsilon \\ &= \frac{1}{2}\lambda_d|B_\delta|^{-2/d}\left(1 - 2\delta|B_\delta|^{1/2}\right) + |B_\delta| - \epsilon \\ &\geq \frac{1}{2}\lambda_d|B_\delta|^{-2/d}\,(1-2\delta) + |B_\delta|\,(1-\delta\lambda_d) - \epsilon \\ &\geq \left(\frac{1}{2}\lambda_d|B_\delta|^{-2/d} + |B_\delta|\right)[1-\delta(2\vee\lambda_d)] - \epsilon \\ &\geq \frac{d+2}{2}\left(\frac{\lambda_d}{d}\right)^{d/(d+2)}[1-\delta(2\vee\lambda_d)] - \epsilon, \end{aligned} \tag{3.6.15}$$

where the second line uses (3.6.4) and the fifth line uses (3.6.5). Now let $c \to \infty$, and subsequently let $\delta, \epsilon \downarrow 0$, to get the lower bound in (1.4.8).

3.7. Proof of Theorems 1.4.3(ii)

Fix $\epsilon \in (0, K/2)$. Then there exists an $R_\epsilon \in (0,\infty)$ such that

$$(K-\epsilon)\kappa^{-1-\gamma} \leq \theta(\kappa) \leq (K+\epsilon)\kappa^{-1-\gamma}, \qquad \kappa \geq R_\epsilon. \tag{3.7.1}$$

By (3.3.1) and (3.3.2),

$$\begin{aligned} &c\langle\kappa\rangle - J_d^{\Pi}(c) \\ &\quad = -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\,G_d^{\Pi}(\phi^2)\colon\ \|\phi\|_2^2 = 1\right\} \\ &\quad \geq -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\int_{\mathbb{R}^d} dx \int_{R_\epsilon}^{\infty} d\kappa\ \theta(\kappa)\right. \\ &\qquad \left.\times\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right)\colon\ \|\phi\|_2^2 = 1\right\} \\ &\quad \geq -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\int_{\mathbb{R}^d} dx \int_{R_\epsilon}^{\infty} d\kappa\ (K-\epsilon)\kappa^{-1-\gamma}\right. \\ &\qquad \left.\times\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right)\colon\ \|\phi\|_2^2 = 1\right\} \\ &\quad = -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c(K-\epsilon)\right. \\ &\qquad \left.\times\int_{\mathbb{R}^d} dx\,|\phi(x)|^{2\gamma}\int_{R_\epsilon\phi^2(x)}^{\infty} d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa} - 1 + \kappa\right)\colon\ \|\phi\|_2^2 = 1\right\}, \end{aligned} \tag{3.7.2}$$

where the second inequality uses the lower bound in (3.7.1). Inserting the scaling $\phi(x) = \delta^{d/2}\psi(\delta x)$, $\delta > 0$, we obtain

$$c\langle\kappa\rangle - J_d^{\Pi}(c) \geq -\inf\Big\{\frac{1}{2}\delta^2\|\nabla\psi\|_2^2 - c(K-\epsilon)\delta^{d(\gamma-1)}\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma} \times \int_{\delta^d R_\epsilon\psi^2(x)}^{\infty} d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right):\ \|\psi\|_2^2=1\Big\}. \tag{3.7.3}$$

We choose δ to be the root of $\frac{1}{2}\delta^2 = c(K-\epsilon)\delta^{d(\gamma-1)}$. Since this root is at least $(cK)^{1/(2-d(\gamma-1))}$, we obtain

$$(2cK)^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^{\Pi}(c)\right] \geq -\frac{1}{2}\left(\frac{K}{K-\epsilon}\right)^{-2/(2-d(\gamma-1))} \times \inf\Big\{\|\nabla\psi\|_2^2 - \int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma}\int_{(cK)^{d/(2-d(\gamma-1))}R_\epsilon\psi^2(x)}^{\infty} d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right):\ \|\psi\|_2^2=1\Big\}. \tag{3.7.4}$$

Next, we note that

$$\int_0^\infty d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right) = \Gamma(-\gamma) \in (0,\infty). \tag{3.7.5}$$

Let $\beta \in (\gamma, 2]$. Since $e^{-\kappa}-1+\kappa \leq \kappa^\beta$, $\kappa \geq 0$, we may estimate

$$\int_0^{(cK)^{d/(2-d(\gamma-1))}R_\epsilon\psi^2(x)} d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right) \leq \frac{1}{\beta-\gamma}\left((cK)^{d/(2-d(\gamma-1))}R_\epsilon\psi^2(x)\right)^{\beta-\gamma}. \tag{3.7.6}$$

By (3.7.4), (3.7.5) and (3.7.6), we obtain that

$$(2cK)^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^{\Pi}(c)\right] \geq -\frac{1}{2}\left(\frac{K}{K-\epsilon}\right)^{-2/(2-d(\gamma-1))} \times \inf\Big\{\|\nabla\psi\|_2^2 - \Gamma(-\gamma)\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma} + E_{\beta,\gamma}(\epsilon,c;\psi^2):\ \|\psi\|_2^2=1\Big\} \tag{3.7.7}$$

with an error term

$$E_{\beta,\gamma}(\epsilon,c;\psi^2) = \frac{1}{\beta-\gamma}R_\epsilon^{\beta-\gamma}(cK)^{(\beta-\gamma)/(2-d(\gamma-1))}\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\beta}. \tag{3.7.8}$$

Furthermore, for $0<\alpha<1$ and conjugate exponents $p, q > 1$, we estimate

$$\int_{\mathbb{R}^d}|\psi|^{2\beta} \leq \left(\int_{\mathbb{R}^d}|\psi|^{2\alpha\beta p}\right)^{1/p}\left(\int_{\mathbb{R}^d}|\psi|^{2(1-\alpha)\beta q}\right)^{1/q}. \tag{3.7.9}$$

Choosing α, β, p, q such that

$$2\alpha\beta p = 2d/(d-2), \qquad p = d/(d-2), \qquad 2(1-\alpha)\beta q = 2, \tag{3.7.10}$$

i.e.,

$$\alpha = d/(d+2), \qquad \beta = (d+2)/d, \qquad p = d/(d-2), \qquad q = d/2, \tag{3.7.11}$$

we obtain from (3.7.9), using $\|\psi\|_2^2 = 1$ and the Sobolev inequality (3.3.15), that

$$(2cK)^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^{\Pi}(c)\right] \geq -\frac{1}{2}\left(\frac{K}{K-\epsilon}\right)^{-2/(2-d(\gamma-1))}$$
$$\times \inf\left\{\big(1+E_\gamma(\epsilon,c)\big)\,\|\nabla\psi\|_2^2 - \Gamma(-\gamma)\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma}\colon\ \|\psi\|_2^2 = 1\right\} \tag{3.7.12}$$

with an error term

$$E_\gamma(\epsilon,c) = R_\epsilon^{(2-d(\gamma-1))/d}\,\frac{d}{S_d(2-d(\gamma-1))}\,cK. \tag{3.7.13}$$

Finally, we insert the scaling $\phi(x) = \eta^{d/2}\psi(\eta x)$, $\eta > 0$, and choose η to be the root of $\eta^2(1+ E_\gamma(\epsilon,c)) = \Gamma(-\gamma)\eta^{d(\gamma-1)}$, to arrive at

$$\{2cK\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^{\Pi}(c)\right] \geq \frac{1}{2}$$
$$\times\left(\frac{K}{K-\epsilon}\,\frac{1}{1+E_\gamma(\epsilon,c)}\right)^{-2/(2-d(\gamma-1))} M_d(\gamma), \tag{3.7.14}$$

where we have used the definition of $M_d(\gamma)$ in (1.4.10). Now let $c \downarrow 0$ and use that $\lim_{c\downarrow 0} E_\gamma(\epsilon,c) = 0$ for all $\epsilon > 0$. Then let $\epsilon \downarrow 0$, to get

$$\liminf_{c\downarrow 0}\{2cK\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^{\Pi}(c)\right] \geq \frac{1}{2}M_d(\gamma), \tag{3.7.15}$$

which is the desired lower bound.

The proof of the upper bound runs as follows. Let ϵ and R_ϵ be as before. We estimate, similarly as in (3.7.2),

$$c\langle\kappa\rangle - J_d^{\Pi}(c)$$
$$\leq -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\int_{\mathbb{R}^d} dx\int_{R_\epsilon}^{\infty} d\kappa\ \theta(\kappa)\right.$$
$$\left.\times\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right) - E_\theta(\epsilon,c;\phi^2)\colon\ \|\phi\|_2^2 = 1\right\} \tag{3.7.16}$$

with an error term

$$E_\theta(\epsilon,c;\phi^2) = c\int_{\mathbb{R}^d} dx\int_0^{R_\epsilon} d\kappa\ \theta(\kappa)\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \tag{3.7.17}$$

Since $e^{-x} - 1 + x \le x^{(d+2)/d}$, $x \ge 0$, we may use the Sobolev inequality (3.3.15) to estimate

$$E_\theta(\epsilon, c; \phi^2) \le c \int_{\mathbb{R}^d} dx \int_0^{R_\epsilon} d\kappa\ \theta(\kappa)\, (\kappa\phi^2(x))^{(d+2)/d} \le c m_\theta(\epsilon)\, S_d^{-1} \|\nabla\phi\|_2^2, \tag{3.7.18}$$

where we abbreviate $m_\theta(\epsilon) = \int_0^{R_\epsilon} d\kappa\, \theta(\kappa)\, \kappa^{(d+2)/d}$. Combining (3.7.16) and (3.7.18), we obtain for c small enough,

$$\begin{aligned} &c\langle\kappa\rangle - J_d^\Pi(c) \\ &\le -\inf\Bigg\{\left(\frac{1}{2} - c m_\theta(\epsilon) S_d^{-1}\right)\|\nabla\phi\|_2^2 \\ &\qquad - c(K+\epsilon)\int_{\mathbb{R}^d} dx \int_{R_\epsilon}^\infty d\kappa\, \kappa^{-1-\gamma}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right) \colon\ \|\phi\|_2^2 = 1\Bigg\} \end{aligned} \tag{3.7.19}$$

$$\begin{aligned} &\le -\inf\Bigg\{\left(\frac{1}{2} - c m_\theta(\epsilon) S_d^{-1}\right)\|\nabla\phi\|_2^2 \\ &\qquad - c(K+\epsilon)\Gamma(-\gamma)\int_{\mathbb{R}^d} dx\, |\phi(x)|^{2\gamma} - E_\gamma(\epsilon, c; \phi^2)\colon\ \|\phi\|_2^2 = 1\Bigg\}, \end{aligned}$$

where in the second inequality we use the upper bound (3.7.1) and the identity (3.7.5), and introduce an error term

$$E_\gamma(\epsilon, c; \phi^2) = c(K+\epsilon)\int_{\mathbb{R}^d} dx \int_0^{R_\epsilon} d\kappa\, \kappa^{-1-\gamma}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \tag{3.7.20}$$

The integral in (3.7.20) can be estimated from above along the lines of the argument connecting (3.7.7), (3.7.8) with (3.7.12), (3.7.13). After some computation, this leads to

$$\begin{aligned} &c\langle\kappa\rangle - J_d^\Pi(c) \\ &\le -\inf\Bigg\{\left(\frac{1}{2} - E_\gamma(\epsilon, c)\right)\|\nabla\phi\|_2^2 - c(K+\epsilon)\Gamma(-\gamma) \\ &\qquad \times \int_{\mathbb{R}^d} dx\, |\phi(x)|^{2\gamma}\colon\ \|\phi\|_2^2 = 1\Bigg\} \end{aligned} \tag{3.7.21}$$

with an error term

$$E_\gamma(\epsilon, c) = c m_\theta(\epsilon) + c(K+\epsilon) R_\epsilon^{(2-d(\gamma-1))/d} \frac{d}{S_d(2 - d(\gamma-1))}. \tag{3.7.22}$$

Via the scaling $\phi(x) = \delta^{d/2}\psi(\delta x)$, $\delta > 0$, with δ the root of $\delta^2(\frac{1}{2} - cE_\gamma(\epsilon, c)) = c(K+\epsilon)\Gamma(-\gamma)\delta^{d(\gamma-1)}$, we arrive at

$$\begin{aligned} &\{2Kc\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \\ &\le \frac{1}{2}\left(\frac{K+\epsilon}{K}\,\frac{1}{1 - 2E_\gamma(\epsilon, c)}\right)^{2/(2-d(\gamma-1))} M_d(\gamma). \end{aligned} \tag{3.7.23}$$

Now let $c \downarrow 0$ and use that $\lim_{c\downarrow 0} E_\gamma(\epsilon, c) = 0$ for all $\epsilon > 0$. Then let $\epsilon \downarrow 0$, to get

$$\limsup_{c\downarrow 0}\{2Kc\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \le \frac{1}{2}M_d(\gamma), \tag{3.7.24}$$

which is the desired upper bound.

It remains to prove that $M_d(\gamma) \in (0,\infty)$ for all $\gamma \in (1,(d+2)/d)$. By scaling we have, for any $\epsilon > 0$,

$$M_d(\gamma) = -\epsilon^2 \inf\left\{\|\nabla\psi\|_2^2 - \epsilon^{-(2-d(\gamma-1))}\int |\psi|^{2\gamma}\colon\ \|\psi\|_2^2 = 1\right\}. \tag{3.7.25}$$

We get a strictly positive lower bound by choosing for ψ the function

$$\psi(x) = \pi^{-d/4}e^{-|x|^2/2} \tag{3.7.26}$$

and by subsequently choosing ϵ sufficiently small.

To prove that $M_d(\gamma)$ is finite for $\gamma \in (1, \frac{d+2}{d})$, we apply the Sobolev inequality (3.3.15) to (1.4.10). This gives

$$M_d(\gamma) \le -\inf\left\{S_d\|\psi\|_{2d/(d-2)}^2 - \int |\psi|^{2\gamma}\colon\ \|\psi\|_2^2 = 1\right\}. \tag{3.7.27}$$

Since $\|\psi\|_2^2 = 1$ and $\gamma \in (1, d/(d-2))$, Hölder's inequality gives

$$\int |\psi|^{2\gamma} \le \left(\int |\psi|^{2d/(d-2)}\right)^{(d-2)(\gamma-1)/2}. \tag{3.7.28}$$

Inserting this into (3.7.27), we get

$$\begin{aligned} M_d(\gamma) &\le \sup\left\{\|\psi\|_{2d/(d-2)}^{d(\gamma-1)} - S_d\|\psi\|_{2d/(d-2)}^2\colon\ \|\psi\|_2^2 = 1\right\} \\ &\le \sup_{\rho\in(0,\infty)}\left\{\rho^{d(\gamma-1)} - S_d\rho^2\right\}. \end{aligned} \tag{3.7.29}$$

The supremum in the right-hand side is finite because $d(\gamma-1) < 2$.

3.8. Proof of Theorem 1.4.4(ii) and Theorem 1.4.4(iv)

In $d = 2$ the analogue of (3.3.3) reads (recall that κ is replaced by 2π)

$$c^* = \inf\left\{\frac{\frac{1}{2}\|\nabla\phi\|_2^2}{G_2(\phi^2)}\colon\ \|\phi\|_2^2 = 1\right\} \tag{3.8.1}$$

with

$$G_2(\phi^2) = \int_{\mathbb{R}^2} dx\,\left(e^{-2\pi\phi^2(x)} - 1 + 2\pi\phi^2(x)\right). \tag{3.8.2}$$

Lemma 3.8.1. (3.8.1) *has no minimiser. If* (ϕ_n) *is a minimising sequence that is* RSNI, *then* $\lim_{n\to\infty}\int_{\{\phi_n>\delta\}} dx = 0$ *for any* $\delta > 0$.

Proof. Suppose that the variational problem in the right-hand side of (3.8.1) has a minimiser, say ψ^*. Then

$$c^* = \frac{\frac{1}{2}\|\nabla\psi^*\|_2^2}{G_2(\psi^{*2})}. \tag{3.8.3}$$

For $\epsilon > 0$, put

$$\psi^*_\epsilon(x) = \epsilon\psi^*(\epsilon x). \tag{3.8.4}$$

Since $\|\psi^*_\epsilon\|_2^2 = 1$, we have

$$c^* \leq \frac{\frac{1}{2}\|\nabla\psi^*_\epsilon\|_2^2}{G_2(\psi^{*2}_\epsilon)} = \frac{\frac{1}{2}\|\nabla\psi^*\|_2^2}{\epsilon^{-4}G_2(\epsilon^2\psi^{*2})}. \tag{3.8.5}$$

Next, we claim that

$$y \mapsto \frac{1}{y^2}\left(e^{-\kappa y} - 1 + \kappa y\right), \qquad y > 0, \tag{3.8.6}$$

is strictly decreasing on $(0,\infty)$ for any $\kappa > 0$. Indeed, its derivative at y equals

$$\frac{2}{y^3}\left[\left(1 - \frac{\kappa y}{2}\right) - \left(1 + \frac{\kappa y}{2}\right)e^{-\kappa y}\right]. \tag{3.8.7}$$

Abbreviate $z = \kappa y/2$ and note that $z \mapsto (1+z)e^{-2z} + z$, $z \geq 0$, is strictly increasing on $[0,\infty)$, and equal to 1 at $z = 0$, to get the claim. Finally, using that (3.8.6) is strictly decreasing, we get from (3.8.2) that $\epsilon \mapsto \epsilon^{-4}G_2(\epsilon^2\psi^{*2})$ is strictly decreasing, which clearly contradicts (3.8.3) and (3.8.5) when $\epsilon < 1$.

To prove the last claim, let (ϕ_n) be a minimising sequence for (3.8.1). Then, for any $\delta > 0$,

$$\lim_{n\to\infty}\int_{\{\phi_n > \delta\}} dx = 0. \tag{3.8.8}$$

Indeed, if (3.8.8) fails, then there exists an $\eta > 0$ and a subsequence (ϕ_{n_j}) such that

$$\int_{\{\phi_{n_j} > \delta\}} dx \geq \eta. \tag{3.8.9}$$

But now the above argument shows that the sequence $(\phi^\epsilon_{n_j})$ with $\phi^\epsilon_{n_j}(x) = \epsilon\phi_{n_j}(\epsilon x)$ yields a strictly lower infimum when $\epsilon < 1$, which is a contradiction. □

Lemma 3.8.2. *Let $d = 2$. Then*

$$c^* = \frac{1}{4\pi^2}\inf\left\{\frac{\|\nabla\phi\|_2^2}{\|\phi\|_4^4} : \|\phi\|_2^2 = 1\right\} \tag{3.8.10}$$

and $c^* \in \left[\frac{27}{64\pi}, \frac{1}{2\pi}\right]$.

Proof. Since $e^{-x} \leq 1 - x + \frac{1}{2}x^2$, $x \geq 0$, we get from (3.8.2) that

$$G_2(\phi^2) \leq 2\pi^2 \|\phi\|_4^4. \tag{3.8.11}$$

Substituting (3.8.11) into (3.8.1), we obtain the desired lower bound

$$c^* \geq \frac{1}{4\pi^2} \inf \left\{ \frac{\|\nabla\phi\|_2^2}{\|\phi\|_4^4} \colon \|\phi\|_2^2 = 1 \right\}. \tag{3.8.12}$$

To prove the converse of (3.8.12), let (ϕ_n) be a minimising sequence for (3.8.10) that is RSNI. Then (ϕ_n^ϵ) with $\phi_n^\epsilon(x) = \epsilon\phi_n(\epsilon x)$ is a minimising sequence too. Replacing ϵ by $\epsilon/\phi_n(0)$, we may assume that $\phi_n(0) = 1$. It suffices to show that

$$\limsup_{\epsilon\downarrow 0} \limsup_{n\to\infty} \left(\frac{\|\nabla\phi_n^\epsilon\|_2^2}{G_2(\phi_n^{\epsilon\,2})} - \frac{\|\nabla\phi_n^\epsilon\|_2^2}{2\pi^2\|\phi_n^\epsilon\|_4^4} \right) \leq 0. \tag{3.8.13}$$

Since (ϕ_n^ϵ) is a minimising sequence, there exists an N such that for $n \geq N$,

$$\phi_n^\epsilon(0) = \epsilon, \quad \|\nabla\phi_n^\epsilon\|_2^2/\|\phi_n^\epsilon\|_4^4 < \infty, \quad \|\phi_n^\epsilon\|_2^2 = 1. \tag{3.8.14}$$

Since $e^{-x} - 1 + x - \frac{1}{2}x^2 \geq -\frac{1}{6}x^3$, $x \geq 0$, it follows from (3.8.2) that

$$G_2(\phi_n^{\epsilon\,2}) \geq \int \left[\frac{1}{2}(2\pi\phi_n^{\epsilon\,2})^2 - \frac{1}{6}(2\pi\phi_n^{\epsilon\,2})^3 \right] \geq 2\pi^2 \left[1 - \frac{2\pi\epsilon^2}{3} \right] \int \phi_n^{\epsilon\,4}, \tag{3.8.15}$$

where we have used that $\phi_n \leq \phi_n(0) = \epsilon$. Hence, for $n \geq N$,

$$\|\nabla\phi_n^\epsilon\|_2^2 \left(\frac{1}{G_2(\phi_n^{\epsilon\,2})} - \frac{1}{2\pi^2\|\phi_n^\epsilon\|_4^4} \right) \leq \frac{1}{2\pi^2} \left(\left[1 - \frac{2\pi\epsilon^2}{3} \right]^{-1} - 1 \right) \frac{\|\nabla\phi_n^\epsilon\|_2^2}{\|\phi_n^\epsilon\|_4^4}. \tag{3.8.16}$$

As $n \to \infty$, the quotient in the right-hand side converges to $2c^*$. Now let $\epsilon \downarrow 0$, to get the claim in (3.8.13).

Finally, the numerical bounds on c^* are obtained as follows. First note that in $d = 2$ we have the Sobolev inequality

$$\|\nabla\phi\|_2^2 \geq S_{2,4}^{-2}\|\phi\|_4^2 - \|\phi\|_2^2 \tag{3.8.17}$$

(see Lieb and Loss [6] page 190). With the substitution $\phi_p(x) = \phi(x/p)$, $p > 0$, this inequality transforms into

$$\|\nabla\phi\|_2^2 \geq S_{2,4}^{-2}\, p\, \|\phi\|_4^2 - p^2\, \|\phi\|_2^2. \tag{3.8.18}$$

After optimisation over p this yields the Sobolev inequality

$$\|\nabla\phi\|_2^2 \geq \tfrac{1}{4} S_{2,4}^{-4}\|\phi\|_4^4\, \|\phi\|_2^{-2}. \tag{3.8.19}$$

Substituting (3.8.19) into (3.8.12), we find the lower bound

$$c^* \geq \frac{1}{16\pi^2} S_{2,4}^{-4}. \tag{3.8.20}$$

This implies that $c^* \geq 27/64\pi$, because $S_{2,4}^{-4} = 27\pi/4$. To obtain the upper bound on c^*, we pick ψ as in (3.7.26) with $d = 2$. Since $\|\nabla\psi\|_2^2 = 1$, $\|\psi\|_2^2 = 1$ and $\|\psi\|_4^4 = \frac{1}{2\pi}$, substitution into (3.8.10) yields that $c^* \leq 1/2\pi$. □

Lemma 3.8.3. *Let $d = 2$. Then* $\lim_{c \downarrow c^*}[J_2(c) - J_2(c^*)]/(c - c^*) = 2\pi$.

Proof. By the concavity of $c \mapsto J_2(c)$ stated in Theorem 1.4.4(i), it suffices to prove that

$$\liminf_{c \downarrow c^*} \frac{J_2(c) - J_2(c^*)}{c - c^*} \geq 2\pi. \tag{3.8.21}$$

Since J_2 does not depend on Π, it is given by the expression we obtained in Theorem 2 (p. 358) and Corollary 2 (p. 363) in vdBBdH, for the case where $\Pi = \delta_{B_a(0)}$ with $a > 0$ arbitrary, namely,

$$J_2(c) = \inf_{0<b\leq 2\pi} [bc + I_2(b)] \tag{3.8.22}$$

with

$$I_2(b) = \left\{ \frac{1}{2}\|\nabla\phi\|_2^2 \colon \ \phi \in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1,\ \int (1 - e^{-2\pi\phi^2}) = b \right\} \tag{3.8.23}$$

(see also Lemma 9 (p. 387) and Lemma 12 (p. 389) in vdBBdH). Now, from Theorems 3(i) and 4(ii) in vdBBdH (pp. 359–360) we know that

$$b \mapsto \frac{I_2(b)}{2\pi - b} \tag{3.8.24}$$

is strictly decreasing on $(0, 2\pi)$, with

$$\lim_{b \uparrow 2\pi} \frac{I_2(b)}{2\pi - b} = \frac{1}{4\pi^2} \inf \left\{ \|\nabla\phi\|_2^2 \colon \ \phi \in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1,\ \|\phi\|_4^4 = 1 \right\} = c^* \tag{3.8.25}$$

(compare with (3.8.10)). Put

$$\Delta(b) = \frac{I_2(b)}{2\pi - b} - c^*. \tag{3.8.26}$$

Using (3.8.22), we may then write

$$\frac{J_2(c) - J_2(c^*)}{c - c^*} = \inf_{0<b\leq 2\pi} \left[b + \frac{(2\pi - b)\Delta(b)}{c - c^*} \right]. \tag{3.8.27}$$

Since $\Delta(b) > 0$ for all $0 < b < 2\pi$, the minimiser in the right-hand side tends to 2π as $c \downarrow c^*$, which yields (3.8.21). □

Lemma 3.8.4. *Let $d = 2$. Then* (1.3.6) *has a minimiser if and only if $c > c^*$. This minimiser has full support.*

Proof. The fact that there is no minimiser for $c = c^*$ is a direct consequence of the first claim in Lemma 3.8.1. The proof that there is a minimiser for $c > c^*$ is the same as that of Lemma 3.5.1. The fact that this minimiser has full support again follows from the analogue of Lemma 11 in vdBBdH (p. 388). □

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