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Stochastic nonlinear beam equations

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Abstract. An extensible beam equation with a stochastic force of a white noise type is studied, Lyapunov functions techniques being used to prove existence of global mild solutions and asymptotic stability of the zero solution.

0. Introduction

The nonlinear beam equation

$$\frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial^4 y}{\partial x^4} = \left[a + b \int_0^l \left(\frac{\partial y}{\partial x} \right)^2 \mathrm{d}x \right] \frac{\partial^2 y}{\partial x^2} \tag{0.1}$$

was proposed by S. Woinowsky-Krieger [34] as a model for the transversal deflection of an extensible beam of natural length l, having the ends fixed at the support, under an axial force. In [11] it was shown that the properties of solutions to (0.1) may be related to the phenomenon of dynamic buckling. An equation in two space variables, analogous to (0.1), has been discussed as a model of nonlinear oscillations of a plate in a supersonic flow of gas (see [9], Chapter 4, and the references therein). (For the physical background, the papers [14], [29] or [18] may be also consulted.) It is not obvious that solutions to (0.1) do not blow up at finite time, however, the equation (0.1) as well as its abstract version discussed below have already attracted considerable attention and their properties are rather well understood nowadays. Let us quote at least the papers [1], [24], [16], [32], [20] and [33], in which nonexplosion results and further references may be found.

Motivated by problems arising in aeroelasticity (the description of large amplitude vibrations of an elastic panel excited by aerodynamic forces), Chow and Menaldi in [8] considered a beam described by the equation (0.1) and subjected to

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a force including random fluctuations. Namely, they studied a stochastic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \left[a + b\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^4 u}{\partial x^4}$$
$$= g\Big(t, x, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\Big) + \sum_{k=1}^{\infty} \sigma_k\Big(t, x, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\Big)\dot{w}_k,\ x \in [0, l],\ t \geq 0,$$
$$u(t,0) = u(t,l) = \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,l) = 0,\ t \geq 0, \tag{0.2}$$

where w_k are independent standard Brownian motions, and they established existence and uniqueness of global solutions. The goal of our paper is threefold. First, we shall treat a wide class of abstract stochastic beam equations with a nonlocal term and nonlinear damping

$$u_{tt} + A^2 u + g(u, u_t) + m(\|B^{1/2}u\|^2)Bu = \sigma(u, u_t)\dot{W} \tag{0.3}$$

in a Hilbert space H, where the operators A and B are positive selfadjoint and W is an (infinite-dimensional) Wiener process (see the next section for details), showing that results concerning the problem (0.3) apply to (0.2) under reasonable assumptions. Secondly, we prove nonexplosion of mild solutions to (0.3) by a relatively straightforward method based on a choice of a suitable Lyapunov function (which may be interpreted as energy). Thirdly, by modifying the Lyapunov function, the same argument will be adapted to yield stability of solutions to (0.3). It should be noted, however, that whilst our nonexplosion results are fully comparable with the deterministic ones, the stability results in the deterministic and stochastic cases are rather different. In particular, to establish asymptotic stability we have to assume that the damping term g is of the form $g(u, u_t) = \beta u_t$, $\beta > 0$. If $\sigma \equiv 0$, many results on the stabilizing effect of nonlinear damping terms are available (see e.g. [7], [9], [13], [15], [20], [33] for various stability results). Stability of the zero solution of stochastic evolution equations has been studied recently in many papers, but mainly in the parabolic case (see e.g. [5], [21] or the survey [22] and the references therein). Stochastic hyperbolic equations were treated in [23], however, the results are not applicable to the equation (0.3). In fact, the stability conditions proposed in the present paper utilize in a substantial way the specific form of the problem (0.3).

The paper is organized as follows: In the next section, the problem is introduced precisely and main results are stated, their proofs being deferred to Sections 2 (existence of global solutions) and 3 (stability of the zero solution). In Section 4 we show that the abstract model we study covers in particular problems of the type (0.2) and in Appendix we provide a lemma justifying the definition of a local mild solution we use.

1. Main results

Let H be a separable Hilbert space, the norm and inner product of which will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Suppose that

(A1) *$A : \mathrm{Dom}(A) \longrightarrow H$ and $B : \mathrm{Dom}(B) \longrightarrow H$ are self-adjoint operators in H, $B > 0$, $A \geq \mu I$ for some $\mu > 0$, $\mathrm{Dom}(B) \supseteq \mathrm{Dom}(A)$ and $B \in \mathscr{L}(\mathrm{Dom}(A), H)$, $\mathrm{Dom}(A)$ being endowed with the graph norm $\|x\|_{\mathrm{Dom}(A)} \equiv \|Ax\|$.*

(A2) *W is a (possibly cylindrical) Wiener process in another real separable Hilbert space U with a covariance operator Q, defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$ such that $\mathscr{F}_0$ contains all $\boldsymbol{P}$-null sets.*

(A3) *$m \in \mathscr{C}^1([0, \infty[)$ is a nonnegative function, $m \geq 0$.*

We shall investigate an equation, written symbolically

$$
\begin{aligned}
&u_{tt} + A^2 u + g(u, u_t) + m(\|B^{1/2}u\|^2)Bu = \sigma(u, u_t)\dot{W}, \\
&u(0) = u_0, \ u_t(0) = u_1.
\end{aligned}
\tag{1.1}
$$

To interpret (1.1) rigorously and to state hypotheses upon the coefficients g and σ, we rewrite (1.1) as a first order system in a standard way. Let the space $\mathrm{Rng}\, Q^{1/2}$ be endowed with its natural Hilbert space structure (see [10], Section 4.2), and let $\mathscr{L}_2$ denote the space of Hilbert-Schmidt operators. Set

$$
\mathscr{H} = \mathrm{Dom}(A) \times H, \quad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{\mathscr{H}}^2 = \|Ax\|^2 + \|y\|^2,
$$

$$
\mathrm{Dom}(\mathfrak{A}) = \mathrm{Dom}(A^2) \times \mathrm{Dom}(A), \quad \mathfrak{A} = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix},
$$

$$
F : \mathscr{H} \longrightarrow \mathscr{H}, \ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ -m(\|B^{1/2}x\|^2)Bx - g(x, y) \end{pmatrix}.
$$

Assume that

(A4) *$\sigma : \mathscr{H} \longrightarrow \mathscr{L}_2(\mathrm{Rng}\, Q^{1/2}, H)$ is Lipschitz continuous on bounded sets in $\mathscr{H}$ and of a linear growth, that is*

$$
\begin{gathered}
\exists L_\sigma < \infty \ \forall \mathfrak{x} \in \mathscr{H} \quad \|\sigma(\mathfrak{x})Q^{1/2}\|_{\mathrm{HS}} \leq L_\sigma\big(1 + \|\mathfrak{x}\|_{\mathscr{H}}\big), \\
\forall N \in \mathbb{N} \ \exists L_\sigma(N) < \infty \ \forall \mathfrak{x}, \mathfrak{y} \in \mathscr{H}, \ \|\mathfrak{x}\|_{\mathscr{H}}, \|\mathfrak{y}\|_{\mathscr{H}} \leq N \\
\|[\sigma(\mathfrak{x}) - \sigma(\mathfrak{y})]Q^{1/2}\|_{\mathrm{HS}} \leq L_\sigma(N)\|\mathfrak{x} - \mathfrak{y}\|_{\mathscr{H}};
\end{gathered}
$$

by $\|\cdot\|_{\mathrm{HS}}$ we denote the norm of both $\mathscr{L}_2(U, H)$ and $\mathscr{L}_2(U, \mathscr{H})$. Obviously, this implies that the mapping

$$
\Sigma : \mathscr{H} \longrightarrow \mathscr{L}_2(\mathrm{Rng}\, Q^{1/2}, \mathscr{H}), \ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ \sigma(x, y) \end{pmatrix}
$$

is Lipschitz on bounded subsets of $\mathscr{H}$ and of a linear growth (with the same constants). Further, suppose that g satisfies the following growth estimate:

(A5) *$g : \mathscr{H} \longrightarrow H$ is Lipschitz on bounded subsets of $\mathscr{H}$ and there exists a $L_g \in]0, \infty[$ such that*

$$
\langle y, g(x, y) \rangle \geq -L_g\big(1 + \|\mathfrak{x}\|_{\mathscr{H}}^2\big) \tag{1.2}
$$

for any $\mathfrak{x} = (x, y)^\top \in \mathscr{H}$.

Finally, set $\mathfrak{u}_0 = (u_0, u_1)^\top$. In this notation, (1.1) reads as

$$d\mathfrak{u} = \big(\mathfrak{A}\mathfrak{u} + F(\mathfrak{u})\big)dt + \Sigma(\mathfrak{u})dW, \tag{1.3}$$

$$\mathfrak{u}(0) = \mathfrak{u}_0. \tag{1.4}$$

The components of the solution $\mathfrak{u}$ will be denoted by $\mathfrak{u} = (u, u_t)^\top$.

The operator $\mathfrak{A}$ is densely defined and skew-symmetric, hence both $\mathfrak{A}$ and $-\mathfrak{A}$ are dissipative. To show that $\mathfrak{A}$ is m-dissipative it suffices to check that $\mathrm{Rng}(I_{\mathscr{H}} - \mathfrak{A}) = \mathscr{H}$. For this we take $(u, v)^\top \in \mathscr{H}$ and we want to find $(x, y)^\top \in \mathrm{Dom}(\mathfrak{A})$ such that $x - y = u$, $y + A^2 x = v$. This system is equivalent to $x - y = u$, $x + A^2 x = u + v$ which has a solution as -1 does not belong to the spectrum of A^2. We can verify in an analogous way that $-\mathfrak{A}$ is m-dissipative, hence the operator $\mathfrak{A}$ is skew-adjoint and generates a C_0-group of unitary operators on $\mathscr{H}$ (see e.g. [6], Corollary 2.4.11 and Theorem 3.2.3). It is easy to check that $F : \mathscr{H} \longrightarrow \mathscr{H}$ is Lipschitz continuous on bounded sets (hence also bounded on bounded sets). Let $\mathfrak{u}_0 : \Omega \longrightarrow \mathscr{H}$ be $\mathscr{F}_0$-measurable. Under the above assumptions, it is possible to find a unique maximal local mild solution $\mathfrak{u}$ to (1.3), (1.4) with lifespan ζ by proceeding in a standard way (compare, for example, [30], Theorem 1.5, or [2], Theorem 4.10). That is, ζ is a stopping time, strictly positive $\boldsymbol{P}$-almost surely, $(\mathfrak{u}(t),\ t < \zeta)$ is a progressively measurable process satisfying

$$\limsup_{t \nearrow \zeta} \|\mathfrak{u}(t)\|_{\mathscr{H}} = +\infty \quad \boldsymbol{P}\text{-almost surely on } \{\zeta < \infty\},$$

and

$$\mathfrak{u}(t \wedge \tau_k) = e^{\mathfrak{A}(t \wedge \tau_k)}\mathfrak{u}(0) + \int_0^{t \wedge \tau_k} e^{\mathfrak{A}(t \wedge \tau_k - r)} F(\mathfrak{u}(r))dr + I_{\tau_k}(\Sigma)(t \wedge \tau_k)$$

for all $t \geq 0$ and $k \in \mathbb{N}$, where we define

$$\tau_k = \inf\big\{t \geq 0;\ \|\mathfrak{u}(t)\|_{\mathscr{H}} \geq k\big\},$$

$$I_{\tau_k}(\Sigma)(t) = \int_0^t \mathbf{1}_{[0,\tau_k[}(r) e^{\mathfrak{A}(t-r)} \Sigma(\mathfrak{u}(r \wedge \tau_k))dW(r).$$

(The choice of the process $I_{\tau_k}(\Sigma)$ is explained in Appendix, see Lemma A.1.) Note that $\tau_k \nearrow \zeta$ as $k \to \infty$. The process $\mathfrak{u}$ has continuous paths, $\mathfrak{u} \in \mathscr{C}([0, \zeta[; \mathscr{H})$ $\boldsymbol{P}$-almost surely.

Now we are ready to state our first main result:

Theorem 1.1. *Suppose that the hypotheses* (A1)–(A5) *are satisfied and* $\mathfrak{u}_0 : \Omega \longrightarrow \mathscr{H}$ *is* $\mathscr{F}_0$*-measurable. Let* $\mathfrak{u}$ *be the unique maximal local mild solution to* (1.3)–(1.4) *with lifespan* ζ. *Then* $\zeta = +\infty$ $\boldsymbol{P}$*-almost surely, i.e. there exists a unique mild solution* $\mathfrak{u}$ *to* (1.3)–(1.4) *on* $[0, \infty[$ *and* $\mathfrak{u} \in \mathscr{C}([0, \infty[; \mathscr{H})$ $\boldsymbol{P}$*-almost surely.*

Set

$$M(s) = \int_0^s m(r)dr, \quad s \geq 0,$$

and define a mapping $\mathscr{E}$ from the set of all random variables $\mathfrak{v} = (v, z)^\top : \Omega \longrightarrow \mathscr{H}$ into $[0, +\infty]$ by

$$\mathscr{E}(\mathfrak{v}) = \boldsymbol{E}\Big\{\|\mathfrak{v}\|^2_{\mathscr{H}} + M(\|B^{1/2}v\|^2)\Big\}.$$

In the course of the proof of Theorem 1.1 we arrive at

Corollary 1.2. *Let the hypotheses* (A1)–(A5) *be satisfied. If we set*

$$C = 2(L_g + L_\sigma^2),$$

then

$$\mathscr{E}(\mathfrak{u}(t)) \leq e^{Ct}\big(2 + \mathscr{E}(\mathfrak{u}_0)\big) \tag{1.5}$$

holds for all $t \geq 0$ *whenever* $\mathfrak{u}$ *is a solution to* (1.3)–(1.4) *the initial datum* $\mathfrak{u}_0$ *of which satisfies* $\mathscr{E}(\mathfrak{u}_0) < \infty$.

Remark 1.1. Theorem 1.1 implies, in particular, that there is a unique global mild solution to (1.3) (with continuous trajectories) for every deterministic initial condition $\mathfrak{u}(0) = \mathfrak{x} \in \mathscr{H}$ and it follows from [26], Theorem 27, that (1.3) defines a Markov process $(\mathfrak{u}, \boldsymbol{P}_{\mathfrak{x}})$ on $\mathscr{H}$.

Corollary 1.3. *Assume* (A1)–(A5)*. Then the Markov process* $(\mathfrak{u}, \boldsymbol{P}_{\mathfrak{x}})$ *associated with* (1.3) *is Feller, the function*

$$\mathfrak{x} \longmapsto \boldsymbol{E}_{\mathfrak{x}}\varphi(\mathfrak{u}(t)) \tag{1.6}$$

being continuous on $\mathscr{H}$ *for every bounded continuous function* $\varphi : \mathscr{H} \longrightarrow \mathbb{R}$ *and for all* $t \geq 0$.

Let us turn to stability results now. The problem (1.1) with the damping term g of the form

$$g(\mathfrak{x}) = \beta y \quad \text{for some } \beta \geq 0 \text{ and all } \mathfrak{x} = (x, y)^\top \in \mathscr{H} \tag{1.7}$$

will be considered. In addition, we strengthen the linear growth hypothesis on σ to

$$\exists R_\sigma < \infty \; \forall \mathfrak{x} \in \mathscr{H} \quad \|\sigma(\mathfrak{x})Q^{1/2}\|_{\mathrm{HS}} \leq R_\sigma \|\mathfrak{x}\|_{\mathscr{H}}. \tag{1.8}$$

This implies, in particular, that (1.3) admits a trivial solution $\mathfrak{u} \equiv 0$.

Theorem 1.4. *Suppose that* (A1)–(A4)*,* (1.7)*,* (1.8) *are satisfied, and*

$$R_\sigma^2 < \beta. \tag{1.9}$$

Let there exist $\alpha > 0$ *such that*

$$ym(y) \geq \alpha M(y) \quad \textit{for all } y \geq 0. \tag{1.10}$$

Then the zero solution to (1.3) *is exponentially mean-square stable and exponentially stable with probability one: there exist constants* $C < \infty$, $\lambda > 0$ *such that if* $\mathfrak{u}$ *is a solution to* (1.3)–(1.4) *satisfying* $\mathscr{E}(\mathfrak{u}_0) < \infty$*, then we have:*

$$(i) \qquad \boldsymbol{E}\|\mathfrak{u}(t)\|^2_{\mathscr{H}} \le Ce^{-\lambda t}\mathscr{E}(\mathfrak{u}_0) \quad \text{for all } t \ge 0,$$

and
(ii) *for every* $\lambda^* \in]0, \lambda[$ *a* $\boldsymbol{P}$*-almost surely finite function* $t_0 : \Omega \longrightarrow [0, \infty]$ *may be found such that*

$$\|\mathfrak{u}(t)\|^2_{\mathscr{H}} \le Ce^{-\lambda^* t}\mathscr{E}(\mathfrak{u}_0) \quad \text{for all } t \ge t_0 \ \boldsymbol{P}\text{-almost surely.}$$

Besides exponential stability our method yields stability in probability.

Theorem 1.5. *Under the hypotheses of Theorem 1.4, the zero solution of* (1.3) *is stable in probability: for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for any solution* $\mathfrak{u}$ *to* (1.3)–(1.4) *with* $\mathscr{E}(\mathfrak{u}_0) < \infty$*,*

$$\boldsymbol{P}\Big\{\|\mathfrak{u}(0)\|_{\mathscr{H}} > \delta\Big\} < \delta \text{ implies } \boldsymbol{P}\left\{\sup_{t\ge 0}\|\mathfrak{u}(t)\|_{\mathscr{H}} > \varepsilon\right\} < \varepsilon.$$

Remark 1.2. Note that if $m(r) = a + br$ for some $a, b \ge 0$ and all $r \ge 0$, as in the equation (0.2), then the condition (1.10) is always satisfied (with $\alpha = 1$).

Remark 1.3. Let $\mathfrak{u}$ be a solution to (1.3) with an arbitrary $\mathscr{F}_0$-measurable initial condition. Set $\Omega_n = \{\omega \in \Omega;\ \|\mathfrak{u}(0,\omega)\|_{\mathscr{H}} \le n\}$, $n \ge 1$, by Theorem 1.1 there exists a solution $\mathfrak{u}_n$ to (1.3) with the initial condition $\mathfrak{u}_n(0) = \mathbf{1}_{\Omega_n}\mathfrak{u}(0)$. By pathwise uniqueness, $\mathfrak{u}_n(t) = \mathfrak{u}(t)$ for all $t \ge 0$ $\boldsymbol{P}$-almost surely on Ω_n. Suppose that the hypotheses of Theorem 1.4 are satisfied and $\lambda^* \in]0, \lambda[$ is chosen. Since $\mathbf{1}_{\Omega_n}\mathfrak{u}(0)$ is bounded, $\mathscr{E}(\mathbf{1}_{\Omega_n}\mathfrak{u}(0)) < \infty$ and there exist almost surely finite random times t_n such that $\|\mathfrak{u}(t)\|^2_{\mathscr{H}} = \|\mathfrak{u}_n(t)\|^2_{\mathscr{H}} \le C\exp(-\lambda^* t)\mathscr{E}(\mathbf{1}_{\Omega_n}\mathfrak{u}(0))$ for all $t \ge t_n$ $\boldsymbol{P}$-almost everywhere on Ω_n. Define $t_0 = t_n$, $K = C\mathscr{E}(\mathbf{1}_{\Omega_n}\mathfrak{u}(0))$ on $\Omega_n \setminus \Omega_{n-1}$ for $n \ge 1$, setting $\Omega_0 = \emptyset$. Then t_0 and K are almost surely finite random functions and $\|\mathfrak{u}(t)\|^2_{\mathscr{H}} \le K\exp(-\lambda^* t)$ for all $t \ge t_0$ $\boldsymbol{P}$-almost surely. Therefore, under the assumptions of Theorem 1.4 the zero solution to (1.3) is exponentially stable with probability one in the class of all solutions of (1.3).

Remark 1.4. A simple analysis of the proof of Theorem 1.4 shows that we may study equations with a more general damping term using exactly the same procedure, if we are interested in ultimate boundedness of solutions instead of in stability. Let us consider the problem (1.1) with $g(\mathfrak{x}) = \tilde{g}(\mathfrak{x}) + \beta y$, $\mathfrak{x} = (x, y)^\top \in \mathscr{H}$, for some $\beta \ge 0$ and a bounded function $\tilde{g} : \mathscr{H} \longrightarrow \mathscr{H}$ Lipschitz continuous on bounded sets. If $L^2_\sigma < \beta$ and the condition (1.10) is satisfied, then solutions to (1.3) are exponentially ultimately bounded in mean square: there exist constants $\bar{C} < \infty$ and $\bar{\lambda} > 0$ such that

$$\boldsymbol{E}\|\mathfrak{u}(t)\|^2_{\mathscr{H}} \le \bar{C}\big(1 + e^{-\bar{\lambda}t}\mathscr{E}(\mathfrak{u}(0))\big) \quad \text{for all } t \ge 0$$

holds for any solution $\mathfrak{u}$ of (1.3) satisfying $\mathscr{E}(\mathfrak{u}(0)) < \infty$.

2. Proofs: Nonexplosion

Proof of Theorem 1.1. First, solutions $\mathfrak{u}$ to (1.3) satisfying $\mathscr{E}(\mathfrak{u}(0)) < \infty$ will be considered. Let $\mathfrak{u}$ be such solution, denote by ζ its lifespan and define

$$\tau_k = \left\{t \geq 0;\ \|\mathfrak{u}(t)\|_{\mathscr{H}} \geq k\right\}, \quad k \in \mathbb{N},$$

as above. According to Khas'minskii's test for nonexplosions (see [19], Theorem III.4.1, for the finite-dimensional case), to prove that $\zeta = +\infty$ $\boldsymbol{P}$-almost surely it suffices to find a Lyapunov function $V : \mathscr{H} \longrightarrow \mathbb{R}$ satisfying

$$V \geq 0 \text{ on } \mathscr{H}, \tag{2.1}$$

$$q_R \equiv \inf_{\|\mathfrak{x}\|_{\mathscr{H}} \geq R} V(\mathfrak{x}) \underset{R\to\infty}{\longrightarrow} +\infty, \tag{2.2}$$

$$\boldsymbol{E}V(\mathfrak{u}(0)) < \infty \tag{2.3}$$

and

$$\boldsymbol{E}V(\mathfrak{u}(t\wedge\tau_k)) \leq \boldsymbol{E}V(\mathfrak{u}(0)) + C\int_0^t \left(1+\boldsymbol{E}V(\mathfrak{u}(s\wedge\tau_k))\right)\mathrm{d}s \tag{2.4}$$

for a constant $C < \infty$ and all $t \geq 0$, $k \in \mathbb{N}$. Let us recall the simple argument: Once a function V satisfying (2.1)–(2.4) is found, we get

$$\boldsymbol{E}V(\mathfrak{u}(t\wedge\tau_k)) \leq e^{Ct}\left(1+\boldsymbol{E}V(\mathfrak{u}(0))\right), \quad t \geq 0, \tag{2.5}$$

by the Gronwall lemma, which implies easily

$$\begin{aligned}\boldsymbol{P}\left\{\tau_k < t\right\} &\leq \frac{1}{q_k}\boldsymbol{E}\mathbf{1}_{\{\tau_k<t\}}V(\mathfrak{u}(t\wedge\tau_k))\\ &\leq \frac{1}{q_k}e^{Ct}\left(1+\boldsymbol{E}V(\mathfrak{u}(0))\right),\end{aligned}$$

so

$$\lim_{k\to\infty}\boldsymbol{P}\left\{\tau_k < t\right\} = 0$$

for each fixed $t \geq 0$, and $\boldsymbol{P}\{\zeta < t\} = 0$ follows.

Set

$$V(\mathfrak{x}) = \frac{1}{2}\|\mathfrak{x}\|^2_{\mathscr{H}} + \frac{1}{2}M(\|B^{1/2}x\|^2), \quad \mathfrak{x} = \begin{pmatrix}x\\y\end{pmatrix} \in \mathscr{H}.$$

Obviously, $V \in \mathscr{C}^2(\mathscr{H})$, V is uniformly continuous on bounded sets and satisfies (2.1), (2.2). Moreover, (2.3) is equivalent to $\mathscr{E}(\mathfrak{u}(0)) < \infty$. To verify (2.4) we use the Itô formula. Although the underlying idea is lucid and simple, the computations are involved and bit cumbersome, so to explain the argument we first derive the estimate (2.4) assuming that there exists a global strong solution $\mathfrak{u}$ of (2.1). Let us denote by DV and D^2V the first and second Fréchet derivative of the function V, respectively. In a straightforward way we may obtain

$$\begin{aligned} DV(\mathfrak{x})\mathfrak{h} &= \langle \mathfrak{x}, \mathfrak{h}\rangle_{\mathscr{H}} + m(\|B^{1/2}x\|^2)\langle B^{1/2}x, B^{1/2}h_1\rangle, \\ D^2V(\mathfrak{x})(\mathfrak{h}, \mathfrak{k}) &= \langle \mathfrak{h}, \mathfrak{k}\rangle_{\mathscr{H}} + m(\|B^{1/2}x\|^2)\langle B^{1/2}k_1, B^{1/2}h_1\rangle \\ &\quad +2m'(\|B^{1/2}x\|^2)\langle B^{1/2}x, B^{1/2}h_1\rangle\langle B^{1/2}x, B^{1/2}k_1\rangle, \end{aligned}$$

whenever $\mathfrak{x} = (x, y)^\top$, $\mathfrak{h} = (h_1, h_2)^\top$, $\mathfrak{k} = (k_1, k_2)^\top \in \mathscr{H}$, that is, representing $DV(\mathfrak{x})$ as an element in $\mathscr{H}$ and $D^2V(\mathfrak{x})$ as an operator in $\mathscr{L}(\mathscr{H})$ we have

$$DV(\mathfrak{x}) = \mathfrak{x} + m(||B^{1/2}x||^2)\begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix}, \tag{2.6}$$

$$\begin{aligned} D^2V(\mathfrak{x}) &= I_{\mathscr{H}} + m(\|B^{1/2}x\|^2)\begin{pmatrix} A^{-2}B & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad +2m'(\|B^{1/2}x\|^2)\begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} \otimes \begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} \end{aligned} \tag{2.7}$$

for $\mathfrak{x} = (x, y)^\top \in \mathscr{H}$. In particular, the functions DV and D^2V are uniformly continuous on bounded subsets of $\mathscr{H}$. From (2.6) and (2.7) we get

$$\langle DV(\mathfrak{x}), \mathfrak{A}\mathfrak{x}\rangle_{\mathscr{H}} = m(\|B^{1/2}x\|^2)\langle Bx, y\rangle$$

for any $\mathfrak{x} = (x, y)^\top \in \mathrm{Dom}(\mathfrak{A})$, and

$$\langle DV(\mathfrak{x}), F(\mathfrak{x})\rangle_{\mathscr{H}} = -m(\|B^{1/2}x\|^2)\langle y, Bx\rangle - \langle y, g(x, y)\rangle$$

for any $\mathfrak{x} = (x, y)^\top \in \mathscr{H}$, whence also

$$\langle DV(\mathfrak{x}), \mathfrak{A}\mathfrak{x} + F(\mathfrak{x})\rangle_{\mathscr{H}} = -\langle y, g(\mathfrak{x})\rangle, \quad \mathfrak{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{Dom}(\mathfrak{A}).$$

Indeed,

$$\begin{aligned} &\langle DV(\mathfrak{x}), \mathfrak{A}\mathfrak{x}\rangle_{\mathscr{H}} \\ &= \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ -A^2x \end{pmatrix} \right\rangle_{\mathscr{H}} + m(\|B^{1/2}x\|^2)\left\langle \begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ -A^2x \end{pmatrix} \right\rangle_{\mathscr{H}} \\ &= \langle Ax, Ay\rangle - \langle y, A^2x\rangle + m(\|B^{1/2}x\|^2)\langle AA^{-2}Bx, Ay\rangle \\ &= m(\|B^{1/2}x\|^2)\langle Bx, y\rangle, \end{aligned}$$

and

$$\begin{aligned} &\langle DV(\mathfrak{x}), F(\mathfrak{x})\rangle_{\mathscr{H}} \\ &\quad = -m(\|B^{1/2}x\|^2)\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\rangle_{\mathscr{H}} \\ &\qquad -m^2(\|B^{1/2}x\|^2)\left\langle \begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\rangle_{\mathscr{H}} \\ &\qquad -\left\langle \mathfrak{x}, \begin{pmatrix} 0 \\ g(\mathfrak{x}) \end{pmatrix} \right\rangle_{\mathscr{H}} - m(\|B^{1/2}x\|^2)\left\langle \begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g(\mathfrak{x}) \end{pmatrix} \right\rangle_{\mathscr{H}} \\ &\quad = -m(\|B^{1/2}x\|^2)\langle y, Bx\rangle - \langle y, g(x, y)\rangle. \end{aligned}$$

Further, we have

$$\operatorname{Tr}\big(Q^{1/2}\Sigma(\mathfrak{x})^* D^2 V(\mathfrak{z})\Sigma(\mathfrak{x})Q^{1/2}\big) = \|\Sigma(\mathfrak{x})Q^{1/2}\|_{\mathrm{HS}}^2, \quad \mathfrak{x},\mathfrak{z}\in\mathscr{H}. \tag{2.8}$$

To check this identity, let us first realize that

$$\Sigma(\mathfrak{x})^* : \mathscr{H} \longrightarrow \operatorname{Rng} Q^{1/2}, \ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \longmapsto \sigma(\mathfrak{x})^* h_2.$$

The operator $\Sigma(\mathfrak{x})Q^{1/2}$ is Hilbert-Schmidt, so $Q^{1/2}\Sigma(\mathfrak{x})^* D^2 V(\mathfrak{z})\Sigma(\mathfrak{x})Q^{1/2}$ is nuclear, its trace is well defined and does not depend on the choice of an orthonormal basis. Hence take an arbitrary orthonormal basis $\{e_i\}_{i\in I}$ in U and $\mathfrak{x},\mathfrak{z}=(w,z)^\top\in\mathscr{H}$. By definition,

$$\begin{aligned}
&\operatorname{Tr}\big(Q^{1/2}\Sigma(\mathfrak{x})^* D^2 V(\mathfrak{z})\Sigma(\mathfrak{x})Q^{1/2}\big)\\
&\quad=\sum_{i\in I}\big\langle Q^{1/2}\Sigma(\mathfrak{x})^* D^2 V(\mathfrak{z})\Sigma(\mathfrak{x})Q^{1/2}e_i, e_i\big\rangle_U\\
&\quad=\sum_{i\in I}\left\langle \Sigma(\mathfrak{x})^* D^2 V(\mathfrak{z})\begin{pmatrix} 0\\ \sigma(\mathfrak{x})Q^{1/2}e_i\end{pmatrix}, Q^{1/2}e_i\right\rangle_U\\
&\quad=\sum_{i\in I}\left\langle \Sigma(\mathfrak{x})^*\left[\begin{pmatrix} 0\\ \sigma(\mathfrak{x})Q^{1/2}e_i\end{pmatrix} + m(\|B^{1/2}w\|^2)\begin{pmatrix} A^{-2}B & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} 0\\ \sigma(\mathfrak{x})Q^{1/2}e_i\end{pmatrix}\right.\right.\\
&\qquad\left.\left.+2m'(\|B^{1/2}w\|^2)\left\langle\begin{pmatrix} A^{-2}Bw\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ \sigma(\mathfrak{x})Q^{1/2}e_i\end{pmatrix}\right\rangle_{\mathscr{H}}\begin{pmatrix} A^{-2}Bw\\ 0\end{pmatrix}\right], Q^{1/2}e_i\right\rangle_U\\
&\quad=\sum_{i\in I}\left\langle \Sigma(\mathfrak{x})^*\begin{pmatrix} 0\\ \sigma(\mathfrak{x})Q^{1/2}e_i\end{pmatrix}, Q^{1/2}e_i\right\rangle_U = \sum_{i\in I}\big\langle\sigma(\mathfrak{x})^*\sigma(\mathfrak{x})Q^{1/2}e_i, Q^{1/2}e_i\big\rangle_U\\
&\quad=\sum_{i\in I}\|\sigma(\mathfrak{x})Q^{1/2}e_i\|^2 = \sum_{i\in I}\|\Sigma(\mathfrak{x})Q^{1/2}e_i\|_{\mathscr{H}}^2\\
&\quad=\|\Sigma(\mathfrak{x})Q^{1/2}\|_{\mathrm{HS}}^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Sigma(\mathfrak{x})^* DV(\mathfrak{z}) &= \Sigma(\mathfrak{x})^*\begin{pmatrix} w\\ z\end{pmatrix} + m(\|B^{1/2}w\|^2)\Sigma(\mathfrak{x})^*\begin{pmatrix} A^{-2}Bw\\ 0\end{pmatrix}\\
&= \sigma(\mathfrak{x})^* z
\end{aligned} \tag{2.9}$$

for all $\mathfrak{x},\mathfrak{z}=(w,z)^\top\in\mathscr{H}$.

Were $\mathfrak{u}$ a global strong solution to (1.3), an application of the Itô formula (see e.g. [10], Theorem 4.17) would yield

$$\begin{aligned}
V(\mathfrak{u}(t\wedge\tau_n)) - V(\mathfrak{u}(0)) = \int_0^{t\wedge\tau_k}&\Big\{\big\langle DV(\mathfrak{u}(s)), \mathfrak{A}\mathfrak{u}(s) + F(\mathfrak{u}(s))\big\rangle_{\mathscr{H}}\\
&+\frac{1}{2}\operatorname{Tr}\big(Q^{1/2}\Sigma(\mathfrak{u}(s))^* D^2 V(\mathfrak{u}(s))\Sigma(\mathfrak{u}(s))Q^{1/2}\big)\Big\}\mathrm{d}s
\end{aligned}$$

$$+\int_0^{t\wedge\tau_k} \Sigma(\mathfrak{u}(s))^* DV(\mathfrak{u}(s))\mathrm{d}W(s)$$
$$= \int_0^{t\wedge\tau_n} \Big\{-\langle u_t(s), g(\mathfrak{u}(s))\rangle + \frac{1}{2}\|\Sigma(\mathfrak{u}(s))Q^{1/2}\|^2_{\mathrm{HS}}\Big\}\mathrm{d}s$$
$$+\int_0^{t\wedge\tau_n} \sigma(\mathfrak{u}(s))^* u_t(s)\mathrm{d}W(s).$$

So

$$\begin{aligned}\boldsymbol{E}V(\mathfrak{u}(t\wedge\tau_n)) &= \boldsymbol{E}V(\mathfrak{u}(0)) + \boldsymbol{E}\int_0^{t\wedge\tau_n}\Big\{-\langle u_t(s), g(\mathfrak{u}(s))\rangle \\ &\quad + \frac{1}{2}\|\Sigma(\mathfrak{u}(s))Q^{1/2}\|^2_{\mathrm{HS}}\Big\}\mathrm{d}s \\ &\le \boldsymbol{E}V(\mathfrak{u}(0)) + (L_g + L_\sigma^2)\int_0^t \big(1 + \boldsymbol{E}\|\mathfrak{u}(s\wedge\tau_n)\|^2_{\mathscr{H}}\big)\mathrm{d}s \\ &\le \boldsymbol{E}V(\mathfrak{u}(0)) + 2(L_g + L_\sigma^2)\int_0^t \big(1 + \boldsymbol{E}V(\mathfrak{u}(s\wedge\tau_n))\big)\mathrm{d}s\end{aligned}$$

and (2.4) holds.

To justify the above considerations, we have to overcome two difficulties: first, the solution $\mathfrak{u}$ may have only a finite lifespan, and second, the Itô formula cannot be applied to mild solutions directly. To treat the first problem, fix $k \in \mathbb{N}$ arbitrarily for a while and set

$$f(t) = \mathbf{1}_{[0,\tau_k[}(t)F(\mathfrak{u}(t\wedge\tau_k)), \quad \kappa(t) = \mathbf{1}_{[0,\tau_k[}(t)\Sigma(\mathfrak{u}(t\wedge\tau_k)), \quad t \ge 0.$$

Then $f : \mathbb{R}_+ \times \Omega \longrightarrow \mathscr{H}$ and $\kappa : \mathbb{R}_+ \times \Omega \longrightarrow \mathscr{L}_2(\mathrm{Rng}\,Q^{1/2}, \mathscr{H})$ are progressively measurable processes, bounded by the very definition of τ_k, thus

$$\boldsymbol{E}\int_0^T \Big\{\|f(t)\|_{\mathscr{H}} + \|\kappa(t)Q^{1/2}\|^2_{\mathrm{HS}}\Big\}\mathrm{d}t < \infty$$

for every $T \ge 0$ and the linear problem

$$\mathrm{d}v(t) = \big(\mathfrak{A}v(t) + f(t)\big)\mathrm{d}t + \kappa(t)\mathrm{d}W(t), \quad v(0) = \mathfrak{u}(0) \tag{2.10}$$

has a unique (global) mild solution. We have

$$v(t\wedge\tau_k) = \mathfrak{u}(t\wedge\tau_k) \quad \text{for all t } \ge 0\ \boldsymbol{P}\text{-almost surely.} \tag{2.11}$$

Indeed,

$$v(t\wedge\tau_k) = e^{\mathfrak{A}(t\wedge\tau_k)}\mathfrak{u}(0) + \int_0^{t\wedge\tau_k} e^{\mathfrak{A}(t\wedge\tau_k - r)}f(r)\mathrm{d}r + I_{\tau_k}(\kappa)(t\wedge\tau_k)$$

for all $t \ge 0$, where

$$\begin{aligned}I_{\tau_k}(\kappa)(t) &= \int_0^t \mathbf{1}_{[0,\tau_k[}(r)e^{\mathfrak{A}(t-r)}\kappa(r)\mathrm{d}W(r) \\ &= \int_0^t \mathbf{1}_{[0,\tau_k[}(r)e^{\mathfrak{A}(t-r)}\Sigma(\mathfrak{u}(r\wedge\tau_k))\mathrm{d}W(r) \\ &= I_{\tau_k}(\Sigma)(t),\end{aligned}$$

so (2.11) follows.

As the next step, we approximate (2.10) by equations having strong solutions. Towards this end, let us consider the Yosida approximations $\mathfrak{A}_n$, $n \geq 1$, to $\mathfrak{A}$:

$$\mathfrak{A}_n = n\mathfrak{A}(nI - \mathfrak{A})^{-1} = n^2(nI - \mathfrak{A})^{-1} - nI.$$

Recall that $\mathfrak{A}_n$'s are dissipative if $\mathfrak{A}$ is dissipative. The operator $\mathfrak{A}_n$ being bounded on $\mathscr{H}$, the equation

$$\mathrm{d}v_n(t) = \big(\mathfrak{A}_n v_n(t) + f(t)\big)\mathrm{d}t + \kappa(t)\mathrm{d}W(t), \quad v_n(0) = \mathfrak{u}(0),$$

has a unique strong solution. Obviously,

$$\begin{aligned} v_n(t) - v(t) = \big[e^{\mathfrak{A}_n t} - e^{\mathfrak{A}t}\big]\mathfrak{u}(0) + \int_0^t \big[e^{\mathfrak{A}_n(t-r)} - e^{\mathfrak{A}(t-r)}\big] f(r)\mathrm{d}r \\ + \int_0^t \big[e^{\mathfrak{A}_n(t-r)} - e^{\mathfrak{A}(t-r)}\big]\kappa(r)\mathrm{d}W(r). \end{aligned}$$

As

$$\boldsymbol{E}\int_0^T \|\kappa(s)Q^{1/2}\|_{\mathrm{HS}}^p \mathrm{d}s < \infty$$

for any $p \in\,]2, \infty[$ and every $T \geq 0$, Proposition 7.3 in [10] implies

$$\boldsymbol{E}\sup_{0\leq t\leq T}\left\|\int_0^t \big[e^{\mathfrak{A}_n(t-r)} - e^{\mathfrak{A}(t-r)}\big]\kappa(r)\mathrm{d}W(r)\right\|_{\mathscr{H}}^p \underset{n\to\infty}{\longrightarrow} 0, \quad T \geq 0,$$

therefore

$$\boldsymbol{E}\sup_{0\leq t\leq T}\|v(t) - v_n(t)\|_{\mathscr{H}}^2 \underset{n\to\infty}{\longrightarrow} 0, \quad T \geq 0, \tag{2.12}$$

by properties of the Yosida approximations (see e.g. [27], Section 1.3).

Since v_n's are strong solutions, we may compute $V(v_n)$ using the Itô formula arriving, for any bounded stopping time ϱ, at

$$\begin{aligned} V(v_n(\varrho)) - V(v_n(0)) &= \int_0^\varrho \langle DV(v_n(s)), \mathfrak{A}_n v_n(s) + f(s)\rangle_{\mathscr{H}} \mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^\varrho \mathrm{Tr}\big\{Q^{1/2}\kappa(s)^* D^2V(v_n(s))\kappa(s)Q^{1/2}\big\}\mathrm{d}s \\ &\quad + \int_0^\varrho \kappa(s)^* DV(v_n(s))\mathrm{d}W(s). \end{aligned}$$

We want to pass $n \to \infty$; as the first step, we simplify the integrand of the first integral on the right hand side. Let us introduce the canonical projections

$$\begin{aligned} &\pi_1 : \mathscr{H} \longrightarrow \mathrm{Dom}(A) \hookrightarrow H, && \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x, \\ &\pi_2 : \mathscr{H} \longrightarrow H, && \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto y. \end{aligned}$$

Using (2.6), dissipativity of $\mathfrak{A}_n$ and the obvious fact $\pi_1 f = 0$ we get

$$\begin{aligned}
&\langle DV(v_n(s)), \mathfrak{A}_n v_n(s) + f(s)\rangle_{\mathscr{H}}\\
&\quad = \langle v_n(s), \mathfrak{A}_n v_n(s)\rangle_{\mathscr{H}} + \langle v_n(s), f(s)\rangle_{\mathscr{H}}\\
&\qquad + m(\|B^{1/2}\pi_1 v_n(s)\|^2)\langle B\pi_1 v_n(s), \pi_1\mathfrak{A}_n v_n(s) + \pi_1 f(s)\rangle\\
&\quad \le \langle v_n(s), f(s)\rangle_{\mathscr{H}} + m(\|B^{1/2}\pi_1 v_n(s)\|^2)\langle B\pi_1 v_n(s), \pi_1\mathfrak{A}_n v_n(s)\rangle.
\end{aligned}$$

To proceed further, we show that

$$\lim_{n\to\infty} \boldsymbol{E} \sup_{0\le t\le T} \|\pi_1\mathfrak{A}_n v_n(t) - \pi_1\mathfrak{A} v(t)\|^2 = 0, \quad T \ge 0. \tag{2.13}$$

Note that $\pi_1\mathfrak{A}((x,y)^\top) = y$, so $\|\pi_1\mathfrak{A}\|_{\mathscr{L}(\mathscr{H},H)} = 1$. Plainly, (2.13) will easily follow from (2.12), if we prove that $\pi_1\mathfrak{A}_n$, $n \ge 1$, are uniformly bounded as operators from $\mathscr{H}$ to H. To see it, we find an explicit formula for $\pi_1\mathfrak{A}_n$. Since $\mathfrak{A}_n = n^2(nI - \mathfrak{A})^{-1} - nI$, we have to compute $(nI - \mathfrak{A})^{-1}$. We claim that

$$(nI - \mathfrak{A})^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}(n^2 I + A^2)^{-1}(n\varphi + \psi)\\ -A^2(n^2 I + A^2)^{-1}\varphi + n(n^2 I + A^2)^{-1}\psi\end{pmatrix}, \quad \begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in \mathscr{H}.$$

By definition,

$$(nI - \mathfrak{A})^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} \text{ if and only if } \begin{pmatrix}\varphi\\ \psi\end{pmatrix} = (nI - \mathfrak{A})\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}nx - y\\ A^2 x + ny\end{pmatrix},$$

i.e. componentwise

$$nx - y = \varphi, \quad A^2 x + ny = \psi.$$

Solving this system of equations we get $(n^2 I + A^2)x = n\varphi + \psi$, consequently

$$\begin{aligned}
y = nx - \varphi &= \big[n^2(n^2 I + A^2)^{-1} - I\big]\varphi + n(n^2 I + A^2)^{-1}\psi\\
&= -A^2(n^2 I + A^2)^{-1}\varphi + n(n^2 I + A^2)^{-1}\psi
\end{aligned}$$

and our claim follows. Taking into account that $\varphi \in \operatorname{Dom}(A)$ we obtain

$$\begin{aligned}
\pi_1\mathfrak{A}_n\begin{pmatrix}\varphi\\ \psi\end{pmatrix} &= -n\varphi + n^2(n^2 I + A^2)^{-1}(n\varphi + \psi)\\
&= n\big[n^2(n^2 I + A^2)^{-1} - I\big]\varphi + n^2(n^2 I + A^2)^{-1}\psi\\
&= -nA^2(n^2 I + A^2)^{-1}\varphi + n^2(n^2 I + A^2)^{-1}\psi\\
&= -nA(n^2 I + A^2)^{-1}A\varphi + n^2(n^2 I + A^2)^{-1}\psi.
\end{aligned}$$

and

$$\pi_2\mathfrak{A}_n\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = -n^2 A(n^2 I + A^2)^{-1}A\varphi - nA^2(n^2 I + A^2)^{-1}\psi.$$

Let $E(\lambda)$ be the spectral measure associated with the positive self-adjoint operator A, by the spectral theorem

$$\begin{aligned}
nA(n^2 I + A^2)^{-1} &= \int_0^\infty \frac{n\lambda}{n^2 + \lambda^2}\,\mathrm{d}E(\lambda),\\
n^2(n^2 I + A^2)^{-1} &= \int_0^\infty \frac{n^2}{n^2 + \lambda^2}\,\mathrm{d}E(\lambda),
\end{aligned}$$

so

$$\sup_{n\geq 1}\|nA(n^2I+A^2)^{-1}\|_{\mathscr{L}(H)} \leq \sup_{n\geq 1}\sup_{\lambda\geq 0}\frac{n\lambda}{n^2+\lambda^2}\leq\frac{1}{2},$$

$$\sup_{n\geq 1}\|n^2(n^2I+A^2)^{-1}\|_{\mathscr{L}(H)} \leq \sup_{n\geq 1}\sup_{\lambda\geq 0}\frac{n^2}{n^2+\lambda^2}\leq 1,$$

which yields

$$\sup_{n\geq 1}\|\pi_1\mathfrak{A}_n\|_{\mathscr{L}(\mathscr{H},H)}\leq\sqrt{2}. \tag{2.14}$$

As the Yosida approximations satisfy $\mathfrak{A}_n\mathfrak{x}\to\mathfrak{A}\mathfrak{x}$ as $n\to\infty$ for every $\mathfrak{x}\in\mathrm{Dom}(\mathfrak{A})$, (2.14) implies $\pi_1\mathfrak{A}_n\mathfrak{x}\to\pi_1\mathfrak{A}\mathfrak{x}$ for all $x\in\mathscr{H}$, the convergence being uniform on compact subsets of $\mathscr{H}$, hence

$$\sup_{0\leq t\leq T}\big\|(\pi_1\mathfrak{A}_n-\pi_1\mathfrak{A})v(t)\big\|\xrightarrow[n\to\infty]{}0\quad \boldsymbol{P}\text{-almost surely}$$

(note that the set $\{v(t,\omega);\ 0\leq t\leq T\}$ is compact for almost all ω by continuity of trajectories). Then

$$\boldsymbol{E}\sup_{0\leq t\leq T}\big\|(\pi_1\mathfrak{A}_n-\pi_1\mathfrak{A})v(t)\big\|^2\xrightarrow[n\to\infty]{}0$$

follows by the dominated convergence theorem and, as

$$\boldsymbol{E}\sup_{0\leq t\leq T}\big\|\pi_1\mathfrak{A}_n(v_n(t)-v(t))\big\|^2\xrightarrow[n\to\infty]{}0$$

is an immediate consequence of (2.12) and (2.14), the proof of (2.13) is completed.

Further, by (2.8) and the definition of κ we obtain

$$\mathrm{Tr}\big\{Q^{1/2}\kappa(s)^*D^2V(v_n(s))\kappa(s)Q^{1/2}\big\}=\big\|\kappa(s)Q^{1/2}\big\|_{\mathrm{HS}}^2.$$

Analogously, (2.9) yields

$$\kappa(s)^*DV(v_n(s))=\mathbf{1}_{[0,\tau_k[}(s)\sigma(\mathrm{u}(s\wedge\tau_k))^*\pi_2v_n(s).$$

Since $\mathbf{1}_{[0,\tau_k[}(s)\big(\sigma(\mathrm{u}(s\wedge\tau_k))Q^{1/2}\big)^*$ is bounded and ϱ is a bounded stopping time,

$$\begin{aligned}&\boldsymbol{E}\bigg|\int_0^\varrho\kappa(s)^*DV(v_n(s))\mathrm{d}W(s)-\int_0^\varrho\kappa(s)^*DV(v(s))\mathrm{d}W(s)\bigg|^2\\ &=\boldsymbol{E}\int_0^\varrho\mathbf{1}_{[0,\tau_k[}(s)\big\|\big(\sigma(\mathrm{u}(s\wedge\tau_k))Q^{1/2}\big)^*\pi_2\big[v_n(s)-v(s)\big]\big\|_U^2\\ &\xrightarrow[n\to\infty]{}0\end{aligned}$$

by (2.12), whence

$$\int_0^\varrho\kappa(s)^*DV(v_n(s))\mathrm{d}W(s)\xrightarrow[n\to\infty]{\boldsymbol{P}}\int_0^\varrho\kappa(s)^*DV(v(s))\mathrm{d}W(s). \tag{2.15}$$

Finally, note that applying (2.12) we may find a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k}(\cdot,\omega) \to v(\cdot,\omega)$ uniformly on $\big[0,\varrho(\omega)\big]$ as $k\to\infty$ for $\boldsymbol{P}$-almost every ω. Accordingly,

$$\begin{aligned} V(v_n(\varrho)) \le\ & V(\mathrm{u}(0)) + \int_0^\varrho \langle v_n(s), f(s)\rangle_{\mathscr{H}} \mathrm{d}s \\ & + \int_0^\varrho m(\|B^{1/2}\pi_1 v_n(s)\|^2)\langle B\pi_1 v_n(s), \pi_1 \mathfrak{A}_n v_n(s)\rangle \mathrm{d}s \\ & + \frac{1}{2}\int_0^\varrho \|\kappa(s)Q^{1/2}\|_{\mathrm{HS}}^2 \mathrm{d}s \\ & + \int_0^\varrho \kappa(s)^* DV(v_n(s))\mathrm{d}W(s) \end{aligned}$$

and using (2.12), (2.13), (2.15) and continuity of B on $\mathrm{Dom}(A)$ we get

$$\begin{aligned} V(v(\varrho)) \le\ & V(\mathrm{u}(0)) + \int_0^\varrho \langle v(s), f(s)\rangle_{\mathscr{H}} \mathrm{d}s \\ & + \int_0^\varrho m(\|B^{1/2}\pi_1 v(s)\|^2)\langle B\pi_1 v(s), \pi_1 \mathfrak{A} v(s)\rangle \mathrm{d}s \\ & + \frac{1}{2}\int_0^\varrho \|\kappa(s)Q^{1/2}\|_{\mathrm{HS}}^2 \mathrm{d}s \\ & + \int_0^\varrho \kappa(s)^* DV(v(s))\mathrm{d}W(s) \end{aligned}$$

$\boldsymbol{P}$-almost surely. Hence

$$\begin{aligned} \boldsymbol{E}V(v(\varrho)) \le\ & \boldsymbol{E}V(\mathrm{u}(0)) + \boldsymbol{E}\int_0^\varrho \langle v(s), f(s)\rangle_{\mathscr{H}} \mathrm{d}s \\ & + \boldsymbol{E}\int_0^\varrho m(\|B^{1/2}\pi_1 v(s)\|^2)\langle B\pi_1 v(s), \pi_1 \mathfrak{A} v(s)\rangle \mathrm{d}s \\ & + \frac{1}{2}\boldsymbol{E}\int_0^\varrho \|\kappa(s)Q^{1/2}\|_{\mathrm{HS}}^2 \mathrm{d}s, \end{aligned}$$

taking $\varrho = t\wedge\tau_k$ and recalling $v(t)=\mathrm{u}(t)$ for $t<\tau_k$ we arrive at

$$\begin{aligned} & \boldsymbol{E}V(\mathrm{u}(t\wedge\tau_k)) \\ & \le \boldsymbol{E}V(\mathrm{u}(0)) + \boldsymbol{E}\int_0^{t\wedge\tau_k} \langle \mathrm{u}(s), F(\mathrm{u}(s))\rangle_{\mathscr{H}} \mathrm{d}s \\ & \quad + \boldsymbol{E}\int_0^{t\wedge\tau_k} m(\|B^{1/2}u(s)\|^2)\langle Bu(s), u_t(s)\rangle \mathrm{d}s \\ & \quad + \frac{1}{2}\boldsymbol{E}\int_0^{t\wedge\tau_k} \|\Sigma(\mathrm{u}(s))Q^{1/2}\|_{\mathrm{HS}}^2 \mathrm{d}s \\ & = \boldsymbol{E}V(\mathrm{u}(0)) + \boldsymbol{E}\int_0^{t\wedge\tau_k} \Big\{-\langle u_t(s), g(\mathrm{u}(s))\rangle + \frac{1}{2}\|\Sigma(\mathrm{u}(s))Q^{1/2}\|_{\mathrm{HS}}^2\Big\}\mathrm{d}s. \end{aligned}$$

Since $k \geq 1$ was arbitrary, the proof of (2.4) may be completed in an obvious way, as we have already shown above.

Therefore, we see that Theorem 1.1 holds under an additional hypothesis $\mathscr{E}(\mathfrak{u}(0)) < \infty$. In particular, a unique global mild solution to (1.3) may be found for every deterministic initial condition $\mathfrak{u}(0) = \mathfrak{x} \in \mathscr{H}$. (Pathwise uniqueness obviously holds for (1.3) owing to the Lipschitz continuity of F and Σ on bounded sets.) Consequently, for any Borel probability measure μ on $\mathscr{H}$ there exists a martingale solution (as defined in [10], Chapter 8) to (1.3) with the initial condition μ by [26], Corollary 22. Using pathwise uniqueness and a suitable version of the Yamada-Watanabe theory (see [25], Theorem 2) we find a (unique) global mild solution to (1.3), defined on the given stochastic basis, for every $\mathscr{F}_0$-measurable initial condition $\mathfrak{u}(0) : \Omega \longrightarrow \mathscr{H}$. □

Remark 2.1. One may avoid the Yamada-Watanabe theory by considering first truncated initial conditions and then passing to the limit, which is possible due to pathwise uniqueness. (Such a procedure was used in Remark 1.3.)

Proof of Corollary 1.2. We know that $\tau_n \nearrow +\infty$ $\boldsymbol{P}$-almost surely. Consequently, (2.5), continuity of trajectories of $\mathfrak{u}$ and the Fatou lemma yield

$$\boldsymbol{E}V(\mathfrak{u}(t)) \leq e^{Ct}\big(1 + \boldsymbol{E}V(\mathfrak{u}(0))\big), \quad t \geq 0.$$

This estimate implies (1.5) immediately due to our choice of V; the computations above show that $C = 2(L_g + L_\sigma^2)$. □

Proof of Corollary 1.3. It suffices to check that the function (1.6) is continuous for every bounded Lipschitz function φ (see e.g. [12], Theorem 11.3.3). So, let us fix an arbitrary $t > 0$, a bounded Lipschitz continuous function $\varphi : \mathscr{H} \longrightarrow \mathbb{R}$, and a convergent sequence $\{\mathfrak{x}_n\}$ in $\mathscr{H}$, $\mathfrak{x}_n \to \mathfrak{x}_0$ as $n \to \infty$. We aim at proving that

$$\lim_{n\to\infty} \boldsymbol{E}_{\mathfrak{x}_n}\varphi(\mathfrak{u}(t)) = \boldsymbol{E}_{\mathfrak{x}_0}\varphi(\mathfrak{u}(t)).$$

Let us denote by $\mathfrak{u}_j$ the solution to (1.3) with the initial condition $\mathfrak{u}_j(0) = \mathfrak{x}_j$, $j \geq 0$. Since $\{\mathfrak{x}_j\}$ is a bounded sequence, there exists a constant $H < \infty$ such that $2 + \mathscr{E}(\mathfrak{u}_j(0)) \leq H$ for all $j \geq 0$, therefore

$$\sup_{j\geq 0} \mathscr{E}(\mathfrak{u}_j(t)) \leq e^{Ct} H$$

by Corollary 1.2. Let $k \geq 1 \vee H$ be fixed for a while, set

$$\varrho_j = \inf\big\{v \geq 0;\ \|\mathfrak{u}_j(v)\|_{\mathscr{H}} \geq k\big\} \wedge \inf\big\{v \geq 0;\ \|\mathfrak{u}_0(v)\|_{\mathscr{H}} \geq k\big\}, \quad j \geq 1.$$

In the first part of the proof of Theorem 1.1 we have already shown that

$$\sup_{j\geq 0} \boldsymbol{P}\big\{\varrho_j < t\big\} \leq \frac{2e^{Ct}H}{q_k}.$$

The function φ being Lipschitz, there exists a constant $L_\varphi < \infty$ such that $|\varphi(\mathfrak{y}) - \varphi(\mathfrak{z})| \leq L_\varphi \|\mathfrak{y} - \mathfrak{z}\|_{\mathscr{H}}$ for all $\mathfrak{y}, \mathfrak{z} \in \mathscr{H}$; set $\|\varphi\|_\infty = \sup_H |\varphi|$. Fix an arbitrary $p \in]2, \infty[$, we may estimate

$$\begin{aligned}
&\left|\boldsymbol{E}_{\mathfrak{x}_n}\varphi(\mathfrak{u}(t)) - \boldsymbol{E}_{\mathfrak{x}_0}\varphi(\mathfrak{u}(t))\right| = \left|\boldsymbol{E}\varphi(\mathfrak{u}_n(t)) - \boldsymbol{E}\varphi(\mathfrak{u}_0(t))\right| \\
&\quad \leq \boldsymbol{E}\left|\varphi(\mathfrak{u}_n(t)) - \varphi(\mathfrak{u}_0(t))\right| \\
&\quad = \boldsymbol{E}\mathbf{1}_{\{\varrho_n < t\}}\left|\varphi(\mathfrak{u}_n(t)) - \varphi(\mathfrak{u}_0(t))\right| + \boldsymbol{E}\mathbf{1}_{\{\varrho_n \geq t\}}\left|\varphi(\mathfrak{u}_n(t)) - \varphi(\mathfrak{u}_0(t))\right| \\
&\quad \leq 2\boldsymbol{P}\left\{\varrho_n < t\right\}\|\varphi\|_\infty + L_\varphi \boldsymbol{E}\mathbf{1}_{\{\varrho_n \geq t\}}\left\|\mathfrak{u}_n(t) - \mathfrak{u}_0(t)\right\|_{\mathscr{H}} \\
&\quad \leq 4\|\varphi\|_\infty \frac{e^{Ct}H}{q_k} + L_\varphi \boldsymbol{E}\left\|\mathfrak{u}_n(t \wedge \varrho_n) - \mathfrak{u}_0(t \wedge \varrho_n)\right\|_{\mathscr{H}} \\
&\quad \leq 4\|\varphi\|_\infty \frac{e^{Ct}H}{q_k} + L_\varphi\left(\boldsymbol{E}\left\|\mathfrak{u}_n(t \wedge \varrho_n) - \mathfrak{u}_0(t \wedge \varrho_n)\right\|^p_{\mathscr{H}}\right)^{1/p}.
\end{aligned}$$

(Note that $\|\mathfrak{u}_n(v \wedge \varrho_n) - \mathfrak{u}_0(v \wedge \varrho_n)\|_{\mathscr{H}}$ is bounded, hence in $L^p(\Omega)$, for all $v \geq 0$ by the definition of ϱ_n.) We have

$$\begin{aligned}
&\boldsymbol{E}\left\|\mathfrak{u}_n(v \wedge \varrho_n) - \mathfrak{u}_0(v \wedge \varrho_n)\right\|^p \\
&\quad \leq 3^{p-1}\boldsymbol{E}\left\|e^{\mathfrak{A}(v\wedge\varrho_n)}(\mathfrak{x}_n - \mathfrak{x}_0)\right\|^p_{\mathscr{H}} \\
&\qquad + 3^{p-1}\boldsymbol{E}\left\|\int_0^{v\wedge\varrho_n} e^{\mathfrak{A}(v\wedge\varrho_n - s)}\left[F(\mathfrak{u}_n(s)) - F(\mathfrak{u}_0(s))\right]\mathrm{d}s\right\|^p_{\mathscr{H}} \\
&\qquad + 3^{p-1}\boldsymbol{E}\left\|J_{\varrho_n}(v \wedge \varrho)\right\|^p_{\mathscr{H}},
\end{aligned}$$

where

$$J_{\varrho_n}(v) = \int_0^v \mathbf{1}_{[0,\varrho_n[}(s)e^{\mathfrak{A}(v-s)}\left[\Sigma(\mathfrak{u}_n(s)) - \Sigma(\mathfrak{u}_0(s))\right]\mathrm{d}W(s).$$

Let us denote by h_i generic constants (that may depend on k and t). By the maximal inequality for stochastic convolutions (see e.g. [10], Proposition 7.3),

$$\begin{aligned}
\boldsymbol{E}\left\|J_{\varrho_n}(v \wedge \varrho_n)\right\|^p_{\mathscr{H}} &\leq \boldsymbol{E}\sup_{0\leq r\leq v}\left\|J_{\varrho_n}(r)\right\|^p_{\mathscr{H}} \\
&\leq h_1\boldsymbol{E}\int_0^v \mathbf{1}_{[0,\varrho_n[}(s)\left\|\Sigma(\mathfrak{u}_n(s))Q^{1/2} - \Sigma(\mathfrak{u}_0(s))Q^{1/2}\right\|^p_{\mathrm{HS}}\mathrm{d}s \\
&\leq h_1\boldsymbol{E}\int_0^v \left\|\Sigma(\mathfrak{u}_n(s \wedge \varrho_n))Q^{1/2} - \Sigma(\mathfrak{u}_0(s \wedge \varrho))Q^{1/2}\right\|^p_{\mathrm{HS}}\mathrm{d}s.
\end{aligned}$$

Taking into account that $\mathfrak{u}_n(s)$, $\mathfrak{u}_0(s)$ remain in the ball with radius k and center at 0 for $s \leq \varrho_n$ and using Lipschitz continuity of F and Σ on this ball we obtain

$$\begin{aligned}
&\boldsymbol{E}\left\|\mathfrak{u}_n(v \wedge \varrho_n) - \mathfrak{u}_0(v \wedge \varrho_n)\right\|^p_{\mathscr{H}} \\
&\quad \leq h_2\|\mathfrak{x}_n - \mathfrak{x}_0\|^p_{\mathscr{H}} + h_3\boldsymbol{E}\int_0^v \left\|\mathfrak{u}_n(s \wedge \varrho_n) - \mathfrak{u}_0(s \wedge \varrho_n)\right\|^p_{\mathscr{H}}\mathrm{d}s
\end{aligned}$$

for $0 \leq v \leq t$. The Gronwall lemma yields

$$\boldsymbol{E}\left\|\mathfrak{u}_n(t \wedge \varrho_n) - \mathfrak{u}_0(t \wedge \varrho_n)\right\|^p_{\mathscr{H}} \leq h_4\|\mathfrak{x}_n - \mathfrak{x}_0\|^p_{\mathscr{H}},$$

thus

$$\left|\boldsymbol{E}_{\mathfrak{x}_n}\varphi(\mathrm{u}(t)) - \boldsymbol{E}_{\mathfrak{x}_0}\varphi(\mathrm{u}(t))\right| \le 4\|\varphi\|_\infty \frac{e^{Ct}H}{q_k} + h_5\left\|\mathfrak{x}_n - \mathfrak{x}_0\right\|_{\mathscr{H}}.$$

The constant h_5 depends on k in general, so given $\varepsilon > 0$ we first realize that $q_k \nearrow \infty$ and find k such that the first term on right hand side is less than $\varepsilon/2$; the second term is then less than $\varepsilon/2$ for all n sufficiently large as $\mathfrak{x}_n \to \mathfrak{x}_0$. □

3. Proofs: Stability

Define an operator P by

$$P : \mathscr{H} \longrightarrow \mathscr{H}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \beta^2 A^{-2}x + 2x + \beta A^{-2}y \\ \beta x + 2y \end{pmatrix},$$

that is,

$$P = \begin{pmatrix} \beta^2 A^{-2} + 2I & \beta A^{-2} \\ \beta I & 2I \end{pmatrix}.$$

The properties of P needed in the sequel are summarized in the following lemma.

Lemma 3.1. *The operator P is a self-adjoint linear isomorphism of $\mathscr{H}$. Moreover,*

$$\|P\|^{-1}_{\mathscr{L}(\mathscr{H})}\langle P\mathfrak{w}, \mathfrak{w}\rangle_{\mathscr{H}} \le \|\mathfrak{w}\|^2_{\mathscr{H}} \le \langle P\mathfrak{w}, \mathfrak{w}\rangle_{\mathscr{H}}, \tag{3.1}$$

$$\left\langle \begin{pmatrix} 0 \\ -\beta y \end{pmatrix}, P\mathfrak{w} \right\rangle_{\mathscr{H}} = -\beta^2\langle x, y\rangle - 2\beta\|y\|^2 \tag{3.2}$$

for all $\mathfrak{w} = (x, y)^\top \in \mathscr{H}$, and

$$\langle \mathfrak{A}\mathfrak{w}, P\mathfrak{w}\rangle_{\mathscr{H}} = -\beta\|Ax\|^2 + \beta^2\langle y, x\rangle + \beta\|y\|^2 \tag{3.3}$$

for every $\mathfrak{w} = (x, y)^\top \in Dom(\mathfrak{A})$.

Proof. It is easy to see that $P \in \mathscr{L}(\mathscr{H})$ and for $\mathfrak{w}_i = (x_i, y_i)^\top \in \mathscr{H}$, $i = 1, 2$, we have

$$\begin{aligned}\langle P\mathfrak{w}_1, \mathfrak{w}_2\rangle_{\mathscr{H}} &= \beta^2\langle x_1, x_2\rangle + 2\langle Ax_1, Ax_2\rangle + \beta\langle y_1, x_2\rangle + \beta\langle x_1, y_2\rangle + 2\langle y_1, y_2\rangle \\ &= \langle \mathfrak{w}_1, P\mathfrak{w}_2\rangle_{\mathscr{H}},\end{aligned}$$

so $P = P^*$. Obviously,

$$\langle P\mathfrak{w}, \mathfrak{w}\rangle_{\mathscr{H}} = 2\|Ax\|^2 + \|y\|^2 + \|\beta x + y\|^2$$

and (3.1) follows. Similar straightforward computations yield (3.2) and (3.3). □

The Lyapunov function $\Phi\colon \mathscr{H} \to \mathbb{R}_+$, which plays a key role in the proofs of both Theorem 1.4 and 1.5 is defined in terms of the operator P by setting

$$\Phi(\mathfrak{w}) = \frac{1}{2}\langle \mathfrak{w}, P\mathfrak{w}\rangle_{\mathscr{H}} + M(\|B^{1/2}x\|^2), \quad \mathfrak{w} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{H}. \tag{3.4}$$

Let us note that a Lyapunov function $\mathfrak{w} \mapsto \langle P\mathfrak{w}, \mathfrak{w}\rangle_{\mathscr{H}}$ appeared for the first time in the paper [28] in connection with a stability analysis of linear deterministic hyperbolic equations and was applied to stochastic hyperbolic problems in [23].

Lemma 3.2. *Suppose that the hypotheses of Theorem 1.4 are satisfied. Then there exists a constant* $\lambda > 0$ *such that if* $\mathfrak{u}$ *is a solution of* (1.3) *with* $\mathscr{E}(\mathfrak{u}(0)) < \infty$*, then the process* $\big(e^{\lambda t}\Phi(\mathfrak{u}(t))\big)_{t\geq 0}$ *is a nonnegative continuous supermartingale.*

Unlike other proofs in this section, the proof of Lemma 3.2 is rather technical, so it is deferred to the end of the section.

Proof of Theorem 1.4. Lemma 3.2 yields

$$\boldsymbol{E}\Phi(\mathfrak{u}(t)) \leq e^{-\lambda t}\boldsymbol{E}\Phi(\mathfrak{u}(0)) \tag{3.5}$$

for all $t \geq 0$, so using (3.1) and nonnegativity of M we obtain

$$\frac{1}{2}\boldsymbol{E}\|\mathfrak{u}(t)\|^2_{\mathscr{H}} \leq e^{-\lambda t}\boldsymbol{E}\Phi(\mathfrak{u}(0))$$

for all $t \geq 0$, and (i) follows, since plainly

$$\boldsymbol{E}\Phi(\mathfrak{u}(0)) \leq \Big(\frac{1}{2}\|P\|_{\mathscr{H}} + 1\Big)\mathscr{E}(\mathfrak{u}(0)).$$

Further, taking an arbitrary $\lambda^* \in \,]0,\lambda[$ and setting $\varepsilon = \lambda - \lambda^* > 0$ we have

$$\begin{aligned}\boldsymbol{P}&\Big\{\sup_{t\in[k,k+1]} e^{\lambda^* t}\Phi(\mathfrak{u}(t)) \geq \boldsymbol{E}\Phi(\mathfrak{u}(0))\Big\}\\ &\leq \boldsymbol{P}\Big\{\sup_{t\in[k,k+1]} e^{\lambda t}\Phi(\mathfrak{u}(t)) \geq e^{\varepsilon k}\boldsymbol{E}\Phi(\mathfrak{u}(0))\Big\}\end{aligned} \tag{3.6}$$

for any $k \in \mathbb{N}$. We may assume that $\boldsymbol{E}\Phi(\mathfrak{u}(0)) > 0$, otherwise there is nothing to prove. In view of Lemma 3.2 we may use Doob's supermartingale inequality and (3.5) to obtain

$$\boldsymbol{P}\left\{\sup_{t\in[k,k+1]} e^{\lambda t}\Phi(\mathfrak{u}(t)) \geq e^{\varepsilon k}\boldsymbol{E}\Phi(\mathfrak{u}(0))\right\} \leq \frac{e^{\lambda k}\boldsymbol{E}\Phi(\mathfrak{u}(k))}{\boldsymbol{E}\Phi(\mathfrak{u}(0))}\, e^{-\varepsilon k} \leq e^{-\varepsilon k}, \tag{3.7}$$

thus

$$\sum_{k=0}^{\infty}\boldsymbol{P}\Big\{\sup_{t\in[k,k+1]} e^{\lambda^* t}\Phi(\mathfrak{u}(t)) \geq \boldsymbol{E}\Phi(\mathfrak{u}(0))\Big\} < \infty$$

by (3.6) and (3.7), hence (ii) follows by the Borel-Cantelli lemma. □

Proof of Theorem 1.5. Owing to the Markov property of solutions to (1.3), we may suppose that $\mathfrak{u}$ is a solution with a deterministic initial condition $\mathfrak{u}(0) \in \mathscr{H}$. For a given $\varepsilon > 0$ denote by σ_ε the first exit time of $\mathfrak{u}$ from the ball in $\mathscr{H}$ centered at the origin with radius ε,

$$\sigma_\varepsilon = \inf\big\{t \geq 0;\ \|\mathfrak{u}(t)\|_{\mathscr{H}} \geq \varepsilon\big\}.$$

By Lemma 3.2 and the optional sampling theorem we have

$$\boldsymbol{E}\Phi(\mathfrak{u}(t\wedge\sigma_\varepsilon)) \leq \Phi(\mathfrak{u}(0))$$

for $t \in \mathbb{R}_+$, and hence in view of (3.1)

$$\frac{\varepsilon^2}{2} \boldsymbol{P}\{\sigma_\varepsilon < t\} \le \Phi(\mathfrak{u}(0))$$

for all $t \in \mathbb{R}_+$. Since $\{\sigma_\varepsilon < t\} \uparrow \{\sigma_\varepsilon < \infty\}$ as $t \to \infty$ we get

$$\boldsymbol{P}\{\sigma_\varepsilon < \infty\} \le \frac{2\Phi(\mathfrak{u}(0))}{\varepsilon^2}$$

or equivalently

$$\boldsymbol{P}\left\{\sup_{t\ge 0} \|\mathfrak{u}(t)\|_{\mathscr{H}} \ge \varepsilon\right\} \le \frac{2\Phi(\mathfrak{u}(0))}{\varepsilon^2}. \tag{3.8}$$

Now we use the fact that $\Phi(0) = 0$ and $\Phi\colon \mathscr{H} \longrightarrow \mathbb{R}_+$ is continuous. Taking $\delta > 0$ such that $\Phi(z) < \frac{1}{2}\varepsilon^3$ for $z \in \mathscr{H}$, $\|z\|_{\mathscr{H}} \le \delta$, the proof is completed by (3.8). □

Remark 3.1. From the proof of Theorem 1.4 it is easily seen that a slightly stronger version of (i), (ii) has been proven: In fact, $\Phi(\mathfrak{u}(t))$ decays exponentially fast to zero (in L^2 and almost surely). In view of (3.1) we have $V \le \Phi$, where V is the Lyapunov function introduced in Section 2, which may be interpreted as the energy of the system. So we have proved exponential dissipation of the energy in the respective sense.

The rest of the present section is devoted to proving Lemma 3.2. Recall that $\mathfrak{A}_n$ is the Yosida approximation of the operator $\mathfrak{A}$, introduced in Section 2.

Lemma 3.3. *There exists a constant $C < \infty$ such that*

$$\sup_{n\ge 1} \langle \mathfrak{A}_n \mathfrak{w}, P\mathfrak{w}\rangle_{\mathscr{H}} \le C\|\mathfrak{w}\|^2_{\mathscr{H}} \tag{3.9}$$

for each $\mathfrak{w} \in \mathscr{H}$. Moreover,

$$\limsup_{n\to\infty} \langle \mathfrak{A}_n \mathfrak{w}_n, P\mathfrak{w}_n\rangle_{\mathscr{H}} \le -\beta\|Ax\|^2 + \beta^2\langle y, x\rangle + \beta\|y\|^2 \tag{3.10}$$

for every sequence $\mathfrak{w}_n \in \mathscr{H}$ such that $\mathfrak{w}_n \to \mathfrak{w} = (x, y)^\top \in \mathscr{H}$.

Proof. Set $R_n = (n^2 I + A^2)^{-1}$ and $J_n = n^2 R_n$. The computations following (2.13) show that

$$\mathfrak{A}_n = \begin{pmatrix} -nA^2R_n & J_n \\ -A^2J_n & -nA^2R_n \end{pmatrix},$$

hence for a $\mathfrak{w} = (x, y)^\top \in \mathscr{H}$ we get

$$\begin{aligned}
\langle \mathfrak{A}_n \mathfrak{w}, P\mathfrak{w}\rangle_{\mathscr{H}} = &-\beta^2 n\langle A^2R_nx, x\rangle - 2n\langle A^3R_nx, Ax\rangle - \beta n\langle A^2R_nx, y\rangle \\
&+\beta^2\langle J_ny, x\rangle + 2\langle A^2J_ny, x\rangle + \beta\langle J_ny, y\rangle \\
&-\beta\langle A^2J_nx, x\rangle - 2\langle A^2J_nx, y\rangle \\
&-\beta n\langle A^2R_ny, x\rangle - 2n\langle A^2R_ny, y\rangle \\
= &-\frac{\beta^2}{n}\langle A^2J_nx, x\rangle - \frac{2}{n}\langle A^2J_nAx, Ax\rangle - \frac{2\beta}{n}\langle A^2J_nx, y\rangle \\
&+\beta^2\langle J_ny, x\rangle + \beta\langle J_ny, y\rangle - \beta\langle A^2J_nx, x\rangle - \frac{2}{n}\langle A^2J_ny, y\rangle,
\end{aligned}$$

because $\langle A^2 J_n y, x\rangle = \langle A^2 J_n x, y\rangle$ due to self-adjointness of A. The operators $-A^2 J_n$ are dissipative as the Yosida approximations to the operator $-A^2$, thus we obtain

$$\langle \mathfrak{A}_n \mathfrak{w}, P\mathfrak{w}\rangle_{\mathscr{H}} \leq -\frac{2\beta}{n}\langle AJ_n y, Ax\rangle + \beta^2\langle J_n y, x\rangle + \beta\langle J_n y, y\rangle \\ -\beta\langle J_n Ax, Ax\rangle. \tag{3.11}$$

The norms $\|J_n\|_{\mathscr{L}(H)}$ are obviously bounded uniformly in n and

$$\sup_{n\geq 1}\left\|\frac{1}{n}AJ_n\right\|_{\mathscr{L}(H)} = \sup_{n\geq 1}\|nA(n^2 I + A^2)^{-1}\|_{\mathscr{L}(H)} \leq \frac{1}{2}, \tag{3.12}$$

as it was verified in Section 2. Therefore,

$$\langle \mathfrak{A}_n \mathfrak{w}, P\mathfrak{w}\rangle_{\mathscr{H}} \leq C(\|Ax\|^2 + \|y\|^2) = C\|\mathfrak{w}\|^2_{\mathscr{H}}$$

for a constant $C < \infty$ and each $\mathfrak{w} = (x, y)^\top \in \mathscr{H}$, which proves (3.9).

Further, let $\mathfrak{w}_n = (x_n, y_n)^\top \in \mathscr{H}$ be a sequence converging to a $\mathfrak{w} = (x, y)^\top$ in $\mathscr{H}$. By (3.11) we have

$$\langle \mathfrak{A}_n \mathfrak{w}_n, P\mathfrak{w}_n\rangle_{\mathscr{H}} \leq -\frac{2\beta}{n}\langle AJ_n Ax_n, y_n\rangle + \beta^2\langle J_n y_n, x_n\rangle + \beta\langle J_n y_n, y_n\rangle \\ -\beta\langle J_n Ax_n, Ax_n\rangle. \tag{3.13}$$

Obviously, $J_n z_n \to z$ in H if $z_n, z \in H$, $z_n \to z$. Moreover, if $z \in \mathrm{Dom}(A)$, $z = A^{-1}v$ then $n^{-1}AJ_n z = n^{-1}J_n v \to 0$, which together with (3.12) and density of $\mathrm{Dom}(A)$ in H yields $n^{-1}AJ_n z \to 0$ for all $z \in H$. So $n^{-1}AJ_n z_n \to 0$ whenever $z_n \to z$ in H and (3.13) implies

$$\limsup_{n\to\infty}\langle \mathfrak{A}_n \mathfrak{w}_n, P\mathfrak{w}_n\rangle_{\mathscr{H}} \leq \beta^2\langle y, x\rangle + \beta\|y\|^2 - \beta\|Ax\|^2.$$

The proof of (3.10) is completed. □

Proof of Lemma 3.2. Obviously we have $\Phi \in \mathscr{C}^2(\mathscr{H})$ and

$$D\Phi(\mathfrak{w})\mathfrak{h} = \langle P\mathfrak{w}, \mathfrak{h}\rangle_{\mathscr{H}} + 2m(\|B^{1/2}x\|^2)\langle B^{1/2}x, B^{1/2}h_1\rangle,$$
$$D^2\Phi(\mathfrak{w})(\mathfrak{h}, \mathfrak{k}) = \langle P\mathfrak{h}, \mathfrak{k}\rangle_{\mathscr{H}} + 4m'(\|B^{1/2}x\|^2)\langle B^{1/2}x, B^{1/2}k_1\rangle\langle B^{1/2}x, B^{1/2}h_1\rangle \\ +2m(\|B^{1/2}x\|^2)\langle B^{1/2}k_1, B^{1/2}h_1\rangle$$

for all $\mathfrak{w} = (x, y)^\top$, $\mathfrak{h} = (h_1, h_2)^\top$, $\mathfrak{k} = (k_1, k_2)^\top \in \mathscr{H}$. The derivative $D^2\Phi$ may be also written in the form

$$D^2\Phi(\mathfrak{w}) = P + 4m'(\|B^{1/2}x\|^2)\begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} \otimes \begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} \\ +2m(\|B^{1/2}x\|^2)\begin{pmatrix} A^{-2}B & 0 \\ 0 & 0 \end{pmatrix}$$

for $\mathfrak{w} = (x, y)^\top \in \mathscr{H}$. Now we compute the terms that would appear in the Itô formula for $\Phi(\mathfrak{u}(t))$, were $\mathfrak{u}(t)$ a strong solution of the equation. Using (3.2) and (3.3) we have

$$\langle \mathfrak{A}\mathfrak{w}, D\Phi(\mathfrak{w})\rangle_{\mathscr{H}} = -\beta\|Ax\|^2 + \beta^2\langle y, x\rangle + \beta\|y\|^2 \\ +2m(\|B^{1/2}x\|^2)\langle y, Bx\rangle$$

for all $\mathfrak{w} = (x, y)^\top \in \mathrm{Dom}(\mathfrak{A})$, and

$$\langle F(\mathfrak{w}), D\Phi(\mathfrak{w})\rangle_{\mathscr{H}} = -\beta^2\langle x, y\rangle - 2\beta\|y\|^2 - \beta m(\|B^{1/2}x\|^2)\langle Bx, x\rangle \\ -2m(\|B^{1/2}x\|^2)\langle Bx, y\rangle,$$

for all $\mathfrak{w} = (x, y)^\top \in \mathscr{H}$. Note that, consequently, $\langle A\mathfrak{w} + F(\mathfrak{w}), D\Phi(\mathfrak{w})\rangle_{\mathscr{H}} \le -\beta\|\mathfrak{w}\|^2_{\mathscr{H}}$ since $\beta > 0$ and $m \ge 0$. The term containing the second order derivative may be computed in a similar manner as (2.8). Namely:

$$\begin{aligned}
&\mathrm{Tr}\big(Q^{1/2}\Sigma(\mathfrak{w})^* D^2\Phi(\mathfrak{z})\Sigma(\mathfrak{w})Q^{1/2}\big)\\
&= \sum_{k\in I}\left\langle Q^{1/2}\Sigma(\mathfrak{w})^* D^2\Phi(\mathfrak{z})\begin{pmatrix}0\\ \sigma(\mathfrak{w})Q^{1/2}e_k\end{pmatrix}, Q^{1/2}e_k\right\rangle\\
&= \sum_{k\in I}\left\langle \Sigma(\mathfrak{w})^* P\begin{pmatrix}0\\ \sigma(\mathfrak{w})Q^{1/2}e_k\end{pmatrix}, Q^{1/2}e_k\right\rangle\\
&= \sum_{k\in I}\left\langle \Sigma(\mathfrak{w})^* \begin{pmatrix}\beta A^{-2}\sigma(\mathfrak{w})Q^{1/2}e_k\\ 2\sigma(\mathfrak{w})Q^{1/2}e_k\end{pmatrix}, Q^{1/2}e_k\right\rangle\\
&= \sum_{k\in I}\langle 2\sigma(\mathfrak{w})^*\sigma(\mathfrak{w})Q^{1/2}e_k, Q^{1/2}e_k\rangle\\
&= 2\|\Sigma(\mathfrak{w})Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}$$

for all $\mathfrak{w}, \mathfrak{z} \in \mathscr{H}$. For the "stochastic" term we have

$$\Sigma(\mathfrak{z})^* D\Phi(\mathfrak{w}) = \sigma(\mathfrak{z})^*\pi_2 D\Phi(\mathfrak{w}) = \sigma(\mathfrak{z})^*(\beta x + 2y)$$

for $\mathfrak{z}, \mathfrak{w} = (x, y)^\top \in \mathscr{H}$. The Itô formula cannot be applied directly to $\Phi(\mathfrak{u}(t))$ and we make use of the approximating strong solution v_n defined in Section 2 changing, however, slightly the definition of the stopping times τ_k. Given $s \ge 0$, let $\tau_k = \inf\{t \ge s;\ \ \|\mathfrak{u}(t)\|_{\mathscr{H}} \ge k\}, k \in \mathbb{N}$. (The definition from Section 1 corresponds to the case $s = 0$; from Theorem 1.1 we already know that the solution $\mathfrak{u}(t)$ is defined globally, so we may consider any $s \ge 0$ now.) Set $f(t) = \mathbf{1}_{[0,\tau_k[}(t)F(\mathfrak{u}(t \wedge \tau_k))$, $\kappa(t) = \mathbf{1}_{[0,\tau_k[}(t)\Sigma(\mathfrak{u}(t \wedge \tau_k))$, $t \ge 0$. For $k \in \mathbb{N}$ fixed we consider the equations

$$\mathrm{d}v(t) = (\mathfrak{A}v(t) + f(t))\mathrm{d}t + \kappa(t)\mathrm{d}W(t), \quad v(0) = \mathfrak{u}(0),$$
$$\mathrm{d}v_n(t) = (\mathfrak{A}_n v_n(t) + f(t))\mathrm{d}t + \kappa(t)\mathrm{d}W(t), \quad v_n(0) = \mathfrak{u}(0).$$

Since v_n are strong solutions we may apply the Itô formula on an interval $[s, t]$, $t > s$, to the process $e^{\lambda r}\Phi(v_n(r))$ (where $\lambda > 0$ will be specified later) to obtain

$$\begin{aligned}\Phi(v_n(t))e^{\lambda t} &= \Phi(v_n(s))e^{\lambda s}\\ &\quad+\int_s^t e^{\lambda r}\Big[\lambda\Phi(v_n(r)) + \langle \mathfrak{A}_n v_n(r), D\Phi(v_n(r))\rangle_{\mathscr{H}}\\ &\qquad\qquad + \langle f(r), D\Phi(v_n(r))\rangle_{\mathscr{H}}\\ &\qquad\qquad +\frac{1}{2}\mathrm{Tr}\big(Q^{1/2}\kappa(r)^* D^2\Phi(v_n(r))\kappa(r)Q^{1/2}\big)\Big]\mathrm{d}r\\ &\quad+\int_s^t e^{\lambda r}\kappa(r)^* D\Phi(v_n(r))\mathrm{d}W(r).\end{aligned}$$

Hence

$$\begin{aligned}\Phi(v_n(t))e^{\lambda t} &= \Phi(v_n(s))e^{\lambda s} + \int_s^t e^{\lambda r}\Big[\lambda\Phi(v_n(r)) + \langle \mathfrak{A}_n v_n(r), P v_n(r)\rangle_{\mathscr{H}}\\ &\quad +2m(\|B^{1/2}\pi_1 v_n(r)\|^2)\Big\langle \mathfrak{A}_n v_n(r), \begin{pmatrix} A^{-2}B\pi_1 v_n(r)\\ 0\end{pmatrix}\Big\rangle_{\mathscr{H}}\\ &\quad +\langle f(r), P v_n(r)\rangle_{\mathscr{H}}\\ &\quad +2m(\|B^{1/2}\pi_1 v_n(r)\|^2)\Big\langle f(r), \begin{pmatrix} A^{-2}B\pi_1 v_n(r)\\ 0\end{pmatrix}\Big\rangle_{\mathscr{H}}\\ &\quad +\|\kappa(r)Q^{1/2}\|_{\mathrm{HS}}^2\Big]\mathrm{d}r\\ &\quad +\int_s^t e^{\lambda r}\kappa(r)^* D\Phi(v_n(r))\mathrm{d}W(r)\end{aligned}$$

and it follows that

$$\begin{aligned}\Phi(v_n(t))e^{\lambda t} &= \Phi(v_n(s))e^{\lambda s} + \int_s^t e^{\lambda r}\Big[\lambda\Phi(v_n(r)) + \langle \mathfrak{A}_n v_n(r), P v_n(r)\rangle_{\mathscr{H}}\\ &\quad +2m(\|B^{1/2}\pi_1 v_n(r)\|^2)\langle \pi_1\mathfrak{A}_n v_n(r), B\pi_1 v_n(r)\rangle\\ &\quad +\langle \pi_2 f(r), \beta\pi_1 v_n(r) + 2\pi_2 v_n(r)\rangle + \|\kappa(r)Q^{1/2}\|_{\mathrm{HS}}^2\Big]\mathrm{d}r\\ &\quad +\int_s^t e^{\lambda r}\kappa(r)^* D\Phi(v_n(r))\mathrm{d}W(r).\end{aligned}\tag{3.14}$$

Now we pass to the limit for $n \to \infty$. Recall the convergence (2.12) (so, possibly for a subsequence, we have $v_n \to v$ in $\mathscr{C}([s, t]; \mathscr{H})$ almost surely). In virtue of Lemma 3.3 and the Fatou lemma we thus have

$$\begin{aligned}&\limsup_{n\to\infty}\int_s^t e^{\lambda r}\langle \mathfrak{A}_n v_n(r), P v_n(r)\rangle_{\mathscr{H}}\mathrm{d}r\\ &\quad\le \int_s^t e^{\lambda r}\big(-\beta\|A\pi_1 v(r)\|^2 + \beta^2\langle \pi_2 v(r), \pi_1 v(r)\rangle + \beta\|\pi_2 v(r)\|^2\big)\mathrm{d}r.\end{aligned}$$

The limit passage in the other terms of (3.14) is justified as in the similar case in the proof of Theorem 1.1 (in particular, cf. (2.13)), so we arrive at

$$\begin{aligned}\Phi(v(t))e^{\lambda t} \le \Phi(v(s))e^{\lambda s} + \int_s^t e^{\lambda r}\Big[&\lambda\Phi(v(r)) - \beta\|A\pi_1 v(r)\|^2\\ &+\beta^2\langle\pi_2 v(r), \pi_1 v(r)\rangle + \beta\|\pi_2 v(r)\|^2\\ &+2m(\|B^{1/2}\pi_1 v(r)\|^2)\langle B\pi_1 v(r), \pi_2 v(r)\rangle\\ &+\beta\langle\pi_2 f(r), \pi_1 v(r)\rangle + 2\langle\pi_2 f(r), \pi_2 v(r)\rangle + \|\kappa(r)Q^{1/2}\|_{\mathrm{HS}}^2\Big]\mathrm{d}r\\ &+\int_s^t e^{\lambda r}\kappa(r)^* D\Phi(v(r))\mathrm{d}W(r).\end{aligned}$$

Since $v(r) = \mathrm{u}(r)$ for $r < \tau_k$, it follows that

$$\begin{aligned}\Phi(\mathrm{u}(t\wedge\tau_k))e^{\lambda(t\wedge\tau_k)} &\le \Phi(\mathrm{u}(s))e^{\lambda s} + \int_s^{t\wedge\tau_k} e^{\lambda r}\Big[\lambda\Phi(\mathrm{u}(r))\\ &-\beta\|Au(r)\|^2 + \beta^2\langle u(r), u_t(r)\rangle + \beta\|u_t(r)\|^2\\ &+2m(\|B^{1/2}u(r)\|^2)\langle Bu(r), u_t(r)\rangle + \beta\langle -\beta u_t(r), u(r)\rangle\\ &+2\langle -\beta u_t(r), u_t(r)\rangle - 2m(\|B^{1/2}u(r)\|^2)\langle Bu(r), u_t(r)\rangle\\ &-\beta m(\|B^{1/2}u(r)\|^2)\langle Bu(r), u(r)\rangle\\ &+\|\sigma(\mathrm{u}(t))Q^{1/2}\|_{\mathrm{HS}}^2\Big]\mathrm{d}r\\ &+\int_s^{t\wedge\tau_k}\sigma(\mathrm{u}(r))^*(\beta u(r) + 2u_t(r))\mathrm{d}W(r)\\ &= \Phi(\mathrm{u}(s))e^{\lambda s} + \int_s^{t\wedge\tau_k} e^{\lambda r}\Big[\lambda\Phi(\mathrm{u}(r)) - \beta\|\mathrm{u}(r)\|_{\mathscr{H}}^2\\ &-\beta m(\|B^{1/2}u(r)\|^2)\langle Bu(r), u(r)\rangle\\ &+\|\sigma(\mathrm{u}(r))Q^{1/2}\|_{\mathrm{HS}}^2\Big]\mathrm{d}r\\ &+\int_s^{t\wedge\tau_k} e^{\lambda r}\sigma(\mathrm{u}(r))^*(\beta u(r) + 2u_t(r))\mathrm{d}W(r)\end{aligned}$$

for all $t \ge s$ $\boldsymbol{P}$-almost surely. Taking into account (3.4), (3.1), (1.8), (1.10) and setting for brevity $c_P = \|P\|_{\mathscr{L}(\mathscr{H})}$ we obtain

$$\begin{aligned}\Phi(\mathrm{u}(t\wedge\tau_k))e^{\lambda(t\wedge\tau_k)} &\le \Phi(\mathrm{u}(s))e^{\lambda s}\\ &+\int_s^{t\wedge\tau_k} e^{\lambda r}\Big[\Big(\frac{1}{2}\lambda c_P + R_\sigma^2 - \beta\Big)\|\mathrm{u}(r)\|_{\mathscr{H}}^2\\ &+\Big(\frac{\lambda}{\alpha} - \beta\Big)m(\|B^{1/2}u(r)\|^2)\|B^{1/2}u(r)\|^2\Big]\mathrm{d}r\\ &+\int_s^{t\wedge\tau_k} e^{\lambda r}\sigma(\mathrm{u}(r))^*(\beta u(r) + 2u_t(r))\mathrm{d}W(r).\end{aligned}$$

Choosing $0 < \lambda < 2c_P^{-1}(\beta - R_\sigma^2) \wedge \alpha\beta$, which is possible by (1.9), we arrive at

$$\begin{aligned}\Phi(\mathrm{u}(t \wedge \tau_k))e^{\lambda(t\wedge\tau_k)} &\leq \Phi(\mathrm{u}(s))e^{\lambda s}\\ &\quad + \int_s^{t\wedge\tau_k} e^{\lambda r}\sigma^*(\mathrm{u}(r))(\beta u(r) + 2u_t(r))\mathrm{d}W(r)\end{aligned}$$

for all $t \geq s$ $\boldsymbol{P}$-almost surely. Passing $k \to \infty$ we have in view of Theorem 1.1

$$\begin{aligned}\Phi(\mathrm{u}(t))e^{\lambda t} &\leq \Phi(\mathrm{u}(s))e^{\lambda s}\\ &\quad + \int_s^t e^{\lambda r}\sigma(\mathrm{u}(r))^*(\beta u(r) + 2u_t(r))\mathrm{d}W(r),\end{aligned}$$

for $t \geq s$, $\boldsymbol{P}$-almost surely, and Lemma 3.2 easily follows by taking into account Corollary 1.2. □

4. Stochastic beam equations

The purpose of this section is to show that results we obtained about the equation (1.3) are applicable to problems like (0.2). Let $D \subseteq \mathbb{R}^n$ be a bounded domain with a $\mathscr{C}^\infty$-boundary ∂D. Let W be a Wiener process in $L^2(D)$ with a nuclear covariance operator Q. Let $G, \Pi : D \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ be Borel functions, $m \in \mathscr{C}^1([0,\infty[)$ a nonnegative function, $\gamma > 0$ a positive constant. We shall consider an equation

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} - m\left(\int_D |\nabla u|^2 \mathrm{d}x\right)\Delta u + \gamma\Delta^2 u + G\Big(x, u, \nabla u, \frac{\partial u}{\partial t}\Big)\\ &\quad = \Pi\Big(x, u, \nabla u, \frac{\partial u}{\partial t}\Big)\dot{W} \qquad (4.1)\end{aligned}$$

with either the clamped boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \qquad (4.2)$$

(by $\partial/\partial\nu$ we denote the outer normal derivative) or the hinged boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial D. \qquad (4.3)$$

First, we have to show that (4.1) may be turned in the form (1.3). To simplify notation, we set $\gamma = 1$. Let $H = L^2(D)$ and let B be the Laplacian with Dirichlet boundary conditions, i.e.

$$\mathrm{Dom}(B) = W^{2,2}(D) \cap W_0^{1,2}(D), \quad B\psi = -\Delta\psi \text{ for } \psi \in \mathrm{Dom}(B).$$

Note that

$$\|B^{1/2}\psi\|_{L^2(D)}^2 = \langle B\psi, \psi\rangle_{L^2(D)} = -\int_D \Delta u{\cdot}u\mathrm{d}x = \int_D |\nabla\psi|^2\mathrm{d}x, \quad \psi \in \mathrm{Dom}(B),$$

since $\psi = 0$ on ∂D.

For the boundary conditions (4.2) we set $A = C^{1/2}$, where

$$\mathrm{Dom}(C) = \left\{\psi \in W^{4,2}(D);\ \psi = \frac{\partial\psi}{\partial\nu} = 0 \ \text{on}\ \partial D\right\},$$
$$C\psi = \Delta^2\psi \ \text{for}\ \psi \in \mathrm{Dom}(C).$$

Then

$$\mathrm{Dom}(A) = \left\{\psi \in W^{2,2}(D);\ \psi = \frac{\partial\psi}{\partial\nu} = 0 \ \text{on}\ \partial D\right\},$$

as follows e.g. by combining Theorems 4.3.3 and 1.15.3 in [31]. For the boundary conditions (4.3) we set $A = B$. In both cases, the hypotheses on A and B, adopted in Section 1, are satisfied. The only assumption that might require a proof is the uniform positivity of the operator C. Take $u \in \mathrm{Dom}(C)$, then the Green formula yields $\langle Cu, u\rangle = \|\Delta u\|^2_{L^2(D)}$. By [17], Lemma 9.17, we have $\|v\|_{W^{2,2}(D)} \le h\|\Delta v\|_{L^2(D)}$ for any $v \in W^{2,2}(D) \cap W_0^{1,2}(D)$, for a constant $h > 0$ dependent only on D. Since $\mathrm{Dom}(C) \subseteq W^{2,2}(D) \cap W_0^{1,2}(D)$, we obtain in particular $\|\Delta u\|^2_{L^2(D)} \ge h^{-2}\|u\|^2_{L^2(D)}$ for all $u \in \mathrm{Dom}(C)$.

Further, we have to find assumptions on G and Π so that the hypotheses on nonlinear terms in (1.3) may be verified. Set

$$g : \mathrm{Dom}(A) \times L^2(D) \longrightarrow L^2(D), \quad (\psi, \varphi) \longmapsto G(\cdot, \psi(\cdot), \nabla\psi(\cdot), \varphi(\cdot)). \tag{4.4}$$

If $G(\cdot, 0, 0, 0) \in L^2(D)$ and $G(x, \cdot, \cdot, \cdot)$ is Lipschitz continuous uniformly in (almost every) $x \in D$, then g is well defined, takes values in $L^2(D)$ and is globally Lipschitz on $\mathrm{Dom}(A) \times L^2(D)$. Global Lipschitz continuity of G is, however, a rather restrictive hypothesis. Suppose instead that G is globally Lipschitz only in the last variable and locally Lipschitz in the second and third ones: there exists $L < \infty$ and for each $N \ge 0$ there exists $L_N < \infty$ such that for almost every $x \in D$ and all $r, \tilde r, z, \tilde z \in \mathbb{R}$ and $s, \tilde s \in \mathbb{R}^n$ we have

$$|G(x, r, s, z) - G(x, \tilde r, \tilde s, \tilde z)|^2 \le L_N|r - \tilde r|^2 + L_N|s - \tilde s|^2 + L|z - \tilde z|^2$$

whenever $|r|, |\tilde r|, |s|, |\tilde s| \le N$. Let $G(\cdot, 0, 0, 0) \in L^2(D)$. Assume moreover that the space dimension $n = 1$. Since $W^{2,2}(D) \hookrightarrow \mathscr{C}^1(\bar D)$ by the Sobolev embedding theorem, there exists a constant $K < \infty$ such that

$$\|\xi\|_{L^\infty(D)} + \|\nabla\xi\|_{L^\infty(D)} \le K\|\xi\|_{W^{2,2}(D)}, \quad \xi \in W^{2,2}(D). \tag{4.5}$$

Now we may check easily that the function g defined by (4.4) maps $\mathrm{Dom}(A) \times L^2(D)$ into $L^2(D)$ and is Lipschitz on bounded sets: Fix $N \ge 0$ and take arbitrary $\psi_1, \psi_2 \in \mathrm{Dom}(A)$, $\varphi_1, \varphi_2 \in L^2(D)$, $\|\psi_i\|_{W^{2,2}(D)} \le N$. Then

$$\begin{aligned}
&\|g(\psi_1, \varphi_1) - g(\psi_2, \varphi_2)\|^2_{L^2(D)} \\
&= \int_D \left|G(x, \psi_1(x), \nabla\psi_1(x), \varphi_1(x)) - G(x, \psi_2(x), \nabla\psi_2(x), \varphi_2(x))\right|^2 \mathrm{d}x \\
&\le L_{KN}\|\psi_1 - \psi_2\|^2_{L^2(D)} + L_{KN}\|\nabla\psi_1 - \nabla\psi_2\|^2_{L^2(D)} + L\|\varphi_1 - \varphi_2\|^2_{L^2(D)}.
\end{aligned}$$

If, in addition, G does not depend on the third variable, we may replace the assumption $n = 1$ by $n \leq 3$ and use the embedding $W^{2,2}(D) \hookrightarrow \mathscr{C}(\bar{D})$ (which is valid for $n \leq 3$) to prove in a similar way that $g : (\psi, \varphi) \mapsto G(\cdot, \psi(\cdot), \phi(\cdot))$ maps $\mathrm{Dom}(A) \times L^2(D)$ to $L^2(D)$ and is Lipschitz on bounded sets.

To establish (1.2), it suffices to suppose

$$G(x, r, s, z)z \geq -L_G\big(1 + |z|^2\big)$$

for some $L_G \geq 0$, all $r, z \in \mathbb{R}$, $s \in \mathbb{R}^n$ and almost every $x \in D$.

Finally, we have to discuss the stochastic term. Again, we set

$$\sigma(\psi, \varphi) = \Pi(\cdot, \psi(\cdot), \nabla\psi(\cdot), \phi(\cdot)), \quad (\psi, \varphi) \in \mathrm{Dom}(A) \times L^2(D).$$

If Π is bounded, then $\sigma(\psi, \varphi)$ acts as a multiplication operator on $L^2(D)$ and

$$\big\|\sigma(\psi, \varphi)\big\|_{\mathscr{L}(L^2(D))} = \big\|\Pi(\cdot, \psi(\cdot), \nabla\psi(\cdot), \varphi(\cdot))\big\|_{L^\infty(D)}.$$

For the Lipschitz continuity of the mapping $(\psi, \varphi) \mapsto \sigma(\psi, \varphi)$ to hold additional restrictions on Π and the space dimension n are needed, as the considerations above indicate. If Π does not depend on the last variable, $n = 1$ and

$$|\Pi(x, r, s) - \Pi(x, \tilde{r}, \tilde{s})| \leq L\big(|r - \tilde{r}| + |s - \tilde{s}|\big)$$

for some $L < \infty$, almost every $x \in D$ and all $r, \tilde{r}, s, \tilde{s} \in \mathbb{R}$, then

$$\begin{aligned}\big\|\sigma(\psi) - \sigma(\tilde{\psi})\big\|_{\mathscr{L}(L^2(D))} &= \operatorname*{ess\,sup}_{x \in D}\big|\Pi(x, \psi(x), \nabla\psi(x)) - \Pi(x, \tilde{\psi}(x), \nabla\tilde{\psi}(x))\big| \\ &\leq L \operatorname*{ess\,sup}_{x \in D}\Big\{|\psi(x) - \tilde{\psi}(x)| + |\nabla\psi(x) - \nabla\tilde{\psi}(x)|\Big\} \\ &\leq KL\|\psi - \tilde{\psi}\|_{W^{2,2}(D)}\end{aligned}$$

for all $\psi, \tilde{\psi} \in \mathrm{Dom}(A)$, the last estimate following from (4.5). Analogously we may proceed if Π depends only on the first and second variable, $\sigma(\psi) = \Pi(\cdot, \psi(\cdot))$, $n \leq 3$ and

$$|\Pi(x, r) - \Pi(x, \tilde{r})| \leq L|r - \tilde{r}|$$

for some $L < \infty$, almost all $x \in D$ and every $r, \tilde{r} \in \mathbb{R}$. Also the locally Lipschitz case may be handled in a similar manner.

The hypotheses on Π, and in particular the boundedness, may be relaxed if we know more about the covariance operator Q. Assume that Q has a representation $Q = \sum_{k=1}^{\infty} \lambda_i e_i \otimes e_i$ for some $\lambda_i \geq 0$ and an orthonormal basis $\{e_i\}$ such that

$$\sup_{i \geq 1} \|e_i\|_{L^\infty(D)} < \infty.$$

Then $\operatorname{Rng} Q^{1/2} \subseteq L^\infty(D)$, since for each $\varphi \in L^2(D)$ we have

$$\begin{aligned}
\|Q^{1/2}\varphi\|_{L^\infty(D)} &= \Big\| \sum_{i=1}^{\infty} \lambda_i^{1/2} \langle \varphi, e_i \rangle e_i \Big\|_{L^\infty(D)} \\
&\le \sum_{i=1}^{\infty} \lambda_i^{1/2} \big| \langle \varphi, e_i \rangle \big| \|e_i\|_{L^\infty(D)} \\
&\le \left(\sum_{i=1}^{\infty} \lambda_i \right)^{1/2} \left(\sum_{i=1}^{\infty} |\langle \varphi, e_i \rangle|^2 \right)^{1/2} \sup_{i \ge 1} \|e_i\|_{L^\infty(D)} \\
&= \sqrt{\operatorname{Tr} Q} \left(\sup_{i \ge 1} \|e_i\|_{L^\infty(D)} \right) \|\varphi\|_{L^2(D)} < \infty.
\end{aligned}$$

Assume that $\sigma(\psi, \varphi) \in L^2(D)$ for $(\psi, \varphi) \in \operatorname{Dom}(A) \times L^2(D)$, then $\sigma(\psi, \varphi)$ acts as a bounded multiplication operator from $L^\infty(D)$ into $L^2(D)$ and

$$\big\| \sigma(\psi, \varphi) \big\|_{\mathscr{L}(L^\infty(D), L^2(D))} = \big\| \Pi(\cdot, \psi(\cdot), \nabla\psi(\cdot), \varphi(\cdot)) \big\|_{L^2(D)}.$$

Hence $\sigma(\psi, \varphi) Q^{1/2} \in \mathscr{L}(L^2(D))$ and

$$\begin{aligned}
\|\sigma(\psi, \varphi) Q^{1/2}\|_{\mathrm{HS}}^2 &= \sum_{i=1}^{\infty} \|\sigma(\psi, \varphi) Q^{1/2} e_i\|_{L^2(D)}^2 \\
&= \sum_{i=1}^{\infty} \lambda_i \big\| \sigma(\psi, \varphi) e_i \big\|_{L^2(D)}^2 \\
&\le \operatorname{Tr} Q \left(\sup_{i \ge 1} \|e_i\|_{L^\infty(D)}^2 \right) \big\| \sigma(\psi, \varphi) \big\|_{\mathscr{L}(L^\infty(D), L^2(D))}^2.
\end{aligned}$$

Just as in the case of the coefficient g we may find easily hypotheses on Π implying that $(\psi, \varphi) \mapsto \sigma(\psi, \varphi)$ is a mapping from $\operatorname{Dom}(A) \times L^2(D)$ into $L^2(D)$, Lipschitz on bounded sets.

Appendix: Stopped stochastic convolutions

In this Appendix we aim at justifying the definition of a local mild solution we adopted in this paper. The approach we follow was used implicitly in several papers (cf., in particular, the paper [3]), but it seems to have been discussed explicitly for the first time only in [4], §4.3 (in a way different from the one presented below).

Let H, U be real separable Hilbert spaces, (S_t) a C_0-semigroup on H, and $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$ a stochastic basis such that $\mathscr{F}_0$ contains all $\boldsymbol{P}$-null sets. Let W be a Q-Wiener process in U defined on this stochastic basis, where $Q \in \mathscr{L}(U)$ is nonnegative and self-adjoint. In this section, by $\|\cdot\|_{\mathrm{HS}}$ the Hilbert-Schmidt norm on $\mathscr{L}(\operatorname{Rng} Q^{1/2}, H)$ will be denoted. Assume that ψ is a progressively measurable $\mathscr{L}(\operatorname{Rng} Q^{1/2}, H)$-valued process such that

$$\int_0^t \|S_{t-s}\psi_s\|_{\mathrm{HS}}^2 \mathrm{d}s < \infty \quad \text{for all } t \ge 0 \ \boldsymbol{P}\text{-almost surely,} \tag{A.1}$$

then the stochastic convolution

$$I(t) = \int_0^t S_{t-s}\psi_s \mathrm{d}W_s, \quad t \geq 0, \tag{A.2}$$

is well defined. Let τ be a stopping time, one often needs to consider the stopped process $I(t \wedge \tau)$ and is tempted to write

$$I(t \wedge \tau) = \int_0^{t\wedge\tau} S_{t\wedge\tau-s}\psi_s \mathrm{d}W_s.$$

This formula, unfortunately, need not make sense since on the right hand side we integrate a process which is not even adapted. To overcome this difficulty, let us set

$$I_\tau(t) = \int_0^t S_{t-s}\big(\mathbf{1}_{[0,\tau[}(s)\psi_{s\wedge\tau}\big)\mathrm{d}W_s, \quad t \geq 0. \tag{A.3}$$

We shall prove

Lemma A.1. *Let ψ be a progressively measurable $\mathscr{L}(\mathrm{Rng}Q^{1/2}, H)$-valued process satisfying* (A.1). *Let τ be an arbitrary stopping time, define processes I and I_τ by* (A.2) *and* (A.3)*, respectively. Suppose that both processes I and I_τ have continuous paths almost surely. Then*

$$S_{t-t\wedge\tau} I(t \wedge \tau) = I_\tau(t) \quad \textit{for all } t \geq 0 \ \boldsymbol{P}\textit{-almost surely.} \tag{A.4}$$

In particular,

$$I(t \wedge \tau) = I_\tau(t \wedge \tau) \quad \textit{for all } t \geq 0 \ \boldsymbol{P}\textit{-almost surely.}$$

Remark A.1. Obviously, $\mathbf{1}_{[0,\tau[}\psi = \mathbf{1}_{[0,\tau[}\psi(\cdot \wedge \tau)$. We have chosen a definition of I_τ that makes sense also for processes ψ with a finite lifespan $\zeta > \tau$. Moreover, if we are interested in behaviour of the process I only on the stochastic interval $[0, \tau[$ we may use also the identity

$$I(t \wedge \tau) = \tilde{I}_\tau(t \wedge \tau) \quad \text{for all } t \in \mathbb{R}_+ \ \boldsymbol{P}\text{-almost surely,} \tag{A.5}$$

where

$$\tilde{I}_\tau(t) = \int_0^t S_{t-s}\psi_{s\wedge\tau}\mathrm{d}W_s.$$

The proof of (A.5) remains the same as that of (A.4).

Remark A.2. Many sufficient conditions are known for the processes I and I_τ to have continuous modifications, see [10] for basic results in this direction. For example, it suffices to assume that $\|\psi\|_{\mathrm{HS}} \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ $\boldsymbol{P}$-almost surely and (S_t) is quasi-contractive, or that $\|\psi\|_{\mathrm{HS}} \in L^p_{\mathrm{loc}}(\mathbb{R}_+)$ $\boldsymbol{P}$-almost surely for some $p > 2$.

Proof. Let us start with recalling the following fact: if ξ, φ are progressively measurable processes, $\|\xi\|_{\mathrm{HS}}, \|\varphi\|_{\mathrm{HS}} \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ $\boldsymbol{P}$-almost surely and there exists $\Omega_0 \in \mathscr{F}$ such that $\xi = \varphi$ on $\mathbb{R}_+ \times \Omega_0$, then

$$\int_0^t \xi \mathrm{d}W = \int_0^t \varphi \mathrm{d}W \quad \text{for all } t \geq 0 \ \boldsymbol{P}\text{-almost everywhere on } \Omega_0. \tag{A.6}$$

First, we consider the case $\tau \equiv a \in \mathbb{R}_+$. If $t < a$ then

$$I(t \wedge a) = I(t) = \int_0^t S_{t-s}\psi_s \mathrm{d}W_s = \int_0^t \mathbf{1}_{[0,a[}(s) S_{t-s}\psi_{s\wedge a}\mathrm{d}W_s = I_a(t).$$

If $t \geq a$ then

$$\begin{aligned} I_a(t) &= S_{t-a}\int_0^a \mathbf{1}_{[0,a[}(s) S_{a-s}\psi_{s\wedge a}\mathrm{d}W_s + \int_a^t \mathbf{1}_{[0,a[}(s) S_{t-s}\psi_a \mathrm{d}W_s \\ &= S_{t-a} I(a) = S_{t-a} I(t \wedge a). \end{aligned}$$

Hence we see that (A.4) holds in this particular case. Consequently, (A.4) holds whenever τ is a stopping time with a discrete range in $[0, \infty]$ owing to (A.6).

Finally, let τ be arbitrary. Then there exist stopping times τ_k having discrete ranges and such that $\tau_k \searrow \tau$ as $k \to \infty$ on Ω. From the continuity of trajectories of the process I we infer that $I(t \wedge \tau_k) \to I(t \wedge \tau)$ for all $t \geq 0$ almost surely. Further, it is easy to check that

$$\int_0^t \left\| \mathbf{1}_{[0,\tau_k[}(s) S_{t-s}\psi_{s\wedge\tau_k} - \mathbf{1}_{[0,\tau[}(s) S_{t-s}\psi_{s\wedge\tau} \right\|^2_{\mathrm{HS}} \mathrm{d}s \xrightarrow[k\to\infty]{} 0$$

almost surely, which yields $I_{\tau_k}(t) \to I_\tau(t)$ in probability as $k \to \infty$ for any $t \geq 0$. The proof may be completed in an obvious way. □

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