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# Weighted BMO and discrete time hedging within the Black-Scholes model

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**Abstract.** The paper combines two objects rather different at first glance: spaces of stochastic processes having weighted bounded mean oscillation (weighted BMO) and the approximation of certain stochastic integrals, driven by the geometric Brownian motion, by integrals over piece-wise constant integrands. The consideration of the approximation error with respect to weighted BMO implies $L_p$ and uniform distributional estimates for the approximation error by a John-Nirenberg type theorem. The general results about weighted BMO are given in the first part of the paper and applied to our approximation problem in the second one.

## 1. Introduction

The approximation of stochastic integrals by integrals over piece-wise constant integrands has, for example, in stochastic finance a natural interpretation: the pay-off of a continuously re-balanced portfolio is replaced by the pay-off of a portfolio, re-balanced at finitely many trading dates only. The approximation error between the stochastic integral, we are starting from, and its approximation can be interpreted as risk.

Usually, the approximation error is measured in a distributional way by limit distributions, like for example in [20], or with respect to $L_2$. The latter $L_2$-approach has some drawbacks: the resulting distributional tail-estimates are rather weak. Secondly, if one considers a sequence of time-nets realizing the asymptotically optimal $L_2$-approximation rate, where the $n$-th net consists of $n + 1$ time-knots, then there are very different such sequences in general (cf. [13]). Hence it might not be clear what sequence of time-nets one should really take.

The present paper approaches both problems. In order to replace the $L_2$-criterion by a stronger one, one would use $L_p$-spaces with $2 < p < \infty$ in a first instance. However, spaces of weighted bounded mean oscillation (weighted BMO) provide much more information for our purpose, while getting the same upper bound for the asymptotics as in the $L_2$-case. The BMO-spaces are of advantage because of the following two reasons: in general estimates with respect to the used BMO-spaces imply $L_p$-estimates (see Theorem 9) and secondly, by a weighted John-Nirenberg

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type theorem we obtain significant better tail-estimates than we would get from $L_2$-estimates (see Theorem 10). Moreover, Theorem 7 gives an example how to single out those time-nets from the asymptotically $L_2$-optimal ones which are optimal with respect to the stronger weighted BMO-condition as well.

The paper is divided into two rather different parts. Section 2 deals with the weighted BMO-spaces. In Section 3 we apply these results to our approximation problem for stochastic integrals.

## 2. Weighted BMO-spaces

In this section we consider some basic properties of the weighted BMO-spaces used later in Section 3. At the same time we introduce a concept of measuring the mean-oscillation of a stochastic process in a pure distributional way which seems to be the right approach in our situation and might be of interest in other situations as well. Parts of this section are the final outline of some results announced in the discrete time setting without proofs in the preprints [9] and [11].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a right-continuous filtration with $\mathcal{F}_T \subseteq \mathcal{F}$ such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. By $\mathcal{CL}(\mathbb{F})$ we denote the set of all $\mathbb{F}$-adapted processes $A = (A_t)_{t\in[0,T]}$ such that all paths are right-continuous and have finite left-hand side limits. The symbol $\mathcal{CL}^+(\mathbb{F})$ stands for the subclass of these processes such that $A_t(\omega) > 0$ on $[0,T] \times \Omega$, whereas $\mathcal{CL}_0(\mathbb{F})$ stands for the $A \in \mathcal{CL}(\mathbb{F})$ with $A_0 \equiv 0$. Given a stopping time $\sigma : \Omega \to [0,T]$ and $A \in \mathcal{CL}_0(\mathbb{F})$ we let $A_{\sigma-} := \lim_{n\to\infty} A_{\left(\left(\sigma - \frac{1}{n}\right)\vee 0\right)}$. For a stochastic process $X = (X_t)_{t\in[0,T]}$ we let $X_t^* := \sup_{u\in[0,t]} |X_u|$. For $a, b \geq 0$ and $c > 0$, the expression $a \sim_c b$ will stand for $a/c \leq b \leq ca$. Finally, we use $\mathbb{P}_B(\cdot) := \mathbb{P}(B \cap \cdot)/\mathbb{P}(B)$ for $B \in \mathcal{F}$ of positive measure and

$$\mathcal{S} := \{\sigma : \Omega \to [0,T] \mid \sigma \text{ stopping time}\}.$$

**Definition 1.** *For $\theta \in (0,1)$, $p \in (0,\infty)$, $A \in \mathcal{CL}_0(\mathbb{F})$, and $\Phi \in \mathcal{CL}^+(\mathbb{F})$ we define*

$$\|A\|_{\mathrm{BMO}_p^\Phi(\mathbb{P})} := \sup_{\sigma\in\mathcal{S}} \left\| \mathbb{E}\left[ \frac{|A_T - A_{\sigma-}|^p}{\Phi_\sigma^p} \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty}^{\frac{1}{p}},$$

$$\|A\|_{\mathrm{BMO}_p^{\Phi,*}(\mathbb{P})} := \sup_{\sigma\in\mathcal{S}} \left\| \mathbb{E}\left[ \frac{\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}|^p}{\Phi_\sigma^p} \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty}^{\frac{1}{p}},$$

$$\|A\|_{\mathrm{BMO}_{0,\theta}^\Phi(\mathbb{P})} := \inf\left\{ c > 0 : \left\| \mathbb{P}\left[ \frac{|A_T - A_{\sigma-}|}{\Phi_\sigma} > c \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty} \leq \theta,\ \sigma \in \mathcal{S} \right\},$$

*and* $\|A\|_{\mathrm{BMO}_{0,\theta}^{\Phi,*}(\mathbb{P})} :=$

$$\inf\left\{ c > 0 : \left\| \mathbb{P}\left[ \frac{\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}|}{\Phi_\sigma} > c \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty} \leq \theta,\ \sigma \in \mathcal{S} \right\},$$

*with* $\inf \emptyset := \infty$ *and where the defined quantities are allowed to be infinite.*

If we are working under $\mathbb{P}$ and there is no risk of confusion we drop the dependence on $\mathbb{P}$ in the notation of the BMO-spaces (and the $L_p$-spaces as well). The definition of $\|\cdot\|_{\mathrm{BMO}_p^{\Phi}}$ and $\|\cdot\|_{\mathrm{BMO}_p^{\Phi,*}}$ models a classical approach to weighted BMO in the probabilistic setting connected to the Garsia-Neveu lemma and related results of Garsia, Stroock, and many others (see for example [7], [19], [18], [17], [1], [5], and [2]). The approach to exploit $\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}}$ and $\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi,*}}$ has at least two sources: firstly, Strömberg [22] measured the mean-oscillation of complex-valued functions defined on $\mathbb{R}^n$ in a distributional way by the sharp function $M_{0,s}^{\sharp} f$. Secondly, in the non-weighted probabilistic setting, which means here $\Phi_t \equiv 1$, the distributional approach can be found in Emery [6] and in [10].

Our results about the spaces introduced in Definition 1 follow immediately from Theorem 1 below. One motivation of this theorem is a result proved by Lépingle [18] (Theorem 1) in the discrete time setting: assuming a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$, an integrable predictable increasing process $(A_n)_{n=0}^{\infty}$ starting in zero such that $A_n \uparrow A_\infty$ a.s. with $A_\infty \in L_1$, one has

$$\mathbb{P}\left(A_\infty > \lambda + \mu\nu\right) \le 2e^{-\frac{\mu}{2}}\mathbb{P}(A_\infty > \lambda) + \mathbb{P}(Z^* > \nu)$$

where $\lambda, \mu, \nu > 0$ and $Z_n := \mathbb{E}(A_\infty - A_n | \mathcal{F}_n)$ is the potential. For instance, in order to get assertions about $\|\cdot\|_{\mathrm{BMO}_p^{\Phi}}$ with $p \in (0,1)$ the usage of the potential $(Z_n)_{n=0}^{\infty}$ yields to an assumption too strong for our purpose. In the following we replace this assumption by a distributional assumption as weak as possible. In this way we transfer Emery [6] (Proposition 2) and [10] (Theorem 4.6) to the weighted case and Strömberg [22] (Lemma 3.4) from the classical setting of functions defined on $\mathbb{R}^n$ to the setting of stochastic processes. What is an appropriate replacement for the potential $(Z_n)_{n=0}^{\infty}$? For example, taking a path-wise continuous $A \in \mathcal{CL}_0(\mathbb{F})$, we are interested in the distribution of $\sup_{u\in[\sigma,T]} |A_u - A_\sigma|$ given $B \in \mathcal{F}_\sigma$, equivalent to $\sup_{u,v\in[\sigma,T]} |A_u - A_v|$ by a factor 2, and look for upper bounds of

$$W_A(B,\lambda;\sigma) := \mathbb{P}\left(B \cap \left\{\sup_{u,v\in[\sigma,T]} |A_u - A_v| > \lambda\right\}\right) \quad \text{for} \quad \lambda \ge 0.$$

What are the abstract properties of $W_A$? Given $0 \le \sigma \le \tau \le T$, $B \subseteq D$, both belonging to $\mathcal{F}_\sigma$, and $0 \le \lambda \le \mu < \infty$ we get that $W_A(B,\mu;\tau) \le W_A(D,\lambda;\sigma)$. This is exactly the inequality we are starting from.

**Definition 2.** *A family* $W(\cdot,\lambda;\sigma) : \mathcal{F}_\sigma \to [0,\infty)$, $\sigma \in \mathcal{S}$, $\lambda \in [0,\infty)$, *belongs to the class* $\mathcal{W}$ *provided that*

$$W(B,\mu;\tau) \le W(D,\lambda;\sigma)$$

*for all* $0 \le \sigma \le \tau \le T$, $0 \le \lambda \le \mu < \infty$, *and* $B, D \in \mathcal{F}_\sigma$ *with* $B \subseteq D$.

**Theorem 1.** *Let* $A \in \mathcal{CL}_0(\mathbb{F})$, $\theta \in (0,1/2)$, *and* $W \in \mathcal{W}$. *Assume that*

$$\mathbb{P}_B\left(|A_T - A_{\sigma-}| > \nu\right) \;\le\; \theta + \frac{W(B,\nu;\sigma)}{\mathbb{P}(B)} \tag{1}$$

*for all* $\nu > 0$, $\sigma \in \mathcal{S}$, *and* $B \in \mathcal{F}_\sigma$ *of positive measure. Then there are constants* $\alpha, a > 0$, *depending on* $\theta$ *only, such that*

$$\mathbb{P}_B\left(\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}| > \lambda + a\mu\nu\right) \leq e^{1-\mu}\,\mathbb{P}_B\left(\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}| > \lambda\right) + \alpha\frac{W(B,\nu;\sigma)}{\mathbb{P}(B)} \tag{2}$$

*for all* $\lambda, \mu, \nu > 0$, $\sigma \in \mathcal{S}$, *and* $B \in \mathcal{F}_\sigma$ *of positive measure. Consequently, for all* $p \in (0,\infty)$ *and* $B \in \mathcal{F}_\sigma$ *one has that*

$$\left(\int_B \sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|^p\, d\mathbb{P}\right)^{\frac{1}{p}} \leq c_p\left(\int_0^\infty p\lambda^{p-1}W(B,\lambda;\sigma)d\lambda\right)^{\frac{1}{p}} \tag{3}$$

*where the right-hand side or both sides may be infinite,* $c_p > 0$ *depends on* $(p,\alpha,a)$ *only, and* $\sup_{1\leq p<\infty}(c_p/p) < \infty$.

*Proof.* Before we start we extend $W(\cdot,\lambda;\sigma)$ to $\mathcal{F}$ by

$$\widetilde{W}(B,\lambda;\sigma) := \inf\{W(A,\lambda;\sigma) : A \supseteq B,\, A \in \mathcal{F}_\sigma\}$$

for $B \in \mathcal{F}$. We get that $\widetilde{W}(B,\lambda;\sigma) = W(B,\lambda;\sigma)$ for $B \in \mathcal{F}_\sigma$ and that

$$\widetilde{W}(B,\mu;\tau) \leq \widetilde{W}(D,\lambda;\sigma) \leq W(\Omega,\lambda;\sigma) < \infty$$

for all $0 \leq \sigma \leq \tau \leq T$, $0 \leq \lambda \leq \mu < \infty$, and $B, D \in \mathcal{F}$ with $B \subseteq D$. After having this we denote $\widetilde{W}$ again by $W$ and start with the proof.

(a) We fix a stopping time $\sigma \in \mathcal{S}$ and $B \in \mathcal{F}_\sigma$ of positive measure. For $\lambda > 0$ we let

$$\sigma_\lambda := \inf\left\{t \in [\sigma,T]\,\middle|\, |A_t - A_{\sigma-}| > \lambda\right\} \wedge T$$

with $\inf\emptyset := \infty$ and

$$B_\lambda := B \cap \left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}| > \lambda\right\} \in \mathcal{F}_{\sigma_\lambda}.$$

Given $\lambda, \nu > 0$ it is known and standard to derive that

$$\mathbb{P}(B_{\lambda+\nu}) \leq \mathbb{P}\left(B_\lambda \cap \left\{\left|A_{\sigma_{\lambda+\nu}} - A_{\sigma_\lambda-}\right| \geq \nu\right\}\right)$$

so that

$$\begin{aligned}
&\mathbb{P}(B_{\lambda+\nu})\\
&\leq \mathbb{P}\left(B_\lambda \cap \left\{\left|A_{\sigma_{\lambda+\nu}} - A_T\right| > \frac{\nu}{2}\right\}\right) + \mathbb{P}\left(B_\lambda \cap \left\{\left|A_T - A_{\sigma_\lambda-}\right| \geq \frac{\nu}{2}\right\}\right)\\
&\leq \liminf_n \mathbb{P}\left(B_\lambda \cap \left\{\left|A_T - A_{((\sigma_{\lambda+\nu}+\frac{1}{n})\wedge T)-}\right| > \frac{\nu}{2}\right\}\right) + \mathbb{P}\left(B_\lambda \cap \left\{\left|A_T - A_{\sigma_\lambda-}\right| \geq \frac{\nu}{2}\right\}\right)\\
&\leq \liminf_n \left[\theta\mathbb{P}(B_\lambda) + W\left(B_\lambda, \frac{\nu}{2}; (\sigma_{\lambda+\nu} + \frac{1}{n}) \wedge T\right)\right] + \theta\mathbb{P}(B_\lambda) + W\left(B_\lambda, \frac{\nu}{3}; \sigma_\lambda\right)\\
&\leq 2\theta\mathbb{P}(B_\lambda) + 2W\left(B_\lambda, \frac{\nu}{3}; \sigma\right).
\end{aligned}$$

(b) Define $g := \sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}|$, $s := 2\theta \in (0,1)$, and

$$V(C,\nu) := 2\frac{W(B\cap C,\nu/3;\sigma)}{\mathbb{P}(B\cap C)} \quad \text{for} \quad C\in\mathcal{F},$$

where $V(C,\nu) := 0$ if $\mathbb{P}(B\cap C)=0$. Step (a) implies that

$$\mathbb{P}_B(g>\lambda+\nu) \;\le\; \Big[s+V(g>\lambda,\nu)\Big]\mathbb{P}_B(g>\lambda) \quad \text{for} \quad \lambda,\nu>0.$$

Now we iterate this estimate (in the non-weighted case iterations can be found, for example, in [21] (p. 154), [6], and [10]). Exploiting the monotonicity $V(C_1,\nu)\mathbb{P}_B(C_1)\le V(C_2,\nu)\mathbb{P}_B(C_2)$ for $C_1\subseteq C_2$ we deduce

$$\mathbb{P}_B(g>\lambda+N\nu) \le \left[s^N + V(g>\lambda,\nu)\left(\sum_{k=1}^N s^{k-1}\right)\right]\mathbb{P}_B(g>\lambda)$$

for $\lambda,\nu>0$ and $N=1,2,\ldots$ by induction over $N$. Hence

$$\mathbb{P}_B(g>\lambda+N\nu) \le s^N\mathbb{P}_B(g>\lambda) + \frac{1}{1-s}V(\Omega,\nu).$$

For $b := 1\vee(-\log s)^{-1}$ and $\lambda,\mu,\nu>0$ this implies

$$\begin{aligned}\mathbb{P}_B(g>\lambda+\mu\nu) &\le e^{1-(\mu/b)}\mathbb{P}_B(g>\lambda) + \frac{1}{1-s}V(\Omega,\nu)\\ &= e^{1-(\mu/b)}\mathbb{P}_B(g>\lambda) + \frac{2}{1-s}\,\frac{W(B,\nu/3;\sigma)}{\mathbb{P}(B)}\end{aligned}$$

so that Formula (2) follows with $\alpha := 2/(1-s)$ and $a := 3b$.

(c) The *consequently*-part: Since $\lambda\to W(B,\lambda;\sigma)$ is monotone it is measurable so that the right-hand side of Formula (3) makes sense. To prove the inequality we proceed like in [4] (Lemma 7.1): for $\delta,\lambda,\mu>0$ and $B\in\mathcal{F}_\sigma$ of positive measure, Formula (2) (with $\lambda$ replaced by $a\lambda/\delta$ and $\nu$ by $\lambda$) implies

$$\mathbb{P}_B\left(g>\lambda\left(\frac{a+a\mu\delta}{\delta}\right)\right) \le e^{1-\mu}\mathbb{P}_B\left(g>\frac{a\lambda}{\delta}\right) + \alpha\frac{W(B,\lambda;\sigma)}{\mathbb{P}(B)}.$$

Letting $c>0$, we integrate the inequality with respect to $p\lambda^{p-1}d\lambda$, where we integrate the left-hand side over $(0, c\delta/(a+a\mu\delta))$, the first term on the right-hand side over $(0,c\delta/a)$, and the remaining one over $(0,\infty)$. After a change of variables we get

$$\begin{aligned}&\left[\left(\frac{\delta}{a+a\mu\delta}\right)^p - e^{1-\mu}\left(\frac{\delta}{a}\right)^p\right]\int_0^c \mathbb{P}_B(g>\lambda)\,p\lambda^{p-1}d\lambda\\ &\quad\le \alpha\int_0^\infty \frac{W(B,\lambda;\sigma)}{\mathbb{P}(B)}p\lambda^{p-1}d\lambda.\end{aligned}$$

Choosing $\mu = p+1$, $\delta = (e-1)/((e+1)(p+1))$, and sending $c\uparrow\infty$ concludes the proof. □

*Remark 1.* (i) Given $\eta \in (0, 1)$ and analyzing the proof of Theorem 1 one realizes that one can replace Formula (1) by

$$\mathbb{P}_B\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \nu\right) \le \eta + \frac{W(B, \nu; \sigma)}{\mathbb{P}(B)}. \quad (4)$$

(ii) In view of [6] it might be of interest to investigate $\theta \in [1/2, 1)$ in Formula (1) and in Corollary 1 below.

*Example 1.* For $\Phi \in \mathcal{CL}^+(\mathbb{F})$, $B \in \mathcal{F}_\sigma$, $\lambda \ge 0$, and $\sigma \in \mathcal{S}$ we set

$$W_\Phi(B, \lambda; \sigma) := \mathbb{P}\left(B \cap \left\{\sup_{u\in[\sigma,T]} \Phi_u > \lambda\right\}\right).$$

For $\|A\|_{\mathrm{BMO}^\Phi_{0,\theta}} \le 1$, $B \in \mathcal{F}_\sigma$ of positive measure, and $\nu, \varepsilon > 0$ this implies via

$$\begin{aligned}\mathbb{P}_B(|A_T - A_{\sigma-}| > (1+\varepsilon)\nu) &\le \mathbb{P}_B(|A_T - A_{\sigma-}| > (1+\varepsilon)\Phi_\sigma) + \mathbb{P}_B(\Phi_\sigma > \nu)\\ &\le \theta + \mathbb{P}_B(\Phi_\sigma > \nu)\end{aligned}$$

Formula (1) with $W = W_\Phi$. In the same way $\|A\|_{\mathrm{BMO}^{\Phi,*}_{0,\eta}} \le 1$ gives (4) and Theorem 1 applies in both cases. For $\theta \in (0, 1/2)$, $A \in \mathcal{CL}_0(\mathbb{F})$, and $\mu, \nu > 0$ we deduce

$$\mathbb{P}\left(A^*_T > a\mu\nu \|A\|_{\mathrm{BMO}^\Phi_{0,\theta}}\right) \le e^{1-\mu} + \alpha\, \mathbb{P}(\Phi^*_T > \nu) \quad (5)$$

where $a, \alpha > 0$ depend at most on $\theta$. Finally, for $p \in (0, \infty)$ and $\Phi^*_T \in L_p$ the right-hand side of Formula (3) can be computed by

$$\int_0^\infty p\lambda^{p-1} W_\Phi(B, \lambda; \sigma) d\lambda = \int_B \sup_{u\in[\sigma,T]} \Phi^p_u d\mathbb{P}$$

so that

$$\|A^*_T\|_{L_p} \le c(p, \theta)\|\Phi^*_T\|_{L_p}\ \|A\|_{\mathrm{BMO}^\Phi_{0,\theta}}. \quad (6)$$

We come back to the introduced BMO-spaces and need

**Definition 3.** *Given $p, d \in (0, \infty)$ and $\Phi \in \mathcal{CL}^+(\mathbb{F})$ with $\Phi_0 \in L_p(\Omega, \mathcal{F}, \mathbb{P})$, we let $\Phi \in \mathcal{SM}_p(\mathbb{P}, d)$ provided that*

$$\mathbb{E}\Big(\sup_{u\in[\sigma,T]} \Phi^p_u \;\Big|\; \mathcal{F}_\sigma\Big) \le d^p\ \Phi^p_\sigma \;\; a.s. \;\; \text{for all} \;\; \sigma \in \mathcal{S}.$$

It is clear that $\mathcal{SM}_p(\mathbb{P}, d) \subseteq \mathcal{SM}_r(\mathbb{P}, d)$ for $0 < r < p$. Typical elements of $\mathcal{SM}_p(\mathbb{P}, d)$ are given by

*Example 2.* Taking a martingale $(M_t)_{t\in[0,T]} \in \mathcal{CL}^+(\mathbb{F})$, $0 < p < q < \infty$, and $\Phi_t := M_t^{1/q}$ gives $(\Phi_t)_{t\in[0,T]} \in \mathcal{SM}_p(\mathbb{P}, d)$ for some $d > 0$. To verify this we use $\left\|\sup_{k=0,\dots,n} N_k\right\|_{L_\eta} \le c_\eta \|N_0\|_{L_\eta}$ for a non-negative martingale $(N_k)_{k=0}^n$ and $\eta \in (0, 1)$.

**Corollary 1.** *Let $\theta \in (0, 1/2)$, $\eta \in (0, 1)$, $p \in (0, \infty)$, and $\Phi \in \mathcal{SM}_p(\mathbb{P}, d)$.*

(i) *There is a constant $c = c(\theta, \eta, p, d) > 0$ such that*

$$\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}} \sim_c \|\cdot\|_{\mathrm{BMO}_{0,\eta}^{\Phi,*}} \sim_c \|\cdot\|_{\mathrm{BMO}_{p}^{\Phi}} \sim_c \|\cdot\|_{\mathrm{BMO}_{p}^{\Phi,*}} . \tag{7}$$

(ii) *Given $A \in \mathcal{CL}_0(\mathcal{F})$, the finiteness of the quantities in (7) is equivalent to each of the following conditions:*

(C1) *There are constants $\beta, b > 0$ such that*

$$\mathbb{P}_B\left(|A_T - A_{\sigma-}| > b\nu\right) \le \theta + \beta \frac{W_\Phi(B, \nu; \sigma)}{\mathbb{P}(B)}$$

*for all $\nu > 0$, $\sigma \in \mathcal{S}$, and $B \in \mathcal{F}_\sigma$ of positive measure.*

(C2) *There are constants $\alpha, a > 0$ such that*

$$\mathbb{P}_B\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > a\mu\nu\right) \le e^{1-\mu} + \alpha \frac{W_\Phi(B, \nu; \sigma)}{\mathbb{P}(B)}$$

*for all $\mu, \nu > 0$, $\sigma \in \mathcal{S}$, and $B \in \mathcal{F}_\sigma$ of positive measure.*

*Proof.* (a) One easily has that

$$\theta^{\frac{1}{p}} \|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}} \le \|\cdot\|_{\mathrm{BMO}_{p}^{\Phi}} \le \|\cdot\|_{\mathrm{BMO}_{p}^{\Phi,*}} \quad \text{and} \quad \eta^{\frac{1}{p}} \|\cdot\|_{\mathrm{BMO}_{0,\eta}^{\Phi,*}} \le \|\cdot\|_{\mathrm{BMO}_{p}^{\Phi,*}}$$

and that conditions (C1) and (C2) are equivalent because of Theorem 1.

(b) If $\|A\|_{\mathrm{BMO}_{0,\theta}^{\Phi}} \le b$ with $b \in (0, \infty)$, then (C1) holds with $\beta = 1$ by the argument given in Example 1. Continuing here with Theorem 1 and using $\Phi \in \mathcal{SM}_p(\mathbb{P}, d)$, we get $\|A/b\|_{\mathrm{BMO}_{p}^{\Phi,*}} \le c_{(1)}(p, \theta)\, d$ and

$$\|\cdot\|_{\mathrm{BMO}_{p}^{\Phi,*}} \le c_{(1)}(p, \theta)\, d\, \|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}}.$$

Remark 1 (i) yields, via the same route,

$$\|\cdot\|_{\mathrm{BMO}_{p}^{\Phi,*}} \le c'_{(1)}(p, \eta)\, d\, \|\cdot\|_{\mathrm{BMO}_{0,\eta}^{\Phi,*}}.$$

(c) The argument from (b) also shows $\|A\|_{\mathrm{BMO}_{p}^{\Phi,*}} < \infty$ if (C1) is true. □

*Remark 2.* In Corollary 1 one cannot replace $\mathcal{SM}_p(\mathbb{P}, d)$ by $\mathcal{SM}_r(\mathbb{P}, d)$ for $0 < r < p$. This follows from adapting a discrete time example, formulated in [11] without proof, to the continuous time case. Since the proof of this example would exceed the scope of this paper and since this result is not needed here, the example will be presented in a forthcoming paper [8].

Part (ii) of Corollary 1 can be viewed as a John-Nirenberg-type theorem. Next we show that $\Phi$ is determined by $\|\cdot\|_{\mathrm{BMO}_{p}^{\Phi}}$ up to a multiplicative constant under the condition $\mathcal{SM}_p(\mathbb{P}, d)$.

**Theorem 2.** *For $p \in (0, \infty)$, $\Phi \in \mathcal{SM}_p(\mathbb{P}, d)$, and $\Phi' \in \mathcal{SM}_p(\mathbb{P}, d')$ the following assertions are equivalent:*

(i) $\|\cdot\|_{\mathrm{BMO}_p^{\Phi}} \sim_{c_1} \|\cdot\|_{\mathrm{BMO}_p^{\Phi'}}$ *for some* $c_1 \ge 1$.

(ii) *There is some* $c_2 \ge 1$ *such that* $\Phi_t \sim_{c_2} \Phi'_t$ *for* $t \in [0,T]$ *a.s.*

*Proof.* We have to prove $(i) \Rightarrow (ii)$ only because the converse is obvious. For $t_0 \in (0,T]$ we define $A \in \mathcal{CL}_0(\mathbb{F})$ by $A_t := 0$ if $t \in [0,t_0)$ and $A_t := \Phi_{t_0}$ if $t \in [t_0, T]$. Then

$$\mathbb{E}\left(|A_T - A_{\sigma-}|^p|\mathcal{F}_\sigma\right) = \mathbb{E}\left(\chi_{\{\sigma\le t_0\}}\Phi_{t_0}^p \;\middle|\; \mathcal{F}_\sigma\right) \le d^p\Phi_\sigma^p \quad \text{a.s.}$$

for a stopping time $\sigma : \Omega \to [0,T]$. Hence

$$\|A\|_{\mathrm{BMO}_p^{\Phi'}} \le c_1\|A\|_{\mathrm{BMO}_p^{\Phi}} \le c_1 d.$$

But this gives

$$\Phi_{t_0}^p = \mathbb{E}\left(|A_T - A_{t_0-}|^p \;\middle|\; \mathcal{F}_{t_0}\right) \le (c_1 d)^p(\Phi'_{t_0})^p \quad \text{a.s.}$$

and $\Phi_t \le c_1 d\Phi'_t$ for $t \in [0,T]$ a.s. The opposite inequality follows in the same way. □

Finally, we show that we can change the underlying measure in a moderate way. We would like to note that, while changing the measure, we keep the whole process. This is different from other settings (see for example [16] (Chapters 3.1 and 3.3)). We recall

**Definition 4** (cf. [15]). *Let* $Q$ *be a probability measure on* $(\Omega, \mathcal{F})$ *with* $Q \sim \mathbb{P}$ *and* $L(\omega) = (dQ/d\mathbb{P})(\omega) > 0$ *for all* $\omega \in \Omega$. *Given* $v \in (1,\infty)$ *and* $c > 0$, *we let* $Q \in \mathcal{RH}_v(\mathbb{P}, c)$ [1] *provided that* $L \in L_v(\Omega, \mathcal{F}, \mathbb{P})$ *and*

$$\sqrt[v]{\mathbb{E}\left(L^v \;\middle|\; \mathcal{F}_\sigma\right)} \;\le\; c\,\mathbb{E}\left(L \;\middle|\; \mathcal{F}_\sigma\right) \quad \text{a.s.} \quad \text{for all} \quad \sigma \in \mathcal{S}.$$

**Theorem 3.** *For* $p \in (0,\infty)$, $\theta \in (0,1)$, *and* $Q \in \mathcal{RH}_v(\mathbb{P}, c)$ *there is a constant* $d = d(p, \theta, v, c) > 0$ *such that*

$$\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}(Q)} \;\le\; d\;\; \|\cdot\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})}\,.$$

*Proof.* Assuming $\|A\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})} < \infty$, $B \in \mathcal{F}_\sigma$ of positive measure for some $\sigma \in \mathcal{S}$, and $\lambda > 0$, we get that

$$\mathbb{P}_B\left(\frac{|A_T - A_{\sigma-}|}{\Phi_\sigma} > \lambda\|A\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})}\right) \;\le\; \frac{1}{\lambda^p}.$$

Since for $1 = (1/u) + (1/v)$, $Q \in \mathcal{RH}_v(\mathbb{P}, c)$, and $D \in \mathcal{F}$ one knows (cf. [3])

$$Q_B(D) \le c\,\|\mathbb{P}(D|\mathcal{F}_\sigma)\|_\infty^{\frac{1}{u}}\,,$$

we conclude

$$Q_B\left(\frac{|A_T - A_{\sigma-}|}{\Phi_\sigma} > \lambda\|A\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})}\right) \;\le\; c\sqrt[u]{\frac{1}{\lambda^p}}.$$

Taking $\lambda_0 > 0$ with $\theta := c\lambda_0^{-p/u}$ we arrive at $\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}(Q)} \le \lambda_0\;\|\cdot\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})}$. □

Having an estimate for $\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}(Q)}$ one can continue with Theorem 1 because of Example 1 or with Corollary 1.

[1] $\mathcal{RH}$ stands for *reverse Hölder inequality.*

## 3. Approximation error for discretizations of certain stochastic integrals

### *3.1. Setting*

We consider a Black-Scholes option pricing model with time-horizon $T > 0$ and constant volatility $\sigma$ (for simplicity, we let $\sigma = 1$) after discounting under the martingale measure. This means, we take $S = (S_t)_{t\in[0,T]}$ with

$$S_t := \exp\left(W_t - \frac{t}{2}\right)$$

as price-process, where $W = (W_t)_{t\in[0,T]}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $W_0 \equiv 0$ and continuous paths for all $\omega \in \Omega$. Having $W$ we agree that $\mathcal{F}$ is the completion of $\sigma(W_t : t \in [0,T])$ and that $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmentation of the natural filtration of $W$. Assume a Borel-function $f : (0,\infty) \to \mathbb{R}$ with $\mathbb{E} f(S_T)^2 < \infty$ (for example the pay-off of an European contingent claim). To obtain the stochastic integral representation of $f(S_T)$ we find an $\varepsilon > 0$ (see for example [13]) such that

$$F(t, y) := \mathbb{E} f(yS_{T-t}) \quad \text{gives} \quad F \in C^\infty((-\varepsilon, T) \times (0,\infty)) \tag{8}$$

(we may think for a moment that $S_t$ is given on $[0, T+\varepsilon]$) and

$$\frac{\partial F}{\partial t}(t, y) + \frac{y^2}{2}\frac{\partial^2 F}{\partial y^2}(t, y) = 0.$$

Itô's formula implies that

$$F(t, S_t) = \mathbb{E} f(S_T) + \int_{(0,t]} \frac{\partial F}{\partial y}(u, S_u) dS_u \quad \text{for} \quad t \in [0,T) \text{ a.s.}$$

Since $\lim_{t\uparrow T} \|F(t, S_t) - f(S_T)\|_{L_2} = 0$, and (for example) with $\frac{\partial F}{\partial y}(T, y) := 0$ for $y > 0$, we end up with the desired representation

$$f(S_T) = \mathbb{E} f(S_T) + \int_{(0,T]} \frac{\partial F}{\partial y}(u, S_u) dS_u \text{ a.s.}$$

To formulate the approximation problem, we are interested in, we introduce the error-processes $C(\tau)$ and $C(\tau, v)$.

**Definition 5.** (i) *We define*

$$\mathcal{T} := \left\{ (t_i)_{i=0}^n \;\middle|\; 0 = t_0 < \cdots < t_n = T, n = 1, 2, ... \right\}$$

*and* $\mathcal{T}_N := \left\{ (t_i)_{i=0}^n \in \mathcal{T} \;\middle|\; n \le N \right\}$ *for* $N \ge 1$.

(ii) *For* $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ *we let* $\|\tau\|_\infty := \sup_i |t_i - t_{i-1}|$ *and*

$$\mathcal{P}(\tau) := \left\{ v = (v_i)_{i=0}^{n-1} \;\middle|\; v_i : \Omega \to \mathbb{R} \quad \mathcal{F}_{t_i}\text{-measurable}, \ \mathbb{E}\left|v_i S_{t_i}\right|^2 < \infty \right\}.$$

(iii) *Given* $\tau \in \mathcal{T}$ *and* $v \in \mathcal{P}(\tau)$, *we define* $C(\tau, v) = (C_t(\tau, v))_{t\in[0,T]}$ *to be*

$$C_t(\tau, v) := \mathbb{E}\left(f(S_T)|\mathcal{F}_t\right) - \mathbb{E}f(S_T) - \sum_{i=1}^{n} v_{i-1}\left(S_{t_i\wedge t} - S_{t_{i-1}\wedge t}\right),$$

*where all paths of* $C(\tau, v)$ *are assumed to be continuous,* $C_0(\cdot, \cdot) \equiv 0$, *and*

$$C(\tau) := C(\tau, v^0) \quad with \quad v^0 := \left(\frac{\partial F}{\partial y}(t_i, S_{t_i})\right)_{i=0}^{n-1}.$$

In the definition above we used that $\mathbb{E}(S_t \frac{\partial F}{\partial y}(t, S_t))^2 < \infty$ for $t \in [0, T)$ (see for example [13]). The definition of $C(\tau)$ gives

$$C_t(\tau) = \int_{(0,t]} \frac{\partial F}{\partial y}(u, S_u)dS_u - \sum_{i=1}^{n} \frac{\partial F}{\partial y}(t_{i-1}, S_{t_{i-1}})\left(S_{t_i\wedge t} - S_{t_{i-1}\wedge t}\right) \text{ a.s.}$$

which is the error-process of a simple approximation of a stochastic integral. There are two results we would like to start from. The first one is due to Zhang and can be formulated in our setting as follows:

**Theorem 4** **([23]).** *Let* $K : [0, \infty) \to \mathbb{R}$ *be a Borel-function and assume that* $\sup_{x\geq 0}(1 + x)^{-m}|K(x)| < \infty$ *for some* $m \geq 1$. *If*

$$f(y) := \int_0^y K(x)dx, \ y \geq 0,$$

*is not linear and if* $\tau_n := (iT/n)_{i=0}^n$, *then there is a constant* $c = c(K, T) \geq 1$ *such that, for* $n = 1, 2, \ldots$,

$$\frac{1}{c\sqrt{n}} \leq \inf_{v^{(n)}\in\mathcal{P}(\tau_n)} \left\| C_T(\tau_n, v^{(n)}) \right\|_{L_2} \leq \|C_T(\tau_n)\|_{L_2} \leq \frac{c}{\sqrt{n}}.$$

Next, it was shown by Gobet and Temam [14] that certain irregular pay-off functions $f$ do not give the approximation rate $1/\sqrt{n}$ for equidistant nets. After that, in [13] the considerations have been extended to general, not necessarily equidistant, time-nets, which yields to the second result we want to mention explicitly:

**Theorem 5** **([13]).** *Let* $f : (0, \infty) \to \mathbb{R}$ *be a Borel function with* $f(S_T) \in L_2$ *and let* $F$ *be given by formula* (8).

(i) *There is an increasing function* $c : [0, \infty) \to [1, \infty)$ *such that*

$$\frac{1}{c(T)}a(f(S_T); \tau) \leq \inf_{v\in\mathcal{P}(\tau)} \|C_T(\tau, v)\|_{L_2} \leq \|C_T(\tau)\|_{L_2} \leq c(T)\, a(f(S_T); \tau)$$

*for all* $T > 0$, $f$, *and* $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ *where*

$$a(f(S_T); \tau) := \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i - u)\mathbb{E}\left|S_u^2 \frac{\partial F^2}{\partial y^2}(u, S_u)\right|^2 du\right)^{\frac{1}{2}}.$$

(ii) *If there are no constants* $c_0, c_1 \in \mathbb{R}$ *such that* $f(S_T) = c_0 + c_1 S_T$ *a.s., then*

$$\inf_{n=1,2,\ldots} \inf_{\tau \in \mathcal{T}_n} \sqrt{n}\, a(f(S_T); \tau) > 0.$$

Item (i) follows from [13] (Theorem 4.4) and item (ii) from the same paper if one uses there Lemma A.3, Theorem 4.6, and the arguments of the proof of Theorem 6.2. In [13] only non-negative $f : (0, \infty) \to [0, \infty)$ are considered because of their interpretation as pay-off functions, however the proofs are valid for $f : (0, \infty) \to \mathbb{R}$ as well without modification. Applying Theorem 5 to $f(y) = (y - K_0)^+$, $K_0 > 0$, gives

$$\frac{1}{c} a((S_T - K_0)^+; \tau) \;\le\; \left( \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{t_i - u}{\sqrt{T-u}} du \right)^{\frac{1}{2}} \;\le\; c\, a((S_T - K_0)^+; \tau), \tag{9}$$

where $c = c(T, K_0) \ge 1$ (see [13] (Section 5)). Formula (9) shows that in this case there are time nets, quite different from the equidistant ones, realizing the optimal rate $1/\sqrt{n}$ in Theorem 4. For instance, exploiting [13] (proof of Theorem 6.2) one could use

$$\tau_n^{\varepsilon} := \left( T - T \left( \frac{n-i}{n} \right)^{\varepsilon} \right)_{i=0}^{n} \quad \text{for} \quad \varepsilon \in \left( \frac{2}{3}, 1 \right] \tag{10}$$

in order to get

$$\inf_{\tau \in \mathcal{T}_n} a((S_T - K_0)^+; \tau) \sim_{c'} a((S_T - K_0)^+; \tau_n^{\varepsilon}) \sim_{c'} \frac{1}{\sqrt{n}} \tag{11}$$

for some $c' = c'(T, K_0, \varepsilon) \ge 1$. From the above, two problems naturally arise: firstly, is it possible to replace in Theorem 4 the $L_2$-norm by a stronger quantity? This is of particular interest if one wants to control uniformly the distribution of $C_T(\tau)$. Up to now, Theorem 4 only implies that

$$\mathbb{P}\left( |C_T(\tau_n)| > \frac{\lambda}{\sqrt{n}} \right) \le \left( \frac{c_{(4)}}{\lambda} \right)^2, \quad \lambda > 0, \tag{12}$$

which is far from being optimal in general, as we shall see later. Secondly, it is not clear which of the time-nets, giving the order $1/\sqrt{n}$ in Formula (9), is the best one in a certain sense. Both problems are connected to each other and will be approached now.

### *3.2. The basic technical estimate*

We provide the main technical upper estimate for the expected quadratic error in Theorem 6, which will be used later on.

**Theorem 6.** *Assume a Borel-function* $K : [0,\infty) \to \mathbb{R}$*, which is integrable over compact intervals, and define*

$$f(y) := \int_0^y K(x)dx \quad \textit{and} \quad \Psi(y) := yK(y) \;\; \textit{for} \;\; y \geq 0. \tag{13}$$

*Assume that*

$$\mathbb{E}\left|\int_0^{S_T} |K(x)|dx\right|^2 + \mathbb{E}\Psi(S_T)^2 < \infty, \tag{14}$$

$\tau = (t_i)_{i=0}^n \in \mathcal{T}$*, and* $a \in [t_{i_0-1}, t_{i_0})$ *for some* $i_0 \in \{1, ..., n\}$*. Then*

$$\mathbb{E}\left(|C_T(\tau) - C_a(\tau)|^2 \;\middle|\; \mathcal{F}_a\right)$$
$$\leq c^2 \; \|\tau\|_\infty \left[\mathbb{E}\left(\Psi(S_T)^2 \;\middle|\; \mathcal{F}_a\right) + \left(S_a/S_{t_{i_0-1}}\right)^2 \left(\mathbb{E}\left(\Psi(S_T) \;\middle|\; \mathcal{F}_{t_{i_0-1}}\right)\right)^2\right] \; a.s.$$

*where* $c > 0$ *depends on* $T$ *only.*

Let us turn to the proof. Through the whole Section 3.2 we assume that Formulas (13) and (14) are satisfied and that $F$ is given by (8), that means $F(t,y) = \mathbb{E}f(yS_{T-t})$. We shall use the Gaussian measure

$$d\gamma(x) := (2\pi)^{-1/2} \exp\left(-x^2/2\right) dx.$$

**Lemma 1.** *For* $u \in [0,T)$ *and* $y > 0$ *one has*

(i) $y\frac{\partial F}{\partial y}(u,y) = \int_{\mathbb{R}} \Psi\left(ye^{\sqrt{T-u}\eta - \frac{T-u}{2}}\right) d\gamma(\eta)$,

(ii) $y^2\frac{\partial^2 F}{\partial y^2}(u,y) = \int_{\mathbb{R}} \Psi\left(ye^{\sqrt{T-u}\eta - \frac{T-u}{2}}\right) \left[\frac{\eta}{\sqrt{T-u}} - 1\right] d\gamma(\eta)$.

*Proof.* For $u^* := T - u \in (0,T]$ we have (see for example [13] (Lemma A.2))

$$\begin{aligned} y\frac{\partial F}{\partial y}(u,y) &= \mathbb{E}f(yS_{u^*})\frac{W_{u^*}}{u^*}, \\ y^2\frac{\partial^2 F}{\partial y^2}(u,y) &= \mathbb{E}f(yS_{u^*})\left(\frac{W_{u^*}^2}{u^{*2}} - \frac{W_{u^*}}{u^*} - \frac{1}{u^*}\right). \end{aligned} \tag{15}$$

We restrict ourself to assertion (ii), the first one can be verified in the same way. We get

$$\begin{aligned} y^2\frac{\partial^2 F}{\partial y^2}(u,y) &= \mathbb{E}\int_{(0,\infty)} \chi_{[\eta,\infty)}(yS_{u^*})K(\eta)d\eta \left(\frac{W_{u^*}^2}{u^{*2}} - \frac{W_{u^*}}{u^*} - \frac{1}{u^*}\right) \\ &= \int_{(0,\infty)} K(\eta)\mathbb{E}\left[\chi_{[\eta,\infty)}(yS_{u^*})\left(\frac{W_{u^*}^2}{u^{*2}} - \frac{W_{u^*}}{u^*} - \frac{1}{u^*}\right)\right] d\eta \\ &= -\int_{(0,\infty)} K(\eta)\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(y/\eta) - u^*/2}{\sqrt{u^*}}\right)^2\right) \\ &\qquad \times \left(1 + \frac{\log(y/\eta) - u^*/2}{u^*}\right) \frac{d\eta}{\sqrt{u^*}} \end{aligned}$$

where the last equality can be checked by using Formula (15). Substituting $\eta' := -\frac{\log(y/\eta)-(u^*/2)}{\sqrt{u^*}}$, assertion (ii) follows. □

Given $\sigma \geq 0$, $(u, x) \in [0, 1) \times \mathbb{R}$, and $\psi \in L_2(\gamma)$, we let

$$(A_\sigma \psi)(u, x) := \int_{\mathbb{R}} \psi\left(\sqrt{u}x + \sqrt{1-u}\eta\right)\left[\frac{\eta}{\sqrt{1-u}} - \sqrt{\sigma}\right] d\gamma(\eta).$$

The operator is related to the Ornstein-Uhlenbeck semi-group and satisfies

**Lemma 2.** $\|A_\sigma \psi\|_{L_2([0,1)\times\mathbb{R},\lambda\times\gamma)} \leq (1 + \sqrt{\sigma})\, \|\psi\|_{L_2(\mathbb{R},\gamma)}$ *where* $\lambda$ *is the Lebesgue measure on* $[0, 1)$.

*Proof.* Let $(h_m)_{m=0}^{\infty}$ be the orthonormal basis of Hermite polynomials in $L_2(\mathbb{R}, \gamma)$. It is easy to check that

$$(A_\sigma h_m)(u, x) = \sqrt{m}\, u^{\frac{m-1}{2}}\, h_{m-1}(x) - \sqrt{\sigma} u^{\frac{m}{2}}\, h_m(x)$$

for $m = 1, 2, ...$ with $0^0 := 1$ and $(A_\sigma h_0)(u, x) = -\sqrt{\sigma} h_0(x)$ so that the claim follows since the functions $g_m(u, x) := \sqrt{m+1} u^{m/2} h_m(x)$, $m = 0, 1, ...$, form an orthonormal system in $L_2([0, 1) \times \mathbb{R}, \lambda \times \gamma)$. □

**Lemma 3.** *Let* $0 \leq a \leq u < T$, $v := \frac{u-a}{T-a} \in [0, 1)$, $y := e^{\sqrt{u-a}x - \frac{u-a}{2}}$, $x \in \mathbb{R}$, *and* $y_0 > 0$, *where* $y_0 = 1$ *in case of* $a = 0$. *Then*

$$(y_0 y)^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 y) = \frac{(A_{T-a}\psi)(v, x)}{\sqrt{T-a}} \quad \text{with} \quad \psi(x) := \Psi(y_0 e^{\sqrt{T-a}x - \frac{T-a}{2}})$$

*where* $\Psi$ *is given in (13).*

*Proof.* Because of $\mathbb{E}\Psi(S_T)^2 < \infty$ one gets $\psi \in L_2(\gamma)$. The rest follows by a computation and Lemma 1 (ii). □

*Proof of Theorem 6.* We recall that $a \in [t_{i_0-1}, t_{i_0})$ for some $i_0 \in \{1, ..., n\}$. Letting $r_{i_0-1} := a$ and $r_k := t_k$ for $i_0 \leq k \leq n$, we obtain, a.s.,

$$\begin{aligned}
&|C_T(\tau) - C_a(\tau)| \\
&\quad \leq \left| \int_{(a,T]} \frac{\partial F}{\partial y}(u, S_u) dS_u - \sum_{k=i_0}^{n} \frac{\partial F}{\partial y}(r_{k-1}, S_{r_{k-1}}) \left(S_{r_k} - S_{r_{k-1}}\right) \right| \\
&\quad\quad + \left| \left[\frac{\partial F}{\partial y}(a, S_a) - \frac{\partial F}{\partial y}(t_{i_0-1}, S_{t_{i_0-1}})\right] \left(S_{t_{i_0}} - S_a\right) \right| \\
&\quad =: I_1 + I_2.
\end{aligned}$$

We first consider $I_2$. From the proof of [13] (Corollary 4.1) we know that $\mathbb{E}\left((\partial F/\partial y)(t, S_t) S_t\right)^2 < \infty$ for all $t \in [0, T)$ (this was shown for $f(S_T) \geq 0$, but the proof remains the same) so that $\mathbb{E}\,((\partial F/\partial y)(t, S_t)S_b)^2 < \infty$ for $0 \leq t \leq b \leq T$ with $t < T$. Consequently, $\mathbb{E}I_2^2 < \infty$. Moreover, a.s.,

$$\begin{aligned}
&\mathbb{E}\left(I_2^2 \mid \mathcal{F}_a\right) \\
&= \left[\overline{\mathbb{E}}\left|\overline{S}_{t_{i_0}-a} - 1\right|^2\right] S_a^2 \left[\frac{\partial F}{\partial y}(a, S_a) - \frac{\partial F}{\partial y}\left(t_{i_0-1}, S_{t_{i_0-1}}\right)\right]^2 \\
&\leq 2e^T |t_{i_0} - a| \left[\left[S_a \frac{\partial F}{\partial y}(a, S_a)\right]^2 \right.\\
&\qquad\qquad \left. + (S_a/S_{t_{i_0-1}})^2 \left[S_{t_{i_0-1}} \frac{\partial F}{\partial y}\left(t_{i_0-1}, S_{t_{i_0-1}}\right)\right]^2\right] \\
&= 2e^T |t_{i_0} - a| \left[\mathbb{E}\left(\Psi(S_T) \mid \mathcal{F}_a\right)^2 + (S_a/S_{t_{i_0-1}})^2 \mathbb{E}\left(\Psi(S_T) \mid \mathcal{F}_{t_{i_0-1}}\right)^2\right]
\end{aligned}$$

where we have used Lemma 1 (i). Now let us consider $I_1$. Take $y_0 > 0$ where $y_0 = 1$ in case of $a = 0$. Defining $\widetilde{f}(y) := f(y_0 y)$, $\widetilde{T} := T - a$, and $\widetilde{F}(t, y) := \mathbb{E}\widetilde{f}(yS_{\widetilde{T}-t})$ for $(t, y) \in [0, \widetilde{T}) \times (0, \infty)$, we get $\widetilde{f}(S_{\widetilde{T}}) \in L_2$ and $\widetilde{F}(t, y) = F(t + a, y_0 y)$. Applying [13] (Theorem 4.4) (see Theorem 5 of this paper) gives that

$$\begin{aligned}
&\mathbb{E}\left|\left[\widetilde{f}(S_{\widetilde{T}}) - \mathbb{E}\widetilde{f}(S_{\widetilde{T}})\right] - \sum_{k=i_0}^{n} \frac{\partial \widetilde{F}}{\partial y}(r_{k-1} - a, S_{r_{k-1}-a})\left(S_{r_k-a} - S_{r_{k-1}-a}\right)\right|^2 \\
&\leq c(\widetilde{T})^2 \sum_{k=i_0}^{n} \int_{r_{k-1}-a}^{r_k-a} (r_k - a - u)\mathbb{E}\left|S_u^2 \frac{\partial^2 \widetilde{F}}{\partial y^2}(u, S_u)\right|^2 du \\
&\leq c(\widetilde{T})^2 \, \|\tau\|_\infty \int_0^{\widetilde{T}} \mathbb{E}\left|S_u^2 \frac{\partial^2 \widetilde{F}}{\partial y^2}(u, S_u)\right|^2 du.
\end{aligned}$$

Substituting $\widetilde{f}$, $\widetilde{F}$, and $\widetilde{T}$, noticing $1 \leq c(\widetilde{T}) \leq c(T)$, letting $S_t(x) := e^{\sqrt{t}x - \frac{t}{2}}$ for $t \geq 0$ and $x \in \mathbb{R}$, and using Lemmas 3 and 2, gives that

$$\begin{aligned}
&\mathbb{E}\Big|\left[f(y_0 S_{T-a}) - F(a, y_0)\right] \\
&\qquad - \sum_{k=i_0}^{n} \frac{\partial F}{\partial y}(r_{k-1}, y_0 S_{r_{k-1}-a})\left(y_0 S_{r_k-a} - y_0 S_{r_{k-1}-a}\right)\Big|^2 \\
&\leq c(T)^2 \, \|\tau\|_\infty \int_a^T \mathbb{E}\left|(y_0 S_{u-a})^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 S_{u-a})\right|^2 du \\
&= c(T)^2 \, \|\tau\|_\infty \int_a^T \int_{\mathbb{R}} \left[(y_0 S_{u-a}(x))^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 S_{u-a}(x))\right]^2 d\gamma(x) du
\end{aligned}$$

$$
\begin{aligned}
&= c(T)^2 \|\tau\|_\infty \\
&\quad \int_{[a,T)} \int_{\mathbb{R}} \frac{1}{T-a} \Big( A_{T-a} \Psi(y_0 S_{T-a}(\cdot)) \Big)^2 \left( \frac{u-a}{T-a}, x \right) d\gamma(x) du \\
&= c(T)^2 \|\tau\|_\infty \int_{[0,1)} \int_{\mathbb{R}} \Big( A_{T-a} \Psi(y_0 S_{T-a}(\cdot)) \Big)^2 (v, x)\, d\gamma(x) dv \\
&\le c(T)^2 \|\tau\|_\infty (1+\sqrt{T})^2 \int_{\mathbb{R}} \Psi(y_0 S_{T-a}(x))^2 d\gamma(x).
\end{aligned}
$$

But this gives, a.s.,

$$
\begin{aligned}
&\mathbb{E}\left( I_1^2 | \mathcal{F}_a \right) \\
&= \mathbb{E}\left( \left| [f(S_T) - F(a, S_a)] - \sum_{k=i_0}^{n} \frac{\partial F}{\partial y}(r_{k-1}, S_{r_{k-1}}) \left( S_{r_k} - S_{r_{k-1}} \right) \right|^2 \; \Big| \; \mathcal{F}_a \right) \\
&\le c(T)^2 \|\tau\|_\infty (1+\sqrt{T})^2 \mathbb{E}\left( \Psi(S_T)^2 \;\big|\; \mathcal{F}_a \right).
\end{aligned}
$$

Combining the estimates for $\mathbb{E}\left(I_1^2|\mathcal{F}_a\right)$ and $\mathbb{E}\left(I_2^2|\mathcal{F}_a\right)$ we are done. □

*3.3. The case $f$ being a Lipschitz function*

Now we start to measure the size of the error processes $C(\tau, v)$ and $C(\tau)$ from Section 3.1 with respect to the BMO-spaces $\|\cdot\|_{\mathrm{BMO}_p^\Phi}$. Motivated by the pay-off functions of the European Call- and Put-Option, $f(y) = (y - K_0)^+$ and $f(y) = (K_0 - y)^+$, we first consider the case of Lipschitz functions $f$. What is the weight process $\Phi$, we take? As pointed out by Example 4 below we cannot use $\Phi \equiv 1$ so that $\mathrm{BMO}_p^\Phi$ becomes the usual $\mathrm{BMO}_p$-space. Instead of $\Phi \equiv 1$ we shall take $\Phi = S$, the geometric Brownian motion, which is natural in view of Theorem 6. Taking this weight we are allowed to change the underlying measure $\mathbb{P}$ in a moderate way: let $dQ = L d\mathbb{P}$ with

$$
L = e^{M_T - \frac{1}{2}\langle M \rangle_T}
$$

where $M = (M_t)_{t\in[0,T]}$ is a (path-wise continuous) $L_2$-martingale starting in zero such that $\|\langle M\rangle_T\|_{L_\infty(\mathbb{P})} < \infty$. It is standard to check that

$$
S \in \mathcal{SM}_p(\mathbb{P}, d_p) \cap \mathcal{SM}_p(Q, d_p') \tag{16}
$$

for all $p \in (0, \infty)$ and appropriate $d_p, d_p' > 0$ and that

$$
Q \in \mathcal{RH}_v(\mathbb{P}, c_v) \quad \text{and} \quad \mathbb{P} \in \mathcal{RH}_v(Q, c_v')
$$

for all $v \in (1, \infty)$ and appropriate $c_v, c_v' > 0$. Corollary 1 and Theorem 3 imply that

$$
\|\cdot\|_{\mathrm{BMO}_p^S(\mathbb{P})} \sim \|\cdot\|_{\mathrm{BMO}_q^S(Q)}
$$

for all $p, q \in (0, \infty)$. Moreover,

$$\|C_T(\tau)\|_{L_q(Q)} \leq \|C(\tau)\|_{\mathrm{BMO}_q^S(Q)}$$

so that measuring $\|C(\tau)\|_{\mathrm{BMO}_2^S(\mathbb{P})}$ is more restrictive than $\|C_T(\tau)\|_{L_q(Q)}$, for example. In the following we restrict ourself to the measure $\mathbb{P}$ and drop, as done earlier, the dependence on $\mathbb{P}$ in the notation of the BMO and $L_p$-spaces. The following result should be compared with the equivalence proved for the $L_2$-error in Theorem 5.

**Theorem 7.** *Let $K : [0, \infty) \to \mathbb{R}$ be a Borel function, $\sup_{x \geq 0} |K(x)| < \infty$, $f(y) := \int_0^y K(x)dx$, $y \geq 0$, and assume that $f$ is non-linear. Then there is a constant $c = c(T, K) \geq 1$ such that for all $\tau \in \mathcal{T}$ one has*

$$\frac{1}{c}\sqrt{\|\tau\|_\infty} \leq \inf_{v \in \mathcal{P}(\tau)} \|C(\tau, v)\|_{\mathrm{BMO}_2^S} \leq \|C(\tau)\|_{\mathrm{BMO}_2^S} \leq c\sqrt{\|\tau\|_\infty}.$$

For the situation of Theorem 7 it turns out that an asymptotically optimal sequence of time nets $\sigma_n \in \mathcal{T}_n$ is characterized by $\sup_{n=1,2,...} n\|\sigma_n\|_\infty < \infty$ or in other words: the nets have to be uniformly dense like the equidistant nets. One might ask whether it is necessary that $f$ is a Lipschitz function in Theorem 7. This is clarified by the second result we prove in this section:

**Theorem 8.** *Let $f : (0, \infty) \to \mathbb{R}$ be a Borel-function such that $f(S_T) \in L_2$, $\theta \in (0, 1/2)$, and $\tau_n := (iT/n)_{i=0}^n$ for $n = 1, 2, ...$ Then the following assertions are equivalent:*

(i) *There is a constant $c > 0$ such that, for $n = 1, 2, ...,$*

$$\inf_{v^{(n)} \in \mathcal{P}(\tau_n)} \left\| C(\tau_n, v^{(n)}) \right\|_{\mathrm{BMO}_2^S} \leq \frac{c}{\sqrt{n}}.$$

(ii) *There is a constant $c > 0$ such that for all $n = 1, 2, ...$ there are $v^{(n)} \in \mathcal{P}(\tau_n)$ with*

$$\mathbb{P}\left(\left\{\frac{\left|C_T(\tau_n, v^{(n)}) - C_\sigma(\tau_n, v^{(n)})\right|}{S_\sigma} > \frac{c}{\sqrt{n}}\right\} \cap B\right) \leq \theta\mathbb{P}(B)$$

*for all stopping times $\sigma : \Omega \to [0, T]$ and $B \in \mathcal{F}_\sigma$.*

(iii) *There is a Borel function $K : [0, \infty) \to \mathbb{R}$ with $\sup_{x \geq 0} |K(x)| < \infty$ and $d \in \mathbb{R}$ such that $f(y) = d + \int_0^y K(x)dx$ for $\lambda$-almost all $y > 0$, where $\lambda$ is the Lebesgue measure.*

In Stochastic Finance Theorem 8 (ii) can be interpreted as upper bound for the shortfall probability that the expected path-wise error, conditioned on $B \in \mathcal{F}_\sigma$ and weighted by $S_\sigma$, exceeds $c/\sqrt{n}$ at time $T$. Before we turn to the proof of Theorem 8 let us compare Theorems 5 and 7 by an example already mentioned in Section 3.1.

*Example 3.* Let $f_{K_0}(y) := (y - K_0)^+$ with $K_0 > 0$ and $\varepsilon \in (2/3, 1]$. Combining Formula (9), Theorems 5 and 7, and using the nets $\tau_n^\varepsilon$ from Formula (10) together with Formula (11), we can compare the approximation with respect to $L_2$ and $\mathrm{BMO}_2^S$ and get, for $\tau = (t_i)_{i=0}^n \in \mathcal{T}$,

$$\begin{aligned}
\|C_T(\tau)\|_{L_2} &\sim_c \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{t_i - u}{\sqrt{T-u}} du \right)^{\frac{1}{2}}, \\
\|C(\tau)\|_{\mathrm{BMO}_2^S} &\sim_d \sqrt{\|\tau\|_\infty}, \\
\inf_{\sigma \in \mathcal{T}_n} \|C_T(\sigma)\|_{L_2} &\sim_{c'} \|C_T(\tau_n^\varepsilon)\|_{L_2} \sim_{c'} \frac{1}{\sqrt{n}}, \\
\inf_{\sigma \in \mathcal{T}_n} \|C(\sigma)\|_{\mathrm{BMO}_2^S} &\sim_{d'} \|C(\tau_n^1)\|_{\mathrm{BMO}_2^S} \sim_{d'} \frac{1}{\sqrt{n}}
\end{aligned}$$

for some $c, d, d' \geq 1$ depending at most on $(T, K_0)$ and $c' \geq 1$ on $(T, K_0, \varepsilon)$. However, the nets $\tau_n^\varepsilon$ with $\varepsilon \in (2/3, 1)$ are not optimal for $\mathrm{BMO}_2^S$ since

$$\|C(\tau_n^\varepsilon)\|_{\mathrm{BMO}_2^S} \sim_d \sqrt{\|\tau_n^\varepsilon\|_\infty} = \frac{\sqrt{T}}{n^{\frac{\varepsilon}{2}}}.$$

*Proof of Theorem 7 and Theorem 8.*

(a) The upper bound for $\|C(\tau)\|_{\mathrm{BMO}_2^S}$ in Theorem 7 follows by Theorem 6 since

$$\mathbb{E}\left( |C_T(\tau) - C_a(\tau)|^2 \;\middle|\; \mathcal{F}_a \right) \leq c_{(6)}^2 \|\tau\|_\infty \sup_{x \geq 0} |K(x)|^2 (\mathbb{E} S_T^2 + 1) S_a^2 \text{ a.s.}$$

and since we can replace in this inequality $a$ by a stopping time.

(b) The implication $(iii) \Rightarrow (i)$ in Theorem 8 is evident because of step (a).

(c) The equivalence $(i) \Leftrightarrow (ii)$ in Theorem 8 follows from Formula (16) and Corollary 1 (i).

(d) Assume that $f : (0, \infty) \to \mathbb{R}$ is a general Borel function with $f(S_T) \in L_2$ and that $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ and $a \in (t_{i_0 - 1}, t_{i_0})$. As in the proof of Theorem 6 we let $y_0 > 0$ with $y_0 = 1$ if $a = 0$, $\widetilde{f}(y) := f(y_0 y)$, $\widetilde{T} := T - a$, and $\widetilde{F}(t, y) := \mathbb{E}\widetilde{f}(y S_{\widetilde{T} - t}) = F(t + a, y_0 y)$ for $(t, y) \in [0, \widetilde{T}) \times (0, \infty)$. Let $\xi \in \mathbb{R}$. Applying [13] (Corollary 3.3) (which is justified by $\widetilde{f}(S_{\widetilde{T}}) \in L_2$ and the arguments given in the proof of [13] (Corollary 4.1), where $t_{i_0} - a = \widetilde{T}$ in Corollary 3.3 is a straightforward limit case) gives

$$\begin{aligned}
&\mathbb{E} \int_0^{t_{i_0} - a} \left( \frac{\partial \widetilde{F}}{\partial y}(u, S_u) - y_0 \xi \right)^2 S_u^2 du \\
&= \mathbb{E} \left| \int_{(0, t_{i_0} - a]} \left( \frac{\partial \widetilde{F}}{\partial y}(u, S_u) - y_0 \xi \right) dS_u \right|^2 \\
&\geq \frac{1}{c(\widetilde{T})^2} (t_{i_0} - a) \left( \frac{\partial \widetilde{F}}{\partial y}(0, 1) - y_0 \xi \right)^2
\end{aligned}$$

with $1 \le c(\widetilde{T}) \le c(T)$. Consequently,

$$\begin{aligned}
&\mathbb{E}\int_a^{t_{i_0}} \left(\frac{\partial F}{\partial y}(u, y_0 S_{u-a}) - \xi\right)^2 (y_0 S_{u-a})^2 \, du \\
&\ge \frac{1}{c(T)^2}(t_{i_0} - a)\left(\frac{\partial F}{\partial y}(a, y_0) - \xi\right)^2 y_0^2.
\end{aligned}$$

For $v = (v_i)_{i=0}^{n-1} \in \mathcal{P}(\tau)$ and $B \in \mathcal{F}_a$ this implies

$$\begin{aligned}
&\int_B |C_T(\tau, v) - C_a(\tau, v)|^2 \, d\mathbb{P} \\
&\ge \int_B \int_a^{t_{i_0}} \left(\frac{\partial F}{\partial y}(u, S_u) - v_{i_0-1}\right)^2 S_u^2 du d\mathbb{P} \\
&= \int_B \overline{\mathbb{E}} \int_a^{t_{i_0}} \left(\frac{\partial F}{\partial y}(u, S_a \overline{S}_{u-a}) - v_{i_0-1}\right)^2 \left(S_a \overline{S}_{u-a}\right)^2 du d\mathbb{P} \\
&\ge \frac{1}{c(T)^2} \int_B (t_{i_0} - a)\left(\frac{\partial F}{\partial y}(a, S_a) - v_{i_0-1}\right)^2 S_a^2 d\mathbb{P}
\end{aligned}$$

and

$$(t_{i_0} - a)\int_B \left(\frac{\partial F}{\partial y}(a, S_a) - v_{i_0-1}\right)^2 S_a^2 d\mathbb{P} \;\le\; c(T)^2 \|C(\tau, v)\|_{\mathrm{BMO}_2^S}^2 \int_B S_a^2 d\mathbb{P}.$$

Consequently,

$$\|C\,(\tau, v)\|_{\mathrm{BMO}_2^S} \;\ge\; \frac{1}{c(T)}\sqrt{t_{i_0} - a}\left|\frac{\partial F}{\partial y}(a, S_a) - v_{i_0-1}\right| \quad \text{a.s.}$$

and, by considering an appropriate product measure $\mathbb{P} \times \overline{\mathbb{P}}$,

$$\|C\,(\tau, v)\|_{\mathrm{BMO}_2^S} \;\ge\; \frac{1}{c(T)}\sqrt{t_{i_0} - a}\left|\frac{\partial F}{\partial y}(a, S_{t_{i_0-1}}(\omega_0)\overline{S}_{a-t_{i_0-1}}) - v_{i_0-1}(\omega_0)\right| \tag{17}$$

$\overline{\mathbb{P}}$-almost surely for some $\omega_0 \in \Omega$ via Fubini's theorem. By the triangle inequality and the continuity of $(\partial F/\partial y)(a, \cdot)$ we conclude that

$$\|C\,(\tau, v)\|_{\mathrm{BMO}_2^S} \;\ge\; \frac{1}{2c(T)}\sqrt{t_{i_0} - a}\sup_{y_1, y_2 > 0}\left|\frac{\partial F}{\partial y}(a, y_1) - \frac{\partial F}{\partial y}(a, y_2)\right|. \tag{18}$$

(e) Implication $(i) \Rightarrow (iii)$ of Theorem 8: Assuming $n \in \{1, 2, ...\}$, $\tau := \tau_n$, $i_0 := n$, and $a_n := T - \frac{T}{2n}$, Formula (18) and our assumption imply

$$\begin{aligned}
\frac{c}{\sqrt{n}} &\ge \inf_{v^{(n)} \in \mathcal{P}(\tau_n)} \left\|C\left(\tau_n, v^{(n)}\right)\right\|_{\mathrm{BMO}_2^S} \\
&\ge \frac{1}{2c(T)}\sqrt{\frac{T}{2n}}\sup_{y_1, y_2 > 0}\left|\frac{\partial F}{\partial y}(a_n, y_1) - \frac{\partial F}{\partial y}(a_n, y_2)\right|. 
\end{aligned} \tag{19}$$

We know that for all $a \in (0,T)$ there is a $y_a > 0$ such that $(\partial F/\partial y)(a, y_a) = (\partial F/\partial y)(0,1)$. For instance, this follows from the continuity of $(\partial F/\partial y)(a,\cdot)$ and the martingale property of

$$\left(S_t \frac{\partial F}{\partial y}(t,S_t)\right)_{t\in[0,T)} \qquad \text{so that} \qquad \mathbb{E}S_a\left(\frac{\partial F}{\partial y}(a,S_a) - \frac{\partial F}{\partial y}(0,1)\right) = 0$$

(for the martingale property use $\left((\partial/\partial t) + (y^2/2)(\partial^2/\partial y^2)\right)(y(\partial F/\partial y)) = 0$ and $\mathbb{E}\sup_{t\in[0,b]} |S_t(\partial F/\partial y)(t,S_t)|^2 < \infty$ for $b \in [0,T)$ coming from the arguments used for example in [13] (Corollary 4.1)). From Formula (19) we may conclude that

$$\sup_{n=1,2,\ldots}\sup_{y_1>0}\left|\frac{\partial F}{\partial y}(a_n,y_1) - \frac{\partial F}{\partial y}(0,1)\right| \le \frac{2\sqrt{2}cc(T)}{\sqrt{T}}$$

and

$$L := \sup_{n=1,2,\ldots}\sup_{y>0}\left|\frac{\partial F}{\partial y}(a_n,y)\right| < \infty.$$

Now let $A$ be the set of all $y > 0$ such that there are $z_1, z_2, \ldots > 0$ with $|F(a_n,z_n) - f(y)| + |z_n - y| \to 0$. The set is Borel measurable and by almost sure martingale convergence we have that $\mathbb{P}(\omega : S_T(\omega) \in A) = 1$. Hence $\mu_T((0,\infty)\backslash A) = 0$, where $\mu_T$ is the law of $S_T$. Consequently, $(0,\infty)\backslash A$ is of Lebesgue measure zero and for all $y_1, y_2 \in A$ there are positive $z_n^{(i)} \to_n y_i$ with

$$\begin{aligned} |f(y_1) - f(y_2)| &= \lim_n |F(a_n, z_n^{(1)}) - F(a_n, z_n^{(2)})| \le L \lim_n |z_n^{(1)} - z_n^{(2)}| \\ &= L|y_1 - y_2|. \end{aligned}$$

Properly redefining $f$ in $(0,\infty)\backslash A$ and extending to $[0,\infty)$ gives a Lipschitz function $\widetilde{f} : [0,\infty) \to \mathbb{R}$ with Lipschitz-constant $L$ for that a representation like in (iii) is known.

(f) Lower bound for $\|C(\tau,v)\|_{\mathrm{BMO}_2^S}$ in Theorem 7: Again we fix $i_0 \in \{1,\ldots,n\}$ and $a \in (t_{i_0-1}, t_{i_0})$. Formula (18) and Lemma 1 imply

$$\begin{aligned} &\|C(\tau,v)\|_{\mathrm{BMO}_2^S} \\ &\ge \frac{1}{2c(T)}\sqrt{t_{i_0} - a}\sup_{y_1,y_2>0}\mathbb{E}S_{T-a}\left(K(y_1S_{T-a}) - K(y_2S_{T-a})\right) \\ &= \frac{1}{2c(T)}\sqrt{t_{i_0} - a}\,\overline{\mathbb{E}}\,\overline{S}_a\sup_{y_1,y_2>0}\mathbb{E}S_{T-a}\left(K(y_1\overline{S}_aS_{T-a}) - K(y_2\overline{S}_aS_{T-a})\right) \\ &\ge \frac{1}{2c(T)}\sqrt{t_{i_0} - a}\sup_{y_1,y_2>0}\mathbb{E}S_T\left(K(y_1S_T) - K(y_2S_T)\right) \end{aligned}$$

so that

$$\|C(\tau,v)\|_{\mathrm{BMO}_2^S} \ge \frac{1}{2c(T)}\sqrt{\|\tau\|_\infty}\sup_{y_1,y_2>0}\mathbb{E}S_T\left(K(y_1S_T) - K(y_2S_T)\right).$$

It remains to verify that

$$\sup_{y_1,y_2>0} \mathbb{E}S_T\left(K(y_1S_T)-K(y_2S_T)\right)>0. \tag{20}$$

Assuming the contrary implies

$$\mathbb{E}\Psi(yS_T)=y\mathbb{E}S_TK(yS_T)=ay \tag{21}$$

for some $a\in\mathbb{R}$ and all $y>0$. Representing $\Psi\left(e^{\sqrt{T}\xi-\frac{T}{2}}\right)=\sum_{m=0}^{\infty}\alpha_m h_m(\xi)$ in $L_2(\gamma)$, where $(h_m)_{m=0}^{\infty}$ are the normalized Hermite polynomials and where $\sum_{m=0}^{\infty}\alpha_m^2<\infty$, a short computation yields to

$$ae^{\sqrt{T}x}=\mathbb{E}\Psi\left(e^{\sqrt{T}x}S_T\right)=\sum_{m=0}^{\infty}\alpha_m\frac{x^m}{\sqrt{m!}} \qquad \text{so that} \qquad \alpha_m=a\frac{T^{\frac{m}{2}}}{\sqrt{m!}}$$

and $\sum_{m=0}^{\infty}\alpha_m h_m(x)=a\exp(\sqrt{T}x-(T/2))$. This implies that $\Psi(S_T)=aS_T$ a.s. and that $K$ is $\lambda$-a.s. constant on $(0,\infty)$ with $\lambda$ being the Lebesgue measure. But this is a contradiction to our assumption that $f$ is non-linear. Hence Formula (21) cannot be true and we get Formula (20). □

*Remark 3.* The paper in its early preprint-form only considered Theorem 7 for $f_{K_0}(y)=(y-K_0)^+$, $K_0>0$, for which the proof can be done more directly. As an application of this, [12] was showing: given one knows the upper estimate for $\|C(\tau)\|_{\mathrm{BMO}_2^S}$ for all $f_{K_0}$, one can deduce estimates when $K$ in Theorem 7 is non-negative, bounded, and *monotone*. However, this argument does not apply to Theorem 7 in our meanwhile general form (the estimate on $\|C(\tau)\|_{\mathrm{BMO}_2^S}$ for $f_{K_0}$ is not shown in [12], but taken from this paper). Moreover, if we know *a-priori* that $f(y)=\int_0^y K(x)dx$ with a monotone $K$, then the reader can find a further equivalence in [12] which can be added to Theorem 8.

We conclude this section by two examples showing that we cannot use the non-weighted BMO-spaces in our considerations.

*Example 4.* We let

$$\mathrm{BMO}_2:=\mathrm{BMO}_2^{\Phi} \quad \text{with} \quad \Phi_t\equiv 1$$

be the non-weighted BMO-space. Given a Borel function $f:(0,\infty)\to\mathbb{R}$ with $f(S_T)\in L_2$, $\tau\in\mathcal{T}$, $v=(v_i)_{i=0}^{n-1}\in\mathcal{P}(\tau)$, and $a\in(t_{i_0-1},t_{i_0})$, the arguments used for Formula (17) imply that

$$\left|\frac{\partial F}{\partial y}\left(a,S_{t_{i_0-1}}(\omega)\overline{S}_{a-t_{i_0-1}}(\overline{\omega})\right)-v_{i_0-1}(\omega)\right|S_{t_{i_0-1}}(\omega)\overline{S}_{a-t_{i_0-1}}(\overline{\omega})\le d \tag{22}$$

$\mathbb{P}\times\overline{\mathbb{P}}$-a.s., where $d:=(t_{i_0}-a)^{-\frac{1}{2}}c(T)\,\|C(\tau,v)\|_{\mathrm{BMO}_2}\in[0,\infty]$. On this estimate the following two examples are based:

(i) For $f(y) := (y - K_0)^+$, $K_0 > 0$, one has $\|C(\tau, v)\|_{\mathrm{BMO}_2} < \infty$ if and only if

$$\mathbb{P}(v_{i-1} = 1) = 1 \ \text{ for } \ i = 1,..., n. \tag{23}$$

*If-part*: We let $g(y) := f(y) - y$ so that $|g(y)| \le K_0$ and obtain

$$\begin{aligned} C_t(\tau) &= [\mathbb{E}(f(S_T)|\mathcal{F}_t) - \mathbb{E}f(S_T)] - [S_t - 1] \\ &= \mathbb{E}(g(S_T)|\mathcal{F}_t) - \mathbb{E}g(S_T) \ \text{ a.s.} \end{aligned}$$

*Only if-part*: Assume $\mathbb{P}(v_{i_0-1} = 1) < 1$ for some $i_0 \in \{1,..., n\}$ and that $\|C(\tau, v)\|_{\mathrm{BMO}_2}$ is finite. Again by Fubini's theorem there is some $\omega_0 \in \Omega$ such that $v_{i_0-1}(\omega_0) \neq 1$ and, $\overline{\mathbb{P}}$-a.s.,

$$\left| \frac{\partial F}{\partial y} \left(a, S_{t_{i_0-1}}(\omega_0)\overline{S}_{a-t_{i_0-1}}\right) - v_{i_0-1}(\omega_0) \right| S_{t_{i_0-1}}(\omega_0)\overline{S}_{a-t_{i_0-1}} \le d.$$

Hence $\left|\frac{\partial F}{\partial y}(a, y) - v_{i_0-1}(\omega_0)\right| y \le d$ for all $y > 0$ since $(\partial F/\partial y)(a, \cdot)$ is continuous. But this contradicts to the known fact $\lim_{y\to\infty} \frac{\partial F}{\partial y}(a, y) = 1$, which can be checked by Lemma 1. Obviously, the case described by Formula (23) is irrelevant for the purpose of this paper.

(ii) For a general Borel function $f : (0, \infty) \to \mathbb{R}$ with $f(S_T) \in L_2$ and a time net $\tau$ with $\tau \neq \{0, T\}$ one has $\|C(\tau)\|_{\mathrm{BMO}_2} < \infty$ if and only if

$$f(S_T) = c_0 + c_1 S_T \ \text{ a.s. } \ \text{for some } c_0, c_1 \in \mathbb{R}.$$

*If-part*: This follows trivially since $C_t(\tau) = 0$ a.s. for $t \in [0, T]$.
*Only if-part*: We use $v_i := (\partial F/\partial y)(t_i, S_{t_i})$ and fix $i_0 \ge 2$ so that $t_{i_0-1} \in (0, T)$. Formula (22) implies

$$\left| \frac{\partial F}{\partial y} \left(a, S_{t_{i_0-1}}\overline{S}_{a-t_{i_0-1}}\right) - \frac{\partial F}{\partial y} \left(t_{i_0-1}, S_{t_{i_0-1}}\right) \right| S_{t_{i_0-1}}\overline{S}_{a-t_{i_0-1}} \ \le \ d$$

$\mathbb{P} \times \overline{\mathbb{P}}$-a.s. and

$$\left| \frac{\partial F}{\partial y} (a, y_0 y_1) - \frac{\partial F}{\partial y} \left(t_{i_0-1}, y_0\right) \right| y_0 y_1 \ \le \ d \tag{24}$$

for all $y_0, y_1 > 0$ (here we use $t_{i_0-1} > 0$ and again the continuity of $\partial F/\partial y$). If $\sup_{y\ge 1} |(\partial F/\partial y)(a, y)| = \infty$, then we get immediately a contradiction. The contrary gives the existence of $y^{(n)} \ge 1$ with $\lim_n y^{(n)} = \infty$ and

$$\lim_n \frac{\partial F}{\partial y}(a, y^{(n)}) = c_1 \in \mathbb{R}.$$

Letting $y_0 y_1 = y^{(n)}$, Formula (24) implies $(\partial F/\partial y)(t_{i_0-1}, y_0) = c_1$ for all $y_0 > 0$ and $F(t_{i_0-1}, y) = c_0 + c_1 y$ for some $c_0 \in \mathbb{R}$. Now it is known that this implies that $f(y) = c_0 + c_1 y$ for $\lambda$-almost all $y > 0$ (for a probabilistic argument see [13] (Lemma 4.8)).

*3.4. The case $K(S_T) \in L_q$*

In this section we extend by Theorem 9 the error estimate of Theorem 4 from $L_2$ to $L_p$ and we improve Formula (12) by Theorem 10. We shall work under the measure $\mathbb{P}$ only. Letting $p \in (0, \infty)$ and $\Phi \in \mathcal{CL}^+((\mathcal{F}_t)_{t\in[0,T]})$, we define

$$\begin{aligned} a_n^{\text{sim}}(f(S_T) \mid L_p) &:= \inf_{\tau \in \mathcal{T}_n} \|C_T(\tau)\|_{L_p}, \\ a_n^{\text{opt}}(f(S_T) \mid L_p) &:= \inf_{\tau \in \mathcal{T}_n} \inf_{v \in \mathcal{P}(\tau)} \|C_T(\tau, v)\|_{L_p}, \\ a_n^{\text{sim}}(f(S_T) \mid \mathrm{BMO}_2^\Phi) &:= \inf_{\tau \in \mathcal{T}_n} \|C(\tau)\|_{\mathrm{BMO}_2^\Phi}, \\ a_n^{\text{opt}}(f(S_T) \mid \mathrm{BMO}_2^\Phi) &:= \inf_{\tau \in \mathcal{T}_n} \inf_{v \in \mathcal{P}(\tau)} \|C(\tau, v)\|_{\mathrm{BMO}_2^\Phi}. \end{aligned}$$

The superscripts *sim* and *opt* are standing for *simple* approximation and *optimal* approximation, respectively. Throughout the whole subsection we again assume that $K : [0, \infty) \to \mathbb{R}$ is a Borel function, integrable over compact intervals, and

$$f(y) := \int_0^y K(x)dx \quad \text{for} \quad y \geq 0.$$

**Theorem 9.** *Let $2 \leq p < q < \infty$ and assume that*

$$\mathbb{E}\left[\left|\int_0^{S_T} |K(x)|dx\right|^2 + |K(S_T)|^q\right] < \infty \quad \textit{and} \quad \Phi_t := S_t(M_t^*)^{\frac{1}{r}},$$

*where $M_t := 1 + \mathbb{E}(|K(S_T)|^r | \mathcal{F}_t)$* [2] *for some $r \in (p, q)$. If the function $f$ is not linear and if $E \in \{L_p, \mathrm{BMO}_2^\Phi\}$, then one has that*

$$\frac{1}{c\sqrt{n}} \leq a_n^{\text{opt}}(f(S_T) \mid E) \leq a_n^{\text{sim}}(f(S_T) \mid E) \leq \frac{c}{\sqrt{n}}$$

*for $n = 1, 2, ...,$ where $c \geq 1$ depends at most on $(T, K, p, r)$. To obtain the optimal asymptotic approximation rate $1/\sqrt{n}$, equidistant nets can be used.*

*Proof.* (a) Using Hölder's inequality and Doob's maximal inequality one quickly checks that $\Phi_T^* \in L_p$.

(b) Recall $\Psi(y) = yK(y)$. Letting $0 \leq t \leq a \leq T$, the conditional Hölder inequality yields to

$$\mathbb{E}\left(\Psi(S_T)^2 \mid \mathcal{F}_a\right) + (S_a/S_t)^2 \left(\mathbb{E}\left(\Psi(S_T) \mid \mathcal{F}_t\right)\right)^2 \leq c_1^2 \Phi_a^2 \text{ a.s.}$$

where $c_1 = c_1(T, r) > 0$. By Theorem 6 we get

$$\mathbb{E}\left(|C_T(\tau) - C_a(\tau)|^2 \mid \mathcal{F}_a\right) \leq c_{(6)}^2 \|\tau\|_\infty c_1^2 \Phi_a^2 \text{ a.s.} \tag{25}$$

[2] For convenience we assume that all paths of the martingale $(M_t)_{t\in[0,T]}$ are continuous and strictly positive.

for $\tau \in \mathcal{T}$ and $a \in [0, T]$. We can replace in the last inequality $a$ by a stopping time to deduce that

$$a_n^{\mathrm{sim}}(f(S_T) \mid \mathrm{BMO}_2^{\Phi}) \le \|C(\tau_n)\|_{\mathrm{BMO}_2^{\Phi}} \le c_{(6)} c_1 \sqrt{\frac{T}{n}}$$

with $\tau_n := (iT/n)_{i=0}^{n}$ being the equidistant net.

(c) Since our assumption and the continuity of $f$ imply that there are no $c_0, c_1 \in \mathbb{R}$ with $f(S_T) = c_0 + c_1 \, S_T$ a.s., Theorem 5 gives, for some $c_2 = c_2(T, K) > 0$, that

$$\frac{1}{c_2\sqrt{n}} \le a_n^{\mathrm{opt}}(f(S_T) \mid L_2) \le a_n^{\mathrm{opt}}(f(S_T) \mid L_p).$$

(d) To conclude the proof we observe that, for $\tau \in \mathcal{T}$ and $v \in \mathcal{P}(\tau)$,

$$\|C_T(\tau, v)\|_{L_p} \le c \, \|\Phi_T^*\|_{L_p} \, \|C(\tau, v)\|_{\mathrm{BMO}_{0,\frac{1}{4}}^{\Phi}} \le 2c \, \|\Phi_T^*\|_{L_p} \, \|C(\tau, v)\|_{\mathrm{BMO}_2^{\Phi}},$$

where $c = c(p, 1/4) > 0$ comes from Formula (6) and the factor 2 from Corollary 1, so that

$$\|C_T(\tau, v)\|_{L_p} \le c_3 \, \|C(\tau, v)\|_{\mathrm{BMO}_2^{\Phi}}$$

with $c_3 = c_3(T, K, p, r) > 0$. (In order to get the last inequality one can also take classical results without using $\mathrm{BMO}_{0,\theta}^{\Phi}$.) □

*Remark 4.* Letting $2 < q < \infty$, $T = 1$,

$$f(y) := \begin{cases} e^{\frac{1}{2q}\left(\log y + \frac{1}{2}\right)^2} - 1 : y \ge e^{-\frac{1}{2}} \\ 0 : y < e^{-\frac{1}{2}} \end{cases}, \quad \text{and} \quad K(x) := f'(x) \ge 0$$

yields to $K(S_1) \in L_q$, $f(S_1) \in L_2$, but $f(S_1) \notin L_q$. The latter implies for instance that $a_1^{\mathrm{opt}}(f(S_1) \mid L_q) = \infty$. Hence one cannot take $p = q$ in Theorem 9.

**Theorem 10.** *Let $\gamma \ge 0$, $c > 0$, and assume that $|K(x)| \le c[1 + x^\gamma]$ for $x \ge 0$. Then*

$$\limsup_{\lambda \to \infty, \lambda \ge e} \frac{\log \left[ \sup_{\tau \in \mathcal{T}} \mathbb{P}\left( C_T^*(\tau) > \|\tau\|_\infty^{\frac{1}{2}} \lambda \right) \right]}{[\log \lambda]^2} \le -\frac{1}{2T(\gamma + 1)^2}.$$

*For all $\gamma \ge 0$ there are $K$ such that one has equality.*

The theorem improves Formula (12) since the assertion of the theorem is equivalent to: for all $\delta > 0$ there is a $\lambda_0 \ge e$ such that

$$\mathbb{P}\left( C_T^*(\tau) > \|\tau\|_\infty^{\frac{1}{2}} \lambda \right) \le \lambda^{\left[ -\frac{1}{2T(\gamma+1)^2} + \delta \right] \log \lambda} \tag{26}$$

for all $\lambda \ge \lambda_0$ and $\tau \in \mathcal{T}$.

*Proof.* Theorem 10. Fix $r = 4$ and let $\Phi$ be defined as in Theorem 9. We get that

$$\Phi_t \le c_1 (S_t^*)^{\gamma+1} \text{ a.s.}$$

for $t \in [0, T]$ and some $c_1 = c_1(T, c, \gamma) > 0$. Using Formula (25) gives

$$\mathbb{E}\left( |C_T(\tau) - C_\sigma(\tau)|^2 \;\middle|\; \mathcal{F}_\sigma \right) \le c_{(6)}^2 \; \|\tau\|_\infty \, c_2^2 (S_\sigma^*)^{2(\gamma+1)} \text{ a.s.}$$

for all $\tau \in \mathcal{T}$ and stopping times $\sigma : \Omega \to [0, T]$, and a $c_2 = c_2(T, c, \gamma) > 0$. Using Formula (5) with the weight process $\widetilde{\Phi}_t := (S_t^*)^{\gamma+1}$, exploiting

$$\| \cdot \|_{\mathrm{BMO}_{0,1/4}^{\widetilde{\Phi}}} \le 2 \| \cdot \|_{\mathrm{BMO}_2^{\widetilde{\Phi}}} \quad \text{and} \quad \mathbb{P}(S_T^* > \eta) \le \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2T} (\log \eta)^2}$$

for $\eta \ge \exp(\sqrt{T})$ (for the latter we used the reflection principle for the Brownian motion), we continue to

$$\begin{aligned} \mathbb{P}\left( C_T^*(\tau) > 2 c_{(6)} c_2 \, \|\tau\|_\infty^{\frac{1}{2}} \, \mu \nu \right) &\le e^{1-\mu} + \alpha \mathbb{P}\left( \left( S_T^* \right)^{\gamma+1} > \frac{\nu}{a} \right) \\ &\le e^{1-\mu} + \alpha \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2T(\gamma+1)^2} (\log \frac{\nu}{a})^2} \end{aligned}$$

for $\mu > 0$ and $\nu \ge a \exp((\gamma + 1)\sqrt{T})$, where $\alpha, a > 0$ are the absolute constants from Formula (5). Given $\varepsilon \in (0, 1)$, we choose $\nu := a \lambda^{1-\varepsilon}$ and $\mu := \lambda^\varepsilon / a$ to arrive at

$$\mathbb{P}\left( C_T^*(\tau) > 2 c_{(6)} c_2 \, \|\tau\|_\infty^{\frac{1}{2}} \, \lambda \right) \le e^{1-\frac{\lambda^\varepsilon}{a}} + \alpha \sqrt{\frac{2}{\pi}} e^{-\frac{(1-\varepsilon)^2}{2T(\gamma+1)^2} (\log \lambda)^2}$$

whenever $\lambda \ge \exp((\gamma + 1)\sqrt{T}/(1 - \varepsilon))$. Now the first assertion of our theorem follows with $\|\tau\|_\infty^{\frac{1}{2}}$ replaced by $2 c_{(6)} c_2 \, \|\tau\|_\infty^{\frac{1}{2}}$ by $\varepsilon \downarrow 0$ after having estimated the lim sup. To end up, one quickly checks that the factor $2 c_{(6)} c_2$ does not change the limit on the left hand side so that we obtain our assertion as stated.

The examples for equality are given by $f(y) = (K_0 - y)^+$ with $K_0 > 0$ for $\gamma = 0$ and $f(y) = y^{\gamma+1}$ for $\gamma > 0$. To check this we take $\tau = \{0, T\}$ so that

$$C_T(\tau) = f(S_T) - \mathbb{E} f(S_T) - v_0 (S_T - 1) \text{ a.s.}$$

with $v_0 := (\partial F / \partial y)(0, 1)$ and assume $A > 1/(2T(\gamma + 1)^2)$ such that (compare Formula (26))

$$\mathbb{P}(|C_T(\tau)| > \sqrt{T} \lambda) \le e^{-A (\log \lambda)^2}$$

for $\lambda \ge \lambda_0 \ge e$. Knowing this distributional estimate, a straightforward computation yields

$$\sup_{p \in [1,\infty)} e^{-\frac{p}{4A}} \, \|C_T(\tau)\|_{L_p} < \infty.$$

On the other hand, from the definition of $C_T(\tau)$ we may deduce directly that

$$\lim_{p\to\infty} e^{-\frac{T}{2}[p(\gamma+1)^2-(\gamma+1)]} \|C_T(\tau)\|_{L_p} \in (0,\infty)$$

where we use $\|S_T^B\|_{L_p} = \exp((T/2)(pB^2 - B))$ for $B \geq 0$ and $v_0 \neq 0$ in case of $\gamma = 0$. But this gives a contradiction. □

*3.5. Concluding remark*

In view of [14] and [13] it might be of interest to investigate more irregular pay-off functions $f$ than considered here. This is supported by the fact that the basic ingredients taken from [13], in order to handle the error-processes $C(\tau, v)$, are available for all Borel-functions $f$ such that $f(S_T) \in L_2$.

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