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Thick points of super-Brownian motion

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Abstract. We determine for a super-Brownian motion $\{X_t : t \geq 0\}$ in $\mathbb{R}^d$, $d \geq 3$, the precise gauge function φ such that, almost surely on survival up to time t,

$$0 < \liminf_{r\downarrow 0} \sup_{x\in \operatorname{supp} X_t} \frac{X_t(B(x,r))}{\varphi(r)} \leq \limsup_{r\downarrow 0} \sup_{x\in \operatorname{supp} X_t} \frac{X_t(B(x,r))}{\varphi(r)} < \infty,$$

improving a result of Barlow, Evans and Perkins about the most visited sites of super-Brownian motion. We also determine upper and lower bounds for the Hausdorff dimension spectrum of thick points refining the multifractal analysis of super-Brownian motion by Taylor and Perkins. The upper bound, conjectured to be sharp, involves a constant which can be characterized in terms of the upper tails of the associated equilibrium Palm distribution.

1. Introduction and statement of results

1.1. Motivation

Distributions with spatially extremely varying intensity are frequently encountered in the natural sciences. Examples include the distributions of resources in the ground and of galaxies in the universe, models from population biology, or the dissipation of energy in a highly turbulent fluid flow. The *multifractal spectrum* is an important means to evaluate the degree of variation in the intensity of such a spatial distribution, or, in other words, to describe quantitatively the irregularities of a fractal measure μ. The value $f(a)$ of the multifractal spectrum is, loosely speaking, the Hausdorff dimension of the set of points x with local dimension

$$\lim_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = a,$$

where $B(x,r)$ denotes the ball of radius r centred in x. In the mid eighties, a so-called *multifractal formalism* emerged in the physics literature, see e.g. Halsey et al. [HJ86], which makes a prediction of the multifractal spectrum based on large-deviation heuristics. Mathematically rigorous statements can be found, for example, in [Ol00].

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Recently, it turned out that this formalism fails for some important (random) measures, which arise naturally in probability theory. The simplest example is the local time of standard Brownian motion, see Dolgopyat and Sidorov [DS95], or Hu and Taylor [HT97]. In [PT98], Perkins and Taylor investigate the multifractal spectrum of the states X_t of a super-Brownian motion[1] in dimension $d \geq 3$, at a fixed time $t > 0$. In this case the multifractal formalism predicts a trivial spectrum. However, Perkins and Taylor show that albeit the lower part of the spectrum *is* trivial in the sense that there are no points of local dimension exceeding 2,

$$\liminf_{r \downarrow 0} \frac{\log X_t\big(B(x,r)\big)}{\log r} = 2 \text{ for all } x \in \operatorname{supp} X_t,$$

the upper part of the spectrum is *nontrivial*,

$$\dim \left\{ x \in \mathbb{R}^d \,:\, \limsup_{r \downarrow 0} \frac{\log X_t\big(B(x,r)\big)}{\log r} = a \right\} = \frac{8}{a} - 2 \text{ for all } a \in [2,4].$$

Note that points with local dimension exceeding two have exceptionally *small* mass in small centred balls, whereas a local dimension below two would indicate exceptionally *large* mass in such balls. Hence, the results show that whilst there are points which are, in certain radii, exceptionally thin, no points are, when seen on a logarithmic scale, exceptionally thick.

In the present paper we initiate a refined multifractal analysis, which provides deeper insight into the lower part of the spectrum by studying *thick points* defined on a more precise scale. The idea of this finer notion of a *dimension spectrum* of thick points is based on identifying a slowly varying gauge function L defined on a small interval $(0, \varepsilon)$, such that

$$0 < \sup_{x \in \mathbb{R}^d} \limsup_{r \downarrow 0} \frac{X_t(B(x,r))}{r^2 L(r)} < \infty.$$

Note that such an L can only exist if the lower part of the multifractal spectrum is trivial. Having found such a function, we now call a point $x \in \mathbb{R}^d$ *thick* if $\limsup_{r \downarrow 0} X_t(B(x,r))/r^2 L(r) > 0$ and ask, how many thick points there are. We give a partial answer in terms of upper and lower bounds for the Hausdorff dimension spectrum for the thick points, which is the function

$$f(a) = \dim \left\{ x \in \mathbb{R}^d \,:\, \limsup_{r \downarrow 0} \frac{X_t(B(x,r))}{r^2 L(r)} \geq a \right\}.$$

Thick points in this sense were studied for some of the most important random measures of probability theory. Examples include the occupation measure of a stable subordinator, studied by Shieh and Taylor [ST98], the occupation measure of Brownian motion, studied by Dembo, Peres, Rosen and Zeitouni [DP00], [DP01], and the intersection local times of several Brownian paths, studied by Dembo, Peres, Rosen and Zeitouni [DP02], and König and Mörters [KM02].

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[1] In this paper we assume that the reader is familiar with the definition and basic properties of super-Brownian motion. A recommended introduction to the subject is [Et00].

Beyond the numerical values provided by these spectra, each thick points analysis offers new insight into the nature of the investigated process. Our analysis in the case of super-Brownian motion requires a study of the genealogy of the branching process, which is performed by means of Le Gall's Brownian snake. In this way we gain insight into the mechanism how the thick points of a super-Brownian motion arise.

1.2. Formulation of the main results

To formulate our results for the thick points of super-Brownian motion, we introduce a parameter θ as follows: Let X_∞^0 be a random measure distributed according to the Palm distribution in 0 of a canonical cluster of the equilibrium measure of a super-Brownian motion in $\mathbb{R}^d$ for $d \geq 3$. This object is carefully introduced in Subsection 1.3. We define the *upper tail exponent*

$$\theta := \theta(d) := -\limsup_{a\uparrow\infty} \frac{1}{a} \log \mathbb{P}\left\{X_\infty^0\,(B(0,1)) > a\right\}. \tag{1.1}$$

We show in Section 4 that θ is strictly positive and finite. The next theorem is our main result.

Theorem 1.1 (Spectrum of thick points)**.** *Suppose $\{X_t \,:\, t \geq 0\}$ is a super-Brownian motion in dimension $d \geq 3$, started in an arbitrary finite measure, and $t > 0$. Then, conditionally on survival up to time t, almost surely,*

$$0 < \sup_{x\in\mathbb{R}^d} \limsup_{r\downarrow 0} \frac{X_t(B(x,r))}{r^2 \log(1/r)} < \infty, \tag{1.2}$$

and, moreover, for all $0 \leq a \leq \frac{2}{\theta}$,

$$\begin{aligned} 2 - \theta_* a \leq \dim &\left\{x \in \mathbb{R}^d \,:\, \limsup_{r\downarrow 0} \frac{X_t(B(x,r))}{r^2\log(1/r)} \geq a\right\} \\ &\leq 2 - \theta a, \ \textit{almost surely,} \end{aligned} \tag{1.3}$$

where θ is as specified in (1.1) and θ_* is a positive, finite constant.

Remark 1. We conjecture that the upper bound given in (1.3) is sharp, but we are unable to identify θ_* with θ. Also, it is an open problem to identify the exact numerical value of the constant $\theta = \theta(d)$.

Our analysis gives some insight how a super-Brownian motion creates exceptionally large mass in a small ball $B(x,r)$. Look at the path of the backbone particle, which ends in x, and follow this path, starting from x, backwards in time. If we only consider particles which split off the backbone up to a time of order r^2, it turns out that the contribution from these particles is enough to produce a point with $X_t(B(x,r)) \approx r^2 \log(1/r)$. Unfortunately, we have not been able to show that this truncation does not affect constants. If this was the case it would be a major step towards our conjecture. □

Remark 2. It is worth noting that the behaviour of thick points is quite different from the behaviour of typical points. Typical points obey the following law of the iterated logarithm, proved by Dawson and Perkins [DP91, Theorem 5.5]: There exists a positive, finite constant c such that,

$$\limsup_{r\downarrow 0} \frac{X_t(B(x,r))}{r^2 \log\log(1/r)} = c, \qquad \text{for } X_t\text{-almost all } x, \text{ almost surely.}$$

It is easy to show that $c \leq 1/\theta$ with θ defined as in (1.1), and we conjecture that equality holds. □

Finally, we have a look at the balls of fixed small radius containing maximal mass. These balls indicate, loosely speaking, the sites most visited by particles at time t. Barlow, Evans and Perkins [BE91, Theorem 4.7] show that

$$\limsup_{r\downarrow 0} \sup_{x\in \operatorname{supp} X_t} \frac{X_t(B(x,r))}{r^2 \log(1/r)} < \infty.$$

In [BE91] it remains open whether this is sharp. We provide an affirmative answer.

Theorem 1.2 (Most visited sites). *Suppose $\{X_t : t \geq 0\}$ is a super-Brownian motion in dimension $d \geq 3$ started in an arbitrary finite measure. Then, almost surely on survival up to time t,*

$$0 < \liminf_{r\downarrow 0} \sup_{x\in \operatorname{supp} X_t} \frac{X_t(B(x,r))}{r^2 \log(1/r)}. \tag{1.4}$$

Remark 3. The results of Theorems 1.1 and 1.2 hold in supercritical dimensions $d \geq 3$ only. It is a challenging open problem to study analogous questions in the critical dimension $d = 2$. No problem arises in $d = 1$, when super-Brownian motion has a well-understood random density with respect to Lebesgue measure, see e.g. [Et00]. □

1.3. The equilibrium measure and its tail behaviour

In this section we review two natural constructions of the equilibrium Palm measure and highlight its role in the study of the states of super-Brownian motion. Fix $d \geq 3$ and denote by $\mathcal{M}(\mathbb{R}^d)$ the set of locally finite measures on $\mathbb{R}^d$.

A natural construction is based on the long-term behaviour of the super-Brownian motion. For this purpose it is important to view super-Brownian motion as a Markov process $\{X_t : t \geq 0\}$ in time. We consider as a state space the space of *p-tempered measures* for $p \geq 1$,

$$\mathcal{M}_p(\mathbb{R}^d) = \left\{\mu \in \mathcal{M}(\mathbb{R}^d) : \int \varphi_p \, d\mu < \infty\right\},$$

for $\varphi_p(x) = (1 + \|x\|^2)^{-p}$, equipped with the *p-vague topology*, generated by the functionals $\mu \mapsto \int \varphi \, d\mu$ for all $\varphi : \mathbb{R}^d \to [0,\infty)$ satisfying $\sup |\varphi(x)/\varphi_p(x)| < \infty$. Then we can start super-Brownian motion in the d-dimensional Lebesgue measure

$\ell^d \in \mathcal{M}_p(\mathbb{R}^d)$, if $p \in (d/2, d/2+1)$, so that we do not have finite time extinction. It is not hard to check (see e.g. [DP91]) that

$$\lim_{t\to\infty} X_t = X_\infty \text{ in law on } \mathcal{M}_p(\mathbb{R}^d),$$

for some random variable X_∞ on the space $\mathcal{M}_p(\mathbb{R}^d)$.

X_∞ is called the *equilibrium random measure* of the super-Brownian motion. This random measure is infinitely divisible without deterministic part and hence we can associate a *canonical cluster measure* $\mathbb{Q}_\infty$ with X_∞, which means that X_∞ can be constructed from a Poisson random field Π on $\mathcal{M}_p(\mathbb{R}^d)$ with intensity measure $\mathbb{Q}_\infty$ as

$$X_\infty = \sum_{X\in\Pi} X.$$

$\mathbb{Q}_\infty$ is a σ-finite and translation invariant measure on $\mathcal{M}(\mathbb{R}^d)$ and hence we can associate a Palm distribution $\mathbb{Q}_\infty^0$ (in the origin). Recall that the Palm distribution is characterized by the formula

$$\int \mathbb{Q}_\infty(dX) \int X(dx)F(x,X) = \int \ell^d(dx) \int \mathbb{Q}_\infty^0(dX)F(x,X_x),$$

for any measurable $F : \mathbb{R}^d \times \mathcal{M}_p(\mathbb{R}^d) \to [0,\infty)$, where the measures $X_x \in \mathcal{M}_p(\mathbb{R}^d)$ are defined by $X_x(A) = X(A-x)$. We call the probability measure $\mathbb{Q}_\infty^0$ the *equilibrium Palm distribution*.

The approach above requires us to look at super-Brownian motion as a process in time. The equilibrium Palm distribution can however also be defined from the law of a single state of super-Brownian motion (irrespective of the chosen time $t > 0$ or starting measure X_0). This second approach is closer in spirit to the use we make of the equilibrium measure in the following sections.

To follow this approach we note that X_t is infinitely divisible and look at the associated canonical cluster measure $\mathbb{Q}_t$, which again is a σ-finite (but not translation invariant) measure on $\mathcal{M}(\mathbb{R}^d)$. We randomly pick a cluster X and choose a point $x \in \mathbb{R}^d$ at random according to X, then we zoom into that point. More precisely, let, for $r > 0$,

$$\mathbb{Q}^r\big(M\big) := \int \mathbb{Q}_t(dX) \int X(dx) 1_M(X_x^r), \text{ for } M \subset \mathcal{M}(\mathbb{R}^d) \text{ Borel},$$

where the measures $X_x^r \in \mathcal{M}(\mathbb{R}^d)$ are defined by $X_x^r(A) = r^{-2}X(rA-x)$. Then

$$\lim_{r\downarrow 0} \mathbb{Q}^r = \mathbb{Q}_\infty^0 \text{ weakly in } \mathcal{M}(\mathbb{R}^d) \text{ equipped with the vague topology.}$$

Here $\mathbb{Q}_\infty^0$ is again the equilibrium Palm distribution, see [Mo01a] for the proof. Recall that $\mathbb{Q}_\infty^0$ is defined only in dimension $d \geq 3$, which is one of the reasons why our methods break down in lower dimensions.

Remark 4. It is easy to check that a random measure X_∞^0 with distribution $\mathbb{Q}_\infty^0$ has the following scale invariance property. For all $r > 0$ and $A \subset \mathbb{R}^d$, we have that

$$X_\infty^0(A) = \frac{1}{r^2} X_\infty^0(rA) \text{ in law.}$$

This scaling relation, together with the Palm property, shows that the random measure X^0_∞ is *statistically self-similar* in the sense of U. Zähle [Za88]. In other words, X^0_∞ is the natural self-similar object associated with super-Brownian motion. □

1.4. Outline of the paper

In Section 2 we discuss the applications of snake constructions in our proofs. Snake constructions are extremely useful for us, as they give a natural parametrisation of the particles in a super-Brownian motion, which enables us to pick particles without size or location bias. We first provide a snake representation for the equilibrium Palm distribution X^0_∞, which is a variant of Le Gall's Brownian snake construction of super-Brownian motion. We introduce the intuitive *backbone picture*, which gives us the right point of view to tackle the problems related to the genealogical structure of the superprocess. We then see that the backbone picture enables us to define easily truncations of X^0_∞ and the states of super-Brownian motion and to establish a coupling link between these measures. Finally, we recall that the support of the random measure X_t is, via the Brownian snake, parametrised by a Brownian level set, and quote a uniform dimension stability property, which reduces our problem to the investigation of the dimension of a random subset of a Brownian level set.

In Section 3 we link the asymptotics of large integer moments and the upper tail asymptotics of $X^0_\infty(B(0,1))$. We provide rough bounds for large moments of $X^0_\infty(B(0,1))$, using an iteration method very similar to techniques in [LP95]. Only the *lower* bound we establish here enters into our result, the *upper* bound ensures that the upper tail exponent $\theta(d)$ is finite. We then use the couplings established in Section 2 to relate the tail asymptotics of $X^0_\infty(B(0,1))$ to corresponding quantities embedded in the Brownian snake, preparing the proofs of our dimension results.

Section 4 contains the main body of the fractal geometry part of the proof of Theorem 1.1, based on the tail estimates provided in the previous section. The upper bounds follow from a fairly standard first moment technique, for the lower bounds a percolation technique is used. This idea as such is not new in fractal geometry, see e.g. [KPX00] and references therein, but its use for super-Brownian motion is new and the fact that here the thick points are themselves a subset of a highly irregular fractal makes some new ideas necessary. Once these tools are provided, we complete the proof of Theorem 1.1 in Section 5 and we give details of the modifications needed to establish Theorem 1.2.

2. The Brownian snake representation

2.1. The snake representation of the equilibrium Palm distribution

In this section we introduce a path-valued process, essentially Le Gall's Brownian snake, which can be used to give a representation of both the equilibrium Palm distribution and the states of the super-Brownian motion. We first explain the former, which, although implicit in [DP91, LG93], is less well-documented in the literature.

Our starting point is a *contour process*, which encodes the underlying branching structure of X^0_∞. In the current setup this process is a two-sided Brownian motion $\zeta = \{\zeta_s : s \in \mathbb{R}\}$ with $\zeta_0 = 0$, though later we will also encounter other contour processes. We denote by L^t the local time at level t of the contour process ζ. The path space $\mathfrak{P}$ is defined by

$$\mathfrak{P} = \Big\{ f : (-\infty, u] \to \mathbb{R}^d \ : \ f \text{ continuous and } u \in \mathbb{R} \Big\}.$$

The *Brownian snake* with contour-process ζ is the Markov process $W = \{W_s : s \in \mathbb{R}\}$ with state space $\mathfrak{P}$, which (given ζ) has the following transition kernel. Given the state $W_s : (-\infty, \zeta_s] \to \mathbb{R}^d$ at time $s \in \mathbb{R}$, we obtain the state $W_t : (-\infty, \zeta_t] \to \mathbb{R}^d$ at time $t \in \mathbb{R}$ by letting

$$W_t(v) = \begin{cases} W_s(v) & \text{if } v \le m := \inf_{r \in [s,t]} \zeta_r, \\ W_s(m) + B(v - m) & \text{if } m \le v \le \zeta_t, \end{cases}$$

where B is an independent Brownian motion started in the origin. We are interested in the process $W = \{W_s : s \in \mathbb{R}\}$ when the initial distribution on $\mathfrak{P}$ is the law of Brownian motion $W_0 = B : (-\infty, 0] \to \mathbb{R}^d$ starting in $B(0) = 0$. Note that, although we have described W as a Markov process conditional on the contour process ζ, the Brownian snake W is also an *unconditional* Markov process.

Proposition 2.1 (Representation of the equilibrium Palm distribution)**.** *The random measure X^0_∞ given by*

$$X^0_\infty(A) = \int_{-\infty}^{\infty} 1_{\{W_s(\zeta_s) \in A\}} \, L^0(ds) \,, \qquad \textit{for } A \subset \mathbb{R}^d \textit{ Borel,} \tag{2.1}$$

is distributed according to the equilibrium Palm distribution. Alternatively, let $\mathbb{N}_x(dW)$ be the 'law' of the path valued process obtained by running a Brownian snake started in the constant process x with the Itô measure of positive excursions as the 'law' of the contour process. We put

$$Z_t = Z_t[W] = \int_0^{\sigma} 1_{\{W_s(\zeta_s) \in \,\cdot\,\}} \, L^t(ds),$$

where σ is the length of the excursion. For a given (backward Brownian) path $B = \{B(-t) : t \ge 0\}$, $B(0) = 0$, we let Λ be a Poisson random measure with intensity $4\,dt\, \mathbb{N}_{B(t)}(dW)$. Then

$$X^0_\infty(A) = \iint_{-\infty}^{0} Z_{-t}[W](A) \, \Lambda(dtdW), \qquad \textit{for } A \subset \mathbb{R}^d \textit{ Borel.} \tag{2.2}$$

Remark 5. The representation (2.2) is based on the decomposition of the contour process into its excursions above the minimum up to date, which provides a very useful and intuitive representation of X^0_∞: First we are sampling a Brownian path $B = \{B(-t) : t \ge 0\}$ with $B(0) = 0$, which can be interpreted as the path of a *backbone particle*. Focusing on positive (snake) times $s > 0$, let $M_s = \min_{0 \le r \le s} \zeta_r$ be the minimum of the contour process up to time s, so that $(\zeta_s - M_s : s \ge 0)$

is a reflected Brownian motion by Lévy's theorem. The decomposition of this process into excursions above zero (formally given, e.g., in [RY94, XII (2.5)]) corresponds to a decomposition of the contour process into excursions representing the branching structure of offspring born at time $t = -M_s$ from the backbone particle. Inserting the path-valued process into these excursions, we obtain measures $Z_{-t}[W]$ representing the family of a particle split off from the backbone t time units before the present. We will frequently refer to this as the *backbone picture* of X^0_∞. This is very close in spirit to Kallenberg's idea of *backward trees* (see [Ka77]) in a different setting. □

Proof of Proposition 2.1. The equivalence of (2.1) and (2.2) follows from the the decomposition of the contour process into its excursions above the minimum up to date, exactly as in [LG94, Proposition 2.5], taking into account that our contour process is two-sided.

By [DP91, 6.1] the Palm distribution $\mathbb{Q}^0_\infty$ is characterised by

$$\mathbb{Q}^0_\infty\Big[e^{-\int \varphi\, dX^0_\infty}\Big] = \mathbb{E}\Big[e^{-4\int_0^\infty U_t\varphi(B(t))\,dt}\Big],$$
$$\text{for } \varphi : \mathbb{R}^d \to [0,\infty) \text{ smooth and bounded,}$$

where $(U_t : t \geq 0)$ is the nonlinear semigroup associated with the super-Brownian motion, and $(B(t) : t \in \mathbb{R})$ is a Brownian motion with $B(0) = 0$. Because the law of $Z_t[W]$ under $\mathbb{N}_x(dW)$ is the canonical cluster measure for X_t started in the Dirac measure δ_x, we have

$$U_t\varphi(x) = -\log \mathbb{E}_{\delta_x}\big[e^{-\int \varphi\, dX_t}\big] = \int 1 - e^{-\int \varphi\, dZ_t[W]}\, \mathbb{N}_x(dW),$$

and therefore, using (2.2),

$$\begin{aligned}
\mathbb{E}\Big[e^{-\int \varphi\, dX^0_\infty}\Big] &= \mathbb{E}\exp\Big(-\iint_{-\infty}^0 1 - e^{-\int \varphi\, dZ_{-t}[W]}\Lambda(dt\,dW)\Big)\\
&= \mathbb{E}\exp\Big(-4\int_{-\infty}^0 dt \int 1 - e^{-\int \varphi\, dZ_{-t}[W]}\,\mathbb{N}_{B(t)}(dW)\Big)\\
&= \mathbb{E}\Big[e^{-4\int_0^\infty U_t\varphi(B(t))\,dt}\Big],
\end{aligned}$$

as required. □

For the lower bound in Theorem 1.1, we also use a *one-sided version* of X^0_∞. Indeed, define, using the notation from above,

$$\bar{X}^0_\infty := \int_0^\infty 1_{\{W_s(\zeta_s)\in\cdot\}}\, L^0(ds),$$

and note that similar to (2.2) we have, for a Poisson random measure $\bar{\Lambda}$ with intensity $2\,dt\,\mathbb{N}_{B(t)}(dW)$,

$$\bar{X}^0_\infty(A) = \iint_{-\infty}^0 Z_{-t}[W](A)\,\bar{\Lambda}(dt\,dW), \text{ for } A \subset \mathbb{R}^d \text{ Borel.}$$

Observe that X^0_∞ stochastically dominates $\bar{X}^0_\infty$, we write $\bar{X}^0_\infty \leq_{\mathrm{st}} X^0_\infty$ for this fact.

2.2. Genealogical truncation and coupling links

One of the crucial advantages of the backbone picture is that *genealogical truncations* may be defined easily. Let $0 \leq S < T$ be stopping times for the Brownian motion $B = \{B(-t) : t \geq 0\}$. Then we define the *truncated measure* X_T^S consisting of all particles splitting from the backbone particle between times S and T by

$$X_T^S(A) = \iint_{-T}^{-S} Z_{-t}[W](A)\, \Lambda(dt\, dW), \text{ for } A \subset \mathbb{R}^d \text{ Borel.} \tag{2.3}$$

One of our main concerns in Section 3 will be to identify stopping times S, T such that the tail behaviour of X_T^S is comparable to the tail behaviour of X_∞^0.

We now explain how the truncation idea helps us to establish a *coupling link* between the states of a super-Brownian motion and the equilibrium Palm distribution. To prepare this, we recall briefly the Brownian snake representation of the state X_t of a super-Brownian motion started with a Dirac mass δ_0. Let $W = \{W_s : s \geq 0\}$ be the Brownian snake process, however with a reflected Brownian motion $\zeta = \{\zeta_s : s \geq 0\}$ as a contour process and start in the constant path $W_0(t) = 0$ for all $t \leq 0 = \zeta_0$. Let $\tau = \inf\{s > 0 : L^0(s) = 1\}$, where L^t is the local time of the contour process ζ at level t. Then the process $\{X_t : t \geq 0\}$ with

$$X_t = \int_0^\tau 1_{\{W_s(\zeta_s)\in \cdot\}}\, L^t(ds) \tag{2.4}$$

is a super-Brownian motion with $X_0 = \delta_0$, see [LG91]. From this representation one can immediately derive the following fact: Suppose that $W = \{W_s : s \geq 0\}$ is the Brownian snake with a reflected Brownian motion $\zeta = \{\zeta_s : s \geq 0\}$, $\zeta_0 = 1$, as a contour process started with a Brownian path $W_0 \colon (-\infty, 1] \to \mathbb{R}^d$ with $W_0(0) = x \in \mathbb{R}^d$. Then, for

$$\sigma = \inf\{s > 0 : \zeta(s) = 0\}, \tag{2.5}$$

the measure

$$Z_1 = \int_0^\sigma 1_{\{W_s(\zeta_s)\in \cdot\}}\, L^1(ds) \tag{2.6}$$

is a super-Brownian cluster at time one emerging from x and conditioned to survive. Now let $0 \leq t \leq 1$. Define the stopping times $\mathcal{T} := \inf\{s > 0 : \zeta_s = t\}$. Then, the law of

$$\int_0^{\mathcal{T}} 1_{B(0,r)}\big(W_s(\zeta_s) - W_0(1)\big)\, L^1(ds) \tag{2.7}$$

equals the law of the truncated *one-sided* measure $\bar{X}^0_{1-t}(B(0,r))$ defined as in (2.3) with $\bar{\Lambda}$ replacing Λ. This idea will be used in various forms later, see in particular Corollary 3.6.

2.3. The dimension stability property

We conclude this section by recalling a result which is crucial for the use of the Brownian snake in dimension calculations. Given a Brownian snake $W = \{W_s : s \geq 0\}$ with a reflected Brownian motion $\zeta = \{\zeta_s : s \geq 0\}$ as a contour process, let $A \subset [0, \infty)$ be a subset of its time domain. We denote by

$$\widehat{W}(A) = \big\{W_s(\zeta_s) \ : \ s \in A\}$$

the image of A under the snake. This provides a parametrization of the range of super-Brownian motion. Most importantly, for each $t > 0$, almost surely,

$$\widehat{W}\big\{s \in [0, \tau] \ : \ \zeta_s = t\big\} = \operatorname{supp} X_t.$$

Lemma 2.2 (Uniform dimension quadrupling). *Let W be the Brownian snake in dimension $d \geq 2$ with a reflected Brownian motion $\zeta = \{\zeta_s \ : \ s \leq \tau\}$ as a contour process. Then, almost surely, for all $t > 0$ and all $A \subset \{s \in [0, \tau] : \zeta_s = t\}$, we have*

$$\dim \widehat{W}(A) = 4 \dim A.$$

The result is due to Serlet [Se95] for $d \geq 4$ and to Mörters [Mo01b] for $d = 2, 3$.

3. Tail asymptotics

3.1. Rough bounds for large moments

In this section we calculate rough bounds for the limit behaviour of $\mathbb{E}\left[X^0_\infty(B(0, 1))^n\right]$ as n tends to infinity, which help us to identify the tail asymptotics of $X^0_\infty(B(0, 1))$, and of related one-sided and truncated measures. Recall the definitions of $\bar{X}^0_\infty$ and $\bar{X}^S_T$ from Section 2.2. We have the following stochastic domination relations:

$$\bar{X}^S_T \leq_{\mathrm{st}} \bar{X}^0_\infty \leq_{\mathrm{st}} X^0_\infty \quad \text{and} \quad \bar{X}^S_T \leq_{\mathrm{st}} X^S_T \leq_{\mathrm{st}} X^0_\infty. \tag{3.1}$$

Lemma 3.1. *Fix $0 \leq S < T \leq \infty$. Then there exist two strictly positive, finite constants $c_1 = c_1(S, T)$, $c_2 = c_2(S, T)$, such that for all $n \geq 1$, we have*

$$c_1 n^{d/2} c_2^n \leq \frac{1}{n!}\mathbb{E}\left[\bar{X}^S_T(B(0, 1))^n\right] \leq \frac{1}{n!}\mathbb{E}\left[X^0_\infty(B(0, 1))^n\right] \leq (n + 1)\left(\frac{2}{d - 2}\right)^n.$$

Proof of the upper bound in Lemma 3.1. We mimic the calculation in the proof of Lemma 3.1 in [LP95] and use the representation of X^0_∞ by a Brownian snake with a two-sided Brownian motion ζ as a contour process.

$$\begin{aligned}
\frac{1}{n!}\mathbb{E}\big[X^0_\infty(B(0, 1))^n\big] &= \frac{1}{n!}\mathbb{E}\left[\left(\int_{-\infty}^{\infty} 1_{B(0,1)}(W_u(\zeta_u))\, L^0(du)\right)^n\right] \\
&= \sum_{j=0}^{n} \mathbb{E} \int\cdots\int_{u_1<\cdots<u_j<0} \prod_{i=1}^{j} 1_{B(0,1)}(W_{u_i}(\zeta_{u_i}))\, L^0(du_1)\cdots L^0(du_j) \\
&\quad \times \int\cdots\int_{0<u_{j+1}<\cdots<u_n} \prod_{i=j+1}^{n} 1_{B(0,1)}(W_{u_i}(\zeta_{u_i}))\, L^0(du_{j+1})\cdots L^0(du_1)\,.
\end{aligned}$$

By the strong Markov property, we can estimate each term as follows,

$$\mathbb{E} \int_{0<u_{j+1}<\cdots<u_n} \cdots \int \prod_{i=j+1}^{n} 1_{B(0,1)}(W_{u_i}(\zeta_{u_i}))\, L^0(du_{j+1}) \cdots L^0(du_n)$$
$$= \int_{0<u_{j+1}<\cdots<u_{n-1}} \cdots \int \prod_{i=j+1}^{n-1} 1_{B(0,1)}(W_{u_i}(\zeta_{u_i}))$$
$$\times \mathbb{E}^{W_{u_{n-1}}}\left[\int_0^\infty 1_{B(0,1)}(W_u(\zeta_u))\, L^0(du)\right] L^0(du_{j+1}) \cdots L^0(du_{n-1}),$$

where $\mathbb{E}^B$ refers to expectation for a Brownian snake started in the path B. To bound the innermost expectation, recall e.g. from [LP95, (2.6)], that

$$\mathbb{N}_x\left[\int \phi\, dZ_t\right] = P_t\phi(x), \tag{3.2}$$

where $(P_t,\ t \geq 0)$ denotes the transition semigroup of standard Brownian motion and ϕ is some positive, measurable function. We have

$$\mathbb{E}^{W_{u_{n-1}}}\left[\int_0^\infty 1_{B(0,1)}(W_u(\zeta_u))\, L^0(du)\right] = 2\int_{-\infty}^0 \mathbb{N}_{W_{u_{n-1}}(t)}[Z_{-t}(B(0,1))]\, dt,$$

recalling the backbone representation and the first moment of a Poisson process. Using (3.2) and Fubini's theorem, we continue

$$\mathbb{E}^{W_{u_{n-1}}}\left[\int_0^\infty 1_{B(0,1)}(W_u(\zeta_u))\, L^0(du)\right] = 2\int_0^\infty P_s[1_{B(0,1)}](W_{u_{n-1}}(-s))\, ds$$
$$\leq 2\int_0^\infty \sup_{x\in\mathbb{R}^d} P_s[1_{B(0,1)}](x)\, ds = 2\int_0^\infty P_s[1_{B(0,1)}](0)\, ds$$
$$= 2\int_{B(0,1)} G(x)\, dx,$$

where G denotes Green's function. Iterating this argument gives

$$\frac{1}{n!}\mathbb{E}\left[X_\infty^0(B(0,1))^n\right] \leq (n+1)\left[2\int_{B(0,1)} G(x)\, dx\right]^n. \tag{3.3}$$

An easy calculation shows that $2\int_{B(0,1)} G(x)\, dx = 2/(d-2)$, which completes the upper bound. $\square$

We now deal with the lower bound. Observe that by the above stochastic domination results we may assume $S > 0$ and that

$$\mathbb{E}\left[X_\infty^0(B(0,1))^n\right] \geq \mathbb{E}\left[\bar{X}_T^S(B(0,1))^n\right]$$
$$\geq 2\mathbb{E}\int_{-T}^{-S} \int Z_{-t}(B(0,1))^n\, \mathbb{N}_{B(t)}(dW)\, dt, \tag{3.4}$$

where the expectation on the right hand side refers to the sampling of the backbone path $B = \{B(-t) : t \geq 0\}$. Therefore, the computation of suitable lower bounds for the moments of $Z_t(B(0,1))$ under the excursion measure $\mathbb{N}$ is sufficient to finish our argument. The proof of the following lemma consists of a calculation similar to that in the proof of [LP95, Corollary 3.3], using a recursive moment formula and induction.

Lemma 3.2. *Fix $\varepsilon > 0$ and let $t > \varepsilon$. Then there exist positive, finite constants $c_3, c_4 = c_4(\varepsilon)$, such that*

$$\frac{1}{n!}\mathbb{N}_x\left[Z_t(B(0,1))^n\right] \geq c_3 n^{d/2} c_4^{n-1} e^{-2n\frac{|x|^2}{t}} t^{-d/2}. \tag{3.5}$$

Proof. We argue by induction. For $n = 1$, we have, by (3.2),

$$\begin{aligned}\mathbb{N}_x\left[Z_t(B(0,1))\right] &= P_t[1_{B(0,1)}](x) = (2\pi t)^{-d/2}\int_{B(0,1)} e^{-\frac{|x-y|^2}{2t}}\,dy \\ &\geq (2\pi t)^{-d/2}\left[e^{-2|x|^2/t} \wedge e^{-2/t}\right] \geq c_3 t^{-d/2} e^{-2|x|^2/t},\end{aligned}$$

as, for the first inequality, if $|x| \geq 1$, we have $|x-y| \leq 2|x|$, and if $|x| < 1$, then $|x-y| < 2$. In the second inequality, we choose $c_3 = (2\pi)^{-d/2}e^{-2}$. For the induction step, use a moment formula (see e.g. [LP95], Proposition 3.2) and the induction hypothesis, which give

$$\begin{aligned}\mathbb{N}_x\left[Z_t(B(0,1)^n\right] &= 2\sum_{j=1}^{n-1}\binom{n}{j}\int_0^t \mathbb{E}_x\left[\mathbb{N}_{B_{t-s}}\left[Z_s(B(0,1))^j\right]\mathbb{N}_{B_{t-s}}\left[Z_s(B(0,1))^{n-j}\right]\right] ds \\ &\geq 2\sum_{j=1}^{n-1} n! c_3^2 c_4^{n-2}(n-j)^{d/2} j^{d/2}\int_{\varepsilon/2}^t s^{-d}\mathbb{E}_x\left[e^{-2n|B_{t-s}|^2/s}\right] ds.\end{aligned}$$

This already looks promising, we only have to deal with the expectation under the integral. Denote by $(p(t,x,y),\ t \geq 0,\ x, y \in \mathbb{R}^d)$ the transition density of Brownian motion and use the equation of Chapman-Kolmogorov to see

$$\begin{aligned}\mathbb{E}_x\left[e^{-2n\frac{|B_{t-s}|^2}{s}}\right] &= (2\pi s/4n)^{d/2}\int_{\mathbb{R}^d} p(t-s,x,y)p(s/4n,y,0)\,dy \\ &= (2\pi s/4n)^{d/2} p(t - s(1-1/4n), x, 0) \\ &= (2\pi s/4n)^{d/2}(2\pi(t - s(1-1/4n)))^{-d/2} e^{-\frac{|x|^2}{2(t-s(1-1/4n))}} \\ &\geq (s/4n)^{d/2} t^{-d/2} e^{-2n\frac{|x|^2}{t}},\end{aligned}$$

since $2(t - s(1-1/4n))$ for $0 \leq s \leq t$ becomes minimal for $s = t$ (in that case it equals $t/2n$). We now restrict our attention to one term in the sum, for which j and $n-j$ are at least $n/4$. Then

$$\begin{aligned}\mathbb{N}_x\left[Z_t(B(0,1)^n\right] &\geq 2n! c_3^2 c_4^{n-2}(n/4)^d\int_{\varepsilon/2}^t s^{-d}(s/4n)^{d/2} t^{-d/2} e^{-2n\frac{|x|^2}{t}}\,ds \\ &\geq n! c_3 n^{d/2} c_4^{n-2} t^{-d/2} e^{-2n\frac{|x|^2}{t}}\left[4^{-d}4^{-d/2}2c_3\int_{\varepsilon/2}^{\varepsilon} s^{-d/2}\,ds\right],\end{aligned}$$

which ends the proof if we choose $c_4 = 2^{1-3d}c_3\int_{\varepsilon/2}^{\varepsilon} s^{-d/2}\,ds$. □

Proof of the lower bound in Lemma 3.1. Plugging (3.5) into (3.4) for $0 < \varepsilon = S$, we see that

$$\frac{1}{n!}\mathbb{E}\left[\bar{X}_T^S(B(0,1))^n\right] \geq 2\mathbb{E}\left[c_3(n/2)^{d/2}c_4^{n-1}\int_S^T e^{-2n\frac{|B(t)|^2}{t}}\,dt\right] \geq c_1 n^{d/2} c_2^n,$$

if we choose $c_1 = 2^{1-d/2}\frac{c_3}{c_4}\int_S^T P\{|B_t|^2 \leq t\}\,dt$, and $c_2 = c_4 e^{-2}$. □

3.2. A Tauberian theorem and tail estimates

To switch from moment asymptotics to tail asymptotics, we make use of the following Tauberian Theorem, which can be proved easily adapting the arguments of [KM02, Lemma 2.3].

Lemma 3.3 (Tauberian Theorem). *Let Y be any nonnegative random variable, and $\kappa \in \mathbb{R}$. Then*

$$\limsup_{n\to\infty}\frac{1}{n}\log E\left[\frac{Y^n}{n!}\right] = -\kappa \qquad \textit{implies} \qquad \limsup_{a\uparrow\infty}\frac{1}{a}\log P\{Y > a\} = -e^{\kappa}.$$

Now we define

$$\kappa := -\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\mathbb{E}\left[X_\infty^0(B(0,1))^n\right], \tag{3.6}$$

and, as κ is *finite*, our Tauberian Theorem shows that

$$\limsup_{a\uparrow\infty}\frac{1}{a}\log P\{X_\infty^0(B(0,1)) > a\} = -e^{\kappa} = -\theta. \tag{3.7}$$

Corollary 3.4. *Fix $0 \leq S < T < \infty$ and abbreviate $\theta_* = \theta_*(S,T) = e^{c_2(S,T)}$ and $\theta^* = e^{2/(d-2)}$. Then*

$$\begin{aligned}-\theta_* &\leq \limsup_{a\uparrow\infty}\frac{1}{a}\log\mathbb{P}\{\bar{X}_T^S(B(0,1)) > a\}\\ &\leq \limsup_{a\uparrow\infty}\frac{1}{a}\log\mathbb{P}\{X_\infty^0(B(0,1)) > a\} \leq -\theta^*.\end{aligned} \tag{3.8}$$

Remark 6. Corollary 3.4 shows that the tail behaviour of $X_\infty^0(B(0,1))$ and of the one-sided truncated version $\bar{X}_T^S(B(0,1))$ are of the same order. To close the gap between the lower and upper bound in Theorem 1.1 it would suffice to strengthen (3.8) and give converging upper and lower bounds as $S \downarrow 0, T \uparrow \infty$. It is however unclear whether this holds true. □

3.3. Tail estimates for measures embedded in the Brownian snake

The following tail estimates are corollaries to the rough bounds and will be useful in the proof of the lower bounds in Section 5. In what follows, we consider $\theta_* = \theta_*(1/2, 1)$ only. From the scaling properties of X^0_∞, see Remark 4, we get that, for all $r > 0$ and $A \subset \mathbb{R}^d$,

$$\bar{X}^0_1(A) = \frac{1}{r^2}\, \bar{X}^0_{r^2}(rA) \text{ in law,}$$

and this immediately implies the following bound.

Corollary 3.5. *Fix $0 \le \beta \le 2$. Then, for all $a > 0$,*

$$-\theta_* a \le \limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \mathbb{P}\left\{\bar{X}^0_{r^{2-\beta}}(B(0,r)) \ge ar^2 \log(1/r)\right\} \le -\theta^* a.$$

We now look at random measures embedded in a Brownian snake $\{W_s \,:\, s \ge 0\}$ with Brownian motion $\{\zeta_s \,:\, s \ge 0\}$, reflected in 0, as contour process started in $W_0\colon (-\infty, 1] \to \mathbb{R}^d$, $W_0(0) = x \in \mathbb{R}^d$. For $0 \le \beta \le 2$ fix a level $t = 1 - r^{2-\beta}$ and recall the definition of the stopping time $\mathcal{T}$ and (2.7) from Section 2.2.

Corollary 3.6. *For all $a > 0$, we have*

$$\begin{aligned}
-\theta_* a &\le \limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \mathbb{P}\left\{\bar{X}^0_{r^{2-\beta}}(B(0,r)) \ge ar^2 \log(1/r)\right\} \\
&= \limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \\
&\quad \mathbb{P}\left\{\int_0^{\mathcal{T}} 1_{B(0,r)}(W_s(1) - W_0(1))\, L^1(ds) \ge ar^2 \log(1/r)\right\} \le -\theta^* a.
\end{aligned}$$

We conclude this section with a tail estimate used in our upper bounds using the same framework as above. Given $s > 0$ we define $\mathcal{S} = \sup\{u < s\colon \zeta_u = 0\}$, with $\mathcal{S} = 0$ if $\zeta_u > 0$ for all $u \in [0, s]$. Also define $\mathcal{T} = \inf\{u > s\colon \zeta_u = 0\}$, and

$$X^{(s)}_1(A) = \int_{\mathcal{S}}^{\mathcal{T}} 1_A(W_u(\zeta_u))\, L^1(du), \qquad \text{for } A \subset \mathbb{R}^d \text{ Borel.} \tag{3.9}$$

The idea is that $X^{(s)}_1$ is the cluster which contains the particle parametrised by s.

Lemma 3.7. *Fix $s > 0$ and let $L\colon (0,1) \to (0,\infty)$ satisfy $\lim_{r\downarrow 0} L(r) \log(1/r)^2 = 0$. Then, for all $a > 0$,*

$$\begin{aligned}
&\limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \\
&\mathbb{P}\Big\{X^{(s)}_1\big(B(W_s(\zeta_s), r)\big) \ge ar^2 \log(1/r) \,\Big|\, |\zeta_s - 1| < r^2 L(r)\Big\} \le -\theta a.
\end{aligned}$$

Proof. We may, if $\zeta_s < 1$, extend the Brownian path W_s by adding an independent piece, so that it is defined on $(-\infty, 1 \vee \zeta_s]$. Denoting by Y a standard normal random variable we obtain that

$$\begin{aligned}
&\limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \mathbb{P}\left\{ \left| W_s(\zeta_s) - W_s(1) \right| > r\sqrt{L(r)} \log(1/r) \,\middle|\, |\zeta_s - 1| < r^2 L(r) \right\} \\
&\quad \le \limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \mathbb{P}\{|Y| > \log(1/r)\} = -\infty.
\end{aligned}$$

On the other hand, if $|W_s(\zeta_s) - W_s(1)| \le r\sqrt{L(r)}\log(1/r)$, we infer that, given $\varepsilon > 0$,

$$\begin{aligned}
X_1^{(s)}\big(B(W_s(\zeta_s), r)\big) &\le X_1^{(s)}\big(B(W_s(1), r(1 + \sqrt{L(r)}\log(1/r)))\big) \\
&\le X_1^{(s)}\big(B(W_s(1), r(1+\varepsilon)\big),
\end{aligned}$$

for sufficiently small values of $r > 0$. Now, irrespective of the fixed value of ζ_s, we have

$$X_1^{(s)}(B(W_s(1), r(1+\varepsilon)) \le_{\text{st}} X_\infty^0(B(0, r(1+\varepsilon))).$$

Indeed, if $\zeta_s < 1$ the left hand side is stochastically dominated by $X_1^{1-\zeta_s}(B(0, r(1+\varepsilon)))$, and if $\zeta_s \ge 1$ it is dominated by $X_1^0(B(0, r(1+\varepsilon)))$. Hence, we have shown that

$$\begin{aligned}
&\limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \mathbb{P}\Big\{ X_1^{(s)}\big(B(W_s(\zeta_s), r)\big) \ge ar^2 \log(1/r) \,\Big|\, |\zeta_s - 1| < r^2 L(r) \Big\} \\
&\le \limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \mathbb{P}\left\{ X_\infty^0\big(B(0, r(1+\varepsilon))\big) \ge ar^2 \log(1/r) \right\} \\
&= \limsup_{r\downarrow 0} \frac{1}{\log(1/r)} \log \mathbb{P}\left\{ X_\infty^0\big(B(0,1)\big) \ge \frac{a}{(1+\varepsilon)^2} \log(1/r) \right\} \\
&= -\theta \frac{a}{(1+\varepsilon)^2},
\end{aligned}$$

using the scaling relation of Remark 4 in the penultimate step and (1.1) in the final step. The result follows, as $\varepsilon > 0$ was arbitrary. □

4. Limsup-subfractals of Brownian level sets

By results of Evans and Perkins [EP91] the laws of X_t for different finite starting measures and times $t > 0$ are mutually absolutely continuous. Therefore it suffices to give the proof for the case of a super-Brownian motion cluster at time 1 emerging from the point 0. We therefore assume that $\{\zeta_s : s \ge 0\}$ is a reflected Brownian motion started in $\zeta_0 = 1$ acting as a contour process for the Brownian snake $\{W_s : s \ge 0\}$ started in a Brownian path $W_0 \colon (-\infty, 1] \to \mathbb{R}^d$ with $W_0(0) = 0$. Then we define Z_1 by (2.6) and we recall that the support of Z_1 is parametrised by $\{s \in [0, \sigma] : \zeta_s = 1\}$, where σ is defined in (2.5). This setup will be in place for all of this and the next section.

Our proof of the dimension spectrum (1.3) indicates a fairly general method for obtaining the Hausdorff dimension of a class of fractal subsets of Brownian level sets constructed by a *limsup mechanism*. It should be pointed out that the theory of limsup-fractals provided recently by Dembo, Peres, Rosen and Zeitouni [DP00, DP01], and Khoshnevisan, Peres and Xiao [KPX00] cannot be applied here, because the embedding of our fractal into the *random environment* given by the Brownian level sets causes too much dependence between various parts. Still, many of the basic ingredients of their method are retained. New ingredients, in particular, exploit the strong regularity features of the Brownian level sets, see e.g. Proposition 4.4 below.

4.1. The upper bounds

Key ingredients for the upper bound are uniform continuity results for the involved processes, which we now recall. Let $b > 0$. By Lévy's modulus of continuity, for every $c > \sqrt{2}$, there exists a random $\Delta_1(c) > 0$ such that, almost-surely,

$$\frac{|\zeta_s - \zeta_t|}{\sqrt{|s-t|\log(1/|s-t|)}} \leq c \text{ for all } s, t \in [0, b] \text{ with } |s-t| < \Delta_1(c), \quad (4.1)$$

and by the Dawson-Perkins modulus of continuity, for every $c > 2$, there exists a random $\Delta_2(c) > 0$ such that, almost-surely,

$$\frac{|W_s(u) - W_s(v)|}{\sqrt{|u-v|\log(1/|u-v|)}} \leq c \text{ for all}$$
$$s \in [0, b], u, v \in [0, \zeta_s) \text{ with } |u-v| < \Delta_2(c). \quad (4.2)$$

We also need a simple uniform continuity result for the Brownian snake.

Lemma 4.1. *For $c > 2^{7/4}$ there exists $\Delta_3(c) > 0$ such that*

$$\frac{|W_s(\zeta_s) - W_t(\zeta_t)|}{|s-t|^{1/4}\log(1/|s-t|)^{3/4}} \leq c, \qquad \textit{for all } s, t \in [0, b] \textit{ with } |s-t| \leq \Delta_3(c).$$

Proof. Pick $c_1 > \sqrt{2}$ and $c_2 > 2$ such that $\sqrt{2}c_2\sqrt{c_1} < c$. Given $s, t \in [0, b]$ (without loss of generality with $s < t$) let $m \in [s, t]$ be the point where the Brownian motion ζ attains its minimum over $[s, t]$. Now, if $t - s$ is small enough and satisfies in particular $t - s < \Delta_1(c_1)$ and $\max_{[s,t]} \zeta - \min_{[s,t]} \zeta < \Delta_2(c_2)$, we get, recalling the definition of the Brownian snake,

$$\begin{aligned} |W_s(\zeta_s) - W_t(\zeta_t)| &\leq |W_s(\zeta_s) - W_s(\zeta_m)| + |W_t(\zeta_m) - W_t(\zeta_t)| \\ &\leq c_2\sqrt{|\zeta_s - \zeta_m|\log(1/|\zeta_s - \zeta_m|)} \\ &\quad + c_2\sqrt{|\zeta_m - \zeta_t|\log(1/|\zeta_m - \zeta_t|)} \\ &\leq 2c_2\sqrt{c_1}(t-s)^{1/4}\log(1/(t-s))^{1/4}\log(1/\sqrt{2(t-s)})^{1/2} \\ &\leq c\,(t-s)^{1/4}\log(1/(t-s))^{3/4}, \end{aligned} \quad (4.3)$$

which completes the proof. □

Recall the definition (3.9) of the cluster $X_1^{(s)}$ and abbreviate, for $a \geq 0$,

$$F(a) = \left\{ s \in [0, \infty) : \zeta_s = 1 \text{ and } \limsup_{r \downarrow 0} \frac{X_1^{(s)}(B(W_s(1), r))}{r^2 \log(1/r)} \geq a \right\}. \quad (4.4)$$

The following lemma is the main step in the proof of the upper bounds.

Lemma 4.2. *Almost surely, we have*

(i) if $a > 2/\theta$*, then* $F(a) = \emptyset$*,*
(ii) if $a \le 2/\theta$*, then* $\dim F(a) \le \frac{1}{2} - \frac{\theta a}{4}$.

Proof. It suffices to give the bounds for $F(a) \cap [\delta, b]$, for any $\delta, b > 0$. Let $\varepsilon > 0$ and fix $r_n = (1+\varepsilon)^{-n}$, $k_n = r_n(1+\varepsilon)$ and

$$s_n = (\varepsilon/4)^4 r_n^4 \log(1/r_n)^{-7}.$$

Hence we may assume that n is sufficiently large to ensure

$$s_n \le \tfrac{1}{4} r_n^4 \log(1/r_n)^{-6} \log(1/s_n)^{-1}, \tag{4.5}$$

and

$$s_n \le \big(\varepsilon/4\big)^4 r_n^4 \log(1/s_n)^{-3}. \tag{4.6}$$

We introduce a collection of intervals of length s_n covering $[\delta, b]$ by

$$\mathcal{I}_n = \Big\{[ks_n, (k+1)s_n) \,:\, \delta/s_n - 1 \le k \le b/s_n,\ k \in \mathbb{N}\Big\},$$

and define a sub-collection

$$\begin{aligned} \mathcal{J}_n \;=& \Big\{ I = [s,t] \in \mathcal{I}_n \,:\, \big|\zeta_s - 1\big| \le r_n^2 \log(1/r_n)^{-3} \text{ and} \\ & X_1^{(s)}(B(W_s(\zeta_s), k_n)) \ge a(1+\varepsilon)^{-5} k_n^2 \log(1/k_n)\Big\}. \end{aligned}$$

We now use our continuity arguments to show that, almost surely, for sufficiently large n,

$$\begin{aligned} &\Big\{ s \in [\delta, b] \,:\, \zeta_s = 1 \text{ and } X_1^{(s)}\big(B(W_s(1), r_n)\big) \\ &\qquad \ge a(1+\varepsilon)^{-3} r_n^2 \log(1/r_n)\Big\} \subset \bigcup_{I \in \mathcal{J}_n} I\,. \end{aligned} \tag{4.7}$$

Indeed, suppose that n is large enough to ensure $s_n < \Delta_1(2) \wedge \Delta_3(4)$ and $\tilde{s} \in I = [s,t] \in \mathcal{I}_n$ is an element of the set on the left hand side of (4.7). Then, using (4.5),

$$\big|\zeta_s - 1\big| = \big|\zeta_s - \zeta_{\tilde{s}}\big| \le 2\sqrt{s_n \log(1/s_n)} \le r_n^2 \log(1/r_n)^{-3},$$

and, using Lemma (4.1) and (4.6),

$$\begin{aligned} X_1^{(s)}\big(B(W_s(\zeta_s), k_n)\big) &\ge X_1^{(s)}\big(B(W_{\tilde{s}}(1), k_n - 4 s_n^{1/4} \log(1/s_n)^{3/4})\big) \\ &\ge X_1^{(\tilde{s})}\big(B(W_{\tilde{s}}(1), r_n)\big) \ge a(1+\varepsilon)^{-3} r_n^2 \log(1/r_n) \\ &\ge a(1+\varepsilon)^{-5} k_n^2 \log(1/k_n), \end{aligned}$$

hence $s \in I \in \mathcal{J}_n$, showing (4.7). If $\tilde{s} \in F(a) \cap [\delta, b]$, we have

$$\limsup_{n \to \infty} \frac{X_1^{(\tilde{s})}\big(B(W_{\tilde{s}}(1), r_n)\big)}{r_n^2 \log(1/r_n)} \ge \limsup_{r \downarrow 0} \frac{X_1^{(\tilde{s})}\big(B(W_{\tilde{s}}(1), r)\big)}{r^2 \log(1/r)} \Big(\frac{1}{1+\varepsilon}\Big)^2 \ge a(1+\varepsilon)^{-2}.$$

and therefore

$$X_1^{(\tilde{s})}\big(B(W_{\tilde{s}}(1), r_n)\big) \geq a(1+\varepsilon)^{-3} r_n^2 \log(1/r_n), \text{ for infinitely many } n\ .$$

In other words, we have established the covering

$$F(a) \cap [\delta, b] \subset \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \bigcup_{I \in \mathcal{J}_n} I. \tag{4.8}$$

For sufficiently large n, we have, for all $I = [s,t] \in \mathcal{I}_n$, that

$$\mathbb{P}\big\{|\zeta_s - 1| \leq r_n^2 \log(1/r_n)^{-3}\big\} \leq r_n^2,$$

where we use that $|\zeta_s - 1|$ has a density which is bounded, uniformly in $s \geq \delta$. Fix a small $\eta > 0$. For all sufficiently large n and any $I \in \mathcal{I}_n$, by Lemma 3.7,

$$\begin{aligned} \mathbb{P}\big\{I \in \mathcal{J}_n\big\} = \mathbb{P}\big\{&X_1^{(s)}(B(W_s(\zeta_s), k_n)) \geq a(1+\varepsilon)^{-5} k_n^2 \log(1/k_n) | \\ &|\zeta_s - 1| \leq r_n^2 \log(1/r_n)^{-3}\big\} \\ &\times \mathbb{P}\{|\zeta_s - 1| \leq r_n^2 \log(1/r_n)^{-3}\} \\ \leq\ &(1+\varepsilon)^{-2}\, k_n^{2+\theta a(1+\varepsilon)^{-5} - \eta}. \end{aligned}$$

For any $\alpha \in \mathbb{R}$, we thus have

$$\mathbb{E}\Big[\sum_{n=1}^{\infty} \#\mathcal{J}_n s_n^{\alpha}\Big] = \sum_{n=1}^{\infty} s_n^{\alpha} \sum_{I \in \mathcal{I}_n} \mathbb{P}\big\{I \in \mathcal{J}_n\big\} \leq \sum_{n=1}^{\infty} \#\mathcal{I}_n\, s_n^{\alpha}\, k_n^{2+\theta a(1+\varepsilon)^{-5} - \eta}. \tag{4.9}$$

As s_n is at most a constant multiple of $k_n^4 \log(1/k_n)^{-7}$, and $\#\mathcal{I}_n \leq b/s_n$ is no bigger than a constant multiple of $k_n^{-4} \log(1/k_n)^7$, the right hand side is finite if $4\alpha - 2 + \theta a(1+\varepsilon)^{-5} - \eta > 0$. Hence, for all $\alpha > 1/2 - \theta a/4$, we can find $\varepsilon > 0$ and $\eta > 0$ such that

$$\sum_{n=1}^{\infty} \#\mathcal{J}_n s_n^{\alpha} < \infty \text{ almost surely.} \tag{4.10}$$

By (4.8), the set $F(a) \cap [\delta, b]$ can be covered, for each j, by the family $\bigcup_{n \geq j} \mathcal{J}_n$ consisting, for each $n \geq j$, of $\#\mathcal{J}_n$ cubes of sidelength s_n. Hence (4.10) implies that $\dim F(a) \leq 1/2 - \theta a/4$. If the right hand side is negative, $F(a)$ must be empty almost surely, so that both parts of Lemma 4.2 are proved. □

4.2. The lower bounds

For the *lower bounds* we use the method of intersection with an independent percolation limit set. Suppose that an interval $[0, b]$ is fixed and denote by $\mathfrak{D}_n$ the collection of compact dyadic intervals (relative to $[0, b]$) of sidelength $b2^{-n}$. We also let $\mathfrak{D} = \bigcup_{n=0}^{\infty} \mathfrak{D}_n$.

Given $\gamma \in [0, 1]$ we construct a random compact set $\Gamma[\gamma] \subset [0, b]$ inductively as follows: We keep each of the two intervals in $\mathfrak{D}_1$ independently with probability $p = 2^{-\gamma}$. Let $\mathfrak{S}_1$ be the collection of intervals kept in this procedure and pass from $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$ by keeping each interval of $\mathfrak{D}_{n+1}$, which is not contained in a previously rejected interval, independently with probability p. Then the random set

$$\Gamma[\gamma] := \bigcap_{n=1}^{\infty} \bigcup_{I \in \mathfrak{S}_n} I$$

is called a *percolation limit set* in $[0, b]$. The usefulness of percolation limit sets in fractal geometry is due to the following lemma, see [Pe96] or [Ha81] for a proof.

Lemma 4.3. *For every $\gamma \in [0, 1]$ and every Borel set $A \subset [0, b]$ the following properties hold*

(i) if $\dim A < \gamma$*, then almost surely,* $A \cap \Gamma[\gamma] = \emptyset$*,*
(ii) if $\dim A > \gamma$*, then* $A \cap \Gamma[\gamma] \neq \emptyset$ *with positive probability,*
(iii) if $\dim A > \gamma$*, then almost surely* $\dim \big(A \cap \Gamma[\gamma]\big) \leq \dim A - \gamma$ *and, for all* $\varepsilon > 0$*, with positive probability* $\dim \big(A \cap \Gamma[\gamma]\big) \geq \dim A - \gamma - \varepsilon$*.*

Observe that the first part of the lemma gives a *lower* bound γ for the Hausdorff dimension of a set A, once we can show that $A \cap \Gamma[\gamma] \neq \emptyset$ with positive probability. This is the trick we use to prove the lower bound in the dimension spectrum.

The key to the investigation of limsup-subfractals of Brownian level sets is the following regularity property of Brownian level sets, which may be of independent interest.

Proposition 4.4. *Suppose that* $\{\zeta_s : s \geq 0\}$ *is a standard Brownian motion with* $\zeta_0 = 1$ *and* $\Gamma = \Gamma[\gamma] \subset [0, b]$ *an independent percolation limit set on* $[0, b]$*, for some* $\gamma < 1/2$*. Let* $L = \{s \in [0, b] : \zeta_s = 1\}$*, then*

$$\mathbb{P}\Big\{ \dim(\Gamma \cap L) = \tfrac{1}{2} - \gamma \,\Big|\, \dim(\Gamma \cap L) > 0 \Big\} = 1. \tag{4.11}$$

Proof. It certainly suffices to show that, for all $0 < \delta < \eta < \frac{1}{2} - \gamma$,

$$\mathbb{P}\big\{\delta < \dim(\Gamma \cap L) < \eta\big\} = 0.$$

For convenience we extend Γ to the whole halfline $[0, \infty)$ by placing independent copies in each interval $[(k-1)b, kb]$, $k \in \mathbb{N}$. We introduce the filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$ by

$$\mathcal{F}_t = \sigma\big(\zeta_s : s \leq t\big) \vee \sigma\big(\{s \in \Gamma\} : s \leq t\big).$$

We fix an $n \in \mathbb{N}$ for the moment and introduce stopping times

$$\begin{aligned} T_1 &= \inf \big\{s \geq 0 : \zeta_s = 1 \text{ and } s \in \Gamma\big\}, \\ T_{k+1} &= \inf \big\{s \geq T_k + b2^{-n} : \zeta_s = 1 \text{ and } s \in \Gamma\big\}, \text{ for } k \geq 0. \end{aligned}$$

Let $N := N(n) := \max\{k : T_k < b\}$. Given $j \in \mathbb{N}$ we choose the smallest $m \geq n$ such that T_j is in the left half of the dyadic interval of length $b2^{-m}$. We denote by $\tilde{I}_j \in \mathfrak{D}_m$ the m-th stage interval containing T_j and by $I_j \in \mathfrak{D}_{m+1}$ the right half of this interval. Crucially, as $T_j \in \tilde{I}_j$, the dyadic interval $\tilde{I}_j$ is kept in the percolation construction and therefore $\Gamma \cap I_j$ is a percolation limit set in I_j and independent of $\mathcal{F}(T_j)$. The events

$$E_j := \Big\{ \dim(\Gamma \cap L \cap I_j) > \eta \Big\}$$

are $\mathcal{F}(T_{j+1})$-measurable. Moreover, by scaling and using Lemma 4.3 (iii) we see that there is a $p > 0$, which does not depend on n, with the property that

$$\mathbb{P}\Big\{ \dim(\Gamma \cap L \cap I_j) \geq \eta \,\Big|\, \mathcal{F}(T_j) \Big\} \geq p.$$

We clearly have, for all k,

$$\begin{aligned} \mathbb{P}\Big\{\delta < \dim(\Gamma \cap L) < \eta\Big\} \leq \mathbb{P}\Big\{ &\text{ neither of } E_1, \dots, E_k \text{ occurs } \Big\} \\ &+ \mathbb{P}\Big\{ N \leq k \text{ and } \dim(\Gamma \cap L) > \delta \Big\}. \end{aligned}$$

Our strategy is to choose $k = k(n)$ on the right hand side such that both terms get arbitrarily close to zero as $n \to \infty$. Let $k(n) = 2^{\delta n}$. For the first term we easily get

$$\mathbb{P}\Big\{ \text{ neither of } E_1, \dots, E_{k(n)} \text{ occurs } \Big\} \leq (1 - p)^{k(n)} \longrightarrow 0 \text{ as } n \to \infty.$$

To study the second term observe that $N \leq 2k(n)$ implies that there exists a covering of $\Gamma \cap L$ by $2k(n)$ intervals of length $b2^{-n}$. If this happened for arbitrarily large values of n we could infer $\dim(\Gamma \cap L) \leq \delta$. Hence,

$$\begin{aligned} 0 &= \mathbb{P}\Big\{ \liminf_{n\to\infty} \frac{N(n)}{k(n)} \leq 1 \text{ and } \dim(\Gamma \cap L) > \delta \Big\} \\ &\geq \lim_{n\to\infty} \mathbb{P}\Big\{ \text{there is } m \geq n \text{ with } N(m) \leq k(m) \text{ and } \dim(\Gamma \cap L) > \delta \Big\} \\ &\geq \limsup_{n\to\infty} \mathbb{P}\Big\{ N(n) \leq k(n) \text{ and } \dim(\Gamma \cap L) > \delta \Big\}. \end{aligned}$$

We thus infer that $\mathbb{P}\{\delta < \dim(\Gamma \cap L) < \eta\} = 0$ and this completes the proof. □

From the lemma we infer that, almost surely, for all open intervals I,

$$\textit{either } \dim\big(I \cap L \cap \Gamma\big) = 0 \textit{ or } \dim\big(I \cap L \cap \Gamma\big) = \frac{1}{2} - \gamma.$$

We can therefore define a *regularisation* of the random set $\Gamma \cap L$ by putting

$$E := (\Gamma \cap L) \setminus \bigcup \Big\{ I \,:\, I \text{ an open interval with } \dim(I \cap \Gamma \cap L) = 0 \Big\}.$$

The regularity features of E are,

- for any open interval I which intersects E, we have $\dim(E \cap I) = \frac{1}{2} - \gamma$;
- $E \neq \emptyset$ with positive probability;
- E is compact.

The main lemma of this section is the following.

Lemma 4.5. *Fix $a \in [0, 2/\theta_*]$ and let $\gamma = \frac{1}{2} - \theta_* a/4$. Define, for $N \in \mathbb{N}$, the open set*

$$U(N) := \Big\{ s \in E \;:\; \text{there exists } 0 < r < 2^{-N} \text{ with } X_1^{(s)}\big(B(W_s(\zeta_s), r)\big)\big) > \big(a - \tfrac{1}{N}\big) r^2 \log(1/r) \Big\}.$$

Then, almost surely, $U(N)$ is dense in E.

Once this is established, the following lemma can be inferred easily with arguments analogous to the argument in [KPX00]. Recall the definition of $F(a)$ from (4.4).

Lemma 4.6. *Suppose $a \in [0, 2/\theta_*]$ and $\gamma = \frac{1}{2} - \theta_* a/4$. Then*

$$\mathbb{P}\big\{F(a) \cap \Gamma[\gamma] \neq \emptyset\big\} = \mathbb{P}\big\{E \neq \emptyset\big\} > 0.$$

Hence, $\dim F(a) \geq \gamma$ *with positive probability.*

Proof. Recall from Lemma 4.5 that, almost surely, $U(N)$ is open and dense in the compact set E. By *Baire's Theorem*

$$\bigcap_{N=1}^{\infty} U(N) = F(a) \cap E$$

is dense in E. Hence, $\mathbb{P}\big\{F(a) \cap \Gamma[\gamma] \neq \emptyset\big\} = \mathbb{P}\big\{E \neq \emptyset\big\} > 0$, and Lemma 4.3 (i) yields $\dim F(a) \geq \gamma$ with positive probability. □

The rest of this section is devoted to the proof of Lemma 4.5. We let $\mathcal{G} = \{\mathcal{G}_t \,:\, t \geq 0\}$ be the right continuous filtration generated by the contour process, and $\mathcal{F} = \{\mathcal{F}_t \,:\, t \geq 0\}$ be the filtration given by $\mathcal{F}_t = \mathcal{G}_t \vee \sigma(\Gamma)$. By the right continuity of $\mathcal{G}$ we have $\{s \in E\} \in \mathcal{F}_s$.

Let $I = (b_0, b_1) \subset [0, b]$ be an open interval, and let $N \in \mathbb{N}$. We need to show that almost surely $E \cap I \neq \emptyset$ implies $U(N) \cap I \neq \emptyset$. Let $\varepsilon > 0$, which we shall fix appropriately later in the proof.

For the moment, fix $n \in \mathbb{N}$, and let $r_n = 2^{-n}$. Define two series of $\mathcal{F}$-stopping times by letting $\mathcal{T}_0 = \inf(E \cap I)$ and, for $k \geq 1$,

$$\mathcal{S}_k = \inf\big\{s \geq \mathcal{T}_{k-1} \,:\, \zeta_s = 1 - r_n^{2-\varepsilon}\big\}, \text{ and } \mathcal{T}_k = \inf\big\{s \geq \mathcal{S}_k \,:\, s \in E\big\}.$$

Let $M = M(n) = \max\{k \in \mathbb{N} \,:\, \mathcal{S}_k < b_1\}$ the number of downcrossings by the contour process of the interval $[1 - r_n^{2-\varepsilon}, 1]$ in the time interval I.

Observe that the path-valued processes

$$\Big((W_{\mathcal{T}_k+s}(1-r_n^{2-\varepsilon}+t) - W_{\mathcal{T}_k}(1) \,:\, t \in [0, \zeta_{\mathcal{T}_k+s} - 1 + r_n^{2-\varepsilon})) \,:\, s \in [0, \mathcal{S}_{k+1} - \mathcal{T}_k]\Big),$$

for $k \geq 1$, are independent and equal in law to a Brownian snake started in a Brownian motion path of length $r_n^{2-\varepsilon}$ ending in 0, and stopped upon hitting the zero path. Therefore, for every $k \in \mathbb{N}$, we may define the *independent* events

$$E_k := E_k(n) := \left\{ \int_{\mathcal{T}_k}^{\mathcal{S}_{k+1}} 1_{B(0,r_n)}\big(W_s(\zeta_s) - W_{\mathcal{T}_k}(1)\big)\, L^1(ds) > r_n^2 \log(1/r_n)\big(a - \tfrac{1}{N}\big) \right\}.$$

From Corollary 3.6 we know that $\mathbb{P}(E_k) \geq r_n^{\theta_*(a-\frac{1}{2N})}$. We use the features of E to get, for all $m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}\Big\{\text{neither of } E_1, \ldots, E_M \text{ holds and } E \cap I \neq \emptyset\Big\} \\ \leq \mathbb{P}\Big\{\text{neither of } E_1, \ldots, E_m \text{ holds }\Big\} \qquad (4.12) \\ + \mathbb{P}\Big\{M \leq m \text{ and } \dim(E \cap I) \geq \tfrac{1}{2} - \gamma\Big\}. \qquad (4.13) \end{aligned}$$

The strategy for the remainder of the proof is to choose m dependent on n as

$$m(n) := \left\lfloor r_n^{-\theta_*(a-\frac{1}{3N})} \right\rfloor,$$

where $\lfloor\,\rfloor$ denotes the integer part of a positive real, and show that both (4.12) and (4.13) vanish as $n \to \infty$. Once this is done we can infer that

$$\begin{aligned} \mathbb{P}\Big\{I \cap U(N) \neq \emptyset\Big\} &\geq \limsup_{n\to\infty} \mathbb{P}\Big(\big\{E \cap I \neq \emptyset\big\} \cap \big(E_1(n) \cup \ldots \cup E_{M(n)}(n)\big)\Big) \\ &\geq \limsup_{n\to\infty} \Big[\mathbb{P}\Big\{E \cap I \neq \emptyset\Big\} - \mathbb{P}\Big\{\text{ neither of } E_1(n), \ldots, E_{m(n)}(n) \text{ holds }\Big\} \\ &\quad - \mathbb{P}\Big\{M(n) \leq m(n) \text{ and } E \cap I \neq \emptyset\Big\}\Big] \\ &= \mathbb{P}\Big\{E \cap I \neq \emptyset\Big\}, \end{aligned}$$

and this is the required statement.

Estimate of (4.12). Note that the events $E_1(n), \ldots, E_{m(n)}(n)$ are independent and identically distributed, and recall that $\mathbb{P}(E_k(n)) \geq r_n^{\theta_*(a-\frac{1}{2N})}$. Hence

$$\begin{aligned} \mathbb{P}\Big\{\text{ neither of } E_1(n), \ldots, E_{m(n)}(n) \text{ holds }\Big\} &\leq \Big(1 - r_n^{\theta_*(a-\frac{1}{2N})}\Big)^{m(n)} \\ &\longrightarrow 0 \text{ as } n \to \infty, \end{aligned}$$

because $m(n) r_n^{\theta_*(a-\frac{1}{2N})} \to \infty$.

Estimate of (4.13). This is slightly more involved. Denote by $K(k,n)$ the number of intervals of length $r_n^{4-3\varepsilon}$ needed to cover the set $L \cap [\mathcal{T}_k, \mathcal{S}_{k+1}]$. We first need to prove that there are constants $A, B > 0$ such that, for $a(n) = A\exp(-B r_n^{-\varepsilon})$,

$$\mathbb{P}\big\{K(k,n) \geq r_n^{-\varepsilon}\big\} \leq a(n). \qquad (4.14)$$

For simplicity we first subject the problem to a Brownian scaling. So let B be a Brownian motion started at $B_0 = 0$ and T be the first hitting time of level -1. We let K be the number of intervals of length $r_n^{-\varepsilon}$ needed to cover $Z \cap [0, T]$ where Z denotes the zero set of B, so that K has the same distribution as $K(k,n)$.

We define stopping times $T_0 = 0$, and for $k \geq 1$,

$$S_k = \inf\big\{s \geq T_{k-1} \,:\, |B_s| = 1\big\}, \text{ and } T_k = \inf\big\{s \geq S_k \,:\, B_s = 0\big\}.$$

Note that the number $N := \max\{k \,:\, S_k \leq T\}$ is geometrically distributed with success parameter $1/2$. Hence

$$\mathbb{P}\big\{N \geq r_n^{-\varepsilon}\big\} \leq (1/2)^{r_n^{-\varepsilon}}. \tag{4.15}$$

Moreover, $S_k - T_{k-1}$ is the exit time from the interval $(-1, 1)$, which is a random variable with finite exponential moments. Hence, by the exponential Chebyshev inequality, for suitable positive constants C, D, we have

$$\mathbb{P}\big\{S_k - T_{k-1} > r_n^{-\varepsilon}\big\} \leq C \exp\big(- D r_n^{-\varepsilon}\big). \tag{4.16}$$

Putting (4.15) and (4.16) together shows that, up to a probability of the right order, the intervals $[T_{k-1}, S_k]$, for $k \leq \lfloor r_n^{-\varepsilon} \rfloor$, form a cover of $Z \cap [0, T]$ by intervals of length at most $r_n^{-\varepsilon}$. This proves (4.14).

We infer from (4.14) that

$$\mathbb{P}\Big\{K(k, n) \geq r_n^{-\varepsilon} \text{ for some } k \leq 2m(n)\Big\} \leq 2m(n)a(n).$$

Hence, we have

$$\mathbb{P}\Big\{\limsup_{n\to\infty} \sup_{k\leq 2m(n)} \frac{K(k, n)}{r_n^{-\varepsilon}} \geq 2\Big\} \leq 2\sum_{i=n}^{\infty} m(i)a(i) \longrightarrow 0. \tag{4.17}$$

Moreover, if $M(n) \leq 2m(n)$ and $K(k, n) \leq 2r_n^{-\varepsilon}$ for all $k \leq M(n)$, then there is a covering of $E \cap I$ by $4m(n)r_n^{-\varepsilon}$ intervals of length $r_n^{4-3\varepsilon}$. Now, for any small $\delta > 0$,

$$4m(n)r_n^{-\varepsilon} \Big[r_n^{4-3\varepsilon}\Big]^{(\frac{1}{2}-\gamma-\delta)} \leq 4r_n^{-\theta_*(a-\frac{1}{3N})+(4-3\varepsilon)(\frac{1}{2}-\gamma-\delta)-\varepsilon}.$$

Depending on N only, we may make the choice of $\varepsilon, \delta > 0$ such that the exponent is positive. If this holds for all large n, it implies an upper bound of $1/2 - \gamma - \delta$ for the dimension of $E \cap I$. Hence the events

$$\liminf_{n\to\infty} \frac{M(n)}{m(n)} \leq 1 \text{ and } \limsup_{n\to\infty} \sup_{k\leq 2m(n)} \frac{K(k, n)}{r_n^{-\varepsilon}} < 2$$

imply

$$\dim(E \cap I) \leq \tfrac{1}{2} - \gamma - \delta.$$

Recalling (4.17) we get

$$\begin{aligned}
0 = \mathbb{P}\Big\{ &\liminf_{n\to\infty} \frac{M(n)}{m(n)} \leq 1,\ \limsup_{n\to\infty} \sup_{k\leq 2m(n)} \frac{K(k, n)}{r_n^{-\varepsilon}} \\
&< 2 \text{ and } \dim(E \cap I) > \tfrac{1}{2} - \gamma - \delta\Big\} \\
= \mathbb{P}\Big\{ &\liminf_{n\to\infty} \frac{M(n)}{m(n)} \leq 1 \text{ and } \dim(E \cap I) > \tfrac{1}{2} - \gamma - \delta\Big\} \\
\geq \limsup_{n\uparrow\infty} &\,\mathbb{P}\left\{\frac{M(n)}{m(n)} \leq 1 \text{ and } \dim(E \cap I) > \tfrac{1}{2} - \gamma - \delta\right\}.
\end{aligned}$$

This completes the estimate of (4.13), and hence the proof of Lemma 4.5. $\square$

5. Completion of the proofs

5.1. The dimension spectrum

We remain in the framework introduced at the start of Section 4. Recall the definition (4.4) of the set $F(a)$ and define $F(a,b) = F(a) \cap [0,b]$. The proof of Theorem 1.1 is completed with the following lemma, an appeal to the snake construction (2.6), and Lemma 2.2.

Lemma 5.1. *For the stopping time σ introduced in (2.5), we have, almost surely,*

$$\frac{1}{2} - \frac{\theta_* a}{4} \le \dim F(a,\sigma) \le \frac{1}{2} - \frac{\theta a}{4},$$

for $a \in [0, 2/\theta]$, and $F(a,\sigma) = \emptyset$, for $a > 2/\theta$.

Proof. For the *upper bounds* we pick $\varepsilon > 0$ arbitrary and choose b such that $\mathbb{P}\{\sigma > b\} < \varepsilon$. Lemma 4.2 yields that $F(a,b) = \emptyset$ if $a > 2/\theta$ and $\dim F(a,b) \le 1/2 - \theta a/4$ otherwise. The upper bounds follow, as $\varepsilon > 0$ was arbitrary.

For the *lower bounds* we fix $a < 2/\theta_*$ and for the moment also a small $b > 0$. Let $\gamma = 1/2 - \theta_* a/4$ and let $\Gamma[\gamma]$ be an independent percolation limit set in $[0,b]$. By Lemma 4.6 we have

$$\mathbb{P}\big\{F(a,b) \cap \Gamma[\gamma] \neq \emptyset\big\} = \mathbb{P}\big\{E \neq \emptyset\} > 0\,.$$

Now note that, by scaling $\mathbb{P}\big\{E \neq \emptyset\}$ is bounded from zero by a constant independent of b. In fact, by scaling this probability only depends on b via the marginal effect coming from a possible reflection of ζ in $[0,b]$, which for decreasing b becomes increasingly unlikely. Together with Lemma 4.3 (i) this implies that

$$\mathbb{P}\big\{\dim F(a,b) \ge \gamma \text{ for all } b > 0\big\} = \lim_{b \downarrow 0} \mathbb{P}\big\{\dim F(a,b) \ge \gamma\big\} > 0.$$

By Blumenthal's zero-one law the event on the left-hand side has probability zero or one, so we infer that

$$\mathbb{P}\big\{\dim F(a,\sigma) \ge \gamma\,\big\} \ge \mathbb{P}\big\{\dim F(a,b) \ge \gamma \text{ for all } b > 0\big\} = 1,$$

which completes the proof. □

5.2. The most visited sites

Some extra work is needed to complete the proof of the lower bound in (1.4). We fix a small $\varepsilon > 0$ and let $r > 0$. Define two series of stopping times by $\mathcal{T}_0 = 0$ and, for $k \ge 1$,

$$\mathcal{S}_k = \inf\big\{s \ge \mathcal{T}_{k-1} \,:\, \zeta_s = 1 - r^{2-\varepsilon}\big\}, \text{ and } \mathcal{T}_k = \inf\big\{s \ge \mathcal{S}_k \,:\, \zeta_s = 1\big\}.$$

Denote by $N := \max\{k \in \mathbb{N} : \mathcal{T}_k < \sigma\}$ the number of downcrossings by the contour process of the interval $[1 - r^{2-\varepsilon}, 1]$ before σ. Recall that N is geometrically distributed with success parameter $p = r^{2-\varepsilon}$. We define events

$$E_k := \Big\{ \int_{\mathcal{T}_k}^{\mathcal{S}_{k+1}} 1_{B(0,r)}(W_s(\zeta_s) - W_{\mathcal{T}_k}(1)\big) \, L^1(ds) \geq ar^2 \log(1/r) \Big\}.$$

Again, these events are independent and identically distributed, moreover they are independent of N. Note that for $k \leq N$ the event E_k implies

$$\sup_{x \in \operatorname{supp} Z_1} Z_1(B(x,r)) \geq ar^2 \log(1/r),$$

where Z_1 is as in (2.6). Corollary 3.6 shows that for $a < 2/\theta_*$ there exists $\delta > 0$ such that $\mathbb{P}(E_k) \geq r^{2-\delta}$. Hence we have,

$$\begin{aligned} \mathbb{P}\Big\{ &\text{ there is no } x \in \operatorname{supp} Z_1 \text{ with } Z_1\big(B(x,r)\big) \geq ar^2 \log(1/r) \Big\} \\ &\leq \mathbb{P}\big\{ E_k \text{ fails for all } 1 \leq k \leq N \big\} \\ &\leq \sum_{k=1}^{\infty} (1-p)^{k-1} p (1 - r^{2-\delta})^k \leq 2r^2 \sum_{k=1}^{\infty} (1 - r^{2-\delta})^k \leq 2r^\delta. \end{aligned}$$

For any $\eta > 1$, we may choose the sequence $r_n = \eta^{-n}$ and the Borel-Cantelli Lemma tells us that there is a random M such that, almost surely, for all $n \geq M$,

$$\sup_{x \in \operatorname{supp} Z_1} \frac{Z_1\big(B(x,r_n)\big)}{r_n^2 \log(1/r_n)} \geq a.$$

And, for all $0 < r < r_M$, we can choose $n \geq M$ with $r_{n+1} \leq r < r_n$ such that

$$\sup_{x \in \operatorname{supp} Z_1} \frac{Z_1\big(B(x,r)\big)}{r^2 \log(1/r)} \geq \sup_{x \in \operatorname{supp} Z_1} \frac{Z_1\big(B(x,r_{n+1})\big)}{r_n^2 \log(1/r_{n+1})} \geq \frac{a}{\eta^2}.$$

Hence, we have

$$\liminf_{r \downarrow 0} \sup_{x \in \operatorname{supp} Z_1} \frac{Z_1\big(B(x,r)\big)}{r^2 \log(1/r)} \geq \frac{a}{\eta^2} > 0.$$

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