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A convex/log-concave correlation inequality for Gaussian measure and an application to abstract Wiener spaces

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Abstract. This paper deals with a generalization of a result due to Brascamp and Lieb which states that in the space of probabilities with log-concave density with respect to a Gaussian measure on $\mathbb{R}^n$, this Gaussian measure is the one which has strongest moments. We show that this theorem remains true if we replace x^α by a general convex function. Then, we deduce a correlation inequality for convex functions quite better than the one already known. Finally, we prove results concerning stochastic analysis on abstract Wiener spaces through the notion of approximate limit.

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1. Introduction

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}^+$ is called log-concave if for $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$:

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

One of the main results of this paper is the following:

Theorem 1.1. *Let g be a convex function on $\mathbb{R}^n$ and f a log-concave function on $\mathbb{R}^n$. Let γ be a Gaussian measure on $\mathbb{R}^n$(not necessarily centered or with density with respect to Lebesgue measure). We suppose that all of the following integrals are well defined, then:*

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$$\int g\,(x+l-m)\,\frac{f(x)\,d\gamma(x)}{\int f\,d\gamma} \le \int g\,d\gamma$$

where

$$l = \int x\,d\gamma \quad , \quad m = \int x\frac{f(x)\,d\gamma(x)}{\int f\,d\gamma}.$$

This theorem generalizes theorem 5.1 of Brascamp and Lieb [6], theorem 7 of Hargé [15] and corollary 6 of Caffarelli [7].

As in corollary 6 of [7], the proof is based on a result of [7] concerning optimal transport of measure. To obtain the general case, we use here the Ornstein-Uhlenbeck semigroup to construct and to study an appropriate function which gives the result. We deduce from theorem 1.1:

Theorem 1.2. *Let f and g be two convex functions on $\mathbb{R}^n$. Let μ be the standard Gaussian measure on $\mathbb{R}^n$(centered and normalized). We suppose that all of the following integrals are well defined, then:*

$$\int fg\,d\mu \ge (1 + \langle m\,(g)\,, m\,(f)\rangle) \int f\,d\mu \int g\,d\mu$$

where

$$m\,(f) = \int x\frac{f(x)\,d\mu(x)}{\int f\,d\mu} \quad , \quad m\,(g) = \int x\frac{g(x)\,d\mu(x)}{\int g\,d\mu}$$

($\langle\ ,\ \rangle$ is the usual scalar product on $\mathbb{R}^n$).

This result generalizes theorem 6.1 of Hu [17] which proves, under the additional hypothesis $m\,(g) = 0$ or $m\,(f) = 0$, that:

$$\int fg\,d\mu \ge \int f\,d\mu \int g\,d\mu\ .$$

Nevertheless, we have to notice it is possible to prove theorem 1.2 by rewriting Hu's proof.

The inequality obtained in this theorem can be compared to the Poincaré inequality which states that:

$$\int f^2\,d\mu - \left(\int f\,d\mu\right)^2 \le \int \|\nabla f\|^2\,d\mu\ .$$

Let choose $f = g$ in theorem 1.2 and note that we have in most cases: $\int xf\,d\mu = \int \nabla f\,d\mu$. Then, we obtain for a convex function f:

$$\left\|\int \nabla f\,d\mu\right\|^2 \le \int f^2\,d\mu - \left(\int f\,d\mu\right)^2.$$

Other correlation inequalities concerning log-concave functions or "decreasing" functions could be found in the papers of Pitt [27], Bakry and Michel [1], Schechtman , Schlumprecht and Zinn [30], Hargé [15], Szarek and Werner [35], Cordero-Erausquin [11] (see those papers for further references).

The first part of this paper is devoted to the proof of theorems 1.1 and 1.2. The second part deals with regularity results for variables on abstract Wiener spaces which are consequences of theorem 1.1. More precisely, let W be an abstract Wiener space and denote by P the Gaussian measure on W. Let H be the Cameron-Martin space of W.

We denote by $\widetilde{N}$ the extension on W of a measurable seminorm N on H (as defined by Gross [13, 14] ; remark that being measurable for a seminorm does not only mean $\widetilde{N}$ is measurable with respect to the σ-algebra of W , see further for the right definition). Then, we obtain in the second part of this paper regularity results as in the following theorem.

Theorem 1.3. *For all measurable seminorm N_1 and measurable norm N_2 on H,*

$$\lim_{\eta\to 0} E\left(\exp\left(\widetilde{N}_1^2\right)\frac{1_{\widetilde{N}_2\le\eta}}{P\left(\widetilde{N}_2\le\eta\right)}\right)=1.$$

This theorem generalizes results of Mayer-Wolf and Zeitouni ([22], lemma 2.5) and of Hargé [16].

2. Proof of theorems 1.1 and 1.2

We will use the Brenier map [5] which gives the optimal mass transport on $\mathbb{R}^n$. Let us recall some terminology. If ν_1 and ν_2 are two Borel probability measures on $\mathbb{R}^n$, a Borel map $T:\mathbb{R}^n\to\mathbb{R}^n$ is said to transport ν_1 on ν_2 if ν_2 is the image of ν_1 by T. It means that for every non-negative Borel function h:

$$\int h\circ T\, d\nu_1=\int h\, d\nu_2.$$

The result of Brenier, as improved by McCann [23] is the following:

Theorem 2.1. *Let ν_1 and ν_2 be two Borel probability measures on $\mathbb{R}^n$ and suppose ν_1 vanishes on subsets of $\mathbb{R}^n$ having Hausdorff dimension $n-1$. Then, a convex function φ on $\mathbb{R}^n$ whose gradient $\nabla\varphi$ transports ν_1 on ν_2 exists. The map $\nabla\varphi$ is uniquely determined ν_1 almost everywhere.*

Caffarelli [7] proves the following result:

Theorem 2.2. *If ν_1 is a gaussian measure and if $d\nu_2 = f d\nu_1$ where f is a log-concave function (such that $\int f d\nu_1 = 1$) then $\nabla\varphi$ is a contraction with respect to the euclidian norm.*

We can now prove theorem 1.1.

It is possible to find an integer $k\le n$ such that for each integrable function h:

$$\int_{\mathbb{R}^n} h\, d\gamma=\int_{\mathbb{R}^k} h\left(\mathcal{L}x+l\right)d\mu$$

where $\mathcal{L} : \mathbb{R}^k \to \mathbb{R}^n$ is a linear map and μ is the standard Gaussian measure on $\mathbb{R}^k$. So, it is sufficient to show the result when $\gamma = \mu$ (the standard Gaussian measure on $\mathbb{R}^n$) and $l = 0$. With:

$$\overline{f}(x) = f(x+m)\exp\left(-\frac{1}{2}\|m\|^2 - \langle x, m\rangle\right),$$

we can assume that $m = 0$.

Then, it is possible to assume:

$$f(x) = \exp(-F)1_{B(0,R)}$$

where F is arbitrarily often differentiable and convex. Then, using a result of Caffarelli [8, 9], if we denote by $\nabla\varphi$ the Brenier map which transports μ on $\frac{f d\mu}{\int f d\mu}$, we obtain $\varphi \in C^{2,\alpha}$ for an $\alpha \in]0, 1[$. The result of Caffarelli [7] is:

$$\varphi = \frac{\|x\|^2}{2} - \psi \quad \text{and} \quad 0 \le Hess\psi \le I.$$

It is possible to assume that g is smooth and ∇g is bounded. This last point is obtained with the following approximation of g:

$$g_n(x) = \sup_{y \in B(0,n)} \langle y, x\rangle - g^*(y),$$

where g^* is the Fenchel-Legendre transform of g.

With the optimal transportation, we can write:

$$\int g(x)\frac{f(x)d\mu(x)}{\int f(x)d\mu(x)} = \int g(x - \nabla\psi(x))\, d\mu(x).$$

Now, let $P_t h$ be the Ornstein-Uhlenbeck semigroup given by:

$$P_t h(x) = \int h\left(e^{-\frac{t}{2}}x + \sqrt{1-e^{-t}}\,y\right) d\mu(y).$$

$P_t h$ is the solution of:

$$\begin{cases} \frac{d}{dt}(P_t(h)) = L(P_t(h)) \\ P_0(h) \quad = h \end{cases}$$

where L is given by:

$$L(h) = \frac{1}{2}(\Delta h - \langle x, \nabla h\rangle).$$

Moreover, we have the following integration by parts formula:

$$\int h_1 L(h_2)\, d\mu = -\frac{1}{2}\int \langle \nabla h_1, \nabla h_2\rangle d\mu.$$

Let θ be the function:

$$\theta(t) = \int g(x - P_t(\nabla\psi)(x))\, d\mu(x)$$

where

$$P_t(\nabla\psi) = \left(P_t\left(\frac{\partial\psi}{\partial x_1}\right), \ldots, P_t\left(\frac{\partial\psi}{\partial x_n}\right)\right).$$

With the following inequality (see for example the proof of proposition 3.1.25 of [29]):

$$\left\| P_t(\nabla\psi)(x) - \int \nabla\psi d\mu \right\| \leq K e^{-t/2} \int \|x-y\| \, d\mu(y)$$

and using the fact that ∇g is bounded, we obtain:

$$\left| g\left(x - P_t(\nabla\psi)(x)\right) - g\left(x - \int \nabla\psi d\mu\right) \right| \leq \widetilde{K} e^{-t/2} \int \|x-y\| \, d\mu(y) \ .$$

Furthermore:

$$\int \nabla\psi d\mu = -\int (x - \nabla\psi(x)) \ d\mu(x) = -\int x \frac{f(x)d\mu(x)}{\int f(x)d\mu(x)} = 0.$$

So, $\theta(t)$ is well defined for each t . Moreover:

$$\lim_{t\to+\infty} \theta(t) = \int g(x) \, d\mu(x) \quad \text{and} \quad \theta(0) = \int g(x - \nabla\psi(x)) \, d\mu(x) \, .$$

Then, it is sufficient to show that θ is an increasing function. Working on $LP_t(\nabla\psi)$ with the same understanding as the one used for $P_t(\nabla\psi)$, we obtain with an integration by parts:

$$\begin{aligned} \theta'(t) &= -\int \langle \nabla g(x - P_t(\nabla\psi)(x)), LP_t(\nabla\psi) \rangle \, d\mu \\ &= \frac{1}{2} \int tr\left[Hessg(x - P_t(\nabla\psi)(x)) \left(I - M^*\right) M\right] \, d\mu, \end{aligned}$$

where $M_{i,j} = \frac{\partial}{\partial x_i} P_t\left(\frac{\partial\psi}{\partial x_j}\right)$. It is easy to see that:

$$M = e^{-\frac{t}{2}} P_t(Hess\psi) \, .$$

So:

$$M = M^* \quad \text{and} \quad 0 \leq M \leq I.$$

Therefore, $(I - M^*) M$ is a symmetric, positive matrix and consequently $\theta'(t) \geq 0$. □

Remark 1. We could rewrite the result of theorem 1.1 in the following way. Consider a centered Gaussian measure γ, denote by N a general norm on $\mathbb{R}^n$, by f a log-concave function such that $\int xf \, d\gamma = 0$ and by $\nabla\varphi$ the Brenier map which transports γ on $\frac{f\,d\gamma}{\int f\,d\gamma}$; then we have:

$$\int N(\nabla\varphi)\,d\gamma \leq \int N\,d\gamma\ . \tag{2.1}$$

The result of Caffarelli says that $\nabla\varphi$ is a contraction with respect to the euclidian norm. Inequality (2.1) says that $\nabla\varphi$ is globally (on average) a contraction with respect to every norm N.

Now, we will deduce from theorem 1.1 the correlation inequality for convex functions given in theorem 1.2.

Proof of theorem 1.2. Recall that μ denotes the standard Gaussian measure on $\mathbb{R}^n$. We define a function $\xi(t)$ for $t \geq 0$ by:

$$\xi(t) = \int g(x - m_t)\exp(-tf)\,d\mu - \int g\,d\mu \int \exp(-tf)\,d\mu$$
$$\text{where } m_t = \frac{\int x\exp(-tf)\,d\mu}{\int \exp(-tf)\,d\mu}\ .$$

We know from theorem 1.1 that $\xi(t) \leq 0$, furthermore $\xi(0) = 0$. So, we have $\xi'(0) \leq 0$. We obtain:

$$\xi'(0) = -\langle \int \nabla g\,d\mu, m_0' \rangle - \int fg\,d\mu + \int f\,d\mu \int g d\mu\,,$$
$$m_0' = -m(f)\int f\,d\mu \quad \text{and} \quad \int \nabla g\,d\mu = m(g)\int g\,d\mu\,.$$

We deduce from this the desired inequality. □

Remark 2. It is possible to obtain a more general inequality if we consider a general Gaussian measure γ. With the same method, we can write in most cases :

$$\int fg\,d\gamma \geq \left(1 + \left\langle K\frac{\int \nabla f\,d\gamma}{\int f\,d\gamma}, \frac{\int \nabla g\,d\gamma}{\int g\,d\gamma}\right\rangle\right)\int f\,d\gamma \int g\,d\gamma$$

where K is the covariance matrix of γ.

3. Approximate limits on abstract Wiener spaces

3.1. Preliminaries

We consider here an abstract Wiener space (W, H, P) as in the introduction. We denote $\langle\,,\,\rangle$ and $\|\ \ \|$ the scalar product and the norm on H and $\mathcal{F}$ the Borel σ -algebra on W. Let us recall some definitions concerning measurable seminorms in the sense of Gross (see Gross [13], [14] and also Kuo [21]) .

Let $Q : H \to H$ be an orthogonal projection such that $\dim QH < \infty$.We have:

$$Qh = \sum_{i=1}^{n} \langle h_i, h\rangle h_i$$

where $(h_1, \ldots, h_n)$ is an orthonormal basis of QH.

Recall that a canonical isometry exists between H and a subspace of $L^2(W)$ (which is the first Wiener Chaos). If $h \in H$, we denote by $\widetilde{h}$ its image in $L^2(W)$. Then we define:

$$Qw = \sum_{i=1}^{n} \widetilde{h_i}\, h_i \quad \text{for} \quad w \in W \ .$$

A sequence of orthogonal projections Q_n is called an approximating sequence of projections if:

$$\dim Q_n H < \infty\ , \qquad Q_n H \text{ increases with } n \quad \text{and}$$
$$Q_n(h) \text{ goes to } h \text{ in } H \text{ for each } h \text{ in } H.$$

In view of corollaries 4.5 and 5.2 of [13], an equivalent definition to that of Gross for measurable seminorm is the following:

Definition 3.1. *A seminorm N on H is said to be measurable if a random variable $\widetilde{N}(w)$ exists such that for each $\eta > 0$, $P(\widetilde{N} < \eta) > 0$ and for all approximating sequence of projections Q_n, the sequence $N\left(Q_n\left(w\right)\right)$ converges in probability to $\widetilde{N}$. If, in addition, N is a norm on H, N is called a measurable norm.*

For example, on the standard Wiener space $W_0 = \left\{f \in C([0,1], \mathbb{R}^d),\ f(0)=0\right\}$, the supremum norm on W_0 defined by:

$$\mid f \mid_\infty = \sup_{s\in[0,1]} \left(\sum_{i=1}^{d} f_i\,(s)^2\right)^{\frac{1}{2}}$$

comes from a measurable norm. This is also the case for Hölder norms with index smaller than $\frac{1}{2}$ ([13], paragraph 5).

For a measurable norm N on H, it is possible to consider the completion $(E, \overline{N})$ of (H, N) and to construct a Gaussian measure γ on $(E, \overline{N})$, but this is not our point here. We are only interested in the "extension " $\widetilde{N}$ of N on W and we would like to compare the behaviour of extensions on W of two measurable seminorms on H. Nevertheless, it is important to notice that the image of γ by $\overline{N}$ is equal to the image of P by $\widetilde{N}$.

Let N be a measurable seminorm on H. According to Cameron-Martin's formula, $P(\widetilde{N}(w-h) < \eta) > 0$ if $h \in H$; so it is possible to define:

Definition 3.2. *Let N be a measurable seminorm on H. For $F \in L^1(W)$, $A \in \mathcal{F}$, $h \in H$ and $\eta > 0$ let:*

$$E^N_{\eta,h}(F) = E\left(F\,\frac{1_{\widetilde{N}(w-h)<\eta}}{P(\widetilde{N}(w-h) < \eta)}\right) \text{ and } P^N_{\eta,h}(A) = \frac{P\left(A, \widetilde{N}(w-h) < \eta\right)}{P\left(\widetilde{N}(w-h) < \eta\right)}\ .$$

If $h = 0$, we will omit the subscript h.

Definition 3.3. *Let $F \in L^1(W)$, N be a measurable seminorm on H, $h \in H$ and $p \in [1, +\infty[$. We say that F possesses an L_N^0, respectively L_N^p, approximate limit at h if there exists a real l such that:*

$$\forall \varepsilon > 0 \quad , \quad \lim_{\eta \to 0} P_{\eta,h}^N(| F - l |> \varepsilon) = 0$$

respectively:

$$\lim_{\eta \to 0} E_{\eta,h}^N(| F - l |^p) = 0$$

l is denoted by $F(h)$.

It is easy to see that if F possesses an L_N^p approximate limit then F possesses L_N^r approximate limit for $r < p$.

The existence of approximate limit may be used to prove existence of Onsager-Machlup functionnals for tubes around every element in H (see for example [19] in the case of the standard Wiener space) or to obtain support theorems for the law of some random variables.
Perhaps the first work on this subject is the one of Stroock and Varadhan [32]. Later, several authors have worked on this notion for particular seminorms ([2], [3], [10], [12], [24], [31], [34], [22]). In the case of the supremum norm on the standard Wiener space, we could in general prove the existence of a limit at h from the existence of a limit at 0 by use of a result of Millet and Nualart ([24]) which is based on a paper of Shepp and Zeitouni [31]. Sugita ([34]) also gave a similar method for general measurable norms. We will use those ideas here.
Furthermore, the notion of approximate limit with respect to N is very sensitive to the choice of N. For example, if F possesses an approximate limit with respect to N, we can say nothing about the existence for F of an approximate limit with respect to another norm equivalent to N. Moreover, Sugita showed in [34] that, if F is the Lévy stochastic area defined on the standard Wiener space, there exists a dense subset A of $\mathbb{R}$ such that for all a in A, we could find a measurable norm N_a such that $F_{N_a}(0) = a$ (where $F_{N_a}(0)$ is the approximate limit of F at 0 with respect to N_a) .
A great number of results concerning approximate limit use correlation inequalities like F.K.G. inequalities or the "strip" version of the Gaussian correlation conjecture. Here, we will use theorem 1.1 instead of those inequalities and we will see that theorem 1.1 is sufficient to prove existence of approximate limits even though it is a weaker result than the Gaussian correlation inequality (which is still a conjecture in the general case). In this part, we will prove that for every measurable seminorm N_1 and measurable norm N_2 and for some functions g, $F = g\left(\widetilde{N_1}\right)$ possesses an approximate limit with respect to N_2 which does not depend on the choice of N_2. This result is not in contradiction with the result of Sugita because the Lévy stochastic area can not be related to a measurable seminorm.

3.2. General results

Theorem 1.1 allows us to obtain:

Theorem 3.4. *Let N_1 and N_2 be two measurable seminorms on H. Let h belong to H . If $g : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and convex function such that:*

$$E\left(\left|g\left(\widetilde{N_1} + N_1(h)\right)\right|\right) < +\infty$$

then:

$$\forall \eta > 0, \quad E\left(g\left(\widetilde{N_1}\left(\cdot + h\right)\right) 1_{\widetilde{N_2} \le \eta}\right) \le E\left(g\left(\widetilde{N_1}\left(\cdot + h\right)\right)\right) P\left(\widetilde{N_2} \le \eta\right).$$

Proof. It is possible to assume $g(0) = 0$.

Denote by $B = (e_n)$ a complete orthonormal system on H such that $e_1 = h/\left\|h\right\|$ (if $h \neq 0$). Let Q_n be the sequence of projections defined by:

$$Q_n(l) = \sum_{i=1}^{n} < l, e_i > e_i \quad \text{where} \quad l \in H.$$

We have $Q_n(h) = h$.

Using subsequences, we obtain (because if $N_2 \neq 0$, $P\left(\widetilde{N_2} = \eta\right) = 0$ for each real η, see [16], corollary 4):

$$\lim_{k \to +\infty} N_1\left(Q_{n_k}(w)\right) = \widetilde{N_1} \quad \text{a.s.} \quad \text{and} \quad \lim_{k \to +\infty} 1_{N_2\left(Q_{n_k}(w)\right) \le \eta} = 1_{\widetilde{N_2} \le \eta} \ \text{a.s..}$$

We deduce from the Cameron-Martin's formula:

$$\lim_{k \to +\infty} g\left(N_1\left(Q_{n_k}(w + h)\right)\right) = g\left(\widetilde{N_1}\left(\cdot + h\right)\right) \quad a.s. \quad .$$

With theorem 5 of [13], we see that:

$$\forall \varepsilon > 0, \quad P\left(N_1\left(Q_{n_k}(w)\right) > \varepsilon\right) \le P\left(\widetilde{N_1} > \varepsilon\right).$$

We deduce:

$$\begin{aligned} \forall x > 0, P\left(g\left[N_1\left(Q_{n_k}(w + h)\right)\right] > x\right) &\le P\left(g\left[N_1\left(Q_{n_k}(w)\right) + N_1(h)\right] > x\right) \\ &\le P\left(g\left[\widetilde{N_1} + N_1(h)\right] > x\right). \end{aligned}$$

$$\begin{aligned} &\Rightarrow E\left(g\left[N_1\left(Q_{n_k}(w + h)\right)\right] 1_{g\left[N_1\left(Q_{n_k}(w+h)\right)\right] > a}\right) \\ &\quad = \int_0^{+\infty} P\left(g\left[N_1\left(Q_{n_k}(w + h)\right)\right] > \max\left(a, t\right)\right) dt \\ &\quad \le \int_0^{+\infty} P\left(g\left[\widetilde{N_1} + N_1(h)\right] > \max\left(a, t\right)\right) dt \\ &\quad \le E\left(g\left[\widetilde{N_1} + N_1(h)\right] 1_{g\left[\widetilde{N_1} + N_1(h)\right] > a}\right). \end{aligned}$$

Consequently:

$$g\left[N_1\left(Q_{n_k}(w+h)\right)\right] \underset{k\to+\infty}{\to} g\left(\widetilde{N_1}\left(\cdot+h\right)\right) \quad \text{in} \quad L^1.$$

So, we obtain:

$$E\left(g\left(\widetilde{N_1}\left(\cdot+h\right)\right)1_{\widetilde{N_2}\le\eta}\right) \le \liminf_{k\to+\infty} E\left(g\left[N_1\left(Q_{n_k}\left(w\right)+h\right)\right]1_{N_2\left(Q_{n_k}(w)\right)\le\eta}\right)$$
$$\lim_{k\to+\infty} E\left(g\left[N_1\left(Q_{n_k}\left(w\right)+h\right)\right]\right) = E\left(g\left(\widetilde{N_1}\left(\cdot+h\right)\right)\right)$$
$$\lim_{k\to+\infty} P\left(N_2\left(Q_{n_k}\left(w\right)\right)\le\eta\right) = P\left(\widetilde{N_2}\le\eta\right).$$

Finally, it is sufficient to prove:

$$E\left(g\left[N_1\left(Q_n\left(w\right)+h\right)\right]1_{N_2\left(Q_n(w)\right)\le\eta}\right)$$
$$\le E\left(g\left[N_1\left(Q_n\left(w\right)+h\right)\right]\right)P\left(N_2\left(Q_n\left(w\right)\right)\le\eta\right).$$

The map: $x \in \mathbb{R}^n \mapsto N_2\left(\sum_{j=1}^{n} x_j e_j\right)$ is even and convex and the map $x \in \mathbb{R}^n \mapsto g\left[N_1\left(\sum_{j=1}^{n} x_j e_j + h\right)\right]$ is convex. Furthermore $(\widetilde{e_1}, \ldots, \widetilde{e_n})$ is a Gaussian vector. We use theorem 1.1 to conclude. □

Remark 3. This theorem remains true if we assume $E\left(\left|g\left((1+\varepsilon)\widetilde{N_1}\right)\right|\right) < +\infty$ for some $\varepsilon > 0$ instead of $E\left(\left|g\left(\widetilde{N_1}+N_1(h)\right)\right|\right) < +\infty$. It is sufficient to notice that:

$$\widetilde{N_1}+N_1(h) = \frac{1}{1+\varepsilon}\left(1+\varepsilon\right)\widetilde{N_1} + \left(1-\frac{1}{1+\varepsilon}\right)\frac{1+\varepsilon}{\varepsilon}N_1(h)$$

and to use the convexity of g.

Remark 4. In this theorem, we can write $1_{\widetilde{N_2}<\eta}$ instead of $1_{\widetilde{N_2}\le\eta}$ because $P\left(\widetilde{N_2}=\eta\right) = 0$ if $\eta > 0$.

Example 3.5. Let N_1 be a measurable seminorm on H. We know from Fernique's theorem that there exists α such that $E\left(\exp\left(\alpha\widetilde{N_1}^2\right)\right) < +\infty$. So, we can apply theorem 3.4 to $g(x) = \exp\left(\alpha x^2\right)$ if $h = 0$ ($g(x) = \exp\left((\alpha-\varepsilon)x^2\right)$ if $h \neq 0$).

Example 3.6. If N_1 is a measurable seminorm on H, using Fernique's theorem, we see that $\widetilde{N_1} \in L^p$ for every $p \ge 1$. So, we can apply theorem 3.4 to $g(x) = x^p$.

This result is essential to study the existence of approximate limits because it gives an uniform bound with respect to η of quantities which will occur:

$$\sup_{\eta>0} E_\eta^{N_2}\left[g\left(\widetilde{N_1}(\cdot+h)\right)\right] \le E\left[g\left(\widetilde{N_1}(\cdot+h)\right)\right].$$

First, let us give a generalization of lemma 1 of [16]. This lemma allows us to construct an approximating sequence of projections well-adapted to a given seminorm.

Lemma 3.7. *Let N be a measurable seminorm on H. There exists a complete orthonormal system (ξ_i) in H such that, for each n in $\mathbb{N}\backslash\{0\}$:*

$$\exists c_n > 0, \quad N(Q_n(w)) \leq c_n \widetilde{N} \quad a.s.$$

where $Q_n(h) = \sum_{i=1}^{n} < \xi_i, h > \xi_i$.

Proof. Let $F = \{x, N(x) = 0\}^{\perp}$ (with respect to the scalar product in $(H, \| \|)$), $\{x, N(x) = 0\}$ is a closed subspace of H because c exists such that $N(x) \leq c \|x\|$ for each x in H (see for example [21] lemma 4.2) .

Let Π be the orthogonal projection on F. We have: $\forall x \in H,\ N(x) = N(\Pi x)$.

Denote $N_1(x) = N(x)$ for $x \in F$ and (E, N_1) the completion of (F, N_1). E' is dense in $(F, \| \|)' = (F, \| \|)$. We choose (ξ_i), a complete orthonormal system in F such that ξ_i belongs to E'.
Denote $\left(\xi_i'\right)$ a complete orthonormal system in $F^{\perp}$. Define:

$$Q_n(g) = \sum_{i=1}^{n} \left(< g, \xi_i > \xi_i + < g, \xi_i' > \xi_i'\right) \quad \text{for} \quad g \in H$$

$$\tilde{Q}_n(x) = \sum_{i=1}^{n} (x, \xi_i)_{E,E'}\ \xi_i \quad \text{for} \quad x \in E$$

(Q_n is an approximating sequence of projections for H) .

There exists $c_n > 0$ such that:

$$\forall x \in E\ ,\ N(\tilde{Q}_n(x)) \ \leq\ c_n N_1(x).$$

Consequently, for $g \in H$:

$$\begin{aligned} N\left(Q_n(g)\right) &= N\left(\Pi Q_n(g)\right) = N(\tilde{Q}_n(\Pi g)) \leq c_n N\left(\Pi g\right) = c_n N\left(g\right). \\ &\Rightarrow \forall q > n\ ,\ N(Q_n(w)) \ \leq\ c_n N(Q_q(w)) \quad \text{a.s.} \\ &\Rightarrow N(Q_n(w)) \ \leq\ c_n \widetilde{N}(w) \quad \text{a.s..} \end{aligned}$$

□

Remark 5. The idea of the proof of this lemma appeared in a paper of Sugita ([34], lemma 2).

Now, we study the existence of a limit at 0 for variables defined with a convex function.

Theorem 3.8. *Let N_1 be a measurable seminorm on H and N_2 be a measurable norm on H. If $g : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and convex function such that:*

$$\exists \varepsilon > 0, \quad E\left(\left|g\left((1+\varepsilon)\widetilde{N_1}\right)\right|\right) < +\infty$$

then

$$\lim_{\eta \to 0} E_{\eta}^{N_2}\left(g\left(\widetilde{N_1}\right)\right) = g(0) \quad .$$

Proof. First, it is possible to assume that $g(0) = 0$. Let Q_n be the approximating sequence of projections given in lemma 3.7 for N_2. We have:

$$\begin{aligned}
\widetilde{N_1} &\leq \frac{1}{1+\varepsilon}(1+\varepsilon)\left|\widetilde{N_1} - N_1(Q_n(w))\right| + \left(1 - \frac{1}{1+\varepsilon}\right)\frac{1+\varepsilon}{\varepsilon}N_1(Q_n(w)) \\
&\Rightarrow E_\eta^{N_2}\left(g\left(\widetilde{N_1}\right)\right) \\
&\leq \frac{1}{1+\varepsilon}E_\eta^{N_2}\left(g\left[(1+\varepsilon)\left|\widetilde{N_1} - N_1(Q_n(w))\right|\right]\right) \\
&\quad + \left(1 - \frac{1}{1+\varepsilon}\right)E_\eta^{N_2}\left(g\left[\frac{1+\varepsilon}{\varepsilon}N_1(Q_n(w))\right]\right).
\end{aligned}$$

For the second term, we have to study the behaviour of $N_1(Q_n(w))$ when $N_2(Q_n(w)) \leq c_n\eta$. Because of the comparison of norms on spaces of finite dimension and because N_2 is a norm, we have:

$$N_1(Q_n(w)) \leq k_n\eta \quad \text{when} \quad N_2(Q_n(w)) \leq c_n\eta.$$

Consequently, we obtain:

$$E_\eta^{N_2}\left(g\left[\frac{1+\varepsilon}{\varepsilon}N_1(Q_n(w))\right]\right) \leq g\left(\frac{1+\varepsilon}{\varepsilon}k_n\eta\right).$$

For the first term, using a subsequence, we obtain:

$$\begin{aligned}
&E_\eta^{N_2}\left(g\left[(1+\varepsilon)\left|\widetilde{N_1} - N_1(Q_n(w))\right|\right]\right) \\
&\leq \liminf_{k\to+\infty} E_\eta^{N_2}\left(g\left[(1+\varepsilon)\left|N_1\left(Q_{n_k}(w)\right) - N_1(Q_n(w))\right|\right]\right) \\
&\quad \leq \liminf_{k\to+\infty} E_\eta^{N_2}\left(g\left[(1+\varepsilon)\left|N_1\left(Q_{n_k}(w) - Q_n(w)\right)\right|\right]\right)
\end{aligned} \tag{3.1}$$

Moreover ([13], corollary 5.2 and theorem 5):

$$N_1(Q_m(w) - Q_n(w)) \underset{n,m\to\infty}{\to} 0 \quad \text{in probability,}$$

$$\forall x > 0, \qquad P(N_1(Q_m(w) - Q_n(w)) > x) \leq P\left(\widetilde{N_1} > x\right).$$

So, we obtain:

$$\begin{aligned}
&\forall x > 0, \quad P(g[(1+\varepsilon)N_1(Q_m(w) - Q_n(w))] > x) \leq P\left(g\left[(1+\varepsilon)\widetilde{N_1}\right] > x\right) \\
&\quad \Rightarrow \sup_{m,n} E\left(g[(1+\varepsilon)N_1(Q_m(w) - Q_n(w))]\,1_{g[(1+\varepsilon)N_1(Q_m(w)-Q_n(w))]>a}\right) \\
&\quad \leq E\left(g\left[(1+\varepsilon)\widetilde{N_1}\right]1_{g\left[(1+\varepsilon)\widetilde{N_1}\right]>a}\right).
\end{aligned}$$

Consequently (because $E\left(g\left[(1+\varepsilon)\widetilde{N_1}\right]\right) < +\infty$):

$$g[(1+\varepsilon)N_1(Q_m(w) - Q_n(w))] \underset{m,n\to+\infty}{\to} 0 \quad \text{in} \quad L^1.$$

Using theorem 3.4 and the fact that $E(g[(1+\varepsilon)N_1(Q_m(w) - Q_n(w))]) < +\infty$, we write:

$$\begin{aligned}
&E_\eta^{N_2}\left(g\left[(1+\varepsilon) N_1\left(Q_m(w)-Q_n(w)\right)\right]\right) \\
&\quad\leq E\left(g\left[(1+\varepsilon) N_1\left(Q_m(w)-Q_n(w)\right)\right]\right). \qquad (3.2)\\
&\quad\Rightarrow \liminf_{n,k\to\infty} \sup_{\eta>0} E_\eta^{N_2}\left(g\left[(1+\varepsilon) N_1\left(Q_{n_k}(w)-Q_n(w)\right)\right]\right)=0 \quad .
\end{aligned}$$

□

Remark 6. It is an open question whether it is essential for N_1 to be a measurable seminorm or not.

Example 3.9. Let $f \in H$ and define $N_1(h) = |\langle f, h\rangle|$, N_1 is a measurable seminorm and $\widetilde{N_1} = \left|\widetilde{f}\right|$. Furthermore, since $\widetilde{f}$ is a Gaussian variable:

$$\forall \varepsilon > 0, \quad E\left(\exp\left((1+\varepsilon)\widetilde{N_1}\right)\right) < +\infty.$$

So, we obtain:

$$\lim_{\eta\to 0} E_\eta^{N_2}\left(\exp\left(\left|\widetilde{f}\right|\right)\right) = 1.$$

This result implies a result of Borell [4] which is:

$$\lim_{\eta\to 0} E_\eta^{N_2}\left(\exp\left(\widetilde{f}\right)\right) = 1.$$

For similar results and improvements on norms N_2 which are not necessarily measurable norms but which verify other hypothesis, see [3], [31].

Corollary 3.10. *For each and every measurable seminorm N_1 and norm N_2 on H,*

$$\forall p \geq 1, \quad \lim_{\eta\to 0} E_\eta^{N_2}\left(\widetilde{N_1}^{\,p}\right) = 0$$

and consequently:

$$\forall \varepsilon > 0, \quad \lim_{\eta\to 0} P_\eta^{N_2}\left(\widetilde{N_1} > \varepsilon\right) = 0.$$

Proof. Using Fernique's theorem, we see that $E\left(\widetilde{N_1}^{\,p}\right) < +\infty$. □

Remark 7. We cannot use theorem 3.8 to obtain $\lim_{\eta\to 0} E_\eta^{N_2}\left(\exp\left(\widetilde{N_1}^{\,2}\right)\right) = 1$ because $E\left(\exp\left(\widetilde{N_1}^{\,2}\right)\right)$ may be infinite. We shall address this problem later.

Now, we are able to give two corollaries of corollary 3.10 which allow us to prove the existence of approximate limits for some variables.

Corollary 3.11. *Let $N_1, \ldots, N_r$ be measurable seminorms and N be a measurable norm. Let $h \in H$ and $f : \mathbb{R}^r \to \mathbb{R}$ be a continuous function at $(N_1(h), \ldots, N_r(h))$. Then $f\left(\widetilde{N_1}, \ldots, \widetilde{N_r}\right)$ possesses an L_N^0 approximate limit at h which is $f(N_1(h), \ldots, N_r(h))$.*

Proof. Using Cameron-Martin's theorem, we write:

$$\begin{aligned}
&P^N_{\eta,h}\left(\mid f\left(\widetilde{N}_1,\ldots,\widetilde{N}_r\right)-f\left(N_1(h),\ldots,N_r(h)\right)\mid>\varepsilon\right)\\
&=\frac{E^N_\eta\left(\exp\left(-\widetilde{h}\right)1_{\mid f\left(\widetilde{N}_1,\ldots,\widetilde{N}_r\right)(\cdot+h)-f(N_1(h),\ldots,N_r(h))\mid>\varepsilon}\right)}{E^{N_2}_\eta\left(\exp\left(-\widetilde{h}\right)\right)}\\
&\le\frac{E^N_\eta\left(\exp\left(-2\widetilde{h}\right)\right)^{1/2}}{E^N_\eta\left(\exp\left(-\widetilde{h}\right)\right)}\sqrt{P^N_\eta\left(\mid f\left(\widetilde{N}_1,\ldots,\widetilde{N}_r\right)(\cdot+h)-f\left(N_1(h),\ldots,N_r(h)\right)\mid>\varepsilon\right)}.
\end{aligned}$$

Furthermore, there exists α such that:

$$\begin{aligned}
&\sum_{i=1}^r\mid\widetilde{N}_i(\cdot+h)-N_i(h)\mid\\
&\quad\le\alpha\Rightarrow\mid f\left(\widetilde{N}_1,\ldots,\widetilde{N}_r\right)(\cdot+h)-f\left(N_1(h),\ldots,N_r(h)\right)\mid\le\varepsilon.
\end{aligned}$$

Moreover, with an approximating sequence of projections Q_n such that $Q_n(h)=h$, we obtain:

$$\begin{aligned}
\left|\widetilde{N}_i(w+h)-N_i(h)\right|&=\lim_{k\to\infty}\left|N_i\left(Q_{n_k}(w+h)\right)-N_i(h)\right|\quad a.s.\\
&=\lim_{k\to\infty}\left|N_i\left(Q_{n_k}(w)+h\right)-N_i(h)\right|\\
&\le\lim_{k\to\infty}N_i\left(Q_{n_k}(w)\right)\\
&\le\widetilde{N}_i(w)\quad a.s..
\end{aligned}$$

Consequently,

$$\begin{aligned}
&P^N_{\eta,h}\left(\mid f\left(\widetilde{N}_1,\ldots,\widetilde{N}_r\right)(\cdot+h)-f\left(N_1(h),\ldots,N_r(h)\right)\mid>\varepsilon\right)\\
&\quad\le\frac{E^N_\eta\left(\exp\left(-2\widetilde{h}\right)\right)^{1/2}}{E^N_\eta\left(\exp\left(-\widetilde{h}\right)\right)}\sqrt{\sum_{i=1}^r P^N_\eta\left(\widetilde{N}_i>\frac{\alpha}{r}\right)}.
\end{aligned}$$

Using a result of Borell [4] (see example 3.9),we have:

$$\lim_{\eta\to0}E^{N_2}_\eta\left(\exp\left(-\widetilde{h}\right)\right)=1\quad\text{and}\quad\lim_{\eta\to0}E^{N_2}_\eta\left(\exp\left(-2\widetilde{h}\right)\right)=1.$$

We conclude with corollary 3.10. □

The following result shows the usefulness of theorem 3.4 to obtain existence of approximate limits.

Corollary 3.12. *Let $N_1,\ldots,N_r$ be measurable seminorms and N be a measurable norm. Let $h\in H$ and $f:\mathbb{R}^r\to\mathbb{R}$ be a continuous function at $(N_1(h),\ldots,N_r(h))$. If there exists $p>1$ such that:*

$$\sup_{\eta\in]0,1]}E^N_\eta\left(\left|f\left(\widetilde{N}_1(\cdot+h),\ldots,\widetilde{N}_r(\cdot+h)\right)\right|^p\right)<+\infty$$

then, for every $q \in [1, p[$*,* $f\left(\widetilde{N_1}, \ldots, \widetilde{N_r}\right)$ *possesses an* L^q_N *approximate limit at* h *which is* $f\left(N_1(h), \ldots, N_r(h)\right)$.

In particular:

$$\lim_{\eta\to 0} E^N_{\eta,h}\left(f\left(\widetilde{N_1}, \ldots, \widetilde{N_r}\right)\right) = f\left(N_1(h), \ldots, N_r(h)\right).$$

Proof. For $\varepsilon > 0$, there exists α such that $\sum_{i=1}^r |x_i - N_i(h)| \leq \alpha \Rightarrow |f(x) - f(N_1(h), \ldots, N_r(h))| \leq \varepsilon$. For $q \in [1, p[$:

$$\begin{aligned}
&E^N_{\eta,h}\left(\mid f\left(\widetilde{N_1}, \ldots, \widetilde{N_r}\right) - f\left(N_1(h), \ldots, N_r(h)\right) \mid^q\right)^{1/q} \\
&= \frac{E^N_\eta\left(\exp\left(-\widetilde{h}\right) \mid f\left(\widetilde{N_1}, \ldots, \widetilde{N_r}\right)(\cdot + h) - f\left(N_1(h), \ldots, N_r(h)\right) \mid^q\right)^{1/q}}{E^N_\eta\left(\exp\left(-\widetilde{h}\right)\right)^{1/q}} \\
&\leq \varepsilon + \frac{E^N_\eta\left(\exp\left(-\widetilde{h}\right) \mid f\left(\widetilde{N_1}, \ldots, \widetilde{N_r}\right)(\cdot + h) - f\left(N_1(h), \ldots, N_r(h)\right) \mid^q 1_{\sum_{i=1}^r |\widetilde{N_i}(\cdot+h) - N_i(h)| > \alpha}\right)^{1/q}}{E^N_\eta\left(\exp\left(-\widetilde{h}\right)\right)^{1/q}}.
\end{aligned}$$

The second term is smaller than:

$$\frac{E^N_\eta\left(\exp\left(-\frac{2p}{p-q}\widetilde{h}\right)\right)^{\frac{p-q}{2pq}} E^N_\eta\left(\mid f\left(\widetilde{N_1}, \ldots, \widetilde{N_r}\right)(\cdot + h) - f\left(N_1(h), \ldots, N_r(h)\right) \mid^p\right)^{1/p}}{E^N_\eta\left(\exp\left(-\widetilde{h}\right)\right)^{1/q}} \times \left(\sum_{i=1}^r P^N_\eta\left(\mid \widetilde{N_i}(\cdot + h) - N_i(h) \mid > \frac{\alpha}{r}\right)\right)^{\frac{p-q}{2pq}}$$

and

$$P^N_\eta\left(\mid \widetilde{N_i}(\cdot + h) - N_i(h) \mid > \frac{\alpha}{r}\right) \leq P^N_\eta\left(\widetilde{N_i} > \frac{\alpha}{r}\right).$$

We conclude as in the previous corollary. □

Remark 8. We could deduce from this corollary a theorem like theorem 3.8 where the hypothesis

$$\exists \varepsilon > 0, \quad E\left(\left|g\left((1+\varepsilon)\widetilde{N_1}\right)\right|\right) < +\infty$$

is replaced by

$$\exists p > 1, \quad E\left(\left|g\left(\widetilde{N_1}\right)\right|^p\right) < +\infty \quad .$$

To do this, combine corollary 3.12 (at $h = 0$ and $r = 1$) with theorem 3.4 (at $h = 0$).

Corollary 3.13. *Let N_1 be a measurable seminorm and N_2 be a measurable norm on H. Then:*

$$\forall p \geq 1\,,\ \forall h \in H,\ \widetilde{N_1} \text{ possesses an } L^p_{N_2} - \text{approximate limit at } h \text{ and } \widetilde{N_1}(h) = N_1(h).$$

Proof. We know from theorem 3.4 that, for every $p \geq 1$:

$$\sup_{\eta>0} E^{N_2}_\eta \left(\left|\widetilde{N_1}\,(\cdot + h)\right|^p \right) \leq E\left(\left|\widetilde{N_1}\,(\cdot + h)\right|^p \right) \leq E\left(\left(\widetilde{N_1} + N_1(h)\right)^p \right) < +\infty.$$

□

3.3. Approximate limits for some particular functions

The following theorems are essentially useful for functions which can be compared with the function $x \mapsto \exp x^2$. More precisely, we will consider a function $g : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\text{for every measurable seminorm } N, \exists \alpha > 0,\ E\left(\left| g\left(\alpha \widetilde{N}\right) \right| \right) < +\infty. \qquad (\mathcal{H})$$

Theorem 3.14. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be an increasing and convex function such that $(\mathcal{H})$ is verified. For every measurable seminorm N_1 and norm N_2 on H, there exist two positive constants c_1 and c_2 such that:*

$$\forall \eta > 0, \quad E^{N_2}_\eta \left(g\left(\widetilde{N_1} \right) \right) \leq c_1 + \frac{1}{2} g\left(c_2 \eta \right).$$

Remark 9. Thus, although $E\left(e^{\widetilde{N_1}^2} \right)$ may be infinite, we obtain $E^{N_2}_\eta \left(e^{\widetilde{N_1}^2} \right) < +\infty$ (using Fernique's theorem).

Proof. Let (ξ_i) be the complete orthonormal system given in lemma 3.7 for N_2.

Here, denote $R_n(h) = \sum_{i=1}^n < \xi_i, h > \xi_i$. In the proof of lemma 1 in [14], it is possible to choose for P_n a subsequence of R_n (in fact, to use this lemma, we have to assume that N_1 is a norm ; actually, this hypothesis is unnecessary, see lemma 4.4 of [21]). With this lemma, we see that the following quantity is a measurable seminorm on H:

$$N(h) = \sum_{n=1}^{\infty} 2^n N_1(Q_n h) \quad \text{where} \quad Q_n = P_{n+1} - P_n \quad \text{and} \quad Q_1 = P_1.$$

Moreover: $\widetilde{N}(w) = \sum_{n=1}^{\infty} 2^n N_1(Q_n w)$.

For $\widetilde{N}$, there exists $\alpha > 0$ such that $E\left(\left| g\left(\alpha \widetilde{N} \right) \right| \right) < \infty$. Choose n such that $2^{-n} \leq \alpha/2$.

If $m > n$:

$$\begin{aligned} N_1(P_m w) &\leq \sum_{i=n}^{m-1} N_1(Q_i w) + N_1(P_n w) \\ &\leq \frac{1}{2^n} \sum_{i=n}^{m-1} 2^i N_1(Q_i w) + N_1(P_n w) \\ &\leq \frac{1}{2^n} \widetilde{N} + N_1(P_n w) \\ &\leq \frac{\alpha}{2} \widetilde{N} + N_1(P_n w). \end{aligned}$$

If $\widetilde{N_2} < \eta$ then $N_2(P_n w) \leq c_n \widetilde{N_2} \leq c_n \eta$. So, because of the comparison of norms on spaces of finite dimension, we have: $N_1(P_n w) \leq k_n \eta$. Consequently: $\widetilde{N_1} \leq \frac{\alpha}{2} \widetilde{N} + k_n \eta$. We deduce :

$$E_\eta^{N_2}\left(g\left(\widetilde{N_1}\right)\right) \leq \frac{1}{2} E_\eta^{N_2}\left(g\left(\alpha \widetilde{N}\right)\right) + \frac{1}{2} g\left(2k_n \eta\right).$$

Then, we apply theorem 3.4 to $g(\alpha x)$. □

Theorem 3.15. *Let $p > 1$. Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function and assume that there exists an increasing and convex function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that f verifies $(\mathcal{H})$ and $|g(x)|^p \leq f(x)$ for all x. Then, for every measurable seminorm N_1 and norm N_2 on H:*

$$\forall h \in H,\ \forall q \in [1, p[, \quad g\left(\widetilde{N_1}\right) \text{ possesses an } L^q_{N_2} \text{ approximate limit at } h$$

which is $g(N_1(h))$.
In particular,

$$\lim_{\eta \to 0} E_{\eta,h}^{N_2}\left(g\left(\widetilde{N_1}\right)\right) = g\left(N_1(h)\right).$$

Proof. $f(2x)$ is an increasing and convex function such that $(\mathcal{H})$ is verified. So, we obtain with theorem 3.14:

$$\sup_{\eta \in]0,1]} E_\eta^{N_2}\left(f(2\widetilde{N_1})\right) < +\infty.$$

Moreover:

$$f\left(\widetilde{N_1}(\cdot + h)\right) \leq f\left(\widetilde{N_1} + N_1(h)\right) \leq \frac{1}{2}\left(f(2\widetilde{N_1}) + f\left(2N_1(h)\right)\right).$$
$$\Rightarrow \sup_{\eta \in]0,1]} E_\eta^{N_2}\left(f(\widetilde{N_1}(\cdot + h))\right) < +\infty.$$

We conclude with corollary 3.12 applied to g. □

Example 3.16. We can apply the previous result to $g(x) = \exp(ax^\beta)$ with $a \in \mathbb{R}$ and $\beta \in [0, 2]$ because:

$$|g(x)|^p \le \exp\left(p\,|a|\,x^\beta\right) \le \exp\left(p\,|a|\right) + \exp\left(p\,|a|\,x^2\right).$$

So, $\forall p \ge 1$, $\exp\left(a\widetilde{N}^\beta\right)$ possesses an $L^p_{N_2}$−approximate limit at h and $\exp\left(a\widetilde{N}^\beta\right)(h) = \exp\left(aN(h)^\beta\right)$.

In particular, we obtain theorem 1.3.

3.4. Consequences for Wiener chaos

In this section, we assume that $H = L^2(T, \mathcal{A}, m)$ where $(T, \mathcal{A}, m)$ is an atomeless separable σ-finite measurable space. It is well-known that the nth Wiener chaos is isomorphic to $H^{\odot n}$, the Hilbert space of symmetric n-tensors over H (see for example [25]).

For $f \in H^{\odot n}$, denote by $\delta^n(f)$ the n-tuple Wiener integral of f (for $h \in H$, we have $\delta(h) = \widetilde{h}$). We will use the notion of k-th limiting trace of f, denote by $\overrightarrow{Tr}^k f$ $\left(\in H^{\odot n-2k}\right)$, introduced by Johnson and Kallianpur [20]. There are many definitions of traces (see for example [20]). Here, we choose this particular definition because of the equivalence between the existence of $\overrightarrow{Tr}^k f$ for $0 \le k \le [n/2]$ and the existence of a limit in L^2 for $\left\langle f, \left(Q_p w\right)^{\otimes n}\right\rangle$ where $\left(Q_p\right)$ is any approximating sequence of projections (theorem 10.2 of [20]). More precisely, let us recall the definition of Johnson and Kallianpur of a $\mathcal{L}^2$ lifting associated to an f in $H^{\odot n}$.

Definition 3.17. *Let f belong to $H^{\odot n}$. The associated n-form $\Psi(f)$ given by $\Psi(f)(h) = \langle f, h^{\otimes n}\rangle$, for h in H, possesses a $\mathcal{L}^2$ lifting if for every approximating sequence of projections $\left(Q_p\right)$, $\Psi(f)\left(Q_p w\right)$ is a Cauchy sequence in L^2. In that case, there exists a random variable X in L^2 such that, for every approximating sequence of projections $\left(Q_p\right)$, $\Psi(f)\left(Q_p w\right)$ converges to X in L^2. X is called the $\mathcal{L}^2$ lifting of $\Psi(f)$.*

The main theorem of this section is the following.

Theorem 3.18. *Let $f \in H^{\odot 2n}$ such that $\Psi(f)$ possesses a $\mathcal{L}^2$ lifting and such that*

$$\forall (h_1, \dots, h_n) \in H^n, \quad \left\langle f, h_1^{\otimes 2} \otimes \cdots \otimes h_n^{\otimes 2}\right\rangle \ge 0. \tag{3.3}$$

Then there exist measurable seminorms $N_1, \dots, N_r$ on H and there exist real numbers $\alpha_1, \dots, \alpha_{r+1}$ and even natural numbers $k_1, \dots, k_r$ such that:

$$\delta^{2n}(f) = \sum_{i=1}^{r} \alpha_i \widetilde{N_i}^{k_i} + \alpha_{r+1}$$

($\max k_i = 2n$ and $|\{i, k_i = 2n\}| = 1 = |\{i, k_i = 2n-2\}|$).

Furthermore, under condition (3.3), the existence of a $\mathcal{L}^2$ lifting for $\Psi(f)$ is equivalent to:

$$\sup_{(e_i)\ \text{CONS of}\ H} \sum_{i_1,\dots,i_n=1}^{\infty} \left|\left\langle f, e_{i_1}^{\otimes 2} \otimes \cdots \otimes e_{i_n}^{\otimes 2}\right\rangle\right| < \infty. \tag{3.4}$$

Remark 10. Johnson and Kallianpur use the notion of $\mathcal{L}^2$ lifting to prove the Hu-Meyer formula ([18]) in the context of abstract Wiener spaces. The lifting of $\Psi(f)$ corresponds to the Stratonovitch integral of f even if, on abstract Wiener space, it is not really an integral. As we will see later in the proof, the formula in theorem 3.18 is simply the statement that the inverse Hu-Meyer formula is valid (in the standard Wiener space, this is the formula which gives an expression of $\delta^{2n}(f)$ in terms of the Stratonovitch integrals of the iterated traces of f).

We begin with a lemma. Let Π denote the set of orthogonal projections of H of finite dimension.

Lemma 3.19. *Let $f \in H^{\odot 2n}$ such that (3.4) is verified and such that:*

$$\forall h \in H, \quad \left\langle f, h^{\otimes 2n} \right\rangle \geq 0$$

then

$$\forall \varepsilon > 0, \exists Q_0 \in \Pi, \forall Q \in \Pi, \quad Q \perp Q_0 \Rightarrow P\left(\left\langle f, (Qw)^{\otimes 2n} \right\rangle > \varepsilon \right) < \varepsilon.$$

Proof. Define

$$l = \sup_{(e_i) \text{ CONS of } H} \sum_{i_1, \dots, i_n = 1}^{\infty} \left| \left\langle f, e_{i_1}^{\otimes 2} \otimes \cdots \otimes e_{i_n}^{\otimes 2} \right\rangle \right|.$$

For $\varepsilon > 0$, there exist a CONS (e_i) of H and a natural number R such that

$$l - \sum_{i_1, \dots, i_n = 1}^{R} \left| \left\langle f, e_{i_1}^{\otimes 2} \otimes \cdots \otimes e_{i_n}^{\otimes 2} \right\rangle \right| < \varepsilon^2.$$

Denote by Q_0 the orthogonal projection on span $(e_1, \dots, e_R)$. Let $Q \in \Pi$ such that $Q \perp Q_0$: it is possible to find a CONS (u_i) of H such that $u_1 = e_1, \dots, u_R = e_R$ and $(u_{R+1}, \dots, u_{R'})$ is an orthonormal basis of QH. Define:

$$Q^{\otimes 2n} f = \sum_{i_1, \dots, i_{2n} = R+1}^{R'} \left\langle f, u_{i_1} \otimes \cdots \otimes u_{i_{2n}} \right\rangle u_{i_1} \otimes \cdots \otimes u_{i_{2n}}.$$

$\overrightarrow{Tr}^k \left(Q^{\otimes 2n} f \right)$ exists for every k (proposition 3.2 of [20]) and we can apply lemma 4.3 of [20] to obtain:

$$\left\langle f, (Qw)^{\otimes 2n} \right\rangle = \sum_{k=0}^{n} \frac{(2n)!}{2^k (2n-2k)! k!} \delta^{2n-2k} \left(\overrightarrow{Tr}^k \left(Q^{\otimes 2n} f \right) \right).$$

We deduce:

$$E\left(\left\langle f, (Qw)^{\otimes 2n} \right\rangle \right) = \frac{(2n)!}{2^n n!} \overrightarrow{Tr}^n \left(Q^{\otimes 2n} f \right).$$
$$\Rightarrow P\left(\left\langle f, (Qw)^{\otimes 2n} \right\rangle > \varepsilon \right) \leq \frac{1}{\varepsilon} \frac{(2n)!}{2^n n!} \overrightarrow{Tr}^n \left(Q^{\otimes 2n} f \right).$$

Furthermore (propositions 3.1 and 3.2 of [20]):

$$\begin{aligned}\overrightarrow{Tr}^n\left(Q^{\otimes 2n}f\right) &= \sum_{i_1,\dots,i_n=R+1}^{R'} \left\langle f, u_{i_1}^{\otimes 2}\otimes\cdots\otimes u_{i_n}^{\otimes 2}\right\rangle \\ &\leq \sum_{i_1,\dots,i_n=R+1}^{R'} \left|\left\langle f, u_{i_1}^{\otimes 2}\otimes\cdots\otimes u_{i_n}^{\otimes 2}\right\rangle\right| \\ &\leq l - \sum_{i_1,\dots,i_n=1}^{R} \left|\left\langle f, e_{i_1}^{\otimes 2}\otimes\cdots\otimes e_{i_n}^{\otimes 2}\right\rangle\right| \\ &\leq \varepsilon^2.\end{aligned}$$

□

Corollary 3.20. *Let $f \in H^{\odot 2n}$ such that (3.4) is verified and such that:*

$$\forall (h_1, h_2) \in H^2, \quad \left\langle f, h_1^{\otimes 2}\otimes h_2^{\otimes 2n-2}\right\rangle \geq 0. \tag{3.5}$$

Then $N_f(h) = \langle f, h^{\otimes 2n}\rangle^{1/2n}$ is a measurable seminorm on H.

Proof. We know from lemma 3.3 of [13] that (3.5) is equivalent to the fact that N_f is a seminorm. Then, with the original definition of Gross for measurable seminorm ([13]), we see, by using the previous lemma, that N_f is a measurable seminorm. □

Remark 11. For $f \in H^{\odot 2n}$, define an operator $K(f)$ from $H^{\odot n}$ into $H^{\odot n}$ by:

$$\forall (u, v) \in H^{\odot n}\times H^{\odot n}, \quad \langle K(f)u, v\rangle = \langle f, u\otimes v\rangle.$$

If we assume that $K(f)$ is of trace class, we obtain:

$$\begin{aligned}&\sup_{(\phi_i)\text{ CONS of } H^{\odot n}} \sum_{i=1}^{\infty}\left|\left\langle f, \phi_i^{\otimes 2}\right\rangle\right| < \infty \\ \Rightarrow &\sup_{(e_i)\text{ CONS of } H} \sum_{i_1,\dots,i_n=1}^{\infty}\left|\left\langle f, e_{i_1}^{\otimes 2}\otimes\cdots\otimes e_{i_n}^{\otimes 2}\right\rangle\right| < \infty.\end{aligned}$$

Consequently, we see that corollary 3.20 generalizes theorem 3 of [13] where the result is proved when $K(f)$ is of trace class.

Remark 12. If $f \in H^{\odot 2n+1}$, it is possible to show that $h \mapsto \langle f, h^{\otimes 2n+1}\rangle$ is the $(2n+1)$th power of a seminorm if and only if, for all h_1, h_2 in H, $\left\langle f, h_1^{\otimes 2}\otimes h_2^{\otimes 2n-1}\right\rangle \geq 0$. But in that case, it implies $\forall h \in H$, $\langle f, h^{\otimes 2n+1}\rangle = 0$. So, we obtain, for any approximating sequence of projections (Q_k) : $\widetilde{N_f}^{2n+1} := \lim_{k\to\infty}\langle f, (Q_k w)^{\otimes 2n+1}\rangle = 0$.

Corollary 3.21. *Let $f \in H^{\odot 2n}$ such that (3.4) and (3.5) are verified then $\overrightarrow{Tr}^k(f)$ exists for every k ($\Psi(f)$ possesses a $\mathcal{L}^2$ lifting) and*

$$\widetilde{N_f}^{2n} = \sum_{k=0}^{n} \frac{(2n)!}{2^k (2n-2k)!k!} \delta^{2n-2k}\left(\overrightarrow{Tr}^k(f)\right).$$

Proof. For every approximating sequence of projections (Q_p), $N_f(Q_p w)$ converges in probability to $\widetilde{N_f}$ (corollary 3.20). Therefore, $N_f(Q_p w)^{2n}$ converges in probability to $\widetilde{N_f}^{2n}$. Furthermore, we know from theorem 5 of [13] that:

$$\forall \varepsilon > 0,\ P\left(N_f\left(Q_p w\right) > \varepsilon\right) \le P\left(\widetilde{N_f} > \varepsilon\right)$$

and from Fernique's theorem that $E\left(\widetilde{N_f}^{4n}\right) < \infty$. So, we obtain that $N_f(Q_p w)^{2n}$ converges to $\widetilde{N_f}^{2n}$ in L^2. That means that $\Psi(f) = N_f^{2n}$ possesses a $\mathcal{L}^2$ lifting and this lifting is $\widetilde{N_f}^{2n}$. Now, the corollary is nothing else but theorem 10.2 of [20]. □

Proof of theorem 3.18. Firstly, for $f \in H^{\odot 2n}$, we will prove that, under condition (3.3), we have:

$$(3.4) \Leftrightarrow \Psi(f) \text{ possesses a } \mathcal{L}^2 \text{ lifting.}$$

For $\Rightarrow$, this is given by corollary 3.21.

For $\Leftarrow$, if we assume that $\Psi(f)$ possesses a $\mathcal{L}^2$ lifting, then, for every k, $\overrightarrow{Tr}^k(f)$ exists (theorem 10.2 in [20]). In that case, $\overrightarrow{Tr}^n(f)$ is given by the following formula (formula 3.20 in [20]), for every CONS (e_i) of H:

$$\overrightarrow{Tr}^n(f) = \sum_{i_1,\dots,i_n=1}^{\infty} \left\langle f, e_{i_1}^{\otimes 2} \otimes \cdots \otimes e_{i_n}^{\otimes 2}\right\rangle.$$

Using condition (3.3), we see that:

$$\sup_{(e_i) \text{ CONS of } H} \sum_{i_1,\dots,i_n=1}^{\infty} \left|\left\langle f, e_{i_1}^{\otimes 2} \otimes \cdots \otimes e_{i_n}^{\otimes 2}\right\rangle\right| < \infty.$$

Now, let us prove theorem 3.18. For $n = 1$, if $f \in H^{\odot 2}$ verifies (3.3) and (3.4), we know from corollary 3.21 that:

$$\widetilde{N_f}^2 = \delta^2(f) + \overrightarrow{Tr} f \qquad (\overrightarrow{Tr} f \in \mathbb{R}).$$

Now, assume that the theorem is true for $1 \le k \le n-1$. With corollary 3.21, we write, for $f \in H^{\odot 2n}$ such that (3.3) and (3.4) are verified:

$$\delta^{2n}(f) = \widetilde{N_f}^{2n} - \sum_{k=1}^{n} \frac{(2n)!}{2^k (2n-2k)!k!} \delta^{2n-2k}\left(\overrightarrow{Tr}^k(f)\right).$$

To obtain the result, it is sufficient to prove that $\overrightarrow{Tr}^k(f)\left(\in H^{\odot 2n-2k}\right)$ verifies (3.3) and (3.4). For this, we use the expression of $\overrightarrow{Tr}^k(f)$ given by formula (3.20) in [20]. For every CONS (e_i) of H,

$$\overrightarrow{Tr}^k(f) = \sum_{i_1,\dots,i_{2n-2k}=1}^{\infty}\left(\sum_{j_1,\dots,j_k=1}^{\infty}\left\langle f, e_{j_1}^{\otimes 2}\otimes\cdots\otimes e_{j_k}^{\otimes 2}\otimes e_{i_1}\otimes\cdots\otimes e_{i_{2n-2k}}\right\rangle\right) \times e_{i_1}\otimes\cdots\otimes e_{i_{2n-2k}}.$$

With this formula, (3.4) is obvious for $\overrightarrow{Tr}^k(f)$. Let $h_1,\dots,h_{n-k}$ belong to h. We choose a CONS (e_i) of H such that span $(h_1,\dots,h_{n-k})\subset$ span $(e_1,\dots,e_{n-k})$, then:

$$\left\langle\overrightarrow{Tr}^k(f), h_1^{\otimes 2}\otimes\cdots\otimes h_{n-k}^{\otimes 2}\right\rangle = \sum_{j_1,\dots,j_k=1}^{\infty}\left\langle f, e_{j_1}^{\otimes 2}\otimes\cdots\otimes e_{j_k}^{\otimes 2}\otimes h_1^{\otimes 2}\otimes\cdots\otimes h_{n-k}^{\otimes 2}\right\rangle \geq 0.$$

So, (3.3) is verified for $\overrightarrow{Tr}^k(f)$. □

Remark 13. For a given f, it is possible to compute $\alpha_1,\dots,\alpha_{r+1},k_1,\dots,k_r$ by using corollary 3.21 but values of those numbers are not easy to write because in general $\overrightarrow{Tr}^k\left(\overrightarrow{Tr}^q(f)\right)\neq\overrightarrow{Tr}^{k+q}(f)$ (see [20]). However, it is an open question whether the assumptions imposed on f in theorem 3.18 imply that $\overrightarrow{Tr}^k\left(\overrightarrow{Tr}^q(f)\right)=\overrightarrow{Tr}^{k+q}(f)$ or not.

Remark 14. Let $f\in H^{\odot 2}(n=1)$ and define $K(f)$ as in remark 11, then:

$$(3.4)\Leftrightarrow K(f) \text{ is of trace class.}$$

In that case, we know there exists a CONS (e_i) of H such that

$$f=\sum_{i=1}^{\infty}\lambda_i e_i\otimes e_i \quad\text{with}\quad \sum_{i=1}^{\infty}|\lambda_i|<\infty.$$

Define $f_1=\sum_{i=1,\lambda_i>0}^{\infty}\lambda_i e_i\otimes e_i$ and $f_2=-\sum_{i=1,\lambda_i<0}^{\infty}\lambda_i e_i\otimes e_i$. For every i, $f_i\in H^{\odot 2}$ and $K(f_i)$ is of trace class. Furthermore:

$$\forall h\in H,\quad \left\langle f_i, h^{\otimes 2}\right\rangle\geq 0.$$

So, if f verifies (3.4) then f_i verifies (3.3) and (3.4). Consequently, since $\delta^2(f)=\delta^2(f_1)-\delta^2(f_2)$, we see, when $n=1$, that condition (3.4) is sufficient to obtain the conclusion in theorem 3.18. The question is open for $n\geq 2$.

Now, we can apply results of sections 3.2 and 3.3 to Wiener chaos. For example, theorem 3.18 with corollary 3.12 allows us to obtain:

Corollary 3.22. *Let $f \in H^{\odot 2n}$ such that (3.3) and (3.4) are verified, then, for every measurable norm N, $\forall p \geq 1$, $\forall h \in H$, $\delta^{2n}(f)$ possesses an L_N^p−approximate limit at h which is, with the notations of theorem 3.18, $\sum_{i=1}^{r} \alpha_i N_i(h)^{k_i} + \alpha_{r+1}$ (this limit is independent of the choice of N).*

Concerning the existence of a limit with the exponential function, we can use theorem 3.15 (example 3.16) and corollary 3.20 to obtain:

Corollary 3.23. *Let $f \in H^{\odot 2n}$ such that (3.4) and (3.5) are verified, then, for every measurable norm N, $\forall h \in H, \forall p \geq 1, \forall a \in \mathbb{R}, \forall \beta \in [0,2]$, $\exp\left(a\widetilde{N_f}^{\beta}\right)$ possesses an L_N^p approximate limit at h which is $\exp\left(aN_f(h)^{\beta}\right)$. In particular, we have:*

$$\lim_{\eta \to 0} E_{\eta,h}^N \left(\exp\left(a\widetilde{N_f}^{\beta}\right)\right) = \exp\left(aN_f(h)^{\beta}\right). \tag{3.6}$$

Example 3.24. Let $g \in H$ and define $f = g \otimes g$. (3.4) and (3.5) are verified. Moreover, $\widetilde{N_f}^2 = \delta(g)^2$. In that case, we obtain ($\beta = 1, a = 1$ and $h = 0$ in (3.6)):

$$\lim_{\eta \to 0} E_\eta^N \left(\exp\left(|\delta(g)|\right)\right) = 1,$$

which is once again example 3.9.

It is not possible to generalize this method for $g \in H^{\odot n}$ (with $n \geq 2$) without an additional hypothesis because in general, $f = g \overset{\wedge}{\otimes} g$ does not verify (3.4) and (3.5) (where $g \overset{\wedge}{\otimes} g$ is the projection on $H^{\odot 2n}$ of $g \otimes g$).

Now, we will obtain a similar result for $\delta^{2n}(f)$ under conditions (3.3) and (3.4).

Theorem 3.25. *Let $f \in H^{\odot 2n}$ and assume that f verifies (3.3) and (3.4). With notations of theorem 3.18, for every measurable norm N, $\forall h \in H, \forall p \geq 1, \forall (a,b) \in \mathbb{R}^2, \forall \beta \in [0,2]$, $\exp\left(a\left|\delta^{2n}(f) + b\right|^{\beta/2n}\right)$ possesses an L_N^p approximate limit at h which is $\exp\left(a\left|\sum_{i=1}^{r} \alpha_i N_i(h)^{k_i} + \alpha_{r+1} + b\right|^{\beta/2n}\right)$. In particular, we have:*

$$\lim_{\eta \to 0} E_{\eta,h}^N \left(\exp\left(a\left|\delta^{2n}(f) + b\right|^{\beta/2n}\right)\right) = \exp\left(a\left|\sum_{i=1}^{r} \alpha_i N_i(h)^{k_i} + \alpha_{r+1} + b\right|^{\beta/2n}\right). \tag{3.7}$$

Proof of the theorem. We will use corollary 3.12. It is sufficient to prove:

$$\sup_{\eta \in]0,1]} E_\eta^N \left(\exp\left(pa\left|\sum_{i=1}^{r} \alpha_i \widetilde{N_i}(\cdot + h)^{k_i} + \alpha_{r+1} + b\right|^{\beta/2n}\right)\right) < +\infty.$$

Because $k_i \frac{\beta}{2n} \in [0,2]$, we have:

$$E_\eta^N \left(\exp\left(pa\left|\sum_{i=1}^{r} \alpha_i \widetilde{N_i}(\cdot + h)^{k_i} + \alpha_{r+1} + b\right|^{\beta/2n}\right)\right)$$
$$\leq E_\eta^N \left(\exp\left(\sum_{i=1}^{r} p\,|a|\,\alpha_i^{\beta/2n} \widetilde{N_i}(\cdot + h)^{k_i\beta/2n} + p\,|a|\,(\alpha_{r+1} + b)^{\beta/2n}\right)\right)$$

and

$$\widetilde{N_i}\,(\cdot + h)^{k_i\beta/2n} \leq \left(\widetilde{N_i} + N_i\,(h)\right)^{k_i\beta/2n} \leq 1 + 2\left(\widetilde{N_i}^{\,2} + N_i\,(h)^2\right).$$

Consequently,

$$\begin{aligned}
&E_\eta^N\left(\exp\left(pa\left|\sum_{i=1}^r \alpha_i \widetilde{N_i}\,(\cdot + h)^{k_i} + \alpha_{r+1} + b\right|^{\beta/2n}\right)\right)\\
&\leq cE_\eta^N\left(\exp\left(\sum_{i=1}^r a_i \widetilde{N_i}^{\,2}\right)\right)\\
&\leq c\prod_{i=1}^r E_\eta^N\left(\exp\left(ra_i\widetilde{N_i}^{\,2}\right)\right)^{\frac{1}{r}}\\
&\Rightarrow \sup_{\eta\in]0,1]} E_\eta^N\left(\exp\left(pa\left|\sum_{i=1}^r \alpha_i \widetilde{N_i}\,(\cdot + h)^{k_i} + \alpha_{r+1} + b\right|^{\beta/2n}\right)\right)\\
&< +\infty \quad \text{(theorem 3.14).}
\end{aligned}$$

□

Remark 15. If $n = 1$, theorem 3.25 remains true under the only condition (3.4), see remark 14.

Example 3.26. Let $f \in H^{\odot 2}$ such that (3.4) is verified, we have: $\delta^2\,(f) = \widetilde{N_f}^{\,2} - \overrightarrow{Tr}\,f$. If we choose in (3.7) $\beta = 2, a = 1, b = \overrightarrow{Tr}\,f, h = 0$ and for N a quadratic norm (that is a norm defined by $N(h) = \left(\left\langle \zeta, h^{\otimes 2}\right\rangle\right)^{1/2}$, where $\zeta \in H^{\odot 2}$, $K(\zeta)$ is of trace class, injective and positive), we obtain:

$$\lim_{\eta\to 0} E_\eta^N\left(\exp\left(\left|\delta^2\,(f) + \overrightarrow{Tr}\,f\right|\right)\right) = 1.$$

So, we recover a result of Mayer-Wolf and Zeitouni ([22], lemma 2.5) which is:

$$\lim_{\eta\to 0} E_\eta^N\left(\exp\left(\delta^2\,(f) + \overrightarrow{Tr}\,f\right)\right) = 1.$$

If we choose $\beta = 2, a = 1, b = \overrightarrow{Tr}\,f, h = 0$ and for N a general measurable norm, we recover a result of [16] (theorem 8).

In those papers, the result is obtained, for the first one, with a F.K.G. inequality and for the second one, with a particular case of the Gaussian correlation conjecture.

References

1. Bakry, D., Michel, Sur les inégalités, F.K.G.: Séminaire de Probabilités XXVI. Lecture Notes in Math. **1526**, 170–188 (1990)
2. Ben Arous, G., Gradinaru, M., Ledoux, M.: Hölder norms and the support theorem for diffusions. Annales de l'I.H.P., Proba. et Stat. **30** (3), 415–436 (1994)

3. Bogachev, V.I.: The Onsager-Machlup functions for Gaussian measures. Doklady Math. **52** (2), 216–218 (1995)
4. Borell, C.: A note on Gauss measures which agree on small balls. Ann. Inst. H. Poincaré **13**, 231–238 (1977)
5. Brenier, Y.: Polar factorization and monotone rearrangement of vector valued functions. Comm. Pure Appl. Math. **44**, 375–417 (1991)
6. Brascamp, H.J., Lieb, E.H.: On extensions of Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation. J. Funct. Anal. **22**, 366–389 (1976)
7. Caffarelli, L.A.: Monotinicity properties of optimal transportation and the FKG and related inequalities. Commun. Math. Phys. **214** (3), 547–563 (2000)
8. Caffarelli, L.A.: The regularity of mappings with a convex potential. J. A. M. S. **5**, 99–104 (1992)
9. Caffarelli, L.A.: Boundary regularity of maps with convex potential. Comm. Pure Appl. Math. **45**, 1141–1151 (1992)
10. Carmona, R.A., Nualart, D.: Traces of random variables on Wiener space and Onsager-Machlup Functional. J. Funct. Anal. **107** (2), 402–438 August 1, 1992
11. Cordero-Erausquin, D.: Some applications of mass transport to Gaussian type inequalities. Arch. Rational Mech. Anal. **161**, 257–269 (2002)
12. Fang, S.: Sur la continuité approximative forte des fonctionnelles d'Itô. Stochastics and Stochastics Reports **36**, 193–204 (1991)
13. Gross, L.: Measurable Functions on Hilbert space. Trans. Am. Math. Soc. **105**, 372–390 (1962)
14. Gross, L.: Abstract Wiener spaces. Proceedings of the Fifth Berkeley Symp. Math. Statist. Prob. II, part I, 31–42 (1967)
15. Hargé, G.: A particular case of correlation inequality for the Gaussian measure. Ann. Probab. **27** (4), 1939–1951 (1999)
16. Hargé, G.: Limites approximatives sur l'espace de Wiener. Potential Anal. **16** (2), 169–191 (2002)
17. Hu, Y.: Itô-Wiener chaos expansion with exact residual and correlation, variance inequalities. J. Th. Probab. **10**, 835–848 (1997)
18. Hu, Y.Z., Meyer, P.A.: Sur les intégrales multiples de Stratonovitch. Séminaire de probabilités **XXII**, 72–81
19. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. North-Holland, 1981
20. Johnson, G.W., Kallianpur, G.: Homogeneous chaos, p-forms, scaling and the Feynman integral. Transactions of the A.M.S. **340** (2), 503–548 (1993)
21. Kuo, H.H.: Gaussian measures in Banach spaces. Lecture Notes in Math. **463**, Springer, 1975
22. Mayer-Wolf, E., Zeitouni, O.: Onsager Machlup functionals for non trace class SPDE'S. Proba. Th. Rel. Fields **95**, 199–216 (1993)
23. McCann, R.J.: Existence and uniqueness of monotone measure-preserving maps. Duke Math. J. **80**, 309–323 (1995)
24. Millet, A., Nualart, D.: Support theorems for a class of anticipating stochastic differential equations. Stochastics and Stochastics Reports **39**, 1–24 (1992)
25. Nualart, D.: The Malliavin Calculus and Related Topics. Springer-Verlag, Berlin, Heidelberg, New-York, 1995
26. Nualart, D., Zakai, M.: Generalized Stochastic Integrals and the Malliavin Calculus. Proba. Th. Rel. Fields **73**, 255–280 (1986)
27. Pitt, L.D.: A gaussian correlation inequality for symetric convex sets. Ann. Probab. **5**, 470–474 (1977)

28. Rosinski, J.: On Stochastic Integration by Series of Wiener Integrals. Appl. Math. Optim. **19**, 137–155 (1989)
29. Royer, G.: Une initiation aux inégalités de Sobolev logarithmiques. Société Mathématique de France, Cours spécialisé **5**, (1999)
30. Schechtman, G., Schlumprecht, T., Zinn, J.: On the Gaussian measure of the intersection of symmetric convex sets. Ann. Probab. **26**, 346–357 (1998)
31. Shepp, L.A., Zeitouni, O.: A note on conditional exponential moments and Onsager Machlup Functionals. Ann. Probab. **20**, 652–654 (1992)
32. Stroock, D.W., Varadhan, S.R.S.: On the support of diffusion processes with applications to the strong maximum principal. Sixth Berkeley Symp. Math. Statist. Prob., 333–359 (1972)
33. Sugita, H.: Hu-Meyer's mutiple Stratonovich integral and essential continuity of multiple Wiener integral. Bull. Sc. Math. **113**, 463–474 (1989)
34. Sugita, H.: Various topologies in the Wiener space and Lévy's stochastic area. Proba. Th. Rel. Fields **91** (3–4), 283–296 (1992)
35. Szarek, S., Werner, E.: A nonsymmetric correlation inequality for Gaussian measure. J. Multivariate Anal. **68**, 193–211 (1999)