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N.V. Krylov

One more square root law for Brownian motion and its application to SPDEs

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Abstract. It is proved that there is a function $p(c) \geq 0$ such that $p(c) > 0$ if c is large enough, and (a.s.) for any $t \in [0, 1]$, the trajectory of Brownian motion after time t is contained in a parallel shift of the box $[0, 2^{-k}] \times [0, c2^{-k/2}]$ for all k belonging to a set with lower density $\geq p(c)$. This law of square root helps show that solutions of one-dimensional SPDEs are Hölder continuous up to the boundary.

1. Introduction

There is a generic saying dating back some 20 years ago that any statement about properties of one-dimensional Brownian motion is either wrong or already known. As far as we understand, the law presented below is true, was not known before, and even more than that, has important applications in the theory of stochastic partial differential equations.

This new law can be described in the following way. Let w_t, $t \geq 0$, be a standard one-dimensional Wiener process and let $c \in (0, \infty)$ be a constant. For any $t \geq 0$ and $k = 0, 1, 2, ...$, we say that $w.$ after time t is contained in a (parabolic) c-box of size 2^{-k} if there is a number a such that

$$a \leq w_{t+s} \leq a + c2^{-k/2} \quad \text{for} \quad 0 \leq s \leq 2^{-k}. \tag{1.1}$$

The law of iterated logarithm applied to $w_{t+2^{-k}}$ implies that for each t (a.s.) there are infinitely many k's such that $w.$ after time t is *not* contained in c-boxes of size 2^{-k}. Actually, the law of large numbers for stationary sequences shows that an even stronger statement is true and this violation of containment has a pattern described in terms of the density of sets $A \subset \{0, 1, 2, ...\}$ defined as $\lim_{n\to\infty} \#(A \cap [0, n])/n$, where $\#B$ is the number of elements in B. Namely, for each c and t, with probability one the density of k's, for which $w.$ after time t is not contained in c-boxes of size 2^{-k}, is strictly bigger than zero (see Remark 2.2).

N.V. Krylov: University of Minnesota, 127 Vincent Hall, Minneapolis, MN 55455, USA.
e-mail: krylov@math.umn.edu

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Furthermore, one can show (see Remark 2.3) that if $c < 1$, then with probability one there is a $t = t(\omega) \in [0, 1]$ such that $w_{\cdot}$ after time t is *not* contained in c-boxes of size 2^{-k} if k is large enough.

If c is large enough the situation is different. Our square root law says, in particular, that *there is a function* $p(c) \geq 0$ *such that* $p(c) > 0$ *if* c *is large enough, and (a.s.) for any* $t \in [0, 1]$, $w_{\cdot}$ *after time* t is *contained in a* c*-box of size* 2^{-k} *for* k*'s in a set with lower density* $\geq p(c)$. The precise statement of the law is to be found in Theorem 2.1.

It is worth noting again that if we are only interested in one particular value of t, then the result is a straightforward consequence of the above mentioned law of large numbers for stationary sequences. One need not even have c large enough, since (a.s.) the density of k's for which $w_{\cdot}$ after time t is contained in a c-box of size 2^{-k} equals the probability $p_0(c)$ that $w_{\cdot}$ after time 0 is contained in a c-box of size 1 and this probability is strictly bigger than zero for any $c > 0$. However, the exceptional set of ω depends on t and for $c < 1$ the union of exceptional sets has probability one, as has been mentioned above. Therefore, the main emphasis of the law is on the fact that for large c the lower density is bounded away from zero by a strictly positive constant depending only on c.

The origin of the name "square root law" lies in the fact that (1.1) is equivalent to

$$\operatorname*{osc}_{[t,t+r]} w_{\cdot} \leq c\sqrt{r} \tag{1.2}$$

with $r = 2^{-k}$. No logarithms or iterated logarithms are involved and this makes such statements quite valuable in connection with second-order elliptic and parabolic partial differential equations where one uses self similarity to a very large extent. We show one of applications of the square root law in Section 4.

Before, one of the square root laws was proved by B. Davis [1]. It says that (a.s.) if $c < 1$, then

(a) there exists a $t \in [0, 1]$ and an $\varepsilon > 0$ such that

$$w_{t+r} - w_t \geq c\sqrt{r} \quad \text{for all} \quad r \in [0, \varepsilon]$$

but if $c > 1$, then the opposite is true, that is

(b) for any $t \in [0, 1]$ and any $\varepsilon > 0$ there is $r \in (0, \varepsilon]$ such that

$$w_{t+r} - w_t < c\sqrt{r}.$$

The author came across the necessity of the square root laws by trying to solve a problem in SPDEs. This problem reduces almost immediately to a problem for the heat equation in a random domain. To be more specific take constants $T, \delta \in (0, \infty)$ and consider the following boundary-value problem

$$\begin{aligned} &D_t u(t,x) + \delta D_x^2 u(t,x) = 0, \quad t \in [0,T),\, w_t < x < w_t + 1, \\ &u(t, w_t) = 0, \quad u(t, w_t + 1) = 1, \quad \text{for} \quad t \in [0, T), \\ &u(T, x) = x - w_T \quad \text{for} \quad w_T \leq x \leq w_T + 1. \end{aligned} \tag{1.3}$$

For each trajectory $w.$ this is a deterministic problem admitting a unique Perron solution (coinciding with the probabilistic one). The solution is infinitely differentiable inside the domain and the hardest issue is if and how the solution agrees with the given values on the lateral boundary. If δ is large enough, then one can use the theory of SPDEs itself to show that the solution is Hölder continuous in the closure of the domain. However, at this moment there are no SPDEs tools to show even the continuity of u up to the boundary if δ is small. We succeeded in doing this in [4] by noticing that for $w.$ being just a deterministic continuous function, the Hölder continuity of the solution at the boundary is possible only if part (b) of Davis's law holds with a finite c (for our deterministic $w.$). Since we knew that for large δ and for $w.$ being a trajectory of the Wiener process the solution is Hölder continuous at the boundary, we concluded that part (b) of Davis's law holds and then the continuity up to the boundary for small δ became a rather trivial exercise on self similarity and Blumenthal's zero-one law. The reader understands that, to get the continuity, in this argument we could skip appealing to the result from SPDEs for large δ. However, the point is that Davis's law is not sufficient to proving any estimate of the modulus of continuity.

In this article, we prove that the solutions are *Hölder* continuous up to the boundary no matter how small $\delta > 0$ is.

Notice that if in part (b) of Davis's law we knew that r is comparable with ε, then the Hölder continuity would follow in almost the same way as the continuity is derived. The matter of fact is that then the boundary would possess a good kind of self similarity. However, the best we could do trying to use Davis's arguments looks like $r \in [\varepsilon^2, \varepsilon]$ and this led only to a logarithmic modulus of continuity no matter δ is large or small. Yet we knew that for large δ the solutions *are* Hölder continuous up to the boundary. The only explanation of this we had was that (1.2) with $r = 2^{-k}$ should occur for large k "on a regular basis" something like for k taken from a kind of arithmetic progression. The main idea always was to rely on self similarity. However, it is rather easy to see that (a.s.) for any $a > 0$

$$\varlimsup_{k\to\infty} \sup_{t\in[0,1]} 2^{ak/2} \operatorname*{osc}_{[t,t+2^{-ak}]} w. \geq \varlimsup_{k\to\infty} 2^{ak/2} \operatorname*{osc}_{[0,2^{-ak}]} w. = \infty.$$

Therefore, no appropriate deterministic arithmetic progression can be found and we decided to interpret "on a regular basis" in the sense of the denseness of the set of k's for which (1.2) holds with $r = 2^{-k}$. This is how SPDEs inspired discovering our square root law.

The article is organized as follows. The law of square root is proved in Section 2. A deterministic version of problem (1.3) is considered in Section 3. Its stochastic version along with an application to SPDEs is given in Section 4.

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2. The law of square root

Let C be the set of real-valued continuous functions on $[0,\infty)$. For $x_\cdot \in C$ set $x_s = x_0$ for $s \le 0$ and for $n = 0, 1, 2, \ldots$ and $t \ge 0$ introduce

$$\Delta_n^{\pm}(x_\cdot, t) = 2^{n/2} \operatorname*{osc}_{[t, t\pm 2^{-n}]} x_\cdot \, .$$

If $c \in (0,\infty)$, then define

$$N_n^{\pm}(x_\cdot, c, t) = \#\{k = 0, \ldots, n : \Delta_k^{\pm}(x_\cdot, t) > c\} = \sum_{k=0}^{n} I_{\Delta_k^{\pm}(x_\cdot, t) > c}.$$

Finally, introduce

$$\Delta_n^{\pm}(t) = \Delta_n^{\pm}(w_\cdot, t), \quad N_n^{\pm}(c,t) = N_n^{\pm}(w_\cdot, c, t),$$

$$\Delta_n(t) = \Delta_n^{+}(t), \quad N_n(c,t) = N_n^{+}(c,t).$$

Theorem 2.1. *For any* $c, T \in (0,\infty)$

$$\overline{\lim_{n\to\infty}} \sup_{t\in[0,T]} \frac{1}{n+1} N_n^{\pm}(c,t) = \alpha(c) \tag{2.1}$$

(a.s.), where $\alpha(c)$ *is a deterministic function of* c *and* $\alpha(c) \to 0$ *as* $c \to \infty$.

The following is a version of a lower estimate of $\alpha(c)$ suggested by M. Safonov. We use the notation $\mathcal{F}_t^w = \sigma(w_s : s \le t)$, $\mathcal{F}_{0+}^w = \bigcap_{t>0} \mathcal{F}_t^w$.

Remark 2.2. The sequence of processes $\xi_t(n) := 2^{n/2} w_{t2^{-n}}$, $t \in [0,1]$, $n = 0, 1, \ldots$, is a stationary $C([0,1])$-valued sequence. Therefore, the sequence $\Delta_n(0)$ is stationary too. The latter sequence is also ergodic since its tail σ-field is contained in $\mathcal{F}_{0+}^w$, which consists only of sets having probability zero or one by Blumenthal's law. Hence, (a.s.)

$$\alpha(c) \ge \lim_{n\to\infty} \frac{1}{n+1} N_n(c,0) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} I_{\Delta_k > c} = P(\operatorname*{osc}_{[0,1]} w_\cdot \ge c).$$

Remark 2.3. It follows from Davis's square root law (a) that if $c \in (0,1)$, then (a.s.) there is a $t \in [0,1]$ and an $m \ge 1$ such that $\Delta_k(c,t) > c$ for all $k \ge m$. Hence,

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \frac{1}{n+1} N_n(c,t) = 1$$

and, in particular, $\alpha(c) = 1$ for $c \in (0,1)$.

Remark 2.4. If $T = \infty$, then (2.1) is wrong, since $\sup_{t\ge 0} N_n(c,t) = n+1$ (a.s.).

We derive Theorem 2.1 from the following estimate, which also plays a central role in the future applications to SPDEs.

Lemma 2.5. *Take* $\alpha \in (0, 1)$ *and* $c \in (0, \infty)$ *and introduce*

$$p = p(c) := P(\sup_{t \le 1} w_t - \inf_{t \le 1} w_t \ge c(\sqrt{2} - 1)/2), \quad r = r(\alpha, c) := \frac{\alpha(1-p)}{p(1-\alpha)},$$

$$\beta = \beta(\alpha, c) := 2\frac{(r-1)p+1}{r^\alpha} = 2\frac{1-p}{1-\alpha} r^{-\alpha}.$$

Assume that $r \ge 1$ *(or equivalently* $\alpha \ge p$*). Then, for any* $n = 1, 2, \dots$*, we have*

$$P_n(c, \alpha) := P(\sup_{t \in [0,T]} N_n^{\pm}(c, t) \ge (n+1)\alpha) \le (T + 2^{-n}) r^{1-\alpha} \beta^n. \tag{2.2}$$

Proof. First observe that $I_{(c,\infty)}(x)$ is a lower semicontinuous function of x, $\Delta_k^{\pm}(t)$ are continuous in t, so that $N_n^{\pm}(c, t)$ is lower semicontinuous, and $\sup_{t \in [0,T]} N_n^{\pm}(c, t)$ are random variables. Therefore, (2.2) makes sense.

Next, notice that $\hat{w}_t := w_T - w_{T-t}$ is a Wiener process on $[0, T]$. Continue it arbitrarily for $t \ge T$ in such a way that $\hat{w}_t$ becomes a Wiener process on $[0, \infty)$. Then as is easy to see for $t \in [0, T]$ it holds that

$$N_n^-(c, t) = N_n^+(\hat{w}_{\cdot \wedge T}, c, T - t) \le N_n^+(\hat{w}_\cdot, c, T - t). \tag{2.3}$$

It follows that it suffices to prove (2.2) for the + sign.

Define $\kappa_n(t) = 2^{-n}[2^n t]$ and notice that, for $m \le n$, we have $\kappa_n(t) \le t$, $t + 2^{-m} \le \kappa_n(t) + 2^{-n} + 2^{-m} \le \kappa_n(t) + 2^{-(m-1)}$. Hence, for $1 \le m \le n$,

$$\Delta_m(t) \le 2^{m/2} \operatorname*{osc}_{[\kappa_n(t), \kappa_n(t) + 2^{-(m-1)}]} w = \sqrt{2}\Delta_{m-1}(\kappa_n(t)),$$

and for $n \ge 1$ and $t \in [0, T]$,

$$N_n(c, t) \le 1 + N_{n-1}(2^{-1/2}c, \kappa_n(t)) \le 1 + \max_{k=0,\dots,[2^n T]} N_{n-1}(2^{-1/2}c, k2^{-n}).$$

Adding to this that the distribution of $N_n(c, t)$ is independent of t and $[2^n T] + 1 \le 2^n T + 1$, we conclude that

$$P_n(c, \alpha) \le (T + 2^{-n}) 2^n P(N_{n-1}(c', 0) \ge \alpha'), \tag{2.4}$$

where

$$c' = 2^{-1/2}c, \quad \alpha' = \alpha(n+1) - 1.$$

Now it is the issue of large deviations for $N_{n-1}(c', 0)$.

Let

$$\tau_k = \inf\{t \ge 0 : \operatorname*{osc}_{[0,t]} w_\cdot \ge 2^{-k/2}c'\}.$$

Then ($\operatorname{osc}_{[0,t]} w_\cdot \le \operatorname{osc}_{[0,s]} w_\cdot + \operatorname{osc}_{[s,t]} w_\cdot$ for $0 \le s \le t$)

$$\{\omega : \Delta_k(0) \ge c'\} = \{\omega : \tau_k \le 2^{-k}\} \subset \{\omega : \Delta_k(\tau_{k+1}) \ge (1 - 2^{-1/2})c'\}.$$

Next we use Chebyshev's inequality, the fact that τ_k are stopping times and that w_t is a strong Markov process. We also notice that (a.s.)

$$E\{\exp(\lambda I_{\Delta_k(\tau_{k+1})\geq(1-2^{-1/2})c'})|\mathcal{F}^w_{\tau_{k+1}}\}$$
$$= E\exp(\lambda I_{\Delta_k(0)\geq(1-2^{-1/2})c'}) = e^{\lambda}p + 1 - p.$$

Then, for any $\lambda \geq 0$, we find

$$P(N_{n-1}(c', 0) \geq \alpha') \leq e^{-\lambda\alpha'}E\exp(\lambda\sum_{k=0}^{n-1} I_{\tau_k\leq 2^{-k}})$$
$$\leq e^{-\lambda\alpha'}E\exp(\lambda\sum_{k=1}^{n-1} I_{\tau_k\leq 2^{-k}})\exp(\lambda I_{\Delta_0(\tau_1)\geq(1-2^{-1/2})c'})$$
$$= e^{-\lambda\alpha'}(e^{\lambda}p + 1 - p)E\exp(\lambda\sum_{k=1}^{n-1} I_{\tau_k\leq 2^{-k}}) \leq ... \leq e^{-\lambda\alpha'}(e^{\lambda}p + 1 - p)^n.$$

By combining this with (2.4) and taking $\lambda = \ln r$, we arrive at (2.2). The lemma is proved. □

Remark 2.6. For any $\alpha \in (0, 1)$, we have $p(c) \to 0, r(\alpha, c) \to \infty$, and $\beta(\alpha, c) \to 0$ as $c \to \infty$.

Proof of Theorem 2.1. First we prove that the left-hand side of (2.1) is a constant (a.s.). The explanation in the beginning of the proof of Lemma 2.5 shows that the left-hand side of (2.1) for the $+$ sign, which we denote by $\xi(T)$, is a random variables.

By using the self-similarity of the Wiener process one easily concludes that the distribution of $\xi(2^k)$, $k = 0, \pm1, \pm2, ...$, is independent of k. In addition obviously, $\xi(T) \geq \xi(S)$ if $S \in (0, T]$. It follows that, for any $S, T > 0$ we have $\xi(S) = \xi(T)$ (a.s.). Furthermore, it is easily seen that $\xi(T)$ is measurable with respect to $\mathcal{F}^w_{2T}$. Since $\xi(S) = \xi(T)$ (a.s.), $\xi(T)$ is also measurable with respect to the completion of $\mathcal{F}^w_{0+} = \bigcup_{S>0}\mathcal{F}^w_{2S}$. By Blumenthal's zero-one law we finally get that $\xi(T) = \text{const}$ (a.s.), where the constant is independent of T. That the same constant suits the $-$ sign as well follows from (2.3) and the equality $N_n^+(\hat{w}_{\cdot\wedge T}, c, T-t) = N_n^+(\hat{w}_{\cdot}, c, T-t)$, which is valid if $T \geq t \geq 1$.

It only remains to show that $\alpha(c) \to 0$ as $c \to \infty$. However, for any $\alpha \in (0, 1)$ and $m \geq 1$,

$$P(\xi(1) > \alpha) \leq P(\sup_{n\geq m}\frac{1}{n+1}\sup_{t\in[0,1]} N_n(c, t) \geq \alpha)$$
$$\leq \sum_{n=m}^{\infty} P(\sup_{t\in[0,1]} N_n(c, t) \geq (n+1)\alpha)$$

and the latter tends to zero as $m \to \infty$, due to Lemma 2.5 if c is so large that $\beta < 1$. The theorem is proved. □

3. The heat equation in curvilinear cylinders

Take a $\delta \in (0, \infty)$ and an $x. \in C$. Introduce

$$Q := Q(x.) := \{(s, y) : s \geq 0, x_s < y < x_s + 1\}$$

and let $u(t, x)$ be the probabilistic solution of

$$\begin{aligned} &D_t u(t, x) + (1/2)\delta D^2_{xx} u(t, x) = 0, \quad (t, x) \in Q, \\ &u(t, x_t) = 0, \quad u(t, x_t + 1) = 1, \quad t \geq 0. \end{aligned} \tag{3.1}$$

Recall that the value of the probabilistic solution u at a point $(t, x) \in Q$ is determined in the following way. Define

$$\tau = \inf\{s > 0 : (t + s, x + w_s\sqrt{\delta}) \notin Q\}.$$

The following argument showing that $\tau < \infty$ (a.s.) was suggested by the referee. We have

$$\begin{aligned} P(\tau > s) &\leq P(x_{t+s} < x + w_s\sqrt{\delta} < x_{t+s} + 1) \leq \sup_y P(y < w_s\sqrt{\delta} < y + 1) \\ &= \sup_y P(y/\sqrt{s\delta} < w_1 < (y + 1)/\sqrt{s\delta}) \leq \frac{1}{\sqrt{2\pi s\delta}} \sup_y e^{-y^2/2} \to 0 \end{aligned}$$

as $s \to \infty$.

Finally, $u(t, x)$ is determined as the probability that $(t, x + w_t\sqrt{\delta})$ meets $(t, x_t + 1)$ before (t, x_t), that is

$$u(t, x) = P(x + w_\tau\sqrt{\delta} = x_{t+\tau} + 1). \tag{3.2}$$

It is well known that u is infinitely differentiable in Q and satisfies (3.1). It is also well known that u is continuous up to the boundary $\{(s, x_s), (s, x_s + 1) : s \geq 0\}$ only at its regular points, that is at those (t, x) at which $P(\tau > 0) = 0$. The following theorem will allow us to give conditions under which u is *Hölder* continuous up to the boundary.

Theorem 3.1. *Take some constants $c \geq 0$, $d > 0$ and for $a \in \mathbb{R}$ define*

$$\tau_{d,a} = \inf\{t \geq 0 : d/\sqrt{2} + w_t = a\},$$

$$\gamma(c, d, \delta) = P(\tau_{d,d} \wedge (\delta/2) < \tau_{d,-c}).$$

Then for $x \in (0, 1)$ and $t \geq 0$ we have

$$u(t, x_t + x) \leq [\gamma(c, d, \delta)]^{M_n(x.,c,t)-k}, \tag{3.3}$$

where $M_n(x., c, t) = n + 1 - N_n(x., c, t)$, $n = n(x/d)$, $k = k(c + d)$, and

$$n(x) = [(-2\log_2 x)_+], \quad k(d) = 2 + [(2\log_2 d)_+].$$

Remark 3.2. The result is reasonable in the following sense. If $x_t \equiv 0$ so that Q is a straight cylinder, then $u(t,x) = x$ and $u(t,x)$ decays to the boundary value on $\{x = 0\}$ linearly. In this case for $c = 0$ we have $M_n(x.,c,t) = n+1$ and as $x \downarrow 0$, dropping obvious values of arguments, we have

$$\gamma^{M_n(t)-k} \asymp \gamma^{\log_2 x^{-2}} = x^{\log_2 \gamma^{-2}}.$$

Furthermore, $\log_2 \gamma^{-2} \to 1$ as $d \downarrow 0$ since $\gamma(0,d,\delta) \to 1/\sqrt{2}$ for any $\delta > 0$.

Remark 3.3. The self-similarity of the Wiener process shows that $\gamma(c,d,\delta) = \gamma(c/\sqrt{\delta}, d/\sqrt{\delta}, 1)$.

In Section 4 we use the following result concerning the probabilistic solution of the equation

$$D_t u = (1/2)\delta D^2_{xx} u$$

in $Q \setminus (\{0\} \times (0,1))$ with boundary condition $u(t, x_t + 1) = 1$ for $t \geq 0$ and zero on the rest of ∂Q. The result is obtained from Theorem 3.1 by reversing time and observing that the right-hand side of (3.2) becomes smaller if we take $Q \cap \{t < T\}$ in place of Q.

Corollary 3.4. *For $t > 0$ and $x \in (0,1)$ we have*

$$P(x_t + x + w_\tau\sqrt{\delta} = x_{t-\tau} + 1) \leq [\gamma(c,d,\delta)]^{n+1-N_n^-(x.,c,t)-k},$$

where $\tau = \inf\{s > 0 : (t-s, x_t + x + w_s\sqrt{\delta}) \notin Q\}$.

One may also be interested in equations like (3.1) when δ is replaced with a function $a(t,x) \geq \delta$. The following theorem addresses this issue and, in particular, implies Theorem 3.1.

Theorem 3.5. *Let w_t be a Wiener process with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete σ-fields and let a_t be a bounded process predictable with respect to $\{\mathcal{F}_t, t \geq 0\}$ and such that $a(t) \geq \delta$. Assume that $x_0 = 0$ and for $x \in (0,1)$ define*

$$\xi_s = \int_0^s \sqrt{a(r)}\, dw_r, \quad \tau(x) = \inf\{s > 0 : (s, x + \xi_s) \notin Q\}.$$

Then $E\tau(x) \leq (2\delta)^{-1}$ and

$$P(x + \xi_{\tau(x)} = x_{\tau(x)} + 1) \leq [\gamma(c,d,\delta)]^{M_n(x.,c,0)-k}. \tag{3.4}$$

Remark 3.6. The function $M_n(x.,c,0)$ is a piecewise constant right-continuous increasing function of c. Moreover, the left-hand side of (3.4) is independent of c and γ and is a continuous function of c. It follows that the inequality

$$P(x + \xi_{\tau(x)} = x_{\tau(x)} + 1) \leq [\gamma(c,d,\delta)]^{M_n(x.,c-,0)-k}, \tag{3.5}$$

which is formally weaker than (3.4) for each particular jump point c, actually is equivalent to (3.4) in the range $c \geq 0$.

To prove Theorem 3.5 we need two lemmas.

Lemma 3.7. *Assume that $x_t \geq -c2^{-p/2}$ for $t \in [0, 2^{-p}]$, where p is an integer. Introduce*

$$Q(p) = \{(t, x) : t \in [0, 2^{-p}], x_t < x < 2^{-p/2}d\}$$

assume that $Q(p+1) \neq \emptyset$ and call $\{(t, x) \in \bar{Q}(p) : x = x_t\}$ the curvilinear lateral boundary of $Q(p)$. Then, for $(t, x) \in Q(p+1)$, the probability that $(t+s, x+\xi_s)$ as a function of s exits from $Q(p)$ before reaching its curvilinear lateral boundary is less than $\gamma(c, d, \delta)$.

Proof. Bearing in mind the possibility of rescaling, we assume that $p = 0$ without losing generality. Then denote by $\alpha(t, x)$ the probability in question. Observe that by assumption $Q(0)$ lies in the box $[0, 1] \times (-c, d)$. Therefore, if $(t, x) \in Q(0)$ and $(t+s, x+\xi_s)$ exits from $Q(0)$ before reaching its curvilinear lateral boundary, then the process $(t + s, x + \xi_s)$ hits the right part of the lateral boundary or the top of the box before reaching the left part of the lateral boundary. The same holds if we change the time scale, that is, for any strictly increasing function $\phi(s)$, such that $\phi(0) = 0$ and $\phi(\infty) = \infty$, we have

$$\alpha(t, x) \leq P\{\theta(t, x, d) \wedge \psi(1-t) < \theta(t, x, -c)\}, \tag{3.6}$$

where ψ is the inverse of ϕ and

$$\theta(t, x, a) = \inf\{s \geq 0 : x + \xi_{\phi(s)} = a\}.$$

It is convenient to take ϕ so that

$$\psi(s) = \int_0^s a(r)\, dr$$

because then the quadratic variation of $\xi_{\phi(s)}$ equals

$$\int_0^{\phi(s)} a(r)\, dr = \psi(\phi(s)) = s,$$

so that $\bar{w}_s := \xi_{\phi(s)}$ is a Wiener process. Also in this case we have $\psi(s) \geq \delta s$ and hence (3.6) yields

$$\alpha(t, x) \leq P\{\tau(d, x) \wedge (\delta - \delta t) < \tau(-c, x)\}, \tag{3.7}$$

where

$$\tau(a, x) = \inf\{r \geq 0 : x + \bar{w}_r = a\}.$$

It only remains to notice that, for $(t, x) \in Q(1)$, the right-hand side of (3.7) is obviously less than its value at $x = d/\sqrt{2}$ and $t = 1/2$ and this value is $\gamma(c, d, \delta)$. The lemma is proved. □

In the following lemma we prove the first assertion of Theorem 3.5. Its proof differs from the original one owing to inspiring comments by the referee.

Lemma 3.8. *We have $E\tau(x) \leq (2\delta)^{-1}$.*

Proof. Fix an $x \in (0, 1)$ and define $\tau_0 = 0$,

$$\tau_{n+1} = \inf\{t \geq \tau_n : |\xi_t - \xi_{\tau_n}| \geq 1/2\}, \quad n \geq 0.$$

Observe that

$$(\xi_t - \xi_{t\wedge\tau_n})^2 - \int_{t\wedge\tau_n}^{t} a(s)\,ds$$

is a martingale implying that

$$E(\tau_{n+1} - \tau_n)I_{\tau_n<\tau(x)} \leq \delta^{-1} E I_{\tau_n<\tau(x)} \int_{\tau_n}^{\tau_{n+1}} a(s)\,ds$$
$$= \delta^{-1} E I_{\tau_n<\tau(x)} (\xi_{\tau_{n+1}} - \xi_{\tau_n})^2 = 4^{-1}\delta^{-1} P(\tau_n < \tau(x)).$$

Furthermore, both points $(\tau_{n+1}, x + \xi_{\tau_n} \pm 1/2)$ cannot be in Q and, since $E(\xi_{\tau_{n+1}} - \xi_{\tau_n}) = 0$ and $\xi_{\tau_{n+1}} - \xi_{\tau_n}$ equals $1/2$ or $-1/2$ with probability $1/2$, we have

$$P(\tau_{n+1} < \tau(x)|\mathcal{F}_{\tau_n}) \leq (1/2) I_{\tau_n<\tau(x)}$$

It follows that $P(\tau_n < \tau(x)) \leq 2^{-n}$,

$$E\tau_{n+1} I_{\tau_n<\tau(x)} = E\tau_n I_{\tau_n<\tau(x)} + E(\tau_{n+1} - \tau_n) I_{\tau_n<\tau(x)}$$
$$\leq E\tau_n I_{\tau_n<\tau(x)} + \delta^{-1} 2^{-n-2},$$

$$E\tau_{n+1} I_{\tau_n<\tau(x)\leq\tau_{n+1}} = E\tau_{n+1} I_{\tau_n<\tau(x)} - E\tau_{n+1} I_{\tau_{n+1}<\tau(x)}$$
$$\leq E\tau_n I_{\tau_n<\tau(x)} - E\tau_{n+1} I_{\tau_{n+1}<\tau(x)} + \delta^{-1} 2^{-n-2}.$$

Finally,

$$E\tau(x) = \sum_{n=0}^{\infty} E\tau(x) I_{\tau_n<\tau(x)\leq\tau_{n+1}} \leq \sum_{n=0}^{\infty} E\tau_{n+1} I_{\tau_n<\tau(x)\leq\tau_{n+1}}$$
$$\leq \delta^{-1} \sum_{n=0}^{\infty} 2^{-n-2} - \lim_{n\to\infty} E\tau_{n+1} I_{\tau_{n+1}<\tau(x)},$$

and the lemma is proved. □

Proof of Theorem 3.5. Step 1. First we reduce the general situation to the one in which $x.$ is infinitely differentiable. We are going to deal with (3.5) which is equivalent to (3.4) as is explained in Remark 3.6.

Observe that if $x.(m) \in C$, $m = 1, 2, \dots$, and $x.(m) \to x.$ uniformly on each bounded interval, then $\Delta_n(x.(m), 0) \to \Delta_n(x., 0)$ and

$$M_n(x., c-, 0) \leq \varliminf_{m\to\infty} M_n(x.(m), c-, 0),$$
$$[\gamma(c, d, \delta)]^{M_n(x.,c-,0)} \geq \varlimsup_{m\to\infty} [\gamma(c, d, \delta)]^{M_n(x.(m),c-,0)}.$$

Then assume that $x_t(m) < x_t$ for all t and define

$$\tau_m(x) = \inf\{s > 0 : x + \xi_s = x_s(m) \quad \text{or} \quad x + \xi_s = x_s(m) + 1\}.$$

Notice that starting from an $x < 1$ for the process $(t, x + \xi_t)$ to cross $(t, x_t + 1)$ it has first to cross $(t, x_t(m) + 1)$ for those m for which $x < x_0(m) + 1$. In addition, if $(t, x + \xi_t)$ crosses $(t, x_t + 1)$ before crossing (t, x_t), then $(t, x + \xi_t)$ crosses $(t, x_t(m) + 1)$ before crossing $(t, x_t(m))$ (always if $x < x_0(m) + 1$). It follows that

$$\{\omega : x + \xi_{\tau(x)} = x_{\tau(x)} + 1\} \subset \varliminf_{m\to\infty} \{\omega : x + \xi_{\tau_m(x)} = x_{\tau_m(x)}(m) + 1\},$$
$$P(x + \xi_{\tau(x)} = x_{\tau(x)} + 1) \leq \varliminf_{m\to\infty} P(x + \xi_{\tau_m(x)} = x_{\tau_m(x)}(m) + 1).$$

The version of (3.5) for $x_\cdot(m)$ in place of $x_\cdot$ is the following inequality

$$P(x + \xi_{\tau_m(x)} = x_{\tau_m(x)}(m) + 1) \leq [\gamma(c, d, \delta)]^{M_{p(m)}(x_\cdot(m),c-,0)-k} \tag{3.8}$$

where $p(m) = n((x-x_0(m))/d)$ is used in place of n since $x_0(m) \neq 0$. Observe that the function $n(x)$ is piecewise constant, left continuous, and decreasing. Therefore, for large m we have $p(m) = n(x/d+)$. Hence, if (3.8) is proved, then by letting $m \to \infty$, we get (3.5) with $n(x/d+)$ in place of $n = n(x/d)$. After that, by using the independence of the left-hand side of (3.5) of d and moving this parameter, we get (3.5) in its original form.

Therefore indeed to prove (3.5) or (3.4), it suffices to concentrate on smooth $x_\cdot$.

Step 2. Below we assume that $x_\cdot$ is smooth and in this step we reduce proving (3.4) to estimating a solution to certain partial differential equation.

For fixed $T \in (0, \infty)$ and $M := \sup_{t,\omega} a$ consider the following equation

$$D_t v(t, x) + \max_{a\in[\delta,M]} [(1/2)aD^2_{xx} v(t, x) - x'_t D_x v(t, x)] = 0 \tag{3.9}$$

in $G_T := [0, T) \times (0, 1)$ with boundary condition

$$v(t, x) = x$$

on $\partial' G_T$, where $\partial' G_T$ is the parabolic boundary of G_T.

It is known (see, for instance, [2]) that this problem admits a unique classical solution, that is there exists a function $v_T(t, x)$ which is continuous in $\bar{G}_T$, satisfies the boundary condition, has continuous bounded derivatives in G_T and satisfies the equation there.

By the maximum principle $0 \leq v_T \leq 1$. Also interior estimates show that for each $S \in (0, \infty)$ the family of functions

$$\{v_T, D_x v_T, D^2_{xx} v_T, D_t v_T; T \geq S + 1\}$$

as functions on $\bar{G}_S$ is uniformly bounded and uniformly continuous. It follows that there is a sequence $T(m) \to \infty$ along which v_T, $D_x v_T$, $D^2_{xx} v_T$, $D_t v_T$ converge uniformly on any bounded subset of $\bar{G}_\infty$ to a function v and its corresponding derivatives. Actually, by using Lemma 3.8 it is not hard to show that v_T, $D_x v_T$, $D^2_{xx} v_T$, $D_t v_T$ converge as $T \to \infty$. In any case we conclude that v is a classical solution of (3.9) in G_∞ such that $v(t, 0) = 0$ and $v(t, 1) = 1$ for $t \geq 0$.

Obviously, the function

$$u(t, x) := v(t, x - x_t)$$

satisfies

$$D_t u(t,x) + (1/2) \max_{a\in[\delta,M]} [a D^2_{xx} u(t,x)] = 0 \tag{3.10}$$

in $Q = Q(x.)$ with boundary conditions

$$u(t,x_t) = 0, \quad u(t,x_t+1) = 1, \quad t \geq 0.$$

Observe that

$$D_t u(t,x+\xi_t) + (1/2)a(t) D^2_{xx} u(t,x+\xi_t) \leq 0$$

for $t \leq \tau(x)$. Therefore, by Itô's formula for any $T > 0$ we obtain

$$\begin{aligned} u(0,x) &\geq E u(\tau(x)\wedge T, x+\xi_{\tau(x)\wedge T}) \\ &= P(x+\xi_{\tau(x)} = x_{\tau(x)}+1, \tau(x) \leq T) + E u(T, x+\xi_T) I_{\tau(x)>T}. \end{aligned}$$

By letting $T \to \infty$ and using Lemma 3.8, we get that $u(0,x) \geq P(x+\xi_{\tau(x)} = x_{\tau(x)}+1)$ and to prove the theorem it suffices to prove that

$$u(0,x) \leq [\gamma(c,d,\delta)]^{M_n(x.,c,0)-k}. \tag{3.11}$$

It is not hard to see that, actually, (3.4) is equivalent to (3.11).

Step 3. Here we prove (3.11). Fix an $x \in (0,1)$ and observe that $n \geq 0$, so that

$$m := M_n(x.,c,0)$$

is well defined and estimate (3.3) is trivial if $m \leq k$. Therefore, we assume that $m \geq k+1$ and let $0 \leq p_1 < ... < p_m \leq n$ be the integers such that

$$\Delta_{p_i} := \Delta_{p_i}(x.,0) \leq c$$

for all i. By the way, obviously $k \geq 2$ and $n+1 \geq m \geq k+1$, so that $n \geq k \geq 2$ and owing to the definition of n, we have $x \leq 2^{-n/2}d \leq 2^{-p_m/2}d$ implying that

$$(0,x) \in Q(p_m). \tag{3.12}$$

Also clearly $p_i \geq i-1$ and $\Delta_{p_i} \leq c$, so that, if $s \in [0, 2^{-p_i}]$, then

$$\begin{aligned} &x_s + 2^{-p_i/2}\Delta_{p_i} \geq 0, \quad x_s + 2^{-p_i/2}c \geq 0, \\ &2^{-p_i/2}d \leq x_s + 2^{-p_i/2}(c+d) \leq x_s + 2^{-(i-1)/2}(c+d). \end{aligned}$$

Since $m \geq k+1$, for some $i = 1,2,...,m$ it holds that $i \geq k$. By the choice of k, for those i's, we have $2^{-(i-1)/2}(c+d) \leq 1, 2^{-p_i/2}d \leq x_s+1$, so that $Q(p_i) \subset Q$. Therefore, the definition

$$u_i = \sup_{Q(p_i)} u, \quad i = k,...,m$$

makes sense.

To relate u_i and u_{i+1} we use the probabilistic representation of u. Denote by $a(s,y)$ a Borel function defined for all (s,y) taking only two values δ and M and providing maximum in (3.10) for in Q, so that in Q

$$D_t u(s,y) + (1/2)a(s,y)D^2_{xx}u(s,y) = 0. \tag{3.13}$$

Then for any $(s,y) \in Q(p_i)$ we can find a probability space carrying a Wiener process w_r such that the equation

$$\eta_t = y + \int_0^t \sqrt{a(s+r,\eta_r)}\,dw_r, \quad t \geq 0$$

has a solution. Owing to (3.13) and to the fact that $u = 0$ on the curvilinear boundary of $Q(p_i)$, by Itô's formula for $i \geq k$ we obtain

$$u(s,y) = EI_{A_i}u(s+\tau_i,\eta_{\tau_i}),$$

where

$$\tau_i = \inf\{t \geq 0 : (s+t,\eta_t) \notin Q(p_i)\}$$

and A_i is the event that the process $(s+t,\eta_t)$ exits from $Q(p_i)$ without touching its curvilinear boundary. It follows by Lemma 3.7 that

$$u(s,y) \leq P(A_i)u_i \leq \gamma u_i$$

if $i \geq k$ and $(s,y) \in Q(p_i+1)$, where $\gamma = \gamma(c,d,\delta)$. In particular, if $k \leq i$ and $i+1 \leq m$, then p_{i+1} exists and is bigger than p_i+1 implying that

$$u_{i+1} \leq \gamma u_i, \quad k \leq i < i+1 \leq m.$$

This means that the sequence $\gamma^{-i}u_i$, $i = k, ..., m$, is decreasing and $\gamma^{-m}u_m \leq \gamma^{-k}u_k \leq \gamma^{-k}$. To finish proving the theorem it only remains to notice that, due to (3.12), we have $u(0,x) \leq u_m$. The theorem is proved. □

4. An application to SPDEs

Let w_t be a Wiener process with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete σ-fields and let a_t and σ_t be bounded real-valued processes predictable with respect to $\{\mathcal{F}_t, t \geq 0\}$ and such that $a_t - \sigma_t^2 \geq \varepsilon\sigma_t^2$, where $\varepsilon \in (0,\infty)$ is a constant, $a_t - \sigma_t^2 > 0$ for all (ω,t) and for all ω

$$\int_0^\infty [a_t - \sigma_t^2]\,dt = \infty.$$

We will be dealing with the SPDE

$$dv(t,x) = (1/2)a_t D^2_{xx}v(t,x)\,dt + \sigma_t D_x v(t,x)\,dw_t$$

in $B = (0,\infty) \times (0,1)$ with boundary conditions

$$v(t,0) = 0, \quad v(t,1) = 1, \quad t > 0, \tag{4.1}$$

$$v(0,x) = 0, \quad 0 < x < 1. \tag{4.2}$$

Theorem 4.1. *There is a function $v(t,x) = v(\omega,t,x)$ defined on $\Omega \times \bar{B}$ such that*

(i) *$v(t,x)$ is $\mathcal{F}_t$-measurable for each $(t,x) \in \bar{B}$,*
(ii) *$v(t,x)$ is bounded and continuous in $\bar{B} \setminus \{(0,1)\}$ for each ω,*
(iii) *derivatives of $v(t,x)$ of any order with respect to x are continuous in $B \cup (\{0\} \times (0,1))$ for each ω,*
(iv) *equations (4.1) and (4.2) hold for each ω,*
(v) *almost surely, for any $(t,x) \in B$*

$$v(t,x) = \int_0^t (1/2) a_s D_{xx}^2 v(s,x)\,ds + \int_0^t \sigma_s D_x v(s,x)\,dw_s,$$

(vi) *for any $T \in (0,\infty)$, $c, d > 0$, such that $\alpha(c\sqrt{\varepsilon}) < 1$, and ν satisfying*

$$0 \le \nu < \nu_0 := (1 - \alpha(c\sqrt{\varepsilon})) \log_2 \gamma^{-2}(c,d,1)$$

we have that with probability one

$$\sup_{x\in(0,1)} \sup_{t\in[0,T]} \frac{v(t,x)}{x^\nu} < \infty, \tag{4.3}$$

(vii) *there exist constants $N, \nu \in (0,\infty)$ depending only on ε such that for any $T \in (0,\infty)$*

$$E \sup_{x\in(0,1)} \sup_{t\in[0,T]} \frac{v(t,x)}{x^\nu} \le N(MT+1), \tag{4.4}$$

where $M := \sup_{\omega,t}(a_t - \sigma_t^2)$.

Proof. On the space C with Wiener measure W introduce the coordinate process $x_t(x.) := x_t$, which is a Wiener process. For $t \ge 0$, $x \in \mathbb{R}$, and $x., y. \in C$ define

$$\tau(t,x,x.,y.) = \inf\{s \ge 0 : x + x_s = y_{t-s}\},$$

where $y_r := y_0$ for $r \le 0$. Then the function

$$u(y.,t,x) := \int_C I_{\tau(t,x,x.,y.+1) < t\wedge\tau(t,x,x.,y.)}\,W(dx.).$$

is the probabilistic solution of the heat equation

$$D_t u = (1/2) D_{xx}^2 u$$

in $Q(y.) \setminus (\{0\} \times (0,1))$ with boundary conditions

$$u(t,y_t) = 0, \quad u(t,y_t+1) = 1, \quad t > 0,$$
$$u(0,x) = 0, \quad y_0 < x < y_0 + 1.$$

Observe that $\tau(t,x,x.,y.)$ is a lower semicontinuous function of its arguments. Therefore by Fubini's theorem $u(y.,t,x)$ is a Borel function of $(y.,t,x)$. Furthermore, $u(y.,t,x)$ will not change if we change y_r for $r > t$. Therefore, $u(y.,t,x)$ is $\mathcal{N}_t$-measurable, where $\mathcal{N}_t = \sigma(y_r : r \le t, y. \in C)$.

Next define

$$\psi_t = \int_0^t (a_s - \sigma_s^2)\,ds, \quad \xi_t = \int_0^{\phi_t} \sigma_s\,dw_s, \quad \tilde{\mathcal{F}}_t = \mathcal{F}_{\phi_t},$$
$$\tilde{u}(t,x) = \tilde{u}(\omega,t,x) = u(\xi.,t,x), \quad \hat{u}(t,x) = \hat{u}(\omega,t,x) = \tilde{u}(\psi_t,x),$$
$$v(t,x) = v(\omega,t,x) = \hat{u}(t,x+\xi_{\psi_t}),$$

where $\phi_t = \inf\{s \geq 0 : \psi_s \geq t\}$ is the inverse function to ψ_t. We are going to prove that v is the function we are looking for.

We will use few well known facts (all of them can be found, for instance, in [3]). Notice that ϕ_t is an $\mathcal{F}_t$-stopping time so that $\tilde{\mathcal{F}}_t$ is well defined. Also ξ_s is $\tilde{\mathcal{F}}_t$-measurable for $s \leq t$. Hence, $\tilde{u}(t,x)$ is $\tilde{\mathcal{F}}_t$-measurable.

Furthermore, it is well known that

$$\sqrt{\varepsilon}\int_0^t \sigma_s\,dw_s = \tilde{w}_{\tilde{\psi}(t)},$$

where $\tilde{w}_t$ is a Wiener process and

$$\tilde{\psi}_t = \varepsilon\int_0^t \sigma_s^2\,ds.$$

Hence $\xi_t = \varepsilon^{-1/2}\tilde{w}_{\tilde{\psi}(\phi_t)}$ with

$$(\tilde{\psi}(\phi_t))' = \varepsilon\sigma_s^2/(a_s - \sigma_s^2)|_{s=\phi_t} \leq 1.$$

It follows that with $c' := c\sqrt{\varepsilon}$ for $n = 0, 1, 2, \ldots$ we have

$$N_n^-(\xi., c, t) \leq N_n^-(\tilde{w}., c', \tilde{\psi}(\phi_t)),$$
$$\sup_{t\leq T} N_n^-(\xi., c, t) \leq \sup_{t\leq T} N_n^-(\tilde{w}., c', t) =: N_n^*(T).$$

By Corollary 3.4 for $t \in [0,T]$, $x \in (0,1)$, and any $c, d > 0$ we have

$$\tilde{u}(t, \xi_t + x) \leq \gamma^{n+1-N_n^*(T)-k}, \tag{4.5}$$

where $\gamma = \gamma(c,d,1)$, $n = n(x/d)$ and $k = k(c+d)$ are taken from Theorem 3.1. If c is so large that $\alpha(c') < 1$, then the right-hand side of (4.5) tends to zero as $x \downarrow 0$ (a.s.) by Theorem 2.1. By using the possibility to change stochastic integrals and Wiener processes on events of zero probability we may assume that this property holds for all ω rather than only almost surely. Then for any ω all points on $\{(t,\xi_t) : t \geq 0\}$ are regular relative to $Q(\xi.) \setminus (\{0\} \times (0,1))$. Similarly all points on $\{(t, \xi_t + 1) : t \geq 0\}$ are regular relative to $Q(\xi.) \setminus (\{0\} \times (0,1))$. It is obvious that all points of $\{0\} \times (0,1)$ are regular relative to $Q(\xi.) \setminus (\{0\} \times (0,1))$. It follows that $\tilde{u}$ is continuous in $\bar{Q}(\xi.) \setminus \{(0,1)\}$. Due to interior estimates of derivatives of solutions to the heat equation, $\tilde{u}$ is infinitely differentiable in $Q(\xi.)$.

Next, the properties of random time change show that v possesses the properties (i)-(iv). Furthermore, $\hat{u}$ possesses similar properties and, for $t > 0$ and $\xi_{\psi_t} < x < \xi_{\psi_t} + 1$

$$d\hat{u}(t,x) = (1/2)(a_t - \sigma_t^2)D_{xx}^2\hat{u}(t,x)\,dt.$$

After that the Itô-Wentzell formula shows that v possesses property (v).

We now prove (vi). Owing to (4.5) we have

$$\overline{\lim_{x\downarrow 0}} \sup_{t\in[0,T]} \frac{v(t,x)}{x^{\nu}} \leq \gamma^{-k} 2^{\kappa}, \tag{4.6}$$

where $\kappa = \lim_{x\downarrow 0} \kappa(x)$ with

$$\kappa(x) := (n+1)\{1 - \sup_{t\in[0,\psi_T]} \frac{1}{n+1} N_n^-(\tilde{w}.,c',t)\} \log_2 \gamma - \nu \log_2 x$$

and $n = [(\log_2(x^{-2}d^2))_+]$. By Theorem 2.1 with probability one, for any $\chi > 0$, if x is sufficiently small we have

$$\kappa(x) \leq (n+1)(1-\alpha(c') - \chi)\log_2 \gamma - \nu \log_2 x.$$

We take $\chi \leq 1 - \alpha(c')$ and again use that $\log_2 \gamma < 0$, $n+1 \geq \log_2(x^{-2}d^2)$. Then we see that for small $x > 0$

$$\begin{aligned} \kappa(x) &\leq (-2\log_2 x + 2\log_2 d)(1-\alpha(c')-\chi)\log_2\gamma - \nu\log_2 x \\ &= \{(1-\alpha(c')-\chi)\log_2\gamma^{-2} - \nu\}\log_2 x + 2(\log_2 d)(1-\alpha(c')-\chi)\log_2\gamma. \end{aligned}$$

If χ is sufficiently close to zero, we see that $\kappa(x) \to -\infty$ as $x \downarrow 0$. Thus $\kappa = -\infty$ and (4.3) follows from (4.6).

To prove (vii), notice that for any $\alpha, x \in (0,1)$ by (4.5) and Lemma 2.5

$$\begin{aligned} E \sup_{t\in[0,T]} v(t,x) &\leq \gamma^{(1-\alpha)(n+1)-k} + P(\sup_{t\in[0,\psi_T]} N_n^-(\tilde{w}.,c',t) \geq (n+1)\alpha) \\ &\leq \gamma^{(1-\alpha)(n+1)-k} + P(\sup_{t\in[0,MT]} N_n^-(\tilde{w}.,c',t) \geq (n+1)\alpha) \\ &\leq \gamma^{(1-\alpha)(n+1)-k} + (MT+1)r^{1-\alpha}\beta^n, \end{aligned}$$

where $r = r(\alpha, c')$ and $\beta = \beta(\alpha, c')$. One can take α and c so that $\beta < 1$ (see Remark 2.6). Furthermore, for any choice of these parameters and d, we have $\gamma < 1$. By recalling what n is, we conclude

$$E \sup_{t\in[0,T]} v(t,x) \leq N(MT+1)x^{2\nu},$$

where N and $\nu > 0$ depend only on ε.

Next, $\tau(t,x,x.,y.)$ is obviously an increasing function of x if $x + x_0 \geq y_t$ and a decreasing function of x if $x + x_0 \leq y_t$. It follows, that $u(y.,t,x)$ is an increasing function of $x \in (y_t, y_t+1)$ and $v(t,x)$ is an increasing function of $x \in (0,1)$. Hence,

$$\begin{aligned} E \sup_{x\in(0,1)} \sup_{t\in[0,T]} \frac{v(t,x)}{x^{\nu}} &\leq \sum_{r=0}^{\infty} E \sup_{t\in[0,T]} \sup_{2^{-r}\geq x\geq 2^{-r-1}} \frac{v(t,x)}{x^{\nu}} \\ &\leq N \sum_{r=0}^{\infty} 2^{\nu(r+1)} E \sup_{t\in[0,1]} v(t,2^{-r}) \leq N(MT+1)\sum_{r=0}^{\infty} 2^{\nu(r+1)} 2^{-2r\nu} \end{aligned}$$

and the theorem is proved. □

Remark 4.2. The largest value of ν in (4.3) is not known. However, Theorem 5.1 and Lemma 4.1 of [4] show that if we take a $\mu > 0$ and

$$\nu = (1+\mu)(2\pi\varepsilon)^{-1/2}e^{-1/(2\varepsilon)},$$

then for ε small enough the left-hand side of (4.3) equals infinity with probability one. Therefore, the largest value of ν is extremely small if ε is small.

Remark 4.3. Since $0 \le v \le 1$, for any $p \ge 1$ we have

$$E \sup_{x\in(0,1)} \sup_{t\in[0,T]} \Big[\frac{v(t,x)}{x^{\mu}}\Big]^p < \infty, \tag{4.7}$$

with $\mu = \nu/p$, where ν is taken from (4.4). Most likely, if we fix $\mu > 0$ and take sufficiently large p, the inequality (4.7) turns wrong even for constant $a > 0$ and $\sigma > 0$.

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