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Central limit theorem and stable laws for intermittent maps

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Abstract. In the setting of abstract Markov maps, we prove results concerning the convergence of renormalized Birkhoff sums to normal laws or stable laws. They apply to one-dimensional maps with a neutral fixed point at 0 of the form $x + x^{1+\alpha}$, for $\alpha \in (0, 1)$. In particular, for $\alpha > 1/2$, we show that the Birkhoff sums of a Hölder observable f converge to a normal law or a stable law, depending on whether $f(0) = 0$ or $f(0) \neq 0$. The proof uses spectral techniques introduced by Sarig, and Wiener's Lemma in non-commutative Banach algebras.

1. Introduction and statement of results

1.1. Introduction

Recently, general methods have been devised to prove the Central Limit Theorem for the Birkhoff averages of mixing dynamical systems. More precisely, if $T : X \to X$ is a map from a space to itself preserving a probability measure ν, and $f : X \to \mathbb{R}$ is a function with zero average, we say that the Central Limit Theorem (CLT) holds for (T, ν, f) if

$$\frac{S_n f}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \to \mathcal{N}(0, \sigma^2),$$

the convergence being in distribution (with respect to the measure ν).

To prove this kind of results, the first step is often to show that the correlations $\mathrm{Cor}(f, f \circ T^n) = \int f \cdot f \circ T^n \, \mathrm{d}\nu$ are summable. Then one may use this to construct a reverse martingale, for which the CLT is known to be true. Clearly, when the correlations are not summable, this method does not work.

When the map T is uniformly expanding, perturbative spectral methods can also be used, giving convergence to normal laws ([GH88]) or other limit laws, namely stable laws ([GLJ93] or [AD01]). In the particular case of the continued

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fraction transformation, Lévy had already obtained this kind of convergence using elementary methods ([Lév52]).

In this paper, we will be interested in cases where T has a neutral fixed point (or in the Markov analogues of this situation), where the classical methods to estimate the decay of correlations fail (however, see [Hu]), and the spectral methods do not work directly, since the expansion is not uniform. We will prove that in some cases, the CLT still holds. Moreover, in situations where it does not hold, we will show the convergence to stable laws, even when the function f is arbitrarily smooth, for example C^∞ or analytic. This kind of result has been announced by Fisher and Lopes (see [FL01] and references therein) for piecewise linear maps.

Our main results (Theorems 1.1 and 1.2) will be stated in the context of abstract Markov maps. Applications to one-dimensional maps are then given: if a map has a neutral fixed point at 0 of the form $x + x^{1+\alpha}$ with $\alpha \in (1/2, 1)$ and is expanding elsewhere, then the Birkhoff sums of a Hölder observable f converge to a normal law or a stable law depending on whether $f(0) = 0$ or $f(0) \neq 0$. Applications are also given for other types of neutral fixed points, and for unbounded observables, for example $x^{-\beta}$, leading to new results even in the case $\alpha \in (0, 1/2)$ (see also [Rau02] in this case, where he studies the decay of correlations for non Hölder observables).

The philosophy of the method is that, if it is possible to induce on a subset Y of the space so that the induced map is uniformly expanding, then the behavior of the Birkhoff sums $\sum_0^{n-1} f(T^i x)$ is in fact dictated by the behavior on Y of the Birkhoff sums $\sum_0^{p-1} f_Y(T_Y^i x)$, where $T_Y = T^\varphi$ is the induced map on Y (φ is the first return time), and $f_Y(x) = \sum_0^{\varphi(x)-1} f(T^i x)$. As convergence to normal laws or stable laws is known in the uniformly expanding case (see [AD01] for example), we get the same kind of results in our nonuniform setting.

1.2. The result for Markov maps

1.2.1. Definition of Markov maps

For these definitions, see for example [ADU93], [Aar97] or [Sar02].

A Markov map is a non-singular transformation T of a Lebesgue space $(X, \mathcal{B}, m)$ of finite mass, together with a measurable partition α of X such that if $a \in \alpha$ then $m(a) > 0$, Ta is a union (mod m) of elements of α, and $T : a \to Ta$ is invertible. Moreover, it is assumed that the completion of $\bigvee_0^\infty T^{-i}\alpha$ is $\mathcal{B}$, i.e. the partition separates the points.

A Markov map is said to be *topologically mixing* if $\forall a, b \in \alpha, \exists N, \forall n \geqslant N, b \subset T^n a$. This corresponds to topological mixing for the topology defined by the cylinders $[a_0, \dots, a_{n-1}] = \bigcap_{i=0}^{n-1} T^{-i} a_i$ (where $a_0, \dots, a_{n-1} \in \alpha$).

If $\gamma \subset \alpha$ is a subset of the partition α, and $Y = \bigcup_{a\in\gamma} a$, the induced map $T_Y : Y \to Y$ is defined as the first return map from Y to Y, i.e. $T_Y = T^{\varphi_Y}$, where $\varphi_Y(x) = \inf\{n \geqslant 1 \mid T^n(x) \in Y\}$ is the return time to Y. Let $\delta = \{[a, \xi_1, \dots, \xi_{n-1}, Y] \mid a \in \gamma, \xi_1, \dots, \xi_{n-1} \in \alpha - \gamma, [a, \xi_1, \dots, \xi_{n-1}, Y] \neq \emptyset\}$: this is a partition of Y, for which T_Y is a Markov map (with the measure $m_Y = m_{|Y}$). The cylinders for this partition will be denoted by $[d_0, \dots, d_{l-1}]_Y$ (with $d_0, \dots, d_{l-1} \in \delta$). If $d = [a, \xi_1, \dots, \xi_{n-1}, Y] \in \delta$, its image is $T_Y d = T\xi_{n-1} \cap Y$ – hence, it is γ-measurable.

For $x, y \in Y$, we will write $s(x, y)$ for the separation time of x and y under T_Y, i.e. $s(x, y) = \inf\{n \in \mathbb{N} \mid T_Y^n x$ and $T_Y^n y$ are not in the same element of $\delta\}$. This separation time is extended to the whole space X: for $x, y \in X$, let x' and y' be their first returns to Y. If $T^i x$ and $T^i y$ stay in the same element of the partition α until the first return to Y, set $s(x, y) = s(x', y') + 1$. Otherwise, $s(x, y) = 0$.

If $\theta < 1$, we will say that a function f on Y (resp. X) is locally θ-Hölder if there exists a constant C such that $\forall x, y \in Y$ (resp. X) with $s(x, y) \geqslant 1$, we have $|f(y) - f(x)| \leqslant C\theta^{s(x,y)}$. The smallest such constant is called the θ-Hölder constant of f. In fact, slightly abusing notation, we shall say that a measurable function is locally θ-Hölder if there exists a locally θ-Hölder version of this function.

The transfer operator $\hat{T}_Y$ associated to T_Y and acting on integrable functions is defined by $\int f \cdot (g \circ T_Y)\,\mathrm{d}m_Y = \int (\hat{T}_Y f) \cdot g\,\mathrm{d}m_Y$. It can be written as $\hat{T}_Y f(x) = \sum_{T_Y y = x} g_{m_Y}(y) f(y)$, where the weight g_{m_Y} is defined on an element d of δ by $g_{m_Y} = \frac{\mathrm{d}m_Y}{\mathrm{d}(m_Y \circ (T_Y)_{|d})}$, i.e. it is the inverse of the jacobian of T_Y. We will say that the distortion is locally Hölder if $\log g_{m_Y}$ is locally θ-Hölder for some $\theta < 1$.

Finally, we will say that T_Y has the *big image property* if there exists a constant $\eta > 0$ such that $\forall d \in \delta, m[T_Y d] \geqslant \eta$. This is in particular the case when the partition γ is finite.

1.2.2. Main results

For the following theorems, $(X, \mathcal{B}, T, m, \alpha)$ will be a topologically mixing probability preserving Markov map. We assume that $\gamma \subset \alpha$ is such that the induced map T_Y on $Y = \bigcup_{a \in \gamma} a$ has a locally θ-Hölder distortion for some θ, and that it has the big image property. The function $\varphi = \varphi_Y$ will denote the return time from Y to itself.

We will write $I(X, Y)$ for the set of functions $f : X \to \mathbb{R}$ which induce "nice" functions on Y. More precisely, $f \in I(X, Y)$ if

1. $\forall N \in \mathbb{N}$, there exist constants C_N and $0 < \theta_N < 1$ such that, on $Y_N = \bigcup_0^{N-1} T^i(Y)$, we have $|f| \leqslant C_N$ and, if $x, y \in Y_N$ satisfy $s(x, y) \geqslant 1$, then $|f(x) - f(y)| \leqslant C_N \theta_N^{s(x,y)}$.
2. Writing $f_Y(x) = \sum_{i=0}^{\varphi(x)-1} f(T^i x)$ for the map induced by f on Y, there exists $\theta < 1$ such that $\sum_{d \in \delta} m[d] D_\theta f_Y(d) < \infty$, where $D_\theta f_Y(d)$ is the least θ-Hölder constant of f_Y restricted to the element d of the partition δ.
3. There exists $\theta < 1$ and $0 < \eta \leqslant 1$ such that $\sum_{d \in \delta} \varphi(d) m[d] D_\theta f_Y(d)^\eta < \infty$.

Note that, increasing θ if necessary, we can assume that the distortion of T_Y is θ-Hölder and that $\sum_{d \in \delta} m[d] D_\theta f_Y(d) < \infty$ and $\sum_{d \in \delta} \varphi(d) m[d] D_\theta f_Y(d)^\eta < \infty$, for the same $\theta < 1$.

For example, if f is bounded and locally θ-Hölder, and the induced map f_Y is also locally θ-Hölder, then these conditions are satisfied, for $\eta = 1$ (note that $\sum \varphi(d) m[d] = m(X) < \infty$, by Kac's Formula). The weaker conditions above are useful in the applications, for example to unbounded observables.

If f is bounded and locally θ-Hölder, and $m[\varphi = n] = O(1/n^{\beta+1})$ for some $\beta > 1$, then $f \in I(X, Y)$: the first condition is clear, the second one is given by Kac's Formula (since $D_\theta f_Y(d) \leqslant C\varphi(d)$), and the third one is satisfied for any $\eta < \min(1, \beta - 1)$.

Theorem 1.1 (Central Limit Theorem)**.** *Let $f \in I(X, Y)$ be integrable with $\int f \, \mathrm{d}m = 0$ and let the induced function f_Y be in $L^2(\mathrm{d}m_Y)$.*

Then there exists a constant $\sigma^2 \geqslant 0$ such that $\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i$ converges in distribution to a normal law $\mathcal{N}(0, \sigma^2)$. Moreover, $\sigma^2 = 0$ if and only if there exists a measurable function $\chi : X \to \mathbb{R}$ satisfying $f = \chi \circ T - \chi$.

For the next theorem, we will need the notion of stable law. The law of the random variable X is said to be stable if there exist i.i.d. random variables X_k and constants $A_n \in \mathbb{R}$ and $B_n > 0$ such that

$$\frac{\sum_{i=0}^{n-1} X_i - A_n}{B_n} \to X \text{ in distribution.}$$

In this case, we say that X_0 is attracted to X, or that the law of X_0 belongs to the domain of attraction of X. The stable laws are completely classified (see for example [Fel66]) and depend on many parameters, the most important of which is certainly the *index* $p \in (0, 2]$. The case $p = 2$ corresponds to the normal law, while if $p \in (1, 2)$ the stable laws are in L^1 but not in L^2. In fact, for $p \in (0, 1) \cup (1, 2)$, the index can be characterized as follows: there exist constants $c_1 \geqslant 0$ and $c_2 \geqslant 0$ with $c_1 + c_2 > 0$, satisfying $P[X > x] = (c_1 + o(1))x^{-p}$ and $P[X < -x] = (c_2 + o(1))x^{-p}$. The case $p = 1$ is problematic and will not be included in the following discussion.

Following [AD01], for $p \in (0, 1) \cup (1, 2]$, $c > 0$ and $\beta \in [-1, 1]$, we will denote by $X_{p,c,\beta}$ the law whose characteristic function is

$$E(e^{itX_{p,c,\beta}}) = e^{-c|t|^p (1 - i\beta \operatorname{sgn}(t) \tan(\frac{p\pi}{2}))}.$$

This is indeed a stable law of index p. When $p \in (1, 2]$, it is centered, i.e. $X_{p,c,\beta}$ is of zero expectation. Note that, when $p = 2$, the value of β is irrelevant.

We recall also that a function $L : (0, \infty) \to \mathbb{R}$ is slowly varying if $\forall x > 0, \lim_{y\to\infty} \frac{L(yx)}{L(y)} = 1$. Then the random variables Z attracted to stable laws of index p can be described as follows ([Fel66]):

- For $p \in (0, 1) \cup (1, 2)$, $P[Z > x] = (c_1 + o(1))x^{-p}L(x)$ and $P[Z < -x] = (c_2 + o(1))x^{-p}L(x)$ for some $c_1, c_2 \geqslant 0$ with $c_1 + c_2 > 0$, and a slowly varying function L.
- For $p = 2$, either $Z \in L^2$, or $P[Z > x] = (c_1 + o(1))x^{-2}l(x)$ and $P[Z < -x] = (c_2 + o(1))x^{-2}l(x)$, for constants $c_1, c_2 \geqslant 0$ with $c_1 + c_2 > 0$, and a slowly varying function l such that $u^{-1}l(u)$ is not integrable at $+\infty$. In this case, we will write $L(x) = \int_{-x}^{x} u^2 \, \mathrm{d}P_Z(u)$, which is an unbounded slowly varying function.

When $Z \notin L^2$, if the normalizing constants B_n and A_n are defined by

$$nL(B_n) = B_n^p, \quad A_n = \begin{cases} 0 & 0 < p < 1 \\ nE(Z) & 1 < p \leqslant 2 \end{cases},$$

then, writing $Z_0, Z_1, \dots$ for independent random variables having the same distribution as Z, we have the convergence

$$\frac{\sum_{i=0}^{n-1} Z_i - A_n}{B_n} \to X_{p,c,\beta}$$

where

$$c = \begin{cases} (c_1 + c_2)\Gamma(1-p)\cos\left(\frac{p\pi}{2}\right) & p \in (0,1) \cup (1,2) \\ \frac{1}{2} & p = 2 \end{cases}, \quad \beta = \frac{c_1 - c_2}{c_1 + c_2}.$$

In the deterministic case, if $f \in I(X, Y)$, we can consider the function f_Y on Y. Since $m(Y) < 1$, its distribution μ (defined by $\mu(A) = m\{y \in Y \mid f_Y(y) \in A\}$) is not a probability measure on $\mathbb{R}$, but we can replace it by the modified distribution $\tilde{\mu} = \mu + (1 - m[Y])\delta_0$. We will say with a little abuse of notation that the distribution of f_Y is attracted to a stable law if the distribution $\tilde{\mu}$ is in the domain of attraction of this stable law.

Theorem 1.2 (Convergence to stable laws)**.** *Let $f \in I(X, Y)$. Assume that the induced function f_Y on Y is not in L^2, and that its distribution is in the domain of attraction of a stable law of index $p \in (0, 1) \cup (1, 2]$, as described previously. If $p > 1$, assume moreover that f is integrable and that $\int f \,\mathrm{d}m = 0$.*

Then $\frac{\sum_{i=0}^{n-1} f \circ T^i}{B_n}$ converges in distribution to the stable law $X_{p,c,\beta}$ where B_n, p, c and β are as in the random case above.

The hypothesis $\int f = 0$ when $p > 1$ ensures that the A_n term vanishes.

Note that, in Theorem 1.1, the constant σ^2 depends on the interactions at different times between f and $f \circ T^n$, while in Theorem 1.2, the limit distribution depends only on the initial distribution (of f_Y).

1.3. Application to one-dimensional maps

Our main theorems can be applied to one-dimensional maps with a neutral fixed point with prescribed behavior. For the sake of simplicity, we will restrict ourselves to a specific case, but the results can in fact be proved for a wide class of maps.

For $0 < \alpha < 1$, we consider the map from $[0, 1]$ to itself introduced in [LSV99] (this paper contains also historical references) and defined by

$$T(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \leqslant x \leqslant 1/2 \\ 2x - 1 & \text{if } 1/2 < x \leqslant 1. \end{cases} \tag{1}$$

This map has a neutral fixed point at 0, and is expanding elsewhere. It is known that it has a unique absolutely continuous invariant measure $\mathrm{d}m$, whose density h is Lipschitz on any interval of the form $(\varepsilon, 1]$ ([LSV99, Lemma 2.3]).

For Hölder observables on $[0, 1]$, Young ([You99]) showed that $\mathrm{Cor}(f, g \circ T^n) = \int f \cdot g \circ T^n \,\mathrm{d}m - \int f \,\mathrm{d}m \int g \,\mathrm{d}m = O(1/n^{1/\alpha-1})$. This rate is in fact optimal for a wide class of functions ([Sar02]). It implies a central limit theorem when $\alpha < 1/2$.

For $\alpha > 1/2$, the above rate is not summable. However, extending the methods of [Sar02], we showed in [Gou02] that, for functions f vanishing in a neighborhood of the fixed point and with zero integral, the correlations decay like $O(n^{-1/\alpha})$, which gives again a central limit theorem. For functions with zero average and equal to a nonzero constant in a neighborhood of the fixed point, the correlations decay exactly like $C/n^{1/\alpha-1}$ ([Gou02]), which is not summable.

Our main theorems yield (see also a list of other consequences after the proof of Theorem 1.3):

Theorem 1.3. *Let* $1/2 < \alpha < 1$, *and* $T : [0, 1] \to [0, 1]$ *be the corresponding Liverani-Saussol-Vaienti map* (1). *Let h be the density of its unique absolutely continuous invariant probability* $\mathrm{d}m$. *Let* $f : [0, 1] \to \mathbb{R}$ *be Hölder, with* $\int f \,\mathrm{d}m = 0$.

If $f(0) \neq 0$, *then* $\frac{S_n f}{n^\alpha}$ *converges in distribution to the stable law* $X_{1/\alpha,c,\mathrm{sgn}(f(0))}$ *with* $c = \frac{h(1/2)}{4(\alpha/|f(0)|)^{1/\alpha}}\Gamma(1-1/\alpha)\cos\left(\frac{\pi}{2\alpha}\right)$.

If $f(0) = 0$, *assume moreover that* $|f(x)| \leqslant Cx^\gamma$ *with* $\gamma > \alpha - 1/2$. *Then there exists a constant* $\sigma^2 \geqslant 0$ *such that* $\frac{S_n f}{\sqrt{n}}$ *tends in distribution to* $\mathcal{N}(0, \sigma^2)$. *Moreover,* $\sigma^2 = 0$ *if and only if there exists a measurable function* χ *with* $f = \chi \circ T - \chi$.

This result does not address the question of the summability of correlations for a function f with $f(0) = 0$. Hu announced recently this summability ([Hu]), which gives another proof of the central limit theorem.

Proof. We construct a Markov structure, starting from $x_0 = 1$ and setting $x_{n+1} = T^{-1}(x_n) \cap [0, 1/2]$. The map T is then Markov for the partition composed of the $I_n := (x_{n+1}, x_n)$, $n \in \mathbb{N}$. We will induce on $Y = (1/2, 1)$. Setting $y_{n+1} = T^{-1}(x_n) \cap (1/2, 1]$, the induced map T_Y is Markov for the partition given by the intervals $J_n = (y_{n+1}, y_n)$, $n \geqslant 1$. The distortion control on T_Y is classical (see for example [LSV99]), it remains to study the properties of f and f_Y. We will write $s(x, y)$ for the separation time of $x, y \in [0, 1]$ under T_Y.

As the induced map on Y is uniformly expanding, with a factor $\lambda > 1$, we obtain $|x - y| \leqslant C\lambda^{-s(x,y)}$. As f is ν-Hölder for some ν, we get $|f(x) - f(y)| \leqslant D|x - y|^\nu \leqslant E\left(\lambda^{-\nu}\right)^{s(x,y)}$, with $\theta := \lambda^{-\nu} < 1$. Thus, f is bounded and locally Hölder. [LSV99] proves that $m[J_n] \sim \frac{C}{n^{1+1/\alpha}}$ for some constant C. Consequently, $f \in I(X, Y)$ (see the examples following the definition of $I(X, Y)$).

We assume now that $f(0) = 0$ and show that $f_Y \in L^2(Y, \mathrm{d}m)$. We may suppose that $\gamma < \alpha$. As $\mathrm{d}m$ and dLeb are equivalent on Y, it is sufficient to show that $f_Y \in L^2(Y, \mathrm{dLeb})$. If $x \in J_n$, then for $1 \leqslant i \leqslant n-1$, $T^i x \in (x_{n+1-i}, x_{n-i})$, whence $|f(T^i x)| \leqslant C|T^i x|^\gamma \leqslant C|x_{n-i}|^\gamma$. This implies that $|f_Y(x)| \leqslant |f(x)| + C\sum_{j=1}^{n-1}|x_j|^\gamma \leqslant C + C\sum_{j=1}^{n-1}\frac{1}{j^{\gamma/\alpha}}$, since $x_n \sim \frac{1}{2}(\alpha n)^{-1/\alpha}$ by [Sar02, Corollary 1]. Hence, $|f_Y| \leqslant Cn^{1-\gamma/\alpha}$ on J_n, with $\mathrm{Leb}(J_n) \asymp x_n - x_{n+1} \asymp 1/n^{1+1/\alpha}$. Thus, $\int |f_Y|^2 \leqslant C\sum \frac{n^{2-2\gamma/\alpha}}{n^{1+1/\alpha}}$. This is summable when $\gamma > \alpha - \frac{1}{2}$. We have proved that $f_Y \in L^2$, whence we can use Theorem 1.1 to conclude.

We assume finally that $f(0) > 0$ (the case $f(0) < 0$ is analogous). Write $f = f(0) + \tilde{f}$, where $\tilde{f}$ is ν-Hölder and satisfies $\tilde{f}(0) = 0$. As in the case $f(0) = 0$, we get that, for $x \in J_n$, $|\tilde{f}_Y(x)| \leqslant |\tilde{f}(x)| + C\sum_{j=1}^{n-1}|x_j|^\nu$, whence $|\tilde{f}_Y| \leqslant Dn^{1-\nu/\alpha}$ on J_n. Take $q > 1/\alpha$, close to $1/\alpha$.

$$\int |\tilde{f}_Y|^q \,\mathrm{dLeb} \leqslant C \sum \frac{n^{q-q\nu/\alpha}}{n^{1+1/\alpha}} < \infty,$$

i.e. $\tilde{f}_Y \in L^q(Y, \mathrm{d}m)$ since m and Leb are equivalent on $(1/2, 1]$. Writing $u(x) = nf(0)$ on J_n, we have $f_Y = \tilde{f}_Y + u$ on Y, and

$$\begin{aligned} m[u > nf(0)] &= m\left[\bigcup_{n+1}^{\infty} J_k\right] \\ &= m(1/2, x_n/2 + 1/2) \sim h(1/2)\frac{x_n}{2} \sim h(1/2)\frac{1}{4}(\alpha n)^{-1/\alpha}. \end{aligned}$$

Thus, we obtain $m[u > x] \sim \frac{h(1/2)}{4(\alpha x/f(0))^{1/\alpha}} = \frac{c_1}{x^{1/\alpha}}$. Adding $\tilde{f}_Y \in L^q$ does not change the asymptotics since $m[|\tilde{f}_Y| > x] \leqslant x^{-q}\int|\tilde{f}_Y|^q = o\left(x^{-1/\alpha}\right)$. We have proved that $m[f_Y > x] = (c_1 + o(1))x^{-p}$ for $p = 1/\alpha$. In the same way, we get $m[f_Y < -x] = o(x^{-p})$. This enables us to use Theorem 1.2, which concludes the proof. □

We enumerate other consequences of Theorems 1.1 and 1.2 in other one-dimensional cases:

1. For $\alpha \in (0, 1/2)$, the function f_Y is always in L^2 and the condition $\gamma > \alpha - \frac{1}{2}$ becomes empty. Thus, we get another proof of the classical central limit theorem, for any Hölder function f.
2. For $\alpha = 1/2$, the condition on γ is also empty, and we get the following result for a Hölder f with zero average:
   - If $f(0) = 0$, there is a central limit theorem.
   - If $f(0) \neq 0$, for example $f(0) > 0$, then we get as above $m[f_Y > x] = (c_1 + o(1))x^{-2}$ for $c_1 = \frac{h(1/2)\sqrt{2f(0)}}{4}$. We are in the nonstandard domain of attraction of the normal law (i.e. f_Y is attracted to a normal law but is not in L^2), for $l(x) = 1$, whence $L(x) \sim \frac{c_1}{2}\log x$. The constant B_n is then $\frac{\sqrt{c_1}}{2}\sqrt{n\log n}$, and we get the convergence

$$\frac{\sum_{i=0}^{n-1} f \circ T^i}{\frac{\sqrt{c_1}}{2}\sqrt{n\log n}} \to \mathcal{N}(0, 1).$$

3. For $\alpha \in (0, 1)$, we can also look at $f(x) = x^{-\beta} + \kappa$ with $\beta > 0$, where κ is taken so that $\int f = 0$. Note that f is integrable with respect to the invariant measure m if and only if $\alpha + \beta < 1$ (since the density of m behaves like $x^{-\alpha}$ close to 0). A small computation shows that $D_\theta f_Y(J_n) \asymp n^{\beta/\alpha}$, whence $\sum nm[J_n]D_\theta f_Y(J_n) < \infty$ as soon as $\alpha + \beta < 1$. Finally, on J_n, $f_Y \sim Cn^{\beta/\alpha+1}$, whence $m[f_Y > x] \sim Cx^{-\frac{1}{\alpha+\beta}}$. Thus, when $\alpha + \beta < 1/2$, we obtain a usual

central limit theorem, while when $\alpha + \beta = 1/2$ we have a central limit theorem with normalization $\frac{1}{\sqrt{n \log n}}$, and for $1/2 < \alpha + \beta < 1$ we obtain a convergence to a stable law for the normalization $\frac{1}{n^{\alpha+\beta}}$. So, even when $\alpha < 1/2$, we can have convergence to a stable law if the observable is unbounded.
Note that Theorems 1.1 and 1.2 can not be used directly in this case, since the function f is not bounded on $\bigcup_0^{N-1} T^i(Y)$. However, what is important in the proof is that we can take an increasing sequence Z_q with $\bigcup Z_q = X$ such that the induction on Z_q has good properties. In this case, we can take $Z_q = (x_q, 1)$ and the proof is exactly the same. In an equivalent way, we can construct an inverse tower $\bigcup_0^{N-1} T^{-i}(Y)$ and work with this inverse tower, the proofs being almost identical.

4. For fixed points such as $x + x^\alpha \log x$ with $\alpha \in (0, 1)$, we also get other slowly varying functions, whence convergence with other normalizing factors.

1.4. Strategy of the proof

Results of convergence to normal laws or stable laws are already known when it is possible to obtain a spectral gap for the transfer operator, for example for uniformly expanding dynamics (see for example [GH88] and [AD01]). In the following, this case will be referred to as the *uniform* case. Such results apply in particular to the induced maps, and they are obtained by perturbing the transfer operators.

The strategy is then to translate the results on the induced map to the whole space. For this, we use *first return transfer operators*, which have been used by Sarig in [Sar02] (see also [Iso00]). We will perturb these operators, and transfer the results in the uniform case to these first return transfer operators, through the abstract Theorem 2.1 which is in fact the main technical part of the proof.

This gives information on the behavior of the Birkhoff sums, but only on the part of the space on which we have induced. We then have to induce on larger and larger parts of the space, to recover more and more information, which will complete the proof.

In Section 2, we will prove the abstract result referred to above, Theorem 2.1. In Section 3, we will establish estimates in the uniform case in our context, which is slightly more general than the results in the literature. Finally, we will prove the main theorems in Section 4.

2. An abstract perturbative theorem

In the following, $\mathbb{D}$ will denote $\{z \in \mathbb{C} \mid |z| < 1\}$, and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$.

2.1. The result

The goal of this section is to prove the following theorem:

Theorem 2.1. *Let $\mathcal{L}$ be a Banach space and $R_n \in \mathrm{Hom}(\mathcal{L}, \mathcal{L})$ be operators on $\mathcal{L}$ with $\|R_n\| \leqslant r_n$ for a sequence r_n such that $a_n = \sum_{k>n} r_k$ is summable. Write $R(z) = \sum R_n z^n$ for $z \in \overline{\mathbb{D}}$. Assume that* 1 *is a simple isolated eigenvalue of* $R(1)$

and that $I - R(z)$ is invertible for $z \in \overline{\mathbb{D}} - \{1\}$. Let P denote the spectral projection of $R(1)$ for the eigenvalue 1, and assume that $PR'(1)P = \mu P$ for some $\mu > 0$.

Let $R_n(t)$ be an operator depending on $t \in [-\alpha, \alpha]$, continuous at $t = 0$, with $R_n(0) = R_n$ and $\|R_n(t)\| \leqslant Cr_n$ $\forall t \in [-\alpha, \alpha]$, for some constant $C > 0$. For $z \in \overline{\mathbb{D}}$ and $t \in [-\alpha, \alpha]$, write

$$R(z,t) = \sum_{n=1}^{\infty} z^n R_n(t).$$

This is a continuous perturbation of $R(z)$. For t small and z close to 1, $R(z,t)$ is close to $R(1)$, whence it admits an eigenvalue $\lambda(z,t)$ close to 1. Assume that $\lambda(1,t) = 1 - (c + o(1))M(|t|)$ for $c \in \mathbb{C}$ with $\mathrm{Re}(c) > 0$, and some continuous function $M : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing only at 0. Then

1. *There exists $\varepsilon_0 > 0$ such that $\forall |t| < \varepsilon_0$, $I - R(z,t)$ is invertible for all $z \in \mathbb{D}$. We can write $(I - R(z,t))^{-1} = \sum T_{n,t} z^n$.*
2. *Furthermore, there exist functions $\varepsilon(t)$ and $\delta(n)$ tending to 0 when $t \to 0$ and $n \to \infty$ such that $\forall |t| < \varepsilon_0$, $\forall n \in \mathbb{N}^*$, we have*

$$\left\| T_{n,t} - \frac{1}{\mu}\left(1 - \frac{c}{\mu} M(|t|)\right)^n P \right\| \leqslant \varepsilon(t) + \delta(n). \tag{2}$$

For example, when $M(t) = t^p$, this gives the convergence $T_{n,t/n^{1/p}} \to \frac{e^{-c|t|^p/\mu}}{\mu} P$.

Remark. The assumption that 1 is a simple isolated eigenvalue of $R(1)$ means that, for λ close to 1 but different from 1, $\lambda I - R(1)$ is invertible (i.e. 1 is isolated in the spectrum of $R(1)$), and that there is a codimension 1 subspace $E \subset \mathcal{L}$ and a non zero vector $\psi \in \mathcal{L}$ such that $\mathcal{L} = E \oplus \mathbb{C}\psi$, $(I - R(1))\psi = 0$, and $I - R(1) : E \to E$ is invertible (i.e. 1 is a simple eigenvalue of $R(1)$).

Remark. There is an analogue of this theorem when the perturbation takes place on $[0, \alpha]$ (with exactly the same proof). Thus, for a perturbation where $\lambda(1,t) = 1 - a|t|^p + ib\,\mathrm{sgn}(t)|t|^p + o(|t|^p)$ for $t \in [-\alpha, \alpha]$, we get once again the same result, using twice this theorem, on $[-\alpha, 0]$ and $[0, \alpha]$.

In the application of Theorem 2.1 to the proof of Theorems 1.1 and 1.2, the operators R_n will describe the returns to the basis Y, and will be easily understood, as well as their perturbations $R_n(t)$. On the other hand, T_n will describe all the iterates at time n, and $T_{n,t}$ will be closely related to the characteristic function $E(e^{itS_n f})$ (see Section 4 for more details). Thus, (2) will enable us to describe precisely $E(e^{itS_n f})$, and this information will be sufficient to get Theorems 1.1 and 1.2.

2.2. Wiener's Lemma in Banach algebras

The classical Wiener's Lemma says that if $f : S^1 \to \mathbb{C}$ has summable Fourier coefficients and is everywhere nonzero, then the Fourier coefficients of $1/f$ are also summable (see for example [Kah70]). We will use an analogue of this result for functions taking their values in non-commutative Banach algebras.

Let $\mathcal{A}$ be a Banach algebra with unit (its norm will be written $\| \ \|_{\mathcal{A}}$). We will denote by $A(\mathcal{A})$ the set of continuous functions $f : S^1 \to \mathcal{A}$ whose Fourier coefficients, defined by

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) \, \mathrm{d}\theta,$$

are summable in norm. Then f coincides with the sum $\sum_{n\in\mathbb{Z}} c_n(f) z^n$ of its Fourier series (see [Gou02, Proposition 2.6]). In particular, if $f, g \in A(\mathcal{A})$, then $c_n(fg) = \sum_{k\in\mathbb{Z}} c_{n+k}(f) c_{n-k}(g)$, whence $fg \in A(\mathcal{A})$. Moreover, $A(\mathcal{A})$ is a Banach algebra for the norm $\|f\|_{A(\mathcal{A})} = \sum \|c_n(f)\|_{\mathcal{A}}$.

For $f : S^1 \to \mathcal{A}$, we will also write $\|f\|_{\mathcal{A}} = \sup\{\|f(z)\|_{\mathcal{A}} \mid z \in S^1\}$. We have of course $\|f\|_{\mathcal{A}} \leqslant \|f\|_{A(\mathcal{A})}$, but the other inequality is false.

The following result can be found in [BP42] (I thank Omri Sarig for this reference).

Theorem 2.2. *Let $f \in A(\mathcal{A})$ be such that for all $z \in S^1$, $f(z)$ is invertible in the Banach algebra $\mathcal{A}$. Then, setting $g(z) = f(z)^{-1}$, we also have $g \in A(\mathcal{A})$.*

Let Ω be an open subset of $\mathbb{C}$ and $h : \Omega \to \mathbb{C}$ an holomorphic function. If $x \in \mathcal{A}$ is such that its spectrum $\sigma(x)$ is included in Ω, it is possible to define $h(x) \in \mathcal{A}$ using the Cauchy formula, by $h(x) := \frac{1}{2i\pi} \int_\gamma \frac{h(u)}{uI - x} \, \mathrm{d}u$, where γ is a path around $\sigma(x)$ in Ω. This is independent of γ (see e.g. [DS57, VII.3]). For example, if $h(z) = z^n$ for $n \in \mathbb{Z}$, then $h(x) = x^n$ (for the multiplication in the algebra $\mathcal{A}$). For another example, take $\mathcal{A} = \mathrm{Hom}(\mathcal{L}, \mathcal{L})$ where $\mathcal{L}$ is a Banach space, and $x \in \mathcal{A}$ for which 1 is isolated in $\sigma(x)$. Then, if h is equal to 1 in a small neighborhood of 1, and to 0 outside of this neighborhood, then $h(x)$ is the spectral projection associated to 1.

Theorem 2.3. *Let Ω be an open subset of $\mathbb{C}$ and $h : \Omega \to \mathbb{C}$ be holomorphic. Let $f \in A(\mathcal{A})$ be such that $\forall z \in S^1$, $\sigma(f(z)) \subset \Omega$. If $g(z) = h(f(z))$, then we also have $g \in A(\mathcal{A})$.*

The previous Wiener's Lemma is a particular case of this result, for $h(z) = 1/z$.

Proof. As the spectrum is semi-continuous, there exists a path γ in Ω around $\sigma(f(z))$ for every $z \in S^1$. Thus, $g(z) = \frac{1}{2i\pi} \int_\gamma \frac{h(u)}{uI - f(z)} \, \mathrm{d}u$. For $u \in \gamma$ fixed, $M_u : z \mapsto \frac{h(u)}{uI - f(z)}$ is in $A(\mathcal{A})$ by Theorem 2.2. As the inversion is continuous in the Banach algebra $A(\mathcal{A})$, $u \mapsto M_u$ is continuous on the compact γ. Since $g = \int_\gamma M_u \, \mathrm{d}u$, this concludes the proof. □

Note that this result applies in particular to spectral projections.

2.3. *Study of $(I - R(z,t))^{-1}$ at $t = 0$*

In this section, we will describe the behavior of the coefficients of $(I - R(z))^{-1}$, and in particular show that they tend to $\frac{1}{\mu} P$. We will use techniques introduced by Sarig in [Sar02]. The main technical difference is that in the proof of Theorem 2.4 (an analogue of his "first main lemma"), we will replace his estimates using $C^{1+\alpha}$ regularity by applications of Wiener's Lemma.

The following result is interesting in itself, and its proof will be instrumental in the proof of Theorem 2.1.

Theorem 2.4. *Let R_n be operators on a Banach space $\mathcal{L}$ such that $a_n = \sum_{k>n} \|R_k\|$ is summable. Write $R(z) = \sum_{n=1}^{\infty} R_n z^n$ and assume that* 1 *is a simple isolated eigenvalue of $R(1)$, while $I - R(z)$ is invertible for $z \in \overline{\mathbb{D}} - \{1\}$. Write P for the spectral projection of $R(1)$ for the eigenvalue* 1, *and assume also that $PR'(1)P = \mu P$ for a nonzero constant μ.*

Then the function $\left(\frac{I-R(z)}{1-z}\right)^{-1}$ can be continuously extended to $\overline{\mathbb{D}}$, and its Fourier coefficients on S^1 are summable. Writing $(I - R(z))^{-1} = \sum_n T_n z^n$, we have $\sum_n \|T_n - T_{n-1}\| < \infty$.

Proof. Write $S(z) = \frac{I-R(z)}{1-z}$ and $\mathcal{A} = \mathrm{Hom}(\mathcal{L}, \mathcal{L})$.

Step 1. $S(z)^{-1}$ can be continuously extended to $\overline{\mathbb{D}}$.

The problem is the extension to 1. For $z \in \overline{\mathbb{D}}$ close to 1, $R(z)$ has a unique eigenvalue $\lambda(z)$ close to 1. We write $P(z)$ for the corresponding spectral projection, and $Q(z) = I - P(z)$. For z close to 1 but different from 1,

$$S(z)^{-1} = \frac{1-z}{1-\lambda(z)} P(z) + (1-z)(I - R(z)Q(z))^{-1} Q(z). \tag{3}$$

As $I - R(z)Q(z)$ is everywhere invertible on $\overline{\mathbb{D}}$, the second term has a continuous extension to 1. As $(I - R(1))P(1) = 0$,

$$\frac{1-\lambda(z)}{1-z} P(z) = P(z)\frac{R(1)-R(z)}{1-z} P(z) + P(z)(I - R(1))\frac{P(z)-P(1)}{1-z}. \tag{4}$$

Note that

$$\frac{R(z)-R(1)}{z-1} = \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} R_k \right) z^n.$$

As $\sum_{k>n} \|R_k\|$ is a summable sequence, the above sum converges in norm, which implies that its limit at 1, denoted by $R'(1)$, is well defined.

If $\delta > 0$ is small enough, then

$$P(z) = \frac{1}{2i\pi} \int_{|u-1|=\delta} \frac{1}{uI - R(z)} \,\mathrm{d}u.$$

Thus,

$$\frac{P(z)-P(1)}{z-1} = \frac{1}{2i\pi} \int_{|u-1|=\delta} \frac{1}{uI - R(z)} \frac{R(z)-R(1)}{z-1} \frac{1}{uI - R(1)} \,\mathrm{d}u. \tag{5}$$

As $\frac{R(z)-R(1)}{z-1}$ converges at 1, we get the convergence of $\frac{P(z)-P(1)}{z-1}$ at 1, to a limit $P'(1)$.

In Equation (4), when z tends to 1, the right hand term tends to $P(1)R'(1)P(1) - P(1)(I - R(1))P'(1) = P(1)R'(1)P(1)$ since $P(1)(I - R(1)) = 0$. Recall that $P(1)R'(1)P(1) = \mu P$ for some nonzero μ.

Let ξ be a bounded linear functional on $\mathrm{Hom}(\mathcal{L}, \mathcal{L})$ such that $\xi(P(1)) = 1$. We apply ξ to both members of Equation (4) and let z go to 1, which gives that

$$\frac{1 - \lambda(z)}{1 - z} \to \mu. \tag{6}$$

As $\mu \neq 0$, this shows that, in Equation (3), both terms on the right hand side converge at 1.

Step 2. Construction of a function $\tilde{R}(z)$ on S^1, coinciding with $R(z)$ for z close to 1, such that it has everywhere a simple eigenvalue $\tilde{\lambda}(z)$ close to 1, but different from 1 if $z \neq 1$, and with $\frac{\tilde{R}(z)-\tilde{R}(1)}{z-1} \in A(\mathcal{A})$.

We construct $\tilde{R}$ in three steps, first defining two approximations F, G and then gluing them together to get $\tilde{R}$.

Fix some $\gamma > 0$, very small. Let $\varphi + \psi$ be a C^∞ partition of unity on S^1 associated to the sets $\{\theta \in [0, \gamma)\}$ and $\{\theta \in (\gamma - \eta, \pi/2]\}$ where θ is the angle on the circle (for some very small $0 < \eta < \gamma$). We define $F(z) = \varphi(z)R(z) + \psi(z)R(e^{i\gamma})$ on $\{\theta \in [0, \pi/2]\}$: F is equal to R on $\{\theta \in [0, \gamma - \eta]\}$ and to $R(e^{i\gamma})$ on $\{\theta \in [\gamma, \pi/2]\}$. In particular, the spectrum of $F(z)$ is "almost the same" as the spectrum of $R(1)$ if γ is small enough.

We define in the same way F on $\{\theta \in [-\pi/2, 0]\}$, equal to $R(e^{-i\gamma})$ on $\{\theta \in [-\pi/2, -\gamma]\}$ and to R on $\{\theta \in [-\gamma + \eta, 0]\}$.

Finally, we construct F on the remaining half-circle by symmetrizing, i.e. $F(e^{i(\pi/2+a)}) = F(e^{i(\pi/2-a)})$, to ensure that everything fits well.

Provided γ is small enough, there is a well defined eigenvalue close to 1 for every $F(z)$, depending continuously on z, which we denote by $\rho(z)$. The problem would be solved if $\rho(z) \neq 1$ for $z \neq 1$, which is not the case since $\rho(-1) = \rho(1) = 1$. Consequently, we have to perturb ρ a little. Let ν be a C^∞ function on $\{\theta \in [\pi/2, 3\pi/2]\}$, arbitrarily close to ρ and which does not take the value 1. On $\{\theta \in [\pi/2 + \eta, 3\pi/2 - \eta]\}$, we define $G(z) = \frac{\nu(z)}{\rho(z)} F(z)$: its eigenvalue close to 1 is $\nu(z) \neq 1$. Finally, we glue F and G together on $\{\theta \in [\pi/2, \pi/2 + \eta]\}$ and $\{\theta \in [3\pi/2 - \eta, 3\pi/2]\}$ with a partition of unity, as above. As the spectrum of $F(e^{i\pi/2}) = R(e^{i\gamma})$ does not contain 1, the gluing will not give an eigenvalue equal to 1 if we choose η small enough and ν close enough to ρ.

Note finally that, since $\frac{R(z)-R(1)}{z-1} \in A(\mathcal{A})$, we also have $\frac{\tilde{R}(z)-\tilde{R}(1)}{z-1} \in A(\mathcal{A})$: to obtain $\tilde{R}(z)$, we have multiplied by C^∞ functions, which are in $A(\mathbb{C})$.

Step 3. Writing $\tilde{S}(z) = \frac{I-\tilde{R}(z)}{1-z}$, then $\tilde{S}(z)^{-1} \in A(\mathcal{A})$.

By construction, $\frac{\tilde{R}(z)-\tilde{R}(1)}{z-1} \in A(\mathcal{A})$, and $\tilde{R}(z) \in A(\mathcal{A})$. Equation (5) (with tildes everywhere) implies that $\frac{\tilde{P}(z)-\tilde{P}(1)}{z-1} \in A(\mathcal{A})$ (use first Wiener's Lemma Theorem 2.2 on $uI - \tilde{R}(z)$, then multiply and integrate, as in the proof of Theorem 2.3). In particular, $\tilde{P}(z) \in A(\mathcal{A})$.

Let ξ be a bounded linear functional on $\mathcal{A}$ satisfying $\xi(P(1)) = 1$ and $\forall z \in S^1$, $\xi(\tilde{P}(z)) \neq 0$ (take γ small enough in the construction of $\tilde{R}$ to ensure this). Then $\xi(\tilde{P}(z)) \in A(\mathbb{C})$, thus $1/\xi(\tilde{P}(z)) \in A(\mathbb{C})$ by Wiener's Lemma 2.2. Applying ξ to both sides of Equation (4) (with tildes) and dividing by $\xi(\tilde{P}(z))$, we obtain that $\frac{1-\tilde{\lambda}(z)}{1-z} \in A(\mathbb{C})$. As this function is everywhere non-vanishing, we can use once more Wiener's Lemma and get that $\frac{1-z}{1-\tilde{\lambda}(z)} \in A(\mathbb{C})$.

Finally, we use Equation (3) (with tildes everywhere). Wiener's Lemma ensures that $(I - \tilde{R}(z)\tilde{Q}(z))^{-1} \in A(\mathcal{A})$ since $I - \tilde{R}(z)\tilde{Q}(z)$ is in $A(\mathcal{A})$ and pointwise invertible. Thus, the right hand side is in $A(\mathcal{A})$, which ends the proof.

Step 4. $S(z)^{-1} \in A(\mathcal{A})$.

Let $\varphi + \psi$ be a C^∞ partition of unity on S^1, such that $\tilde{R} = R$ on the support of φ. Then

$$\left(\frac{I - R(z)}{1-z}\right)^{-1} = \varphi(z)\left(\frac{I - \tilde{R}(z)}{1-z}\right)^{-1} + \psi(z)\left(\frac{I - R(z)}{1-z}\right)^{-1}.$$

The first term on the right hand side is in $A(\mathcal{A})$ according to Step 3. For the second term, we can modify $\frac{I-R(z)}{1-z}$ outside of the support of ψ so that it is everywhere defined and invertible, using the same techniques as in the construction of $\tilde{R}$. Wiener's Lemma gives that its inverse is in $A(\mathcal{A})$, which concludes the proof.

Step 5. conclusion.

$I - R(z)$ is a power series that is invertible everywhere on $\mathbb{D}$. According to [DS57, Lemma VII.6.13], its pointwise inverse is also analytic on $\mathbb{D}$, whence it is a power series, that can be written as $\sum T_n z^n$. Multiplying by $1 - z$, we get $S(z)^{-1} = \sum (T_n - T_{n-1}) z^n$. Thus, if $r < 1$, we have $T_n - T_{n-1} = \frac{1}{2\pi r^n} \int e^{-in\theta} S(re^{i\theta})^{-1} \, d\theta$. As $S(z)^{-1}$ is continuous on the whole disk $\overline{\mathbb{D}}$, we obtain by letting r tend to 1 that $T_n - T_{n-1}$ is the nth Fourier coefficient of $S(z)^{-1}$ on S^1. But we already know that $S(z)^{-1} \in A(\mathcal{A})$, which proves that $\sum \|T_n - T_{n-1}\| < \infty$. □

Remark. Theorem 2.4 is in fact the main technical ingredient in the proof of Sarig in [Sar02], but he had to assume $a_n \sim \frac{1}{n^\beta}$ with $\beta > 2$. Thus, it is possible using Theorem 2.4 to extend Sarig's results on lower bounds for the decay of correlations (Theorem 2 and Corollaries 1 and 2 in [Sar02]), even when the mass of the set of points returning at time n is not of the order $1/n^\beta$. For example, it is not hard to generalize his results to maps of the interval with a very neutral fixed point, such as $x(1 + x \log^2 x)$. With a little work, this shows that all the upper bounds on the decay of correlations obtained by Holland in [Hol02] are in fact optimal.

The following lemma will be needed later on in the proof of Theorem 2.1.

Lemma 2.5. *Under the same hypotheses as in Theorem 2.4, we have* $\left(\frac{I-R(z)}{1-z}\right)^{-1} = \frac{1}{\mu} P + (1-z)A(z)$, *where* $A(z) = \sum_{n=0}^{\infty} A_n z^n$ *with* $A_n \to 0$ *as* $n \to \infty$.

Proof. We use the method of Sarig in [Sar02], starting from our analogue of his first main lemma, Theorem 2.4. More precisely, Sarig constructs a polynomial $R_B(z)$, approximating $R(z)$, such that $\frac{R(1)-R_B}{1-z}$ and $\frac{1}{1-z}\left(\frac{R(1)-R_B}{1-z} - R'(1)\right)$ are polynomials. Moreover, $I - R_B(z)$ is invertible for $z \in \mathbb{D}$, and $\left(\frac{I-R_B(z)}{1-z}\right)^{-1} = \frac{1}{\mu}P + (1-z)B(z)$ where $B(z) = \sum B_n z^n$ and $B_n = o(\kappa^n)$ for some $\kappa < 1$. The proof of this result ([Sar02, Lemma 6]) relies only on his first main lemma, which we have already proved in our context.

Then, he uses the following perturbative expansion of $S(z)^{-1} = \left(\frac{I-R(z)}{1-z}\right)^{-1}$ around $S_B(z)^{-1} = \left(\frac{I-R_B(z)}{1-z}\right)^{-1}$:

$$S(z)^{-1} = S_B(z)^{-1} + S_B(z)^{-1}(S_B(z) - S(z))S(z)^{-1}.$$

To prove the lemma, we only have to show that $S_B(z)^{-1}(S_B(z) - S(z))S(z)^{-1}$ can be written as $(1-z)C(z)$ for some $C(z) = \sum C_n z^n$ with $C_n \to 0$ (indeed, $A_n = B_n + C_n$).

We first study $\frac{S_B - S}{1-z}$. We have

$$S_B - S = \left(\frac{R(1) - R_B}{1-z} - R'(1)\right) + \left(R'(1) - \frac{R(1) - R}{1-z}\right) = I + II.$$

As $I/(1-z)$ is a polynomial (by construction of R_B), we can forget this term. For the other term,

$$II = \sum_{k=0}^{\infty}(1-z^k)\sum_{n=k+1}^{\infty} R_n.$$

Thus, writing $D_n = \sum_{k=n+1}^{\infty}\sum_{l=k+1}^{\infty} R_l$, we have $\frac{II}{1-z} = \sum D_n z^n$. The summability of $a_k = \sum_{l=k+1}^{\infty}\|R_l\|$ implies that $D_n \to 0$. We have shown that

$$\frac{S_B - S}{1-z} = \sum E_n z^n$$

where $E_n \to 0$.

We already know that $S_B(z)^{-1}$ and $S(z)^{-1}$ have summable coefficients. Thus, the lemma will be proved if we show that, if $E_n \to 0$ and F_n is summable, then the coefficient $(EF)_n$ of z^n in $\left(\sum E_n z^n\right)\left(\sum F_n z^n\right)$ tends to 0 as $n \to \infty$.

We have $(EF)_n = \sum_{k=0}^{n} E_k F_{n-k}$. We fix $\varepsilon > 0$, and N such that if $k \geqslant N$, $\|E_k\| \leqslant \varepsilon$. If K is greater than $\|E_k\|$ and $\sum\|F_k\|$, we get for $n \geqslant N$

$$\|(EF)_n\| \leqslant \sum_{k=0}^{N-1}\|E_k\|\,\|F_{n-k}\| + \sum_{k=N}^{n}\varepsilon\,\|F_{n-k}\| \leqslant K\sum_{l=n-N}^{\infty}\|F_l\| + K\varepsilon.$$

For n large enough, this is less than $2K\varepsilon$. □

2.4. Invertibility of $I - R(z, t)$

From this point on, and until the end of the proof of Theorem 2.1, the notations and hypotheses of all the results will be those of Theorem 2.1.

In this section, we prove the invertibility of $I - R(z, t)$ when $(z, t) \neq (1, 0)$ and t is small enough. The problem occurs for z close to 1, where $R(z, t)$ has a well defined eigenvalue $\lambda(z, t)$ close to 1. Let us first study this eigenvalue.

Lemma 2.6. *In a neighborhood of* $(1, 0)$*, it is possible to write* $\lambda(z, t) = 1 + (z - 1)(\mu + a(z, t)) - (c + b(t))M(|t|)$ *where the functions* a *and* b *tend to* 0 *respectively at* $(1, 0)$ *and* 0.

Proof. As

$$\frac{R(z, t) - R(1, t)}{z - 1} = \sum_{n=0}^{\infty} z^n \left(\sum_{k>n} R_k(t) \right),$$

the summability assumption implies that $(z, t) \mapsto \frac{R(z,t)-R(1,t)}{z-1}$ is continuous at $t = 0$. Writing

$$\begin{aligned} &\frac{P(z, t) - P(1, t)}{z - 1} \\ &= \frac{1}{2i\pi} \int_{|u-1|=\delta} \frac{1}{uI - R(z, t)} \frac{R(z, t) - R(1, t)}{z - 1} \frac{1}{uI - R(1, t)} \, \mathrm{d}u, \end{aligned}$$

we also get the continuity of $(z, t) \mapsto \frac{P(z,t)-P(1,t)}{z-1}$ at $t = 0$. Thus, applying a bounded linear functional to the equation

$$\begin{aligned} \frac{\lambda(z, t) - \lambda(1, t)}{z - 1} P(z, t) = {} & \frac{R(z, t) - R(1, t)}{z - 1} P(z, t) \\ & + (R(1, t) - \lambda(1, t)) \frac{P(z, t) - P(1, t)}{z - 1}, \end{aligned}$$

we get also the continuity of $F(z, t) = \frac{\lambda(z,t)-\lambda(1,t)}{z-1}$. Moreover, $F(z, 0) \to \mu$ when $z \to 1$ (Equation (6)), which makes it possible to write $F(z, t) = \mu + a(z, t)$ with $a(z, t)$ vanishing at $(1, 0)$. Moreover, by assumption, $\lambda(1, t) = 1 - (c + b(t))M(|t|)$ where b vanishes at 0. Hence,

$$\begin{aligned} \lambda(z, t) &= \lambda(1, t) + (z - 1)F(z, t) \\ &= 1 - (c + b(t))M(|t|) + (z - 1)(\mu + a(z, t)). \qquad \square \end{aligned}$$

Proposition 2.7. *There exists* $\varepsilon_0 > 0$ *such that* $\forall (z, t) \in (\overline{\mathbb{D}} \times [-\varepsilon_0, \varepsilon_0]) - \{(1, 0)\}$*,* $I - R(z, t)$ *is invertible.*

Proof. Assume that there is a neighborhood V of $(1, 0)$ such that, in $V - \{(1, 0)\}$, $I - R(z, t)$ is invertible. We can assume that $V = U \times [-\varepsilon_1, \varepsilon_1]$. For $w \in \overline{\mathbb{D}} - U$, $I - R(w)$ is invertible, thus $I - R(z, t)$ is invertible in a small neighborhood of $(w, 0)$, which can be taken of the form $U_w \times [-\varepsilon_w, \varepsilon_w]$. By compactness, $\overline{\mathbb{D}} - U$ is covered by a finite number of the U_w, say $U_{w_1}, \dots, U_{w_n}$. To conclude, just take $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_{w_1}, \dots, \varepsilon_{w_n})$.

So, we just have to construct a neighborhood V as above. We have to show that the eigenvalue $\lambda(z,t)$ of $R(z,t)$ is different from 1 when (z,t) is close to but different from $(1,0)$. This essentially comes from the asymptotic expansion of $\lambda(z,t)$ given by Lemma 2.6, but we will write it more carefully.

Let K be such that $\frac{\mu}{4} > \frac{3|c|}{2K}$. The neighborhood V will be the set of (z,t) such that $|b(t)| \leqslant \mathrm{Re}(c)/2$ and $|a(z,t)| \leqslant \min\left(\frac{\mu}{4}, \frac{\mathrm{Re}(c)}{2K}\right)$.

We fix $(z,t) \in V \cap (\overline{\mathbb{D}}\times\mathbb{R})$. Assuming that $\lambda(z,t) = 1$, we show that $(z,t) = (1,0)$. There are two possibilities, namely $|\mathrm{Im}(z-1)| \geqslant |z-1|/2$ or $|\mathrm{Re}(z-1)| \geqslant |z-1|/2$. The latter case, being easier, is left to the reader. We can assume for example that $\mathrm{Im}(z-1) \geqslant |z-1|/2$.

As $\mathrm{Re}(\mu(z-1)) \leqslant 0$ since $\mu > 0$ and $z \in \overline{\mathbb{D}}$, we have

$$\begin{aligned}
1 &= \mathrm{Re}(\lambda(z,t)) = 1 + \mathrm{Re}(\mu(z-1)) + \mathrm{Re}(a(z,t)(z-1)) \\
&\quad -(\mathrm{Re}(c) + \mathrm{Re}(b(t))M(|t|) \\
&\leqslant 1 + |a(z,t)||z-1| - (\mathrm{Re}(c) - |b(t)|)M(|t|) \\
&\leqslant 1 + \frac{\mathrm{Re}(c)}{2K}|z-1| - \frac{\mathrm{Re}(c)}{2}M(|t|).
\end{aligned}$$

Thus, $|z-1| \geqslant KM(|t|)$. The imaginary part then satisfies

$$\begin{aligned}
0 = \mathrm{Im}(\lambda(z,t)) &\geqslant (\mu - |a(z,t)|)\,\mathrm{Im}(z-1) - (|c| + |b(t)|)M(|t|) \\
&\geqslant \frac{\mu}{2}\frac{|z-1|}{2} - \frac{3|c|}{2}\frac{|z-1|}{K}.
\end{aligned}$$

The definition of K implies that $|z-1| = 0$, which in turn gives $t = 0$ since $|z-1| \geqslant KM(|t|)$ and M vanishes only at 0. □

In particular, for $|t| < \varepsilon_0$, $I - R(z,t)$ is a power series invertible everywhere on $\mathbb{D}$, whence its inverse can be written as $\sum T_{n,t} z^n$ (see Step 5 of the proof of Theorem 2.4). Moreover, for $t \neq 0$, everything can be continuously extended to $\partial\mathbb{D}$, which implies that the $T_{n,t}$ are the Fourier coefficients of the function $(I - R(z,t))^{-1}$ (considered as a function from S^1 to $\mathcal{A}$).

2.5. *Convergence of $\tilde{R}(z,t)$*

In this section, we will work exclusively on S^1 (z will thus denote a point in $S^1 \subset \mathbb{C}$ and *not* in $\mathbb{D}$).

To apply Fourier series methods to $R(z,t)$, we will need it to have a well defined eigenvalue close to 1, which is not *a priori* true outside of a neighborhood of $(1,0)$.

Construction of $\tilde{R}(z,t)$

We use the function $\tilde{R}(z)$ that has been constructed in Step 2 of the proof of Theorem 2.4. We set $\tilde{R}(z,t) = \tilde{R}(z) - \left(R(z) - \sum_{n=1}^{\infty} z^n R_n(t)\right)$. The map $t \mapsto \tilde{R}(\cdot,t)$ is then continuous at $t = 0$ (for the $A(\mathcal{A})$-norm, whence for the supremum norm), and $\tilde{R}(z,t)$ has a well defined eigenvalue $\tilde{\lambda}(z,t)$ close to 1. Moreover, for z in a neighborhood of 1, $\tilde{\lambda}(z,t) = \lambda(z,t)$, which is different from 1 when $(z,t) \neq (1,0)$, and outside of this neighborhood $\tilde{\lambda}(z,t)$ is close to $\tilde{\lambda}(z,0)$ and is thus different from 1 if t is small enough.

The main result of this section will be the following proposition (recall that the hypotheses are those of Theorem 2.1):

Proposition 2.8. *We have*

$$\left\| \left(\frac{I - \tilde{R}(z,t)}{1 - (1 - \frac{c}{\mu} M(|t|))z} \right)^{-1} - \left(\frac{I - \tilde{R}(z)}{1 - z} \right)^{-1} \right\|_{A(\mathcal{A})} \xrightarrow[t \to 0]{} 0.$$

Proof. Write $\tilde{P}(z,t)$ for the spectral projection associated to the eigenvalue $\tilde{\lambda}(z,t)$, and $\tilde{Q}(z,t) = I - \tilde{P}(z,t)$. We then have

$$I - \tilde{R}(z,t) = (1 - \tilde{\lambda}(z,t))\tilde{P}(z,t) + (I - \tilde{R}(z,t)\tilde{Q}(z,t))\tilde{Q}(z,t).$$

Thus,

$$\begin{aligned} \left(\frac{I - \tilde{R}(z,t)}{1 - (1 - \frac{c}{\mu} M(|t|))z} \right)^{-1} &= \frac{1 - (1 - \frac{c}{\mu} M(|t|))z}{1 - \tilde{\lambda}(z,t)} \tilde{P}(z,t) \\ &+ \left(1 - (1 - \frac{c}{\mu} M(|t|))z \right) (I - \tilde{R}(z,t)\tilde{Q}(z,t))^{-1} \tilde{Q}(z,t). \end{aligned}$$

As $\tilde{R}(z,t) \to \tilde{R}(z)$ as $t \to 0$, Wiener's Lemma Theorem 2.3 gives the convergence of $\tilde{P}(z,t)$ and $\tilde{Q}(z,t)$ respectively to $\tilde{P}(z)$ and $\tilde{Q}(z)$ (in $A(\mathcal{A})$). We also obtain (again by Wiener's Lemma) the convergence of $(I - \tilde{R}(z,t)\tilde{Q}(z,t))^{-1}$ to $(I - \tilde{R}(z)\tilde{Q}(z))^{-1}$, since $I - \tilde{R}(z)\tilde{Q}(z)$ is invertible for every $z \in S^1$.

It only remains to prove that $\frac{1-(1-\frac{c}{\mu}M(|t|))z}{1-\tilde{\lambda}(z,t)} \to \frac{1-z}{1-\tilde{\lambda}(z)}$ (in $A(\mathbb{C})$), which is given by the following lemma and Wiener's Lemma (since inversion is continuous in the Banach algebra $A(\mathbb{C})$). □

Lemma 2.9. *We have*

$$\left\| \frac{1 - \tilde{\lambda}(z,t)}{1 - (1 - \frac{c}{\mu} M(|t|))z} - \frac{1 - \tilde{\lambda}(z)}{1 - z} \right\|_{A(\mathbb{C})} \xrightarrow[t \to 0]{} 0.$$

Proof. We will usually work with the circle S^1 embedded in $\mathbb{C}$, but it will sometimes be convenient to consider the circle as $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, in which case the variable will be denoted by θ.

We recall some results on Fourier series. For $|\theta| \leqslant \pi$ and small $\varepsilon > 0$, write $\Delta_\varepsilon(\theta) = \sup\left(0, 1 - \frac{|\theta|}{\varepsilon}\right)$: this function is supported in $\{\theta \in [-\varepsilon, \varepsilon]\}$. We write also $V_\varepsilon = 2\Delta_{2\varepsilon} - \Delta_\varepsilon$: this function is equal to 1 on $\{\theta \in [-\varepsilon, \varepsilon]\}$ and vanishes outside of $\{\theta \in [-2\varepsilon, 2\varepsilon]\}$. We will use V_ε and $1 - V_\varepsilon$ as a C^0 partition of unity. By [Kah70, page 56], we have $\|V_\varepsilon\|_{A(\mathbb{C})} \leqslant 3$. Note that V_ε is piecewise C^1, its derivative being bounded by $1/\varepsilon$.

If $g : \mathbb{T} \to \mathbb{C}$ is continuous and piecewise C^1, then ([Kah70, page 56, Equation (1)])

$$\|g\|_{A(\mathbb{C})} \leqslant |c_0(g)| + \left(\frac{\pi}{12}\int_{-\pi}^{\pi} |g'(\theta)|^2 \,\mathrm{d}\theta\right)^{1/2}. \tag{7}$$

Finally, if $f : S^1 \to \mathbb{C}$ is in $A(\mathbb{C})$ and satisfies $f(1) = 0$, then ([Kah70, page 56])

$$\|f V_\varepsilon\|_{A(\mathbb{C})} \xrightarrow[\varepsilon\to 0]{} 0. \tag{8}$$

Fix $|t| < \varepsilon_0$ and write $\psi_t = V_{G(|t|)}$ for $G(t) = M(t)^{1/3}$, and $\varphi_t = 1 - \psi_t$, on S^1 or $\mathbb{T}$. Setting $A_t(z) = \frac{1-\tilde{\lambda}(z,t)}{1-(1-\frac{c}{\mu}M(|t|))z}$, we have to check that $A_t(z)$ tends (in $A(\mathbb{C})$) to $B(z) = \frac{1-\tilde{\lambda}(z)}{1-z}$. We will prove that $A_t(z)\psi_t(z) - B(z)\psi_t(z) \to 0$, and $A_t(z)\varphi_t(z) - B(z)\varphi_t(z) \to 0$.

In this proof, we denote by $C(z)$ a function whose norm in $A(\mathbb{C})$ is bounded by a constant. For example, as $\|V_t\|_{A(\mathbb{C})} \leqslant 3$, we sometimes replace $V_t(z)$ by $C(z)$. As $\frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z} = 1 - \frac{c}{\mu}M(|t|)\sum_{n=1}^{\infty}(1-\frac{c}{\mu}M(|t|))^{n-1}z^n$, we also have $\left\|\frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z}\right\|_{A(\mathbb{C})} \leqslant 1 + \frac{\frac{|c|}{\mu}M(|t|)}{1-\left|1-\frac{c}{\mu}M(|t|)\right|}$, which is bounded when $t \to 0$ (since $\operatorname{Re} c > 0$). Thus, we may also replace $\frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z}$ by $C(z)$.

Using exactly the same equations as in the proof of Lemma 2.6 (with tildes everywhere) and Wiener's Lemma, we prove that $K(z,t) = \frac{\tilde{\lambda}(z,t)-\tilde{\lambda}(1,t)}{z-1} - \frac{\tilde{\lambda}(z)-\tilde{\lambda}(1)}{z-1}$ tends to 0 in $A(\mathbb{C})$ when $t \to 0$.

Step 1 (close to $z = 1$). $A_t(z)\psi_t(z) - B(z)\psi_t(z)$ tends to 0 in $A(\mathbb{C})$-norm.

We proved in the third step of the proof of Theorem 2.4 that $\frac{\tilde{\lambda}(z)-\tilde{\lambda}(1)}{z-1} \in A(\mathbb{C})$. As this quotient tends to μ when $z \to 1$, we can write $\frac{\tilde{\lambda}(z)-\tilde{\lambda}(1)}{z-1} = \mu + F(z)$ where $F \in A(\mathbb{C})$ and $F(1) = 0$. We also write $\tilde{\lambda}(1,t) = 1 - (c + b(t))M(|t|)$ (since $\tilde{\lambda}(1,t) = \lambda(1,t)$, as $\tilde{R}(z) = R(z)$ in a neighborhood of 1). This gives

$$\tilde{\lambda}(z,t) = 1 - (c + b(t))M(|t|) + (z-1)\left(K(z,t) + F(z) + \mu\right).$$

Thus,

$$\begin{aligned} A_t(z) = \frac{1-\tilde{\lambda}(z,t)}{1-(1-\frac{c}{\mu}M(|t|))z} &= \mu + \frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z}(F(z) \\ &+K(z,t) + cM(|t|)) + \frac{M(|t|)}{1-(1-\frac{c}{\mu}M(|t|))z}b(t). \end{aligned}$$

Therefore,

$$\begin{aligned} A_t(z)\psi_t(z) &= \mu\psi_t(z) + C(z)F(z)\psi_t(z) + C(z)(K(z,t) + M(|t|)) \\ &+\frac{M(|t|)}{1-(1-\frac{c}{\mu}M(|t|))z}b(t)\psi_t(z). \end{aligned}$$

A small computation shows that $\frac{M(|t|)}{1-(1-\frac{c}{\mu}M(|t|))z}$ has a bounded norm in $A(\mathbb{C})$. Moreover, $\mu = B(z) - F(z)$, and the term $F(z)\psi_t(z)$ can be included in $C(z)F(z)\psi_t(z)$. Finally,

$$A_t(z)\psi_t(z) - B(z)\psi_t(z) = C(z)F(z)\psi_t(z) + C(z)(M(|t|) + K(z,t) + b(t)).$$

In the rightmost term, everything tends to 0 as $t \to 0$. Moreover, Equation (8) ensures that $F(z)\psi_t(z) \to 0$. This concludes the first step.

Step 2 (away from $z = 1$). $A_t(z)\varphi_t(z) - B(z)\varphi_t(z)$ tends to 0 in $A(\mathbb{C})$-norm.

We start from

$$\frac{1-\tilde{\lambda}(z,t)}{1-(1-\frac{c}{\mu}M(|t|))z} - \frac{1-\tilde{\lambda}(z)}{1-z} = \frac{1-\tilde{\lambda}(z)}{1-z}\left(\frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z} - 1\right) + \frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z}\,\frac{\tilde{\lambda}(z)-\tilde{\lambda}(z,t)}{1-z}.$$

We will see that each term on the right, multiplied by φ_t, tends to 0 in $A(\mathbb{C})$ when $t \to 0$.

We will use that, on the support of φ_t, we have $|z-1| \geqslant CG(|t|)$, which implies that $|1-(1-\frac{c}{\mu}M(|t|))z| \geqslant CG(|t|) - DM(|t|) \geqslant CG(|t|)$ (for a smaller constant C) since $M(t) = o(G(t))$. In the following, C will denote a generic constant.

The first term

As $\frac{1-\tilde{\lambda}(z)}{1-z} \in A(\mathbb{C})$, it is enough to prove the convergence of $f = \left(\frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z} - 1\right)$ $\varphi_t(z)$ to 0. We will use Equation (7).

We have

$$|f(z)| \leqslant \frac{\frac{|c|}{\mu}M(|t|)}{CG(t)} \leqslant CM(|t|)^{2/3}.$$

In particular, $|c_0(f)| \leqslant CM(|t|)^{1/3}$.

For the derivative, writing $f(\theta) = h(\theta)\varphi_t(\theta)$, we have $f' = h'\varphi_t + h\varphi_t'$. As $|h| \leqslant CM(|t|)^{2/3}$ and $|\varphi_t'| \leqslant 1/G(|t|)$, we get $|h\varphi_t'| \leqslant CM(|t|)^{1/3}$.

To control h', we write

$$h(\theta) = -\frac{\frac{c}{\mu}M(|t|)}{1-\frac{c}{\mu}M(|t|)} + \frac{\frac{c}{\mu}M(|t|)}{1-\frac{c}{\mu}M(|t|)}\,\frac{1}{1-(1-\frac{c}{\mu}M(|t|))e^{i\theta}}.$$

Thus,

$$h'(\theta) = i\,\frac{cM(|t|)}{\mu}\,\frac{1}{(1-(1-\frac{c}{\mu}M(|t|))e^{i\theta})^2}.$$

On the support of φ_t, the modulus of the denominator on the right is $\geqslant (CG(|t|))^2$, which gives $|h'| \leqslant CM(|t|)^{1/3}$.

We have shown that $|f'| \leqslant CM(|t|)^{1/3}$. As $|c_0(f)| \leqslant CM(|t|)^{1/3}$, Equation (7) gives that $\|f\|_{A(\mathbb{C})} \leqslant CM(|t|)^{1/3}$.

The second term

As $\frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z}$ is bounded in $A(\mathbb{C})$, it is enough to show that $g(z) = \frac{\tilde{\lambda}(z,t)-\tilde{\lambda}(z)}{1-z}\varphi_t(z)$ tends to 0 in $A(\mathbb{C})$.

Writing again $K(z,t) = \frac{\tilde{\lambda}(z,t)-\tilde{\lambda}(1,t)}{z-1} - \frac{\tilde{\lambda}(z)-\tilde{\lambda}(1)}{z-1}$ (this difference tends to 0 when $t \to 0$), we have

$$g(z) = K(z,t)\varphi_t(z) + (\tilde{\lambda}(1) - \tilde{\lambda}(1,t))\frac{\varphi_t(z)}{1-z}.$$

The first term $K(z,t)\varphi_t(z)$ tends to 0. For the second term, if $h(z) = \frac{\varphi_t(z)}{1-z}$, we have $|c_0(h)| \leqslant \frac{1}{CG(|t|)}$ and

$$|h'(\theta)| = \left| i\frac{\varphi_t(\theta)}{(1-e^{i\theta})^2} + \frac{1}{1-e^{i\theta}}\varphi_t'(\theta)\right| \leqslant \frac{1}{C^2G(|t|)^2} + \frac{1}{CG(|t|)}\frac{1}{G(|t|)}.$$

Equation (7) gives that $\|h\|_{A(\mathbb{C})} \leqslant \frac{C}{G(|t|)^2} = O(M(|t|)^{-2/3})$. As $\tilde{\lambda}(1) - \tilde{\lambda}(1,t) = O(M(|t|))$, we finally get $\left\|(\tilde{\lambda}(1) - \tilde{\lambda}(1,t))\frac{\varphi_t(z)}{1-z}\right\|_{A(\mathbb{C})} = O(M(|t|)^{1/3})$, which concludes the proof of Step 2 of the proof of Lemma 2.9. □

2.6. Conclusion of the proof of Theorem 2.1

Proposition 2.10. *We have*

$$\left\| \left(\frac{I-R(z,t)}{1-(1-\frac{c}{\mu}M(|t|))z}\right)^{-1} - \left(\frac{I-R(z)}{1-z}\right)^{-1} \right\|_{A(\mathcal{A})} \xrightarrow[t\to 0]{} 0.$$

Proof. Let χ_1, χ_2 be a partition of unity such that $R = \tilde{R}$ on the support of χ_1, and set $A(z,t) = \left(\frac{I-R(z,t)}{1-(1-\frac{c}{\mu}M(|t|))z}\right)^{-1}$.

Proposition 2.8 shows that $\chi_1(z)A(z,t) \to \chi_1(z)A(z,0)$. For the term $\chi_2(z)A(z,t)$, we can safely modify R on a small neighborhood of 1 so that $I - R(z)$ is everywhere invertible (this does not change $\chi_2(z)A(z,t)$). Wiener's Lemma and the continuity of inversion then give that $\chi_2(z)A(z,t) \to \chi_2(z)A(z,0)$. □

Proof of Theorem 2.1. We now conclude the proof of Theorem 2.1, i.e. we construct functions $\varepsilon(t)$ and $\delta(n)$ satisfying Equation (2).

Proposition 2.10 shows that it is possible to write

$$\left(\frac{I-R(z,t)}{1-(1-\frac{c}{\mu}M(|t|))z}\right)^{-1} = \left(\frac{I-R(z)}{1-z}\right)^{-1} + F_t(z)$$

where $F_t(z) \in A(\mathcal{A})$ tends to 0 when $t \to 0$. Moreover, Lemma 2.5 gives that

$$\left(\frac{I - R(z)}{1 - z}\right)^{-1} = \frac{1}{\mu} P + (1 - z) A(z)$$

where $A(z) = \sum A_n z^n$ with $A_n \to 0$. We get

$$\sum T_{n,t} z^n = (I - R(z,t))^{-1} = \frac{1}{1 - (1 - \frac{c}{\mu} M(|t|))z} \frac{1}{\mu} P$$
$$+ \frac{1 - z}{1 - (1 - \frac{c}{\mu} M(|t|))z} A(z) + \frac{1}{1 - (1 - \frac{c}{\mu} M(|t|))z} F_t(z).$$

In the first term, the coefficient of z^n is $\frac{1}{\mu}\left(1 - \frac{c}{\mu} M(|t|)\right)^n P$ and corresponds to (2). We have to check that the other terms are smaller than $\varepsilon(t) + \delta(n)$.

Let us denote by $\delta(n)$ the maximum for $t \in [-\varepsilon_0, \varepsilon_0]$ of the coefficient $c_n(t)$ of z^n in $\frac{1-z}{1-(1-\frac{c}{\mu}M(|t|))z} A(z)$. We will prove that $\delta(n) \to 0$ as $n \to \infty$. We have

$$\frac{1 - z}{1 - (1 - \frac{c}{\mu} M(|t|))z} = 1 - \frac{c}{\mu} M(|t|) \sum_{n=1}^{\infty} \left(1 - \frac{c}{\mu} M(|t|)\right)^{n-1} z^n.$$

Thus,

$$c_n(t) = A_n - \frac{c}{\mu} M(|t|) \sum_{k=0}^{n-1} \left(1 - \frac{c}{\mu} M(|t|)\right)^{n-k-1} A_k.$$

Note that, since $\mathrm{Re}(c) > 0$ and $M(t) \to 0$ when $t \to 0$, we have $\left|1 - \frac{c}{\mu} M(|t|)\right| \leqslant 1 - \frac{\mathrm{Re}\, c}{2\mu} M(|t|)$ when t is small enough. Let C be greater than the $\|A_n\|$, $n \in \mathbb{N}$. We fix $\varepsilon > 0$, and N such that $\forall k \geqslant N$, $\|A_k\| \leqslant \varepsilon$. Then, if $n \geqslant N$,

$$\|c_n(t)\| \leqslant \|A_n\| + \frac{|c|}{\mu} M(|t|) \sum_{k=0}^{N-1} \left(1 - \frac{\mathrm{Re}\, c}{2\mu} M(|t|)\right)^{n-k-1} \|A_k\|$$
$$+ \frac{|c|}{\mu} M(|t|) \sum_{k=N}^{n-1} \left(1 - \frac{\mathrm{Re}\, c}{2\mu} M(|t|)\right)^{n-k-1} \varepsilon$$
$$\leqslant \varepsilon + N \frac{|c|}{\mu} C M(|t|) \left(1 - \frac{\mathrm{Re}\, c}{2\mu} M(|t|)\right)^{n-N} + E\varepsilon.$$

As $M(|t|)\left(1 - \frac{\mathrm{Re}\, c}{2\mu} M(|t|)\right)^{n-N}$ tends uniformly to 0 on $[-\varepsilon_0, \varepsilon_0]$ (if ε_0 is small enough) when $n \to \infty$, we get that for n large enough, $\|c_n(t)\| \leqslant (2 + E)\varepsilon$, and $\delta(n) \leqslant (2 + E)\varepsilon$.

Let us finally bound the coefficients of $\frac{1}{1-(1-\frac{c}{\mu}M(|t|))z} F_t(z)$, which will give the $\varepsilon(t)$. The Fourier coefficients of $\frac{1}{1-(1-\frac{c}{\mu}M(|t|))z}$ are $\leqslant 1$, which implies that the coefficient of z^n in the product is at most $\sum_{k=0}^{n} \|F_t(z)_k\| \leqslant \|F_t\|_{A(\mathcal{A})}$. Since $\|F_t\|_{A(\mathcal{A})} \to 0$ when $t \to 0$, we can take $\varepsilon(t) = \|F_t\|_{A(\mathcal{A})}$ to conclude. □

3. Results in the uniform case

In Sections 3.1 and 3.2, we state some technical estimates on Markov maps, and we use them in Sections 3.3 and 3.4 to derive an expansion of the eigenvalue of a perturbed transfer operator, which will subsequently be used in the proofs of Theorems 1.1 and 1.2.

We work in the setting of Markov maps of Section 1.2.1, and we will use freely the definitions introduced here.

3.1. Some technical estimates

In this section, $(Z, \mathcal{B}, T, m, \delta)$ will be a Markov map preserving a measure m of finite mass. Write $t(x, y)$ for the separation time of the points x and y (under the iteration of T) and assume that the distortion of T is τ-Hölder for some $\tau < 1$. For $f : Z \to \mathbb{C}$, we will denote by $D_\tau f$ the least τ-Hölder constant of f for points in the same elements of the partition, i.e. $D_\tau f = \inf\{c \mid \forall x, y \in Z, t(x, y) \geqslant 1 \Rightarrow |f(x) - f(y)| \leqslant c\tau^{t(x,y)}\}$. Let also $\mathcal{L}_\tau$ be the space of bounded functions on Z with $D_\tau f < \infty$. It is a Banach space for the norm $\|f\|_{\mathcal{L}_\tau} = \|f\|_\infty + D_\tau f$.

We will need a property slightly weaker than the big image property from Section 1.2.1. We say that Z is a *bounded tower* if there exists $q \in \mathbb{N}$, a subset $\delta_0 \subset \delta$ (corresponding to the basis of the tower) and a function $R : \delta_0 \to \{0, \dots, q-1\}$, such that Z is isomorphic to the set $\{(x, i) \mid x \in a \subset \delta_0, i < R(a)\}$, with the partition into the sets $a \times \{i\}$ for $a \in \delta_0$ and $i < R(a)$. We require T to map $a \times \{i\}$ isomorphically onto $a \times \{i+1\}$ if $i + 1 < R(a)$, while if $i + 1 = R(a)$ then $T(a \times \{i\}) \subset \bigcup_{d \in \delta_0} d$, i.e. the points get out of the tower through the top and fall down to the basis. We will write $\Delta_k = \{(x, k) \mid x \in a, R(a) > k\}$, i.e. this is the set of points at height k in the tower.

We will say that T has the *bounded tower big image property* if Z is a bounded tower, and the returns to the basis have measure bounded away from 0, i.e. $\forall a \in \delta$, $Ta \subset \Delta_0 \Rightarrow m[Ta] > \eta$ for a constant $\eta > 0$. Note that, when $q = 1$, i.e. there are no floors in the tower, we recover the usual big image property.

In the rest of Section 3.1, T will have the bounded tower big image property and the distortion $\log g_m$ will be τ-Hölder.

Lemma 3.1 (Distortion Lemma). *Writing $g_m^{(n)}(x) = g_m(x) \cdots g_m(T^{n-1}x)$ (this is the inverse of the jacobian of T^n), there exists a constant C such that $t(x, y) \geqslant n \Rightarrow \left|1 - \frac{g_m^{(n)}(x)}{g_m^{(n)}(y)}\right| \leqslant C\tau^{t(x,y)-n}$.*

In particular, there exists a constant D such that, if $\underline{d} = [d_0, \dots, d_{n-1}]$ is a cylinder of length n and $x \in \underline{d}$, then $D^{-1} g_m^{(n)}(x) \leqslant \frac{m[\underline{d}]}{m[T^n \underline{d}]} \leqslant D g_m^{(n)}(x)$.

Proof. Let x, y be such that $t(x,y) \geqslant n$. Then

$$\begin{aligned}\left|\log(g_m^{(n)}(x)) - \log(g_m^{(n)}(y))\right| &\leqslant \sum_{i=0}^{n-1} \left|\log(g_m(T^i x)) - \log(g_m(T^i y))\right| \\ &\leqslant \sum_{i=0}^{n-1} D_\tau(\log(g_m))\tau^{t(x,y)-i} \\ &\leqslant \frac{D_\tau(\log(g_m))}{1-\tau}\tau^{t(x,y)-n+1}.\end{aligned}$$

Taking exponentials, we get the first statement of the lemma.

In particular, if $\underline{d}$ is a cylinder of length n and $x, y \in \underline{d}$, we have $\frac{g_m^{(n)}(x)}{g_m^{(n)}(y)} \leqslant D$ for some D. Using that $1/g_m^{(n)}(y)$ it the jacobian of T^n, we integrate on $y \in \underline{d}$ and get that $g_m^{(n)}(x)m[T^n\underline{d}] \leqslant Dm[\underline{d}]$. The other inequality is analogous. □

If ψ is a function on Z and $\hat{T}$ denotes the transfer operator associated to T, then $\hat{T}^n\psi(x) = \sum_{T^n(y)=x} g_m^{(n)}(y)\psi(y)$. There is at most one preimage of x in every non-empty cylinder $\underline{d} = [d_0, \dots, d_{n-1}]$, which we denote by $d_0 \dots d_{n-1}x$. Set $M_{\underline{d}}\psi(x) = g_m^{(n)}(d_0 \dots d_{n-1}x)\psi(d_0 \dots d_{n-1}x)$ if this point is defined, and 0 otherwise. Then $\hat{T}^n = \sum M_{\underline{d}}$ where the sum is over all non-empty cylinders $\underline{d}$ of length n.

Lemma 3.2. *There exists a constant B such that $\forall \underline{d} = [d_0, \dots, d_{n-1}]$, $\forall \psi : Z \to \mathbb{C}$,*

$$\left\| M_{\underline{d}}\psi \right\|_{\mathcal{L}_\tau} \leqslant B\frac{m[\underline{d}]}{m[T^n\underline{d}]}\left(\tau^n D_\tau\psi + \frac{1}{m[\underline{d}]}\int_{[\underline{d}]}|\psi|\,\mathrm{d}m\right).$$

Proof. Let x and y be in the same partition element a. If a is not in the image of $T(d_{n-1})$, $(M_{\underline{d}}\psi)_{|a} = 0$ and there is nothing to prove. Otherwise, write $x' = d_0 \dots d_{n-1}x$, and $y' = d_0 \dots d_{n-1}y$.

First note that, for $z' \in \underline{d}$, we have $|\psi(y') - \psi(z')| \leqslant D_\tau\psi\ \tau^n$, which gives, after integrating on z', that $|\psi(y')| \leqslant D_\tau\psi\ \tau^n + \frac{1}{m[\underline{d}]}\int_{\underline{d}}|\psi|\,\mathrm{d}m$. The inequality $g_m^{(n)}(y') \leqslant D\frac{m[\underline{d}]}{m[T^n\underline{d}]}$ from Lemma 3.1 gives then the claimed bound on $\left\|M_{\underline{d}}\psi\right\|_\infty$.

Using Lemma 3.1, we have

$$\begin{aligned}|M_{\underline{d}}\psi(x) - M_{\underline{d}}\psi(y)| &\leqslant |g_m^{(n)}(x')||\psi(x') - \psi(y')| \\ &\quad + |\psi(y')||g_m^{(n)}(y')|\left|1 - \frac{g_m^{(n)}(x')}{g_m^{(n)}(y')}\right| \\ &\leqslant D\frac{m[\underline{d}]}{m[T^n\underline{d}]}D_\tau\psi\ \tau^{t(x,y)+n} + |\psi(y')|D\frac{m[\underline{d}]}{m[T^n\underline{d}]}C\tau^{t(x,y)}\end{aligned}$$

which gives the conclusion, using the aforementioned bound on $|\psi(y')|$. □

Corollary 3.3. *If T has the bounded tower big image property and is ergodic, then the associated transfer operator $\hat{T}$ acts continuously on $\mathcal{L}_\tau$, and it has a simple isolated eigenvalue at* 1*, the eigenspace being the constant functions.*

Proof. The proof will use Lemma 3.2 and the fact that $\hat{T}^n = \sum M_{\underline{d}}$, where the sum is over all cylinders of length n.

We first show the continuity of $\hat{T}$. Decompose $\hat{T} = 1_{\Delta_0}\hat{T} + (1 - 1_{\Delta_0})\hat{T}$, where the first term sees the points on the basis and the second one in the floors. If x is in the basis, the elements a of the partition such that $x \in Ta$ return to the basis, which implies that they have a big image. We can thus sum the estimates given by Lemma 3.2 for cylinders of length 1, and forget the $m[T\underline{d}]$, to get that $\left\| 1_{\Delta_0}\hat{T}\psi \right\|_{\mathcal{L}_\tau} \leqslant B(\tau \left\|\psi\right\|_{\mathcal{L}_\tau} + \left\|\psi\right\|_1)$. On the other hand, if x is not in the basis and x' denotes the point just below, then $\hat{T}\psi(x) = \psi(x')$. This implies that $\left\|(1 - 1_{\Delta_0})\hat{T}\right\|_{\mathcal{L}_\tau} \leqslant 1$, and concludes the proof of the continuity of $\hat{T}$.

For $n \geqslant q$ (where q is the height of the tower), we can use the same argument on the basis and get that

$$\left\|(\hat{T}^n\psi)_{|\Delta_0}\right\|_{\mathcal{L}_\tau} \leqslant B\left(\tau^n \left\|\psi\right\|_{\mathcal{L}_\tau} + \left\|\psi\right\|_1\right).$$

If $x \in \Delta_k$, let x' denote the corresponding point in the basis of the tower. Then $\hat{T}^n\psi(x) = \hat{T}^{n-k}\psi(x')$, and the inequality on the basis gives

$$\left\|(\hat{T}^n\psi)_{|\Delta_k}\right\|_{\mathcal{L}_\tau} \leqslant B\left(\tau^{n-k} \left\|\psi\right\|_{\mathcal{L}_\tau} + \left\|\psi\right\|_1\right).$$

As the tower is bounded, we finally get that

$$\left\|\hat{T}^n\psi\right\|_{\mathcal{L}_\tau} \leqslant \frac{B}{\tau^q}\left(\tau^n \left\|\psi\right\|_{\mathcal{L}_\tau} + \left\|\psi\right\|_1\right).$$

This is a so-called Doeblin-Fortet inequality ([ITM50]), since the inclusion $\mathcal{L}_\tau \to L^1$ is compact. Thus, by Hennion's Theorem ([Hen93]) the essential spectral radius of $\hat{T}$ is bounded by τ, and 1 is an isolated eigenvalue of $\hat{T}$. The constant functions are eigenfunctions, and ergodicity of T shows that they are the only eigenfunctions ([Aar97, Theorem 1.4.8]). Moreover, there is no nilpotent part since $\|\hat{T}^n\|$ is bounded. □

The following lemma will be useful later, applied to unbounded functions in L^2. We recall that, if d is an element of the partition δ, then $D_\tau h(d)$ is the least τ-Hölder constant of h restricted to d.

Lemma 3.4. *Assume that T has the bounded tower big image property. Let $h \in L^1(m)$ satisfy $\sum_{d\in\delta} m[d]D_\tau h(d) < \infty$. Assume that h is bounded on the set of points whose image is not in the basis of the tower, and that h is uniformly τ-Hölder on this set. Then $\hat{T}h \in \mathcal{L}_\tau$.*

Proof. For $\underline{d} = [d_0]$ of length 1, the estimates given by Lemma 3.2 depend in fact only on $D_\tau \psi(d_0)$. Thus, summing these estimates on the basis, and using the big image property, we get

$$\left\| (\hat{T}h)_{|\Delta_0} \right\|_{\mathcal{L}_\tau} \leqslant B \left(\sum_{d \in \delta} \tau m[d] D_\tau h(d) + \|h\|_1 \right) < \infty.$$

If x is in the floors, and x' is just below x, then $\hat{T}h(x) = h(x')$. As h is bounded and Hölder on the set of such points x' by hypothesis, we get that $\hat{T}h$ is in $\mathcal{L}_\tau$. □

3.2. Some technical estimates: the perturbative case

In this section, $(Z, \mathcal{B}, T, m, \delta)$ will satisfy the same assumptions as in the previous section, i.e. T is a Markov map, preserving the finite measure m, with Hölder distortion. Moreover, T has the bounded tower big image property. Additionally, we will use a function $h : Z \to \mathbb{C}$ which, on the set of points whose image is not in the basis, will be bounded and uniformly τ-Hölder, and satisfying $\sum_{d \in \delta} m[d] D_\tau h(d) < \infty$.

Let $\underline{d} = [d_0]$ be a non-empty cylinder in Z of length 1. For small t, we define a perturbation $M_{\underline{d}}(t)$ of $M_{\underline{d}}$ by

$$M_{\underline{d}}(t)\psi(x) = \begin{cases} e^{ith(d_0 x)} g_m(d_0 x) \psi(d_0 x) & \text{if } d_0 x \text{ is defined} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Lemma 3.5. *Take* $0 < \eta \leqslant 1$. *Then there exists a constant* B *such that* $\forall \underline{d} = [d_0]$, $\forall t \in [-1, 1]$,

$$\left\| M_{\underline{d}}(t) - M_{\underline{d}} \right\|_{\mathcal{L}_{\tau^\eta}} \leqslant B \frac{m[\underline{d}]}{m[T\underline{d}]} \left(|t|^\eta D_\tau h(\underline{d})^\eta + \frac{1}{m[\underline{d}]} \int_{\underline{d}} |e^{ith} - 1| \, \mathrm{d}m \right).$$

Note that the norm in this lemma is the $\mathcal{L}_{\tau^\eta}$ norm.

Proof. Let $\psi \in \mathcal{L}_{\tau^\eta}$. If x, y are in the same element of the partition, we have to study $(M_{\underline{d}}(t) - M_{\underline{d}})\psi(x) - (M_{\underline{d}}(t) - M_{\underline{d}})\psi(y)$. If $d_0 x$ is not defined, then $d_0 y$ is not defined either, and there is nothing to prove. Otherwise, denote $x' = d_0 x$ and $y' = d_0 y$. Then

$$\begin{aligned} &|(M_{\underline{d}}(t) - M_{\underline{d}})\psi(x) - (M_{\underline{d}}(t) - M_{\underline{d}})\psi(y)| \\ &\quad \leqslant \left| e^{ith(x')} - 1 \right| |M_{\underline{d}}\psi(x) - M_{\underline{d}}\psi(y)| + |M_{\underline{d}}\psi(y)| \left| e^{ith(x')} - e^{ith(y')} \right|. \end{aligned}$$

Take C such that $|e^{is} - 1| \leqslant C|s|^\eta$ for all real s. Then

$$\left| e^{ith(x')} - e^{ith(y')} \right| \leqslant C|t|^\eta \left| h(x') - h(y') \right|^\eta \leqslant C|t|^\eta \left(D_\tau h(\underline{d}) \tau^{t(x,y)+1} \right)^\eta.$$

If $\chi = e^{ith} - 1$, we see that $|\chi(x')| \leqslant |\chi(y')| + C|t|^\eta D_\tau h(\underline{d})^\eta$ on the cylinder $\underline{d}$, whence $|e^{ith(x')} - 1| \leqslant \frac{1}{m[\underline{d}]} \int_{\underline{d}} |e^{ith} - 1| \,\mathrm{d}m + C|t|^\eta D_\tau h(\underline{d})^\eta$. The estimates on $M_{\underline{d}}\psi$ from Lemma 3.2 (in the space $\mathcal{L}_{\tau^\eta}$) imply that

$$
\begin{aligned}
&|(M_{\underline{d}}(t) - M_{\underline{d}})\psi(x) - (M_{\underline{d}}(t) - M_{\underline{d}})\psi(y)| \\
&\quad \leqslant \left(\frac{1}{m[\underline{d}]} \int_{\underline{d}} |e^{ith} - 1| \,\mathrm{d}m + C|t|^\eta D_\tau h(\underline{d})^\eta \right) \\
&\qquad \times B\tau^{\eta t(x,y)} \frac{m[\underline{d}]}{m[T\underline{d}]} \left(\tau^\eta D_{\tau^\eta}\psi + \frac{1}{m[\underline{d}]} \int_{[\underline{d}]} |\psi| \,\mathrm{d}m \right) \\
&\qquad + B \frac{m[\underline{d}]}{m[T\underline{d}]} \left(\tau^\eta D_{\tau^\eta}\psi + \frac{1}{m[\underline{d}]} \int_{[\underline{d}]} |\psi| \,\mathrm{d}m \right) C|t|^\eta D_\tau h(\underline{d})^\eta \tau^{\eta t(x,y)}.
\end{aligned}
$$

This gives the estimate on $D_{\tau^\eta}\big((M_{\underline{d}}(t) - M_{\underline{d}})\psi\big)$, using $\left(\tau^\eta D_{\tau^\eta}\psi + \frac{1}{m[\underline{d}]} \int_{[\underline{d}]} |\psi| \,\mathrm{d}m\right) \leqslant \|\psi\|_{\mathcal{L}_{\tau^\eta}}$. The bound on the supremum norm is analogous. □

Corollary 3.6. *Assume that T has the bounded tower big image property. Writing $R_t = \hat{T}(e^{ith}\cdot)$, then $t \mapsto R_t$ is continuous at* 0, *and*

$$
\|R_t - R_0\|_{\mathcal{L}_\tau} = O(|t| + E(|e^{ith} - 1|)).
$$

In particular, when $h \in L^1$, $\|R_t - R_0\|_{\mathcal{L}_\tau} = O(|t|)$.

The notation E denotes the expectation, i.e. the integral with respect to the measure $\mathrm{d}m$.

Proof. We will use that $R_t = \hat{T}(e^{ith}\cdot) = \sum M_{\underline{d}}(t)$, where the sum is over the cylinders of length 1.

To prove continuity of $t \mapsto R_t$ at $t = 0$, we write $R_t = 1_{\Delta_0} R_t + (1 - 1_{\Delta_0}) R_t$. On the basis, summing the estimates of Lemma 3.5 (with $\eta = 1$) and using the big image property, we get

$$
\left\| 1_{\Delta_0} R_t - 1_{\Delta_0} R_0 \right\|_{\mathcal{L}_\tau} \leqslant B \left(|t| \sum m[\underline{d}] D_\tau h(\underline{d}) + \int |e^{ith} - 1| \right).
$$

In the floors, if x' is the point just below x, then $R_t\psi(x) = e^{ith(x')}\psi(x')$, which yields that $\left\|(1 - 1_{\Delta_0})R_t - (1 - 1_{\Delta_0})R_0\right\|$ is bounded by the $\mathcal{L}_\tau$-norm of $e^{ith} - 1$ restricted to the set of points not returning to the basis at the next step. As h is uniformly bounded and Hölder on these points, we get that this $\mathcal{L}_\tau$-norm is $O(|t|)$. This proves that $\|R_t - R_0\| = O(|t| + \int |e^{ith} - 1|)$. By the dominated convergence theorem, the integral tends to 0 when $t \to 0$, and this proves the continuity of R_t.

Moreover, when h is integrable, then $\int |e^{ith} - 1| \leqslant |t| \int |h|$. □

3.3. The classical method: central limit theorem case

In this section, we recall classical results on the central limit theorem, obtained in the uniform case by perturbing the transfer operator. For bounded h, the method dates back to [Nag57], and is developed in [RE83] and [GH88]. The result we prove here is slightly more complicated to obtain, since the function h we will consider is generally unbounded, so that the perturbation of the transfer operator is not analytic, and we can not use an *a priori* expansion of the eigenvalue. In view of applications, we give the result for Markov maps with the bounded tower big image property.

Theorem 3.7. *Let $(Z, \mathcal{B}, T, m, \delta)$ be a Markov map, preserving a finite ergodic measure m. Assume that $\log g_m$ is τ-Hölder for some $\tau < 1$, and that T has the bounded tower big image property.*

Let $h : Z \to \mathbb{C}$ with $\sum_{d\in\delta} m[d]D_\tau h(d) < \infty$ and $h \in L^2$, with $\int h \,\mathrm{d}m = 0$. Assume that, restricted to the set of points that do not return to the basis of the tower at the next iteration, h is bounded and uniformly τ-Hölder.

Denote by R the transfer operator associated to T (acting on $\mathcal{L}_\tau$), and $R_t\psi = R(e^{ith}\psi)$. Then, for small t, R_t has an eigenvalue $\lambda(t)$, close to 1, satisfying $\lambda(t) = 1 - \frac{\sigma^2}{2m(Z)}t^2 + o(t^2)$. Write $a = (I-R)^{-1}(Rh)$ (where we consider $I - R$ acting on the space of functions of $\mathcal{L}_\tau$ with 0 average, on which it is invertible) and $u = (I-R)^{-1}(h)$. Then

$$\sigma^2 = \int h^2 \,\mathrm{d}m + 2\int ah \,\mathrm{d}m = \int R(u^2) - (Ru)^2 \,\mathrm{d}m.$$

Moreover, $\sigma^2 = 0$ if and only if there exists a function $\psi \in \mathcal{L}_\tau$ with $h = \psi\circ T - \psi$.

Note that Lemma 3.4 ensures that $Rh \in \mathcal{L}_\tau$, which implies that it makes sense to consider $a = (I-R)^{-1}(Rh)$, since $I - R$ is invertible on the functions of $\mathcal{L}_\tau$ of zero integral, according to Corollary 3.3. However, $u = (I-R)^{-1}(h)$ is *a priori* not defined, since $h \notin \mathcal{L}_\tau$. In fact, we simply set $u = h + a$, and indeed $(I-R)u = h - Rh + (I-R)a = h$.

Proof. We can replace m by $m/m(Z)$ and assume that m is a probability measure (the greater generality in the statement of the theorem will be useful in the applications).

Corollary 3.3 gives that $R = \hat{T}$ is continuous on $\mathcal{L}_\tau$, and 1 is a simple isolated eigenvalue. According to Corollary 3.6, $t \mapsto R_t$ is continuous at 0. As simple isolated eigenvalues depend continuously on the operator, R_t has a unique eigenvalue $\lambda(t)$ close to 1 for t small. Write P_t for the corresponding spectral projection, and ξ_t for the eigenfunction (with $\int \xi_t = 1$).

Since eigenvalues and eigenfunctions depend holomorphically on operators (Lipschitz would be enough), $\|\xi_t - 1\| = O(\|R_t - R\|) = O(|t|)$ using Corollary 3.6. In the same way, $\lambda(t) = 1 + O(t)$.

Note for future use that

$$E(e^{ith}) = 1 - \frac{t^2}{2}\int h^2 \,\mathrm{d}m + o(t^2). \tag{10}$$

Indeed, if we consider $\Phi(t) = E(e^{ith})$ as the characteristic function of the random variable h, it is classical that if $h \in L^2$, then Φ is C^2 with $\Phi(0) = 1$, $\Phi'(0) = iE(h)$ and $\Phi''(0) = -E(h^2)$ ([Fel66, Corollary to Lemma XV.4.2]).

As $\lambda(t)\xi_t = R_t\xi_t$, we get after integration that

$$\begin{aligned}\lambda(t) &= \int R_t\xi_t = \int (R_t - R)(\xi_t - 1) + \int (R_t - R)(1) + \int R\xi_t \\ &= \int (R_t - R)(\xi_t - 1) + E(e^{ith} - 1) + 1. \qquad (11)\end{aligned}$$

As $R_t - R = o(1)$ and $\xi_t - 1 = O(t)$, we get that $\lambda(t) = 1 + o(t)$ (using Equation (10) to estimate $E(e^{ith} - 1)$).

Thus,

$$\begin{aligned}\frac{\xi_t - \xi_0}{t} &= \frac{\lambda(t)\xi_t - \xi_0}{t} + o(1) = \frac{R_t\xi_t - R_0\xi_0}{t} + o(1) \\ &= (R_t - R_0)\frac{\xi_t - \xi_0}{t} + R_0\frac{\xi_t - \xi_0}{t} + \frac{R_t - R_0}{t}\xi_0 + o(1).\end{aligned}$$

But $(R_t - R_0)(\xi_0) = R(e^{ith} - 1)$. Moreover, $\frac{\xi_t - \xi_0}{t}$ is bounded and $R_t - R_0$ tends to 0, whence $(R_t - R_0)\frac{\xi_t - \xi_0}{t} = o(1)$. Thus,

$$(I - R)\frac{\xi_t - \xi_0}{t} = R\left(\frac{e^{ith} - 1}{t}\right) + o(1).$$

The sequence $(\xi_t - \xi_0)/t$ is bounded in $\mathcal{L}_\tau$. By compactness, we can extract a subsequence converging in L^2 to a function ia, which will still be in $\mathcal{L}_\tau$ (we have multiplied by i for convenience). In L^2, $\frac{e^{ith}-1}{t} \to ih$ by the dominated convergence theorem (as $\left|\frac{e^{ith}-1}{t}\right|^2 \leqslant |h|^2$ integrable), which implies that $(I - R)a = R(h)$. Moreover, according to Lemma 3.4, $Rh \in \mathcal{L}_\tau$. We also have $\int Rh = \int h = 0$. As $I - R$ is invertible on the set of functions in $\mathcal{L}_\tau$ with zero integral, we get $a = (I - R)^{-1}(Rh)$. Thus, $(\xi_t - \xi_0)/t$ has a unique cluster value, whence it converges (in L^2) to $ia = i(I - R)^{-1}(Rh)$.

By continuity of the product $L^2 \times L^2 \to L^1$, we obtain the convergence of $\int \frac{e^{ith}-1}{t}\frac{\xi_t - \xi_0}{t}$ to $\int (ih)\cdot(ia)$. Thus,

$$\begin{aligned}\int (R_t - R)(\xi_t - 1) &= \int R((e^{ith} - 1)(\xi_t - 1)) \\ &= \int (e^{ith} - 1)(\xi_t - 1) = -t^2\int ha + o(t^2).\end{aligned}$$

Equation (11) then gives that

$$\begin{aligned}\lambda(t) &= 1 + E(e^{ith} - 1) + \int (R_t - R)(\xi_t - 1) \\ &= 1 - \frac{t^2}{2}\int h^2 - t^2\int ha + o(t^2).\end{aligned}$$

We transform this result a little to obtain the other claimed expression for $\lambda(t)$. Recall that $h = u - Ru$ and $a = u - h$.

$$\begin{aligned}-\lambda''_{|t=0} &= \int h^2 + 2\int ah = \int h^2 + 2(u-h)h \\ &= \int (u - Ru)^2 + \int 2Ru \cdot (u - Ru) \\ &= \int (u + Ru)(u - Ru) = \int u^2 - \int (Ru)^2 = \int R(u^2) - (Ru)^2.\end{aligned}$$

We have $R(u^2)(x) = \sum_{Ty=x} g_m(y)u^2(y)$ and $(Ru)^2(x) = \left(\sum_{Ty=x} g_m(y)u(y)\right)^2$. Moreover, $\sum_{Ty=x} g_m(y) = 1$ since $R1 = 1$. The convexity of $w \mapsto w^2$ then gives that $(Ru)^2(x) \leqslant R(u^2)(x)$, which implies that $-\lambda'' \geqslant 0$. We can thus write $\lambda'' = -\sigma^2$.

Finally, if $\sigma^2 = 0$, then there is equality in the convexity inequality, whence all the $u(y)$ are equal (since g_m is everywhere nonzero as $\log g_m$ is defined everywhere). Writing $\psi = Ru$ (with $\psi \in \mathcal{L}_\tau$ since $Ru = Ra + Rh$) we get $u = \psi \circ T$. As $h = u - Ru$, this gives $h = \psi \circ T - \psi$. □

3.4. The classical method: stable laws case

The following theorem has essentially been proved by Aaronson and Denker in [AD01].

Theorem 3.8. *Let $(Z, \mathcal{B}, T, m, \delta)$ be a Markov map preserving the finite ergodic measure m. Assume that $\log g_m$ is τ-Hölder for some $\tau < 1$ and that T has the bounded tower big image property.*

Let $p \in (0,1) \cup (1,2)$ and h be a function on Z with $\sum_{d\in\delta} m[d] D_\tau h(d) < \infty$, such that, on the elements of the partition not returning to the basis of the tower at the next iteration, h is bounded and uniformly Hölder. Moreover, assume that $m[h > x] = (c_1 + o(1))x^{-p}L(x)$ and $m[h < -x] = (c_2 + o(1))x^{-p}L(x)$ for constants $c_1, c_2 \geqslant 0$ with $c_1 + c_2 > 0$, and a slowly varying function L. Finally, assume $\int h = 0$ if $p > 1$.

Let R be the transfer operator associated to T, acting on $\mathcal{L}_\tau$, and $R_t\psi = R(e^{ith}\psi)$. Then, for small enough t, R_t has a unique eigenvalue $\lambda(t)$ close to 1, and this eigenvalue satisfies

$$\begin{aligned}\lambda(t) = 1 &- \frac{c}{m(Z)}|t|^p L(1/|t|) + i\frac{c}{m(Z)}|t|^p L(1/|t|)\beta \operatorname{sgn}(t) \tan\left(\frac{p\pi}{2}\right) \\ &+ o(|t|^p L(1/|t|))\end{aligned}$$

for constants $\beta = \frac{c_1 - c_2}{c_1 + c_2}$ and $c = (c_1 + c_2)\Gamma(1-p)\cos\left(\frac{p\pi}{2}\right)$.

Proof. Replacing m by $m/m(Z)$, we can assume that m is a probability measure.

This theorem is essentially Theorem 5.1 in [AD01], with some differences in the hypotheses: we use the bounded tower big image property, whence their proof has to be slightly modified, as we explain now.

Corollary 3.3 gives that 1 is a simple eigenvalue of R acting continuously on $\mathcal{L}_\tau$. Corollary 3.6 ensures that $t \mapsto R_t$ is continuous at 0, which implies that R_t also has a simple isolated eigenvalue $\lambda(t)$ close to 1. The proof of Theorem 5.1 in [AD01] applies then literally if we can prove that, for $p > 1$, the eigenfunction ξ_t of R_t (normalized to have integral 1) satisfies $\|\xi_t - 1\| = O(|t|)$.

As ξ_t is an eigenfunction of R_t and eigenfunctions depend holomorphically on operators, $\|\xi_t - 1\| = O(\|R_t - R\|)$. Corollary 3.6 gives that this is a $O(|t|)$, which concludes the proof. □

An analogous theorem for $p = 2$ and $h \notin L^2$ may be proved using [AD98].

4. Proof of the main theorems

In this section, we will prove Theorems 1.1 and 1.2 using the results of Sections 2 and 3. $(X, \mathcal{B}, T, m, \alpha)$ will be a probability preserving topologically mixing Markov map, such that the induced map on $Y \subset X$ has the big image property and Hölder distortion. The function $\varphi = \varphi_Y$ will denote the first return time from Y to itself.

Note first that, since T is topologically mixing, T_Y is topologically transitive. Theorem 4.6.3 in [Aar97] implies that T_Y is ergodic, whence T is also ergodic.

4.1. Construction of an extension

In order to change variables between $\{x \in Y \mid \varphi(x) > i\}$ and its image by T^i, we have to construct an extension of the system since it is possible that two different points in Y are sent on the same image in X before they return to Y.

We use a tower extension, sometimes called a Kakutani tower (the towers in Section 3 will actually correspond to making a finite height cutoff of the Kakutani tower). It is built as follows: we set $X' = \{(x, i) \mid x \in Y, 0 \leqslant i < \varphi(x)\}$. Y is identified with $Y' := \{(x, 0) \mid x \in Y\} \subset X'$. The space X' is endowed with a measure m', equal to m on Y' and then lifted in the tower, i.e. if $A \subset Y$ and $\varphi > i$ on A, then $m'(A \times \{i\}) = m(A)$.

We define T' on X' by $T'(x, i) = (x, i + 1)$ if $i < \varphi(x) - 1$, and $(T^{\varphi(x)}x, 0)$ otherwise. We have a projection $\pi : X' \to X$ defined by $\pi(x, i) = T^i x$ (which is not necessarily injective). The following diagram is then commutative, and all applications are measure preserving:

$$\begin{tikzcd} X' \arrow[r, "T'"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi"] \\ X \arrow[r, "T"] & X \end{tikzcd}$$

All this is classical material, but I have not been able to locate a satisfying reference in the literature. Thus, we give for completeness a proof of the fact that the map π is measure-preserving.

Proposition 4.1. *$\forall K \subset X$, we have $m'(\pi^{-1}K) = m(K)$.*

Proof. Let $A_0 = K - Y$, $B_0 = K \cap Y$. We define inductively $A_k = T^{-1}(A_{k-1}) - Y$ and $B_k = T^{-1}(A_{k-1}) \cap Y$: the points of A_k have not been treated up to time k, while the points in B_k are treated at time k. Then we have $\pi^{-1}(K) = \{(x, i) \mid i \in \mathbb{N}, x \in B_i\}$. Thus, $m'(\pi^{-1}K) = \sum m(B_k)$.

As $m(K) = m(A_0) + m(B_0)$ and, by construction, $m(A_{k-1}) = m(T^{-1}A_{k-1}) = m(A_k) + m(B_k)$, we get that for any n, $m(K) = m(B_0) + \ldots + m(B_n) + m(A_n)$. To conclude, we just have to show that $m(A_n) \to 0$ when $n \to \infty$.

Let $C_n = \{x \mid \forall i \leqslant n, T^i x \notin Y\}$. Then $A_n \subset C_n$. Moreover, C_n is decreasing. Let $C = \bigcap C_n$. It suffices to show that $m(C) = 0$, since this will prove that $m(C_n) \to 0$. As $C \subset T^{-1}C$ and T is measure preserving, $C = T^{-1}C$ mod 0. By ergodicity of T, $m(C) = 0$ or 1. As $C \cap Y = \emptyset$ and $m(Y) > 0$, we obtain $m(C) = 0$. □

If χ is a function on X, we define χ' on X' by $\chi' = \chi \circ \pi$. Then $\int_{X'} \chi' \, \mathrm{d}m' = \int_X \chi \, \mathrm{d}m$.

To construct a partition on X', we consider first the partition δ on Y for which the induced map T_Y is Markov. Then, we lift it in the tower, i.e. the partition at height i is the same as the partition of the projection of this floor on the basis. Then T' is Markov for this new partition α'. Note that the partition for the induced map $T'_{Y'}$ is then simply the restriction of the partition α' to the basis of the tower.

Moreover, the inverse of the jacobian of $T'_{Y'}$ is the same as the inverse of the jacobian of T_Y, whence it is locally θ-Hölder (note that the separation times are the same in Y and in Y').

Finally, T' is still topologically mixing: if $a \in \delta$, then one of its iterates $T^n a$ contains an element b of α. If $c \in \delta$, it is contained in some $c' \in \alpha$, and there exists N such that $\forall p \geqslant N$, $c' \subset T^p(T^n a)$. This proves that $\forall p \geqslant n + N$, $c \subset T^p(a)$. This easily implies topological mixing for T'.

In fact, all the hypotheses on (X, T, f, f_Y) are carried over to (X', T', f', f'_Y). If we prove the central limit theorem or the convergence to a stable law on X', this will give the same result on X since $(S_n f)' = S_n(f')$. Thus, it is sufficient to prove the result for X'.

From this point on, we will thus work in a tower. It is Markov, and the returns to the basis have a big image (but are not necessarily surjective). We will first prove Theorem 1.1 on the Central Limit Theorem, and we will indicate in a last section the modifications to be done for Theorem 1.2 (the stable case).

4.2. Local result

The hypotheses of the main theorems are tailor-made so that the induction on the basis Y of the tower has good properties. However, we will have to induce on larger parts of the space. In this section, we describe properties of the induced maps on bounded parts of the tower. In this section, the hypotheses will be those of Theorem 1.1.

4.2.1. The space

Let $q \geqslant 1$. We write $Z = Z_q$ for the union of the first q floors of the Kakutani tower from Section 4.1. We will induce on Z. We will write δ for the partition on Z for which the induced map T_Z is Markov – this is in fact simply the restriction of the original partition α to Z, because of the particular combinatorics of the tower. For $k < q$, we will also denote by Δ_k the k^{th} floor of the tower. Note that the hypothesis $f \in I(X, Y)$ guarantees that the function f is bounded and Hölder on Z.

If s is the separation time with respect to the basis Y of the tower, and t is the separation time for T_Z, then an iteration of T_Y corresponds to at most q iterations of T_Z, which implies that $t(x, y) \leqslant qs(x, y)$. Thus, if ψ satisfies $|\psi(x) - \psi(y)| \leqslant C\theta^{s(x,y)}$, it will satisfy $|\psi(x) - \psi(y)| \leqslant C\tau^{t(x,y)}$ for $\tau = \theta^{1/q}$.

We write $g_{m_Z} = \frac{\mathrm{d}m_Z}{\mathrm{d}m_Z \circ T_Z}$. For $(x, i) \in Z$, if $i < \varphi(x) - 1$ then $g_{m_Z}(x) = 1$, and if $i = \varphi(x) - 1$ then $g_{m_Z}(x, i) = g_{m_Y}(x, 0)$ (the inverse of the jacobian of T_Y with respect to m_Y). As $\log g_{m_Y}$ is locally θ-Hölder (for the separation time s), we obtain that $\log g_{m_Z}$ is locally τ-Hölder (for the separation time t).

Since T is ergodic, so is T_Z ([Aar97, Proposition 1.5.2]). Moreover, T_Z has the bounded tower big image property. Thus, the transfer operator $\hat{T}_Z$ acts continuously on $\mathcal{L}_\tau(Z)$ according to Corollary 3.3, and has a simple isolated eigenvalue at 1.

4.2.2. First return transfer operators

Following [Sar02], we define operators R_n and T_n: R_n is a first return transfer operator, i.e. it sees only the first returns to Z at time n, while T_n sees all returns at time n. They are defined by $R_n\psi = 1_Z\hat{T}^n(1_{\{\varphi_Z=n\}}\psi)$ (where φ_Z is the first return time from Z to itself) and $T_n\psi = 1_Z\hat{T}^n(1_Z\psi)$. They can also be written in the following way: if $x \in Z$, then $R_n\psi(x) = \sum g_m^{(n)}(y)\psi(y)$ where the sum is over all points y in Z such that $Ty, \dots, T^{n-1}y \notin Z$ and $T^n y = x$, while $T_n\psi(x) = \sum g_m^{(n)}(y)\psi(y)$ where the sum is over all points $y \in Z$ with $T^n y = x$.

If $y \in Z$ with $T^n y = x$, we can consider all its iterates between 0 and n that fall into Z. To go from one of these iterates to the following corresponds to iterating one of the R_ks. Thus, we get $T_n = \sum_{k_1+\dots+k_l=n} R_{k_1} \dots R_{k_l}$: this is the *renewal equation*, which will make it possible to understand the T_n if the R_n are well understood.

Lemma 4.2. *The operators R_n and T_n act continuously on $\mathcal{L}_\tau(Z)$, and $\|R_n\|_{\mathcal{L}_\tau(Z)} = O(m[\varphi_Z = n])$ (where φ_Z is the return time from Z to itself).*

Proof. As $\hat{T}_Z$ acts continuously on $\mathcal{L}_\tau$ and $R_1\psi = \hat{T}_Z(1_{\{\varphi_Z=1\}}\psi)$, the operator R_1 is continuous.

For $n \geqslant 2$, we estimate $\|R_n\|$ using the notations of Section 3.1 (applied to Z and T_Z):

$$R_n = \sum_{\underline{d}=[d_0]_Z, d_0=[a_0, \dots, a_{n-1}, Z]} M_{\underline{d}}.$$

For $n \geqslant 2$, the cylinders appearing in R_n correspond to points coming back to Z in n steps: they get out of the tower Z through the top, spend $n-1$ iterates in the higher floors of the Kakutani tower, and then get back to the basis of the tower, with a big image. Thus, we can forget the terms $m[T_Z d_0]$ for these cylinders in the estimates of Lemma 3.2, and we get $\|R_n\| \leqslant B(1+\tau)\sum m[d_0] = B(1+\tau)m[\varphi_Z = n]$.

The R_n are continuous and $T_n = \sum_{k_1+\ldots+k_l=n} R_{k_1}\ldots R_{k_l}$, whence T_n is also continuous. □

Lemma 4.3. *For $z \in \overline{\mathbb{D}}$, write $R(z) = \sum R_n z^n$. Then $R(1)$ acting on $\mathcal{L}_\tau(Z)$ has a simple isolated eigenvalue equal to 1, and the corresponding spectral projection is given by $\psi \mapsto \frac{1}{m[Z]}\int_Z \psi\, \mathrm{d}m$.*

Moreover, for $z \neq 1$, $I - R(z)$ is invertible.

Note that, since $\|R_n\| = O(m[\varphi_Z = n])$ is summable by Kac's Formula, the series $R(z)$ is converging for all $z \in \overline{\mathbb{D}}$.

Proof. Let $z \in \overline{\mathbb{D}}$. We will show a Doeblin-Fortet inequality for $R(z)^s$ when $s \geqslant q$. Let $\psi \in \mathcal{L}_\tau(Z)$. Then

$$R(z)^s = \sum_{\underline{d}=[d_0,\ldots,d_{s-1}]_Z} z^{l(\underline{d})} M_{\underline{d}}$$

where $l(\underline{d})$ is the sum of the lengths of the d_is seen as cylinders in X. In particular, $l(\underline{d}) \geqslant s$, whence $|z|^{l(\underline{d})} \leqslant |z|^s$. On the basis, summing the inequalities given by Lemma 3.2 (and using the big image property for the returns to the basis), we obtain that $\left\|(R(z)^s\psi)_{|\Delta_0}\right\|_{\mathcal{L}_\tau} \leqslant B|z|^s \left(\tau^s \|\psi\|_{\mathcal{L}_\tau} + \int|\psi|\right)$.

On Δ_k the set of points at height k, $R(z)^s\psi(x) = z^k R(z)^{s-k}\psi(x')$ where x' is the point corresponding to x but in the basis. Thus, $\left\|(R(z)^s\psi)_{|\Delta_k}\right\| \leqslant B|z|^s \left(\tau^{s-k}\|\psi\|_{\mathcal{L}_\tau} + \int|\psi|\right)$.

Summing on all the floors, we have proved that

$$\forall s \geqslant q, \forall \psi \in \mathcal{L}_\tau, \left\|R(z)^s\psi\right\|_{\mathcal{L}_\tau} \leqslant \frac{B}{\tau^q}|z|^s\left(\tau^s\|\psi\|_{\mathcal{L}_\tau} + \int|\psi|\right).$$

As the injection from $\mathcal{L}_\tau$ to L^1 is compact, this is a Doeblin-Fortet inequality, which implies that the essential spectral radius of $R(z)$ is $\leqslant \tau|z| < |z|$. Moreover, this inequality also gives that the spectral radius of $R(z)$ is $\leqslant |z|$.

If $|z| < 1$, then $R(z)$ has spectral radius $\leqslant |z| < 1$, whence $I - R(z)$ is invertible.

If $z = 1$, as $R(1)$ counts the first returns to Z, it is not hard to check that $R(1) = \hat{T}_Z$ is the transfer operator associated to T_Z. Hence, Corollary 3.3 ensures that it has a simple isolated eigenvalue at 1.

If $|z| = 1$ but $z \neq 1$, we have to show that $I - R(z)$ is invertible. We write $z = e^{it}$ for some $0 < t < 2\pi$. Since the essential spectral radius of $R(z)$ is < 1, it is enough to show that 1 is not an eigenvalue of $R(z)$. Suppose on the contrary that $R(z)a = a$ for some nonzero $a \in \mathcal{L}_\tau(Z)$.

For $\psi, \xi \in L^2(m_Z)$, write $\langle\psi, \xi\rangle = \int \overline{\psi}\xi\, \mathrm{d}m_Z$. Define the operator $W : L^\infty(m_Z) \to L^\infty(m_Z)$ by $W\psi = e^{-it\varphi_Z}\psi \circ T_Z$. As $R(z)\xi = R(1)(e^{it\varphi_Z}\xi)$, the operator W satisfies

$$\begin{aligned}\langle\psi, R(z)\xi\rangle &= \int \overline{\psi}\, R(z)\xi = \int \overline{\psi}\, R(1)(e^{it\varphi_Z}\xi) = \int \overline{\psi}\circ T_Z\, e^{it\varphi_Z}\xi \\ &= \int \overline{W\psi}\cdot\xi = \langle W\psi, \xi\rangle.\end{aligned}$$

We show that a is an eigenfunction of W for the eigenvalue 1:

$$\begin{aligned}\|Wa-a\|_2^2 &= \|Wa\|_2^2 - 2\,\mathrm{Re}\langle Wa, a\rangle + \|a\|_2^2 \\ &= \|Wa\|_2^2 - 2\,\mathrm{Re}\langle a, R(z)a\rangle + \|a\|_2^2 \\ &= \|Wa\|_2^2 - 2\,\mathrm{Re}\langle a, a\rangle + \|a\|_2^2 = \|Wa\|_2^2 - \|a\|_2^2.\end{aligned}$$

As T_Z preserves the measure m_Z, we have $\|Wa\|_2^2 = \int |a|^2\circ T_Z = \int |a|^2 = \|a\|_2^2$, which gives $\|Wa-a\|_2^2 = 0$. Hence, the function $Wa-a$ is zero m_Z-almost everywhere. As $a \in \mathcal{L}_\tau$ and m_Z is nonzero on every cylinder, the function a is continuous, whence $Wa-a=0$ everywhere, i.e. $e^{-it\varphi_Z}a\circ T_Z = a$.

Let b be the restriction of a to the basis Y of the tower. Then the previous equality implies that $e^{-it\varphi_Y}b\circ T_Y = b$. Taking the modulus, ergodicity of T_Y gives that $|b|$ is constant almost everywhere, hence everywhere by continuity. As $b \not\equiv 0$, this constant is nonzero, and we get $e^{-it\varphi_Y} = b/b\circ T_Y$. We can apply Theorem 3.1. in [AD01] since T_Y has the big image property (this is not the case for T_Z, which is why we have to restrict a to the basis of the tower). Writing β for the induced partition on the basis, this theorem gives that b is β^*-measurable, where β^* is the smallest partition such that $\forall d \in \beta$, $T_Y d$ is contained in an atom of β^*. In particular, b is constant on each set of β.

Let $d \in \beta$. On $[d]$, b is equal to a constant c. As T is topologically mixing, there exists N such that, $\forall n \geqslant N$, $[d] \subset T^n[d]$. Let $n \geqslant N$, and $x \in [d]$ be such that $T^n x \in [d]$. Let $T^{k_1}x, T^{k_2}x, \dots, T^{k_p}x$ be the successive returns of x to Y, with $k_p = n$. Then $T^n x = T_Y^p x$ and $n = \sum_{k=0}^{p-1}\varphi_Y(T_Y^k x)$. Thus,

$$e^{-itn} = e^{-it\sum_{k=0}^{p-1}\varphi_Y(T_Y^k x)} = \frac{b(x)}{b(T_Y x)}\frac{b(T_Y x)}{b(T_Y^2 x)}\cdots\frac{b(T_Y^{p-1}x)}{b(T_Y^p x)} = \frac{b(x)}{b(T^n x)} = \frac{c}{c} = 1.$$

This is true for any $n \geqslant N$. Taking for example $n = N$ and $N+1$ and quotienting, we obtain $e^{it} = 1$, which is a contradiction and concludes the proof. □

Lemma 4.4. *Writing $R'(1) = \sum_{n=1}^{\infty} nR_n$ and P for the spectral projection associated to the eigenvalue* 1 *of $R(1)$, we have $PR'(1)P = \mu P$ for $\mu = \frac{1}{m[Z]} > 0$.*

Proof. Since $Pf = \frac{1}{m[Z]}\int_Z f\,\mathrm{d}m$, we have $PR_nP = \frac{m[\varphi_Z = n]}{m[Z]}P$, whence

$$PR'(1)P = \left(\sum_{n=1}^{\infty} nm[\varphi_Z = n]\right)\frac{1}{m[Z]}P = \frac{1}{m[Z]}P$$

by Kac's Formula. □

4.2.3. Perturbation of the transfer operators

Recall that we are in the setting of Theorem 1.1: f is a function on X belonging to $I(X, Y)$ such that the induced function $f_Y = \sum_0^{\varphi_Y - 1} f \circ T^k$ is square integrable on Y.

Recall also that φ_Z is the first return time to our truncated tower Z, and define a function f_Z on Z by $f_Z(x) = \sum_{k=0}^{\varphi_Z(x)-1} f(T^k x)$. If x does not go out of the bounded tower Z through the top, we have $f_Z(x) = f(x)$, while otherwise $f_Z(x)$ is the sum of the $f(y)$ for y above x in the tower. Since $f \in L^1(m)$, we also have $f_Z \in L^1(m_Z)$ and $\int_Z f_Z = \int_X f = 0$. This function f_Z is interesting since we can study the Birkhoff sums of f for T by looking at the Birkhoff sums of f_Z for T_Z: if $x \in Z$ and $T^n x \in Z$, and if k is the number of returns of x to Z between times 0 and n, then $\sum_{i=0}^{n-1} f(T^i x) = \sum_{j=0}^{k-1} f_Z(T_Z^j x)$.

We show that $f_Z \in L^2(Z)$ (recall that we are proving the Central Limit Theorem, i.e. we assume that $f_Y \in L^2(Y)$). Let $\Delta = \Delta_{q-1}$ be the highest floor of Z, and Λ its projection on the basis. Then f_Z is bounded on $Z - \Delta$. Moreover, if $x \in \Delta$ and x' is the corresponding point in Λ, then $f_Y(x') = f_Z(x) + \sum_0^{q-2} f(T^j x')$, and the sum is bounded by $q \left\| f_{|Z} \right\|_\infty$, whence $f_Y(x') = f_Z(x) \pm C$. As $f_{Y|\Lambda}$ is in L^2, we obtain also $f_{Z|\Delta} \in L^2$.

We now perturb the first return transfer operators, setting $R_n(t) = R_n(e^{itf_Z} \cdot)$ and $T_{n,t} = T_n(e^{itS_n f} \cdot)$. Since the Birkhoff sums of f for T and of f_Z for T_Z are equal, the renewal equation still holds for these operators, i.e. $T_{n,t} = \sum_{k_1+\ldots+k_l=n} R_{k_1}(t) \ldots R_{k_l}(t)$. This equation can also be written as $\sum T_{n,t} z^n = \left(I - \sum R_n(t) z^n\right)^{-1}$.

For technical reasons, we will let the operators act on $\mathcal{L}_{\tau^\eta}(Z)$ instead of $\mathcal{L}_\tau(Z)$, where η is as in the definition of $I(X, Y)$. Since $\tau^\eta \geqslant \tau$, the distortion is also τ^η-Hölder. Thus, the results of the previous paragraphs still apply. In particular, Lemma 4.2 implies that $\|R_n\|_{\mathcal{L}_{\tau^\eta}} = O(m[\varphi_Z = n])$.

Lemma 4.5. *There exist constants r_n and $\varepsilon_0 > 0$ such that, $\forall n \geqslant 1, \forall t \in [-\varepsilon_0, \varepsilon_0]$, $\|R_n(t)\|_{\mathcal{L}_{\tau^\eta}} \leqslant r_n$, and satisfying $\sum n r_n < \infty$.*

Moreover, for every $n \geqslant 1$, the map $t \mapsto R_n(t)$ is continuous at $t = 0$.

Proof. Let $n \geqslant 2$. We have $R_n(t) = \sum M_{\underline{d}}(t)$ where the sum is over all $\underline{d} = [d_0]_Z$ with $d_0 = [a_0, \ldots, a_{n-1}, Z]$ (we use here the operators $M_{\underline{d}}(t)$ introduced in (9), for the function $h = f_Z$). The returns to Z do not take place at the first iteration, so they have to be returns to the basis, whence they have a big image. Summing the estimates given by Lemma 3.5, we obtain

$$\|R_n(t)\|_{\mathcal{L}_{\tau^\eta}} \leqslant \|R_n\|_{\mathcal{L}_{\tau^\eta}} + B\left(\sum_{d \in \delta, \varphi_Z(d)=n} m[d] D_\tau f_Z(d)^\eta + \int_{\{\varphi_Z = n\}} |e^{itf_Z} - 1| \,\mathrm{d}m \right)$$

$$\leqslant Cm[\varphi_Z = n] + B \sum_{d\in\delta,\varphi_Z(d)=n} m[d]D_\tau f_Z(d)^\eta$$
$$+2Bm[\varphi_Z = n] =: r_n.$$

We know that $\sum nm[\varphi_Z = n] = m(X) < \infty$, by Kac's formula. Thus, to show that $\sum nr_n < \infty$, it is sufficient to prove that $\sum_{d\in\delta} \varphi_Z[d]m[d]D_\tau f_Z(d)^\eta < \infty$. Note that this inequality holds for f_Y by assumption, since $f \in I(X, Y)$.

Let A be the set of elements of the partition of Z that do not return to the basis of Z at the next iteration, and B the set of those that come back to the basis. If $d \in A$, then $\varphi_Z = 1$ on d, and $f_Z = f$ is uniformly τ-Hölder, whence

$$\sum_{d\in A} \varphi_Z[d]m[d]D_\tau f_Z(d)^\eta \leqslant C\sum_{d\in A} m[d] \leqslant Cm[Z] < \infty.$$

Let $d \in B$ and $x, y \in d$. Let x' and y' be the points in the basis corresponding to x and y, and d' be the corresponding partition element in the basis: $x = T^i x'$ and $y = T^i y'$ for some $i < q$. Then $f_Y(x') = \sum_0^{i-1} f(T^j x') + f_Z(x)$, and we have the same expression for y, whence

$$|f_Z(x) - f_Z(y)| \leqslant |f_Y(x') - f_Y(y')| + \sum_{j=0}^{i-1} |f(T^j x') - f(T^j y')|$$
$$\leqslant CD_\theta f_Y(d')\theta^{s(x',y')} + Dq\theta^{s(x',y')} \leqslant (CD_\theta f_Y(d') + Dq)\tau^{t(x,y)}$$

using $s(x', y') = s(x, y) \geqslant t(x, y)/q$ and $\tau = \theta^{1/q}$. Since $\varphi_Z(d) \leqslant \varphi_Y(d')$, we get

$$\sum_{d\in B} \varphi_Z[d]m[d]D_\tau f_Z(d)^\eta \leqslant \sum_{d'} \varphi_Y[d']m[d'](CD_\theta f_Y(d') + Dq)^\eta < \infty$$

by Kac's formula and the assumption on f_Y. This proves that $\sum nr_n < \infty$.

For $n = 1$, $R_1(t) = R_t(1_{\{\varphi_Z=1\}}\cdot)$ and R_t depends continuously on t at $t = 0$ according to Corollary 3.6 (in the space $\mathcal{L}_{\tau^\eta}$).

Finally, to prove the continuity of $t \mapsto R_n(t)$ at $t = 0$ for $n \geqslant 2$, we sum once again the estimates of Lemma 3.5 (in the space $\mathcal{L}_{\tau^\eta}$) and we get $\|R_n(t) - R_n\|_{\mathcal{L}_{\tau^\eta}} \leqslant B|t|$ since the sum $\sum m[d]D_{\tau^\eta} f_Z(d)$ is finite (this is as above a consequence of the fact that $\sum m[d']D_\theta f_Y(d') < \infty$, which implies that $\sum m[d']D_{\theta^\eta} f_Y(d') < \infty$ since $D_{\theta^\eta} f_Y(d') \leqslant D_\theta f_Y(d')$). □

4.2.4. The local result

We will apply the abstract Theorem 2.1 to the $R_n(t)$ acting on $\mathcal{L}_{\tau^\eta}(Z)$. Lemmas 4.2, 4.3, 4.4 and 4.5 ensure that the spectral hypotheses of this theorem are satisfied.

The only hypothesis that remains to be checked is the behavior of the eigenvalue $\lambda(1, t)$ of $R(1, t) = \sum R_n(t) = \hat{T}_Z(e^{itf_Z}\cdot)$. As T_Z has the bounded tower big image property and $f_Z \in L^2$, Theorem 3.7 gives $\lambda(t) = 1 - \frac{\sigma^2}{2m(Z)}t^2 + o(t^2)$.

If $\sigma^2 \neq 0$, we can use the abstract result Theorem 2.1 for $M(t) = t^2$ and $c = \frac{\sigma^2}{2m(Z)}$, and we get:

Proposition 4.6. *Assume $\sigma^2 > 0$. Then, $\forall t \in \mathbb{R}$, we have*

$$\int_{T^{-n}Z\cap Z} e^{itS_n f/\sqrt{n}}\,\mathrm{d}m \to m(Z)^2 e^{-\sigma^2 t^2/2}.$$

Proof. Note that, if t is fixed, $t/\sqrt{n} \in [-\varepsilon_0, \varepsilon_0]$ for n large enough, whence Theorem 2.1 applies. The constant μ is equal to $1/m(Z)$ according to Lemma 4.4.

On the one hand,

$$\begin{aligned}\int T_{n,t/\sqrt{n}}1\,\mathrm{d}m &= \int 1_Z \hat{T}^n(e^{itS_n f/\sqrt{n}}1_Z)\,\mathrm{d}m = \int 1_Z\circ T^n \cdot e^{itS_n f/\sqrt{n}}1_Z\,\mathrm{d}m\\ &= \int_{Z\cap T^{-n}Z} e^{itS_n f/\sqrt{n}}\,\mathrm{d}m\end{aligned}$$

and on the other hand,

$$\begin{aligned}\int T_{n,t/\sqrt{n}}1\,\mathrm{d}m &= \int \left(1-\frac{\sigma^2}{2\mu m(Z)}\frac{t^2}{n}\right)^n \frac{1}{\mu}P(1)\,\mathrm{d}m + o(1)\\ &= \frac{m[Z]}{\mu}e^{-\sigma^2 t^2/2} + o(1)\end{aligned}$$

using $P(1) = 1_Z$, and $1/\mu = m(Z)$ whence $\mu m(Z) = 1$. □

If $\sigma^2 = 0$, Theorem 2.1 can not be used, but we have nevertheless the following result:

Proposition 4.7. *Assume that $\sigma^2 = 0$. Then f is a coboundary on X, i.e. it is possible to write $f = \chi\circ T - \chi$ for some measurable function χ on X.*

Proof. Theorem 3.7 gives that, if $\sigma^2 = 0$, then f_Z is a coboundary, i.e. it can be written as $f_Z = \psi\circ T_Z - \psi$ for some function ψ on Z. We have to go from f_Z to f.

We extend ψ to the whole space X by $\psi(x) = 0$ if $x \notin Z$. Then, writing $\tilde{f} = f - \psi\circ T + \psi$, we have for $x \in Z$ that

$$\sum_0^{\varphi_Z(x)-1} \tilde{f}(T^i x) = f_Z(x) - \psi(T_Z x) + \psi(x) = 0.$$

We then set $\delta(x) = -\sum_0^{\varphi_Z(x)-1}\tilde{f}(T^i x)$, where φ_Z denotes the first return (resp. entry) time to Z if $x \in Z$ (resp. $x \notin Z$). Then, if $Tx \notin Z$, we have $\tilde{f}(x) = \delta(Tx) - \delta(x)$, while if $Tx \in Z$, then $\delta(Tx) = 0$ and $\delta(x) = -\tilde{f}(x)$, whence $\tilde{f}(x) = \delta(Tx) - \delta(x)$. Thus, $\tilde{f} = \delta\circ T - \delta$, whence $f = (\psi+\delta)\circ T - (\psi+\delta)$. □

4.3. Proof of the Central Limit Theorem 1.1

A priori, the constant σ associated to Z_q depends on q, and should be written σ_q. We first prove that this is not the case:

Proposition 4.8. *σ_q is independent of q.*

Proof. We fix $q > 1$, and we want to prove that $\sigma_q = \sigma_1$. We will write $Z = Z_q$ and $f_Z(x) = \sum_0^{\varphi_Z(x)-1} f(x)$ (recall that f_Y is the same map, but on the basis Y and with the return time φ_Y). We will write $\tilde{\varphi}$ for the return time from Y to itself for the map T_Z induced by T on Z. Thus, with these notations, we have for $x \in Y$ that $f_Y(x) = \sum_{j<\tilde{\varphi}(x)} f_Z(T_Z^j x)$.

Theorem 3.7 gives that $\sigma_1^2 = \int f_Y^2 + 2\int a f_Y$ where $a = (I - \hat{T}_Y)^{-1}(\hat{T}_Y f_Y)$ and $\hat{T}_Y$ is the transfer operator associated to T_Y. In the same way, $\sigma_q^2 = \int f_Z^2 + 2\int b f_Z$ where $b = (I - \hat{T}_Z)^{-1}(\hat{T}_Z f_Z)$.

We define a function c on Z by $c(x,i) = a(x) + \sum_{j=0}^{i-1} f_Z(T_Z^j x)$ for x in the basis of the tower and $i < \tilde{\varphi}(x)$. Then, if ψ is a bounded function on Z, we will check that

$$\int_Z c \cdot (\psi - \psi \circ T_Z) = \int_Z f_Z \cdot \psi \circ T_Z. \tag{12}$$

Indeed, note that the definition of a implies that for a function χ on Y, $\int_Y a \cdot (\chi - \chi \circ T_Y) = \int_Y f_Y \cdot \chi \circ T_Y$. Thus,

$$\begin{aligned}
&\int_Z c \cdot (\psi - \psi \circ T_Z)\\
&\quad = \int_Y \sum_{i<\tilde{\varphi}(x)} \left(a(x) + \sum_{j<i} f_Z(T_Z^j x) \right) (\psi(T_Z^i x) - \psi(T_Z^{i+1} x))\\
&\quad = \int_Y a(x)(\psi(x) - \psi(T_Y x)) + \sum_{j<i<\tilde{\varphi}(x)} f_Z(T_Z^j x)(\psi(T_Z^i x) - \psi(T_Z^{i+1} x))\\
&\quad = \int_Y f_Y(x)\psi(T_Y x) + \sum_{j<\tilde{\varphi}(x)-1} f_Z(T^j x)(\psi(T_Z^{j+1} x) - \psi(T_Y x))\\
&\quad = \int_Y \sum_{j<\tilde{\varphi}(x)} f_Z(T_Z^j x)\psi(T_Y x) + \sum_{j<\tilde{\varphi}(x)-1} f_Z(T^j x)(\psi(T_Z^{j+1} x) - \psi(T_Y x))\\
&\quad = \int_Y \sum_{j<\tilde{\varphi}(x)-1} f_Z(T_Z^j x)\psi(T_Z^{j+1} x) + f_Z(T_Z^{\tilde{\varphi}(x)-1} x)\psi(T_Y x)\\
&\quad = \int_Y \sum_{j<\tilde{\varphi}(x)} f_Z(T_Z^j x)\psi(T_Z^{j+1} x) = \int_Z f_Z(x)\psi(T_Z x).
\end{aligned}$$

Equation (12) implies that $(I - \hat{T}_Z)c = \hat{T}_Z f_Z$. By definition of the function b, we also have $(I - \hat{T}_Z)b = \hat{T}_Z f_Z$, whence $c - b$ is in the kernel of $I - \hat{T}_Z$, i.e. it is a constant K, and $c = b + K$.

We now compute σ_1^2, using $\int f_Z = 0$:

$$\begin{aligned}
\sigma_1^2 &= \int_Y f_Y^2 + 2 f_Y a = \int_Y \left(\sum_{i<\tilde{\varphi}(x)} f_Z(T_Z^i x) \right)^2 + 2 \sum_{i<\tilde{\varphi}(x)} f_Z(T_Z^i x) a(x) \\
&= \int_Y \sum_{i<\tilde{\varphi}(x)} f_Z(T_Z^i x)^2 + 2 \sum_{j<i<\tilde{\varphi}(x)} f_Z(T_Z^i x) f_Z(T_Z^j x) \\
&\quad + 2 \sum_{i<\tilde{\varphi}(x)} f_Z(T_Z^i x) \left(c(T_Z^i x) - \sum_{j<i} f_Z(T_Z^j x) \right) \\
&= \int_Y \sum_{i<\tilde{\varphi}(x)} f_Z(T_Z^i x)^2 + 2 \sum_{i<\tilde{\varphi}(x)} f_Z(T_Z^i x) c(T_Z^i x) = \int_Z f_Z(x)^2 + 2 f_Z(x) c(x) \\
&= \int_Z f_Z(x)^2 + 2 f_Z(x)(b(x) + K) = \int_Z f_Z(x)^2 + 2 f_Z(x) b(x) = \sigma_q^2.
\end{aligned}$$

□

Conclusion of the proof of Theorem 1.1. Let σ^2 be given by Proposition 4.8.

If $\sigma^2 = 0$, then Proposition 4.7 ensures that f can be written as $\chi \circ T - \chi$ for some measurable function χ. Hence, $S_n f/\sqrt{n}$ tends to 0 in probability, and the theorem is proved.

Assume now that $\sigma^2 > 0$. Let $t \in \mathbb{R}$, we want to prove that $\int_X e^{itS_n f/\sqrt{n}} \to e^{-\sigma^2 t^2/2}$. Fix $\varepsilon > 0$. Let $Z = Z_q$ with q large enough so that $m(Z) \geqslant 1 - \varepsilon$. Proposition 4.6 gives N such that $\forall n \geqslant N$

$$\left| \int_{T^{-n} Z \cap Z} e^{itS_n f/\sqrt{n}} - m(Z)^2 e^{-\sigma^2 t^2/2} \right| \leqslant \varepsilon.$$

Moreover,

$$\begin{aligned}
\left| \int_X e^{itS_n f/\sqrt{n}} - \int_{T^{-n} Z \cap Z} e^{itS_n f/\sqrt{n}} \right| &\leqslant m(X - (Z \cap T^{-n} Z)) \\
&\leqslant m(X - Z) + m(X - T^{-n} Z) \\
&= 2m(X - Z) \leqslant 2\varepsilon.
\end{aligned}$$

Finally,

$$\left| e^{-\sigma^2 t^2/2} - m(Z)^2 e^{-\sigma^2 t^2/2} \right| \leqslant 1 - m(Z)^2 \leqslant 2\varepsilon.$$

The above three inequalities conclude the proof of Theorem 1.1. □

4.4. Proof of Theorem 1.2 on stable laws

The proof is almost the same as that of the Central Limit Theorem. We consider only the case $p \in (0,1) \cup (1,2)$, the case $p = 2$ being analogous using [AD98].

We work on $Z = Z_q$, the union of the q first floors of the Kakutani tower from Section 4.1. Setting $f_Z(x) = \sum_0^{\varphi_Z(x)-1} f(x)$, we show as in the proof of the central limit theorem that $\sum \varphi_Z[d] m[d] D_\tau f_Z(d)^\eta < \infty$ and $\sum m[d] D_\tau f_Z(d) < \infty$. To apply Theorem 3.8, we have to estimate $m[f_Z > x]$ and $m[f_Z < -x]$ for large x.

On the basis, c_1 is characterized by $m[f_Y > x] = (c_1 + o(1))x^{-p}L(x)$. We will show that we also have $m[f_Z > x] = (c_1 + o(1))x^{-p}L(x)$. Let Δ be the highest floor in Z, and Λ its projection on the basis Y. Then f_Z is bounded on $Z - \Delta$ and f_Y is bounded on $Y - \Lambda$, whence it suffices to consider Δ and Λ. Let $x \in \Lambda$ and $y = T^{q-1}x$ its image in Δ. Then $f_Z(y) = f_Y(x) + \sum_{i=0}^{q-2} f(T^i x)$. As the sum is bounded by $C = q \left\| f_{|Z} \right\|_\infty$, we get $f_Z(y) = f_Y(x) \pm C$. Thus, for x large enough, $m[f_Y > x - C] \geqslant m[f_Z > x] \geqslant m[f_Y > x + C]$. As $m[f_Y > x] = (c_1 + o(1))x^{-p}L(x)$, we obtain the same estimate for $m[f_Z > x]$, using the slow variation of L. In the same way, $m[f_Z < -x] = (c_2 + o(1))x^{-p}L(x)$.

Using Theorem 3.8, we obtain that $\lambda(t) = 1 - a|t|^p L(1/|t|) + ib \operatorname{sgn}(t)|t|^p L(1/|t|) + o(|t|^p L(1/|t|))$, for $a = \frac{c}{m(Z)}$ and $b = \frac{c}{m(Z)}\beta \tan\left(\frac{p\pi}{2}\right)$.

We use then the remark following Theorem 2.1, with $M(t) = t^p L(1/t)$, to get

$$\left\| T_{n,t} - \frac{1}{\mu}\left(1 - c|t|^p L(1/|t|) + ic\beta \tan\left(\frac{p\pi}{2}\right) \operatorname{sgn}(t)|t|^p L(1/|t|)\right)^n P \right\| \\ \leqslant \varepsilon(t) + \delta(n).$$

If B_n is defined by $B_n^p = nL(B_n)$, we have

$$n\frac{|t|^p}{B_n^p} L\left(\frac{B_n}{|t|}\right) = |t|^p \frac{L(B_n/|t|)}{L(B_n)} \to |t|^p$$

since L is slowly varying. Thus,

$$T_{n,t/B_n} \to \frac{1}{\mu} e^{-c|t|^p + ic\beta \tan\left(\frac{p\pi}{2}\right) \operatorname{sgn}(t)|t|^p} P$$

Applying this result to the function 1 and integrating, we get

$$\int_{T^{-n}Z \cap Z} e^{itS_n f/B_n} \, \mathrm{d}m \to m(Z)^2 e^{-c|t|^p (1 - i\beta \operatorname{sgn}(t) \tan(\frac{p\pi}{2}))},$$

the exponential being the characteristic function of the stable law $X_{p,c,\beta}$.

We then let $q \to \infty$ and conclude the proof as for the Central Limit Theorem. □

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